III Appendix: λ-rings

Here is a survey of λ-rings following nLab. One can think this is an expanding of it.

Motivation from representation theory

Typically one can form direct sums of representations of some algebraic structure. The decategorification to isomorphism classes of such representations then inherits the structure of a commutative monoid. But nobody likes commutative monoids: we all have an urge to subtract. So, we throw in formal negatives and get an abelian group – the ***Grothendieck group***.

In many situations, we can also take tensor products of representations. Then the Grothendieck group becomes something better than an abelian group. It becomes a ring: the ***representation ring***. Moreover, in many situations we can also take exterior and symmetric powers of representations; indeed, we can often apply any *Young diagram* to a representation and get a new representation. Then the representation ring becomes something better than a ring: it becomes a λ-*ring*.

More generally, the Grothendieck group of a monoidal abelian category is always a ring, called a ***Grothendieck ring***. If we start with a braided monoidal abelian category, this ring is commutative. But if we start with a symmetric monoidal abelian category, we get a λ-*ring*.

So, λ-*rings* are all about getting the most for your money when you decategorify a symmetric monoidal abelian category – for example the category of representations of a group, or the category of vector bundles on a topological space.

Unsurprisingly, the Grothendieck group of the free symmetric monoidal abelian category on one generator is the free λ-ring on one generator. This category is very important in representation theory. Objects in this category are called ***Schur functors***, because for obvious reasons they act as functors on any symmetric monoidal abelian category. The irreducible objects in this category are called "***Young diagrams***". Elements of the free λ-ring on one generator are called ***symmetric functions***.

1 Preliminaries

A.1 (pre-λ-ring) A ***pre-λ-structure*** on a commutative ring R is a sequence of maps

$$\lambda^n \colon R \to R$$

for each $n \geqslant 0$ satisfying the relations for all $x \in R$:

$$\lambda^0(x) = 1, \quad \lambda^1(x) = x.$$

and for all integers $n \geqslant 0$, and $x, y \in R$,

$$\lambda^n(x+y) = \sum_{k=0}^{n} \lambda^k(x)\lambda^{n-k}(y).$$

A ***pre-λ-ring*** is a commutative ring equipped with a pre-λ-structure on it.

For a pre-λ-ring (R, λ), we define the formal power series

$$\lambda_t(x) = \sum_{n=0}^{\infty} \lambda^n(x)t^n = 1 + xt + \text{higher terms},$$

called the ***generating function*** of the pre-λ-structure $\lambda^\bullet$. One can see the map $x \mapsto \lambda_t(x)$ induces a homomorphism from the additive group of R into the principal unit group $1 + tR[[t]]$ of power series over R. Conversely, any such a homomorphism such that $\lambda_t(x) = 1 + xt+$ higher terms gives rise to a pre-λ-structure on R.

When $\lambda_t(x)$ is a polynomial of degree n, we say x is of ***degree*** n, write $\deg_\lambda(x) = n$. From $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$, one has

$$\deg_\lambda(x+y) \leqslant \deg_\lambda(x) + \deg_\lambda(y).$$

To further define the notion of λ-rings, we need some knowledge from symmetric polynomials.

Recall (Symmetric polynomial): The n-th symmetric group $\mathfrak{S}_n$ acts on $R[x_1, \cdots, x_n]$ by

$$\sigma.f(x_1, \cdots, x_n) = f(x_{\sigma(1)}, \cdots, x_{\sigma(n)}).$$

A polynomial is called ***symmetric*** if $\sigma.f = f$ for all $\sigma \in \mathfrak{S}_n$. The ***weight*** of a monomial $x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}$ is $\nu_1 + 2\nu_2 + \cdots + n\nu_n$. The weight of a polynomial is the maximum of the weights of its monomials. One has

A.2 Lemma (Fundamental Theorem of Symmetric Polynomials) *Let f be a symmetric polynomial of indeterminates $x_1, x_2, \cdots, x_n$ and of degree d. Then there exists a polynomial $g(X_1, \cdots, X_n)$ of weight $\leqslant d$ such that*

$$f(t) = g(s_1, s_2, \cdots, s_n).$$

Here s_i is the i-th elementary symmetric polynomial of $x_1, x_2, \cdots, x_n$.

Proof: Refer [Lan02, IV, Theorem 6.1]. □

For any positive integers m and n, consider the polynomial

$$g(t) = \prod_{1 \leqslant i_1 < \cdots < i_m \leqslant nm} (1 + x_{i_1} \cdots x_{i_m} t).$$

One can see that the coefficient of each t^j in $g(t)$ is a symmetric polynomial of $x_1, \cdots, x_{nm}$ over $\mathbb{Z}$. Specially, the coefficient of t^n is such a symmetric polynomial. Hence by Lemma A.2, there exists a polynomial $P_{n,m}$ in nm indeterminates with integer coefficients such that the coefficient of t^n in $g(t)$ is $P_{n,m}(s_1, \cdots, s_{nm})$.

Recall (Symmetric polynomials in two sets of variables): The result of Lemma A.2 can be generalized to multi sets of variables. A polynomial $f(x; y) \in R[x_1, \cdots, x_n; y_1, \cdots, y_m]$ is said to be ***symmetric*** if

$$f(x; y) = f(x_{\sigma(1)}, \cdots, x_{\sigma(n)}; y_{\tau(1)}, \cdots, y_{\tau(m)}), \quad \forall \sigma \in \mathfrak{S}_n, \tau \in \mathfrak{S}_m.$$

Let s_i, σ_i denote the i-th elementary symmetric polynomial of $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_m$ respectively, one has

A.3 Corollary *Every symmetric polynomial $f(x; y) \in R[x_1, \cdots, x_n; y_1, \cdots, y_m]$ can be written uniquely as a polynomial of $s_1, s_2, \cdots, s_n; \sigma_1, \sigma_2, \cdots, \sigma_m$ with coefficients in R.*

For any positive integer n, consider the polynomial

$$h(t) = \prod_{i,j=1}^{n} (1 + x_i y_j t).$$

One can see that the coefficient of each t^k in h is a symmetric polynomial in $\mathbb{Z}[x_1, \cdots, x_n; y_1, \cdots, y_n]$. Specially, the coefficient of t^n is such a symmetric polynomial. Hence by Corollary A.3, there exists a polynomial $P_n \in \mathbb{Z}[X_1, \cdots, X_n; Y_1, \cdots, Y_n]$ such that the coefficient of t^n in $h(t)$ is $P_n(s_1, s_2, \cdots, s_n; \sigma_1, \sigma_2, \cdots, \sigma_n)$.

2 The "orthodox" definition of λ-rings

Now, we can give the "orthodox" definition of λ-rings.

A.4 A ***(special) λ-structure*** on a commutative ring R is a pre-λ-structure $\lambda^\bullet$ satisfying

(i) $\lambda^n(1) = 0$, for $n \geqslant 1$;

(ii) $\lambda^n(xy) = P_n(\lambda^1(x), \ldots, \lambda^n(x); \lambda^1(y), \ldots, \lambda^n(y))$ for all $x, y \in R$;

(iii) $\lambda^m(\lambda^n(x)) := P_{m,n}(\lambda^1(x), \ldots, \lambda^{mn}(x))$, for all $x \in R$.

Here, the polynomials $P_n, P_{m,n}$ have been given in the above recalls on symmetric polynomials. A commutative ring equipped with a λ-structure on it is called a ***(special) λ-ring***.

The following is a useful lemma in the theory of λ-rings.

A.5 Lemma *Let (R, λ) be a λ-ring, and let x and y be elements in R. If both x and y are of degree 1, then so is xy.*

Proof: Consider the polynomial for $n \geqslant 2$

$$h(t) = \prod_{i,j=1}^{n} (1 + x_i y_j t).$$

set $x_2, \cdots, x_n, y_2, \cdots, y_n$ to 0. Then one can see the coefficient of t^n is 0. This implies that

$$P_n(s_1, 0, 0, \cdots; \sigma_1, 0, 0, \cdots) = 0.$$

Thus

$$\lambda^n(xy) = P_n(\lambda^1(x), 0, 0, \cdots; \lambda^1(y), 0, 0, \cdots) = 0,$$

which shows $\deg_\lambda(xy) = 1$. $\square$

The usual properties and constructions of rings extend in an obvious way to λ-rings.

A.6 Let R, S be two (pre-)λ-rings, then

- a ***(pre-)λ-homomorphism*** is a ring homomorphism $f\colon R \to S$ such that $f \circ \lambda^n = \lambda^n \circ f$ for all n;
- a ***(pre-)λ-ideal*** of R is an ideal I of R such that $\lambda^n(x) \in I$ for $n \geqslant 1$ and $x \in I$;
- a ***(pre-)λ-subring*** of R is a subring R' of R such that $\lambda^n(x) \in R'$ for all n and $x \in R'$.

A.7 Proposition *Let $f\colon R \to S$ be a λ-homomorphism between λ-rings, then*

(i) The kernel of f is a λ-ideal in R;

(ii) The image of f is a λ-subring in S;

(iii) The quotient R/I of R by a λ-ideal I is naturally a λ-ring, and the projection map $R \to R/I$ is a λ-homomorphism.

(iv) An ideal I of R is a λ-ideal if and only if $\lambda^n(z_j) \in I$ holds for $n \geqslant 1$ and every element of a set of generators $\{z_j\}$ of I;

(v) The direct product $R \times S$ is a λ-ring, in which

$$\lambda_t(r,0) = (1,1) + \sum_{n=1}^{\infty} (\lambda^n(r), 0)t^n,$$
$$\lambda_t(0,s) = (1,1) + \sum_{n=1}^{\infty} (0, \lambda^n(s))t^n.$$

(vi) The tensor product $R \otimes S$ is a λ-ring, in which

$$\lambda^n(r \otimes 1) = \lambda^n(r) \otimes 1,$$
$$\lambda^n(1 \otimes s) = 1 \otimes \lambda^n(s).$$

(vii) If $\{R_i\}$ is an inverse system of λ-rings, then the inverse limit $\varprojlim R_i$ is naturally a λ-ring.

Given a λ-ring R, it is convenient to know how to put a λ-structure on the polynomial ring $R[x]$ or the power series ring $R[[x]]$. This is illustrated by the following result.

A.8 Proposition *Let R be a λ-ring. Then there exists a unique λ-structure on the polynomial ring $R[x]$ such that $\lambda_t(x) = 1 + xt$, i.e., $\deg_\lambda(x) = 1$. If R is augmented, then so is $R[x]$ with $\varepsilon(x) = 0$ or 1. The same holds for the power series ring $R[[x]]$.*

Proof: It suffices to extend $\lambda_t(x) = 1 + xt$ to all of $R[x]$ or $R[[x]]$. First, λ^n can be extended to the powers x^k using the axiom for $\lambda^n(xy)$ repeatedly. Then, use this axiom again we extend λ^n to monomials of the form ax^k. Finally, we extend λ^n to polynomials (or formal power series) $f = \sum a_k x^k$ by setting

$$\lambda_t(f) = \prod \lambda_t(a_k x^k).$$

The same reasoning shows that the λ-structure is uniquely determined by the condition $\lambda_t(x) = 1 + xt$. The assertion about the augmentation is clear. □

Apply Proposition A.8 repeatedly, one obtains

A.9 Corollary *Let R be a λ-ring. Then there exists a unique λ-structure on the polynomial ring $R[x_1, \cdots, x_n]$ such that $\lambda_t(x_i) = 1 + x_i t$, i.e., $\deg_\lambda(x_i) = 1$ for $i = 1, 2, \cdots, n$. If R is augmented, then so is $R[x_1, \cdots, x_n]$ with $\varepsilon(x_i) = 0$ or 1 for each i. The same holds for the power series ring $R[[x_1, \cdots, x_n]]$.*

3 Examples

The followings are typical examples.

A.10 Example (The initial $\boldsymbol{\lambda}$-ring) The simplest λ-ring is the ring of integers $\mathbb{Z}$ with the λ-structure

$$\lambda^n(m) = \binom{m}{n}.$$

Or, in other words, its λ-structure is given by the generating function

$$\lambda_t(m) = (1+t)^m.$$

In fact, this is the unique λ-structure on $\mathbb{Z}$. Indeed, in any λ-ring R, one has $\lambda_t(1) = 1 + t$, hence

$$\lambda_t(m) = (1+t)^m, \quad \forall m \in \mathbb{Z}.$$

This also shows that Every λ-ring has characteristic 0 and contains a λ-subring that is isomorphic to $\mathbb{Z}$ as λ-rings. If conversely R comes equipped with a λ-homomorphism $\varepsilon\colon R \to \mathbb{Z}$, then we say R is an ***augmented λ-ring*** and ε is an ***augmentation***.

A.11 Proposition *A λ-ring R is augmented if and only if there exists a λ-ideal I such that $R = \mathbb{Z} \oplus I$ as an abelian group.*

Proof: Taking I to be $\ker \varepsilon$. □

A.12 Example (The cofree $\boldsymbol{\lambda}$-ring $\boldsymbol{\Lambda}$) It is not obvious that the principal unit group $1 + tR[[t]]$ of formal power series is a λ-ring:

- *addition* on $1 + tR[[t]]$ is defined to be multiplication of power series,
- *multiplication* is defined by

  $$(1 + \sum_{n=1}^{\infty} r_n t^n) * (1 + \sum_{n=1}^{\infty} s_n t^n) := 1 + \sum_{n=1}^{\infty} P_n(r_1, \dots, r_n; s_1, \dots, s_n) t^n,$$

  with identity $1 + t$;
- the λ-*structure* is defined by

  $$\lambda^n(1 + \sum_{m=1}^{\infty} r_m t^m) = 1 + \sum_{m=1}^{\infty} P_{m,n}(r_1, \dots, r_{mn}) t^m.$$

One should notice that to verify properties about the multiplication, it suffices to prove equalities of the integral polynomials P_n, which is enough to verify the equalities in the case the indeterminates are taking to be elementary symmetric polynomials. For instance, the associativity of multiplication follows from considering the coefficient of t^n of both sides of the equalities

$$\prod(1+x_iy_jt)*\prod(1+z_kt)=\prod(1+x_iy_jz_kt)=\prod(1+x_it)*\prod(1+y_jz_kt).$$

In similar way, one can show that $1+tR[[t]]$ is a λ-ring. This λ-ring is denoted by $\Lambda(R)$.

The following proposition gives another definition of λ-rings.

A.13 Proposition *A pre-λ-ring is a λ-ring if and only if*

$$\lambda_t\colon R\longrightarrow\Lambda(R)=1+tR[[t]]$$

is a pre-λ-homomorphism. If this is the case, λ_t is a split monomorphism in both categories **CRing** *and* λ **Ring**.

Proof: We have seen λ_t is always a homomorphism of additive groups. Then λ_t is a ring homomorphism if and only if

$$\lambda_t(1)=1+t,\quad \lambda_t(xy)=\lambda_t(x)*\lambda_t(y).$$

Furthermore, λ_t is a pre-λ-homomorphism if and only if

$$\lambda_t(\lambda^n(x))=\lambda^n(\lambda_t(x)).$$

Expanding all the three conditions, one obtain (i)-(iii) in A.4.

To show that λ_t is a split monomorphism, we give its retraction by

$$\begin{aligned}\alpha_1\colon\Lambda(R)&\longrightarrow R\\ 1+\sum_{n=1}^{\infty}r_nt^n&\longmapsto r_1.\end{aligned}$$

One can see that this is the retraction of λ_t in **Ab**. Moreover, α_1 is a ring homomorphism, thus λ_t is a split monomorphism in **CRing**. To show α_1 is a λ-homomorphism when R is a λ-ring, one only need to prove the corresponding equalities of symmetric polynomials. In our case, it is

$$P_{1,n}(s_1,\cdots,s_n)=s_n,$$

where the symmetric polynomials are in n indeterminates. Consider the polynomial $g(t)$ after Lemma A.2, this is obvious. Thus λ_t is a split monomorphism in λ **Ring**. $\square$

The following theorem gives the universal property of the functor Λ.

A.14 Theorem *The functor* $\Lambda\colon \mathbf{CRing} \to \lambda\,\mathbf{Ring}$ *is the right adjoint of the forgetful functor* $F\colon \lambda\,\mathbf{Ring} \to \mathbf{CRing}$.

Proof: Given λ-ring R and a ring S, we need to show

$$\mathrm{Hom}_{\mathbf{CRing}}(R, S) \cong \mathrm{Hom}_{\lambda\,\mathbf{Ring}}(R, \Lambda(S)).$$

Any ring homomorphism $f\colon R \to S$ induces a canonical λ-homomorphism $\Lambda(f)\colon \Lambda(R) \to \Lambda(S)$ by mapping t to t. By Proposition A.13, $\lambda_t\colon R \to \Lambda(R)$ is a λ-homomorphism, thus we get a canonical λ-homomorphism $\Lambda(f) \circ \lambda_t$. Note that the following diagram is commutative.

$$\begin{tikzcd}
\Lambda(R) \arrow[r, "\Lambda(f)"] & \Lambda(S) \arrow[d, "\alpha_1"] \\
R \arrow[u, "\lambda_t"] \arrow[r, "f"] & S
\end{tikzcd}$$

We now show that $f \mapsto \Lambda(f) \circ \lambda_t$ is injective. Considering two ring homomorphisms f and g, if $\Lambda(f) \circ \lambda_t = \Lambda(g) \circ \lambda_t$, then, $\alpha_1 \circ \Lambda(f) \circ \lambda_t = \alpha_1 \circ \Lambda(g) \circ \lambda_t$, i.e., $f = g$.

Finally, we show that $f \mapsto \Lambda(f) \circ \lambda_t$ is surjective. Let $\widehat{f}\colon R \to \Lambda(S)$ be a λ-homomorphism, then $\alpha_1 \circ \widehat{f}\colon R \to S$ is a ring homomorphism. Apply the mapping $f \mapsto \Lambda(f) \circ \lambda_t$ on it, we have

$$\begin{aligned}
\Lambda(\alpha_1 \circ \widehat{f}) \circ \lambda_t(x) &= \Lambda(\alpha_1 \circ \widehat{f})(1 + \sum_{n=1}^{\infty} \lambda^n(x) t^n) \\
&= 1 + \sum_{n=1}^{\infty} (\alpha_1 \circ \widehat{f})(\lambda^n(x)) t^n \\
&= 1 + \sum_{n=1}^{\infty} \alpha_1(\lambda^n(\widehat{f}(x))) t^n.
\end{aligned}$$

Let $\widehat{f}(x) = 1 + \sum_{n=1}^{\infty} f_n(x) t^n$, then we have

$$\begin{aligned}
\alpha_1(\lambda^n(\widehat{f}(x))) &= \alpha_1 \left(1 + \sum_{m=1}^{\infty} P_{m,n}(f_1(x), \ldots, f_{mn}(x)) t^m \right) \\
&= P_{1,n}\left(f_1(x), \ldots, f_n(x)\right) \\
&= f_n(x).
\end{aligned}$$

Therefore

$$\Lambda(\alpha_1 \circ \widehat{f}) \circ \lambda_t(x) = 1 + \sum_{n=1}^{\infty} f_n(x) t^n = \widehat{f}(x).$$

Now, $f \mapsto \Lambda(f) \circ \lambda_t$ is bijective and the naturality is obvious, thus Λ is right adjoint to F. $\square$

Remark: Recall that a ***comonad*** on a category $\mathcal{C}$ is a *comonoid* in the category of endofunctors of $\mathcal{C}$. In other words, a ***comonad*** is a functor $T\colon \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\delta\colon T \to T^2 := T \circ T$ and $\varepsilon\colon T \to \mathrm{id}$ such that the following diagrams commute.

$$\begin{tikzcd} T^2 \arrow[r, "\delta T"] & T^3 \\ T \arrow[u, "\delta"] \arrow[r, "\delta"'] & T^2 \arrow[u, "T\delta"'] \end{tikzcd} \qquad \begin{tikzcd} T^2 \arrow[r, "\varepsilon T"] & T \\ T \arrow[u, "\delta"] \arrow[ur, equal] \arrow[r, "\delta"'] & T^2 \arrow[u, "T\varepsilon"'] \end{tikzcd}$$

A ***coalgebra*** over a comonad (T, δ, ε) on a category $\mathcal{C}$ is an object A in $\mathcal{C}$ together with a morphism $\alpha\colon A \to TA$ such that the following diagrams commute.

$$\begin{tikzcd} T(A) \arrow[r, "\delta_A"] & T^2(A) \\ A \arrow[u, "\alpha"] \arrow[r, "\alpha"'] & T(A) \arrow[u, "T(\alpha)"'] \end{tikzcd} \qquad \begin{tikzcd} TA \arrow[r, "\varepsilon_A"] & A \\ A \arrow[u, "\alpha"] \arrow[ur, equal] & \end{tikzcd}$$

The category of *coalgebras* over a comonad is called its ***co-Eilenberg-Moore category*** and denoted by $T\,\mathbf{Alg}$.

Given a pair $L \dashv R\colon \mathcal{C} \to \mathcal{D}$ of *adjoint functors*, with *counit* ε and *unit* η, then there is a natural comonad $T = (L \circ R, L\eta R, \varepsilon)$ and a natural *comparison functor*

$$\begin{aligned} K\colon \mathcal{C} &\longrightarrow T\,\mathbf{Alg} \\ A &\longmapsto (L(A), L(A) \overset{L(\eta_A)}{\longrightarrow} LRL(A)). \end{aligned}$$

The adjunction $L \dashv R$ is said to be a ***comonadic adjunction*** if K is an equivalence of categories.

By, Theorem A.14, $(F \circ \Lambda, \lambda_t, \alpha_1)$ is a comonad on $\mathbf{CRing}$. Moreover, from the proof, one can see that $F \dashv \Lambda$ is a comonadic adjunction.

A.15 Example ($\boldsymbol{\Omega}$) Let $\Omega_n(R)$ denote the ring $R[x_1, \cdots, x_n]$ of polynomials in n indeterminates over R. For every n there is a surjective ring homomorphism

$$\rho_n\colon \Omega_{n+1}(R) \longrightarrow \Omega_n(R),$$

defined by setting the last indeterminate x_{n+1} to 0.

The, we have an inverse system

$$R = \Omega_0(R) \overset{\rho_0}{\longleftarrow} \Omega_1(R) \overset{\rho_1}{\longleftarrow} \Omega_2(R) \longleftarrow \cdots$$

Its inverse limit is denoted by $\Omega(R)$.

Let ϕ_n denote the structure map $\Omega(R) \to \Omega_n(R)$. An element of $\Omega(R)$ is then a power series f in infinite indeterminates $x_1, x_2, \cdots$ such that for any $n \geqslant 1$,

$$\phi_n(f) = f(x_1, \cdots, x_n, 0, 0, \cdots)$$

is a polynomial of $x_1, \cdots, x_n$.

Let $s_k(x_1, \cdots, x_n)$ denote the k-th elementary symmetric polynomial of $x_1, \cdots, x_n$. Since

$$\phi_n(s_k(x_1, \cdots, x_{n+1})) = s_k(x_1, \cdots, x_n, 0) = s_k(x_1, \cdots, x_n),$$

the sequence $\{s_k(x_1, \cdots, x_n)\}_{n \geqslant 0}$ determines an element

$$s_k := \varprojlim_n s_k(x_1, \cdots, x_n)$$

in $\Omega(R)$, called the k-th ***elementary symmetric function*** over R. The sub R-algebra $\mho(R)$ of $\Omega(R)$ generated by all the elementary symmetric functions is called the ***ring of symmetric functions*** over R.

Now let R be a λ-ring. By Corollary A.9, $\Omega_n(R)$ are λ-rings. Straight forward calculation shows ρ_n are λ-homomorphisms. Therefore, by (vii) of Proposition A.7, $\Omega(R)$ is a λ-ring and ϕ_n are λ-homomorphisms.

A.16 Lemma *$\lambda^n(s_1) = s_n$ for all $n \geqslant 1$.*

Proof: For any $k \geqslant n$, we have

$$\begin{aligned}
\phi_k \lambda^n(s_1) &= \lambda^n(\phi_k(s_1)) \\
&= \lambda^n(x_1 + \cdots + x_k) \\
&= \sum_{1 \leqslant i_1 < \cdots < i_n \leqslant k} x_{i_1} \cdots x_{i_n} \\
&= \phi_k(s_n).
\end{aligned}$$

Therefore $\lambda^n(s_1) = s_n$. □

In the case $R = \mathbb{Z}$, all the λ-structures are uniquely determined. The λ-rings $\Omega(\mathbb{Z}), \mho(\mathbb{Z})$ are simply denoted by Ω and $\mho$. The above lemma implies that $\mho$ is the smallest λ-subring of Ω which contains s_1.

A.17 Example (The free $\boldsymbol{\lambda}$-ring $\mho$) We now give the universal property of $\mho$: it is the *free λ-ring on one generator* s_1. That means for any λ-ring R and an element $x \in R$, there exists a unique λ-homomorphism $u_x : \mho \to R$ such that $x = u_x(s_1)$. This follows immediately from Lemma A.16 and the fact that $\mho$ is generated by $\{s_n\}_{n \geqslant 1}$.

There is another approach to the ***ring of symmetric functions*** $\mho(R)$ as follows.

First, let $\mho_n(R)$ denote the ring $\Omega_n(R)^{\mathfrak{S}_n}$ of symmetric polynomials in n indeterminates over R. It is not difficult to see that it is the smallest λ-sub-R-algebra of $\Omega_n(R)$ containing s_1. For every n the surjective λ-homomorphism $\rho_n : \Omega_{n+1}(R) \to \Omega_n(R)$ induces the surjective λ-homomorphism

$$\rho_n : \mho_{n+1}(R) \longrightarrow \mho_n(R).$$

Then, we have

$$\mho(R) = \varprojlim_{\rho_n} \mho_n(R).$$

On the other hand, we can interpret this inverse limit as a direct limit as follows. First note that the nonzero elements of the kernel of ρ_n have degree at least $n+1$ (in fact, they are multiples of $x_1 x_2 \cdots x_{n+1}$). Therefore the restriction of ρ_n to elements of degree at most n is a bijective linear map, and we have

$$\rho_n(s_k(x_1, \cdots, x_{n+1})) = s_k(x_1, \cdots, x_n),$$

for all $k \leqslant n$. Thus, by Lemma A.2, the inverse of this restriction can be extended uniquely to a ring homomorphism

$$\varphi_n \colon \mho_n(R) \longrightarrow \mho_{n+1}(R).$$

Since the images $\varphi_n(s_k(x_1, \cdots, x_n)) = s_k(x_1, \cdots, x_{n+1})$ for $k = 1, \cdots, n$ are still algebraically independent over R, the homomorphisms φ_n are injective. Applying φ_n to a polynomial amounts to adding all monomials containing the new indeterminate obtained by symmetry from monomials already present.

The ring $\mho(R)$ is then the direct limit

$$\mho(R) := \varinjlim_{\varphi_n} \mho_n(R).$$

As a direct limit of monomorphisms, an element F of $\mho(R)$ can be uniquely determined by an element, denoted by $F(x_1, \cdots, x_n)$, of $\mho_n(R)$ for enough large n. Note that φ_n are compatible with the total degree of polynomials, hence one can define the ***degree*** of F as the total degree of $F(x_1, \cdots, x_n)$. This gives $\mho(R)$ the structure of a graded ring

$$\mho(R) = \bigoplus_{n=1}^{\infty} \mho(R)_n.$$

The following are fundamental examples of symmetric functions.

- The ***monomial symmetric functions*** m_ν. Let $\nu = (\nu_1, \nu_2, \cdots)$ be a sequence of non-negative integers, only finitely many of which are non-zero. Then we can consider the monomial defined by ν:

  $$x^\nu := x_1^{\nu_1} x_2^{\nu_2} \cdots .$$

  Then m_ν is the symmetric function determined by x^ν, i.e. the sum of all monomials obtained from x^ν by symmetry. A formal definition of this is

  $$m_\nu = \sum_{\sigma \in \mathfrak{S}} x^{\sigma\nu}.$$

Since any symmetric function containing any of the monomials of some m_ν must contain all of them with the same coefficient, the distinct monomial symmetric functions therefore form a basis of $\mho(R)$ as graded R-module.

- The ***elementary symmetric functions*** s_k, for any natural number k. They are $s_k = m_\nu$ where $x^\nu = \prod_{i=1}^k x_i$.
- The ***power sum symmetric functions*** p_k, for any positive integer k. They are $p_k = m_\nu$, where $\nu = (k, 0, 0, \cdots)$.
- The ***complete homogeneous symmetric functions*** h_k, for any natural number k. They are the sum of all monomial symmetric functions m_ν where ν varies over all partitions of k.
- The ***Schur functions*** $s_\nu := m_\nu$ for any partition ν. Note that the set of Schur functions also form a basis of $\mho(R)$ as graded R-module.

The following are examples of expressions, which provide symmetric polynomials for all n but do not define symmetric functions.

- $p_0(x_1, \cdots, x_n) = \sum_{k=1}^n x_k^0 = n$.
- The "discriminant" $d(x_1, \cdots, x_n) = \prod\limits_{1 \leqslant i < j \leqslant n} (x_i - x_j)^2$.

Important properties of $\mho(R)$ include the following.

A.18 Theorem (Fundamental Theorem of Symmetric Functions) *There is a canonical isomorphism of graded R-algebras:*

$$\begin{aligned} \mho(R) &\longrightarrow R[y_1, y_2, \cdots] \\ s_k &\longmapsto y_k. \end{aligned}$$

Proof: Follows from Lemma A.2. □

A.19 Corollary *Theorem A.18 implies the following.*

(i) The subring of $\mho(R)$ generated by its elements of degree $\leqslant n$ is isomorphic to the ring $\mho_n(R)$.

(ii) For every $n > 0$, $\mho(R)_n$ modulo its intersection with the subring generated by its elements of degree $< n$, is free of rank 1, and is generated by the image of s_n.

(iii) The **Hilbert–Poincaré series** *of $\mho(R)$ is*

$$P_{\mho(R)}(t) := \sum_{n=0}^{\infty} \dim_R(\mho(R)_n) t^n = \prod_{k=1}^{\infty} \frac{1}{1 - t^k},$$

the generating function of the integer partitions.

Proof: (i),(ii) are obvious. As for (iii), just notice that the Hilbert–Poincaré series is additive, hence

$$P_{R[y_1,y_2,\cdots]}(t) = \prod_{k=1}^{\infty} P_{R[y_k]}(t).$$

In our case y_k are algebraic independent and $\deg(y_k) = \deg(s_k) = k$. Note that a the only monomials in $R[y_k]$ are those of the form y_k^n, thus we have

$$P_{R[y_k]}(t) = \sum_{n=0}^{\infty} t^{kn} = \frac{1}{1-t^k}.$$

The statement then follows. Note that $P_{\mho(R)}(t)$ is the generating function of the integer partitions since the set of Schur functions form a basis of $\mho(R)$ as graded R-module. $\square$

Remark: Theorem A.18 and hence Corollary A.19 still hold if one replace the family $\{s_n\}_{n>0}$ by any family $\{f_n\}_{n>0}$ of symmetric functions satisfying $\deg(f_n) = n$ and the above (ii). The set of complete homogeneous symmetric functions $\{h_n\}_{n>0}$, for instance, is such a family.

A.20 Corollary *There is an involutory automorphism ω of $\mho(R)$ such that*

(i) it interchanges the elementary symmetric functions s_k and the complete homogeneous symmetric function h_k for all k;

(ii) it sends each power sum symmetric function p_k to $(-1)^{k-1}p_k$;

(iii) it permutes the Schur functions among each other: interchanging s_ν and s_{ν^t} where ν^t is the transpose partition of ν.

Proof: The composite $s_k \mapsto y_k \mapsto h_k$ gives the automorphism ω. To prove the properties of ω, one only need to show the following identities.

- The *symmetry between elementary and complete homogeneous symmetric functions*:

$$\sum_{i=0}^{k}(-1)^i s_i h_{k-i} = 0 = \sum_{i=0}^{k}(-1)^i h_i s_{k-i}, \quad \forall k > 0.$$

- The *Newton identities*:

$$ks_k = \sum_{i=1}^{k}(-1)^{i-1} p_i s_{k-i}, \quad \forall k > 0,$$

$$kh_k = \sum_{i=1}^{k} p_i h_{k-i}, \quad \forall k > 0.$$

They can be proved by, for instance, using generating functions. □

A.21 Remark The generating functions.

- for the elementary symmetric functions:

$$S(t) = \sum_{k=0}^{\infty} s_k(x)t^k = \prod_{i=1}^{\infty}(1 + x_i t).$$

- for complete homogeneous symmetric functions

$$H(t) = \sum_{k=0}^{\infty} h_k(x)t^k = \prod_{i=1}^{\infty}\left(\sum_{k=0}^{\infty}(x_i t)^k\right) = \prod_{i=1}^{\infty}\frac{1}{1 - x_i t}.$$

- for the power sum symmetric functions

$$P(t) = \sum_{k=1}^{\infty} p_k(x)t^k = \sum_{k,i=1}^{\infty}(x_i t)^k = \sum_{i=1}^{\infty}\frac{x_i t}{1 - x_i t}.$$

One has

$$P(-t) = t\,\frac{\mathrm{d}}{\mathrm{d}\,\mathrm{t}}\log\left(S(t)\right) = t\frac{S'(t)}{S(t)}, \qquad P(t) = t\,\frac{\mathrm{d}}{\mathrm{d}\,\mathrm{t}}\log\left(H(t)\right) = t\frac{H'(t)}{H(t)}.$$

The generating functions are related to Hirzebruch polynomials as follows. Let $\phi(t)$ be a integral power series, then we may form the power series (we assume that $\phi(0) = 0$ in the first case, resp. that $\phi(0) = 1$ in the second):

$$ch_\phi(t) := \sum_{k=1}^{\infty}\phi(x_i t), \qquad td_\phi(t) := \prod_{k=1}^{\infty}\phi(x_i t).$$

The coefficient of t^n in $ch_\phi(t)$ (resp. $td_\phi(t)$) is a symmetric function $H^{+}_{\phi,n}$ (resp. $H^{\times}_{\phi,n}$) and is called the (n-th) additive (resp. multiplicative) ***Hirzebruch polynomial*** associated to ϕ.

A.22 Theorem *The functor* $\Lambda\colon$ **CRing** $\to \lambda\,$**Ring** *is representable by* $\mho$.

Proof: Given a ring R, we need to show

$$\Lambda(R) \cong \mathrm{Hom}_{\mathbf{CRing}}(\mho, R).$$

Consider the mapping $f \mapsto 1 + \sum_{n=1}^{\infty} f(s_n)t^n$. It is obviously bijective since $\mho \cong \mathbb{Z}[s_1, s_2, \cdots]$.

Note that this identification gives $\mathrm{Hom}_{\mathbf{CRing}}(\mho, R)$ a λ-ring structure, which is different from the obvious one. Since $\mho \cong \mathbb{Z}[s_1, s_2, \cdots]$, one can identify $\mathrm{Hom}_{\mathbf{CRing}}(\mho, R)$ with the set $\mathcal{A}(R)$ of arithmetic functions over R (cf. 3.3). □

4 Verification Principle and Splitting Principle

A.23 (Natural operation on λ-rings) A natural operation on λ-rings is a rule that assigns to each λ-ring R a function $\mu_R \colon R \to R$ such that, for any λ-homomorphism $f \colon R \to S$, the following square commutes.

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\mu_R"'] & S \arrow[d, "\mu_S"] \\ R \arrow[r, "f"] & S \end{tikzcd}$$

The addition and multiplication of natural operations on λ-rings are:

$$(\mu + \nu)_R = \mu_R + \nu_R,$$
$$(\mu\nu)_R = \mu_R \nu_R.$$

Then, one can verify that the set of all natural operations on λ-rings is a ring, which is denoted by Op^λ.

For instance, each λ^n is a natural operation. Thus so is any polynomial of them. Then we get a homomorphism

$$\alpha \colon \mathbb{Z}[\lambda^1, \lambda^2, \lambda^3, \cdots] \longrightarrow Op^\lambda$$

by setting $\alpha(f(\lambda^1, \cdots, \lambda^n))_R(x) = f(\lambda^1(x), \cdots, \lambda^n(x))$ for an element x in a λ-ring R.

A.24 Theorem (Verification Principle) *The homomorphism α is an isomorphism. Moreover, one has*

$$\mu = f(\lambda^1, \cdots, \lambda^n)$$

if and only if this equality holds when applied to finite sums of elements of degree 1.

Proof: Obviously, α is a ring homomorphism.

α is injective. Indeed, if $\alpha(f(\lambda^1, \cdots, \lambda^n)) = 0$, then we have

$$\begin{aligned} 0 &= \alpha(f(\lambda^1, \cdots, \lambda^n))_{\mho}(s_1) \\ &= f(\lambda^1(s_1), \cdots, \lambda^n(s_1)) \\ &= f(s_1, \cdots, s_n). \end{aligned}$$

But $s_1, \cdots, s_n$ are algebraically independent, thus $f = 0$.

α is surjective. Let μ be a natural operation, then $\mu_{\mho}(s_1)$, as an element of $\mho$, is of the form $g(s_1, \cdots, s_n)$ for some integral polynomial g. We claim that $\mu = \alpha(g(\lambda^1, \cdots, \lambda^n))$.

Let x be an arbitrary element of a λ-ring R. Then, by the universal property (cf. A.17) of $\mho$, there exists a unique λ-homomorphism $u_x\colon \mho \to R$ such that $u_x(s_1) = x$. Therefore, we have

$$\begin{aligned}\mu_R(x) &= \mu_R(u_x(s_1))\\ &= u_x(\mu_\mho(s_1))\\ &= u_x(g(s_1, \cdots, s_n))\\ &= g(u_x(s_1), \cdots, u_x(s_n))\\ &= g(\lambda^1(x), \cdots, \lambda^n(x)).\end{aligned}$$

Consequently, α is an isomorphism.

As for the second statement, note that the above argument shows that to prove $\mu = f(\lambda^1, \cdots, \lambda^n)$, it suffices to show $\mu_\mho(s_1) = f(s_1, \cdots, s_n)$. This can be done by checking it on Ω, which is the inverse limit of Ω_k. Thus it suffices to show

$$\mu_{\Omega_k}(s_1(x_1, \cdots, x_k)) = f(s_1(x_1, \cdots, x_k), \cdots, s_n(x_1, \cdots, x_k))$$

for all k. Note that $s_1(x_1, \cdots, x_k) = x_1 + \cdots + x_k$ and each x_i is of degree 1, the above equalities are precisely the conditions. $\square$

Remark: We may also consider a verification principle for more than one variable. The notion of a natural map in two variables is evident and the proof of the verification principle is similar.

We have seen the important of elements of degree 1, the following is a splitting principle which allows one to write a element of finite degree in a λ-ring as a sum of elements of degree 1 in a possibly larger λ-ring.

A.25 Theorem (Splitting Principle) *Let x be an element of degree n in a λ-ring R. Then there exists a λ-ring S containing R such that*

$$x = x_1 + \cdots + x_n$$

in S, in which each x_i is of degree 1. Moreover, if R is augmented with $\varepsilon(x) = m$, then the augmentation can be extended to S in such a way that

$$\varepsilon(x_i) = \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant m, \\ 0 & \text{if } m < i \leqslant n. \end{cases}$$

To prove this theorem, we need some lemmas

A.26 Lemma *The polynomial $P_{n,m}$ in A.4 satisfies*

$$P_{n,m}(s_1, \cdots, s_{m-1}, 0, \cdots, 0) = 0.$$

Therefore every nonzero term in $P_{n,m}(s_1, \cdots, s_{nm})$ contains a factor of s_i for some $i \geqslant m$.

Proof: Recall that $P_{n,m}$ is the polynomial with integer coefficients in nm indeterminates such that the coefficient of t^n in

$$g(t) = \prod_{1 \leqslant i_1 < \cdots < i_m \leqslant nm} (1 + x_{i_1} \cdots x_{i_m} t).$$

is $P_{n,m}(s_1, \cdots, s_{nm})$, where s_i is the i-th elementary symmetric polynomial of $x_1, \cdots, x_{nm}$. Setting $x_m = x_{m+1} = \cdots = x_{nm} = 0$ in $g(t)$, it follows that the coefficient of t^n is 0. This proves the statement. $\square$

A.27 Lemma *Let x be an element of degree n in a λ-ring R. In the polynomial λ-ring $R[\xi]$ in which ξ has degree 1, the ideal I generated by the element*

$$z = \xi^n - \lambda^1(x)\xi^{n-1} + \cdots + (-1)^{n-1}\lambda^{n-1}(x)\xi + (-1)^n\lambda^n(x)$$

is a λ-ideal.

Proof: By (iv) of Proposition A.7, it suffices to show that $\lambda^m(z) \in I$ for $m \geqslant 1$. To do this, we first claim that I is also generated by $\lambda^n(x - \xi)$. Considering

$$\lambda_t(x - \xi) = (1 + \lambda^1(x)t + \cdots + \lambda^n(x)t^n)(1 + \xi t)^{-1},$$

we have for $r \geqslant 0$,

$$\lambda^{n+r}(x - \xi) = (-1)^{n+r}\xi^r(\xi^n - \lambda^1(x)\xi^{n-1} + \cdots + (-1)^n\lambda^n(x)),$$

in particular, $\lambda^n(x - \xi) = (-1)^n z$ and so I is also generated by $\lambda^n(x - \xi)$.

To prove the Lemma, it suffices to show that $\lambda^m(\lambda^n(x - \xi)) \in I$ for $m \geqslant 1$. But we have

$$\lambda^m(\lambda^n(x - \xi)) = P_{m,n}(\lambda^1(x - \xi), \cdots, \lambda^{mn}(x - \xi)).$$

By Lemma A.26, the right side of above is a sum of terms, each one containing a factor of $\lambda^{n+r}(x - \xi) = (-1)^{n+r}\xi^r z \in I$ for some $r \geqslant 0$, thus $\lambda^m(\lambda^n(x - \xi)) \in I$. $\square$

A.28 Lemma *Let x be an element of degree n in a λ-ring R. Then there exists a λ-ring $R[x_1]$ containing R such that $\deg_\lambda(x_1) = 1$ and $\deg_\lambda(x - x_1) = n - 1$. Moreover, if R is augmented with $\varepsilon(x) = m$, then the augmentation can be extended to $R[x_1]$ in such a way that*

$$\varepsilon(x_1) = \begin{cases} 1 & \textit{if } m > 0, \\ 0 & \textit{if } m = 0. \end{cases}$$

and then

$$\varepsilon(x - x_1) = \begin{cases} m - 1 & \textit{if } m > 0, \\ 0 & \textit{if } m = 0. \end{cases}$$

Proof: Let ξ and I be as in Lemma A.27, and let $R[x_1] = R[\xi]/I$. Then x_1 is of degree 1 since so is ξ. Moreover, the element $x - x_1$ is of degree $n-1$ since $\lambda^{n+r}(x-\xi) \in I$ for some $r \geqslant 0$ and $\lambda^{n-1}(x-\xi) \notin I$.

Now, let R be augmented with $\varepsilon(x) = m$. Then, by Proposition A.8, $R[\xi]$ is augmented with $\varepsilon(\xi) = 0$ or 1. Note that if $\varepsilon(I) = 0$, then $R[x_1]$ is naturally augmented by $\varepsilon(\omega + I) = \varepsilon(\omega)$ for $\omega \in R[\xi]$. If $\varepsilon(x) = m$, then

$$\varepsilon(\lambda^r(x)) = \lambda^r(\varepsilon(x)) = \binom{m}{r}.$$

Now if $m > 0$, choose $\varepsilon(\xi) = 1$. Then

$$\begin{aligned}\varepsilon(z) &= \varepsilon(\xi^n - \lambda^1(x)\xi^{n-1} + \cdots + (-1)^{n-1}\lambda^{n-1}(x)\xi + (-1)^n\lambda^n(x)) \\ &= 1 - \binom{m}{1} + \cdots + (-1)^m\binom{m}{m} = (1-1)^m = 0,\end{aligned}$$

and thus $\varepsilon(I) = 0$. Then $R[x_1]$ can be augmented by $\varepsilon(x_1) = 1$.

If $m = 0$, choose $\varepsilon(\xi) = 0$. Then $\varepsilon(z) = 0$, thus $\varepsilon(I) = 0$. Then $R[x_1]$ can be augmented by $\varepsilon(x_1) = 0$. □

Proof (of Theorem A.25): By downward induction on $n = \deg_\lambda(x)$, we obtain a λ-ring $S = R[x_1, \cdots, x_n]$ in which each x_i is of degree 1 and $\deg_\lambda(x - x_1 - \cdots - x_n) = 0$, i.e., $x = x_1 + \cdots + x_n$ as desired. □

5 γ-filtration and Adams operations

A.29 **(γ-ring)** Let R be a λ-ring. We define the corresponding ***γ-structure*** γ^n and ***Grotbendieck power series*** γ_t as

$$\gamma^n(x) = \lambda^n(x + n - 1) \quad \text{and} \quad \gamma_t(x) = \sum_{n=0}^{\infty} \gamma^n(x)t^n.$$

One can see

$$\gamma_t(x) = \lambda_{t/(1-t)}(x) \quad \text{and} \quad \lambda_s(x) = \gamma_{s/(1+s)}(x).$$

The following statements are easy to check:

(i) $\gamma^n(x+y) = \sum \gamma^i(x)\gamma^{n-i}(y)$.

(ii) $\gamma_t(1) = \frac{1}{1-t}$. More generally, $\gamma_t(m) = (1-t)^{-m}$

(iii) The map $\gamma_t \colon R \to \Lambda(R)$ is also a group homomorphism.

(iv) $\gamma^n(x) = \sum_{i=0}^{\infty} \binom{n-1}{i} \lambda^{n-i}(x)$.

(v) $\lambda^n(x) = \sum_{i=0}^{\infty} (-1)^i \binom{n-1}{i} \gamma^{n-i}(x)$.

(vi) If $\deg_\lambda(x) = 1$, then $\gamma_t(x-1) = 1 + (x-1)t$ and thus $\deg_\gamma(x-1) \leqslant 1$. More generally, $\deg_\lambda(x) = n$ implies $\deg_\gamma(x-n) \leqslant n$.

(vii) Similar to Theorem A.24, there is a **verification principle** for γ-structures.

Now, suppose R is augmented with $\varepsilon\colon R \to \mathbb{Z}$ and $I = \ker \varepsilon$. We define a decreasing filtration $F_\gamma^\bullet$ on R as follows. First, let F_γ^n be the additive subgroup of R generated by monomials $\gamma^{n_1}(a_1)\cdots\gamma^{n_r}(a_r)$ with $a_i \in I$ and $\sum n_i \geqslant n$. We have

A.30 Proposition *$F_\gamma^\bullet$ is a decreasing filtration of the λ-ring R, which means*

(i) $F_\gamma^0 = R \supset F_\gamma^1 = I \supset F_\gamma^2 \supset \cdots$;

(ii) $F_\gamma^m F_\gamma^n \subset F_\gamma^{m+n}$;

(iii) each F_γ^n is a λ-ideal of R for $n \geqslant 1$.

Proof: (i), (ii) are obvious. By Proposition A.11, $R = \mathbb{Z} \oplus F_\gamma^1$ and so F_γ^n is an ideal for $n \geqslant 1$. To show F_γ^n is a λ-ideal, it suffices to show $\lambda^m(\gamma^n(x)) \in F_\gamma^n$ for all $x \in I$.

Indeed, we have

$$\begin{aligned}\lambda^m(\gamma^n(x)) &= \lambda^m(\lambda^n(x+n-1)) \\ &= P_{m,n}(\lambda^1(x+n-1), \cdots, \lambda^{mn}(x+n-1)).\end{aligned}$$

By Lemma A.26, $\lambda^m(\gamma^n(x))$ is a sum of monomials each a multiple of some $\lambda^i(x+n-1)$ for $i \geqslant n$. It remains to show them belong to F_γ^n.

Let $s = i - n$, then we have

$$\begin{aligned}\lambda^i(x+n-1) &= \gamma^{n+s}(x-s) \\ &= \sum_{r=0}^{n+s} \gamma^{n+s-r}(x)\gamma^r(-s).\end{aligned}$$

Since $\gamma^r(-s) = 0$ for $r > s \geqslant 0$, we have

$$\lambda^i(x+n-1) = \sum_{r=0}^{s} \gamma^{n+s-r}(x)\gamma^r(-s) \in F_\gamma^n.$$

Thus F_γ^n is a λ-ideal. □

This filtration is called the ***γ-filtration*** of R. We write $\mathrm{Gr}(R)$ for the graded ring associated to the γ-filtration, i.e.

$$\mathrm{Gr}_n(R) := F_\gamma^n / F_\gamma^{n+1}.$$

For any $x \in R$, one can see that $x - \varepsilon(x) \in I$, hence $\gamma^n(x - \varepsilon(x)) \in F_\gamma^n$. We define the n-th ***algebraic Chern class*** of x to be

$$c^n(x) := \gamma^n(x - \varepsilon(x)) \bmod F_\gamma^{n+1}.$$

6 Adams operations

A.31 (Adams operation) Let R be a λ-ring. The ***Adams operations*** ψ^n on R is defined by the generating function

$$\psi_t(x) = \sum_{n=1}^{\infty} \psi^n(x) t^n$$

satisfying

$$\psi_{-t}(x) = -t \frac{\mathrm{d}}{\mathrm{d}\,\mathrm{t}} \log\left(\lambda_t(x)\right) = -t \frac{\lambda_t'(x)}{\lambda_t(x)}.$$

One can verify from Remark A.21 that

$$\psi^n(x) = \nu_n(\lambda^1(x), \cdots, \lambda^n(x)),$$

where ν_n is the integral polynomial satisfying

$$\nu_n(s_1, \cdots, s_r) = x_1^n + \cdots + x_r^n.$$

A.32 Lemma *ψ^n is a λ-homomorphism. Moreover, $\psi^m\psi^n = \psi^{mn} = \psi^n\psi^m$. Let p be a prime number, then*

$$\psi^{p^r}(x) \equiv x^{p^r} \bmod p.$$

Proof: By Theorem A.25, we can write any $x, y \in R$ as sum $\sum x_i, \sum y_j$ of elements of degree 1 in a larger λ-ring. Then we have

$$\begin{aligned}
\psi^n(x+y) &= \sum x_i^n + \sum y_j^n = \psi^n(x) + \psi^n(y);\\
\psi^n(xy) &= \psi^n(\sum x_i y_j) = \sum (x_i y_j)^n = \sum x_i^n \sum y_j^n = \psi^n(x)\psi^n(y);\\
\psi^n(\lambda^m(x)) &= \psi^n(s_m(x_1, \cdots, x_r)) = s_m(x_1^n, \cdots, x_r^n)\\
&= \lambda^m(\sum x_i^n) = \lambda^m(\psi^n(x)).
\end{aligned}$$

Therefore, ψ^n is a λ-homomorphism.

As for the rest, by Theorem A.24, it suffices to check them for the case $x = s_1$, which is obvious. $\square$

A.33 Lemma (Newton formula for Adams operations) *In any λ-ring R, the equality*

$$\psi^k(x) - \lambda^1(x)\psi^{k-1}(x) + \cdots + (-1)^{k-1}\lambda^{k-1}(x)\psi^1(x) = (-1)^{k+1}k\lambda^k(x)$$

holds for $x \in R$ and $k \geqslant 1$.

Proof: First, by the definition, we have

$$\psi_{-t}(x)\lambda_t(x) + t\lambda_t'(x) = 0.$$

Expanding it, we obtain

$$\left(\sum_{m=1}^{\infty} \psi^m(x)(-t)^m\right)\left(\sum_{n=0}^{\infty} \lambda^n(x)t^n\right) + \sum_{n=1}^{\infty} k\lambda^k(x)t^k.$$

Compare the coefficient of t^k, we obtain the required equality. $\square$

By straightforward calculation, the Newton formula implies the following formulas.

$$\psi^n(x) = \det\begin{pmatrix} \lambda^1(x) & 1 & 0 & \dots & 0 \\ 2\lambda^2(x) & \lambda^1(x) & 1 & \ddots & \vdots \\ 3\lambda^3(x) & \lambda^2(x) & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \lambda^1(x) & 1 \\ n\lambda^n(x) & \lambda^{n-1}(x) & \dots & \lambda^2(x) & \lambda^1(x) \end{pmatrix},$$

$$n!\lambda^n(x) = \det\begin{pmatrix} \psi^1(x) & 1 & 0 & \dots & 0 \\ \psi^2(x) & \psi^1(x) & 2 & \ddots & \vdots \\ \psi^3(x) & \psi^2(x) & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \psi^1(x) & n-1 \\ \psi^n(x) & \psi^{n-1}(x) & \dots & \psi^2(x) & \psi^1(x) \end{pmatrix}.$$

One can see when R is ***$\mathbb{Z}$-torsion-free***, which means there exists no nonzero $x \in R$ such that $nx = 0$ for some $n \in \mathbb{Z}$, the above formulas allow us to write λ^n in terms of $\psi^1, \cdots, \psi^n$. Therefore

A.34 Theorem *In a $\mathbb{Z}$-torsion-free λ-ring, the λ-structure can be uniquely determined by its Adams operations.*

A.35 Corollary *Let $f\colon R \to S$ be a ring homomorphism between λ-rings in which S is $\mathbb{Z}$-torsion-free. If f commutes with Adams operators, then it is a λ-homomorphism.*

A.36 Lemma *Let R be an augmented λ-ring with augmentation $\varepsilon\colon R \to \mathbb{Z}$ with $I = \ker\varepsilon$ and γ-filtration $F_\gamma^\bullet$. If $x \in F_\gamma^n$, then $\psi^k(x) - k^n x \in F_\gamma^{n+1}$.*

Proof: First, it suffices to show $\psi^k\gamma^n x - k^n\gamma^n x \in F_\gamma^{n+1}$ for $x \in I$. Since $\psi^k\gamma^n - k^n\gamma^n$ is a natural operation on R, by verification principle for γ, we may assume $x = x_1 + \cdots + x_r$ with $\deg_\gamma(x_i) = 1$. Note that for those x_i, $\psi^k(x_i) = (1 + x_i)^k - 1$. Then

$$\begin{aligned}\psi^k\gamma^n x - k^n\gamma^n x &= \gamma^n\psi^k(x_1 + \cdots + x_r) - k^n s_n(x_1, \cdots, x_r)\\ &= s_n((x_1+1)^k - 1, \cdots, (x_r+1)^k - 1) - k^n s_n(x_1, \cdots, x_r),\end{aligned}$$

which is a symmetric polynomial of degree $\geqslant n+1$. □

A.37 Corollary *The Adams operation ψ^k acts as k^n on* $\mathrm{Gr}_n(R)$.

7 The "heterodox" definition of λ-rings

Now, we give the "heterodox" definition of λ-rings.

A.38 Let A be an $\mathbb{F}_p$-algebra, then its ***Frobenius endomorphism*** is given by

$$\begin{aligned}F_p \colon A &\longrightarrow A\\ x &\longmapsto x^p.\end{aligned}$$

A ***p-typical ψ-ring*** is a commutative ring R equipped with a ***lift of Frobenius***, which means an endomorphism $F \colon R \to R$ satisfying

$$F \otimes \mathbb{F}_p = F_p.$$

A ***ψ-ring*** is a commutative ring equipped with a lift of Frobenius for each prime number p such that those lifts commute.

Lemma A.32 shows that any λ-ring R is a ψ-ring as ψ^p is a lift of Frobenius for prime number p.

Conversely, let R be a ψ-ring with lifts ψ^p and has characteristic 0. We define $\psi^1 = \mathrm{id}$,

$$\psi^{p^\nu} := \underbrace{\psi^p \circ \cdots \circ \psi^p}_{\nu},$$

and ψ^n for $n = p_1^{\nu_1} \cdots p_r^{\nu_r}$ by

$$\psi^n := \psi^{p_1^{\nu_1}} \circ \cdots \circ \psi^{p_r^{\nu_r}}.$$

Define

$$\psi_t(x) = \sum_{n=1}^{\infty} \psi^n(x) t^n,$$

and let

$$\lambda_t(x) = \exp\left(\oint \frac{\psi_{-t}(x)}{t}\,\mathrm{d}\,t\right).$$

Then, this gives a λ-structure on R whose Adams operations are ψ^n.