Algebraic Geometry

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Sheaves

Sheaves are the abstract of how to glue local information into global.

§ 1 Presheaves

In this section, we give the notion of *presheaves*: they are contravariant functors. Then, we show presheaves are colimits of the *representable ones* (Theorem 2.e). In this sense, presheaves are generalized objects. Next, we describe the *monomorphisms, epimorphisms and isomorphisms of presheaves*, defines the notions of *subpresheaves* and *image* and points out that *limits and colimits of presheaves are computed object by object*. This makes presheaves more like kind of generalized objects. Finally, we show that a functor u between categories induces a chain of adjunctions: $u_p \dashv u^p \dashv {}_pu$.

1 (Presheaves are contravariant functors) A **presheaf** on a category $\mathcal{C}$ with values in another category $\mathcal{A}$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{A}$. In the case $\mathcal{A} = \mathbf{Set}$, we simply call it a presheaf. *Morphisms* between presheaves are natural transformations.

Notations:

- $\mathbf{PSh}_{\mathcal{A}}(\mathcal{C}) = [\mathcal{C}^{\text{opp}}, \mathcal{A}]$: the category of presheaves on $\mathcal{C}$ with values in $\mathcal{A}$.
- $\mathbf{PSh}(\mathcal{C}) = \mathbf{PSh}_{\mathbf{Set}}(\mathcal{C})$: the category of presheaves on $\mathcal{C}$.
- An element $s \in \mathscr{F}(U)$ is called a **section** of $\mathscr{F}$ on U. For a morphism $f : V \to U$, we denote $\mathscr{F}(f)(s)$ by $s|_V$ or $s|_f$.

2 Example (Representable presheaves) For any object $U \in \mathcal{C}$, the **functor of points** $h_U : X \mapsto \operatorname{Hom}(X, U)$ is a presheaf. For any presheaf $\mathscr{F}$, a **representation** of it is a natural isomorphism from h_U to $\mathscr{F}$ for some object U. If this is the case, we say $\mathscr{F}$ is **representable** and is represented by U.

2.a Theorem (Yoneda lemma) *There is a canonical bijection*

$$\begin{aligned} \operatorname{Hom}_{\mathbf{PSh}(\mathcal{C})}(h_U, \mathscr{F}) &\longrightarrow \mathscr{F}(U) \\ s &\longmapsto s_U(\mathrm{id}_U) \end{aligned}$$

natural in both the object U and the presheaf $\mathscr{F}$.

2.b Corollary *The functor $\Upsilon : U \mapsto h_U$ is a* **full embedding***, which means Υ is fully faithful and injective on object. This functor is called the* **Yoneda embedding**.

2.c Corollary *A representation of a presheaf $\mathscr{F}$ on $\mathcal{C}$ is precisely the terminal object in the comma category $(\Upsilon \downarrow \mathrm{const}_{\mathscr{F}})$, which means a pair (U, u) of an object $U \in \operatorname{ob}\mathcal{C}$ and an element $u \in \mathscr{F}(U)$ satisfies the following universal property:*

For every pair (X, x) of $X \in \operatorname{ob}\mathcal{C}$ and $x \in \mathscr{F}(X)$, there is a unique morphism $f\colon X \to U$ such that $\mathscr{F}(f)(u) = x$.

Remark The comma category $(\Upsilon \downarrow \mathrm{const}_{\mathscr{F}})$ is isomorphic to the comma category $(* \downarrow \mathscr{F})$, where $*$ denote the constant functor mapping any object to the the singleton. This is indeed another expression of the Yoneda lemma. This comma category is called **the category of sections** of $\mathscr{F}$ and is denoted by $\mathcal{C}_{\mathscr{F}}$.

2.d Lemma (The set of global sections) *Let $\mathscr{F}$ be a presheaf on a small category $\mathcal{C}$, then*

$$\varprojlim_{\mathcal{C}} \mathscr{F} = \operatorname{Hom}_{\mathbf{PSh}(\mathcal{C})}(*, \mathscr{F}).$$

where $*$ denote the presheaf mapping any object in $\mathcal{C}$ to the singleton. This set is called the **set of global sections** *of $\mathscr{F}$.*

Proof: Just note that a natural transformation from $*$ to $\mathscr{F}$ is the same thing as a compatible data of the system $\{\mathscr{F}(U)\}_{U \in \operatorname{ob}\mathcal{C}}$, which is an element in the limit of $\mathscr{F}$. □

2.e Theorem (Every presheaf is a colimit of representable ones) *Let $\mathscr{F}$ be a presheaf on a small category $\mathcal{C}$, then*

$$\mathscr{F} \cong \varinjlim_{h_U \to \mathscr{F}} h_U := \varinjlim_{\mathcal{C}_{\mathscr{F}}} \Upsilon(U).$$

Proof (by Urs Schreiber): Notice that for every $\mathscr{G} \in \mathbf{PSh}(\mathcal{C})$, we have

$$\begin{aligned}
\operatorname{Hom}_{\mathbf{PSh}(\mathcal{C})}(\varinjlim_{\mathcal{C}_{\mathscr{F}}} \Upsilon(U), \mathscr{G}) &= \varprojlim_{\mathcal{C}_{\mathscr{F}}} \operatorname{Hom}_{\mathbf{PSh}(\mathcal{C})}(\Upsilon(U), \mathscr{G}) \\
&= \varprojlim_{\mathcal{C}_{\mathscr{F}}} \mathscr{G}(U) \\
&= \operatorname{Hom}_{\mathbf{PSh}(\mathcal{C}_{\mathscr{F}})}(*, \mathscr{G}),
\end{aligned}$$

where the last equality follows from Lemma 2.d.

Now, notice that an $\alpha \in \operatorname{Hom}_{\mathbf{PSh}(\mathcal{C}_{\mathscr{F}})}(*, \mathscr{G})$ gives each objects $h_U \to \mathscr{F}$ in $\mathcal{C}_{\mathscr{F}}$, which is equivalent to an element of $\mathscr{F}(U)$ by the Yoneda lemma, a map $*(U) \to \mathscr{G}(U)$, i.e. an element of $\mathscr{G}(U)$. Therefore, we have

$$\operatorname{Hom}_{\mathbf{PSh}(\mathcal{C}_{\mathscr{F}})}(*, \mathscr{G}) = \operatorname{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathscr{F}, \mathscr{G}).$$

Then the conclusion follows. □

3 ¶Remark All the above notions and results can be generalized to enriched categories, cf. [Kel05]. Note that, in this case, a *presheaf* should mean a contravariant $\mathcal{A}$-functor to $\mathcal{A}$; the *functor of points* h_U should mean the contravariant $\mathcal{A}$-functor $X \mapsto \hom(X, U)$, where the notation hom emphasize that this is the internal hom-object in $\mathcal{A}$ rather than a hom-set.

3.a Theorem (Weak Yoneda lemma) *There is a canonical bijection*

$$\begin{aligned}\mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(h_U, \mathscr{F}) &\longrightarrow \mathrm{Hom}_{\mathcal{A}}(I, \mathscr{F}(U))\\ s &\longmapsto \left(I \xrightarrow{1_U} \hom(U,U) \xrightarrow{s_U} \mathscr{F}(U)\right)\end{aligned}$$

natural in both the object U and the presheaf $\mathscr{F}$.

The strong form of Yoneda lemma reqiures the completeness of $\mathcal{A}$. Then, given a small $\mathcal{A}$-enriched category $\mathcal{C}$ and $\mathcal{A}$-enriched functors $\mathscr{F}, \mathscr{G} \colon \mathcal{C} \to \mathcal{A}$, one may construct the object of $\mathcal{A}$-*natural transformations* as an *enriched end*:

$$\mathcal{A}^{\mathcal{C}}(\mathscr{F}, \mathscr{G}) := \int_{X \in \mathrm{ob}\, \mathcal{C}} \hom_{\mathcal{A}}(\mathscr{F}(X), \mathscr{G}(X)).$$

This is the hom-object in the enriched functor category $\mathcal{A}^{\mathcal{C}}$.

3.b Theorem (Strong Yoneda lemma) *There is a $\mathcal{A}$-natural isomorphism*

$$\mathcal{A}^{\mathcal{C}^{\mathrm{opp}}}(h_U, \mathscr{F}) \xrightarrow{\sim} \mathscr{F}(U).$$

$\mathcal{A}$-natural in both the object U and the presheaf $\mathscr{F}$.

3.c Corollary (Ninja Yoneda lemma) *(following T. Leinster's comment in MathOverflow) Let $\mathscr{F}$ be a presheaf, then*

$$\mathscr{F} \cong \int_{X \in \mathrm{ob}\, \mathcal{C}} \hom_{\mathcal{A}}(\hom_{\mathcal{C}}(X, -), \mathscr{F}(X)) \cong \int^{X \in \mathrm{ob}\, \mathcal{C}} \Upsilon(X) \otimes \mathscr{F}(X).$$

4 (Injective and surjective sheaf morphisms) A morphism $\varphi \colon \mathscr{F} \to \mathscr{G}$ of presheaves is said to be **injective** (resp. **surjective**) if $\varphi_U \colon \mathscr{F}(U) \to \mathscr{G}(U)$ is injective (resp. surjective) for every $U \in \mathrm{ob}\, \mathcal{C}$.

4.a Lemma *Let $f \colon X \to Y$ be a morphism in a category $\mathcal{C}$, then*

f is	*if and only if the induced map*
monic	$\mathrm{Hom}(U,X) \xrightarrow{f_*} \mathrm{Hom}(U,Y)$ *is injective for all* $U \in \mathrm{ob}\, \mathcal{C}$*;*
epic	$\mathrm{Hom}(X,U) \xrightarrow{f^*} \mathrm{Hom}(Y,U)$ *is injective for all* $U \in \mathrm{ob}\, \mathcal{C}$*;*
split epic	$\mathrm{Hom}(U,X) \xrightarrow{f_*} \mathrm{Hom}(U,Y)$ *is surjective for all* $U \in \mathrm{ob}\, \mathcal{C}$*;*
split monic	$\mathrm{Hom}(X,U) \xrightarrow{f^*} \mathrm{Hom}(Y,U)$ *is surjective for all* $U \in \mathrm{ob}\, \mathcal{C}$*.*

4.b Proposition (Monic/epic is pointwise) *The injective (resp. surjective) morphisms of presheaves are precisely the monomorphism (resp. epimorphism) in* $\mathbf{PSh}(\mathcal{C})$*. In particular, the isomorphisms in* $\mathbf{PSh}(\mathcal{C})$ *are those both injective and surjective.*

Remark The injective part of this statement is straightly from Theorem 2.e and lemma 4.a, while the surjective part requires the slogan "*limits in functor categories are computed pointwise*" and the fact that a morphism is epic if and only if its *cokernel pair* is trivial.

5 (Subpresheaves) We say $\mathscr{F}$ is a **subpresheaf** of $\mathscr{G}$ if for every $U \in \operatorname{ob} \mathcal{C}$, $\mathscr{F}(U) \subset \mathscr{G}(U)$ and the inclusion maps glue together to give an injective morphism of presheaves $\mathscr{F} \to \mathscr{G}$.

5.a Proposition (Image of a morphism) *For any morphism of presheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$, there exists a unique subpresheaf $\mathscr{G}' \subset \mathscr{G}$ such that φ can be factorized into $\mathscr{F} \to \mathscr{G}' \to \mathscr{G}$ and that the first morphism is surjective. Such a subpresheaf $\mathscr{G}'$ is called the* **(presheaf) image** *of φ.*

6 Proposition (Limits and colimits of presheaves) *Limits and colimits exist in the category* $\mathbf{PSh}(\mathcal{C})$*. Indeed, they are computed pointwise. Moreover, for every $U \in \operatorname{ob} \mathcal{C}$, the* **section functors**

$$\begin{aligned} \Gamma(U, -)\colon \mathbf{PSh}(\mathcal{C}) &\longrightarrow \mathbf{Set} \\ \mathscr{F} &\longmapsto \mathscr{F}(U) \end{aligned}$$

commutes with limits and colimits.

As a result of this, statements about limits and colimits of presheaves can be deduced to the similar statements for sets. See my note *BMO* or refer [Bor94] for more details.

6.a Corollary *If $\mathcal{C}$ is a small category. Then the Yoneda embedding $\Upsilon\colon \mathcal{C} \to$ $\mathbf{PSh}(\mathcal{C})$ commutes with limits.*

Proof: Let $\varprojlim U_i = U$ in $\mathcal{C}$ and $V \in \mathcal{C}$. Then

$$\begin{aligned} (\varprojlim \Upsilon(U_i))(V) &= \varprojlim \Upsilon(U_i)(V) \\ &= \varprojlim \operatorname{Hom}(V, U_i) \\ &= \operatorname{Hom}(V, \varprojlim U_i) \\ &= \operatorname{Hom}(V, U) = \Upsilon(U)(V). \end{aligned}$$

Thus $\varprojlim \Upsilon(U_i) = \Upsilon(\varprojlim U_i)$. □

Remark However, the Yoneda embedding does not commute with colimits in general.

7 (Changing the base space) Let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor. Let u^p denote the functor

$$\begin{aligned} u^p\colon \mathbf{PSh}(\mathcal{D}) &\longrightarrow \mathbf{PSh}(\mathcal{C}) \\ \mathscr{G} &\longmapsto \mathscr{G} \circ u. \end{aligned}$$

Note that this functor commutes with limits and colimits.

Now, we are going to introduce a *left adjoint* to this functor. Before we do so, we introduce a category $\mathcal{I}_V$ for every $V \in \operatorname{ob}\mathcal{D}$ as follows. The objects in $\mathcal{I}_V$ are pairs (U, ϕ) where $U \in \operatorname{ob}\mathcal{C}$ and $\phi\colon V \to u(U)$. A morphism between (U, ϕ) and (U', ϕ') is a morphism $f\colon U \to U'$ such that $u(f) \circ \phi = \phi'$. in other words, $\mathcal{I}_V$ is the *comma category* $(\mathrm{const}_V \downarrow u)$.

Before going forward, we recall the notion of *filtered colimite*.

7.a A category $\mathcal{I}$ is said to be **filtered** if it is nonempty and if every *finite diagram* in which has a *cocone*, in other words, if every functor from a finite category to $\mathcal{I}$ admits a natural transformation to a constant functor.

Like the equivalence condition of cocompleteness, we have

7.b Proposition *A category $\mathcal{I}$ is filtered, if and only if*

1. *$\mathcal{I}$ is nonempty;*
2. *For any two objects $A, B \in \operatorname{ob}\mathcal{I}$, there exists an object $C \in \operatorname{ob}\mathcal{I}$ and morphisms $A \to C$ and $B \to C$;*
3. *For any two parallel morphisms $f, g\colon A \rightrightarrows B$ in $\mathcal{I}$, there exists a morphism $h\colon B \to C$ such that $h \circ f = h \circ g$.*

We have a slogan "*in* **Set***, filtered colimits commute with finite limits*". The proof of this statement is technical, one can either refer Theorem 2.12.11 in my note *BMO*, or directly refer [Bor94].

Now, we go back to the category $\mathcal{I}_V$.

7.c Lemma *Let $\mathcal{C}$ be a finite-complete category, which means it has all finite limits, and $u\colon \mathcal{C} \to \mathcal{D}$ be a functor commutes with all those finite limits, then $\mathcal{I}_V^{\mathrm{opp}}$ are filtered.*

Proof: First, we show that $\mathcal{I}_V^{\mathrm{opp}}$ is nonempty. Indeed, let X be a terminal object in $\mathcal{C}$. Then $u(X)$ is a terminal object in $\mathcal{D}$. Thus there exists a morphism $V \to u(X)$, and therefore $\mathcal{I}_V$ has at least one object.

Then we verify condition *2* in Proposition 7.b. Let $(A, \phi), (B, \psi) \in \operatorname{ob}\mathcal{I}_V$. Let C be the product of A and B in $\mathcal{C}$. Then $u(C)$ is the product of $u(A)$ and $u(B)$. Hence there exists a unique morphism $\theta\colon V \to u(C)$ compatible with ϕ and ψ. Then (C, θ) is the required object.

Finally, we verify condition *3*. Let $f, g\colon (A, \phi) \rightrightarrows (B, \psi)$ be two parallel morphisms in $\mathcal{I}_V^{\mathrm{opp}}$. Then $f, g\colon B \rightrightarrows A$ are two parallel morphisms in $\mathcal{C}$. Let $h\colon C \to B$ be the equalizer of them, then $u(h)$ is the equalizer of $u(f)$ and $u(g)$. Hence there exists a unique morphism $\theta\colon V \to u(C)$ such that $u(h) \circ \theta = \psi$. Then $u(h)$ is the required morphism. □

Given a presheaf $\mathscr{F}$ on $\mathcal{C}$, we have a functor

$$\begin{aligned} \mathscr{F}_V \colon \mathcal{I}_V^{\mathrm{opp}} &\longrightarrow \mathbf{Set} \\ (U, \phi) &\longmapsto \mathscr{F}(U). \end{aligned}$$

So, we define

$$u_p\mathscr{F}(V) := \varinjlim \mathscr{F}_V.$$

Given a morphism $g \colon V' \to V$, by the functorality of comma category, we have a functor

$$\begin{aligned} \bar{g} \colon \mathcal{I}_V &\longrightarrow \mathcal{I}_{V'} \\ (U, \phi) &\longmapsto (U, \phi \circ g), \end{aligned}$$

such that $\mathscr{F}_{V'} \circ \bar{g} = \mathscr{F}_V$. Therefore, there exists a unique map $g^* \colon u_p\mathscr{F}(V) \to u_p\mathscr{F}(V')$ compatible with this relation, i.e. the following diagram commutes.

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "c(\phi)"] \arrow[d, "\mathrm{id}"'] & u_p\mathscr{F}(V) \arrow[d, "g^*"] \\ \mathscr{F}(U) \arrow[r, "c(\phi \circ g)"] & u_p\mathscr{F}(V') \end{tikzcd}$$

The uniqueness of those g^* implies that we obtain a presheaf on $\mathcal{D}$, denoted by $u_p\mathscr{F}$. Note that any morphism of presheaves $\mathscr{F} \to \mathscr{F}'$ gives rise to compatible systems of morphisms between functors $\mathscr{F}_V \to \mathscr{F}'_V$, and hence to a morphism of presheaves $u_p\mathscr{F} \to u_p\mathscr{F}'$. In this way, we have defined a functor

$$u_p \colon \ \mathbf{PSh}(\mathcal{C}) \longrightarrow \mathbf{PSh}(\mathcal{D}).$$

7.d Theorem *The functor u_p is a left adjoint to the functor u^p.*

Proof: Let $\mathscr{G}$ be a presheaf on $\mathcal{D}$ and let $\mathscr{F}$ be a presheaf on $\mathcal{C}$. We need to show the following one-one corresponding:

$$\mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathscr{F}, u^p\mathscr{G}) \cong \mathrm{Hom}_{\mathbf{PSh}(\mathcal{D})}(u_p\mathscr{F}, \mathscr{G}).$$

First, given a morphism $\alpha \colon u_p\mathscr{F} \to \mathscr{G}$, we get $u^p\alpha \colon u^p u_p\mathscr{F} \to u^p\mathscr{G}$. Since there already exists a morphism $\mathscr{F} \to u^p u_p\mathscr{F}$ given by the canonical maps $c(\mathrm{id}_{u(U)}) \colon \mathscr{F}(U) \to u_p\mathscr{F}(u(U))$, we find the corresponding morphism $\mathscr{F} \to u^p u_p\mathscr{F} \to u^p\mathscr{G}$.

Then, given a morphism $\beta \colon \mathscr{F} \to u^p\mathscr{G}$, we get $u_p\beta \colon u_p\mathscr{F} \to u_p u^p\mathscr{G}$. For every $V \in \mathcal{D}$, consider the set $u_p u^p\mathscr{G}(V) := \varprojlim u^p\mathscr{G}_V$. Now, for each $(U, \phi) \in \mathcal{I}_V$, its value under $u^p\mathscr{G}_V$ is $\mathscr{G}(u(U))$ which admits a map $\mathscr{G}(\phi) \colon \mathscr{G}(u(U)) \to \mathscr{G}(V)$. These maps form a natrual transformation from $u^p\mathscr{G}_V$ to the constant functor $\mathrm{const}_{\mathscr{G}(V)}$ on $\mathcal{I}_V$. Then there exists a map $u_p u^p\mathscr{G}(V) \to \mathscr{G}(V)$. These maps form a morphism of presheaves $u_p u^p\mathscr{G} \to \mathscr{G}$. Then we obtain the required morphism $u_p\mathscr{F} \to_p u^p\mathscr{G} \to \mathscr{G}$.

Finally, one can verify the above are mutually inverse. $\square$

Remark Note that if $\mathcal{A}$ is a category such that any diagram $\mathcal{I}_V^{\mathrm{opp}} \to \mathcal{A}$ has a limit, then the functors u^p and u_p can be defined on the categories of presheaves with values in $\mathcal{A}$. Moreover, the adjointness of the pair u^p and u_p continues to hold in this setting.

7.e Corollary *Let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor between categories. Then, for any $U \in \operatorname{ob}\mathcal{C}$ we have $u_p h_U = h_{u(U)}$.*

Proof: By the adjointness and Yoneda lamma, we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{PSh}(\mathcal{D})}(u_p h_U, \mathscr{G}) \cong \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(h_U, u^p\mathscr{G}) \cong u^p\mathscr{G}(U) = \mathscr{G}(u(U)),\\
&\mathrm{Hom}_{\mathbf{PSh}(\mathcal{D})}(h_{u(U)}, \mathscr{G}) \cong \mathscr{G}(u(U)).
\end{aligned}$$

Therefore, $u_p h_U = h_{u(U)}$. □

7.f Remark (Kan extensions) One can similar define a *right adjoint* ${}_pu$ of u^p as follows. First, construct the category ${}_V\mathcal{I}$ as the comma category $(u \downarrow \mathrm{const}_V)$. Then construct the functor ${}_V\mathscr{F}$ as

$$\begin{aligned}
{}_V\mathscr{F} : {}_V\mathcal{I}^{\mathrm{opp}} &\longrightarrow \mathbf{Set}\\
(U, \phi) &\longmapsto \mathscr{F}(U).
\end{aligned}$$

Finally, the functor is given by

$${}_pu\mathscr{F}(V) := \varprojlim {}_V\mathscr{F}.$$

The functor u_p (resp. ${}_pu$) is called the **left (resp. right) Kan extension operation along** u and $u_p\mathscr{F}$ (resp. ${}_pu\mathscr{F}$) is called the **left (resp. right) Kan extension of** $\mathscr{F}$ **along** u. More details can be found in §4.4 of my note *BMO* or refer [Bor94].

7.g Lemma *Let $u \dashv v\colon \mathcal{C} \to \mathcal{D}$ be an* adjoint pair, *which means a pair of functors $u\colon \mathcal{C} \to \mathcal{D}$ and $v\colon \mathcal{D} \to \mathcal{C}$ such that u is left adjoint to v. Then*

1. *$u^p h_V = h_{v(V)}$ for any $V \in \operatorname{ob}\mathcal{D}$;*
2. *the category $\mathcal{I}_U^v$ has an initial object;*
3. *the category ${}_V^u\mathcal{I}$ has a terminal object;*
4. *${}_pu = v^p$;*
5. *$u^p = v_p$.*

Proof: *1.* Let $V \in \operatorname{ob}\mathcal{D}$, then

$$u^p h_V(U) = h_V(u(U)) = \mathrm{Hom}(u(U), V) = \mathrm{Hom}(U, v(V)) = h_{v(V)}(U).$$

2. Let $\eta_U : U \to v(u(U))$ be the map adjoint to the map $\mathrm{id}_{u(U)}$. Then $(u(U), \eta_U)$ is an initial object of $\mathcal{I}_U^v$.

3. Let $\epsilon_V : u(v(V)) \to V$ be the map adjoint to the map $\mathrm{id}_{v(V)}$. Then $(v(V), \epsilon_V)$ is a terminal object of ${}^u_V\mathcal{I}$.

4. Indeed, for any presheaf $\mathscr{F}$ on $\mathcal{C}$, we have

$$\begin{aligned} v^p\mathscr{F}(V) &= \mathscr{F}(v(V)) \\ &= \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(h_{v(V)}, \mathscr{F}) \\ &= \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(u^p h_V, \mathscr{F}) \\ &= \mathrm{Hom}_{\mathbf{PSh}(\mathcal{D})}(h_V, {}_pu\mathscr{F}) \\ &= {}_pu\mathscr{F}(V). \end{aligned}$$

5. u^p is right adjoint to ${}_pu$, v_p is right adjoint to v^p. By the uniqueness of adjoint functor, ${}_pu = v^p$ implies $u^p = v_p$. ☐

7.h Remark (Kan extensions) The Kan extension operations u_p and ${}_pu$ above are special case of the following general notions.

Let $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ and $p \colon \mathcal{C} \to \mathcal{C}'$ be two functors. The **left Kan extension** of $\mathscr{F}$ along p, if it exists, is a pair $(\mathscr{G}, \alpha)$ where

- $\mathscr{G} : \mathcal{C}' \to \mathcal{D}$ is a functor,
- $\alpha \colon \mathscr{F} \Rightarrow \mathscr{G} \circ p$ is a natural transformation,

satisfying the following universal property: if $(\mathscr{H}, \beta)$ is another pair with

- $\mathscr{H} : \mathcal{C}' \to \mathcal{D}$ is a functor,
- $\beta \colon \mathscr{F} \Rightarrow \mathscr{H} \circ p$ is a natural transformation,

then there exists a unique natural transformation $\gamma \colon \mathscr{G} \Rightarrow \mathscr{H}$ such that $(\gamma * p) \circ \alpha = \beta$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathscr{F}"] \arrow[d, "p"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathcal{D} \\ \mathcal{C}' \arrow[ur, "\mathscr{G}"] \arrow[ur, bend right, "\mathscr{H}"'] \arrow[ur, phantom, bend right=20, "\Downarrow\gamma"] & \end{tikzcd}$$

We shall use the notation $\mathrm{Lan}_p \mathscr{F}$ to denote the **left Kan extension** of $\mathscr{F}$ along p. The notation $\mathrm{Ran}_p \mathscr{F}$ is used for the dual notion of **right Kan extension**.

7.i Example (Yoneda extension) Let $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ be a functor, then its **Yoneda extension** $\widetilde{\mathscr{F}} \colon \mathbf{PSh}(\mathcal{C}) \to \mathcal{D}$ is the left Kan extension of $\mathscr{F}$ along the Yoneda embedding, i.e. $\widetilde{\mathscr{F}} = \mathrm{Lan}_\Upsilon \mathscr{F}$. Note that one has $\widetilde{\mathscr{F}} \circ \Upsilon = \mathscr{F}$ and the formula

$$\widetilde{\mathscr{F}}(\mathscr{G}) = \varinjlim_{(h_U \to \mathscr{G}) \in \mathrm{ob}(\Upsilon \downarrow \mathrm{const}_{\mathscr{G}})} \mathscr{F}(U).$$

§ 2 Sites and sheaves

In this section, we define *sheaves on a site*. One slogan about the topologies used in algebraic geometry is that "it is the covering does matter, not the open set". Therefore, a convention approach to the notion of sites is using coverings. So we list the definition of *coverings* and their *equivalence relations*. On this foundation, we define sheaves as presheaves satisfying *descent condition* respect to coverings. Then, we briefly introduce the notion sheaves with values in an algebraic category. Finally, we show equivalent sites define the same sheaves.

1 (Sites as categpries equipped with a Grothendieck pretopology) A **site** is a category $\mathcal{C}$ equipped with a **Grothendieck pretopology**, that is a collection $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target $\{U_i \to U\}$, called **coverings** on $\mathcal{C}$, satisfying the following axioms

Cov1. If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$;

Cov2. the collection of coverings is stable under pullback: if $\{U_i \to U\}_{i\in I}$ is a covering and $f\colon V \to U$ is any morphism in $\mathcal{C}$, then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I}$ is a covering;

Cov3. if $\{U_i \to U\}_{i\in I}$ is a covering and for each i, $\{U_{i,j} \to U_i\}_{j\in J_i}$ is also a covering, then the family of compositions $\{U_{i,j} \to U_i \to U\}_{i\in I, j\in J_i}$ is a covering.

Remark One may hope $\mathrm{Cov}(\mathcal{C})$ to be a set. But this may not be true even if $\mathcal{C}$ is a small category. Usually, we need to shrink the Grothendieck pretopology to make it become a set.

1.a Example (Topological space) Let X be a topological space and $\mathcal{T}_X$ the category whose objects are all the open subsets of X and morphisms are the inclusion maps. Then there is a standard Grothendieck pretopology on $\mathcal{T}_X$ given by

$$\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{T}_X) \iff \bigcup U_i = U.$$

We should point out that in this site, $U \times V = U \cap V$ and that empty covering of the empty set is a covering.

However, this Grothendieck pretopology is too big: the collection $\mathrm{Cov}(\mathcal{T}_X)$ is not a set as we allow arbitrary set as the index set I. But this can be avoid if we exclude those coverings having duplicative members. But then, this set is not a Grothendieck pretopology unless we modify the axioms as:

0' $\mathrm{Cov}(\mathcal{C}) \subset \mathcal{P}(\mathrm{Hom}(\mathcal{C}))$;

1' If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$;

2' the collection of coverings is stable under pullback: if $\{U_i \to U\}_{i\in I}$ is a covering and $f\colon V \to U$ is any morphism in $\mathcal{C}$, then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I}$ is tautologically equivalent to an element of $\mathrm{Cov}(\mathcal{C})$;

3' if $\{U_i \to U\}_{i\in I}$ is a covering and for each i, $\{U_{i,j} \to U_i\}_{j\in J_i}$ is also a covering, then the family of compositions $\{U_{i,j} \to U_i \to U\}_{i\in I, j\in J_i}$ is tautologically equivalent to an element of $\mathrm{Cov}(\mathcal{C})$.

Here $\mathcal{P}$ denotes the power set and $\mathrm{Hom}(\mathcal{C})$ denotes the union of all hom-sets in $\mathcal{C}$.

1.b Example (G-sets) Let G be a group and $G\,\mathbf{Set}$ the category whose objects are sets X with a left G-action and whose morphisms are G-equivariant maps. Now, define

$$\{\varphi_i\colon U_i \to U\}_{i\in I} \in \mathrm{Cov}(G\,\mathbf{Set}) \iff \bigcup \varphi_i(U_i) = U.$$

One can verify this $\mathrm{Cov}(G\,\mathbf{Set})$ satisfies the axioms. However, since both $G\,\mathbf{Set}$ and $\mathrm{Cov}(G\,\mathbf{Set})$ are too big (they are proper classes), one may prefer to work with some smaller substitutes.

First, for any G-set X_0, there exists a suitable universe $\mathcal{U}$ such that the full subcategory $G\,\mathbf{Set}_{\mathcal{U}}$ of $\mathcal{U}$-small G-sets contains X_0 and, up to isomorphism, every G-sets smaller than those in this subcategory. Then replace $\mathrm{Cov}(G\,\mathbf{Set}_{\mathcal{U}})$ by a smaller one $\mathrm{Cov}(G\,\mathbf{Set}_{\mathcal{U}})_s$, which contains the coverings we care about and every covering in $\mathrm{Cov}(G\,\mathbf{Set}_{\mathcal{U}})$ is *combinatorially equivalent* to a covering in $\mathrm{Cov}(G\,\mathbf{Set}_{\mathcal{U}})_s$. This site $(G\,\mathbf{Set}_{\mathcal{U}}, \mathrm{Cov}(G\,\mathbf{Set}_{\mathcal{U}})_s)$ is denoted by $\mathcal{T}_G$.

1.c Example Any category $\mathcal{C}$ admits a canonical Grothendieck pretopology by setting $\{\mathrm{id}_U\colon U \to U\}$ as the coverings. *Sheaves* on this site are the presheaves on $\mathcal{C}$. The corresponding topology is called the *chaotic* or *indiscrete topology*.

1.d Remark (Coverages) In [Joh02], Johnstone introduced a more general concept called **coverage**, which is basically the same as a Grothendieck pretopology except the second axiom may not be satisfied. In his text, a *site* is a category equipped with a coverage, not necessary a Grothendieck pretopology. Many constructions and results still hold in this setting.

2 (Morphisms and refinements) Let $\mathfrak{U} = \{U_i \to U\}_{i\in I}$ and $\mathfrak{V} = \{V_j \to V\}_{j\in J}$ be two coverings on a category $\mathcal{C}$, a **morphism** from $\mathfrak{U}$ to $\mathfrak{V}$ consists of a morphism $f\colon U \to V$, a map $\alpha\colon I \to J$ and for each $i \in I$, a morphism $U_i \to V_{\alpha(i)}$ making the following diagram commute.

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_{\alpha(i)} \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

When $U = V$ and $U \to V$ is the identity, we call $\mathfrak{U}$ a **refinement** of $\mathfrak{V}$.

Remark If $\mathfrak{V}$ is the empty covering, i.e. $J = \varnothing$, then no nonempty covering $\mathfrak{U}$ can refine $\mathfrak{V}$.

Now, we define the equivalence relation of coverings, so that we can shrink the Grothendieck pretopology in the case that we still have all the coverings up to equivalence.

3 (Equivalence relations of coverings) Let $\mathfrak{U} = \{\phi_i \colon U_i \to U\}_{i\in I}$ and $\mathfrak{V} = \{\psi_j \colon V_j \to U\}_{j\in J}$ be two coverings on a category $\mathcal{C}$.

1. We say $\mathfrak{U}$ and $\mathfrak{V}$ are **combinatorially equivalent** if there exist maps $\alpha\colon I \to J$ and $\beta\colon J \to I$ such that $\phi_i = \psi_{\alpha(i)}$ and $\psi_j = \phi_{\beta(j)}$.

2. We say $\mathfrak{U}$ and $\mathfrak{V}$ are **tautologically equivalent** if there exist maps $\alpha\colon I \to J$ and $\beta\colon J \to I$ such that for all $i \in I$ and $j \in J$ the following diagrams commute.

$$\begin{tikzcd} U_i \arrow[rr, "\cong"] \arrow[dr] & & V_{\alpha(i)} \arrow[dl] \\ & U & \end{tikzcd} \qquad \begin{tikzcd} V_j \arrow[rr, "\cong"] \arrow[dr] & & U_{\beta(j)} \arrow[dl] \\ & U & \end{tikzcd}$$

3.a Lemma *Let $\mathfrak{U} = \{\phi_i \colon U_i \to U\}_{i\in I}$ and $\mathfrak{V} = \{\psi_j \colon V_j \to U\}_{j\in J}$ be two coverings on a category $\mathcal{C}$.*

1. *If $\mathfrak{U}$ and $\mathfrak{V}$ are combinatorially equivalent then they are tautologically equivalent.*

2. *If $\mathfrak{U}$ and $\mathfrak{V}$ are tautologically equivalent then $\mathfrak{U}$ is a refinement of $\mathfrak{V}$ and $\mathfrak{V}$ is a refinement of $\mathfrak{U}$.*

3. *The relation "being combinatorially equivalent" is an equivalence relation.*

4. *The relation "being tautologically equivalent" is an equivalence relation.*

5. *The relation "$\mathfrak{U}$ refines $\mathfrak{V}$ and $\mathfrak{V}$ refines $\mathfrak{U}$" is an equivalence relation.*

4 (Sheaves are gluing presheaves) Let $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$ be a site. A **sheaf** on it, or a *sheaf* on $\mathcal{C}$ respect to $\mathrm{Cov}(\mathcal{C})$ is a presheaf $\mathscr{F}$ satisfying the following **gluing axiom**:

For any covering $\{U_i \to U\}_{i\in I}$ and sections $s_i \in \mathscr{F}(U_i)$ such that

$$s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$$

in $\mathscr{F}(U_i \times_U U_j)$ for all $i, j \in I$, there exists a unique $s \in \mathscr{F}(U)$ such that $s_i = s|_{U_i}$.

Remark If in the above definition there is at most one such s, we say that $\mathscr{F}$ is a **separated presheaf**.

The above component-wise definition can be written into a more abstract way: A presheaf $\mathscr{F}$ is called a **sheaf** if for every covering $\{U_i \to U\}_{i\in I}$, the diagram

$$\mathscr{F}(U) \longrightarrow \prod_{i\in I} \mathscr{F}(U_i) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod_{i_0,i_1\in I} \mathscr{F}(U_{i_0} \times_U U_{i_1}) \tag{2.1}$$

is *exact*, which means the first arrow is an equalizer of pr_0^* and pr_1^*. This condition is also called the **descent condition**.

Remark By this definition, if there exists an empty covering $\{U_i \to U\}_{i\in I}$, which means $I = \varnothing$, then $\mathscr{F}(U)$ is a singleton, the terminal object in **Set**.

The morphisms between sheaves are the morphisms between their underlying presheaves. In this way, the category of sheaves $\mathbf{Sh}(\mathcal{C})$ is a *full subcategory* of the category of presheaves $\mathbf{PSh}(\mathcal{C})$.

4.a Example (Sheaves on topological spaces) Let X be a topological space and let $\mathcal{T}_X$ be the site in Example 1.a. Then the sheaves on $\mathcal{T}_X$ is called sheaves on the topological space X. Actually, this is the original notion of sheaves.

4.b Example Let X be a topological space and let $\mathcal{T}'_X$ be the site basically the same as $\mathcal{T}_X$ except it excludes empty coverings. The sheaves on $\mathcal{T}'_X$ are the same as sheaves on the space $X \sqcup \{\eta\}$ whose open sets are the empty set and union of open sets in X with $\{\eta\}$.

5 (Sheaves with values in a category) Since the *descent condition* (2.1) makes sense for arbitrary category, thus we can easily generalize the notion of sheaves to allow values in an arbitrary category $\mathcal{A}$.

Let $\mathscr{F}$ be a presheaf with values in $\mathcal{A}$. For any $X \in \operatorname{ob}\mathcal{A}$, We define presheaves $\mathscr{F}_X$ as

$$\mathscr{F}_X(U) := \operatorname{Hom}_{\mathcal{A}}(X, \mathscr{F}(U)).$$

Then, the Yoneda lemma tells us that $\mathscr{F}$ is a *sheaf with values in* $\mathcal{A}$ if and only if for all $X \in \operatorname{ob}\mathcal{A}$, $\mathscr{F}_X$ is a sheaf.

5.a Theorem *Presheaves (resp. sheaves) with values in the category* **Ab** *of abelian groups are precisely the abelian group objects in the category of presheaves (resp. sheaves). They are also called* **abelian presheaves** *(resp.* **abelian sheaves***). The category of abelian presheaves (resp. abelian sheaves) is also denoted by* $\mathbf{PAb}(\mathcal{C})$ *(resp.* $\mathbf{Ab}(\mathcal{C})$*).*

5.b Let $\mathcal{A}$ be a **concrete category**, which is a category equipped with a faithful functor $F\colon \mathcal{A} \to \mathbf{Set}$ called the **forgetful functor**. Then a presheaf with values in $\mathcal{A}$ gives rise to a presheaf of sets $F \circ \mathscr{F}$ called the **underlying presheaf of sets** of $\mathscr{F}$.

In practice, a concrete category often appears as a category of *structured sets.* Sheaves of structured sets can be checked by their underlying presheaves of sets.

5.c Lemma *Let $\mathcal{A}$ be a complete concrete category with forgetful functor F which commutes with all limits and reflects isomorphisms. Then for any presheaf $\mathscr{F}$ with values in $\mathcal{A}$, $\mathscr{F}$ is a sheaf with values in $\mathcal{A}$ if and only if its underlying presheaf of sets is a sheaf.*

Proof: Apply F to the diagram (2.1) and one can check the requirements by the properties of F. $\square$

6 (Algebraic categories) A **category of algebraic structures**, or **algebraic category** is a concrete category $\mathcal{A}$ equipped with a forgetful functor F satisfying the following conditions:

1. $\mathcal{A}$ is *complete*, meaning it has limits, and F commutes with limits;
2. $\mathcal{A}$ has filtered colimits and F commutes with them;
3. F reflects isomorphisms.

6.a Example The following categories, equipped with the obvious forgetful functor, are algebraic categories:

- The category $*\,\mathbf{Set}$ of pointed sets.
- The category $\mathbf{Ab}$ of abelian groups.
- The category $\mathbf{Grp}$ of groups.
- The category $\mathbf{Mon}$ of monoids.
- The category $\mathbf{Ring}$ of rings.
- The category $R\,\mathbf{Mod}$ of R-modules over a fixed ring R.
- The category of Lie algebras over a fixed field.

6.b Proposition *Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$. Then f is monic (resp. epic) if so is $F(f)$. Moreover, F reflects monomorphisms.*

Proof: Note that f is monic if and only if its *kernel pair* $X \times_Y X$ is trivial, i.e. $X \to X \times_Y X$ is an isomorphism. $\square$

6.c Lemma *Let $f\colon X \to Y$ and $g\colon X' \to Y$ be two morphisms in $\mathcal{A}$. If $F(g)$ is injective and $\operatorname{im}(F(f)) \subset \operatorname{im}(F(g))$, then there exists a morphism $h\colon X \to X'$ such that $f = g \circ h$.*

Proof: Note that the assumptions imply that $F(X) \times_{F(Y)} F(X') = F(X)$. Then the conclusion follows. $\square$

7 Theorem (Equivalent sites provide same sheaves) *Let $\mathcal{C}$ be a category and $\mathrm{Cov}_1, \mathrm{Cov}_2$ two Grothendieck pretopologies.*

1. *If each $\mathfrak{U} \in \mathrm{Cov}_1$ is tautologically equivalent to some $\mathfrak{V} \in \mathrm{Cov}_2$, then $\mathbf{Sh}(\mathcal{C}, \mathrm{Cov}_2) \subset \mathbf{Sh}(\mathcal{C}, \mathrm{Cov}_1)$.*
2. *If for each $\mathfrak{U} \in \mathrm{Cov}_1$, there exists a $\mathfrak{V} \in \mathrm{Cov}_2$ refining $\mathfrak{U}$, then $\mathbf{Sh}(\mathcal{C}, \mathrm{Cov}_2) \subset \mathbf{Sh}(\mathcal{C}, \mathrm{Cov}_1)$.*

Proof: Let $\mathfrak{U} = \{U_i \to U\}_{i \in I}$ be a covering in Cov_1 and $\mathfrak{V} = \{V_j \to U\}_{j \in J}$ a refinement of $\mathfrak{U}$ in Cov_2 given by the map $\alpha\colon J \to I$ and the morphisms $f_j\colon V_j \to U_{\alpha(j)}$. Let $\mathscr{F}$ be a presheaf on $\mathcal{C}$. We need to show that the *descent condition* (2.1) for $\mathscr{F}$ with respect to all coverings in Cov_2 implies the one with respect to $\mathfrak{U}$.

The uniqueness is easy to prove. Indeed, let $s, s' \in \mathscr{F}(U)$ such that $s|_{U_i} = s'|_{U_i}$ for all $i \in I$. Then we also have $s|_{V_j} = s'|_{V_j}$ for all V_j. Thus $s = s'$ by the descent condition respect to $\mathfrak{V}$.

Now we turn to the gluing condition. Let $s_i \in \mathscr{F}(U_i)$ be a family of sections satisfying $s_i|_{U_i \times_U U_{i'}} = s_{i'}|_{U_i \times_U U_{i'}}$ for all $i, i' \in I$. Let $s_j := \mathscr{F}(f_j)(s_{\alpha(i)}) \in \mathscr{F}(V_j)$. Then from the following Cartesian diagrams,

$$\begin{tikzcd}
V_j \times_U V_{j'} \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & V_{j'} \arrow[d] \\
\cdot \arrow[r] \arrow[d] & U_{\alpha(j)} \times_U U_{\alpha(j')} \arrow[r] \arrow[d] & U_{\alpha(j')} \arrow[d] \\
V_j \arrow[r] & U_{\alpha(j)} \arrow[r] & U
\end{tikzcd}$$

we obtain $s_j|_{V_j \times_U V_{j'}} = s_{j'}|_{V_j \times_U V_{j'}}$. By the descent condition respect to $\mathfrak{V}$, there exists a section $s \in \mathscr{F}(U)$ such that $s_j = s|_{V_j}$ for all $j \in J$. We remain to show that $s_i = s|_{U_i}$ for all $i \in I$.

Now we have to consider some other coverings. Let $i_0 \in I$, then $\mathfrak{U}' = \{U_i \times_U U_{i_0} \to U_{i_0}\}_{i \in I}$ is a covering in Cov_1 and $\mathfrak{V}' = \{V_j \times_U U_{i_0} \to U_{i_0}\}_{j \in J}$ is a covering in Cov_2 which refines $\mathfrak{U}'$ via α and $f'_j := f_j \times \mathrm{id}_{U_0}$. Then consider $s_{i_0}|_{V_j \times_U U_{i_0}}$ given by the composition

$$\mathscr{F}(U_{i_0}) \longrightarrow \mathscr{F}(U_{\alpha(j)} \times_U U_{i_0}) \longrightarrow \mathscr{F}(V_j \times_U U_{i_0}).$$

Since $s_i|_{U_i\times_U U_{i_0}} = s_{i_0}|_{U_i\times_U U_{i_0}}$ for all $i\in I$ and $s_j = \mathscr{F}(f_j)(s_{\alpha(i)})$, we have $s_{i_0}|_{V_j\times_U U_{i_0}} = s_j|_{V_j\times_U U_{i_0}}$ for all $j\in J$. Now, from the following Cartesian diagrams,

$$\begin{tikzcd}
V_j \times_U U_{i_0} \arrow[r] \arrow[d] & U_{\alpha(j)} \times_U U_{i_0} \arrow[r] \arrow[d] & U_{i_0} \arrow[d] \\
V_j \arrow[r] & U_{\alpha(j)} \arrow[r] & U
\end{tikzcd}$$

we have $s_{i_0}|_{V_j\times_U U_{i_0}} = s|_{U_{i_0}}\Big|_{V_j\times_U U_{i_0}}$ for all $j\in J$. Hence $s_{i_0} = s|_{U_{i_0}}$. □

§ 3 Sheaves on topological spaces

We have seen sheaves on topological spaces in Example 2.4.a. Now we discuss something more about them.

1 (Sheaves) Recall that a presheaf on a topological space X is called a **sheaf** if it satisfies the *descent condition*. By the description of limits in **Set**, this means for any covering $\{U_i \subset U\}_{i\in I}$,

1. the canonical map $\mathscr{F}(U) \to \prod_{i\in I} \mathscr{F}(U_i)$ is injective,
2. the image of $\mathscr{F}(U)$ under the canonical map equals

$$\left\{(s_i)\in \mathscr{F}(U_i) \middle| \forall i_0, i_1 \in I\ s_{i_0}|_{U_{i_0}\cap U_{i_1}} = s_{i_1}|_{U_{i_0}\cap U_{i_1}}\right\}.$$

More elementarily, the descent condition equals the following two axioms:

Identity axiom For any sections $s, s' \in \mathscr{F}(U)$, if there exists a covering $\{U_i \subset U\}_{i\in I}$ such that $s|_{U_i} = s'|_{U_i}$, then $s = s'$.

Gluing axiom For any **system of compatible sections** $(s_i)_{i\in I}$ respect to a covering $\{U_i \subset U\}_{i\in I}$, namely an element $(s_i) \in \prod_{i\in I}\mathscr{F}(U_i)$ satisfying $s_{i_0}|_{U_{i_0}\cap U_{i_1}} = s_{i_1}|_{U_{i_0}\cap U_{i_1}}$ for all $i_0, i_1 \in I$.

In this case, we call the image of inclusion maps under a presheaf $\mathscr{F}$ the **restriction maps**.

1.a (global sections) Let $\mathscr{F}$ be a presheaf on a topological space X. Since X itself is the terminal object in $\mathcal{T}_X$, we have $\mathscr{F}(X) \cong \mathrm{Hom}_{\mathbf{PSh}(X)}(*, \mathscr{F})$. Therefore, we call an element $s\in\mathscr{F}(X)$ a **global section**.

A significant fact about a topological space is that it has points.

2 (Stalks and germs) Let X be a topological space and $x\in X$ be a point. Let $\mathscr{F}$ be a presheaf on X. The **stalk** of $\mathscr{F}$ at x is the colimit

$$\mathscr{F}_x := \varinjlim_{x\in U} \mathscr{F}(U).$$

2.a Remark (Taking stalk is exact) One can see this gives rise to a functor

$$\begin{aligned} \Gamma_x\colon \mathbf{PSh}(X) &\longrightarrow \mathbf{Set} \\ \mathscr{F} &\longmapsto \mathscr{F}_x \end{aligned}$$

called the **stalk functor**.

Moreover, note that the system of neighborhoods of x is filtered, thus the stalks are filtered colimits, thus commute with colimits and finite limits. In particular, *the stalk functor is exact.*

It is easy to describe the set $\mathscr{F}_x$. It is the quotient

$$\mathscr{F}_x = \{(U, s) | x \in U, s \in \mathscr{F}(U)\} \,/\, \sim$$

where the equivalence relation $\sim$ is given by $(U, s) \sim (U', s')$ if and only if there exists an open $U'' \subset U \cap U'$ such that $x \in U''$ and that $s|_{U''} = s'|_{U''}$. The equivalence class of (U, s) will be denoted by s_x and called the **germ** of s at x.

From this description, we get a canonical map for every open set $U \subset X$:

$$\begin{aligned} \mathscr{F}(U) &\longrightarrow \prod_{x\in U} \mathscr{F}_x \\ s &\longmapsto \prod_{x\in U} s_x. \end{aligned}$$

2.b Lemma (Sections are determined by germs) *Let X be a topological space. A presheaf $\mathscr{F}$ on X is* separated *if and only if for every open $U \subset X$ the above canonical map is injective.*

Proof: Let $\mathscr{F}$ be a separated presheaf. Let s, s' be two sections of $\mathscr{F}$ on some open set U such that they have the same germ at each $x \in U$. Then, for each $x \in U$, there exists a open neighborhood U_x of x such that $s|_{U_x} = s'|_{U_x}$. Note that $\{U_x \subset U\}_{x\in U}$ is a covering, thus $s = s'$.

Conversely, let $\mathscr{F}$ be a presheaf satisfying the condition in statement. Let $\{U_i \subset U\}_{i\in I}$ be a covering. Let s, s' be two sections of $\mathscr{F}$ on U such that $s|_{U_i} = s'|_{U_i}$ for all $i \in I$. Note that this implies that s and s' have the same germ at every point in each U_i, thus in whole U. Then, by the injectivity of the canonical map, $s = s'$. $\square$

2.c (Compatible germs) Now we turn to the image of the canonical map. The elements in the image are called **systems of compatible germs** of $\mathscr{F}$ over U. To be explicit, a *system of compatible germs* is an element $\prod_{x\in U} s_x \in \prod_{x\in U} \mathscr{F}_x$ such that for any $x \in U$, there exists some representative (U_x, s^x) of s_x with $U_x \subset U$ such that the germ of s^x at any point $y \in U_x$ is s_y.

2.d Example Let X be a topological space. For each $x \in X$, give a set S_x. Then we have a presheaf given by $\mathscr{F}(U) = \prod_{x \in U} S_x$ with obvious restriction maps. This is a sheaf. But, usually $\mathscr{F}_x \neq S_x$. We only have a map $\mathscr{F}_x \to S_x$.

3 (Support of a section) Let $\mathscr{F}$ be an abelian sheaf on X and s be a global section. The **support** of s, denoted by $\mathrm{Supp}(s)$, is the subset of X consisting of points of X where s has nonzero germ:

$$\mathrm{Supp}(s) := \{x \in X | s_x \neq 0 \in \mathscr{F}_x\}.$$

3.a Proposition $\mathrm{Supp}(s)$ *is a closed subset of* X*.*

Proof: Consider any point y in the closure $\overline{\mathrm{Supp}(s)}$. Then for any neighborhood U of y, there exists a point x contained in $U \cap \mathrm{Supp}(s)$. Then $s_x \neq 0$, thus $s|_U \neq 0$. Vary U through all neighborhood of y, we get $s_y \neq 0$. □

The followings are some important examples.

4 Example (Restriction of a sheaf) Let $\mathscr{F}$ be a sheaf on X and U be a open subset of X. Then there is a natural sheaf $\mathscr{F}|_U$ on U given by $\mathscr{F}|_U(V) := \mathscr{F}(V)$ for all open subsets $V \subset U$, called the **restriction** of $\mathscr{F}$ to U.

5 Example (Skyscraper sheaves) Let X be a topological space with $x \in X$ and S a set. Let $i_x \colon x \to X$ be the inclusion. The **skyscraper sheaf** $i_{x,*}S$ is given by

$$i_{x,*}S(U) = \begin{cases} S & \text{if } x \in U; \\ * & \text{if } x \notin U. \end{cases}$$

with obvious restriction maps. Here $*$ denote a *singleton*, or more generally a *terminal object.*

Note that it may be true that there are nontrivial stalks of a skyscraper sheaf other than the one at x. Indeed, we have

$$(i_{x,*}S)_y = \begin{cases} S & \text{if } y \in \overline{\{x\}}; \\ * & \text{if } y \notin \overline{\{x\}}. \end{cases}$$

One can see that taking skyscraper sheaf at a point is a functor, moreover, we have

5.a Theorem (Stalk is left adjoint to skyscraper sheaf) *Let X be a topological space and $x \in X$. Then there exists a bijection*

$$\mathrm{Hom}_{\mathcal{A}}(\mathscr{F}_x, S) \cong \mathrm{Hom}_{\mathbf{Sh}(X)}(\mathscr{F}, i_{x,*}S)$$

natural in both the sheaf $\mathscr{F}$ of algebraic structures and the algebraic structure $S \in \mathrm{ob}\,\mathcal{A}$.

Proof: Note that stalk functor can be viewed as the *pullback functor* for the inclusion map $i_x \colon x \to X$, thus this property is a corollary of theorem, which will appear later. $\square$

6 Example (Sheaf of continuous maps) Let X, Y be two topological spaces. Define $\mathscr{F}$ as follows. $\mathscr{F}(U)$ consists of all continuous maps from U to Y, and the restriction maps are the obvious ones.

Note that for S a set, regarded as a topological space with discrete topology, the continuous maps to S are precisely the locally constant maps to S. Here we define

6.a Example (Constant sheaves) Let X be a topological space and S be a set. The **constant sheaf** with value S, denoted by $\underline{S}$ or $\underline{S}_X$, is the sheaf of locally constant maps to S.

One may feel confusion about the name. Maybe a constant sheaf should be a presheaf with constant values, i.e. $\mathscr{F}(U) = S$ for all open subsets U. This presheaf is called the **constant presheaf** with value S, denoted by const_S. But this is rarely a sheaf. Even when one remember that the value of a sheaf on the empty set should be a terminal object and modify the definition of const_S, it is still far from being a sheaf.

The relationship between the constant sheaves and constant presheaves will be discussed later.

6.b Example (Sheaf of sections of a map) Let $f \colon Y \to X$ be a continuous map. Define $\mathscr{F}(U)$ to be the set of **sections** of f over U, which are continuous maps $s \colon U \to Y$ such that $f \circ s = \mathrm{id}|_U$.

When Y is further a topological group, one can see that $\mathscr{F}$ is a sheaf of groups.

7 Example (Sheaf of differential functions) Let X be a *differential manifold*. One can consider the sheaf $\mathscr{O}$ of differential functions on X similar as the sheaf of continuous maps. Since functions having same germ at a point are locally the same, it makes sense to define the **value of a germ** s_x at a point x as the value of a representative $s \in \mathscr{F}(U_x)$ at x.

Obviously, $\mathscr{O}_x$ is a ring for all $x \in X$. Moreover, it is a local ring. Let $\mathfrak{m}_x$ denote the ideal of $\mathscr{O}_x$ consisting of germs vanishing at x. Then one can check the following is an exact sequence.

$$0 \longrightarrow \mathfrak{m}_x \longrightarrow \mathscr{O}_x \xrightarrow{f \mapsto f(x)} \mathbb{R} \longrightarrow 0.$$

As any germ in $\mathscr{O}_x \setminus \mathfrak{m}_x$ is invertible, this shows that $\mathscr{O}_x$ is a local ring with the maximal ideal $\mathfrak{m}_x$.

Note that $\mathfrak{m}_x/\mathfrak{m}_x^2$ is a vector space over the residue field $\mathscr{O}_x/\mathfrak{m}_x \cong \mathbb{R}$. This vector space is called the **cotangent space** of X at x.

8 (Sheafification) Let $\mathscr{F}$ be a preshaef on a topological space X. Then a morphism $\mathscr{F} \to \mathscr{F}^{\#}$ is called a **sheafification** of $\mathscr{F}$ if for any sheaf $\mathscr{G}$ and presheaf morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$, there exists a unique sheaf morphism $\varphi^{\#}\colon \mathscr{F}^{\#} \to \mathscr{G}$ making the following digram commute.

$$\begin{tikzcd}
\mathscr{F} \arrow[r] \arrow[dr, "\varphi"'] & \mathscr{F}^{\#} \arrow[d, "\varphi^{\#}"] \\
 & \mathscr{G}
\end{tikzcd}$$

One can see that the sheafification is unique up to unique isomorphism and that sheafifications, if they exist, give rise to a functor:

$$\begin{aligned}
\#\colon \mathbf{PSh}(X) &\longrightarrow \mathbf{Sh}(X) \\
\mathscr{F} &\longmapsto \mathscr{F}^{\#}.
\end{aligned}$$

Now, we give the construction. First, recall we have canonical maps $\mathscr{F}(U) \to \prod_{x\in U} \mathscr{F}_x$, which induce a canonical presheaf morphism

$$\mathscr{F} \longrightarrow \Pi(\mathscr{F}),$$

here $\Pi(\mathscr{F})$ is defined as $\Pi(\mathscr{F})(U) := \prod_{x\in U} \mathscr{F}_x$. Like in Example 2.d, this presheaf is a sheaf but usually $\mathscr{F}_x \neq (\Pi\mathscr{F})_x$.

8.a Lemma (Sheafification through stalks) *The sheafification $\mathscr{F}^{\#}$ of $\mathscr{F}$ is given by the fibre product in the following Cartesian diagram:*

$$\begin{tikzcd}
\mathscr{F}^{\#} \arrow[r] \arrow[d] & \Pi(\mathscr{F}) \arrow[d] \\
\Pi(\mathscr{F}) \arrow[r] & \Pi(\Pi(\mathscr{F}))
\end{tikzcd}$$

where the right vertical map is the canonical map for the sheaf $\Pi(\mathscr{F})$, while the bottom horizontal map come from the maps

$$\prod_{x\in U} \mathscr{F}_x \longrightarrow \prod_{x\in U} (\Pi(\mathscr{F}))_x,$$

which is the product of the canonical maps for the presheaf $\mathscr{F}$.

Proof: The universal property of fibre product gives us a canonical morphism $\theta\colon \mathscr{F} \to \mathscr{F}^{\#}$. Then the only things one needs to verify are *1,* $\mathscr{F}^{\#}$ is a sheaf;. *2,* $\mathscr{F}$ is a sheaf if and only if θ is an isomorphism. If so, then the functorality of Π implies the functorality of $\#$ and the universal property of sheafification follows from this and *2*.

1. By *limits commutes with limits*, we can deduce the descent condition of the fibre product $\mathscr{F}^{\#}$ from those of $\Pi(\mathscr{F})$ and $\Pi(\Pi(\mathscr{F}))$. Therefore $\mathscr{F}^{\#}$ is a sheaf.

2. We show that $\mathscr{F}^{\#}(U)$ is precisely the set of all *systems of compatible germs* of $\mathscr{F}$ over U.

For any pair (t, t') with $t, t' \in \Pi(\mathscr{F})(U)$ which have same image in $\Pi(\Pi(\mathscr{F}))$ through the bottom horizontal map and the right vertical map respectively, let $t = (s_x)_{x \in U}$, $t' = (s'_x)_{x \in U}$ and their images in $\mathscr{F}(U)$ be $(t_x)_{x \in U}$ and $(t'_x)_{x \in U}$. For each s_x, let (U_x, s^x) be a representative of it, then the germ s^x_y of s^x at each $y \in U_x$ form a representative $(s^x_y)_y$ of t_x. On the other hand, (U, t') is definitely a representative of t'_x. Then since $t_x = t'_x$, there exists a neighborhood U'_x of x such that $s^x_y = s'_y$ and thus that $s_x = s'_x$.

We have seen in 2.c that $\mathscr{F}^{\#}$ is precisely the image of $\mathscr{F}$ in $\Pi(\mathscr{F})$, thus θ is an isomorphism when $\mathscr{F}$ is a sheaf. □

8.b Lemma (Sheafification preserves stalks) *Let $\mathscr{F}$ be a presheaf on a topological space X, then for any $x \in X$, $\mathscr{F}_x \cong \mathscr{F}^{\#}_x$.*

Proof: First, let s_x, s'_x be two germs of $\mathscr{F}$ at x sharing the same image under the canonical map $\mathscr{F}_x \to \mathscr{F}^{\#}_x$. Then, for some representatives (U, s) (resp.(U', s')) of s_x (resp. s'_x), we have $(U, (s_y)) \sim (U', (s'_y))$ in $\mathscr{F}^{\#}_x$. Then there exists a neighborhood U'' of x contained in $U \cap U'$ such that $(s_y)|_{U''} = s'_y|_{U''}$, i.e. $s_y = s'_y$ for all $y \in U''$. Particularly, $s_x = s'_x$.

To show the surjectivity, consider a germ $\bar{t} \in \mathscr{F}^{\#}_x$. Taking any representative (U, t) of this germ, then $t = (s_y)$ is a system of compatible germs of $\mathscr{F}$ over U. Therefore, there exists a representative (V, s^x) of s_x such that $V \subset U$ and the germ of s^x at any point $y \in V$ is s_y. Thus $t|_V$ is the image of s^x under the canonical map $\mathscr{F}(V) \to \mathscr{F}^{\#}(V)$. Then passing to the stalks, the germ of s^x at x will be a preimage of $\bar{t}$. □

Proof: A more simple proof is this: apply Π to the Cartesian diagram in Lemma 8.a, then we get $\Pi(\mathscr{F}^{\#}) = \Pi(\mathscr{F})^{\#} \cong \Pi(\mathscr{F})$. □

8.c Proposition (Sheafification is free) *The sheafification functor $\#$ is* left adjoint *to the forgetful functor from sheaves on X to presheaves on X.*

This is nothing but the universal property of sheafification. But its corollaries are very useful.

8.d Corollary *The sheafification preserves colimits and hence is* right exact*; the forgetful functor preserves limits and hence is* left exact*. In particular, the presheaf kernel is already the sheaf kernel.*

Proof: This follows from the property of adjoint functors. One can refer my note *BMO* or [Bor94] directly. □

8.e Example (Constant sheaves) Let S be a set. Then the constant sheaf $\underline{S}$ is precisely the sheafification of the constant presheaf const_S. Indeed, define maps $S \to \underline{S}(U)$ by mapping $s \in S$ to the constant map $x \mapsto s$ for all $x \in U$. Then we get a morphism $\mathrm{const}_S \to \underline{S}$, which induces a morphism $\mathrm{const}_S^{\#} \to \underline{S}$. One can see this is an isomorphism.

8.f Theorem (Sheafification of presheaves of algebraic structures) *Let $\mathscr{F}$ be a presheaf on a topological space X with values in an algebraic category $\mathcal{A}$. Then there exists a unique morphism $\mathscr{F} \to \mathscr{F}^{\#}$ of presheaves with values in $\mathcal{A}$ such that the corresponding morphism of underlying presheaves of sets is a sheafification and that it satisfying the universal property of a sheafification.*

Proof: The main idea is to define $\mathscr{F}^{\#}(U)$ as the fibre product:

$$\begin{tikzcd}
\mathscr{F}^{\#}(U) \arrow[r] \arrow[d] & \prod(\mathscr{F})(U) \arrow[d] \\
\prod_{x\in U} \mathscr{F}_x \arrow[r] & \prod_{x\in U} \prod(\mathscr{F})_x
\end{tikzcd}$$

Now, we check the conditions. First, apply the forgetful functor F to the above Cartesian diagram. Then the first statement follows. Next, let $\mathscr{G}$ be a sheaf on X with values in $\mathcal{A}$ and $\varphi\colon \mathscr{F} \to \mathscr{G}$ a presheaf morphism. Then the following diagram satisfies the assumptions in Lemma 2.6.c:

$$\begin{tikzcd}
\mathscr{F}^{\#}(U) \arrow[d] & \mathscr{G}(U) \arrow[d] \\
\prod(\mathscr{F})(U) \arrow[r] & \prod(\mathscr{G})(U)
\end{tikzcd}$$

the underlying map of the right vertical morphism is injective since $\mathscr{G}$ is a sheaf; the image of the composition of left and bottom morphism lies in the image of the right vertical morphism since the sections in $\mathscr{F}^{\#}(U)$ are systems of compatible germs and there already exists a morphism $\mathscr{F} \to \mathscr{G}$. Thus, by Lemma 2.6.c, there exists a morphism $\mathscr{F}^{\#}(U) \to \mathscr{G}(U)$ making the diagram commute. The uniqueness of such a morphism comes from the injectivity of the right vertical morphism. $\square$

9 (Sheaves on bases) Let X be a topological space and $\mathcal{B}$ a *base* of it. Recall that a **base** for the topology on X is a full subcategory $\mathcal{B}$ of $\mathcal{T}_X$ such that every object of $\mathcal{T}_X$, i.e. open subset of X, is a colimit of diagrams in $\mathcal{B}$, i.e. a union of sets in $\mathcal{B}$. Moreover, $\mathcal{B}$ inherits a *coverage* (not a Grothendieck pretopology since pullbacks do not exist in $\mathcal{B}$) from $\mathcal{T}_X$, thus becomes a site. Now, we can define the notion of **(pre)sheaves on** $\mathcal{B}$ as (pre)sheaves on the site $\mathcal{B}$.

Let x be a point in X, then the **stalk** of a (pre)sheaf $\mathscr{F}$ on $\mathcal{B}$ at x is the colimit

$$\mathscr{F}_x := \varinjlim_{x\in U, U\in\mathcal{B}} \mathscr{F}(U).$$

We still call the elements in $\mathscr{F}_x$ **germs** at x. Note that the notion of *compatible germs* still works for (pre)sheaves on $\mathcal{B}$ and since neighborhoods of x in $\mathcal{B}$ are *cofinal* in the system of neighborhoods of x, one can actually define *compatible germs* for arbitrary subsets of X. Next, one can define and state the notions and facts about (pre)sheaves and stalks like on a topological space.

From the inclusion functor $\mathcal{B} \to \mathcal{T}_X$, we get two canonical functors $\mathbf{PSh}(X) \to \mathbf{PSh}(\mathcal{B})$ and $\mathbf{Sh}(X) \to \mathbf{Sh}(\mathcal{B})$. However, since we can not glue things from smaller subsets, the notion of presheaves on base and on topological space are very different. Luckily, things are good for sheaves.

On one hand, any sheaf $\mathscr{F}$ on X can be viewed as a sheaf on $\mathcal{B}$ through the canonical functor $\mathbf{Sh}(X) \to \mathbf{Sh}(\mathcal{B})$. In this case, since neighborhoods in $\mathcal{B}$ are *cofinal* in those in $\mathcal{T}_X$, the stalks of $\mathscr{F}$, viewed as a sheaf on $\mathcal{B}$ and on X respectively, are the same.

On the other hand, for any sheaf $\mathscr{F}$ on $\mathcal{B}$, its stalks induces a sheaf $\Pi(\mathscr{F})$ on X:

$$\Pi(\mathscr{F})(U) := \prod_{x \in U} \mathscr{F}_x.$$

Then the fibre product $\mathscr{F}^{\#} := \Pi(\mathscr{F}) \times_{\Pi(\Pi(\mathscr{F}))} \Pi(\mathscr{F})$ defines a functor

$$\#\colon \mathbf{Sh}(\mathcal{B}) \longrightarrow \mathbf{Sh}(X).$$

One can see this sheaf $\mathscr{F}^{\#}$ is just the *sheaf of compatible germs of* $\mathscr{F}$.

9.a Theorem *The functor # is a weak inverse of the canonical functor from* $\mathbf{Sh}(X)$ *to* $\mathbf{Sh}(\mathcal{B})$*. In other words, it provides an equivalence between the two categories. Moreover, # commutes with taking stalks, i.e. there are canonical bijection*

$$\mathscr{F}_x = \mathscr{F}^{\#}_x$$

for all $x \in X$*.*

Remark The notions and statements also work for sheaves of algebraic structures.

§ 4 Sheaves on topological spaces: morphisms

Now we turn to consider the category $\mathbf{Sh}(X)$ of sheaves on a topological space X.

We start with the following important fact.

1 Lemma (Morphisms are determined by stalks) *Let* φ_1, φ_2 *be two morphisms from a presheaf* $\mathscr{F}$ *to a sheaf* $\mathscr{G}$*, which induce the same maps on every stalk. Then* $\varphi_1 = \varphi_2$*.*

Proof: Consider following commutative diagrams:

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi_i(U)"] \arrow[d] & \mathscr{G}(U) \arrow[d, hook] \\
\prod_{x\in U} \mathscr{F}_x \arrow[r] & \prod_{x\in U} \mathscr{G}_x
\end{tikzcd}$$

where the vertical morphisms are the canonical maps. Then, since the right canonical map is injective, $\varphi_1 = \varphi_2$. □

1.a Corollary *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Then φ is a monomorphism (resp. epimorphism, isomorphism) if and only if for all $x \in X$, $\varphi_x\colon \mathscr{F}_x \to \mathscr{G}_x$ is injective (resp. surjective, bijective).*

Proof: The "if" part for monomorphisms and epimorphisms follow from Lemma 1, while the "only if" part follow from the exactness of the stalk functor. As for the isomorphisms, let $\psi_x\colon \mathscr{G}_x \to \mathscr{F}_x$ be the inverse of each $\varphi_x\colon \mathscr{F}_x \to \mathscr{G}_x$ induced by φ. We need to glue them into a morphism $\psi\colon \mathscr{G} \to \mathscr{F}$. To do this, consider the following diagram.

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[d, hook] & \mathscr{F}(U) \arrow[d, hook] \\
\prod_{x\in U} \mathscr{G}_x \arrow[r, "\prod \psi_x"] & \prod_{x\in U} \mathscr{F}_x
\end{tikzcd}$$

Since the vertical canonical maps are injective and the bottom map is bijective, the assumptions of Lemma 2.6.c are satisfied. Then there exists a map $\mathscr{F}(U) \to \mathscr{G}(U)$ making the diagram commute. The commutativity also implies that those maps form a morphism $\psi\colon \mathscr{G} \to \mathscr{F}$ of sheaves. Since for all $x \in X$, $\varphi_x \circ \psi_x = \mathrm{id}, \psi_x \circ \varphi_x = \mathrm{id}$, by Lemma 1, $\varphi \circ \psi = \mathrm{id}, \psi \circ \varphi = \mathrm{id}$. This shows that φ is an isomorphism. □

Proof: Note that saying for all $x \in X$, $\varphi_x\colon \mathscr{F}_x \to \mathscr{G}_x$ is bijective is equal to say $\Pi(\varphi)\colon \Pi(\mathscr{F}) \to \Pi(\mathscr{G})$ is an isomorphism. In this case, let $\underline{\psi}$ be the inverse of $\Pi(\varphi)$. Then we have the following commutative diagram

$$\begin{tikzcd}
\mathscr{G} \arrow[rr] \arrow[dr, dashed, "\exists!\psi"] \arrow[dd] & & \Pi(\mathscr{G}) \arrow[dr, "\overline{\psi}"] & & \\
& \mathscr{F} \arrow[rr] \arrow[dr, "\varphi"] \arrow[dd] & & \Pi(\mathscr{F}) \arrow[dr, "\Pi(\varphi)"] \arrow[dd] & \\
\Pi(\mathscr{G}) \arrow[dr, "\overline{\psi}"'] & & \mathscr{G} \arrow[rr] \arrow[dd] & & \Pi(\mathscr{G}) \arrow[dd] \\
& \Pi(\mathscr{F}) \arrow[rr] \arrow[dr, "\Pi(\varphi)"'] & & \Pi(\Pi(\mathscr{F})) \arrow[dr, "\Pi(\Pi(\varphi))"] & \\
& & \Pi(\mathscr{G}) \arrow[rr] & & \Pi(\Pi(\mathscr{G}))
\end{tikzcd}$$

where the front two layers are obtained by apply the canonical Cartesian diagram for sheafification to the morphism φ. Then the universal property of fibre product implies the unique existence of the inverse ψ of φ. $\square$

1.b Example (Distinct sheaves may have isomorphic stalks) Note that Corollary 1.a doesn't imply that sheaves having isomorphic stalks are isomorphic. This is because the isomorphisms for stalks may not be induced from a morphism of sheaves.

For instance, let X be the set $\{a, b\}$ with the topology $\{X, U = \{a\}, \varnothing\}$. Define $\mathscr{F}$ and $\mathscr{G}$ as $\mathscr{F}(X) = \mathscr{F}(U) = \mathscr{G}(X) = \mathscr{G}(U) = X$ with the restriction maps $\mathscr{F}(X) \to \mathscr{F}(U)$ the identity and $\mathscr{G}(X) \to \mathscr{G}(U)$ mapping both a and b to a. Then $\mathscr{F}$ and $\mathscr{G}$ are non-isomorphic sheaves with the same stalks.

1.c Corollary (Subsheaves) *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Then the followings are equivalent.*

1. *φ is monic;*
2. *for all $x \in X$, $\varphi_x\colon \mathscr{F}_x \to \mathscr{G}_x$ is injective;*
3. *for all open subsets $U \subset X$, $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is injective.*

If this is the case, $\mathscr{F}$ is called a **subsheaf** *of $\mathscr{G}$.*

Proof: We have seen *1*$\Leftrightarrow$*2* before. *3*$\Rightarrow$*1* is obvious. It remains to show *2*$\Rightarrow$*3*. Suppose *2* and consider the following diagram.

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, hook] & \mathscr{G}(U) \arrow[d, hook] \\
\prod_{x\in U} \mathscr{F}_x \arrow[r] & \prod_{x\in U} \mathscr{G}_x
\end{tikzcd}$$

Since the vertical canonical maps and the bottom map are injective, $\varphi(U)$ must be also injective. $\square$

1.d Corollary (quotient sheaves) *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Then the followings are equivalent.*

1. *φ is epic;*
2. *for all $x \in X$, $\varphi_x\colon \mathscr{F}_x \to \mathscr{G}_x$ is surjective;*
3. *for any open subset $U \subset X$ and any section $s \in \mathscr{G}(U)$, there exists a covering $\{U_i \to U\}$ such that $s|_{U_i}$ lies in the image of $\mathscr{F}(U_i) \to \mathscr{G}(U_i)$.*

If this is the case, $\mathscr{G}$ is called a **quotient sheaf** *of $\mathscr{F}$.*

Proof: We have seen *1⇔2* before. Suppose *3* and let $\psi_1, \psi_2 \colon \mathscr{G} \rightrightarrows \mathscr{T}$ be two parallel morphisms of sheaves such that $\psi_1 \circ \varphi = \psi_2 \circ \varphi$. We need to show $\psi_1 = \psi_2$. For any open subset $U \subset X$ and any section $s \in \mathscr{G}(U)$, there exists a covering $\{U_i \to U\}$ and for each i, there exists a section $t_i \in \mathscr{F}(U_i)$ such that $\varphi(U_i)(t_i) = s|_{U_i}$. Then

$$\begin{aligned}\psi_1(U)(s)|_{U_i} = \psi_1(U_i)(s|_{U_i}) = \psi_1\varphi(U_i)(t_i) \\ = \psi_2\varphi(U_i)(t_i) = \psi_2(U_i)(s|_{U_i}) = \psi_2(U)(s)|_{U_i}\end{aligned}$$

Since $\{U_i \to U\}$ is a covering, this shows $\psi_1(U)(s) = \psi_2(U)(s)$. Thus $\psi_1 = \psi_2$.

Now suppose *2*. For any open subset $U \subset X$ and any section $s \in \mathscr{G}(U)$, consider its germs s_x at each point $x \in U$. Since φ_x are surjective, there exists some $t_x \in \mathscr{F}_x$ such that $\varphi_x(t_x) = s_x$. Let (U_x, t^x) be a representative of t_x such that $U_x \subset U$. Then $\varphi(U_x)(t^x)$ and s have the same germ s_x at x. Thus we can shrink U_x so that $\varphi(U_x)(t^x) = s|_{U_x}$. Note that $\{U_x \subset U\}_{x \in U}$ is a covering, this shows *3*. □

Remark Recall the notions of injective and surjective presheaf morphisms, we find that a sheaf morphism is monic if and only if its underlying presheaf morphism is injective, in other word, the forgetful functor is left exact. However, the similar statement fails to be true for epimorphisms.

All the above statements can be easily generalized to *sheaves of algebraic structures*. One can also use the following lemma.

1.e Lemma *Let $\mathscr{F}, \mathscr{G}$ be two sheaves on a topological space X with values in an algebraic category $\mathcal{A}$. Let $\varphi \colon \mathscr{F} \to \mathscr{G}$ be a morphism of underlying sheaves of sets. If for every $x \in X$, $\varphi_x \colon \mathscr{F}_x \to \mathscr{G}_x$ induces a morphism in $\mathcal{A}$, then φ induces a morphism of sheaves with values in $\mathcal{A}$.*

Proof: Consider the following commutative digram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, hook] & \mathscr{G}(U) \arrow[d, hook] \\ \prod_{x\in U} \mathscr{F}_x \arrow[r] & \prod_{x \in U} \mathscr{G}_x \end{tikzcd}$$

in which all maps but $\varphi(U)$ induce morphisms in $\mathcal{A}$. By Lemma 2.6.c and the uniqueness of $\varphi(U)$, it must also induce a morphism in $\mathcal{A}$. □

2 (Abelian sheaves form an abelian category) Now we turn to look at abelian sheaves.

Since presheaves are nothing but contravariant functor, the category of abelian sheaves, or more generally sheaves with values in an abelian category, form an abelian category, and subpresheaves, quotient presheaves, presheaf kernels, presheaf cokernels and presheaf images are computed open sets by open sets. In other words, *the section functor* $\Gamma(-,U)$ *is exact and the combination of all section functors reflects exact sequences.*

But things are not so easy for abelian sheaves. Since sheafification is left adjoint to the forgetful functor from sheaves to presheaves, it is true that for a sheaf morphism its *presheaf kernel* is already the **sheaf kernel**. The problem is the cokernel.

2.a Example (Holomorphic logarithms) Let X be the complex plane, $\underline{\mathbb{Z}}$ the constant sheaf with values $\mathbb{Z}$, $\mathscr{O}_X$ the sheaf of *holomorphic functions*, and $\mathscr{F}$ the presheaf of functions admitting a *holomorphic logarithm*. Then there is an exact sequence of abelian presheaves on X:

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\times 2\pi i} \mathscr{O}_X \xrightarrow{\exp} \mathscr{F} \longrightarrow 0.$$

However, $\mathscr{F}$ is not a sheaf since there are functions that don't have a logarithm but locally have a logarithm. For instance, the function $f(z) = z$ has no logarithm in an annular region round 0, while it has logarithm in any simply connected part of this region.

So the presheaf cokernel $\operatorname{coker}^p \varphi$ is not automatically the sheaf cokernel in general. But, from the universal properties of cokernel and sheafification, the **sheaf cokernel** $\operatorname{coker}\varphi$ should be the sheafification of the *presheaf cokernel*.

2.b Example (Holomorphic logarithms) We turn back to Example 2.a. Now we have known that the correct cokernel of the map $\underline{\mathbb{Z}} \to \mathscr{O}_X$ should be the sheafification of $\mathscr{F}$. Now we describe it.

Let $\mathscr{O}_X^*$ denote the presheaf of *invertible* (nowhere zero) holomorphic functions. One can see it is a sheaf of abelian groups under multiplications.

Here we have a theorem

> *a holomorphic function f on a simply connected domain D is invertible if and only if f has logarithm on D.*

Indeed, the logarithm is given by the integral

$$\log f(z) := \int_\gamma \frac{df}{f},$$

where γ is a path from a fixed point z_0 to z in D. Refer [Pri03] for more details.

Now, for each germ of $\mathscr{F}$ at a point x, which is also germ of $\mathscr{F}^\#$ at x, one can always find a representative of it in a simply connected neighborhood of x. In this way, we have $\mathscr{F}_x = \mathscr{O}_x^*$. As we have seen that $\mathscr{O}_X^*$ is a sheaf, it is the sheafification of $\mathscr{F}$.

Consequently, there is an exact sequence:

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\times 2\pi i} \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^* \longrightarrow 1.$$

Now we summarize the results about abelian sheaves into the following theorem.

2.c Theorem *Let X be a topological spaces. Then $\mathbf{PAb}(X)$ (resp. $\mathbf{Ab}(X)$) is an abelian category with a family of exact functors $\{\Gamma(-,U) | U \in \operatorname{ob} \mathcal{T}_X\}$ (resp. $\{\Gamma_x | x \in X\}$)* reflecting the exactness *in the sense that a sequence of abelian presheaves (resp. sheaves) is exact if and only if it is also exact after applying every functor $\Gamma(-,U)$ (resp. Γ_x).*

Proof: We have seen the statement is trivially true for presheaves. As for the statement about sheaves, we only need to show that the stalks reflect sheaf kernels and cokernels. Since the presheaf kernels are already sheaf kernels, our claim about kernels follows from the exactness of colimits (note that stalks are colimits). As for the cokernel, just note that the stalks of the sheafification of a presheaf are equal to the stalks of itself, then the argument for kernel works. □

This theorem, as with as similar statements we have seen before, can be summarized into the following slogan:

> *Presheaves can be checked at the level of open sets, while sheaves at the level of stalks.*

2.d Corollary (Section functor is left exact) *The section functor $\Gamma(-,U)$ on $\mathbf{Ab}(X)$ is left exact, but is not exact.*

3 (Limits and colimits) As in the general case, the category $\mathbf{PSh}(X)$ of *presheaves is complete and cocomplete*, and that *the limits and colimits are computed open sets by opensets* (refer Proposition 1.6). As for the stalks, one can easily check that *taking stalks commutes with all colimits and all finite limits*. But note that taking stalks in general can not commute with an arbitrary limit.

Recall that in a finite-complete and finite-cocomplete category, a functor is said to be **left exact** (resp. **right exact**) if it commutes with all finite limits (resp. colimits). A functor is said to be **exact** if it is both left exact and right exact. Note that in an abelian category those notions coincide with the usual notion of exact functors.

3.a Theorem (Limits and colimits in Sh(X)) *Let X be a topological space.*

1. *$\mathbf{Sh}(X)$ is complete and cocomplete.*
2. *The forgetful functor $F\colon \mathbf{Sh}(X) \to \mathbf{PSh}(X)$ commutes with all limits. In particular, the section functors $\Gamma(-,U)$ are left exact.*

3. *The forgetful functor $F\colon \mathbf{Sh}(X) \to \mathbf{PSh}(X)$ does* NOT *commute with colimits. However, we have*

$$\varinjlim \mathscr{F}_i = \left(\varinjlim F(\mathscr{F}_i)\right)^{\#}.$$

4. *The sheafification $\#\colon \mathbf{PSh}(X) \to \mathbf{Sh}(X)$ commutes with all colimits and all finite limits. In particular, the stalk functors Γ_x are exact.*

3.b Lemma *Let X be a topological space. Let $\{\mathscr{F}_j\}_{j\in \operatorname{ob}\mathcal{J}}$ be a filtered system of sheaves of sets. Consider the canonical map*

$$\Phi\colon \varinjlim_{\mathcal{J}} \mathscr{F}_j(U) \longrightarrow (\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U).$$

1. *If all the transition morphisms are injective then Φ is injective.*
2. *If U is quasi-compact, then Φ is injective.*
3. *If U is quasi-compact and all the transition morphisms are injective then Φ is an isomorphism.*
4. *If any covering of U can be refined by some coverings $\{U_i \to U\}_{i\in I}$ with I finite and $U_i \cap U_{i'}$ quasi-compact, then Φ is bijective.*

Proof: *1.* Assume all the transition morphisms are injective. First, the canonical maps

$$\mathscr{F}_j(U) \longrightarrow \prod_{x\in U} \mathscr{F}_{j,x}$$

are injective, thus so is the colimit

$$\varinjlim_{\mathcal{J}} \mathscr{F}_j(U) \longrightarrow \varinjlim_{\mathcal{J}} \prod_{x\in U} \mathscr{F}_{j,x}.$$

It remains to show the canonical map

$$\varinjlim_{\mathcal{J}} \prod_{x\in U} \mathscr{F}_{j,x} \longrightarrow \prod_{x\in U} \varinjlim_{\mathcal{J}} \mathscr{F}_{j,x}$$

is injective.

Let $\bar{s}$ and $\bar{t}$ be two elements of $\varinjlim_{\mathcal{J}} \prod_{x\in U} \mathscr{F}_{j,x}$ having the same image in $\prod_{x\in U} \varinjlim_{\mathcal{J}} \mathscr{F}_{j,x}$ and $s = (s_x), t = (t_x)$ be their representatives in some $\prod_{x\in U} \mathscr{F}_{j,x}$. Now, the image of $\bar{s}$ and $\bar{t}$ in $\prod_{x\in U} \varinjlim_{\mathcal{J}} \mathscr{F}_{j,x}$ can be written as $(\bar{s_x})$ and $(\bar{t_x})$, where each $\bar{s_x}$ or $\bar{t_x}$ is the image of s_x or t_x in $\varinjlim_{\mathcal{J}} \mathscr{F}_{j,x}$. Since $\bar{s_x} = \bar{t_x}$, there exists some $j_x \in \operatorname{ob}\mathcal{J}$ such that the image of s_x and t_x in $\mathscr{F}_{j_x,x}$ are the same. Then, since the transition morphism $\mathscr{F}_j \to \mathscr{F}_{j_x}$ is injective, so is the transition map $\mathscr{F}_{j,x} \to \mathscr{F}_{j_x,x}$. Therefore $s_x = t_x$. Then, we get $s = t$ and *a fortiori* $\bar{s} = \bar{t}$.

2. Assume U is quasi-compact. Let $\bar{s}$ and $\bar{t}$ be two elements of $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$ having the same image under Φ and s, t be their representatives in some $\mathscr{F}_j(U)$. Now, the image of $\bar{s}$ and $\bar{t}$ under Φ can be written as systems of compatible germs $(\bar{s}_x)$ and $(\bar{t}_x)$. For each $x \in U$, $\bar{s}_x = \bar{t}_x$ implies that there exists some open neighborhood U_x of x such that $\bar{s}|_{U_x} = \bar{t}|_{U_x}$. Then, there exists $j_x \in \operatorname{ob} \mathcal{J}$ such that the image of $s|_{U_x}$ and $t|_{U_x}$ in $\mathscr{F}_{j_x}(U_x)$ are the same. Since U is quasi-compact, the covering $\{U_x \to U\}_{x \in U}$ has a finite subcovering $\{U_i \to U\}_{i \in I}$. For this covering, we can take j_0 to be the index such that there are arrows $j_i \to j_0$. Now, the image of $s|_{U_i}$ and $t|_{U_i}$ in $\mathscr{F}_{j_0}(U_i)$ are the same. Then s and t maps to the same element in $\mathscr{F}_{j_0}(U)$, *a fortiori* in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$.

3. Assume U is quasi-compact and all the transition morphisms are injective. Then Φ is injective. It suffices to show it is surjective. Let $(\bar{s_x})$ be an element in $(\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U)$, where each $\bar{s_x}$ belongs to the stalk $\varinjlim_{\mathcal{J}} \mathscr{F}_{j,x}$. Then for each $\bar{s_x}$, let s_x be its representative in some $\mathscr{F}_{j_x,x}$ and (U_x, s^x) be the representative of s_x. Note that the image of $s^x|_{U_x \cap U_y}$ and $s^y|_{U_x \cap U_y}$ in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_x \cap U_y)$ have the same image under Φ. Thus, by *1.*, they are the same.

Since U is quasi-compact, the covering $\{U_x \to U\}_{x \in U}$ has a finite subcovering $\{U_i \to U\}_{i \in I}$. For this covering, we can take j_0 to be the index such that there are arrows $j_i \to j_0$. Then the sections $s^i \in \mathscr{F}_{j_i}(U_i)$ give rise to sections $s_i \in \mathscr{F}_{j_0}(U_i)$. Since $s_i|_{U_i \cap U_{i'}}$ and $s_{i'}|_{U_i \cap U_{i'}}$ have the same image in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \cap U_{i'})$, by the similar argument in *1.*, they are the same. Therefore (s_i) is a system of compatible sections, and thus gives a section $s \in \mathscr{F}_{j_0}(U)$ such that $s|_{U_i} = s_i$. Then, this s gives an element $\bar{s}$ of $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$ and one can see it maps to $(\bar{s_x})$ under Φ.

4. Assume the hypothesis of *4.* It is obvious that U is quasi-compact. It suffices to show Φ is surjective. Let $(\bar{s_x})$ be an element in $(\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U)$, where each $\bar{s_x}$ belongs to the stalk $\varinjlim_{\mathcal{J}} \mathscr{F}_{j,x}$. Then for each $\bar{s_x}$, let s_x be its representative in some $\mathscr{F}_{j_x,x}$ and (U_x, s^x) be the representative of s_x.

Now, the covering $\{U_x \to U\}_{x \in U}$ can be refined by a finite covering $\{U_i \to U\}_{i \in I}$ such that $U_i \cap U_{i'}$ are quasi-compact. Since the image of $s^i|_{U_i \cap U_{i'}}$ and $s^{i'}\Big|_{U_i \cap U_{i'}}$ in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \cap U_{i'})$ have the same image under Φ, by *2.*, they are the same and thus there exists $j_{ii'} \in \operatorname{ob} \mathcal{J}$ such that $s^i|_{U_i \cap U_{i'}}$ and $s^{i'}\Big|_{U_i \cap U_{i'}}$ have the same image in $\mathscr{F}_{j_{ii'}}(U_i \cap U_{i'})$.

Now, we can take j_0 to be the index such that there are arrows $j_{ii'} \to j_0$. Then the sections $s^i \in \mathscr{F}_{j_i}(U_i)$ give rise to sections $s_i \in \mathscr{F}_{j_0}(U_i)$ and furthermore, they form a system of compatible sections. Thus we get a section $s \in \mathscr{F}_{j_0}(U)$ such that $s|_{U_i} = s_i$. Then, this s gives an element $\bar{s}$ of

$\varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$ and one can see it maps to $(\bar{s_x})$ under Φ. $\square$

3.c Example Let $X = I \cup \mathbb{N}$, where $I = \{x_1, \cdots, x_k\}$ is a finite set. Given a topology on X as following: U is an open subset if and only if it is a subset of $\mathbb{N}$ or a union of $\mathbb{N}$ with some subset of I. Write $n \in \mathbb{N}$ as ξ_n. Let $U_n = \{\xi_n, \xi_{n+1}, \cdots\}$ and $j_n\colon U_n \to X$ be the inclusions. Set $\mathscr{F}_n = j_{n,*}\underline{S}$ (refer 5.1 and 3.6.a) and transition morphisms induced by inclusions $U_n \to U_m$. This gives a filtered system of sheaves indexed by $\mathbb{N}$. Let $\mathscr{F} = \varinjlim_{\mathbb{N}} \mathscr{F}_n$.

For $m < n$, we have $\mathscr{F}_{n,\xi_m} = *$ since $\{\xi_m\}$ is a open neighborhood of ξ_m missing U_n. Therefore, passing to the colimit, we have $\mathscr{F}_{\xi_m} = *$ for all $m \in \mathbb{N}$. On the other hand, since for any open neighborhood U of x_i, we have

$$\mathscr{F}_n(U) = \mathscr{F}_n(\mathbb{N}) = \underline{S}(U_n) = \prod_{m \geqslant n} S,$$

so $\mathscr{F}_{n,x_i} = \prod_{m \geqslant n} S$ and thus $\mathscr{F}_{x_i}$ is the colimit

$$M := \varinjlim_{n \in \mathbb{N}} \prod_{m \geqslant n} S.$$

Now, by Theorem 3.a, we can see that $\mathscr{F}$ is the direct sum of the *skyscraper sheaves* with value M at the closed points $x_1, \cdots, x_k$.

Now, $\mathscr{F}(X) = \bigoplus_{i=1}^k M$, while $\varinjlim_{\mathbb{N}} \mathscr{F}_n(X) = \varinjlim_{\mathbb{N}} \underline{S}(U_n) = M$.

4 (Sheaf Hom) Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ a sheaf on a topological space X. Define the presheaf $\mathcal{H}om(\mathscr{F}, \mathscr{G})$ by

$$\mathcal{H}om(\mathscr{F}, \mathscr{G})(U) := \mathrm{Hom}_{\mathbf{PSh}(U)}(\mathscr{F}|_U, \mathscr{G}|_U).$$

By the following Lemma 4.a, this is indeed a sheaf, called the **sheaf Hom**.

4.a Lemma (Gluing morphisms) *Let X be a topological space with a covering $\{U_i \subset X\}_{i \in I}$. Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ a sheaf on X. Suppose that there are morphisms*

$$\varphi_i\colon \mathscr{F}|_{U_i} \longrightarrow \mathscr{G}|_{U_i}$$

such that for all $i, j \in I$, the restrictions of φ_i and φ_j to $U_i \cap U_j$ are the same morphism $\varphi_{ij}\colon \mathscr{F}|_{U_i \cap U_j} \to \mathscr{G}|_{U_i \cap U_j}$. Then there exists a unique morphism

$$\varphi\colon \mathscr{F} \longrightarrow \mathscr{G}$$

whose restriction to each U_i is φ_i.

4.b Lemma *There exist a canonical map $\mathcal{H}om(\mathscr{F}, \mathscr{G})_x \to \mathrm{Hom}(\mathscr{F}_x, \mathscr{G}_x)$.*

Proof: By the functorality of stalks, there are canonical maps

$$\mathrm{Hom}_{\mathbf{PSh}(U)}(\mathscr{F}|_U, \mathscr{G}|_U) \to \mathrm{Hom}(\mathscr{F}_x, \mathscr{G}_x)$$

for all neighborhood U of x. Thus the existence of required canonical map follows from the universal property of colimits. $\square$

4.c Example (Sheaf Hom doesn't commute with taking stalks) Let X be a topological space and x a non-isolated closed point in X. Let S be a *nontrivial*, meaning neither empty or singleton, set.

First, $\mathcal{H}om(i_{x,*}S, \underline{S})_x \to \mathrm{Hom}((i_{x,*}S)_x, \underline{S}_x)$ *is not surjective.* Indeed, since any section s of $i_{x,*}S$ is trivial away from x, thus so is its image under any morphism $i_{x,*}S \to \underline{S}$. But this implies that the image of s is the trivial section. Thus $\mathcal{H}om(i_{x,*}S, \underline{S})$ is the trivial sheaf and thus $\mathcal{H}om(i_{x,*}S, \underline{S})_x$ is a singleton. On the other hand, $\mathrm{Hom}((i_{x,*}S)_x, \underline{S}_x) = \mathrm{Hom}(S, S)$, which is definitely nontrivial.

Secondly, let $V = X \setminus \{x\}$ and $\mathscr{F}$ be the sheaf satisfying $\mathscr{F}|_V = \underline{S}_V$ and $\mathscr{F}(U) = \varnothing$ if $U \not\subset V$. Then $\mathcal{H}om(\mathscr{F}, \mathscr{F})_x \to \mathrm{Hom}(\mathscr{F}_x, \mathscr{F}_x)$ *is not injective.* Indeed, as $\mathscr{F}_x = \varnothing$, so $\mathrm{Hom}(\mathscr{F}_x, \mathscr{F}_x)$ is a singleton. On the other hand, $\mathrm{Hom}(\mathscr{F}|_U, \mathscr{F}|_U)$ is nontrivial, thus so is the colimit $\mathcal{H}om(\mathscr{F}, \mathscr{F})_x$.

4.d Lemma *Let $\mathscr{F}$ be a sheaf on a topological space X, then $\mathcal{H}om(\underline{*}, \mathscr{F}) \cong \mathscr{F}$.*

Proof: For any $\varphi \in \mathrm{Hom}_{\mathbf{PSh}(U)}(\underline{*}_U, \mathscr{F}|_U)$, its corresponding element in $\mathscr{F}(U)$ is the image of the singleton under $\varphi(U)$. This gives rise to a morphism $\Phi\colon \mathcal{H}om(\underline{*}, \mathscr{F}) \to \mathscr{F}$. To show it is an isomorphism, we check it at the level of stalks.

Let φ_x, ψ_x be two germs of $\mathcal{H}om(\underline{*}, \mathscr{F})$ at x having the same image under Φ_x. Then taking representatives (U_φ, φ) and (U_ψ, ψ) of φ_x and ψ_x respectively, there exists a neighborhood U_x of x such that $U_x \subset U_\varphi \cap U_\psi$ and that $\varphi(U_\varphi)(*)|_{U_x} = \Phi(U_\varphi)(\varphi)|_{U_x} = \Phi(U_\psi)(\psi)|_{U_x} = \psi(U_\psi)(*)|_{U_x}$. Then for any open subset V of U_x, we have $\varphi(V)(*) = \varphi(U_\varphi)(*)|_V = \psi(U_\psi)(*)|_V = \psi(V)(*)$. Therefore $\varphi|_{U_x} = \psi|_{U_x}$ and thus $\varphi_x = \psi_x$.

Let s_x be any germ of $\mathscr{F}$ at x and take a representative (U, s) of it. Define $\varphi\colon \underline{*}_U \to \mathscr{F}|_U$ as $\varphi(V)(*) = s|_V$ for all open subsets $V \subset U$. Then the germ of φ at x will be mapped to s_x under Φ_x. $\square$

Remark The notion of sheaf Hom also works for sheaves of algebraic structures and the above results still hold.

4.e Lemma (Sheaf Hom is left exact) *Let $\mathscr{F}$ be a sheaf on a topological space X, then $\mathcal{H}om(\mathscr{F}, -)$ is a left exact covariant functor and $\mathcal{H}om(-, \mathscr{F})$ is a left exact contravariant functor.*

4.f Lemma (Sheaf Hom is the internal Hom) *For any sheaves $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ on a topological space X, there is a canonical bijection*

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(\mathscr{F} \times \mathscr{G}, \mathscr{H}) \cong \mathrm{Hom}_{\mathbf{Sh}(X)}(\mathscr{F}, \mathcal{H}om(\mathscr{G}, \mathscr{H})).$$

5 (Gluing data) Let X be a topological space and $\mathfrak{U} = \{U_i \subset X\}_{i \in I}$ a covering on X. A **gluing data for sheaves of sets with respect to the covering** $\mathfrak{U}$ consists of the following stuff:

- For each $i \in I$, a sheaf $\mathscr{F}_i$ of sets on U_i;
- For each pair $i, j \in I$, an isomorphism $\varphi_{ij} \colon \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j|_{U_i \cap U_j}$,

satisfying the **cocycle condition**:

For any $i, j, k \in I$, the following diagram commutes.

$$\begin{tikzcd}
\mathscr{F}_i|_{U_i \cap U_j \cap U_k} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & \mathscr{F}_k|_{U_i \cap U_j \cap U_k} \\
& \mathscr{F}_j|_{U_i \cap U_j \cap U_k} \arrow[ur, "\varphi_{jk}"'] &
\end{tikzcd}$$

One can see this definition can be easily generalized to **gluing data for sheaves of algebraic structures**.

5.a Lemma *Let X be a topological space and $\mathfrak{U} = \{U_i \subset X\}_{i \in I}$ a covering on X. Let $(\mathscr{F}_i, \varphi_{ij})$ be a gluing data for sheaves of sets with respect to the covering $\mathfrak{U}$. Then there exists a sheaf $\mathscr{F}$ on X together with isomorphisms*

$$\varphi_i \colon \mathscr{F}|_{U_i} \longrightarrow \mathscr{F}_i$$

such that the following diagram commutes.

$$\begin{tikzcd}
\mathscr{F}|_{U_i \cap U_j} \arrow[r, "\varphi_i"] \arrow[d, equal] & \mathscr{F}_i|_{U_i \cap U_j} \arrow[d, "\varphi_{ij}"] \\
\mathscr{F}|_{U_i \cap U_j} \arrow[r, "\varphi_j"] & \mathscr{F}_j|_{U_i \cap U_j}
\end{tikzcd}$$

The similar statement holds for sheaves of algebraic structures.

Proof: For any open subset W of X, the object $\mathscr{F}(W)$ is given as the equalizer of the morphisms:

$$\textstyle\prod_{i \in I} \mathscr{F}_i(W \cap U_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}_i(W \cap U_i \cap U_j).$$

For sheaves of sets, this set can be written as

$$\mathscr{F}(W) = \left\{ (s_i) \in \prod_{i \in I} \mathscr{F}_i(W \cap U_i) \,\middle|\, \varphi_{ij}(\, s_i|_{W \cap U_i \cap U_j}) = s_j|_{W \cap U_i \cap U_j} \right\}.$$

As for the isomorphism, just note that a section in $\mathscr{F}|_{U_i}(W)$ is nothing but a system of compatible sections $s_j \in \mathscr{F}_i(W \cap U_i \cap U_j)$, which gives rise to a section $s \in \mathscr{F}_i(W)$. Thus the lemma follows. $\square$

Obviously, any sheaf $\mathscr{F}$ admits a gluing data $(\mathscr{F}_i, \varphi_{ij})$, where $\mathscr{F}_i$ is the restriction $\mathscr{F}|_{U_i}$ and φ_{ij} is the induced morphism

$$\mathscr{F}|_{U_i}\big|_{U_i \cap U_j} \longrightarrow \mathscr{F}|_{U_j}\Big|_{U_i \cap U_j}.$$

Moreover, this construction is functoral, meaning it gives rise to a functor from $\mathbf{Sh}(X)$ to the category of gluing data.

5.b Theorem (Sheaf = gluing data) *The above functor induces an equivalence of category between* $\mathbf{Sh}(X)$ *and the category of gluing data. The similar statement holds for sheaves of algebraic structures.*

Proof: The functor is fully faithful by Lemma 4.a and essentially surjective by Lemma 5.a. □

§ 5 Sheaves on topological spaces: continuous maps

In this section, we give the *direct images* and *inverse images* induced by a continuous map. In some special cases, the inverse image even has a left adjoint.

1 (Direct images of sheaves) Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{F}$ a presheaf on X. Define $f_*\mathscr{F}$ by

$$f_*\mathscr{F}(V) := \mathscr{F}(f^{-1}(V))$$

with obvious restriction maps. These data form a presheaf on Y, called the **direct image** or **pushforward** of $\mathscr{F}$ by f. This construction is functoral, thus we get a functor:

$$f_*\colon\ \mathbf{PSh}(X) \longrightarrow \mathbf{PSh}(Y).$$

1.a Lemma *Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{F}$ a sheaf on X. Then $f_*\mathscr{F}$ is a sheaf on Y.*

Proof: Note that if $\{V_i \subset V\}$ is a covering in Y, then $\{f^{-1}(V_i) \subset f^{-1}(V)\}$ is a covering in X. Thus the descent condition for $f_*\mathscr{F}$ follows from the descent condition for $\mathscr{F}$. □

As a consequence, we get the functor

$$f_*\colon\ \mathbf{Sh}(X) \longrightarrow \mathbf{Sh}(Y).$$

1.b Lemma *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous maps of topological spaces. Then the functors $(g \circ f)_*$ and $g_* \circ f_*$ are equal.*

Proof: This is because $(g \circ f)^{-1}(W) = f^{-1}g^{-1}(W)$. □

1.c Example (Skyscraper sheaves) The *skyscraper sheaf* $i_{x,*}S$ is the direct image of the constant sheaf $\underline{S}$ on a one-point space x, under the inclusion morphism $i_x\colon x \to X$.

1.d Lemma *Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{F}$ a sheaf on X. If $f(x) = y$, then there is a canonical map $(f_*\mathscr{F})_y \to \mathscr{F}_x$.*

Proof: Note that

$$(f_*\mathscr{F})_y = \varinjlim_{y\in V} f_*\mathscr{F}(V) = \varinjlim_{y\in V} \mathscr{F}(f^{-1}(V)).$$

and that $\{f^{-1}(V)|y \in V\} \subset \{U|x \in U\}$. Then there exists a unique map $(f_*\mathscr{F})_y \to \mathscr{F}_x$ compatible with the restriction maps.

Let s_y be a germ of $f_*\mathscr{F}$ at $y \in Y$. Then its image under this canonical map can be describe as follows. Let (V, s) be a representative of s_y. Since $s \in \mathscr{F}(f^{-1}(V))$, it represents a germ s_x of $\mathscr{F}$ at x. One can see this s_x is independent of the choice of representative and is the image of s_y. $\square$

2 (Inverse images of presheaves) Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{G}$ a presheaf on Y. Define

$$f_p\mathscr{G}(U) := \varinjlim_{f(U)\subset V} \mathscr{G}(V)$$

with the restriction maps induces from the canonical maps between colimits. These data form a presheaf on X, called the **inverse image** or **pullback** of $\mathscr{G}$ by f.

2.a Theorem (Inverse and direct images are adjoint) *Let $f\colon X \to Y$ be a continuous map of topological spaces, then $f_p \dashv f_*$ form an* adjoint pair*.*

Proof: Recall that a pair of functors L and R is called a **adjoint pair** or **adjunction** if they admit two natural transformations $\eta\colon \mathrm{id} \Rightarrow RL$ and $\epsilon\colon LR \Rightarrow \mathrm{id}$ satisfy the **triangle identities**.

$$\begin{tikzcd} & LRL \arrow[dr, "\epsilon * L"] & \\ L \arrow[ur, "L*\eta"] \arrow[rr, "\mathrm{id}"] & & L \end{tikzcd} \qquad \begin{tikzcd} & RLR \arrow[dr, "R*\epsilon"] & \\ R \arrow[ur, "\eta * R"] \arrow[rr, "\mathrm{id}"] & & R \end{tikzcd}$$

For details, refer §4.1 in my note *BMO*, or [Bor94] directly.

Let $\mathscr{F}$ be a presheaf on X and $\mathscr{G}$ a preshaef on Y.

First, note that the index system of the colimit

$$f_p\mathscr{G}(f^{-1}(V)) = \varinjlim_{V\subset V'} \mathscr{G}(V')$$

contains V itself. Thus we get a map $\mathscr{G}(V) \to f_p\mathscr{G}(f^{-1}(V))$, which induces a canonical morphism $\eta_{\mathscr{G}}\colon \mathscr{G} \to f_* f_p \mathscr{G}$.

Next, consider the colimit

$$f_p f_* \mathscr{F}(U) = \varinjlim_{f(U)\subset V} f_*\mathscr{F}(V) = \varinjlim_{f(U)\subset V} \mathscr{F}(f^{-1}(V)).$$

Since for each $V \supset f(U)$, there is a restriction map $\mathscr{F}(f^{-1}(V)) \to \mathscr{F}(U)$, we obtain a canonical map $f_p f_* \mathscr{F}(U) \to \mathscr{F}(U)$, which induces a canonical morphism $\epsilon_{\mathscr{F}}\colon f_p f_* \mathscr{F} \to \mathscr{F}$.

One can check that η and ϵ are natural transformations and that the following compositions are identities.

$$f_p\mathscr{G} \xrightarrow{f_p(\eta_{\mathscr{G}})} f_p f_* f_p \mathscr{G} \xrightarrow{\epsilon_{f_p\mathscr{G}}} f_p\mathscr{G}, \qquad f_*\mathscr{F} \xrightarrow{\eta_{f_*\mathscr{F}}} f_* f_p f_* \mathscr{F} \xrightarrow{f_*(\epsilon_{\mathscr{F}})} f_*\mathscr{F}.$$

This shows the triangle identities and thus $f_p \dashv f_*$ are an adjoint pair. $\square$

Remark One may except to show that f_p is *left adjoint* to f_*, i.e. to prove the following natural bijection

$$\mathrm{Hom}_{\mathbf{PSh}(X)}(f_p\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathbf{PSh}(Y)}(\mathscr{G}, f_*\mathscr{F}).$$

This follows directly from the adjoint pair: for $\phi\colon f_p\mathscr{G} \to \mathscr{F}$ a morphism of presheaves on X, the corresponding morphism is the composition

$$\mathscr{G} \xrightarrow{\eta_{\mathscr{G}}} f_* f_p \mathscr{G} \xrightarrow{f_*\phi} f_*\mathscr{F};$$

for $\psi\colon \mathscr{G} \to f_*\mathscr{F}$ a morphism of presheaves on Y, the corresponding morphism is the composition

$$f_p\mathscr{G} \xrightarrow{f_p(\psi)} f_p f_* \mathscr{F} \xrightarrow{\epsilon_{\mathscr{F}}} \mathscr{F}.$$

2.b Corollary *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous maps of topological spaces. Then the functors $(g \circ f)_p$ and $g_p \circ f_p$ are equal.*

Proof: This follows from the uniqueness of adjoint functor and Lemma 1.b. $\square$

2.c Lemma *Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{G}$ a sheaf on Y. Then there is a canonical bijection $(f_p\mathscr{G})_x \cong \mathscr{G}_{f(x)}$.*

Proof: This can be shown as follows.

$$\begin{aligned}(f_p\mathscr{G})_x &= \varinjlim_{x\in U} f_p\mathscr{G}(U) \\ &= \varinjlim_{x\in U} \varinjlim_{f(U)\subset V} \mathscr{G}(V) \\ &= \varinjlim_{f(x)\in V} \mathscr{G}(V) \\ &= \mathscr{G}_{f(x)}\end{aligned}$$

$\square$

3 **(Inverse images of sheaves)** Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{G}$ a sheaf on Y. Then we already has a presheaf $f_p\mathscr{G}$, which is called the *inverse image* of $\mathscr{G}$ by f. However, this $f_p\mathscr{G}$ is rarely a sheaf. So we define the **inverse image** or **pullback** of $\mathscr{G}$ by f as the sheafification of $f_p\mathscr{G}$, i.e.

$$f^*\mathscr{G} := (f_p\mathscr{G})^{\#}.$$

3.a **Theorem (Inverse and direct images are adjoint)** *Let $f\colon X \to Y$ be a continuous map of topological spaces, then $f^* \dashv f_*$ form an* adjoint pair*.*

Proof: Consider the following commutative digram.

$$\begin{tikzcd}
\mathbf{Sh}(X) \arrow[r, shift left, "f_*"] \arrow[d, shift left, "F"] & \mathbf{Sh}(Y) \arrow[l, shift left, "f^*"] \arrow[d, shift left, "F"] \\
\mathbf{PSh}(X) \arrow[r, shift left, "f_*"] \arrow[u, shift left, "\#"] & \mathbf{PSh}(Y) \arrow[l, shift left, "f_p"] \arrow[u, shift left, "\#"]
\end{tikzcd}$$

Except the upper one, all pairs are adjoint pairs, thus so is the upper one.

More precisely, this can be shown by

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Sh}(X)}(f^*\mathscr{G}, \mathscr{F}) &= \mathrm{Hom}_{\mathbf{PSh}(X)}(f_p\mathscr{G}, \mathscr{F}) \\
&= \mathrm{Hom}_{\mathbf{PSh}(Y)}(\mathscr{G}, f_*\mathscr{F}) \\
&= \mathrm{Hom}_{\mathbf{Sh}(Y)}(\mathscr{G}, f_*\mathscr{F}).
\end{aligned}$$ □

3.b **Corollary** *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous maps of topological spaces. Then the functors $(g \circ f)^*$ and $g^* \circ f^*$ are equal.*

Proof: This follows from the uniqueness of adjoint functor and Lemma 1.b. □

3.c **Lemma** *Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{G}$ a sheaf on Y. Then there is a canonical bijection $(f^*\mathscr{G})_x \cong \mathscr{G}_{f(x)}$.*

Proof: This follows from Lemma 3.8.b and Lemma 2.c. □

4 **(f-maps)** Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{F}$ a sheaf on X, $\mathscr{G}$ a sheaf on Y. Combine the inverse and direct image functors, we define the **f-map** $\xi\colon \mathscr{G} \to \mathscr{F}$ as a morphism from $\mathscr{G}$ to $f_*\mathscr{F}$.

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous maps of topological spaces and $\mathscr{F}$ a sheaf on X, $\mathscr{G}$ a sheaf on Y, $\mathscr{H}$ a sheaf on Z. Let $\phi\colon \mathscr{G} \to \mathscr{F}$ be a f-map and $\psi\colon \mathscr{H} \to \mathscr{G}$ a g-map, then the *composition* $\psi \circ \phi$ of them is the $(g \circ f)$-map defined as the composition

$$\mathscr{H} \xrightarrow{\psi} g_*\mathscr{G} \xrightarrow{g_*\phi} g_*f_*\mathscr{F} = (g \circ f)_*\mathscr{F}.$$

Any f-map $\phi\colon \mathscr{G} \to \mathscr{F}$ gives rise to canonical maps at stalks:

$$\phi_x\colon \mathscr{G}_{f(x)} \longrightarrow \mathscr{F}_x,$$

which are given by the compositions:

$$\mathscr{G}_{f(x)} \longrightarrow (f_*\mathscr{F})_{f(x)} \longrightarrow \mathscr{F}_x.$$

5 Remark Now all the above constructions also appear in general case. Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{F}$ a sheaf on X, $\mathscr{G}$ a sheaf on Y, both with values in an algebraic category $\mathcal{A}$. Let $\mathbf{PSh}(X, \mathcal{A})$ (resp. $\mathbf{Sh}(X, \mathcal{A})$) denote the category of presheaves (resp. sheaves) on X with values in $\mathcal{A}$. Then we have the following functors

$$\begin{aligned}
f_*\colon\ & \mathbf{PSh}(X, \mathcal{A}) \longrightarrow \mathbf{PSh}(Y, \mathcal{A}) \\
f_*\colon\ & \mathbf{Sh}(X, \mathcal{A}) \longrightarrow \mathbf{Sh}(Y, \mathcal{A}) \\
f_p\colon\ & \mathbf{PSh}(Y, \mathcal{A}) \longrightarrow \mathbf{PSh}(X, \mathcal{A}) \\
f^*\colon\ & \mathbf{Sh}(Y, \mathcal{A}) \longrightarrow \mathbf{Sh}(X, \mathcal{A})
\end{aligned}$$

which are compatible with the forgetful functor $F\colon \mathcal{A} \to \mathbf{Set}$.

We also have some formulas:

$$\begin{aligned}
f_*\mathscr{F}(V) &= \mathscr{F}(f^{-1}(V)), \\
f_p\mathscr{G}(U) &= \varinjlim_{f(U)\subset V} \mathscr{G}(V), \\
f^*\mathscr{G} &= (f_p\mathscr{G})^{\#}, \\
(f_p\mathscr{G})_x &= \mathscr{G}_{f(x)}, \\
(f^*\mathscr{G})_x &= \mathscr{G}_{f(x)}.
\end{aligned}$$

What's most important is the adjoint pairs:

$$\begin{aligned}
f_p \dashv f_*\colon\ & \mathbf{PSh}(X, \mathcal{A}) \leftrightarrows \mathbf{PSh}(Y, \mathcal{A}), \\
f^* \dashv f_*\colon\ & \mathbf{Sh}(X, \mathcal{A}) \leftrightarrows \mathbf{Sh}(Y, \mathcal{A})
\end{aligned}$$

Finally, the notion of *f-maps* also works.

Now we turn to consider a special kind of continuous maps. They are the *immersions* of subspaces.

6 Lemma (Inverse images by a open immersion) *Let X be a topological space and $j\colon U \to X$ an inclusion map of a open subset U into X.*

1. *Let $\mathscr{G}$ be a presheaf on X. Then the presheaf $j_p\mathscr{G}$ is given by $V \mapsto \mathscr{G}(V)$ for all open subsets V of U.*

2. *Let $\mathscr{G}$ be a sheaf on X. Then the sheaf $j^*\mathscr{G}$ is given by $V \mapsto \mathscr{G}(V)$ for all open subsets V of U.*

3. *For any point $u \in U$ and any sheaf $\mathscr{G}$ on X we have a canonical identification of stalks*

$$j^*\mathscr{G}_u = (\mathscr{G}|_U)_u = \mathscr{G}_u.$$

4. *We have $j_p j_* = \mathrm{id}$ in $\mathbf{PSh}(U)$ and $j^* j_* = \mathrm{id}$ in $\mathbf{Sh}(U)$.*

The same description holds for (pre)sheaves of algebraic structures.

Proof: Note that V is *cofinal* in the system $\{W|V \subset W\}$, thus the first two results follow. Then, *3* follows from the fact that neighborhoods of u which are contained in U is *cofinal* in the system of all open neighborhoods of u in X. Finally, *4* follows from direct computing. □

Remark One can see the (pre)sheaves in *1* and *2* are precisely the *restrictions* $\mathscr{G}|_U$ of $\mathscr{G}$ on a open subset U.

In the case of open immersions, there is a left adjoint functor to f^*.

7 **(Extension by zero)** Let X be a topological space and $j\colon U \to X$ an inclusion map of a open subset U into X.

- Let $\mathscr{F}$ be a presheaf on U. Define the **extension of $\mathscr{F}$ by empty** $j_{p!}\mathscr{F}$ as the presheaf given by

$$j_{p!}\mathscr{F}(V) := \begin{cases} \mathscr{F}(V) & \text{if } V \subset U \\ \varnothing & \text{if } V \not\subset U \end{cases}$$

  with obvious restriction maps.

- Let $\mathscr{F}$ be a sheaf on U. Define the **extension of $\mathscr{F}$ by empty** $j_!\mathscr{F}$ as the sheafification of the presheaf $j_{p!}\mathscr{F}$.

For sheaves of algebraic structures, there are similar notions. Let 0 denote the initial object in an algebraic category $\mathcal{A}$.

- Let $\mathscr{F}$ be a presheaf on U with values in $\mathcal{A}$. Define the **extension of $\mathscr{F}$ by zero** $j_{p!}\mathscr{F}$ as the presheaf given by

$$j_{p!}\mathscr{F}(V) := \begin{cases} \mathscr{F}(V) & \text{if } V \subset U \\ 0 & \text{if } V \not\subset U \end{cases}$$

  with obvious restriction maps.

- Let $\mathscr{F}$ be a sheaf on U with values in $\mathcal{A}$. Define the **extension of $\mathscr{F}$ by zero** $j_!\mathscr{F}$ as the sheafification of the presheaf $j_{p!}\mathscr{F}$.

Remark Although we can define the extension by zero for general sheaves of algebraic structures, but this construction depends on what the initial object is. For instance, the extension by zero of a sheaf of rings in the category of sheaves of rings is different from the one in the category of abelian sheaves. In particular, the functor $j_!$ *doesn't commute with taking underlying sheaves of sets* as other functors!

7.a Example ($j_{p!}\mathscr{F}$ is not a sheaf) Let U be the an open interval in $X = \mathbb{R}$ and $\mathscr{F}$ the sheaf of continuous functions on U. Then $j_{p!}\mathscr{F}$ is not a sheaf since one can definitely glue a nonzero function with zeros near the boundary of U with zero functions outside U to get a nonzero function on X, which does not lie in $j_{p!}\mathscr{F}(X)$.

7.b Theorem (Extension by zero is left adjoint to restriction) *Let X be a topological space and $j\colon U \to X$ an inclusion map of a open subset $U \subset X$.*

1. *$j_{p!}$ is left adjoint to the restriction j_p.*

2. *$j_!$ is left adjoint to the restriction j^*.*

3. *The stalks of the sheaf $j_!\mathscr{F}$ are described as follows*

$$j_!\mathscr{F}_x = \begin{cases} \mathscr{F}_x & \text{if } x \in U \\ \varnothing & \text{if } x \notin U. \end{cases}$$

4. *We have $j_p j_{p!} = \mathrm{id}$ in $\mathbf{PSh}(U)$ and $j^* j_! = \mathrm{id}$ in $\mathbf{Sh}(U)$.*

The same results hold for (pre)sheaves of algebraic structures except that $\varnothing$ should be replaced by an initial object 0.

Proof: Let $\mathscr{F}$ be a presheaf on U and $\mathscr{G}$ a presheaf on X. First, as $j_{p!}\mathscr{F}$ vanishes outside U, a morphism from $j_{p!}\mathscr{F}$ to $\mathscr{G}$ is determined by its components on open subsets of U, which form a morphism from $\mathscr{F}$ to $\mathscr{G}|_U$. This shows the adjointness of $j_{p!} \dashv j_p$. Then the adjointness of $j_! \dashv j^*$ follows from this and the adjointness of $\# \dashv F$. The rests are from direct computing. $\square$

We say that a sheaf $\mathscr{F}$ **vanishes** at a point x if $\mathscr{F}_x$ is an initial object.

7.c Corollary *Let X be a topological space and $j\colon U \to X$ an inclusion map of a open subset $U \subset X$. Then the functor*

$$j_!\colon \mathbf{Sh}(U) \longrightarrow \mathbf{Sh}(X)$$

is fully faithful. Moreover, this functor induces an equivalence between $\mathbf{Sh}(U)$ and the full subcategory of $\mathbf{Sh}(X)$ consisting of sheaves vanishing outside of U. The same result holds for sheaves of algebraic structures.

Proof: $j_!$ is fully faithful since $j^*j_! = \mathrm{id}$.As for the second statement, just note that the canonical morphism $\epsilon_{\mathscr{G}} \colon f_!f^*\mathscr{G} \to \mathscr{G}$ is an isomorphism if $\mathscr{G}$ vanishes outside of U. $\square$

8 Lemma (Direct image by a closed immersion) *Let X be a topological space and $i \colon Z \to X$ an inclusion map of a closed subset $Z \subset X$.*

1. *Let $\mathscr{F}$ be a sheaf on Z. Then the stalks of the sheaf $i_*\mathscr{F}$ on X can be described as*
$$i_*\mathscr{F}_x = \begin{cases} \mathscr{F}_x & \text{if } x \in Z \\ * & \text{if } x \notin Z. \end{cases}$$
2. *We have $i^*i_* = \mathrm{id}$ in $\mathbf{Sh}(Z)$.*

The same results hold for sheaves of algebraic structures.

Proof: Note that as a sheaf $\mathscr{F}$ should map empty set to a terminal object. Then the results follow. $\square$

8.a Theorem *Let X be a topological space and $i \colon Z \to X$ an inclusion map of a closed subset $Z \subset X$. Then the functor*

$$i_* \colon \mathbf{Sh}(Z) \longrightarrow \mathbf{Sh}(X)$$

is fully faithful. Moreover, this functor induces an equivalence between $\mathbf{Sh}(Z)$ and the full subcategory of $\mathbf{Sh}(X)$ consisting of sheaves $\mathscr{G}$ satisfying $\mathscr{G}_x = *$ for all $x \in X \setminus Z$. The same result holds for sheaves of algebraic structures.

Proof: i_* is fully faithful since $i^*i_* = \mathrm{id}$.As for the second statement, just note that the canonical morphism $\eta_{\mathscr{G}} \colon \mathscr{G} \to i_*i^*\mathscr{G}$ is an isomorphism if $\mathscr{G}_x = *$ for all $x \in X \setminus Z$. $\square$

8.b Remark (Direct image has no right adjoint) Let X be a topological space and $i \colon Z \to X$ an inclusion map of a closed subset $Z \subset X$. Let $x \in X \setminus Z$ and $\mathscr{F}$ be a sheaf on Z. Then $i_*\mathscr{F}_x = *$. Let $\mathscr{F} = \underline{*} \sqcup \underline{*}$, then $i_*\mathscr{F}_x = * \neq i_*(\underline{*})_x \sqcup i_*(\underline{*})_x$. This shows that the functor i_* is *NOT* right exact, hence can not has a right adjoint functor.

However, this is not the case for abelian sheaves. In fact, i_* on abelian sheaves is exact and does have right adjoint.

§ 6 Sheafification

In this section, we construct the *sheafification* for presheaves on a site through the *zeroth Čech cohomology*.

1 **(Zeroth Čech cohomology)** Recall that a presheaf $\mathscr{F}$ is a sheaf respect to the coverage Cov if for every covering $\mathfrak{U} = \{U_i \to U\}_{i\in I} \in \mathrm{Cov}$, the $\mathscr{F}(U)$ is the equalizer of the maps:

$$\prod\nolimits_{i\in I} \mathscr{F}(U_i) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod\nolimits_{i_0,i_1\in I} \mathscr{F}(U_{i_0} \times_U U_{i_1}).$$

In general, the equalizer is not $\mathscr{F}(U)$ and its value depends on the covering $\mathfrak{U}$. This set is called the **zeroth Čech cohomology** of the presheaf $\mathscr{F}$ respect to the covering $\mathfrak{U}$ and is denoted by $\check{H}^0(\mathfrak{U}, \mathscr{F})$.

By the universal property of equalizers, there is a canonical map

$$\mathscr{F}(U) \to \check{H}^0(\mathfrak{U}, \mathscr{F}).$$

Then the *descent condition* turns out to say a presheaf $\mathscr{F}$ is a sheaf if and only if this canonical map is bijective for every covering.

Now we focus on the zeroth Čech cohomology $\check{H}^0(\mathfrak{U}, \mathscr{F})$. Let $\mathcal{J}_U$ denote the category of coverings of U.

First of all, any morphism $f\colon \mathfrak{U} \to \mathfrak{V}$ of coverings induces a map

$$f^*\colon \check{H}^0(\mathfrak{V}, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{U}, \mathscr{F}),$$

compatible with the map $\mathscr{F}(V) \to \mathscr{F}(U)$. In this way, the zeroth Čech cohomology $\check{H}^0(\mathfrak{U}, \mathscr{F})$ is a functor from $\mathcal{J}_U$ to **Set**. One may wish $\mathcal{J}_U^{\mathrm{opp}}$ to be filtered. However, this is not true in general. But luckily, we still have

1.a **Lemma** *The diagram $\check{H}^0(-, \mathscr{F})\colon \mathcal{J}_U^{\mathrm{opp}} \to$ **Set** is filtered.*

Proof: First, since $\{\mathrm{id}\colon U \to U\}$ is a covering, the category is nonempty.

Next, for any two coverings $\mathfrak{U} = \{U_i \to U\}_{i\in I}$ and $\mathfrak{V} = \{V_j \to U\}_{j\in I}$, there is a covering

$$\mathcal{W} := \{U_i \times_U V_j \to U\}_{(i,j)\in I\times J}$$

refines both $\mathfrak{U}$ and $\mathfrak{V}$.

But now the troubles appear when we try to check the last axiom for filtered category. However, we still have the following Lemma 1.b. $\square$

1.b **Lemma** *Any two morphisms $f, g\colon \mathfrak{U} \rightrightarrows \mathfrak{V}$ of coverings inducing the same morphism $U \to V$ induce the same map $\check{H}^0(\mathfrak{V}, \mathscr{F}) \to \check{H}^0(\mathfrak{U}, \mathscr{F})$.*

Proof: Let f (resp. g) is given by the map α (resp. β) and the morphisms $U_i \to V_{\alpha(i)}$ (resp. $U_i \to V_{\beta(i)}$). Then we have the following commutative diagram.

$$\begin{tikzcd}
& V_{\alpha(i)} \arrow[dr] & \\
U_i \arrow[ur, "f_i"] \arrow[r, "\varphi"] \arrow[dr, "g_i"'] & V_{\alpha(i)} \times_V V_{\beta(i)} \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \arrow[r] & V \\
& V_{\beta(i)} \arrow[ur] &
\end{tikzcd}$$

Then, for any $s = (s_j) \in \check{H}^0(\mathfrak{V}, \mathscr{F})$, we have $\mathrm{pr}_1^*(s_{\alpha(i)}) = \mathrm{pr}_2^*(s_{\beta(i)})$ by the definition of $\check{H}^0(\mathfrak{V}.\mathscr{F})$. Therefore, we have

$$(f^*s)_i = f_i^*(s_{\alpha(i)}) = \varphi^* \mathrm{pr}_1^*(s_{\alpha(i)}) = \varphi^* \mathrm{pr}_2^*(s_{\beta(i)}) = g_i^*(s_{\beta(i)}) = (g^*s)_i.$$

Thus $f^* = g^*$ as desired. □

Now, since the diagram $\check{H}^0(-, \mathscr{F}) \colon \mathcal{J}_U^{\mathrm{opp}} \to \mathbf{Set}$ is filtered, we can apply the following lemma.

1.c Lemma *If $D \colon \mathcal{I} \to \mathbf{Set}$ is filtered, then*

$$\varinjlim_{\mathcal{I}} D = (\bigsqcup_{i \in \mathrm{ob}\, \mathcal{I}} D(i)) / \sim,$$

where the equivalence relation is given as following: two elements $s_i \in D(i)$ and $s_j \in D(j)$ are equivalent if there exists morphisms $f \colon i \to k$ and $g \colon j \to k$ such that $D(f)(s_i) = D(g)(s_j)$.

Now, we define

$$\check{H}^0(U, \mathscr{F}) := \varinjlim_{\mathcal{J}_U^{\mathrm{opp}}} \check{H}^0(\mathfrak{U}, \mathscr{F}),$$

which is called the **zeroth Čech cohomology** of $\mathscr{F}$ on U.

Now, let $U \to V$ be a morphism, then it induces a functor

$$\begin{aligned} \mathcal{J}_V &\longrightarrow \mathcal{J}_U \\ \{V_i \to V\} &\longmapsto \{V_i \times_V U \to U\}. \end{aligned}$$

Then, this functor induces a canonical map $\check{H}^0(V, \mathscr{F}) \to \check{H}^0(U, \mathscr{F})$. In this way, $\check{H}^0(-, \mathscr{F})$ becomes a presheaf, denoted by $\mathscr{F}^+$.

Now, notice that since $\mathfrak{U}_0 = \{\mathrm{id} \colon U \to U\}$ is a covering of U, there is a canonical map $\check{H}^0(\mathfrak{U}_0, \mathscr{F}) \to \check{H}^0(U, \mathscr{F})$. But $\check{H}^0(\mathfrak{U}_0, \mathscr{F})$ is nothing but $\mathscr{F}(U)$, thus we get a canonical map $\mathscr{F}(U) \to \mathscr{F}^+(U)$, which induces a canonical morphism

$$\mathscr{F} \longrightarrow \mathscr{F}^+.$$

Now, we claim that the corresponding $\mathscr{F} \mapsto (\mathscr{F} \to \mathscr{F}^+)$ forms a functor.

Given a presheaf morphism $\varphi \colon \mathscr{F} \to \mathscr{G}$, it induces a map

$$\check{H}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{U}, \mathscr{G})$$

for every covering $\mathfrak{U}$, and thus also induces a map

$$\check{H}^0(U, \mathscr{F}) \longrightarrow \check{H}^0(U, \mathscr{G})$$

for every object $U \in \operatorname{ob} \mathcal{C}$. In this way, we obtain a morphism $\varphi^+ \colon \mathscr{F}^+ \to \mathscr{G}^+$ and a commutative diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[d] & \mathscr{G} \arrow[d] \\ \mathscr{F}^+ \arrow[r, "\varphi^+"] & \mathscr{G}^+ \end{tikzcd}$$

and thus show that $\mathscr{F} \mapsto (\mathscr{F} \to \mathscr{F}^+)$ forms a functor.

1.d Lemma *The functor $+$ is left exact, i.e. commutes with finite limits.*

Proof: First, *the functor $\check{H}^0(\mathfrak{U}, -)$ commutes with limits.* Indeed, by the definition, for any diagram $\mathcal{J} \to \mathbf{PSh}(\mathcal{C}) \colon j \mapsto \mathscr{F}_j$ and any covering $\mathfrak{U} = \{U_i \to U\}_{i \in I}$, we have

$$\begin{aligned} \varprojlim_{\mathcal{J}} \check{H}^0(\mathfrak{U}, \mathscr{F}_j) &= \varprojlim_{\mathcal{J}} \ker \left(\prod_{i \in I} \mathscr{F}_j(U_i) \rightrightarrows \prod_{i_0, i_1 \in I} \mathscr{F}_j(U_{i_0} \times_U U_{i_1}) \right) \\ &= \ker \left(\prod_{i \in I} \varprojlim_{\mathcal{J}} (\mathscr{F}_j(U_i)) \rightrightarrows \prod_{i_0, i_1 \in I} \varprojlim_{\mathcal{J}} (\mathscr{F}_j(U_{i_0} \times_U U_{i_1})) \right) \\ &= \ker \left(\prod_{i \in I} \varprojlim_{\mathcal{J}} \mathscr{F}_j(U_i) \rightrightarrows \prod_{i_0, i_1 \in I} \varprojlim_{\mathcal{J}} \mathscr{F}_j(U_{i_0} \times_U U_{i_1}) \right), \end{aligned}$$

where the second equality comes from the commutativity of limits and the third form the fact that limits in $\mathbf{PSh}(\mathcal{C})$ are computed pointwise.

Therefore, for any finite diagram $\mathcal{I} \to \mathbf{PSh}(\mathcal{C}) \colon i \mapsto \mathscr{F}_i$ and any object $U \in \operatorname{ob} \mathcal{C}$, since filtered colimits commute with finite limits, we have

$$\begin{aligned} \varprojlim_{\mathcal{I}} \mathscr{F}_i^+(U) = \varprojlim_{\mathcal{I}} (\mathscr{F}_i^+(U)) &= \varprojlim_{\mathcal{I}} \varinjlim_{\mathcal{J}_U^{\mathrm{opp}}} \check{H}^0(\mathfrak{U}, \mathscr{F}_i) \\ &= \varinjlim_{\mathcal{J}_U^{\mathrm{opp}}} \varprojlim_{\mathcal{I}} \check{H}^0(\mathfrak{U}, \mathscr{F}_i) = \varinjlim_{\mathcal{J}_U^{\mathrm{opp}}} \check{H}^0(\mathfrak{U}, \varprojlim_{\mathcal{I}} \mathscr{F}_i) = (\varprojlim_{\mathcal{I}} \mathscr{F}_i)^+(U). \ \square \end{aligned}$$

2 Theorem (Sheafification) *Let $\mathscr{F}$ be a presheaf.*

1. *$\mathscr{F}^+$ is separated.*
2. *If $\mathscr{F}$ is separated, then $\mathscr{F}^+$ is a sheaf and the morphism of presheaves $\mathscr{F} \to \mathscr{F}^+$ is injective.*
3. *If $\mathscr{F}$ is a sheaf, then $\mathscr{F} \to \mathscr{F}^+$ is an isomorphism.*
4. *The presheaf $\mathscr{F}^{++}$ is always a sheaf.*

Proof: *1.* Let $\mathfrak{U} = \{U_i \to U\}_{i\in I}$ be a covering. Let $\bar{s}$ and $\bar{s}'$ be two elements of $\mathscr{F}^+(U)$ having the same image under the canonical map

$$\mathscr{F}^+(U) \to \prod_{i\in I} \mathscr{F}^+(U_i).$$

Let s (resp. s') be a representative of $\bar{s}$ (resp. $\bar{s}'$) in some $\check{H}^0(\mathfrak{V}, \mathscr{F})$, where $\mathfrak{V}$ is a covering of U. Since s and s' have the same image under the compose

$$\begin{tikzcd}
\check{H}^0(\mathfrak{V}, \mathscr{F}) \arrow[r] \arrow[d] & \check{H}^0(\mathfrak{V} \times_U U_i, \mathscr{F}) \arrow[d] \\
\mathscr{F}^+(U) \arrow[r] & \mathscr{F}^+(U_i)
\end{tikzcd}$$

there exists a covering $\mathfrak{W}_i$ of U_i such that s and s' have the same image under the compose

$$\check{H}^0(\mathfrak{V}, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{V} \times_U U_i, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{W}_i, \mathscr{F}).$$

But now those $\mathfrak{W}_i$ give a covering $\mathfrak{W}$ of U by compose each $\mathfrak{W}_i$ with $U_i \to U$. Then s and s' have the same image under the compose

$$\check{H}^0(\mathfrak{V}, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{V} \times_U U_i, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{W}_i, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{W}, \mathscr{F}).$$

Then as s and s' have the same image in the component $\check{H}^0(\mathfrak{W}, \mathscr{F})$, *a fortiori* the colimit $\check{H}^0(U, \mathscr{F})$. Thus $\bar{s} = \bar{s}'$.

2. Now for every covering $\mathfrak{U} = \{U_i \to U\}_{i\in I}$, the canonical map

$$\mathscr{F}(U) \longrightarrow \prod_{i\in I} \mathscr{F}(U_i)$$

is injective, thus so is

$$\mathscr{F}(U) \longrightarrow \check{H}^0(\mathfrak{U}, \mathscr{F}),$$

As the diagram $\check{H}^0(-, \mathscr{F})\colon \mathcal{J}_U^{\text{opp}} \to \mathbf{Set}$ is filtered, we further obtain the injectivity of the canonical map

$$\mathscr{F}(U) \longrightarrow \mathscr{F}^+(U).$$

Therefore, *$\mathscr{F} \to \mathscr{F}^+$ is injective.*

Now, let's prove $\mathscr{F}^+$ is a sheaf by checking the canonical map

$$\mathscr{F}^+(U) \longrightarrow \check{H}^0(\mathfrak{U}, \mathscr{F}^+)$$

is bijective for all coverings $\mathfrak{U}$. By *1*, it's already injective. Therefore we only need to show it's *surjective*. Let $\bar{s} = (\bar{s}_i)$ be an element of $\check{H}^0(\mathfrak{U}, \mathscr{F}^+)$. For each $\bar{s}_i \in \mathscr{F}^+(U_i)$, chose a representative $s_i = (s_{i\alpha}) \in \check{H}^0(\mathfrak{U}_i, \mathscr{F})$, where $\mathfrak{U}_i$ is a covering of U_i. Now, compose those coverings with $\mathfrak{U}$, we get another covering $\mathfrak{W} = \{U_{i\alpha} \to U\}$. Then $(s_{i\alpha})$ forms an element of $\prod \mathscr{F}(U_{i\alpha})$. Now, we wish *it lies in* $\check{H}^0(\mathfrak{W}, \mathscr{F})$. In other words, for any $i, j \in I$ and $\alpha \in I_i, \beta \in I_j$, we need to show $s_{i\alpha} \in \mathscr{F}(U_{i\alpha})$ *and* $s_{j\beta} \in \mathscr{F}(U_{j\beta})$ *have the same image under the maps*

$$\mathscr{F}(U_{i\alpha}) \longrightarrow \mathscr{F}(U_{i\alpha} \times_U U_{j\beta}) \longleftarrow \mathscr{F}(U_{j\beta}).$$

To do this, consider the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{U}_i \times_U \mathfrak{U}_j \arrow[r] \arrow[d] & U_i \times_U \mathfrak{U}_j \arrow[r] \arrow[d] & \mathfrak{U}_j \arrow[d] \\
\mathfrak{U}_i \times_U U_j \arrow[r] \arrow[d] & U_i \times_U U_j \arrow[r] \arrow[d] & U_j \arrow[d] \\
\mathfrak{U}_i \arrow[r] & U_i \arrow[r] & U
\end{tikzcd}$$

Let s^1_{ij} and s^2_{ij} denote the images of s_i and s_j on $\mathfrak{U}_i \times_U U_j$ and $U_i \times_U \mathfrak{U}_j$ respectively. Since $(\bar{s}_i) \in \check{H}^0(\mathfrak{U}, \mathscr{F}^+)$, s^1_{ij} and s^2_{ij} have the same image in $\mathscr{F}^+(U_i \times_U U_j)$. Then there exists a covering $\mathfrak{V}$ refining both $\mathfrak{U}_i \times_U U_j$ and $U_i \times_U \mathfrak{U}_j$ such that s^1_{ij} and s^2_{ij} have the same image in $\check{H}^0(\mathfrak{V}, \mathscr{F})$. Now, let

$$\mathfrak{U}_{ij} = \mathfrak{U}_i \times_U \mathfrak{U}_j,$$

which is a common refinement of $\mathfrak{U}_i \times_U U_j$ and $U_i \times_U \mathfrak{U}_j$. Then, by Lemma 2.a below, s^1_{ij} and s^2_{ij} have the same image in $\check{H}^0(\mathfrak{U}_{ij}, \mathscr{F})$. Thus s_i and s_j have the same image in $\check{H}^0(\mathfrak{U}_{ij}, \mathscr{F})$. In particular, $s_{i\alpha}$ and $s_{j\beta}$ have the same image in $\mathscr{F}(U_{i\alpha} \times_U U_{j\beta})$.

Now $(s_{i\alpha}) \in \check{H}^0(\mathfrak{W}, \mathfrak{F})$, so it represents an element $\bar{s}'$ of $\mathscr{F}^+(U)$. Since $\bar{s}'|_{U_i}$ and $\bar{s}_i$ have the same representative $s_i \in \check{H}^0(\mathfrak{U}_i, \mathfrak{F})$, they have to be the same. In this way, we see that the canonical map $\mathscr{F}^+(U) \to \check{H}^0(\mathfrak{U}, \mathscr{F}^+)$ maps $\bar{s}'$ to $\bar{s}$ as desired.

3. Now, assume $\mathscr{F}$ is a sheaf. Since for every covering $\mathfrak{U}$, the canonical map $\mathscr{F}(U) \to \check{H}^0(\mathfrak{U}, \mathscr{F})$ is bijective, the diagram $\check{H}^0(-, \mathscr{F}) \colon \mathscr{J}_U^{\mathrm{opp}} \to \mathbf{Set}$ is constant. Thus passing to the colimit, $\mathscr{F}(U) \to \mathscr{F}^+(U)$ is also bijective. Therefore, $\mathscr{F} \to \mathscr{F}^+$ is an isomorphism.

4. It is obvious now. □

2.a Lemma *Let $\mathscr{F}$ be a separated presheaf.*

1. *If there is a refinement $f \colon \mathfrak{V} \to \mathfrak{U}$, then the map*

$$f^* \colon \check{H}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{V}, \mathscr{F})$$

is injective.

2. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be two coverings of U and $s_{\mathfrak{U}} \in \check{H}^0(\mathfrak{U}, \mathscr{F})$, $s_{\mathfrak{V}} \in \check{H}^0(\mathfrak{V}, \mathscr{F})$. If there exists a common refinement $\mathfrak{W}_0$ of them such that $s_{\mathfrak{U}}$ and $s_{\mathfrak{V}}$ have the same image in $\check{H}^0(\mathfrak{W}_0, \mathscr{F})$, then for any common refinement $\mathfrak{W}$ of them, $s_{\mathfrak{U}}$ and $s_{\mathfrak{V}}$ have the same image in $\check{H}^0(\mathfrak{W}, \mathscr{F})$.*

Proof: *1.* Let $\mathfrak{W}$ denote the covering $\mathfrak{U} \times_U \mathfrak{V}$ obtained by fibre products. $\mathfrak{W}$ admits two morphisms $\mathrm{pr}_1\colon \mathfrak{W} \to \mathfrak{U}$ and $\mathrm{pr}_2\colon \mathfrak{W} \to \mathfrak{V}$ via projections. Now, for each $U_i \to U$,

$$\mathscr{F}(U_i) \longrightarrow \prod_{j\in J} \mathscr{F}(U_i \times_U V_j)$$

is injective. Thus so is the product

$$\prod_{i\in I} \mathscr{F}(U_i) \longrightarrow \prod_{(i,j)\in I\times J} \mathscr{F}(U_i \times_U V_j).$$

Then, by the definition of zeroth Čech cohomology,

$$\mathrm{pr}_1^*\colon \check{H}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{W}, \mathscr{F})$$

is injective. Now, note that since there is a refinement $f\colon \mathfrak{V} \to \mathfrak{U}$, thus $\mathrm{pr}_1 = f \circ \mathrm{pr}_2$. Then since $\mathrm{pr}_2^* \circ f^* = \mathrm{pr}_1^*$ is injective, so is

$$f^*\colon \check{H}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{V}, \mathscr{F}).$$

2. Assumptions as in *2*, let $\mathfrak{W}$ be another common refinement of $\mathfrak{U}$ and $\mathfrak{V}$. Then there is a common refinement $\mathfrak{W}'$ of $\mathfrak{W}_0$ and $\mathfrak{W}$. Now, $s_{\mathfrak{U}}$ and $s_{\mathfrak{V}}$ have the same image in $\check{H}^0(\mathfrak{W}_0, \mathscr{F})$, *a fortiori* in $\check{H}^0(\mathfrak{W}', \mathscr{F})$. But since $\mathfrak{W}$ is a refinement of $\mathfrak{U}$, the map $\check{H}^0(\mathfrak{U}, \mathscr{F}) \to \check{H}^0(\mathfrak{W}_0, \mathscr{F})$ factors through $\check{H}^0(\mathfrak{W}, \mathscr{F})$. For $\mathfrak{V}$, the story is similar. Now, $s_{\mathfrak{U}}$ and $s_{\mathfrak{V}}$ have the same image under the following composes

$$\begin{tikzcd}
\check{H}^0(\mathfrak{U}, \mathscr{F}) \arrow[dr] & & \\
& \check{H}^0(\mathfrak{W}, \mathscr{F}) \arrow[r] & \check{H}^0(\mathfrak{W}', \mathscr{F}) \\
\check{H}^0(\mathfrak{V}, \mathscr{F}) \arrow[ur] & &
\end{tikzcd}$$

where the last map is injective, thus have the same image in $\check{H}^0(\mathfrak{W}, \mathscr{F})$. $\square$

3 **(Sheafification)** Let $\mathscr{F}$ be a presheaf on a site $\mathcal{C}$. Then the sheaf $\mathscr{F}^{\#} := \mathscr{F}^{++}$ together with the canonical morphism $\mathscr{F} \to \mathscr{F}^{\#}$ is called the **sheafification** of $\mathscr{F}$.

The sheafification has the following universal property:

> For any sheaf $\mathscr{G}$ and presheaf morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$, there exists a unique sheaf morphism $\varphi^{\#}\colon \mathscr{F}^{\#} \to \mathscr{G}$ making the following digram commute.
>
> $$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[dr, "\varphi"'] & \mathscr{F}^{\#} \arrow[d, "\varphi^{\#}"] \\ & \mathscr{G} \end{tikzcd}$$

In other words,

3.a Proposition (Sheafification is free) *The sheafification functor $\#$ is* left adjoint *to the forgetful functor from sheaves on $\mathcal{C}$ to presheaves on $\mathcal{C}$.*

Proof: Indeed, for any presheaf morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$ with $\mathscr{G}$ a sheaf, the unique sheaf morphism factors it is its sheafification $\varphi^{\#}$. □

3.b Corollary *The sheafification preserves colimits and hence is* right exact*; the forgetful functor preserves limits and hence is* left exact*. In particular, the presheaf kernel is already the sheaf kernel.*

Proof: This follows from the property of adjoint functors. One can refer my note *BMO* or [Bor94] directly. □

3.c Proposition *The sheafification is exact.*

Proof: The right exactness comes from the freeness. As for the left exactness, just note that the colimit used to construct the functor $+$ is filtered, thus commutes with finite limits.

More precisely, since Corollary 3.b, the limits in $\mathbf{Sh}(\mathcal{C})$ are computed in the category $\mathbf{PSh}(\mathcal{C})$. Then, for any finite diagram $\mathcal{I} \to \mathbf{PSh}(\mathcal{C})\colon i \mapsto \mathscr{F}_i$, by Lemma 1.d, we have

$$\varprojlim_{\mathcal{I}} \mathscr{F}_i^{\#} = \varprojlim_{\mathcal{I}} \mathscr{F}_i^{++} = (\varprojlim_{\mathcal{I}} \mathscr{F}_i^{+})^{+} = (\varprojlim_{\mathcal{I}} \mathscr{F}_i)^{++} = (\varprojlim_{\mathcal{I}} \mathscr{F}_i)^{\#}.$$

□

3.d (Compatible sections) The sheafification can also be described by *compatible sections*. Let $\mathscr{F}$ be a presheaf on a site $\mathcal{C}$ and $\mathfrak{U} = \{U_i \to U\}_{i\in I}$ a covering. A **system of compatible sections** respect to $\mathfrak{U}$ is an element $(s_i) \in \prod \mathscr{F}(U_i)$ satisfying the following property:

> For every $i, j \in I$, there exists a covering $\{U_{ijk} \to U_i \times_U U_j\}$ such that the pullbacks of s_i and s_j to each U_{ijk} agrees.

One can verify that given an element $s \in \mathscr{F}^{\#}(U)$ is equivalent to giving a system of compatible sections (s_i) for every coverings $\mathfrak{U}$ of U such that $s|_{U_i}$ is the image of s_i under the canonical map $\mathscr{F}(U_i) \to \mathscr{F}^{\#}(U_i)$.

§ 7 The category of sheaves

1 (Morphisms of sheaves) Before going forward, we briefly recall the morphisms in $\mathbf{PSh}(\mathcal{C})$.

1. The monomorphisms (resp. epimorphisms, isomorphisms) in $\mathbf{PSh}(\mathcal{C})$ are precisely the *injective* (resp. *surjective*, *bijective*) morphisms, meaning injective (resp. surjective, bijective) on each object $U \in \operatorname{ob}\mathcal{C}$.

2. As a corollary, the category $\mathbf{PSh}(\mathcal{C})$ is *balanced*, meaning isomorphism = monomorphism + epimorphism.

3. Moreover, since the limits and colimits in $\mathbf{PSh}(\mathcal{C})$ are computed pointwise, $\mathbf{PSh}(\mathcal{C})$ inherits many wonderful properties of $\mathbf{Set}$, such as

   (a) The monomorphisms are *regular*, meaning they are equalizers of some parallel morphisms. Indeed, in $\mathbf{Set}$, a map $f\colon X \to Y$ is monic if and only if it is the equalizer of the *characteristic function*

   $$\chi_f(y) = \begin{cases} 1 & \text{if } y \in \operatorname{im}(f) \\ 0 & \text{others} \end{cases}$$

   and the constant function $1(y) = 1$. In $\mathbf{PSh}(\mathcal{C})$, the story is the same: just replace sets by presheaves and $\{0, 1\}$ by the constant presheaf $\Delta_{\{0,1\}}$ and notice that the equalizer is computed pointwise.

   Note that *a regular monomorphism is an isomorphism if and only if it is also epic.* Therefore this leads to another proof of *iso = monic + epic.*

   (b) The epimorphisms are *effective*, meaning they are *coequalizers* of their *kernel pair*. Indeed, in $\mathbf{Set}$, a map $f\colon X \to Y$ is epic if and only if it is the coequalizer of its kernel pair, i.e. the two projections $\mathrm{pr}_1\,\mathrm{pr}_2$ in the following Cartesian diagram.

   $$\begin{tikzcd} \ker(f) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

   In $\mathbf{PSh}(\mathcal{C})$, the story is the same: just replace sets by presheaves and notice that the coequalizer and kernel pair are computed pointwise.

   Note that *an effective epimorphism is an isomorphism if and only if it is also monic.* Therefore this leads to another proof of *iso = monic + epic.*

1.a Proposition *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a site $\mathcal{C}$. Then, the following are equivalent.*

1. *φ is a monomorphism.*
2. *φ is an injective presheaf morphism, i.e. for any object $U \in \operatorname{ob}\mathcal{C}$ the map $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is injective.*
3. *φ is a regular monomorphism.*

Proof: *2⇒1* and *3⇒1* are obvious. *1⇒2* comes form the left exactness of the forgetful functor $\mathbf{Sh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{C})$ and the exactness of the section functors $\Gamma(U,-)$.

As for *2⇒3*, consider the presheaf morphisms $\chi_\varphi, 1\colon \mathscr{G} \rightrightarrows \Delta_{\{0,1\}}$. Then φ, viewed as a presheaf morphism, is the equalizer of them. Apply sheafification to them, by Proposition 6.3.c, we see that the sheafification of φ, i.e. itself, is the equalizer of the sheafification of χ_φ and 1. □

1.b Proposition *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a site $\mathcal{C}$. Then, the following are equivalent.*

1. *φ is an isomorphism.*
2. *φ is an bijective presheaf morphism, i.e. for any object $U \in \operatorname{ob}\mathcal{C}$ the map $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is bijective.*
3. *φ is both monic and epic.*

Proof: *1⇒3* is trivial. Recall that $\mathbf{Sh}(\mathcal{C})$ is a full subcategory of $\mathbf{PSh}(\mathcal{C})$. Thus a sheaf morphism is an isomorphism if and only if it is an isomorphism in $\mathbf{PSh}(\mathcal{C})$. Therefore *1⇔2*.

As for *3⇒1*, just notice that *iso = regular monic + epic* is always true and then use Proposition 1.a. □

1.c Proposition *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a site $\mathcal{C}$. Then, the following are equivalent.*

1. *φ is an epimorphism.*
2. *φ is a* **locally surjective** *presheaf morphism, which means that for any object $U \in \operatorname{ob}\mathcal{C}$ and any section $s \in \mathscr{G}(U)$, there exists a covering $\{U_i \to U\}$ such that $s|_{U_i}$ lies in the image of $\varphi(U_i)\colon \mathscr{F}(U_i) \to \mathscr{G}(U_i)$.*
3. *φ is an effective epimorphism.*

Proof: *3⇒1* is trivial.

2⇒1: Let $\psi_1, \psi_2\colon \mathscr{G} \rightrightarrows \mathscr{T}$ be two parallel morphisms of sheaves such that $\psi_1 \circ \varphi = \psi_2 \circ \varphi$. We need to show $\psi_1 = \psi_2$. For any object $U \in \operatorname{ob}\mathcal{C}$

and any section $s \in \mathscr{G}(U)$, there exists a covering $\{U_i \to U\}$ such that for each i, there exists a section $t_i \in \mathscr{F}(U_i)$ such that $\varphi(U_i)(t_i) = s|_{U_i}$. Then

$$\begin{aligned}\psi_1(U)(s)|_{U_i} = \psi_1(U_i)(s|_{U_i}) = \psi_1\varphi(U_i)(t_i) \\ = \psi_2\varphi(U_i)(t_i) = \psi_2(U_i)(s|_{U_i}) = \psi_2(U)(s)|_{U_i}\end{aligned}$$

Since $\{U_i \to U\}$ is a covering, this shows $\psi_1(U)(s) = \psi_2(U)(s)$. Thus $\psi_1 = \psi_2$.

1⇒2: Define a subpresheaf $\mathscr{G}'$ of $\mathscr{G}$ as follows:

> $s \in \mathscr{G}'(U)$ if and only if there exists a covering $\{U_i \to U\}$ such that $s|_{U_i}$ lies in the image of $\varphi(U_i)\colon \mathscr{F}(U_i) \to \mathscr{G}(U_i)$.

It remains to show $\mathscr{G}' = \mathscr{G}$.

First of all, this $\mathscr{G}'$ is actually a sheaf. Indeed, we only need to verify the gluing condition. Let $(s_i) \in \prod \mathscr{G}'(U_i)$ be a system of compatible sections respect to a covering $\{U_i \to U\}$. Then, it corresponds to a section $s \in \mathscr{G}(U)$. It remains to show $s \in \mathscr{G}'(U)$. Indeed, for each s_i, there exists a covering $\{U_{ij} \to U_i\}$ such that $s_i|_{U_{ij}}$ lies in the image of $\varphi(U_{ij})\colon \mathscr{F}(U_{ij}) \to \mathscr{G}(U_{ij})$. Now, combine all those coverings, we obtain a covering $\{U_{ij} \to U\}$ such that $s|_{U_{ij}} = s_i|_{U_{ij}}$ lies in the image of $\varphi(U_{ij})\colon \mathscr{F}(U_{ij}) \to \mathscr{G}(U_{ij})$. Therefore $s \in \mathscr{G}'(U)$.

Remark This sheaf $\mathscr{G}'$ is called the *sheaf image* of φ.

Now, we have sheaf morphisms

$$\mathscr{F} \longrightarrow \mathscr{G}' \xrightarrow{i} \mathscr{G}.$$

Since the compose φ is epic, so is $i\colon \mathscr{G}' \to \mathscr{G}$. But this i is the inclusion morphism from the subsheaf $\mathscr{G}'$ to $\mathscr{G}$, thus is monic. Now, i is both monic and epic in $\mathbf{Sh}(\mathcal{C})$, thus is an isomorphism by Proposition 1.b. This shows $\mathscr{G}' = \mathscr{G}$.

1⇒3: Let $\mathscr{T}$ be a sheaf and $\psi\colon \mathscr{F} \to \mathscr{T}$ a morphism coequalize the *kernel pair* of φ:

$$\mathscr{F} \times_{\mathscr{G}} \mathscr{F} \rightrightarrows \mathscr{F}.$$

Let $\mathscr{G}' \subset \mathscr{G}$ be the presheaf image of φ. Since the sheafification is exact, it maps $\mathscr{G}'$ to the sheaf image of φ, here which is $\mathscr{G}$ itself. Therefore, the above kernel pair can be obtained by applying sheafification to the kernel pair of $\varphi'\colon \mathscr{F} \to \mathscr{G}'$ in $\mathbf{PSh}(\mathcal{C})$. But since sheafification is exact, the two kernel pairs are the same.

Now, ψ coequalize the kernel pair of $\varphi'\colon \mathscr{F} \to \mathscr{G}'$ in $\mathbf{PSh}(\mathcal{C})$. Since φ' is epic, hence effective epic in $\mathbf{PSh}(\mathcal{C})$, there exists a unique presheaf morphism $\tau'\colon \mathscr{G} \to \mathscr{T}$ such that $\tau' \circ \varphi' = \psi$. Apply sheafification to them, we obtain a unique sheaf morphism $\tau\colon \mathscr{G} \to \mathscr{T}$ such that $\tau \circ \varphi = \psi$. □

1.d Lemma *The sheaf image is the sheafification of the presheaf image.*

Recall that the presheaf image $\mathrm{im}^p \varphi$ of a presheaf morphism $\varphi \colon \mathscr{F} \to \mathscr{G}$ is the unique subpresheaf of the codomain $\mathscr{G}$ such that the morphism factors through it and that $\mathscr{F} \to \mathrm{im}^p \varphi$ is an epimorphism. Then one can verify that this epimorphism is nothing but the coequalizer of the kernel pair of φ, i.e. the **category-theoretic image** in $\mathbf{PSh}(\mathcal{C})$.

1.e Corollary *The sheaf image is the category-theoretic image in* $\mathbf{Sh}(\mathcal{C})$*.*

The locally surjectivity can also be defined for presheaf morphisms. Obviously, surjective morphisms are locally surjective and the converse is false. Likely, the sheaf image can be defined for presheaves. But now, it may not be a sheaf and thus should have another name, **local image**.

1.f Corollary *A presheaf morphism is locally surjective if and only if its sheafification is an epimorphism.*

Proof: First, assume $\varphi \colon \mathscr{F} \to \mathscr{G}$ is locally surjective. For any $\bar{s} \in \mathscr{G}^{\#}(U)$, let $(s_i) \in \prod \mathscr{G}(U_i)$ be a family of compatible sections respect to the a covering $\{U_i \to U\}$. For each s_i, there exists a covering $\{U_{ij} \to U_i\}$ such that $s_i|_{U_{ij}}$ has a preimage $t_{ij} \in \mathscr{F}(U_{ij})$. Let $\bar{t_{ij}}$ be the image of t_{ij} in $\mathscr{F}^{\#}(U_{ij})$, then $\varphi^{\#}(U_{ij})(\bar{t_{ij}}) = \bar{s}|_{U_{ij}}$. This shows $\varphi^{\#}$ is locally surjective, hence epic.

Next, assume the sheafification of $\varphi \colon \mathscr{F} \to \mathscr{G}$ is an epimorphism. For any section $s \in \mathscr{G}(U)$ of $\mathscr{G}$, consider its image under the canonical map $\mathscr{G}(U) \to \mathscr{G}^{\#}(U)$, saying $\bar{s}$. Since $\varphi^{\#}$ is epic, by Proposition 1.c, there exists a covering $\mathfrak{U} = \{U_i \to U\}$ such that each $\bar{s}|_{U_i}$ lies in the image under $\varphi^{\#}(U_i)$. Let $\bar{t_i} \in \mathscr{F}^{\#}(U_i)$ be the preimage of $\bar{s}|_{U_i}$. Now, by 6.3.d, each $\bar{t_i}$ is equivalent to a system of compatible sections $(t_{ij}) \in \mathscr{F}(U_{ij})$ for each covering $\{U_{ij} \to U_i\}$. Now, consider the covering $\{U_{ij} \to U\}$, one can verify that $\varphi(U_{ij})(t_{ij}) = s|_{U_{ij}}$. This shows φ is locally surjective. $\square$

1.g Corollary *The sheafification preserves local images.*

Proof: Indeed, let $\varphi \colon \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves and $\mathrm{im}^p \varphi$ the local image of it. Apply sheafification to them, we have the following commutative diagram.

$$\begin{tikzcd}
\mathscr{F} \arrow[r] \arrow[d] & \mathrm{im}^p \varphi \arrow[r] \arrow[d] & \mathscr{G} \arrow[d] \\
\mathscr{F}^{\#} \arrow[r] & (\mathrm{im}^p \varphi)^{\#} \arrow[r] & \mathscr{G}^{\#}
\end{tikzcd}$$

By the exactness of $\#$, $(\mathrm{im}^p \varphi)^{\#}$ is a subsheaf of $\mathscr{G}^{\#}$. By Corollary 1.f, $\mathscr{F}^{\#} \to (\mathrm{im}^p \varphi)^{\#}$ is epic. Since the sheaf image $\mathrm{im}\, \varphi^{\#}$ of $\varphi^{\#}$ is the unique subsheaf of the codomain $\mathscr{G}^{\#}$ such that $\varphi^{\#}$ factors through it and $\mathscr{F}^{\#} \to \mathrm{im}\, \varphi^{\#}$ is epic, we have $(\mathrm{im}^p \varphi)^{\#} = \mathrm{im}\, \varphi^{\#}$. $\square$

2 **(Quasi-compactness)** Let $\mathcal{C}$ be a site. An object U of $\mathcal{C}$ is said to be **quasi-compact** if every covering of U can be refined by a finite covering.

2.a Lemma *Let $\mathcal{C}$ be a site. Let $\mathcal{J} \to \mathbf{Sh}(\mathcal{C})$ be a filtered diagram of sheaves of sets. Let $U \in \operatorname{ob}\mathcal{C}$. Consider the canonical map*

$$\Phi\colon \varinjlim_{\mathcal{J}} \mathscr{F}_j(U) \longrightarrow (\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U).$$

1. *If all the transition morphisms are injective then Φ is injective.*
2. *If U is quasi-compact, then Φ is injective.*
3. *If U is quasi-compact and all the transition morphisms are injective then Φ is an isomorphism.*
4. *If any covering of U can be refined by some coverings $\{U_i \to U\}_{i\in I}$ with I finite and $U_i \times_U U_{i'}$ quasi-compact, then Φ is bijective.*

Proof: *1.* Assume all the transition morphisms are injective. First of all, we show the presheaf $\mathscr{F}_{\mathcal{J}}\colon U \mapsto \varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$ is *separated*. Indeed the canonical maps

$$\mathscr{F}_j(U) \longrightarrow \prod \mathscr{F}_j(U_i)$$

are injective, thus so is the colimit

$$\varinjlim_{\mathcal{J}} \mathscr{F}_j(U) \longrightarrow \varinjlim_{\mathcal{J}} \prod \mathscr{F}_j(U_i).$$

It remains to show the canonical map

$$\varinjlim_{\mathcal{J}} \prod \mathscr{F}_j(U_i) \longrightarrow \prod \varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i)$$

is injective.

Let $\bar{s}$ and $\bar{t}$ be two elements of $\varinjlim_{\mathcal{J}} \prod \mathscr{F}_j(U_i)$ having the same image in $\prod \varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i)$ and $s = (s_i), t = (t_i)$ be their representatives in some $\prod \mathscr{F}_j(U_i)$. Now, the image of $\bar{s}$ and $\bar{t}$ in $\prod \varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i)$ can be written as $(\bar{s_i})$ and $(\bar{t_i})$, where each $\bar{s_i}$ or $\bar{t_i}$ is the image of s_i or t_i in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i)$. Since $\bar{s_i} = \bar{t_i}$, there exists some $j_i \in \operatorname{ob}\mathcal{J}$ such that the image of s_i and t_i in $\mathscr{F}_{j_i}(U_i)$ are the same. Then, since the transition morphism $\mathscr{F}_j \to \mathscr{F}_{j_i}$ is injective, we have $s_i = t_i$. Then, we get $s = t$ and *a fortiori* $\bar{s} = \bar{t}$.

By Proposition 6.3.c, $\varinjlim_{\mathcal{J}} \mathscr{F}_j$ is the sheafification of the presheaf $\mathscr{F}_{\mathcal{J}}$. Then, by Theorem 6.2, Φ is injective.

2. Assume U is quasi-compact. Let $\bar{s}$ and $\bar{t}$ be two elements of $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$ having the same image under Φ and s, t be their representatives in some $\mathscr{F}_j(U)$. Now, for any covering $\mathfrak{U} = \{U_i \to U\}_{i\in I}$, the image of $\bar{s}$ and $\bar{t}$

under Φ can be written as systems of compatible sections $(\bar{s}_i)$ and $(\bar{t}_i)$ of the presheaf$\mathscr{F}_{\mathcal{J}}$. Then, there exists $j_i \in \operatorname{ob} \mathcal{J}$ such that the image of $s|_{U_i}$ and $t|_{U_i}$ in $\mathscr{F}_{j_i}(U_i)$ are the same. Since U is quasi-compact, the covering $\mathfrak{U}$ can be refined by a finite covering $\mathfrak{V} = \{V_i \to U\}_{i \in I'}$ with the index transformation $\alpha \colon I' \to I$. For this covering, we can take j_0 to be the index such that there are arrows $j_{\alpha(i)} \to j_0$. Now, the image of $s|_{V_i}$ and $t|_{V_i}$ in $\mathscr{F}_{j_0}(V_i)$ are the same. Then s and t maps to the same element in $\mathscr{F}_{j_0}(U)$, *a fortiori* in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$.

3. Assume U is quasi-compact and all the transition morphisms are injective. Then Φ is injective. It suffices to show it is surjective. Let $\bar{s}$ be a section in $(\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U)$. For any covering $\mathfrak{U} = \{U_i \to U\}_{i \in I}$, $\bar{s}$ can be written as a system of compatible sections $(\bar{s}_i)$ of the presheaf $\mathscr{F}_{\mathcal{J}}$. Let s^i be the representative of $\bar{s}_i$ in some $\mathscr{F}_{j_i}$. Then the images of $s^i|_{U_i \times_U U_{i'}}$ and $s^{i'}\Big|_{U_i \times_U U_{i'}}$ in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \times_U U_{i'})$ have the same image under the composition

$$\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \times_U U_{i'}) \xrightarrow{\Phi} (\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U_i \times_U U_{i'}) \longrightarrow (\varinjlim_{\mathcal{J}} \mathscr{F}_j)(V_{ii'})$$

for each $V_{ii'k} \to U_i \times_U U_{i'}$ in some covering $\mathfrak{V}_{ii'}$. Thus, by *1.*, the images of $s^i|_{V_{ii'}}$ and $s^{i'}\Big|_{V_{ii'}}$ in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(V_{ii'})$ are the same.

Since U is quasi-compact, the covering $\mathfrak{U}$ can be refined by a finite covering. Thus we may assume $\mathfrak{U}$ is finite. Now, we can take j_0 to be the index such that there are arrows $j_i \to j_0$. Then the sections $s^i \in \mathscr{F}_{j_i}(U_i)$ give rise to sections $s_i \in \mathscr{F}_{j_0}(U_i)$. For any $V_{ii'}$, $s_i|_{V_{ii'}}$ and $s_{i'}|_{V_{ii'}}$ have the same image in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(V_{ii'})$, by the similar argument in *1.*, they are the same. Therefore (s_i) is a system of compatible sections of $\mathscr{F}_{j_0}$, and thus gives a section $s \in \mathscr{F}_{j_0}(U)$ such that $s|_{U_i} = s_i$. Then, this s gives an element of $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$ which maps to $\bar{s}$ under Φ.

4. Assume the hypothesis of *4.* It is obvious that U is quasi-compact and thus Φ is injective. It suffices to show Φ is surjective. Let $\bar{s}$ be an element in $(\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U)$. For any covering $\mathfrak{U} = \{U_i \to U\}_{i \in I}$, $\bar{s}$ can be written as a system of compatible sections $(\bar{s}_i)$ of the presheaf $\mathscr{F}_{\mathcal{J}}$. We may assume that the covering $\mathfrak{U}$ is finite and that $U_i \times_U U_{i'}$ are quasi-compact. Let s^i be the representative of $\bar{s}_i$ in some $\mathscr{F}_{j_i}$. Then the images of $s^i|_{U_i \times_U U_{i'}}$ and $s^{i'}\Big|_{U_i \times_U U_{i'}}$ in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \times_U U_{i'})$ have the same image under the composition

$$\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \times_U U_{i'}) \xrightarrow{\Phi} (\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U_i \times_U U_{i'}) \longrightarrow (\varinjlim_{\mathcal{J}} \mathscr{F}_j)(V_{ii'})$$

for each $V_{ii'k} \to U_i \times_U U_{i'}$ in some covering $\mathfrak{V}_{ii'}$. Since $\varinjlim_{\mathcal{J}} \mathscr{F}_j$ is a sheaf, this shows the images of $s^i|_{U_i \times_U U_{i'}}$ and $s^{i'}\Big|_{U_i \times_U U_{i'}}$ in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \times_U U_{i'})$

have the same image under Φ. Since $U_i \times_U U_{i'}$ is quasi-compact, by *2.*, this shows $s^i\big|_{U_i \times_U U_{i'}}$ and $s^{i'}\Big|_{U_i \times_U U_{i'}}$ have the same image in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \times_U U_{i'})$, and thus in $\mathscr{F}_{j_{ii'}}(U_i \times_U U_{i'})$ for some $j_{ii'}$.

Now, we can take j_0 to be the index such that there are arrows $j_{ii'} \to j_0$. Then the sections $s^i \in \mathscr{F}_{j_i}(U_i)$ give rise to sections $s_i \in \mathscr{F}_{j_0}(U_i)$ and furthermore, they form a system of compatible sections. Thus we get a section $s \in \mathscr{F}_{j_0}(U)$ such that $s|_{U_i} = s_i$. Then, this s gives an element of $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$ which maps to $\bar{s}$ under Φ. $\square$

We also give another proof.

Proof: *2.* Assume U is quasi-compact. First of all, for a finite covering $\mathfrak{U}$ of U, since filtered colimits commute with finite limits, the canonical map $\mathscr{F}_J(U) \to \check{H}^0(\mathfrak{U}, \mathscr{F}_J)$ is injective. As for an arbitrary covering $\mathfrak{V}$ of U, let $\mathfrak{U}$ be a finite covering refining $\mathfrak{V}$, then we have the following commutative diagram

$$\begin{tikzcd} \mathscr{F}_J(U) \arrow[r] \arrow[d, equal] & \check{H}^0(\mathfrak{V}, \mathscr{F}_J) \arrow[d] \\ \mathscr{F}_J(U) \arrow[r] & \check{H}^0(\mathfrak{U}, \mathscr{F}_J) \end{tikzcd}$$

where the bottom $\mathscr{F}_J(U) \to \check{H}^0(\mathfrak{U}, \mathscr{F}_J)$ is injective, thus so is the upper $\mathscr{F}_J(U) \to \check{H}^0(\mathfrak{V}, \mathscr{F}_J)$. Therefore the canonical map $\mathscr{F}_J(U) \to \mathscr{F}_J^+(U)$ is injective. By Theorem 6.2, $\mathscr{F}_J^+$ is separated, thus $\mathscr{F}_J^+(U) \to \mathscr{F}_J^{++}(U)$ is injective. Since Φ is the canonical map $\mathscr{F}_J(U) \to \mathscr{F}_J^{\#}(U)$, which equals the composition $\mathscr{F}_J(U) \to \mathscr{F}_J^+(U) \to \mathscr{F}_J^{++}(U)$, it is also injective.

3. Assume U is quasi-compact and all the transition morphisms are injective. First of all, by the proof of *1.*, $\mathscr{F}_J$ is separated. Thus $\varinjlim_{\mathcal{J}} \mathscr{F}_j = \mathscr{F}_J^+$ and the canonical map Φ is just the map $\mathscr{F}_J(U) \to \mathscr{F}_J^+(U)$. Since U is quasi-compact, the finite coverings are cofinal in all coverings of U. Thus

$$(\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U) = \mathscr{F}_J^+(U) = \varinjlim_{\mathfrak{U} \text{ is a finite covering of } U} \check{H}^0(\mathfrak{U}, \mathscr{F}_J).$$

For any finite covering $\mathfrak{U} = \{U_i \to U\}$ of U, since filtered colimits commute with finite limits, we have

$$\begin{aligned} \check{H}^0(\mathfrak{U}, \mathscr{F}_J) &= \ker\left(\prod \varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i) \rightrightarrows \prod \varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \times_U U_{i'})\right) \\ &= \varinjlim_{\mathcal{J}} \ker\left(\prod \mathscr{F}_j(U_i) \rightrightarrows \prod \mathscr{F}_j(U_i \times_U U_{i'})\right) \\ &= \varinjlim_{\mathcal{J}} \mathscr{F}_j(U). \end{aligned}$$

Therefore
$$\mathscr{F}_J^+(U) = \varinjlim_{\mathfrak{U} \text{ is a finite covering of } U} \mathscr{F}_J(U)$$
and thus the map $\mathscr{F}_J(U) \to \mathscr{F}_J^+(U)$ is an isomorphism as desired.

4. Assume the hypothesis of *4.* It is obvious that U is quasi-compact. By the proof of *3.*, we have $\mathscr{F}_J(U) \cong \mathscr{F}_J^+(U)$. Let $\mathfrak{U} = \{U_i \to U\}$ be a covering satisfying the assumption in *4.*, we claim that each U_i is quasi-compact. Indeed, consider any covering $\mathfrak{U}_i = \{U_{ij} \to U_i\}$ of U_i. Then $\mathfrak{U}_i \times_U U_{i'}$ is a covering of $U_i \times_U U_{i'}$ for all i'. Since $U_i \times_U U_{i'}$ is quasi-compact, $\mathfrak{U}_i \times_U U_{i'}$ has a finite refinement $\mathfrak{V}_{i'}$. Combining all those $\mathfrak{V}_{i'}$, we get a finite covering $\{V_{i'k} \to U_i \times_U U_{i'} \to U_i\}$ which refines $\mathfrak{U}_i$.

The coverings satisfying the assumption in *4.* are cofinal in all coverings of U. Thus
$$(\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U) = \mathscr{F}_J^{++}(U) = \varinjlim_{\mathfrak{U} \text{ satisfies the assumption in } 4.} \check{H}^0(\mathfrak{U}, \mathscr{F}_J^+).$$
But for such a covering $\mathfrak{U} = \{U_i \to U\}$, since all U_i and $U_i \times_U U_{i'}$ are quasi-compact, we have
$$\begin{aligned}\check{H}^0(\mathfrak{U}, \mathscr{F}_J^+) &= \ker\Big(\prod \mathscr{F}_J^+(U_i) \rightrightarrows \prod \mathscr{F}_J^+(U_i \times_U U_{i'})\Big)\\ &\cong \ker\Big(\prod \mathscr{F}_J(U_i) \rightrightarrows \prod \mathscr{F}_J(U_i \times_U U_{i'})\Big)\\ &= \mathscr{F}_J(U).\end{aligned}$$
Therefore
$$\mathscr{F}_J^{++}(U) \cong \varinjlim_{\mathfrak{U} \text{ satisfies the assumption in } 4.} \mathscr{F}_J(U)$$
and thus the map $\mathscr{F}_J(U) \to \mathscr{F}_J^{++}(U)$ is an isomorphism as desired. □

3 (Canonical topology) A site is said to have **subcanonical topology** if all representable presheaves on this site are sheaves. A pretopology defines such a site is called a **subcanonical pretopology**. The largest subcanonical pretopology is called the **canonical pretopology**.

3.a (Effective epimorphisms) In a category $\mathcal{C}$, a family of morphisms with fixed target $\mathfrak{U} = \{U_i \to U\}$ is **effective-epic** if all the morphisms $U_i \to U$ have pullbacks and for any $X \in \operatorname{ob}\mathcal{C}$, the presheaf h_X satisfies the descent condition respect to $\mathfrak{U}$ (refer (2.1)). We say $\mathfrak{U}$ is **universal effective-epic** if its pullback along any morphism $V \to U$ is effective-epic.

Obviously, we have

3.b Proposition *A pretopology on $\mathcal{C}$ is subcanonical if and only if its coverings are all universal effective-epic. Moreover, if $\mathcal{C}$ has fibre products, then the canonical pretopology contains precisely all the universal effective-epic families.*

3.c Example The canonical topology on **Set** is given by the coverage Cov as follows.

$$\{\varphi_i\colon U_i \to U\}_{i\in I} \in \mathrm{Cov} \iff \bigcup_{i\in I} \varphi_i(U_i) = U.$$

To show this, it suffices to show that for any set S, $h_S(\bigcup_{i\in I} \varphi_i(U_i))$ is the equalizer of

$$\prod h_S(U_i) \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} \prod h_S(U_i \times_U U_{i'}).$$

Indeed, for any $f \in \mathrm{Hom}(\bigcup_{i\in I} \varphi_i(U_i), S)$, the compositions $f \circ \varphi_i$ give rise to an element in $\prod h_S(U_i)$, which has same image under pr_1 and pr_2. Conversely, for any element $(g_i) \in \prod h_S(U_i)$ having same image under pr_1 and pr_2, we have the following commutative diagram.

$$\begin{tikzcd} U_i \times_U U_{i'} \arrow[r] \arrow[d] & U_{i'} \arrow[d, "g_{i'}"] \\ U_i \arrow[r, "g_i"] & S \end{tikzcd}$$

For any element of $s \in \bigcup_{i\in I} \varphi_i(U_i)$, taking any preimage of it, say $u_i \in U_i$, we define $f(s) = g_i(u_i)$. The above commutative diagram guarantees the map $f\colon \bigcup_{i\in I} \varphi_i(U_i) \to S$ is well-defined.

3.d (Representable sheaves) Let $\mathcal{C}$ be a site having subcanonical topology. Then a **representable sheaf** is a sheaf of the form h_U for some $U \in \mathrm{ob}\,\mathcal{C}$. In this case, h_U is also denoted by $\widetilde{U}$.

Note that for any sheaf $\mathscr{F}$, we have

$$\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(h_U^{\#}, \mathscr{F}) \cong \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(h_U, \mathscr{F}) \cong \mathscr{F}(U).$$

In the case that $\mathcal{C}$ has subcanonical topology, $h_U^{\#} = h_U$ and then the Yoneda embedding identifies $\mathcal{C}$ with the full subcategory of $\mathbf{Sh}(\mathcal{C})$ consisting of representable sheaves.

3.e Lemma *Let $\mathcal{C}$ be a site. If $\{U_i \to U\}_{i\in I}$ is a covering, then the morphism of presheaves of setas*

$$\coprod_{i\in I} h_{U_i} \longrightarrow h_U$$

becomes surjective after sheafification.

Proof: We need to show $\coprod_{i\in I} h^{\#}_{U_i} \to h^{\#}_U$ is epic, which is equivalent to that for any sheaf $\mathscr{F}$,

$$\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(h^{\#}_U, \mathscr{F}) \to \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\coprod_{i\in I} h^{\#}_{U_i}, \mathscr{F})$$

is injective. But $\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(h^{\#}_U, \mathscr{F}) = \mathscr{F}(U)$ and $\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\coprod_{i\in I} h^{\#}_{U_i}, \mathscr{F}) = \prod_{i\in I} \mathscr{F}(U_i)$. Thus this is just the descent condition of $\mathscr{F}$. □

The following construct will be used later.

3.f Lemma *Let $\mathcal{C}$ be a site and $\mathcal{E}$ be a set of objects in $\mathcal{C}$ such that any object U in $\mathcal{C}$ has a covering by members of $\mathcal{E}$. Then for any sheaf of sets $\mathscr{F}$, there exists a diagram of sheaves of sets*

$$\mathscr{F}_1 \rightrightarrows \mathscr{F}_0 \longrightarrow \mathscr{F}$$

Where $\mathscr{F}_0$ and $\mathscr{F}_1$ are coproducts of sheaves of the form $h^{\#}_U$ with $U \in \mathcal{E}$ and $\mathscr{F}$ is the coequalizer.

Proof: First, for any $U \in \mathcal{E}$, a section $s \in \mathscr{F}(U)$ corresponds to a morphism $h^{\#}_U \to \mathscr{F}$. Taking the coproduct of them, we get a morphism

$$\mathscr{F}_0 = \coprod_{U\in\mathcal{E}, s\in\mathscr{F}(U)} h^{\#}_U \longrightarrow \mathscr{F}.$$

This is an epimorphism since for any section $s \in \mathscr{F}(V)$, choosing a covering $\{U_i \to V\}$ with $U_i \in \mathcal{E}$, we see each $s|_{U_i}$ has preimage $\mathrm{id}_{U_i} \in h^{\#}_{U_i}(U_i)$.

Let $\mathscr{G} = \mathscr{F}_0 \times_{\mathscr{F}} \mathscr{F}_0$, then $\mathscr{F}$ is the coequalizer of the kernel pair $\mathscr{G} \rightrightarrows \mathscr{F}_0$ of the above epimorphism.

Now, construct an epimorphism $\mathscr{F}_1 = \mathscr{G}_0 \to \mathscr{G}$ as above. Then, since compositing with an epimorphism does not change the coequalizer, this $\mathscr{F}_1$ is the required one. □

§ 8 Topoi and geometric morphisms

There are many distinct kinds of the notion of morphisms of sites, but they all induce the *geometric morphisms* between the categories of sheaves on them. Remember what crucial is the category of sheaves not the site.

1 (Topoi) A **(Grothendieck) topos** is a category $\mathbf{Sh}(\mathcal{C})$ of sheaves on a site $\mathcal{C}$. A **geometric moprhism** $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ between topoi is an *adjunction*:

$$f^* \dashv f_*\colon \mathbf{Sh}(\mathcal{D}) \leftrightarrows \mathbf{Sh}(\mathcal{C}),$$

in which the left adjoint f^* is left exact. The left adjoint f^* is called the **inverse image** and the right adjoint is called the **direct image**.

The *composition of geometric morphisms* are just the composition of adjunctions. More precisely, if $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ and $g\colon \mathbf{Sh}(\mathcal{D}) \to \mathbf{Sh}(\mathcal{E})$ are two geometric morphisms, then their composition $g \circ f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{E})$ is given by $(g \circ f)^* = f^* \circ g^*$ and $(g \circ f)_* = g_* \circ f_*$.

1.a (2-morphisms of topoi) Let $f, g\colon \mathbf{Sh}(\mathcal{C}) \rightrightarrows \mathbf{Sh}(\mathcal{D})$ be two geometric morphisms, a **2-morphism** from f to g is given by a natural transformation $\alpha\colon f_* \Rightarrow g_*$. Usually we denote it as

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{Sh}(\mathcal{D}). \end{tikzcd}$$

Note that since f^* is left adjoint to f_* and g^* is left adjoint to g_*, α also induces a natural transformation from g^* to f^* uniquely characterized by the commutative diagram.

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(f^*\mathscr{F}, \mathscr{G}) \arrow[r, equal] \arrow[d, "\alpha\circ -"'] & \mathrm{Hom}_{\mathbf{Sh}(\mathcal{D})}(\mathscr{F}, f_*\mathscr{G}) \arrow[d, "-\circ\alpha"] \\ \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(g^*\mathscr{F}, \mathscr{G}) \arrow[r, equal] & \mathrm{Hom}_{\mathbf{Sh}(\mathcal{D})}(\mathscr{F}, g_*\mathscr{G}) \end{tikzcd}$$

The *vertical and horizontal compositions of of 2-morphisms* are just the vertical and horizontal compositions of natural transformations. In this way, the topoi together with geometric morphisms between them and 2-morphisms between geometric morphisms from a *strict* 2-*category*. Note that this is a big category since the geometric morphisms between two topoi may not form a set.

1.b Remark Obviously the notion of geometric morphisms does not only work for topoi. For instance, let $F\colon \mathcal{C} \to \mathcal{D}$ be an equivalence of categories with weak inverse G, then $G \dashv F$ is a geometric morphism. Moreover, if both $\mathcal{C}$ and $\mathcal{D}$ are topoi, then this geometric morphism is an equivalence in the 2-category of topoi. Therefore, it makes sense to extend our definition by saying a category is a topos when it is equivalent to a category of sheaves on a site.

1.c Example (Empty topos) Let $*$ denote the category consisting of only one object $*$ with only one morphism id_*. It has two possible site structure on it. One is to treat it as the category of open sets of $\varnothing$, so this site is denoted as $\varnothing$. Then, since any sheaf on it must sent the initial object $*$ to a singleton, $\mathbf{Sh}(\varnothing)$ consists only one sheaf with only one morphism. In this way, $\mathbf{Sh}(\varnothing)$ is equivalent to $*$ itself.

1.d Example (Punctual topos) Another coverage on $*$ consists only one covering, namely the identity covering $\{\mathrm{id}_*\}$. Denote this site by $*$. Then, it is clearly that $\mathbf{Sh}(*) = \mathbf{PSh}(*) = \mathbf{Set}$.

But $*$ is not the only site giving the punctual topos. For instance, let $\mathcal{S}$ be a small full subcategory of **Set** which contains at least one nonempty set and has fibre products. Define coverings on $\mathcal{S}$ as surjective families of maps and shrink this coverage so that it defines a site. Now, we claim that $\mathbf{Sh}(\mathcal{S})$ is equivalent to **Set**.

First, if $\mathcal{S}$ contains a singleton, then the equivalence $i\colon \mathbf{Sh}(*) \to \mathbf{Sh}(\mathcal{S})$ is given by

$$i_* S = \mathrm{Hom}_{\mathbf{Set}}(-, S) \qquad \text{and} \qquad i^* \mathscr{F} = \mathscr{F}(*).$$

Indeed, if $\mathscr{F}$ is a sheaf on $\mathcal{S}$, then for any $U \in \operatorname{ob} \mathcal{S}$, there is a covering $\{\varphi_u\colon * \to U\}_{u \in U}$. The descent condition for this covering implies that

$$\mathscr{F}(U) \cong \prod_{u \in U} \mathscr{F}(*) \cong \mathrm{Hom}_{\mathbf{Set}}(U, \mathscr{F}(*)).$$

Moreover, this equality is compatible with restriction maps. Therefore the geometric morphism i is an equivalence of topoi.

Next, for general $\mathcal{S}$, we still have i_*. Let $\widetilde{*}$ be a nonempty set in $\mathcal{S}$ and $\varphi\colon \widetilde{*} \to \widetilde{*}$ is a map whose image is a singleton. Let

$$i^* \mathscr{F} = \operatorname{im} \mathscr{F}(\varphi).$$

Then this i^* is also a weak inverse of i_*.

1.e Example (Geometric morphisms between presheaf topoi) Any functor $u\colon \mathcal{C} \to \mathcal{D}$ induces a geometric morphism

$$u^p \dashv {}_p u\colon\ \mathbf{PSh}(\mathcal{D}) \leftrightarrows \mathbf{PSh}(\mathcal{C}),$$

where $u^p \mathscr{F} = \mathscr{F} \circ u$ for any $\mathscr{F} \in \operatorname{ob} \mathbf{PSh}(\mathcal{D})$ and ${}_p u$ is the *right Kan extension operator along* u. Moreover, if u itself is left exact, then there is a second geometric morphism

$$u_p \dashv u^p\colon\ \mathbf{PSh}(\mathcal{C}) \leftrightarrows \mathbf{PSh}(\mathcal{D}),$$

where u_p is the *left Kan extension operator along* u.

Proof: This first statement follows from Theorem 1.7.d and Remark 1.7.f. To show the second one, it suffices to show u_p is left exact when u is left exact. This follows from Lemma 1.7.c and the fact that filtered colimit commutes with finite limits. □

1.f (Geometric embeddings) A geometric morphism $f = (f^* \dashv f_*)$ is called a **geometric embedding** if f_* is fully faithful.

1.g Example Let $\mathcal{C}$ be a site. The sheafification $\#$ and the forgetful functor F from $\mathbf{Sh}(\mathcal{C})$ to $\mathbf{PSh}(\mathcal{C})$ forms a geometric embedding $\# \dashv F$.

One approach to geometric morphisms is induced by the morphisms of sites. But there are many candidates.

For the classical ones, we introduce the notions of *continuous functors* and *cocontinuous functors*.

2 (Continuous functors) Let $\mathcal{C}, \mathcal{D}$ be two sites. A functor $u \colon \mathcal{C} \to \mathcal{D}$ is called **continuous** if for every $\mathfrak{U} = \{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ we have

1. $u(\mathfrak{U}) := \{u(U_i) \to u(U)\}_{i \in I} \in \mathrm{Cov}(\mathcal{D})$,
2. For any morphism $T \to U$ in $\mathcal{C}$, $u(T \times_U U_i) \to u(T) \times_{u(U)} u(U_i)$ is an isomorphism.

Remark Do *NOT* confuse this "continuous functor" with the one used in category theory, which means a functor commuting with all limits.

2.a Lemma *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u \colon \mathcal{C} \to \mathcal{D}$ be a continuous functor. If $\mathscr{F}$ is a sheaf on $\mathcal{D}$, then $u^p \mathscr{F}$ is a sheaf on $\mathcal{C}$.*

Proof: The descent condition for $u^p \mathscr{F}$ respect to a covering $\mathfrak{U}$ is the same as the descent condition for $\mathscr{F}$ respect to the covering $u(\mathfrak{U})$. $\square$

Therefore, if $u \colon \mathcal{C} \to \mathcal{D}$ is continuous, then the restriction of u^p on $\mathbf{Sh}(\mathcal{D})$ gives a functor

$$u^s \colon \mathbf{Sh}(\mathcal{D}) \longrightarrow \mathbf{Sh}(\mathcal{C}).$$

2.b Theorem (The left adjoint of $\boldsymbol{u^s}$) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u \colon \mathcal{C} \to \mathcal{D}$ be a continuous functor. Then*

1. *$u_s := \# \circ u_p \circ F$ is a left adjoint to u^s;*
2. *For any presheaf $\mathscr{G}$ on $\mathcal{C}$, we have $(u_p \mathscr{G})^\# = (u_p(\mathscr{G}^\#))^\#$;*
3. *For any object U of $\mathcal{C}$, we have $u_s h_U^\# = h_{u(U)}^\#$.*

Proof: *1.* By Theorem 1.7.d and Proposition 6.3.a.

2. For any sheaf $\mathscr{F}$ on $\mathcal{D}$ we have

$$\begin{aligned}
\mathrm{Hom}_{Sh(\mathcal{D})}((u_p \mathscr{G})^\#, \mathscr{F}) &\cong \mathrm{Hom}_{\mathbf{PSh}(\mathcal{D})}(u_p \mathscr{G}, \mathscr{F}) \\
&\cong \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathscr{G}, u^p \mathscr{F}) \\
&\cong \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathscr{G}^\#, u^s \mathscr{F}) \\
&\cong \mathrm{Hom}_{\mathbf{Sh}(\mathcal{D})}(u_s(\mathscr{G}^\#), \mathscr{F}).
\end{aligned}$$

3. By Corollary 1.7.e and *2.* $\square$

2.c (Morphisms of sites) Let $\mathcal{C}, \mathcal{D}$ be two sites. A **morphism of sites** $f\colon \mathcal{D} \to \mathcal{C}$ is given by a continuous functor $u\colon \mathcal{C} \to \mathcal{D}$ (*NOT* $\mathcal{D} \to \mathcal{C}$!) such that the functor u_s is exact.

A morphism of sites $f\colon \mathcal{D} \to \mathcal{C}$ gives a geometric morphism $f = (f^*, f_*)$ as

$$f^* := u_s\colon \mathbf{Sh}(\mathcal{C}) \longrightarrow \mathbf{Sh}(\mathcal{D}), \qquad f_* := u^s\colon \mathbf{Sh}(\mathcal{D}) \longrightarrow \mathbf{Sh}(\mathcal{C}).$$

2.d Example (Continuous maps) Let $f\colon X \to Y$ be a continuous map between topological spaces. Recall that in Example 2.1.a, we defines the sites $\mathcal{T}_X$ and $\mathcal{T}_Y$. Then one can see the functor

$$\begin{aligned} u\colon \mathcal{T}_Y &\longrightarrow \mathcal{T}_X \\ V &\longmapsto f^{-1}(V) \end{aligned}$$

is a continuous functor. Moreover, u^s equals the *direct image functor* f_* defined in 5.1 and thus its left adjoint u_s is isomorphic to the *inverse image functor* f^* defined in 5.3. Since the inverse image functor is exact, so is u_s. Thus u induces a morphism of sites $f\colon \mathcal{T}_X \to \mathcal{T}_Y$.

2.e Lemma (Composition of continuous functors) *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be sites and $u\colon \mathcal{D} \to \mathcal{C}$ and $v\colon \mathcal{E} \to \mathcal{D}$ be continuous functors which induce morphisms of sites. Then the functor $u \circ v\colon \mathcal{E} \to \mathcal{C}$ is continuous and defines a morphism of sites $\mathcal{C} \to \mathcal{E}$.*

2.f (Composition of morphisms of sites) Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be three sites and $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D} \to \mathcal{E}$ be morphisms of sites given by continuous functors $u\colon \mathcal{D} \to \mathcal{C}$ and $v\colon \mathcal{E} \to \mathcal{D}$. The *composition* $g \circ f$ is the morphism of sites induced by the continuous functor $u \circ v$.

The following lemmas give some conditions for a continuous functor to define a morphism of sites.

2.g Lemma *Let $\mathcal{C}$ and $\mathcal{D}$ be sites and $u\colon \mathcal{C} \to \mathcal{D}$ be continuous functor. Assume all the categories $\mathcal{I}_V^{\mathrm{opp}}$ in 1.7 are filtered. Then u defines a morphism of sites $\mathcal{D} \to \mathcal{C}$, in other words u_s is exact.*

Proof: It suffices to show u_s is left exact, which follows from the fact that filtered colimit commutes with finite limits plus Proposition 6.3.a. □

2.h Lemma *Let $\mathcal{C}$ and $\mathcal{D}$ be sites and $u\colon \mathcal{C} \to \mathcal{D}$ be continuous functor. Assume $\mathcal{C}$ is finite-complete and u is left exact. Then u defines a morphism of sites $\mathcal{D} \to \mathcal{C}$, in other words u_s is exact.*

Proof: It suffices to show u_s is left exact, which follows from Lemma 1.7.c and Lemma 2.g. □

2.i Lemma *A continuous functor between sites which has a continuous left adjoint defines a morphism of sites.*

Proof: Let $u\colon \mathcal{D} \to \mathcal{C}$ be a continuous functor and $v\colon \mathcal{C} \to \mathcal{D}$ its continuous left adjoint. By Lemma 1.7.g, $u_p = v^p$, and hence $u_s = v^s$ has both left and right adjoint, whence is exact. $\square$

3 (Cocontinuous functors) Let $\mathcal{C}, \mathcal{D}$ be two sites. A functor $u\colon \mathcal{C} \to \mathcal{D}$ is called **cocontinuous** if for every $U \in \operatorname{ob}\mathcal{C}$ and every covering $\mathfrak{V} = \{V_j \to u(U)\}_{j\in J} \in \operatorname{Cov}(\mathcal{D})$, there exists a covering $\mathfrak{U} = \{U_i \to U\}_{i\in I} \in \operatorname{Cov}(\mathcal{C})$ such that $u(\mathfrak{U})$ refines the covering $\mathfrak{V}$.

Remark Note that this $u(\mathfrak{U})$ is in general *NOT* a covering.

Remark Do *NOT* confuse this "cocontinuous functor" with the one used in category theory, which means a functor commuting with all colimits.

3.a Lemma *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. If $\mathscr{F}$ is a sheaf on $\mathcal{C}$, then ${}_pu\mathscr{F}$ is a sheaf on $\mathcal{D}$.*

Proof: Let $\mathfrak{V} = \{V_j \to V\}$ be a covering in $\mathcal{D}$. We need to show the sequence

$$
{}_pu\mathscr{F}(V) \longrightarrow \prod {}_pu\mathscr{F}(V_j) \rightrightarrows \prod {}_pu\mathscr{F}(V_{j_0} \times_V V_{j_1})
$$

is exact. But ${}_pu\mathscr{F}$ is right adjoint to u^p, thus

$$
\begin{aligned}
{}_pu\mathscr{F}(V) &= \operatorname{Hom}_{\mathbf{PSh}(\mathcal{D})}(h_V, {}_pu\mathscr{F}) \\
&= \operatorname{Hom}_{\mathbf{PSh}(\mathcal{C})}(u^p h_V, \mathscr{F}) = \operatorname{Hom}_{\mathbf{Sh}(\mathcal{C})}((u^p h_V)^{\#}, \mathscr{F}).
\end{aligned}
$$

Therefore, it suffices to show that the sheafification of

$$
\coprod u^p h_{V_{j_0} \times_V V_{j_1}} \rightrightarrows \coprod u^p h_{V_j} \longrightarrow u^p h_V
$$

is exact.

First, we show the sheafification of $\coprod u^p h_{V_j} \to u^p h_V$ is an epimorphism. To do this, we use Corollary 7.1.f. Thus it suffices to show it is locally surjective. Let $s \in u^p h_V(U)$ be a section, which is a morphism $u(U) \to V$. Then $\mathfrak{V} \times_V u(U) = \{V_j \times_V u(U) \to u(U)\}$ is a covering in $\mathcal{D}$. As u is cocontinuous, there is a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that $\{u(U_i) \to u(U)\}$ refines $\mathfrak{V} \times_V u(U)$. This means that each restriction $s|_{U_i} : u(U_i) \to V$ factors through a morphism $s_i\colon u(U_i) \to V_j$ for some j. Thus $s|_{U_i}$ lies in the image of $u^p h_{V_j} \to u^p h_V$ as desired.

$$
\begin{tikzcd}
u(U_i) \arrow[r] \arrow[dr] & V_j \times_V u(U) \arrow[r] \arrow[d] & V_j \arrow[d] \\
& u(U) \arrow[r] & V
\end{tikzcd}
$$

Now we show the exactness. To do this, we use Proposition 7.1.c. Thus, note that sheafification is exact, it suffices to show $\coprod u^p h_{V_{j_0} \times_V V_{j_1}} \rightrightarrows \coprod u^p h_{V_j}$ is the kernel pair of $\coprod u^p h_{V_j} \to u^p h_V$. Let $s\colon u(U) \to V_j$ and $s'\colon u(U) \to V_{j'}$ be two sections of $\coprod u^p h_{V_j}$ having the same image in $u^p h_V$. Then we have the following commutative diagram

$$\begin{tikzcd} u(U) \arrow[r, "s'"] \arrow[d, "s"'] & V_{j'} \arrow[d] \\ V_j \arrow[r] & V \end{tikzcd}$$

and thus get a morphism $t = (s, s')\colon u(U) \to V_j \times_V V_{j'}$. This is a section of $\coprod u^p h_{V_{j_0} \times_V V_{j_1}}$ mapping to s and s' as desired. $\square$

Therefore, if $u\colon \mathcal{C} \to \mathcal{D}$ is cocontinuous, then the restriction of ${}_p u$ on $\mathbf{Sh}(\mathcal{C})$ gives a functor

$$ {}_s u\colon\ \mathbf{Sh}(\mathcal{C}) \longrightarrow \mathbf{Sh}(\mathcal{D}). $$

3.b Theorem (The left adjoint of ${}_s u$) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Then*

1. *$\# \circ u^p \circ F$ is a left adjoint to ${}_s u$ and is exact;*
2. *For any presheaf $\mathscr{G}$ on $\mathcal{C}$, we have $(u^p \mathscr{G})^\# = (u^p(\mathscr{G}^\#))^\#$.*

Proof: *1.* The adjunction follows from Remark 1.7.f and Proposition 6.3.a. The left exactness follows from the exactness of $\#$, u^p and the left exactness of F.

2.: For any sheaves $\mathscr{F}$ on $\mathcal{C}$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}((u^p\mathscr{G})^\#, \mathscr{F}) &= \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(u^p\mathscr{G}, \mathscr{F}) \\ &= \mathrm{Hom}_{\mathbf{PSh}(\mathcal{D})}(\mathscr{G}, {}_p u\mathscr{F}) \\ &= \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathscr{G}^\#, {}_s u\mathscr{F}) \\ &= \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}((u^p(\mathscr{G}^\#))^\#, \mathscr{F}). \end{aligned}$$ $\square$

3.c Corollary *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Then $g_* = {}_s u$ and $g^* = (u^p)^\#$ define a geometric morphism $g\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$.*

3.d Lemma (Composition of cocontinuous functors) *Let $u\colon \mathcal{C} \to \mathcal{D}$ and $v\colon \mathcal{D} \to \mathcal{E}$ be two cocontinuous functors inducing geometric morphisms $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ and $g\colon \mathbf{Sh}(\mathcal{D}) \to \mathbf{Sh}(\mathcal{E})$. Then $v \circ u$ is a cocontinuous functor and induces the geometric morphism $g \circ f$.*

Proof: Let $U \in \operatorname{ob}\mathcal{C}$. Let $\mathfrak{W} = \{W_i \to v(u(U))\}$ be a covering in $\mathcal{E}$. As v is cocontinuous, there exists a covering $\mathfrak{V} = \{V_j \to u(U)\}$ in $\mathcal{D}$ such that $v(\mathfrak{V})$ refines $\mathfrak{W}$. As u is cocontinuous, there exists a covering $\mathfrak{U} = \{U_k \to U\}$ in $\mathcal{C}$ such that $u(\mathfrak{U})$ refines $\mathfrak{V}$. Then $v(u(\mathfrak{U}))$ refines $\mathfrak{W}$. This shows $v \circ u$ is cocontinuous. As for the last assertion, it suffices to show ${}_sv \circ {}_su = {}_sv \circ u$, which suffices to show ${}_pv \circ {}_pu = {}_pv \circ u$. By Remark 1.7.f, it suffices to show $u^p \circ v^p = (v \circ u)^p$, which is obvious. $\square$

3.e Example (Open immersion) Let X be a topological space and $j\colon U \to X$ an inclusion map of a open subset U into X. Recall that we have sites $\mathcal{T}_X$ and $\mathcal{T}_U$ and continuous functor $u\colon \mathcal{T}_X \to \mathcal{T}_U$ as in Example 2.d. Next, consider the functor

$$\begin{aligned} v\colon \mathcal{T}_U &\longrightarrow \mathcal{T}_X \\ V &\longmapsto V. \end{aligned}$$

It is a cocontinuous functor, thus it induces a geometric morphism $(v^p)^\# \dashv {}_sv$. One can see $(v^p)^\# = j^*$ and ${}_sv = j_*$ by Lemma 5.6. In other words, the cocontinuous functor v induces the same geometric morphism as the continuous functor u.

3.f Example (Open map) Let $f\colon X \to Y$ be a open map between topological spaces. Then we have sites $\mathcal{T}_X$ and $\mathcal{T}_Y$ and continuous functor $u\colon \mathcal{T}_Y \to \mathcal{T}_X$ as in Example 2.d. Next, consider the functor

$$\begin{aligned} v\colon \mathcal{T}_X &\longrightarrow \mathcal{T}_Y \\ U &\longmapsto f(U). \end{aligned}$$

It is a cocontinuous functor, thus it induces a geometric morphism $(v^p)^\# \dashv {}_sv$. One can see $(v^p)^\# = f^*$ and ${}_sv = f_*$. Indeed, for any sheaf $\mathscr{G}$ on Y, we have

$$v^p\mathscr{G}(U) = \mathscr{G}(f(U)) \qquad \text{and} \qquad u_p\mathscr{G}(U) = \varinjlim_{f(U)\subset V} \mathscr{G}(V) = \mathscr{G}(f(U)).$$

Therefore, the cocontinuous functor v induces the same geometric morphism as the continuous functor u.

Remark Note that the functor v in Example 3.e is both cocontinuous and continuous, while the functor v in Example 3.f is not continuous in general.

3.g Remark (Property P) Let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor between categories. Given morphisms $g\colon u(U) \to V$ and $f\colon W \to V$ in $\mathcal{D}$ we can define a presheaf on $\mathcal{C}$ by

$$T \longmapsto \operatorname{Hom}_{\mathcal{C}}(T, U) \times_{\operatorname{Hom}_{\mathcal{D}}(u(T),V)} \operatorname{Hom}_{\mathcal{D}}(u(T), W).$$

If this presheaf is representable, then we denote the representative object by $U \times_{g,V,f} W$ or simply $U \times_V W$ or $g \times_V f$. Obviously this is a generalization of fibre product.

Assume $\mathcal{C}$ and $\mathcal{D}$ are sites. A functor u is said to have **property P** if for every covering $\mathfrak{V}$ and any morphism $g\colon u(U) \to V$ in $\mathcal{D}$, $U \times_V \mathfrak{V}$ exists and is a covering in $\mathcal{C}$. One can see this property is similar to the definition of continuous functor and implies that u is cocontinuous.

4 Lemma (Continuous and cocontinuous functors) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a functor which is both continuous and cocontinuous. Let $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be the geometric morphism induced by u, then*

1. *$f^* = u^p$;*

2. *f^* has a left adjoint $f_! := (u_p)^\#$.*

In this case, we have a sequence of adjunctions:

$$f_! \dashv f^* \dashv f_*.$$

Note that the functor $f_!$ is *NOT* exact in general.

4.a Lemma *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a functor which is both continuous and cocontinuous. Let $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be the geometric morphism induced by u. Assume $\mathcal{C}$ has fibre products and equalizers and u commutes with them. Then $f_!$ commutes with finite connected limits.*

Proof: By Lemma 1.7.c and the fact that *coproducts commute with connected limits* (refer §2.12 in *BMO*) It suffices to show the opposite of $\mathcal{I}_V^u$ is a disjoint union of filtered categories. To do this, it suffices to show

1. For any $f\colon (A, \phi) \to (C, \chi)$ and $g\colon (B, \psi) \to (C, \chi)$, there exists some $f'\colon (D, \theta) \to (A, \phi)$ and $g'\colon (D, \theta) \to (B, \psi)$.

2. For any $f, g\colon (A, \phi) \rightrightarrows (B, \psi)$, there exists a $h\colon (C, \theta) \to (A, \phi)$ such that $f \circ h = g \circ h$.

For *1.*, let D be the fibre product of A and B over C in $\mathcal{C}$, then $u(D) = u(A) \times_{u(C)} u(B)$. Thus there exists a unique morphism $\theta\colon V \to u(D)$ compatible with ϕ and ψ. Then the two projections $(D, \theta) \to (A, \phi)$ and $(D, \theta) \to (B, \psi)$ are the required ones.

The prove for *2.* is the same in Lemma 1.7.c. □

4.b Lemma *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a functor which is both continuous and cocontinuous. Let $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be the geometric morphism induced by u. If u is fully faithful, then $\eta\colon \mathrm{id}_{\mathbf{Sh}(\mathcal{C})} \Rightarrow f^* f_!$ and $\epsilon\colon f^* f_* \Rightarrow \mathrm{id}_{\mathbf{Sh}(\mathcal{C})}$ are natural isomorphisms.*

Proof: Let $U \in \operatorname{ob}\mathcal{C}$. We have

$$f^* f_* \mathscr{F}(U) = f_* \mathscr{F}(u(U)) = \varprojlim_{{}_{u(U)}\mathcal{I}^{\mathrm{opp}}} \mathscr{F}(V).$$

For any $(V, \phi) \in \operatorname{ob}{}_{u(U)}\mathcal{I}$, since u is fully faithful, there exists a morphism $\psi\colon V \to U$ such that $u(\psi) = \phi$. Therefore, ${}_{u(U)}\mathcal{I}$ has a terminal object $(U, \mathrm{id}_{u(U)})$. Thus $f^* f_* \mathscr{F}(U) = \mathscr{F}(U)$.

On the other hand, we have

$$f^* f_! \mathscr{F}(U) = f_! \mathscr{F}(u(U)) = (u_p \mathscr{F})^{\#}(u(U)),$$

and

$$u_p \mathscr{F}(u(U)) = \varinjlim_{\mathcal{I}_{u(U)}^{\mathrm{opp}}} \mathscr{F}(V).$$

For any $(V, \phi) \in \operatorname{ob}\mathcal{I}_{u(U)}$, since u is fully faithful, there exists a morphism $\psi\colon U \to V$ such that $u(\psi) = \phi$. Therefore, $\mathcal{I}_{u(U)}$ has an initial object $(U, \mathrm{id}_{u(U)})$. Thus $u_p \mathscr{F}(u(U)) = \mathscr{F}(U)$. Since u is both continuous and cocontinuous, any covering of $u(U)$ in $\mathcal{D}$ can be refined by a covering $\{u(U_i) \to u(U)\}$ in $\mathcal{D}$, where $\{U_i \to U\}$ is a covering in $\mathcal{C}$. Therefore $(u_p \mathscr{F})^{+}(u(U)) = \mathscr{F}(U)$. Thus $(u_p \mathscr{F})^{\#}(u(U)) = \mathscr{F}(U)$ as desired. □

4.c Lemma (Functors inducing two geometric morphisms) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a functor which is both continuous and cocontinuous. Let $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be the geometric morphism induced by u. Assume*

- *u is fully faithful;*
- *$\mathcal{C}$ is finite-complete and u is left exact.*

Then u induces a morphism of sites $g\colon \mathcal{D} \to \mathcal{C}$ such that

1. *$g^* = f_!$ and $g_* = f^*$;*
2. *the composition $g \circ f$ of f and g is isomorphic to the identity geometric morphism $\mathrm{id}_{\mathbf{Sh}(\mathcal{C})}$;*
3. *g^* is fully faithful.*

Proof: *1.* follows from Lemma 2.h, *2.* from Lemma 4.b. As for *3.*, consider any sheaves $\mathscr{F}$ and $\mathscr{G}$ on $\mathcal{C}$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathscr{F}, \mathscr{G}) &\cong \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(g_* g^* \mathscr{F}, \mathscr{G}) \\ &\cong \mathrm{Hom}_{\mathbf{Sh}(\mathcal{D})}(g^* \mathscr{F}, g^* \mathscr{G}).\end{aligned}$$

□

4.d Lemma (Cocontinuous functors with a right adjoint) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Let $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be the geometric morphism induced by u. Assume u has a right adjoint v. Then $f_* = v^s$, $f^* = v_s := (v_p)^\#$. Therefore, when v is continuous, it defines a morphism of sites which induces the same geometric morphism with u.*

Proof: Let $\mathscr{F}$ be a sheaf on $\mathcal{C}$, it suffices to show $f_*\mathscr{F}(V) = \mathscr{F}(v(V))$ for any object $V \in \operatorname{ob}\mathcal{D}$. First, we have $u^p h_V = h_{v(V)}$ by Lemma 1.7.g. Then, by Theorem 3.b, we have

$$f^*(h_V^\#) = (u^p h_V)^\# = h_{v(V)}^\#.$$

Therefore

$$\begin{aligned} f_*\mathscr{F}(V) &= \operatorname{Hom}_{\mathbf{Sh}(\mathcal{D})}(h_V^\#, f_*\mathscr{F}) \\ &= \operatorname{Hom}_{\mathbf{Sh}(\mathcal{C})}(f^* h_V^\#, \mathscr{F}) \\ &= \operatorname{Hom}_{\mathbf{Sh}(\mathcal{C})}(h_{v(V)}^\#, \mathscr{F}) \\ &= \mathscr{F}(v(V)). \end{aligned}$$

□

4.e Lemma (Cocontinuous functors with a left adjoint) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a functor which is both continuous and cocontinuous. Let $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be the geometric morphism induced by u. Assume u has a left adjoint w.*

1. *$f_! = (w^p)^\#$ and it is exact.*
2. *If w is continuous, then $f_!$ has a left adjoint.*
3. *If w is cocontinuous and induces a geometric morphism $g\colon \mathbf{Sh}(\mathcal{D}) \to \mathbf{Sh}(\mathcal{C})$, then $g^* = f_!$ and $g_* = f^*$.*

Proof: *1.* follows from Lemma 1.7.g. If w is continuous, then $f_! = w^p = w^s$, which has a left adjoint w_s as in Theorem 2.b. If w is cocontinuous, then the statement is nothing but Lemma 4.d. □

5 Lemma (Existence of lower shriek) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be a geometric morphism. Assume $\mathcal{C}$ has a subcategory $\mathcal{E}$ such that*

- *for every $U \in \operatorname{ob}\mathcal{E}$, there exists a sheaf $\mathscr{G}_U$ on $\mathcal{D}$ such that $g^*\mathscr{F}(U) = \operatorname{Hom}_{\mathbf{Sh}(\mathcal{D})}(\mathscr{G}_U, \mathscr{F})$ functorally for $\mathscr{F} \in \operatorname{ob}\mathbf{Sh}(\mathcal{D})$;*
- *every object in $\mathcal{C}$ has a covering by objects in $\mathcal{E}$.*

Then f^* has a left adjoint $f_!$.

Proof: Since $f^*\mathscr{F}(U) = \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(h_U^\#, f^*\mathscr{F})$, we have the bijection

$$\mathrm{Hom}_{\mathbf{Sh}(\mathcal{D})}(\mathscr{G}_U, \mathscr{F}) \cong \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(h_U^\#, f^*\mathscr{F})$$

natural on $U \in \mathrm{ob}\,\mathcal{E}$ and $\mathscr{F} \in \mathrm{ob}\,\mathbf{Sh}(\mathcal{D})$.

Therefore, we already have a functor $f_!\colon h_U^\# \mapsto \mathscr{G}_U$ from the full subcategory of $\mathbf{Sh}(\mathcal{C})$ consisting of sheaves of the form $h_U^\#$ with $U \in \mathrm{ob}\,\mathcal{E}$ to $\mathbf{Sh}(\mathcal{D})$. It remains to extend it to a functor from $\mathbf{Sh}(\mathcal{C})$ to $\mathbf{Sh}(\mathcal{D})$.

First, since $\prod \mathrm{Hom}(-,-) = \mathrm{Hom}(\coprod, -)$, we have natural bijection

$$\mathrm{Hom}_{\mathbf{Sh}(\mathcal{D})}(\coprod \mathscr{G}_{U_j}, \mathscr{F}) \cong \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\coprod h_{U_j}^\#, f^*\mathscr{F}).$$

This extends $f_!$ to coproducts of some $h_U^\#$ with each $U \in \mathrm{ob}\,\mathcal{E}$.

Now, for any sheaf $\mathcal{H}$ on $\mathcal{C}$, by Lemma 7.3.f, we have a coequalizer diagram

$$\mathcal{H}_1 \rightrightarrows \mathcal{H}_0 \longrightarrow \mathcal{H}.$$

where $\mathcal{H}_1$ and $\mathcal{H}_0$ are coproducts of some $h_U^\#$ with each $U \in \mathrm{ob}\,\mathcal{E}$. Apply $f_!$ to $\mathcal{H}_1 \rightrightarrows \mathcal{H}_0$, then the coequalizer of it gives $f_!\mathcal{H}$. One can see this functor $f_!\colon\ \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ is left adjoint to f^*. □

§ 9 Localization

1 (Localization) Let $\mathcal{C}$ be a site and $X \in \mathrm{ob}\,\mathcal{C}$. Then the *slice category* $\mathcal{C}/X$ (whose objects are morphisms $U \to X$ in $\mathcal{C}$ with fixed target X, usually denoted as U/X, and morphisms are morphisms in $\mathcal{C}$ compatible with them) inherits a coverage from $\mathcal{C}$: a family $\mathfrak{U}/X = \{U_i/X \to U/X\}$ in $\mathcal{C}/X$ is a covering when $\mathfrak{U} = \{U_i \to U\}$ is a covering in $\mathcal{C}$. This site $\mathcal{C}/X$ is called the **localization** of $\mathcal{C}$ at X.

Consider the forgetful functor $j_X\colon \mathcal{C}/X \to \mathcal{C}$. It is both continuous and cocontinuous. Therefore, by Lemma 8.4, it induces a geometric morphism

$$j_X\colon\ \mathbf{Sh}(\mathcal{C}/X) \longrightarrow \mathbf{Sh}(\mathcal{C}),$$

called the **localization morphism**. This geometric morphism is given by the *direct image functor* j_{X*} and the *inverse image functor* j_X^*. For any sheaf $\mathscr{F}$ on $\mathcal{C}$, we call the sheaf $j_X^*\mathscr{F}$ the **restriction of $\mathscr{F}$ on** X and denoted by $\mathscr{F}|_X$. Obviously, we have $\mathscr{F}|_X(U/X) = \mathscr{F}(U)$. Moreover, j_X^* has a left adjoint $j_{X!}$. For any sheaf $\mathscr{G}$ on $\mathcal{C}/X$, we call the sheaf $j_{X!}\mathscr{G}$ the **extension of $\mathscr{G}$ by empty**.

1.a Lemma (Description of $j_{X!}$) *Let $\mathcal{C}$ be a site and $X \in \mathrm{ob}\,\mathcal{C}$. Let $\mathscr{G}$ be a presheaf on $\mathcal{C}/X$. Then $j_{X!}(\mathscr{G}^\#)$ is the sheafification of the presheaf*

$$V \mapsto \coprod_{\phi \in \mathrm{Hom}_{\mathcal{C}}(U,X)} \mathscr{G}(U \xrightarrow{\phi} X).$$

Proof: First of all, by definition,

$$j_{Xp}\mathscr{G}(U) = \varinjlim_{(V/X, U\to V)\in \mathrm{ob}\,\mathcal{I}_U^{\mathrm{opp}}} \mathscr{G}(V).$$

By the family $\{(U/X, \mathrm{id}_U) | U/X \in \mathrm{Hom}_{\mathcal{C}}(U, X)\}$ is final in $\mathcal{I}_U^{\mathrm{opp}}$, thus

$$j_{Xp}\mathscr{G}(U) = \coprod_{\phi\in \mathrm{Hom}_{\mathcal{C}}(U,X)} \mathscr{G}(U \xrightarrow{\phi} X).$$

By Lemma 8.4, $j_{X!}(\mathscr{G}^{\#}) = (j_{Xp}(\mathscr{G}^{\#}))^{\#}$, which equals to $(j_{Xp}\mathscr{G})^{\#}$ by Theorem 8.2.b. □

1.b Lemma *Let $\mathcal{C}$ be a site and $X \in \mathrm{ob}\,\mathcal{C}$. Let $U/X \in \mathrm{ob}\,\mathcal{C}/X$. Then we have $j_{X!}(h^{\#}_{U/X}) = h^{\#}_U$.*

Proof: Since $h_{U/X}(V/X) = \mathrm{Hom}_{\mathcal{C}/X}(V/X, U/X)$ and

$$\mathrm{Hom}_{\mathcal{C}}(V, U) = \coprod_{V/X\in \mathrm{Hom}_{\mathcal{C}}(V,X)} \mathrm{Hom}_{\mathcal{C}/X}(V/X, U/X),$$

the statement follows from Lemma 1.a. □

1.c Theorem *Let $\mathcal{C}$ be a site and $X \in \mathrm{ob}\,\mathcal{C}$. The functor $j_{X!}$ gives an equivalence of categories*

$$\mathbf{Sh}(\mathcal{C}/X) \longrightarrow \mathbf{Sh}(\mathcal{C})/h^{\#}_X.$$

Proof: First, any topos has a terminal object, namely the constant sheaf $\underline{*}$ of singleton. But in $\mathbf{Sh}(\mathcal{C}/X)$, we have $\underline{*} = h^{\#}_{X/X}$. Thus for any sheaf $\mathscr{G}$ on $\mathcal{C}/X$, we have a canonical morphism

$$j_{X!}(\mathscr{G}) \longrightarrow j_{X!}(\underline{*}) = j_{X!}(h^{\#}_{X/X}) = h^{\#}_X.$$

This gives a functor from $\mathbf{Sh}(\mathcal{C}/X)$ to $\mathbf{Sh}(\mathcal{C})/h^{\#}_X$.

Conversely, for any sheaf $\mathscr{F}$ on $\mathcal{C}$ and morphism $\varphi\colon \mathscr{F} \to h^{\#}_X$, we define $\mathscr{F}_\varphi(U/X)$ to be the fiber of $U/X \in h_X(U) \to h^{\#}_X(U)$ along $\mathscr{F}(U) \to h^{\#}_X(U)$.

$$\begin{tikzcd}
\mathscr{F}_\varphi(U/X) \arrow[rr, hook] \arrow[d] & & \mathscr{F}(U) \arrow[d, "\varphi"] \\
U/X \quad \in & h_X(U) \arrow[r] & h^{\#}_X(U)
\end{tikzcd}$$

Then one can see this $\mathscr{F}_\varphi$ is a sheaf on $\mathcal{C}/X$ and further the functor $\varphi \mapsto \mathscr{F}_\varphi$ gives a weak inverse of the functor above. □

Note that this lemma factors $j_{X!}$ as

$$\mathbf{Sh}(\mathcal{C}/X) \longrightarrow \mathbf{Sh}(\mathcal{C})/h_X^{\#} \longrightarrow \mathbf{Sh}(\mathcal{C}),$$

where the first functor is an equivalence of categories and the second is a localization again! We sometimes also denote the equivalence as $j_{X!}$.

1.d Lemma *Let $\mathcal{C}$ be a site and $X \in \operatorname{ob}\mathcal{C}$. The functor $j_{X!}$ commutes with finite connected limits. In particular, if $\mathscr{F} \subset \mathscr{F}'$ in $\mathbf{Sh}(\mathcal{C}/X)$, then $j_{X!}\mathscr{F} \subset j_{X!}\mathscr{F}'$.*

Proof: One can see the forgetful functor $\mathbf{Sh}(\mathcal{C})/h_X^{\#} \to \mathbf{Sh}(\mathcal{C})$ commutes with fibre products and equalizers. Therefore it commutes with finite connected limits and so does $j_{X!}$. $\square$

Proof: Note that coproducts commute with finite connected limites, then by Lemma 1.a, so is $j_{X!}$. $\square$

1.e Lemma *Let $\mathcal{C}$ be a site and $X \in \operatorname{ob}\mathcal{C}$. For any sheaf $\mathscr{F}$ on $\mathcal{C}$, we have $j_{X!}j_X^*\mathscr{F} = \mathscr{F} \times h_X^{\#}$.*

Proof: By Lemma 1.a, $j_{X!}j_X^*\mathscr{F}$ is the sheafification of $U \mapsto \mathscr{F}(U) \times h_X(U)$. Thus $j_{X!}j_X^*\mathscr{F} = \mathscr{F} \times h_X^{\#}$. $\square$

1.f Lemma *Let $\mathcal{C}$ be a site and $X \in \operatorname{ob}\mathcal{C}$. Assume $\mathcal{C}$ has subcanonical topology. Then for any sheaf $\mathscr{G}$ on $\mathcal{C}/X$, we have*

$$j_{X!}\mathscr{G}(U) = \coprod_{\phi \in \operatorname{Hom}_{\mathcal{C}}(U,X)} \mathscr{G}(U \xrightarrow{\phi} X).$$

Proof: One needs to show that $U \mapsto \coprod_{\phi \in \operatorname{Hom}_{\mathcal{C}}(U,X)} \mathscr{G}(U \xrightarrow{\phi} X)$ defines a sheaf $\mathscr{H}$ on $\mathcal{C}$. Let $\mathfrak{U} = \{U_i \to U\}_{i \in I}$ be a covering in $\mathcal{C}$. We need to show $\mathscr{H}(U) \cong \check{H}^0(\mathfrak{U}, \mathscr{H})$. Note that there is a canonical morphism $\mathscr{H} \to h_X$ given by $(s, \phi) \mapsto \phi$, where $\phi\colon U \to X$ is an object in $\mathcal{C}/X$ and $s \in \mathscr{G}(\phi)$.

Let $(s_i, \phi_i)_{i \in I} \in \check{H}^0(\mathfrak{U}, \mathscr{H})$. Then $(\phi_i)_{i \in I} \in \check{H}^0(\mathfrak{U}, h_X)$. Since $\mathcal{C}$ has subcanonical topology, h_X is a sheaf and thus $\check{H}^0(\mathfrak{U}, h_X) = h_X(U)$. Therefore there exists a unique $\phi\colon U \to X$ such that ϕ_i are compositions of ϕ with $U_i \to U$. Then, $\mathfrak{U}/X$ is a covering of ϕ in $\mathcal{C}/X$. In this case, $(s_i)_{i \in I}$ lies in $\check{H}^0(\mathfrak{U}/X, \mathscr{G})$ and thus defines a section $s \in \mathscr{G}(\phi)$. One can see the pair $(s, \phi) \in \mathscr{H}(U)$ is unique the preimage of $(s_i, \phi_i)_{i \in I}$ under the canonical map $\mathscr{H}(U) \to \check{H}^0(\mathfrak{U}, \mathscr{H})$. This shows the desired isomorphism. $\square$

1.g Lemma *Let $\mathcal{C}$ be a site and $X \in \operatorname{ob}\mathcal{C}$. Assume $\mathcal{C}$ has finite products. Then*

1. *j_X has a continuous right adjoint v given by $v(U) = U \times X/X$;*

2. *the functor v defines a morphism of sites $\mathcal{C}/X \to \mathcal{C}$ which induces the same geometric morphisms as j_X;*

3. $j_{X*}\mathscr{F}(U) = \mathscr{F}(U \times X/X)$.

Proof: *1.* To show the adjunction, we only need to verify that for any $U/X \in \mathcal{C}/X$ and $V \in \mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(U, V) \cong \mathrm{Hom}_{\mathcal{C}/X}(U/X, V \times X/X),$$

which is clear. To see v is continuous, let $\mathfrak{U} = \{U_i \to U\}$ be a covering of $\mathcal{C}$, then $\mathfrak{U} \times_U (U \times X)$ is a covering of $U \times X$ in $\mathcal{C}$ and therefore $\mathfrak{U} \times_U (U \times X)/X$ is a covering in $\mathcal{C}/X$.

2. and *3.* follow from Lemma 8.4.d. □

1.h Lemma *Let $\mathcal{C}$ be a site and $X \in \mathrm{ob}\,\mathcal{C}$. Assume every $U \in \mathrm{ob}\,\mathcal{C}$ has at most one morphism to X. Then $\eta\colon \mathrm{id}_{\mathbf{Sh}(\mathcal{C}/X)} \Rightarrow j_X^* j_{X!}$ and $\epsilon\colon j_X^* j_{X*} \Rightarrow \mathrm{id}_{\mathbf{Sh}(\mathcal{C}/X)}$ are natural isomorphisms.*

Proof: Note that the assumption implies j_X is fully faithful. Then the statement follows from Lemma 8.4.b. □

2 Lemma (Changes of base object) *Let $\mathcal{C}$ be a site and $f\colon Y \to X$ be a morphism of $\mathcal{C}$. Then there exists a commutative diagram*

$$\begin{tikzcd} \mathcal{C}/Y \arrow[rr, "j"] \arrow[dr, "j_Y"'] & & \mathcal{C}/X \arrow[dl, "j_X"] \\ & \mathcal{C} & \end{tikzcd}$$

of cocontinuous functors. Here $j\colon \mathcal{C}/Y \to \mathcal{C}/X$ is identified with the functor $j_{Y/X}\colon (\mathcal{C}/X)/(Y/X) \to \mathcal{C}/X$ via the identification $(\mathcal{C}/X)/(Y/X) = \mathcal{C}/Y$. Moreover, we have $j_{Y!} = j_{X!} \circ j_!$, $j_Y^* = j^* \circ j_X^*$, and $j_{Y*} = j_{X*} \circ j_*$.

Proof: The identification is obvious and the statements then follow from Lemma 8.3.d. □

2.a Lemma *Notations as in Lemma 2. Through the identifications $\mathbf{Sh}(\mathcal{C}/Y) = \mathbf{Sh}(\mathcal{C})/h_Y^{\#}$ and $\mathbf{Sh}(\mathcal{C}/X) = \mathbf{Sh}(\mathcal{C})/h_X^{\#}$ of Theorem 1.c, the functor j^* has the following description*

$$j^*(\mathscr{H} \xrightarrow{\varphi} h_X^{\#}) = (\mathscr{H} \times_{h_X^{\#}} h_Y^{\#} \longrightarrow h_Y^{\#}).$$

Proof: Let $\varphi\colon \mathscr{H} \to h_X^{\#}$ be an object in $\mathbf{Sh}(\mathcal{C})/h_X^{\#}$. Let θ denote the canonical morphism induced by sheafification. By Theorem 1.c, it corresponds to the sheaf $\mathscr{H}_\varphi$ on $\mathcal{C}/X$ and we have

$$\mathscr{H}_\varphi(U/X) = \{s \in \mathscr{H}(U) | \varphi(s) = \theta(U/X)\}.$$

Write f as Y/X, we have

$$\begin{aligned} j^*\mathscr{H}_\varphi(U/Y) &= j^*_{Y/X}\mathscr{H}_\varphi((Y/X \circ U/Y)/(Y/X)) \\ &= \mathscr{H}_\varphi(Y/X \circ U/Y) \\ &= \{s \in \mathscr{H}(U) | \varphi(s) = \theta(Y/X \circ U/Y)\}\,. \end{aligned}$$

On the other hand, the sheaf on $\mathcal{C}/Y$ corresponding to

$$\mathscr{H}' = \mathscr{H} \times_{h_X^\#} h_Y^\# \xrightarrow{\varphi'} h_Y^\#,$$

is given by

$$\begin{aligned} \mathscr{H}'_{\varphi'}(U/Y) &= \left\{s' \in \mathscr{H}'(U) \middle| \varphi'(s') = \theta(U/Y)\right\} \\ &= \left\{(s,a) \in \mathscr{H}(U) \times h_Y^\#(U) \middle| a = \theta(U/Y), \varphi(s) = \theta(Y/X \circ U/Y)\right\} \\ &= \{s \in \mathscr{H}(U) | \varphi(s) = \theta(Y/X \circ U/Y)\}\,. \end{aligned}$$

Therefore $j^*\mathscr{H}_\varphi = \mathscr{H}'_{\varphi'}$. □

2.b Lemma *The localization j satisfies property P (ref. Remark 8.3.g).*

Proof: Let $\mathfrak{U}/X$ be a covering and $g\colon j(V/Y) \to (U/X)$ be a morphism in $\mathcal{C}/X$. Then $(V/Y) \times_{(U/X)} (\mathfrak{U}/X) = (V \times_U \mathfrak{U}/X)$ and is a covering in $\mathcal{C}/Y$.□

2.c Lemma *Let $\mathcal{C}$ be a site and $f\colon Y \to X$ be a morphism of $\mathcal{C}$. Assume $\mathcal{C}$ has fibre products. Then*

1. *j has a continuous right adjoint v given by $v(U/X) = U \times_X Y/Y$;*
2. *the functor v defines a morphism of sites $\mathcal{C}/Y \to \mathcal{C}/X$ which induces the same geometric morphisms as j;*
3. *$j_*\mathscr{F}(U) = \mathscr{F}(U \times_X Y/Y)$.*

Proof: Follows from the identification $j = j_{Y/X}$ and Lemma 1.g . □

3 Lemma (Localization of a morphism of sites) *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a morphism of sites defined by a continuous functor $u\colon \mathcal{D} \to \mathcal{C}$. Let $Y \in \operatorname{ob}\mathcal{D}$ and $X = u(Y)$. Then the functor*

$$\begin{aligned} u'\colon \mathcal{D}/Y &\longrightarrow \mathcal{C}/X \\ V/Y &\longmapsto u(V)/X \end{aligned}$$

defines a morphism of sites $f'\colon \mathcal{C}/X \to \mathcal{D}/Y$, which satisfies a commutative diagram of topoi

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}/X) \arrow[r, "j_X"] \arrow[d, "f'"'] & \mathbf{Sh}(\mathcal{C}) \arrow[d, "f"] \\ \mathbf{Sh}(\mathcal{D}/Y) \arrow[r, "j_Y"] & \mathbf{Sh}(\mathcal{D}) \end{tikzcd}$$

Through the identifications $\mathbf{Sh}(\mathcal{C}/X) = \mathbf{Sh}(\mathcal{C})/h_X^\#$ *and* $\mathbf{Sh}(\mathcal{D}/Y) = \mathbf{Sh}(\mathcal{D})/h_Y^\#$ *of Theorem 1.c, the functor* f'^* *has the following description*

$$f'^*(\mathscr{H} \xrightarrow{\varphi} h_Y^\#) = (f^*\mathscr{H} \xrightarrow{f^*\varphi} h_X^\#).$$

Finally, we have $f'_* \circ j_X^* = j_Y^* \circ f_*$.

Proof: It is clear that u' is also continuous. Then u' induces an adjunction

$$f'^* \dashv f'_*\colon \mathbf{Sh}(\mathcal{C}/X) \leftrightarrows \mathbf{Sh}(\mathcal{C}/Y)$$

by $f'_* = u'^s$ and $f'^* = u'_s$. Then for any sheaf $\mathscr{F}$ on $\mathcal{C}$, we have

$$f'_* j_X^* \mathscr{F}(V/Y) = j_X^*\mathscr{F}(u(V)/X) = \mathscr{F}(u(V)) = f_*\mathscr{F}(V) = j_Y^* f_*\mathscr{F}(V/Y).$$

Thus $f'_* \circ j_X^* = j_Y^* \circ f_*$. It remains to show u' is exact, the diagram of topoi is commutative and the description of f'^* is as given.

First, $j_{X!} \circ f'^* = f^* \circ j_{Y!}$. Indeed, for any sheaf $\mathscr{H}$ on $\mathcal{D}/Y$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Sh}(\mathcal{D}/Y)}(j_{X!}f'^*\mathscr{H}, \mathscr{F}) &= \mathrm{Hom}_{\mathbf{Sh}\,\mathcal{C}}(\mathscr{H}, f'_* j_X^*\mathscr{F}) \\ &= \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathscr{H}, j_Y^* f_*\mathscr{F}) \\ &= \mathrm{Hom}_{\mathbf{Sh}(\mathcal{D}/Y)}(f^* j_{Y!}\mathscr{H}, \mathscr{F}). \end{aligned}$$

Moreover, this equality factors through the equivalences in Theorem 1.c:

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}/X) \arrow[r, "j_{X!}"] & \mathbf{Sh}(\mathcal{C})/h_X^\# \arrow[r] & \mathbf{Sh}(\mathcal{C}) \\ \mathbf{Sh}(\mathcal{D}/Y) \arrow[r, "j_{Y!}"] \arrow[u, "f'^*"] & \mathbf{Sh}(\mathcal{D})/h_Y^\# \arrow[r] \arrow[u, "f^*"] & \mathbf{Sh}(\mathcal{D}) \arrow[u, "f^*"] \end{tikzcd}$$

Indeed, this follows from that $f^*h_Y^\# = h_{u(Y)}^\# = h_X^\#$ by Theorem 8.2.b.

This commutative diagram shows the description of f'^*. Since f^* is exact, the equivalences shows so is f'^*.

Now, we show $f'^* \circ j_Y^* = j_X^* \circ f^*$. Indeed, for any sheaf $\mathscr{G}$ on $\mathcal{D}$, by Lemma 1.e, we have

$$j_{X!}f'^* j_Y^*\mathscr{G} = f^* j_{Y!} j_Y^*\mathscr{G} = f^*(\mathscr{G} \times h_Y^\#) = f^*\mathscr{G} \times h_X^\# = j_{X!} j_X^* f^*\mathscr{G}.$$

Since $j_{X!}$ is an equivalence, we have $f'^* \circ j_Y^* = j_X^* \circ f^*$. The commutativity of the diagram of topoi then follows. $\square$

3.a Lemma *Let* $u\colon \mathcal{D} \to \mathcal{C}$ *be a continuous functor. Let* $Y \in \operatorname{ob}\mathcal{D}$, $X = u(Y)$ *and* u' *be*

$$\begin{aligned} u'\colon \mathcal{D}/Y &\longrightarrow \mathcal{C}/X \\ V/Y &\longmapsto u(V)/X. \end{aligned}$$

Assume $\mathcal{C}$ and $\mathcal{D}$ are finite-complete and u is left exact. Then there exists a commutative diagram of morphisms of sites

$$\begin{tikzcd} \mathcal{C}/X \arrow[r, "j_X"] \arrow[d, "f'"'] & \mathcal{C} \arrow[d, "f"] \\ \mathcal{D}/Y \arrow[r, "j_Y"] & \mathcal{D} \end{tikzcd}$$

where the horizontal morphisms are defined as in Lemma 2.c and the vertical morphisms are defined by u and u'. Moreover, this commutative diagram induces the same commutative diagram of topoi as in Lemma 3. In particular, $f'_* \circ j_X^* = j_Y^* \circ f_*$.

Proof: By Lemma 8.2.h, u defines a morphism of sites $f\colon \mathcal{C} \to \mathcal{D}$. Then by Lemma 3, u' defines a morphism of sites $f'\colon \mathcal{C}/X \to \mathcal{D}/Y$. It remains to show $f \circ j_X = j_Y \circ f'$ as morphisms of sites, which follows from Lemma 2.c and that

$$u(V) \times X = u(V) \times u(Y) = u(V \times Y). \qquad \square$$

Combining Lemma 2 and Lemma 3, we have

3.b Lemma *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a morphism of sites defined by a continuous functor $u\colon \mathcal{D} \to \mathcal{C}$. Let $Y \in \operatorname{ob}\mathcal{D}$, $X \in \operatorname{ob}\mathcal{C}$ and $c\colon X \to u(Y)$ a morphism in $\mathcal{C}$. Then there exists a commutative diagram of topoi*

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}/X) \arrow[r, "j_X"] \arrow[d, "f_c"'] & \mathbf{Sh}(\mathcal{C}) \arrow[d, "f"] \\ \mathbf{Sh}(\mathcal{D}/Y) \arrow[r, "j_Y"] & \mathbf{Sh}(\mathcal{D}) \end{tikzcd}$$

where $f_c = f' \circ j_c$, $f'\colon \mathbf{Sh}(\mathcal{C}/u(Y)) \to \mathbf{Sh}(\mathcal{D}/Y)$ is defined in Lemma 3, $j_c\colon \mathbf{Sh}(\mathcal{C}/X) \to \mathbf{Sh}(\mathcal{C}/u(Y))$ is defined in Lemma 2. Through the identifications $\mathbf{Sh}(\mathcal{C}/X) = \mathbf{Sh}(\mathcal{C})/h_X^\#$ and $\mathbf{Sh}(\mathcal{D}/Y) = \mathbf{Sh}(\mathcal{D})/h_Y^\#$ of Theorem 1.c, the functor f_c^* has the following description

$$f_c^*(\mathscr{H} \xrightarrow{\varphi} h_Y^\#) = (f^*\mathscr{H} \times_{h^\#_{u(Y)}} h_X^\# \to h_X^\#).$$

Finally, for any morphisms $a\colon X \to X'$, $b\colon Y \to Y'$ and $c'\colon X' \to u(Y')$ making the diagram

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "a"'] & u(Y) \arrow[d, "u(b)"] \\ X \arrow[r, "c'"] & u(Y') \end{tikzcd}$$

the diagram

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}/X) \arrow[r, "j_a"] \arrow[d, "f_c"'] & \mathbf{Sh}(\mathcal{C}/X') \arrow[d, "f_{c'}"] \\ \mathbf{Sh}(\mathcal{D}/Y) \arrow[r, "j_b"] & \mathbf{Sh}(\mathcal{D}/Y') \end{tikzcd}$$

commutes.

Proof: The first commutative diagram follows from those in Lemma 2 and Lemma 3. The description follows from those in Lemma 2.a and Lemma 3. As for the last statement, we use the identifications and the descriptions. Let $\mathscr{H}$ be a sheaf on $\mathcal{D}$, then

$$\begin{aligned} j_a^* f_{c'}^*(\mathscr{H} \to h^{\#}_{Y'}) &= j_a^*(f^*\mathscr{H} \times_{h^{\#}_{u(Y')}} h^{\#}_{X'} \to h^{\#}_{X'}) \\ &= (f^*\mathscr{H} \times_{h^{\#}_{u(Y')}} h^{\#}_{X'} \times_{h^{\#}_{X'}} h^{\#}_{X} \to h^{\#}_{X}) \\ &= (f^*\mathscr{H} \times_{h^{\#}_{u(Y')}} h^{\#}_{X} \to h^{\#}_{X}) \\ &= (f^*\mathscr{H} \times_{h^{\#}_{u(Y')}} h^{\#}_{u(Y)} \times_{h^{\#}_{u(Y)}} h^{\#}_{X} \to h^{\#}_{X}) \\ &= (f^*(\mathscr{H} \times_{h^{\#}_{Y'}} h^{\#}_{Y}) \times_{h^{\#}_{u(Y)}} h^{\#}_{X} \to h^{\#}_{X}) \\ &= f_c^*(\mathscr{H} \times_{h^{\#}_{Y'}} h^{\#}_{Y} \to h^{\#}_{Y}) \\ &= f_c^* j_b^*(\mathscr{H} \to h^{\#}_{Y'}). \end{aligned}$$

This shows the required commutativity. $\square$

Unsurprised, there is a *cocontinuous* version of Lemma 3:

3.c Lemma (Localization of a cocontinuous functor) *Let $u\colon \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Let $X \in \operatorname{ob}\mathcal{C}$ and $Y = u(X)$. Then the functor*

$$\begin{aligned} u'\colon \mathcal{C}/X &\longrightarrow \mathcal{D}/Y \\ U/X &\longmapsto u(U)/Y \end{aligned}$$

satisfies the commutative digram

$$\begin{tikzcd} \mathcal{C}/X \arrow[r, "j_X"] \arrow[d, "u'"'] & \mathcal{C} \arrow[d, "u"] \\ \mathcal{D}/Y \arrow[r, "j_Y"] & \mathcal{D} \end{tikzcd}$$

Moreover, u' is cocontinuous and we have a commutative diagram

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}/X) \arrow[r, "j_X"] \arrow[d, "f'"'] & \mathbf{Sh}(\mathcal{C}) \arrow[d, "f"] \\ \mathbf{Sh}(\mathcal{D}/Y) \arrow[r, "j_Y"] & \mathbf{Sh}(\mathcal{D}) \end{tikzcd}$$

where f (resp. f') is induced by u (resp. u').

Proof: The commutativity of the first diagram is clear. Once we proved u' is cocontinuous, the second commutative diagram follows.

To show u' is cocontinuous, let $\mathfrak{V}/Y = \{V_j/Y \to u(U)/Y\}$ be a covering in $\mathcal{D}/Y$. Since u is cocontinuous, there is a covering $\mathfrak{U} = \{U_i \to U\}$ in $\mathcal{C}$ such that $u(\mathfrak{U})$ refines $\mathfrak{V} = \{V_j \to u(U)\}$. Then $\mathfrak{U}/X = \{U_i/X \to U/X\}$ is a covering in $\mathcal{C}/X$ refining $\mathfrak{V}/Y$. Hence u' is cocontinuous. $\square$

Remark In general, the equality $f'_* \circ j_X^* = j_Y^* \circ f_*$ does not hold in the case of Lemma 3.c.

4 (Sheaf Hom) Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ a sheaf on a site $\mathcal{C}$. Define the presheaf $\mathcal{H}om(\mathscr{F},\mathscr{G})$ by

$$\mathcal{H}om(\mathscr{F},\mathscr{G})(U) := \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C}/U)}(\mathscr{F}|_U, \mathscr{G}|_U).$$

By the following Lemma 4.a, this is indeed a sheaf, called the **sheaf Hom**.

4.a Lemma (Gluing morphisms) *Let $\mathcal{C}$ be a site and $\{U_i \to X\}_{i\in I}$ a covering. Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ a sheaf on $\mathcal{C}$. Suppose that there are morphisms*

$$\varphi_i\colon \mathscr{F}|_{U_i} \longrightarrow \mathscr{G}|_{U_i}$$

such that for all $i, j \in I$, the restrictions of φ_i and φ_j to $U_i \times_X U_j$ are the same morphism $\varphi_{ij}\colon \mathscr{F}|_{U_i\times_X U_j} \to \mathscr{G}|_{U_i\times_X U_j}$. Then there exists a unique morphism

$$\varphi\colon \mathscr{F} \longrightarrow \mathscr{G}$$

whose restriction to each U_i is φ_i.

4.b Lemma *Let $\mathscr{F}$ be a sheaf on a site $\mathcal{C}$, then $\mathcal{H}om(\underline{*}, \mathscr{F}) \cong \mathscr{F}$.*

Proof: Let $U \in \mathrm{ob}\,\mathcal{C}$. For any $\varphi \in \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C}/U)}(\underline{*}|_U, \mathscr{F}|_U)$, its corresponding section in $\mathscr{F}(U)$ is the image of the singleton under $\varphi(U)$. This gives rise to a morphism $\Phi\colon \mathcal{H}om(\underline{*}, \mathscr{F}) \to \mathscr{F}$. Conversely, any section $s \in \mathscr{F}(U)$ gives a map $* \to \mathscr{F}(U)$, and furthermore a map $* \to \mathscr{F}(U) \to \mathscr{F}(V)$ for any $V/U \in \mathcal{C}/U$. We can see this gives a morphism $\underline{*} \to \mathscr{F}|_U$. In this way we find the inverse of Φ, thus it's an isomorphism. $\square$

The following two lemmas are easy to verify.

4.c Lemma (Sheaf Hom is left exact) *Let $\mathscr{F}$ be a sheaf on a site $\mathcal{C}$, then $\mathcal{H}om(\mathscr{F}, -)$ is a left exact covariant functor and $\mathcal{H}om(-, \mathscr{F})$ is a left exact contravariant functor.*

4.d Lemma (Sheaf Hom is the internal Hom) *For any sheaves $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ on a site $\mathcal{C}$, there is a canonical bijection*

$$\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathscr{F} \times \mathscr{G}, \mathscr{H}) \cong \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathscr{F}, \mathcal{H}om(\mathscr{G}, \mathscr{H})).$$

5 (Gluing data) Let $\mathcal{C}$ be a site and $\mathfrak{U} = \{U_i \to U\}_{i \in I}$ a covering. A **gluing data for sheaves of sets with respect to the covering** $\mathfrak{U}$ consists of the following stuff:

- For each $i \in I$, a sheaf $\mathscr{F}_i$ of sets on $\mathcal{C}/U_i$;
- For each pair $i, j \in I$, an isomorphism $\varphi_{ij} \colon \mathscr{F}_i|_{U_i \times_U U_j} \to \mathscr{F}_j|_{U_i \times_U U_j}$,

satisfying the **cocycle condition**:

For any $i, j, k \in I$, the following diagram commutes.

$$\begin{tikzcd}
\mathscr{F}_i|_{U_i \times_U U_j \times_U U_k} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & \mathscr{F}_k|_{U_i \times_U U_j \times_U U_k} \\
& \mathscr{F}_j|_{U_i \times_U U_j \times_U U_k} \arrow[ur, "\varphi_{jk}"'] &
\end{tikzcd}$$

One can see this definition can be easily generalized to **gluing data for sheaves of algebraic structures**.

5.a Lemma *Let $\mathcal{C}$ be a site and $\mathfrak{U} = \{U_i \to U\}_{i \in I}$ a covering. Let $(\mathscr{F}_i, \varphi_{ij})$ be a gluing data for sheaves of sets with respect to the covering $\mathfrak{U}$. Then there exists a sheaf $\mathscr{F}$ on $\mathcal{C}/U$ together with isomorphisms*

$$\varphi_i \colon \mathscr{F}|_{U_i} \longrightarrow \mathscr{F}_i$$

such that the following diagram commutes.

$$\begin{tikzcd}
\mathscr{F}|_{U_i \times_U U_j} \arrow[r, "\varphi_i"] \arrow[d, equal] & \mathscr{F}_i|_{U_i \times_U U_j} \arrow[d, "\varphi_{ij}"] \\
\mathscr{F}|_{U_i \times_U U_j} \arrow[r, "\varphi_j"] & \mathscr{F}_j|_{U_i \times_U U_j}
\end{tikzcd}$$

The similar statement holds for sheaves of algebraic structures.

Proof: For any object $V/U \in \mathrm{ob}\,\mathcal{C}/U$, the object $\mathscr{F}(V/U)$ is given as the equalizer of the morphisms:

$$\textstyle\prod_{i \in I} \mathscr{F}_i(V \times_U U_i/U_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}_i(V \times_U U_i \times_U U_j/U_i).$$

For sheaves of sets, this set can be written as

$$\mathscr{F}(V/U) = \left\{ (s_i) \in \prod_{i\in I} \mathscr{F}_i(V \times_U U_i/U_i) \middle| \varphi_{ij}(s_i|_{V\times_U U_i \times_U U_j}) = s_j|_{V\times_U U_i\times_U U_j} \right\}.$$

As for the isomorphism, just note that a section in $\mathscr{F}|_{U_i}(V/U_i)$ is nothing but a system of compatible sections $(s_j) \in \prod_{j\in I} \mathscr{F}_i(V \times_U U_i \times_U U_j/U_i)$, which gives rise to a section $s \in \mathscr{F}_i(V/U_i)$. Thus the lemma follows. □

Let $\mathcal{C}$ be a site and $\mathfrak{U} = \{U_i \to U\}_{i\in I}$ a covering. Any sheaf $\mathscr{F}$ on $\mathcal{C}/U$ admits a canonical gluing data $(\mathscr{F}_i, \varphi_{ij})$, where $\mathscr{F}_i$ is the restriction $\mathscr{F}|_{U_i}$ and φ_{ij} is the induced morphism

$$\mathscr{F}|_{U_i}\big|_{U_i\times_U U_j} \longrightarrow \mathscr{F}|_{U_j}\big|_{U_i\times_U U_j}.$$

Moreover, this construction is functoral, meaning it gives rise to a functor from $\mathbf{Sh}(\mathcal{C}/U)$ to the category of gluing data.

5.b Theorem (Sheaf = gluing data) *The above functor induces an equivalence of category between* $\mathbf{Sh}(\mathcal{C}/U)$ *and the category of gluing data. The similar statement holds for sheaves of algebraic structures.*

Proof: The functor is fully faithful by Lemma 4.a and essentially surjective by Lemma 5.a. □

§ 10 Geometric morphisms

In this section we show that any geometric morphism is equivalent to one comes from a morphism of sites.

1 (Special cocontinuous functors) Let $\mathcal{C}, \mathcal{D}$ be two sites. A **special cocontinuous functor** $u\colon \mathcal{C} \to \mathcal{D}$ is a functor which is both continuous and cocontinuous and satisfies the followings.

SC1. For any $a, b\colon U \rightrightarrows U'$ in $\mathcal{C}$ such that $u(a) = u(b)$, there exists a covering $\{\phi_i\colon U_i \to U\}$ in $\mathcal{C}$ such that $a \circ \phi_i = b \circ \phi_i$.

SC2. For any $U, U' \in \operatorname{ob}\mathcal{C}$ and a morphism $c\colon u(U) \to u(U')$ in $\mathcal{D}$, there exists a covering $\{\phi_i\colon U_i \to U\}$ and morphisms $c_i\colon U_i \to U'$ in $\mathcal{C}$ such that $u(c_i) = c \circ u(\phi_i)$.

SC3. Any $V \in \operatorname{ob}\mathcal{D}$ admits a covering of the form $\{u(U_i) \to V\}$.

1.a Lemma (Special cocontinuous functors induce equivalences of topoi) *Let* $\mathcal{C}, \mathcal{D}$ *be two sites and* $u\colon \mathcal{C} \to \mathcal{D}$ *be a special cocontinuous functors. Let* $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ *be the geometric morphism induced by* u*. Then* f *is an equivalence of topoi.*

1.b Lemma *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a special cocontinuous functor. Then we have a commutative diagram*

$$\begin{tikzcd} \mathcal{C}/X \arrow[r, "j_X"] \arrow[d] & \mathcal{C} \arrow[d, "u"] \\ \mathcal{D}/u(X) \arrow[r, "j_{u(X)}"] & \mathcal{D} \end{tikzcd}$$

as in Lemma 9.3.c. The vertical arrows are special cocontinuous functors. Hence in the commutative diagram of topoi,

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}/X) \arrow[r, "j_X"] \arrow[d] & \mathbf{Sh}(\mathcal{C}) \arrow[d, "f"] \\ \mathbf{Sh}(\mathcal{D}/u(X)) \arrow[r, "j_{u(X)}"] & \mathbf{Sh}(\mathcal{D}) \end{tikzcd}$$

vertical arrows are equivalences of topoi.

1.c Lemma *Let $\mathcal{C}$ be a small site. Let $\mathcal{C}' \subset \mathbf{Sh}(\mathcal{C})$ be a small full subcategory such that*

- $h_U^{\#} \in \operatorname{ob}\mathcal{C}'$ *for all* $U \in \operatorname{ob}\mathcal{C}$,
- $\mathcal{C}'$ *is preserved under pullbacks.*

Define a covering of $\mathcal{C}'$ as a family $\{\mathscr{F}_i \to \mathscr{F}\}$ such that $\coprod \mathscr{F}_i \to \mathscr{F}$ is an epimorphism. Then

1. *$\mathcal{C}'$ is a site;*
2. *$\mathcal{C}'$ has subcanonical topology;*
3. *$u\colon U \mapsto h_U^{\#}$ is a special cocontinuous functor, hence induces an equivalence of topoi $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C}')$;*
4. *for any $\mathscr{F} \in \operatorname{ob}\mathcal{C}'$, we have $f^* h_{\mathscr{F}} = \mathscr{F}$;*
5. *for any $U \in \operatorname{ob}\mathcal{C}$, we have $f_* h_U^{\#} = h_{u(U)} = h_{h_U^{\#}}$.*

1.d Lemma *Let $\mathcal{C}$ be a small site. Let $\{\mathscr{F}_i\}$ be a set of sheaves on $\mathcal{C}$. Then there exists an equivalence of topoi $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C}')$ induced by a special cocontinuous functor $u\colon \mathcal{C} \to \mathcal{C}'$ of sites such that*

1. *$\mathcal{C}'$ has subcanonical topology;*
2. *a family $\{V_i \to V\}$ of morphisms of $\mathcal{C}'$ is (combinatorially equivalent to) a covering of $\mathcal{C}'$ if and only if $\coprod h_{V_i} \to h_V$ is epic;*

3. *$\mathcal{C}'$ is finite-complete;*

4. *every subsheaf of a representable sheaf on $\mathcal{C}'$ is representable;*

5. *each $f_*\mathscr{F}_i$ is a representable sheaf.*

2 Theorem (Geometric morphisms are induced by morphisms of sites) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be a geometric morphism. Then there exists a site $\mathcal{C}'$ and a diagram of functors*

$$\mathcal{C} \xrightarrow{u} \mathcal{C}' \xleftarrow{v} \mathcal{D}$$

such that

1. *u is a special cocontinuous functor;*

2. *v commutes with fibre products, is continuous and defines a morphism of sites $\mathcal{C}' \to \mathcal{D}$;*

3. *the geometric morphism f agree with the composition*

$$\mathbf{Sh}(\mathcal{C}) \longrightarrow \mathbf{Sh}(\mathcal{C}') \longrightarrow \mathbf{Sh}(\mathcal{D}).$$

which is induced by u and v.

2.a Remark Notation as in Theorem 2. Assume $\mathcal{D}$ is finite-complete. Then, by the construction of v, it is left exact. Then, by Lemma 8.2.h, v defines a morphism of sites. Apply Lemma 1.d to $\mathcal{D}$ to get $\mathcal{D}'$, then v' induces a morphism of site $v'\colon \mathcal{C}' \to \mathcal{D}'$. let f' be the geometric morphisms induced by v'. Next, apply Theorem 2 to the geometric morphism $\mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D}')$. We finally have the following statement:

2.b Corollary *Notation and assumptions as above. Then the geometric morphism satisfies a commutative diagram of topoi.*

$$\begin{tikzcd}
\mathbf{Sh}(\mathcal{C}) \arrow[r, "f"] \arrow[d] & \mathbf{Sh}(\mathcal{D}) \arrow[d] \\
\mathbf{Sh}(\mathcal{C}') \arrow[r, "f'"] & \mathbf{Sh}(\mathcal{D}')
\end{tikzcd}$$

Moreover, we have

1. *the vertical arrows are equivalence of topoi induced by the special cocontinuous functors $\mathcal{C} \to \mathcal{C}'$ and $\mathcal{D} \to \mathcal{D}'$;*

2. *the sites $\mathcal{C}'$ and $\mathcal{D}'$ are finite-complete and have subcanonical topologies;*

3. *given any set of sheaves $\{\mathscr{F}_i\}$ (resp. $\{\mathscr{G}_i\}$) on $\mathcal{C}$ (resp. $\mathcal{D}$), we may assume each of them is a representable sheaf on $\mathcal{C}'$ (resp. $\mathcal{D}'$).*

§ 11 Points

§ 12 Flatness

For the first, we follows [Kar04].

1 (Flatness respect to a site) Let $\mathcal{C}$ be a site and $\mathcal{D}$ a category. A functor $u\colon \mathcal{D} \to \mathcal{C}$ is said to be **flat respect to** $\mathcal{C}$ if for any finite diagram $\mathscr{D}\colon \mathcal{I} \to \mathcal{D}$ and any cone $\alpha\colon \Delta_U \Rightarrow u \circ \mathscr{D}$ over $u \circ \mathscr{D}$ in $\mathcal{C}$, the family

$$S_\alpha := \{\varphi\colon V \to U | \exists \beta\colon \Delta_W \Rightarrow \mathscr{D} \text{ s.t. } \alpha \circ \varphi \text{ factors through } u * \beta\}$$

is a covering in $\mathcal{C}$.

Remark This family is in fact a **sieve**, which means a family of morphisms with fixed target S which is *closed under precomposition*: if $\varphi\colon V \to U \in S$ and $\psi\colon W \to V$ is a morphism, then $\varphi \circ \psi \in S$. Thus any family of morphisms with fixed target $\mathfrak{U}$ can be extended into a sieve $S_{\mathfrak{U}}$ by added up with precompositions. The notion of sieves will be used in the definition of *Grothendieck topology*.

To see why this property is called flatness, we introduce the notion of tensor products of **Set**-valued functors. Who doesn't care this can skip the following few pages.

1.a Lemma (Tensor products) *Let $\mathcal{C}$ be a small category. Let $\mathscr{F}\colon \mathcal{C}^{\mathrm{opp}} \to$ **Set** and $\mathscr{G}\colon \mathcal{C} \to$ **Set** be a pair of a presheaf and a functor. Let Υ' denote the presheaf*

$$\begin{aligned} \Upsilon'\colon \mathcal{C}^{\mathrm{opp}} &\longrightarrow [\mathcal{C}, \mathbf{Set}] \\ U &\longmapsto \mathrm{Hom}_{\mathcal{C}}(U, -). \end{aligned}$$

Then, there is a natural bijection

$$(\mathrm{Lan}_{\Upsilon'} \mathscr{F})(\mathscr{G}) \cong (\mathrm{Lan}_{\Upsilon} \mathscr{G})(\mathscr{F}).$$

Proof: Since (refer Theorem 1.2.e and Example 1.7.i)

$$\begin{aligned}
(\mathrm{Lan}_{\Upsilon'} \mathscr{F})(\mathscr{G}) &\cong \varinjlim_{(\Upsilon'(U)\to\mathscr{G})\in\mathrm{ob}(\Upsilon'\downarrow\mathrm{const}_{\mathscr{G}})} \mathscr{F}(U), \\
(\mathrm{Lan}_{\Upsilon} \mathscr{G})(\mathscr{F}) &\cong \varinjlim_{(\Upsilon(U)\to\mathscr{F})\in\mathrm{ob}(\Upsilon\downarrow\mathrm{const}_{\mathscr{F}})} \mathscr{G}(U), \\
\mathscr{G} &\cong \varinjlim_{(\Upsilon'(U)\to\mathscr{G})\in\mathrm{ob}(\Upsilon'\downarrow\mathrm{const}_{\mathscr{G}})} \Upsilon'(U), \\
\mathscr{F} &\cong \varinjlim_{(\Upsilon(U)\to\mathscr{F})\in\mathrm{ob}(\Upsilon\downarrow\mathrm{const}_{\mathscr{F}})} \Upsilon(U),
\end{aligned}$$

we have

$$
\begin{aligned}
(\operatorname{Lan}_{\Upsilon'} \mathscr{F})(\mathscr{G}) &\cong \varinjlim_{(\Upsilon'(U)\to\mathscr{G})\in\operatorname{ob}(\Upsilon'\downarrow\operatorname{const}_{\mathscr{G}})} \mathscr{F}(U) \\
&\cong \varinjlim_{(\Upsilon'(U)\to\mathscr{G})\in\operatorname{ob}(\Upsilon'\downarrow\operatorname{const}_{\mathscr{G}})} \varinjlim_{(\Upsilon(V)\to\mathscr{F})\in\operatorname{ob}(\Upsilon\downarrow\operatorname{const}_{\mathscr{F}})} \operatorname{Hom}_{\mathcal{C}}(U,V) \\
&\cong \varinjlim_{(\Upsilon(V)\to\mathscr{F})\in\operatorname{ob}(\Upsilon\downarrow\operatorname{const}_{\mathscr{F}})} \varinjlim_{(\Upsilon'(U)\to\mathscr{G})\in\operatorname{ob}(\Upsilon'\downarrow\operatorname{const}_{\mathscr{G}})} \operatorname{Hom}_{\mathcal{C}}(U,V) \\
&\cong \varinjlim_{(\Upsilon(U)\to\mathscr{F})\in\operatorname{ob}(\Upsilon\downarrow\operatorname{const}_{\mathscr{F}})} \mathscr{G}(U) \\
&\cong (\operatorname{Lan}_{\Upsilon} \mathscr{G})(\mathscr{F}). \qquad \square
\end{aligned}
$$

1.b (Tensor products) The construction in the lemma is called the **tensor product** of $\mathscr{F}$ and $\mathscr{G}$, and is denoted by $\mathscr{F} \otimes_{\mathcal{C}} \mathscr{G}$ or $\mathscr{F} \otimes \mathscr{G}$.

Recall Example 1.7.i, we see that $- \otimes_{\mathcal{C}} \mathscr{F}$ is nothing but the *Yoneda extension* $\widetilde{\mathscr{F}}$. This functor has a right adjoint, which is $\Upsilon_{\mathbf{Set}}(-)\circ\mathscr{F}$. Indeed, we have a bijection

$$\operatorname{Hom}_{\mathbf{Set}}(\mathscr{G} \otimes_{\mathcal{C}} \mathscr{F}, S) \cong \operatorname{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathscr{G}, \operatorname{Hom}_{\mathbf{Set}}(\mathscr{F}(-), S))$$

natural in $\mathscr{G} \in \operatorname{ob}\mathbf{PSh}(\mathcal{C})$ and $S \in \operatorname{ob}\mathbf{Set}$.

1.c ¶Example Let $\mathcal{C}$ be a small category and $\mathscr{F}\colon \mathcal{C} \to \mathbf{Set}$ a functor. View **Set** as a site via its canonical topology, then the followings are equivalent.

1. $\mathscr{F}$ is flat respect to **Set**;
2. the opposite of its **category of elements** $\mathbf{El}(\mathscr{F}) := (* \downarrow \mathscr{F})$, is filtered;
3. its *Yoneda extension* $\widetilde{\mathscr{F}}\colon \mathbf{PSh}(\mathcal{C}) \to \mathbf{Set}$ is left exact.

Proof: *1.⇒2.*: For any finite diagram $\mathscr{D}\colon \mathcal{I} \to \mathcal{C}$, consider the limit cone $\lambda\colon \Delta_{\varprojlim \mathscr{F}\circ\mathscr{D}} \Rightarrow \mathscr{F}\circ\mathscr{D}$ of $\mathscr{F}\circ\mathscr{D}$, then the sieve S_λ is generated by the family

$$\mathfrak{U}_\lambda := \left\{ \varphi\colon \mathscr{F}(V) \to \varprojlim \mathscr{F}\circ\mathscr{D} \,\middle|\, \exists\beta\colon \Delta_V \Rightarrow \mathscr{D} \text{ s.t. } \mathscr{F} * \beta = \lambda\circ\varphi \right\}.$$

Then *1.* implies that such kind of families are surjective. Next, we show that the opposite of $\mathbf{El}(\mathscr{F})$ is filtered in this case. Recall that the opposite of a category is filtered if and only if any finite diagram in that category has a cone.

Let $\mathscr{D}'\colon \mathcal{I} \to \mathbf{El}(\mathscr{F})$ be a finite diagram. Then, itself gives a cone $\alpha\colon \Delta_* \Rightarrow \mathscr{F}\circ\mathscr{D}$ over the image of a diagram $\mathscr{D}\colon \mathcal{I} \to \mathcal{C}$ under $\mathscr{F}$. This cone uniquely factors through the limit cone $\lambda\colon \Delta_{\varprojlim \mathscr{F}\circ\mathscr{D}} \Rightarrow \mathscr{F}\circ\mathscr{D}$ and thus gives an element $a \in \varprojlim \mathscr{F} \circ \mathscr{D}$. Therefore, by the surjectivity of the family above, there exists a map $\varphi\colon \mathscr{F}(V) \to \varprojlim \mathscr{F} \circ \mathscr{D}$ and a cone $\beta\colon \Delta_V \Rightarrow \mathscr{D}$ such that $\mathscr{F} * \beta = \lambda \circ \varphi$ and that $a \in \operatorname{im} \varphi$. Now taking any preimage s of a in $\mathscr{F}(V)$, viewing it as a map from $*$ to $\mathscr{F}(V)$, we have $\varphi \circ s = a$. Therefore, we have a commutative diagram

$$\begin{tikzcd} \Delta_* \arrow[r, "s"] \arrow[dr, "\alpha"'] & \Delta_{\mathscr{F}(V)} \arrow[d, "\mathscr{F}*\beta"] \\ & \mathscr{F} \circ \mathscr{D} \end{tikzcd}$$

which induces a cone $\Delta_{s\colon * \to \mathscr{F}(V)} \Rightarrow \mathscr{D}'$ over $\mathscr{D}'$.

2.⇒3.: Let $\mathscr{G}$ be any presheaf on $\mathcal{C}$, then

$$\begin{aligned} \widetilde{\mathscr{F}}(\mathscr{G}) &\cong \varinjlim_{(\Upsilon'(U) \to \mathscr{F}) \in \operatorname{ob}(\Upsilon' \downarrow \operatorname{const}_{\mathscr{F}})} \mathscr{G}(U) \\ &\cong \left(\varinjlim_{(\Upsilon'(U) \to \mathscr{F}) \in \operatorname{ob}(\Upsilon' \downarrow \operatorname{const}_{\mathscr{F}})} \Gamma(U, -) \right)(\mathscr{G}). \end{aligned}$$

Since $(\Upsilon' \downarrow \operatorname{const}_{\mathscr{F}})$ is the opposite of $\mathbf{El}(\mathscr{F})$, it is filtered. Note that the functor $\Gamma(U, -)$ is exact. Now $\mathscr{F}$ is a filtered colimit of exact functors, thus is left exact.

3.⇒1.: For any finite diagram $\mathscr{D}\colon \mathcal{I} \to \mathcal{C}$, let $\lambda\colon \Delta_{\varprojlim \mathscr{F} \circ \mathscr{D}} \Rightarrow \mathscr{F} \circ \mathscr{D}$ be the limit cone of $\mathscr{F} \circ \mathscr{D}$. Then it suffices to show the family $\mathfrak{U}_\lambda$ is surjective. Indeed, if so, then the sieve S_λ is surjective, thus a covering. Now, for an arbitrary cone $\alpha\colon \Delta_U \Rightarrow \mathscr{F} \circ \mathscr{D}$, pullback S_λ along the canonical map $U \to \varprojlim \mathscr{F} \circ \mathscr{D}$, we obtain a covering of U which is contained in the sieve S_α, thus S_α is also surjective.

Now, note that $\varprojlim \mathscr{F} \circ \mathscr{D} = \varprojlim \widetilde{\mathscr{F}} \circ \Upsilon \circ \mathscr{D} = \widetilde{\mathscr{F}}(\varprojlim \Upsilon \circ \mathscr{D})$ and that a section $s \in (\varprojlim \Upsilon \circ \mathscr{D})(U)$ is a cone $s\colon \Delta_U \Rightarrow \mathscr{D}$. Denote $\varprojlim \Upsilon \circ \mathscr{D}$ by $\mathscr{L}$. For any object $t\colon h_V \to \mathscr{L}$ in $(\Upsilon \downarrow \operatorname{const}_{\mathscr{L}})$, denote the image of id_V by β, which is a cone $\beta\colon \Delta_V \Rightarrow \mathscr{D}$. Now $\Upsilon * \beta\colon \Delta_{h_V} \Rightarrow \Upsilon \circ \mathscr{D}$ is nothing but the composition of t with λ. Therefore $\mathscr{F} * \beta = \lambda \circ \widetilde{\mathscr{F}}(t)$. Therefore

$$\left\{ \widetilde{\mathscr{F}}(t)\colon \mathscr{F}(V) \to \varprojlim \mathscr{F} \circ \mathscr{D} \,\middle|\, t\colon h_V \to \varprojlim \Upsilon \circ \mathscr{D} \right\} \subset \mathfrak{U}_\lambda$$

but the left family is already surjective, *a fortiori* $\mathfrak{U}_\lambda$. □

Remark Recall that $- \otimes_{\mathcal{C}} \mathscr{F}$ has a right adjoint and thus is already right exact. Therefore $\mathscr{F}$ is flat respect to **Set** if and only if the tensor product functor $- \otimes_{\mathcal{C}} \mathscr{F}$ is exact.

1.d Corollary *If $\mathcal{C}$ is finite complete, then $\mathscr{F}$ is flat respect to* **Set** *if and only if it is left exact.*

Proof: *only if*: Note that $\mathscr{F} = \widetilde{\mathscr{F}} \circ \Upsilon$ and that Υ commutes with limits.

if: similarly to Lemma 1.7.c, one can show that $\mathbf{El}(\mathscr{F})^{\mathrm{opp}}$ is filtered if $\mathscr{F}$ is left exact. $\square$

II

Ringed spaces and $\mathscr{O}$-modules

Basically this chapter is talking about the commutative algebra theory in a topos.

§ 1 ¶Algebraic structures

Recall that a *sheaf* with value in an algebraic category $\mathcal{A}$ is a presheaf satisfying suitable descent condition (ref. I.2.5 and I.2.6). Of course one can give another description (Theorem 1.c) of this notion by introducing the notion of *algebraic structures*. Here we do a bit.

1 (Lawvere theory) We begin with declare some notions about algebraic objects in an elementary way. Let $\mathcal{C}$ be a category. Let $*$ denote the (possibly existed) terminal object. For $X \in \operatorname{ob}\mathcal{C}$, let X^n denote the n-fold product of X. In particular, $X^0 = *$.

- A **n-ary operation** on X is a morphism from X^n to X. Conversely, a **n-ary cooperation** on X is a morphism from X to X^n. More generally, a **(m, n)-operation** on X is a morphism from X^n to X^m.

In this way, the notion of operations on a set has been generalized into an arbitrary category. The relations of operations on sets are described by equations. Now, they become communities of diagrams.

1.a Example (group object) A **group object** in a category $\mathcal{C}$ is an object G with a 0-ary operation e called *unit*, a 1-ary operation $\imath$ called *inverse* and a 2-ary operation m called *multiplication* satisfying the following commutative diagrams.

$$\begin{tikzcd} G^3 \arrow[r, "\mathrm{id}\times m"] \arrow[d, "m\times\mathrm{id}"'] & G^2 \arrow[d, "m"] \\ G^2 \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(e{,}\mathrm{id})"] \arrow[d, "(\mathrm{id}{,}e)"'] \arrow[dr, equal] & G^2 \arrow[d, "m"] \\ G^2 \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(\imath{,}\mathrm{id})"] \arrow[d, "(\mathrm{id}{,}\imath)"'] \arrow[dr, "e"] & G^2 \arrow[d, "m"] \\ G^2 \arrow[r, "m"] & G \end{tikzcd}$$

where $e\colon G \to G$ is the composition $G \to * \xrightarrow{e} G$.

The followings are common examples:

- A group object in **Set** is a *group*.
- A group object in **Top** is a *topological group*.
- A group object in **Diff** is a *Lie group*.
- A group object in **Grp** is an *abelian group*.
- A group object in **Ab** is an *abelian group* again.
- A group object in **Cat** is a *strict* 2-*group*.
- A group object in **Grpd** is a *strict* 2-*group* again.
- A group object in $\mathbf{CRing}^{\mathrm{opp}}$ is a *commutative Hopf algebra*.

- A group object in a functor category is a *group functor*.
- A group object in the category of schemes is a *group scheme*.

The data of operations and relations of them can be encoded into a category $\mathbb{T}$ which has finite products and all its objects are finite products of a distinguished object x called the **generic object**. Such a category is called a **Lawvere theory**. A *morphism* of Lawvere theories is a functor preserving finite products.

Use this notion, we define

- An **algebraic object of type** $\mathbb{T}$ in $\mathcal{C}$ is an object X with a functor, also denoted by X, from $\mathbb{T}$ to $\mathcal{C}$ which maps x to X and preserves finite products. A **homomorphism** between algebraic objects X and Y is a morphism $f\colon X \to Y$ inducing a natural transformation $X \Rightarrow Y$.

1.b Example ($\mathbb{T}$-algebras) Specially, we call an algebraic object of type $\mathbb{T}$ in **Set** a $\mathbb{T}$-algebra. Let $\mathbb{T}\,\mathbf{Alg}$ denote the category of $\mathbb{T}$-algebras.

1.c Theorem (Presheaves of $\mathbb{T}$-algebras) *The algebraic objects of type $\mathbb{T}$ in $\mathbf{PSh}(\mathcal{C})$ for some category $\mathcal{C}$ having finite products are equivalent to the presheaves of $\mathbb{T}$-algebras. In particular, algebraic objects of type $\mathbb{T}$ in $\mathcal{C}$ are equivalent to the representable presheaves of $\mathbb{T}$-algebras.*

Proof: Given an algebraic object $(\mathscr{X}, X)$ of type $\mathbb{T}$ in $\mathbf{PSh}(\mathcal{C})$, since the section functor $\Gamma(U, -)$ is exact, we obtain $\mathbb{T}$-algebras $(\mathscr{X}(U), \Gamma(U,-)\circ X)$. For any morphism $V \to U$ in $\mathcal{C}$, the transition map $\mathscr{X}(U) \to \mathscr{X}(V)$ is a homomorphism by the functorality of $\mathscr{X}^n$. This shows $\mathscr{X}$ is a presheaf of $\mathbb{T}$-algebras. Conversely, let $\mathscr{X}$ be a presheaf of $\mathbb{T}$-algebras, then $X\colon x^n \mapsto \mathscr{X}^n$ defines an algebraic object $(\mathscr{X}, X)$ of type $\mathbb{T}$ in $\mathbf{PSh}(\mathcal{C})$. □

1.d Proposition (Limits and colimits in $\mathbb{T}\,\mathbf{Alg}$) $\mathbb{T}\,\mathbf{Alg}$ *has limits and filtered colimits, and they are computed pointwise.*

Proof: Since $\mathbb{T}\,\mathbf{Alg}$ can be viewed as a full sub category of $[\mathbb{T}, \mathbf{Set}]$, it suffices to show the limits and filtered colimits of finite-products-preserving functors also preserve finite products. This comes from that limits and filtered colimits commute with finite limits. □

1.e Proposition $\mathbb{T}\,\mathbf{Alg}$ *is an algebraic category.*

Proof: For any $\mathbb{T}$-algebra X, the set X is called the *underlying set* of X. Taking underlying set forms a functor $F\colon \mathbb{T}\,\mathbf{Alg} \to \mathbf{Set}$. This functor is faithful and reflects isomorphisms. By Proposition 1.d, $\mathbb{T}\,\mathbf{Alg}$ has limits and filtered colimits, and they are computed pointwise. Thus F commutes with them. □

1.f Proposition *The forgetful functor* $F\colon \mathbb{T}\,\mathbf{Alg} \to \mathbf{Set}$ *has a left adjoint.*

Proof: Let $[n]$ denote a finite set with n elements. For any $\mathbb{T}$-algebra X, we have

$$\mathrm{Hom}_{\mathbf{Set}}([n], F(X)) = X^n = X(x^n).$$

Let $\widetilde{n}$ denote the functor $\mathrm{Hom}_{\mathbb{T}}(x^n, -)$. One can see it defines a $\mathbb{T}$-algebra. By the Yoneda lemma, we have

$$\mathrm{Hom}_{\mathbb{T}\,\mathbf{Alg}}(\widetilde{n}, X) = X(x^n).$$

Therefore we have

$$\mathrm{Hom}_{\mathbb{T}\,\mathbf{Alg}}(\widetilde{n}, X) = \mathrm{Hom}_{\mathbf{Set}}([n], F(X)).$$

For any set S, let $\mathbf{Sub}(S)$ be the category of finite subsets of S and their inclusions. Then we have

$$S = \varinjlim_{U \in \mathbf{Sub}(S)} U,$$

and thus

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Set}}(S, F(X)) &= \mathrm{Hom}_{\mathbf{Set}}(\varinjlim_{U \in \mathbf{Sub}(S)} U, F(X)) \\
&= \varprojlim_{U \in \mathbf{Sub}(S)} \mathrm{Hom}_{\mathbf{Set}}(U, F(X)) \\
&= \varprojlim_{U \in \mathbf{Sub}(S)} \mathrm{Hom}_{\mathbb{T}\,\mathbf{Alg}}(\widetilde{\mathrm{Card}(U)}, X) \\
&= \mathrm{Hom}_{\mathbb{T}\,\mathbf{Alg}}(\varinjlim_{U \in \mathbf{Sub}(S)} \widetilde{\mathrm{Card}(U)}, X).
\end{aligned}$$

In this way we get a functor $T\colon S \mapsto \varinjlim_{U \in \mathbf{Sub}(S)} \widetilde{\mathrm{Card}(U)}$ which is left adjoint to F. This $T(S)$ is called the **free $\mathbb{T}$-algebra generated by** S. One can further write down the underlying set of $T(S)$, which is

$$\{f(s_1, \cdots, s_n) | n \in \mathbb{N}, f \in \mathrm{Hom}_{\mathbb{T}}(x^n, x), s_1, \cdots, s_n \in S\}. \qquad \square$$

2 ¶Remark (PROPs) Any category with finite products can be considered as a *cartesian monoidal category*. So we may generalize the above into *monoidal categories*. Let $\mathcal{C}$ be a monoidal category with tensor product $\otimes$ and unit I. For any $X \in \mathrm{ob}\,\mathcal{C}$, let $X^{\otimes n}$ denote the n-fold tensor product of X. In particular, $X^{\otimes 0} = I$.

- A n-**ary operation** on X is a morphism from $X^{\otimes n}$ to X. Conversely, a n-**ary cooperation** on X is a morphism from X to $X^{\otimes n}$. More generally, a (m, n)-**operation** on X is a morphism from $X^{\otimes n}$ to $X^{\otimes m}$.

2.a Example (Monoids) A **monoid** in a monoidal category $\mathcal{C}$ is an object M with a 0-ary operation e called *unit* and a 2-ary operation m called *multiplication* satisfying the following commutative diagrams.

$$\begin{tikzcd} M^{\otimes 3} \arrow[r, "\mathrm{id}\otimes m"] \arrow[d, "m\otimes\mathrm{id}"'] & M^{\otimes 2} \arrow[d, "m"] \\ M^{\otimes 2} \arrow[r, "m"] & M \end{tikzcd} \qquad \begin{tikzcd} M^{\otimes 2} \arrow[dr, "m"] & I \otimes M \arrow[l, "e\otimes\mathrm{id}"'] \arrow[d, "\cong"] \\ M \otimes I \arrow[u, "\mathrm{id}\otimes e"] \arrow[r, "\cong"] & M \end{tikzcd}$$

The followings are common examples:

- A monoid in **Set** is a *monoid*.
- A monoid in **Top** is a *topological monoid*.
- A monoid in **Mon** is an *abelian monoid*.
- A monoid in **Ab** (with the tensor products of abelian groups) is a *ring*.
- A monoid in $A\,$**Mod** (with the tensor products of A-modules) for some commutative algebra A is an A-*algebra*.
- A monoid in $\mathrm{End}(\mathcal{C})$ for some category $\mathcal{C}$ is a *monad* on $\mathcal{C}$.

A *homomorphism* of monoids is then a morphism commutes with unit and multiplication. The category of monoid is denoted by **Mon**$(\mathcal{C})$.

2.b Example (Commutative monoid) Let $\mathcal{C}$ be a symmetric monoidal category, then there is a natural transformation $\gamma_{X,Y} : X \otimes Y \to Y \otimes X$ called *braiding*. Then a monoid (M, m, e) in $\mathcal{C}$ such that m commutes with $\gamma_{M,M}$ is called a **commutative monoid**. The category of commutative monoids is denoted by **CMon**$(\mathcal{C})$.

The data of operations and relations of them can be encoded into a *monoidal category* $\mathbb{T}$ in which all objects are finite tensor products of a distinguished object x called the **generic object**. Such a monoidal category is called a **PROP** (abbreviating of "products and permutations category"). A *morphism* of PROP is a *monoidal functor*.

Use this notion, we define

- An **algebraic object of type** $\mathbb{T}$ in a monoidal category $\mathcal{C}$ is an object X with a monoidal functor, also denoted by X, from $\mathbb{T}$ to $\mathcal{C}$ which maps x to X and preserves finite products. A **homomorphism** between algebraic objects X and Y is a morphism $f : X \to Y$ inducing a *monoidal natural transformation* $X \Rightarrow Y$.

3 **(Monads)** We now turn to another way to encode an algebraic theory. Recall that a **monad** on a category $\mathcal{C}$ is a monoid in the category $\mathrm{End}(\mathcal{C})$ of endofunctors. More precisely, a **monad** on $\mathcal{C}$ is an endofunctor $\mathfrak{T} \colon \mathcal{C} \to \mathcal{C}$ with two natural transformations $\eta \colon \mathrm{id}_{\mathcal{C}} \Rightarrow \mathfrak{T}$ (called *unit*) and $\mu \colon \mathfrak{T} \circ \mathfrak{T} \Rightarrow \mathfrak{T}$ (called *multiplication*) satisfying the following commutative diagrams.

$$
\begin{tikzcd}
\mathfrak{T} \circ \mathfrak{T} \circ \mathfrak{T} \arrow[r, "\mathrm{id} * \mu"] \arrow[d, "\mu * \mathrm{id}"'] & \mathfrak{T} \circ \mathfrak{T} \arrow[d, "\mu"] \\
\mathfrak{T} \circ \mathfrak{T} \arrow[r, "\mu"] & \mathfrak{T}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{T} \circ \mathfrak{T} \arrow[dr, "\mu"] & \mathrm{id}_{\mathcal{C}} \circ \mathfrak{T} \arrow[l, "\eta * \mathrm{id}"'] \arrow[d, "\cong"] \\
\mathfrak{T} \circ \mathrm{id}_{\mathcal{C}} \arrow[u, "\mathrm{id} * \eta"] \arrow[r, "\cong"] & \mathfrak{T}
\end{tikzcd}
$$

3.a **Lemma** *Any adjunction induces a monad.*

Proof: Let $L \dashv R \colon \mathcal{D} \to \mathcal{C}$ be an adjunction. Let $\eta \colon \mathrm{id}_{\mathcal{C}} \Rightarrow R \circ L$ be the unit and $\epsilon \colon L \circ R \Rightarrow \mathrm{id}_{\mathcal{C}}$ the counit. Then the counit induces a natural transformation $\mu \colon R \circ L \circ R \circ L \Rightarrow R \circ L$. Let $\mathfrak{T} = R \circ L$, then the commutative diagram comes from the triangle identities for adjunction. $\square$

Recall that any Lawvere theory $\mathbb{T}$ admits a forgetful functor $F \colon \mathbb{T}\,\mathbf{Alg} \to \mathbf{Set}$ and its left adjoint T. Then the adjunction $T \dashv F$ provides a monad $\mathfrak{T}$ on $\mathbf{Set}$.

3.b **(Modules over a monad)** Let $\mathfrak{T}$ be a monad on a category $\mathcal{C}$, a **left $\mathfrak{T}$-module** is a functor $M \colon \mathcal{D} \to \mathcal{C}$ with a natural transformation $\lambda \colon \mathfrak{T} \circ M \Rightarrow M$ satisfying the following commutative diagrams.

$$
\begin{tikzcd}
\mathfrak{T} \circ \mathfrak{T} \circ M \arrow[r, "\mathfrak{T} * \lambda"] \arrow[d, "\mu * M"'] & \mathfrak{T} \circ M \arrow[d, "\lambda"] \\
\mathfrak{T} \circ M \arrow[r, "\lambda"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{id}_{\mathcal{C}} \circ M \arrow[r, "\eta * M"] \arrow[dr, "\cong"'] & \mathfrak{T} \circ M \arrow[d, "\lambda"] \\
 & M
\end{tikzcd}
$$

A **homomorphism** of left $\mathfrak{T}$-modules is a natural transformation preserving λ. Similarly, one can define the notion of **right $\mathfrak{T}$-modules**. In the case M is a constant functor, we call it a **$\mathfrak{T}$-algebra**. Let $\mathfrak{T}\,\mathbf{Alg}$ denote the category of $\mathfrak{T}$-algebras.

3.c **Proposition** *Let $\mathfrak{T}$ be a monad associate to a Lawvere theory $\mathbb{T}$. Then the $\mathfrak{T}$-algebras are equivalent to the $\mathbb{T}$-algebras.*

Proof: Given a $\mathbb{T}$-algebra X. Define $M = X(x)$ and $\lambda = F * \epsilon$. Then this (M, λ) is a $\mathfrak{T}$-algebra by the triangle identities. A homomorphism of $\mathbb{T}$-algebras induces a homomorphism of $\mathfrak{T}$-algebras by the functorality of F and ϵ. Therefore, we have a functor Φ from $\mathbb{T}\,\mathbf{Alg}$ to $\mathfrak{T}\,\mathbf{Alg}$. In particular, each $\widetilde{n}$ induces a $\mathfrak{T}$-algebra, also denoted by $\widetilde{n}$.

Conversely, any $\mathfrak{T}$-algebra (M, λ) admits a functor

$$\begin{aligned} X\colon \mathbb{T} &\longrightarrow \mathbf{Set} \\ x^n &\longmapsto \mathrm{Hom}_{\mathfrak{T}\,\mathbf{Alg}}(\widetilde{n}, M). \end{aligned}$$

We claim that

$$\mathrm{Hom}_{\mathfrak{T}\,\mathbf{Alg}}(\widetilde{n}, M) \cong \mathrm{Hom}_{\mathbf{Set}}([n], M).$$

Indeed, any $\varphi \in \mathrm{Hom}_{\mathfrak{T}\,\mathbf{Alg}}(\widetilde{n}, M)$ induces a map $\varphi\colon \widetilde{n} \to M$. Then we define $\alpha(\varphi)$ as the composition

$$[n] \xrightarrow{\eta} \mathfrak{T}([n]) = \widetilde{n} \xrightarrow{\varphi} M.$$

Conversely, for any $\psi \in \mathrm{Hom}_{\mathbf{Set}}([n], M)$, we define $\beta(\psi)$ as the composition

$$\widetilde{n} = \mathfrak{T}([n]) \xrightarrow{\mathfrak{T}(\psi)} \mathfrak{T}(M) \xrightarrow{\lambda} M.$$

Obviously, α and β are inverse to each other.

Note that $\mathrm{Hom}_{\mathbf{Set}}([n], M) = M^n$. Therefore $X(x^n) = M^n$ and thus X is a finite-product preserving functor. In this way, we get a functor

$$\begin{aligned} \Psi\colon \mathfrak{T}\,\mathbf{Alg} &\longrightarrow \mathbb{T}\,\mathbf{Alg} \\ (M, \lambda) &\longmapsto X. \end{aligned}$$

Obviously, Φ and Ψ are weakly inverse to each other and thus $\mathfrak{T}\,\mathbf{Alg}$ is equivalent to $\mathbb{T}\,\mathbf{Alg}$. □

4 Remark (Modules over a monoid) Recall that monads on a category $\mathcal{C}$ are monoids in $\mathrm{End}(\mathcal{C})$. The definition in 3.b can be easily generalized. Let A be a monoid in a monoidal category $\mathcal{C}$, a **left A-module**, or simply called **A-module**, is an object $M \in \mathrm{ob}\,\mathcal{C}$ with a morphism $l\colon A \otimes M \to M$, called *action*, satisfying the following commutative diagrams.

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "A\otimes l"] \arrow[d, "m\otimes M"'] & A \otimes M \arrow[d, "l"] \\ A \otimes M \arrow[r, "l"] & M \end{tikzcd} \qquad \begin{tikzcd} I \otimes M \arrow[r, "e\otimes M"] \arrow[dr, "\cong"'] & A \otimes M \arrow[d, "l"] \\ & M \end{tikzcd}$$

A **homomorphism** of A-modules is a morphism commuting with actions. The category of A-modules is denoted by $A\,\mathbf{Mod}$. Similarly, one can define the notion of **right A-modules**. Let A, B be two monoids, then an **(A, B)-bimodule** is an object equipped both a left A-module structure and a right B-module structure. An (A, A)-bimodule is called an **A-bimodule**.

4.a Remark This definition actually generalize nothing. Indeed, one can see that a monoid in $\mathcal{C}$ admits a monad $A \otimes -$ and that the modules over a monoid A are precisely the algebras over the monad $A \otimes -$.

4.b (Algebras over a commutative monoid) Let $\mathcal{C}$ be a symmetric monoidal category and A a commutative monoid in it, then any left A-module is also a right A-module and thus an A-bimodule. Then, just like in commutative algebras, we can define the **tensor product** of two A-modules M and N as follows.

First, the object $M \otimes_A N$ is the coequalizer of

$$M \otimes A \otimes N \underset{M\otimes l}{\overset{l\otimes N}{\rightrightarrows}} M \otimes N$$

in $\mathcal{C}$. Then action $A \otimes (M \otimes_A N) \to M \otimes_A N$ is unique morphism making the following commutative diagram whose uniqueness and existence are guaranteed by the universal property of coequalizer.

$$\begin{tikzcd}
A \otimes M \otimes A \otimes N \arrow[r, shift left, "l\otimes N"] \arrow[r, shift right, "M\otimes l"'] \arrow[d, "l\otimes A\otimes N"'] & A \otimes M \otimes N \arrow[r] \arrow[d, "l\otimes N"'] & A \otimes (M \otimes_A N) \arrow[d, dashed] \\
M \otimes A \otimes N \arrow[r, shift left, "l\otimes N"] \arrow[r, shift right, "M\otimes l"'] & M \otimes N \arrow[r] & M \otimes_A N
\end{tikzcd}$$

In this way, $A\,\mathbf{Mod}$ becomes a symmetric monoidal category with tensor product $\otimes_A$, unit A and braiding $M \otimes_A N \to N \otimes_A M$ induced by the braiding γ of $\mathcal{C}$. Now, a monoid in $A\,\mathbf{Mod}$ is called an **A-algebra**.

5 ¶ (Bimonoids and Hopf monoids) A **comonoid** in a monoidal category $\mathcal{C}$ is an object which is a monoid in $\mathcal{C}^{\mathrm{opp}}$. Note that the category $\mathbf{CoMon}(\mathcal{C})$ of comonoids is $(\mathbf{Mon}(\mathcal{C}^{\mathrm{opp}}))^{\mathrm{opp}}$, not $\mathbf{Mon}(\mathcal{C}^{\mathrm{opp}})$.

A **bimonoid** is then an object which is both a monoid and a comonoid in a compatible way. In the case $\mathcal{C}$ is symmetric monoidal, this compatibility is easy to describe: a **bimonoid** is an object B with a 0-ary operation e called *unit*, a 0-ary cooperation ϵ called *counit*, a 2-ary operation m called *multiplication* and a 2-ary cooperation δ called *comultiplication* such that

- (B, m, e) is a monoid;
- (B, δ, ϵ) is a comonoid;

and satisfying one of the following equivalent compatible conditions:

1. δ and ϵ are homomorphisms of monoids;
2. m and e are homomorphisms of comonoids.

A *homomorphism* of bimonoids $f\colon (B, m, e, \delta, \epsilon) \to (B', m', e', \delta', \epsilon')$ is then a morphism $f\colon B \to B'$ which is both a homomorphism of monoids $f\colon (B, m, e) \to (B', m', e')$ and a homomorphism of comonoids $f\colon (B, \delta, \epsilon) \to (B', \delta', \epsilon')$. The category of bimonoids is denoted by $\mathbf{BiMon}(\mathcal{C})$.

5.a Lemma (Monoids form a monoidal category) *Let $\mathcal{C}$ be a symmetric monoidal category, then the category* $\mathbf{Mon}(\mathcal{C})$ *of monoids in $\mathcal{C}$ is again a symmetric monoidal category, in which the tensor product of two monoids (M_1, m_1, e_1) and (M_2, m_2, e_2) is given by $(M_1 \otimes M_2, m, e)$ where m is the composition*

$$(M_1 \otimes M_2) \otimes (M_1 \otimes M_2) \xrightarrow{\gamma} (M_1 \otimes M_1) \otimes (M_2 \otimes M_2) \xrightarrow{m_1 \otimes m_2} M_1 \otimes M_2,$$

and e is the composition

$$I \cong I \otimes I \xrightarrow{e_1 \otimes e_2} M_1 \otimes M_2.$$

Moreover, the coproduct in $\mathbf{Mon}(\mathcal{C})$ *is given by the tensor product.*

5.b Proposition *The following three symmetric monoidal categories are equivalent:* $\mathbf{Mon}(\mathbf{CoMon}(\mathcal{C}))$, $\mathbf{CoMon}(\mathbf{Mon}(\mathcal{C}))$ *and* $\mathbf{BiMon}(\mathcal{C})$.

5.c A bimonoid $(B, m, e, \delta, \epsilon)$ is said to be **commutative** (resp. **cocommutative**) if (B, m, e) is commutative (resp. (B, δ, ϵ) is cocommutative).

A **Hopf monoid** H is a bimonoid with an extra 1-ary operation called **antipode** such that the following diagram commutes.

$$\begin{tikzcd} H \arrow[r, "e"] \arrow[d, "\delta"'] & I \arrow[r, "\epsilon"] & H \\ H \otimes H \arrow[rr, shift left, "H\otimes s"] \arrow[rr, shift right, "s\otimes H"'] & & H \otimes H \arrow[u, "m"'] \end{tikzcd}$$

5.d Example (Groups) In a cartesian monoidal category, for example **Set**, every monoid object is a bimonoid in a unique way. Such a bimonoid is a Hopf monoid if and only if it is a group object. In this way *Hopf monoid is a generalization of group objects.*

5.e Proposition *A commutative Hopf monoid in $\mathcal{C}$ is the same thing as a group object in* $\mathbf{CMon}(\mathcal{C})$

5.f (Modules over Hopf monoids) A **module** over a bimonoid B is a module over the monoid B. We have seen $B\,\mathbf{Mod}$ is a symmetric monoidal category. However, we give $B\,\mathbf{Mod}$ another monoidal structure which directly inherits from $\mathcal{C}$. Indeed, if B is a bimonoid, then for any two B-modules M and N, their tensor product $M \otimes N$ admits an action as the composition

$$B \otimes (M \otimes N) \xrightarrow{\delta} (B \otimes B) \otimes (M \otimes N) \xrightarrow{\gamma} (B \otimes M) \otimes (B \otimes N) \xrightarrow{l \otimes l} M \otimes N.$$

Similarly, the counit $\epsilon \colon B \to I$ induces a B-module structure on I. In this way, we see the forgetful functor $B\,\mathbf{Mod} \to \mathcal{C}$ is a *strong monoidal functor*, meaning it preserves the monoidal structures.

5.g (Tensor-Hom adjunction) A monoidal category is said to be **closed** if every functor $- \otimes M$ with $M \in \operatorname{ob}\mathcal{C}$ has a right adjoint, denoted by $\hom(M, -)$. If this is the case, we get a bifunctor $\hom(-,-)\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the **internal Hom** in $\mathcal{C}$.

5.h Theorem (Closed monoidal structure on modules) *If H is a Hopf monoid in a closed symmetric monoidal category $\mathcal{C}$, then the category $H\,$***Mod** *is also closed and the forgetful functor is strong closed, meaning it preserves internal hom.*

Proof: It suffices to give an H-module structure on $\hom(M,N)$ for any H-modules M, N. Indeed, since

$$\operatorname{Hom}_{\mathcal{C}}(H \otimes \hom(M,N), \hom(M,N)) = \operatorname{Hom}_{\mathcal{C}}(H \otimes \hom(M,N) \otimes M, N),$$

it suffices to give a morphism $H \otimes \hom(M,N) \otimes M \to N$. Here it is

$$\begin{aligned} H \otimes \hom(M,N) \otimes M &\xrightarrow{\delta} (H \otimes H) \otimes \hom(M,N) \otimes M \\ &\xrightarrow{\gamma} (H \otimes \hom(M,N)) \otimes (H \otimes M) \\ &\xrightarrow{l \otimes l} \hom(M,N) \otimes M \\ &\xrightarrow{\mathrm{ev}} N, \end{aligned}$$

where ev is the morphism corresponding to $\mathrm{id}_{\hom(M,N)}$ under the tensor-Hom adjunction, called the **evaluation**. $\square$

§ 2 Ringed spaces

1 (Ringed spaces) A **ringed space** is a pair $(X, \mathscr{O}_X)$ of a topological space X and a sheaf of rings on X. A *morphism of ringed spaces* is is a pair $(f, f^\sharp)$ of a continuous map $f\colon X \to Y$ and an *f-map* (ref. I.5.4) of sheaves of rings $f^\sharp\colon \mathscr{O}_Y \to \mathscr{O}_X$.

Let $(f, f^\sharp)\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ and $(g, g^\sharp)\colon (Y, \mathscr{O}_Y) \to (Z, \mathscr{O}_Z)$ be two morphisms of ringed spaces. The *composition of morphisms of ringed spaces* is given by

$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$

Note that here the composition $f^\sharp \circ g^\sharp$ follows I.5.4.

1.a ($\mathscr{O}$-modules) Let $\mathscr{F}$ and $\mathscr{G}$ be two presheaves of abelian groups, then their **tensor product** $\mathscr{F} \otimes \mathscr{G}$ is given by

$$(\mathscr{F} \otimes \mathscr{G})(U) := \mathscr{F}(U) \otimes \mathscr{G}(U).$$

In this way, $\mathbf{PAb}(X)$ becomes an abelian symmetric monoidal category. Let $\mathscr{O}$ be a presheaf of rings on X, then it is a monoid in $\mathbf{PAb}(X)$.

Apply sheafification, we get a symmetric monoidal structure on $\mathbf{Ab}(X)$. By a special case of Theorem 1.1.c or direct check, a sheaf of rings is the same as a monoid in $\mathbf{Ab}(X)$. Then, we call a module over such a monoid $\mathscr{O}_X$ as a **$\mathscr{O}_X$-module**. Written out, a $\mathscr{O}_X$ *module* is an abelian sheaf $\mathscr{F}$ with a abelian sheaf morphism $\mathscr{O}_X \otimes \mathscr{F} \to \mathscr{F}$, called the *action*, such that each $\mathscr{O}_X(U) \otimes \mathscr{F}(U) \to \mathscr{F}(U)$ gives $\mathscr{F}(U)$ an $\mathscr{O}_X(U)$-module structure.

A *morphism* of $\mathscr{O}_X$-modules

§ 3 Manifolds, bundles and étalé spaces

1 (Manifolds) Recall that a manifold M is a topological space locally like $\mathbb{R}^n$. Usually, there are some extra requirements such as *second countability* and *Hausdorff property*. More precisely, for every point $x \in M$, there exists a neighborhood U of x equipped with an embedding $\phi\colon \mathbb{R}^n$, called a *chart*, and those charts are *compatible*. Here two charts (U, ϕ) and (V, ψ) are said to be *compatible* if the *transition function* $\psi \circ \phi^{-1}$ is a continuous (or k-differential, smooth, etc. depending on what kind of manifold is considered) map in $\mathbb{R}^n$.

1.a (Structure sheaf) Any manifold M admits a canonical sheaf $\mathscr{O}_M$, called its **structure sheaef**. For a topological manifold, it is just the sheaf $\mathscr{C}_M$ of continuous maps to $\mathbb{R}$.

Next, we consider the differential manifolds. But before that, let's recall that there are many subsheaves of $\mathscr{C}$ on the Euclidean space $\mathbb{R}^n$ such as $\mathscr{C}^k$, the sheaf of k-differential functions, $\mathscr{C}^\infty$, the sheaf of smooth functions, $\mathscr{C}^\omega$, the sheaf of real analytic functions, et cetera.

Anyhow, let $\mathscr{O}$ denotes one of those subsheaf, for instance $\mathscr{C}^\infty$. Let's see how the definition of a manifold translate them to the manifold M. Recall a chart is nothing but a embedding $\phi\colon U \to \mathbb{R}^n$, this embedding translates the sheaf $\mathscr{O}$ to U via inverse image ϕ^{-1}. Then the compatible condition of charts (U, ϕ) and (V, ψ) require the transition functions $\psi \circ \phi^{-1}$ being smooth. In this way the transition functions provides isomorphisms between $\phi^{-1}\mathscr{O}\big|_{U\cap V}$ and $\psi^{-1}\mathscr{O}\big|_{U\cap V}$ via

$$\begin{aligned} \psi^{-1}\mathscr{O}\big|_{U\cap V}(W) = \mathscr{O}(\psi(W)) &\longrightarrow \phi^{-1}\mathscr{O}\big|_{U\cap V}(W) = \mathscr{O}(\phi(W)) \\ f &\longmapsto f \circ \psi \circ \phi^{-1}. \end{aligned}$$

Now, we have a system of sheaves on open sets of M together with isomorphisms on their overlaps. Obviously, this system can be extended into a gluing data. Then, by Theorem I.4.5.b, we obtain a sheaf $\mathscr{O}_M$ on M. This sheaf is called the **structure sheaef** of M and in the smooth case it is called the **sheaf of smooth functions** on M and denoted by $\mathscr{C}^\infty_M$.

Conversely, we have

1.b Theorem *A manifold M is equivalent to a locally ringed space $(M, \mathscr{O}_M)$, which is locally isomorphic to an open subset of $\mathbb{R}^n$.*

1.c Lemma *Let $(f, \psi)\colon X \to Y$ be a morphism of locally ringed spaces, where X and Y are smooth manifolds with their sheaves of smooth functions. If $\psi\colon \mathscr{C}_Y^\infty \to f_*\mathscr{C}_X^\infty$ is a morphism of sheaves of $\mathbb{R}$-algebras, then f is smooth and $\psi = f^\sharp$.*

Now, we give another definition of manifolds using the language of sheaves. To begin with, we need a standard model consisting of a space like $\mathbb{R}^n$, $\mathbb{C}^n$, etc. and a sheaf of special type of maps on it.

1.d Let X be a topological space. A **transformation group** G on X is a subsheaf of the sheaf of continuous functions.

2 (Espace étalé) Let **Top** denote the category of topological spaces and continuous maps. Define a Grothendieck pretopology Cov on **Top** given by

$$\{\phi_i\colon U_i \to U\}_{i\in I} \in \mathrm{Cov} \Longleftrightarrow \bigcup_{i\in I} \phi_i(U_i) = U \text{ and } \phi_i \text{ are injective and open.}$$

One can see the representable presheaves on **Top** are sheaves. In this way, *sheaves can be thought as generalized spaces.*

Let X be a topological space. Then the category $\mathbf{Top}/X$ of continuous maps to X has an inherited Grothendieck pretopology and form a site $\mathbf{Top}_X$. Now, one can see that $\mathcal{T}_X$ forms a *subsite* of $\mathbf{Top}_X$. So there is a functor $\mathbf{Sh}(\mathbf{Top}_X) \to \mathbf{Sh}(X)$. Compositing it with the Yoneda embedding, we get the following functor

$$\mathbf{Top}/X \xrightarrow{\Upsilon} \mathbf{Sh}(\mathbf{Top}_X) \longrightarrow \mathbf{Sh}(X).$$

To simplify notations, we still use Υ denote this functor. Now, what surprising is this functor has a right adjoint, meaning for every sheaf $\mathscr{F}$ on X, there is a canonical topological space $E_{\mathscr{F}}$ with a canonical map $\pi_{\mathscr{F}}\colon E_{\mathscr{F}} \to X$ such that

$$\mathrm{Hom}(Y, E_{\mathscr{F}}) = \mathrm{Hom}(\Upsilon(Y), \mathscr{F}).$$

One may wants to conversely extend every sheaf $\mathscr{F}$ on X to a sheaf on $\mathbf{Top}_X$. Let $f\colon Y \to X$ be an arbitrary object in $\mathbf{Top}/X$, one attempt is to define $\mathscr{F}(f)$ as the same with $\mathscr{F}(f(Y))$. However, $f(Y)$ is in general not an open set, thus $\mathscr{F}(f(Y))$ is still non-defined. So, one may try to restrict to a suitable subsite of $\mathbf{Top}_X$. The first candidate is the subcategory of $\mathbf{Top}/X$ consisting of only open maps.

More precisely, let

III

Schemes

IV
Summary

§ 1 Schemes

1 A **scheme** is a locally ringed space which is locally isomorphic to an affine scheme.

1.a Theorem (Fundamental theorem) *The functor*

$$\begin{aligned}\mathbf{Sch} &\longrightarrow \mathbf{PSh}(\mathbf{CRing}^{\mathrm{opp}})\\ (X, \mathscr{O}_X) &\longmapsto \mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}(-), X)\end{aligned}$$

is fully faithful and identifies schemes with those presheaves on $\mathbf{CRing}^{\mathrm{opp}}$ *such that*

1. *are sheaves with respect to the* Zariski Grothendieck topology *on* $\mathbf{CRing}^{\mathrm{opp}}$,
2. *have a cover by Zariski-open immersions of affine schemes.*

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2 ¶Remark (Local isomorphisms) The sheafification can also be described in terms of *local isomorphism.* However, here we do not introduce the true theory of local isomorphisms. Instead, we just pick up a point. A **local isomorphism**, in idea, should be such kind of presheaf morphisms whose sheaffication are isomorphisms.

Now, assume we have known what are local isomorphisms.

2.a Lemma *1. If $\mathscr{F} \to \mathscr{G}$ is a local isomorphism and $\mathscr{H} \to \mathscr{G}$ is a morphism, then $\mathscr{F} \times_{\mathscr{G}} \mathscr{H} \to \mathscr{H}$ is a local isomorphism.*

$$\begin{tikzcd} \mathscr{F} \times_{\mathscr{G}} \mathscr{H} \arrow[r] \arrow[d] & \mathscr{H} \arrow[d] \\ \mathscr{F} \arrow[r] & \mathscr{G} \end{tikzcd}$$

2. Let $\mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H}$ be presheaf morphisms. Then if two of the three morphisms $f, g, f \circ g$ are local isomorphisms, then all of them are local isomorphisms.

Proof: *1.* Since sheafification is exact, it maps the Cartesian diagram above to the corresponding Cartesian diagram below.

$$\begin{tikzcd} \mathscr{F}^\sharp \times_{\mathscr{G}^\sharp} \mathscr{H}^\sharp \arrow[r] \arrow[d] & \mathscr{H}^\sharp \arrow[d] \\ \mathscr{F}^\sharp \arrow[r] & \mathscr{G}^\sharp \end{tikzcd}$$

Now the bottom is an isomorphism, thus so is its pullback.

2. Apply the sheafification to the diagram

$$\mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H}.$$

Then the claim is clear. □

Let $\mathcal{LI}_U$ denote the subcategory of $\mathbf{PSh}(\mathcal{C})/h_U$ consisting of only local isomorphisms. It is a category since lemma 2.a. Then we will construct the sheafification via a colimit over this category. So, we should study this category first.

2.b Lemma *The category $\mathcal{LI}_U$ is finite-complete, a fortiori filtered.*

Proof: First, $\mathcal{LI}_U$ has a terminal object $\mathrm{id}\colon h_U \to h_U$. Next, if $\mathscr{F} \to h_U$, $\mathscr{G} \to h_U$ and $\mathscr{H} \to h_U$ are objects in $\mathcal{LI}_U$, then so is $\mathscr{F} \times_{\mathscr{H}} \mathscr{G} \to h_U$ by lemma 2.a. Therefore $\mathcal{LI}_U$ has fibre products. □

Let

$$\mathcal{L}\mathscr{F}(U) := \varinjlim_{\mathcal{LI}_U} \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathscr{U}, \mathscr{F}).$$

If there is a morphism $V \to U$, then we have a canonical map by

$$\begin{aligned}
\mathcal{L}\mathscr{F}(U) &= \varinjlim_{\mathcal{LI}_U} \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathscr{U}, \mathscr{F}) \\
&\longrightarrow \varinjlim_{\mathcal{LI}_U} \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathscr{U} \times_U V, \mathscr{F}) \\
&\longrightarrow \varinjlim_{\mathcal{LI}_V} \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathscr{V}, \mathscr{F}) = \mathcal{L}\mathscr{F}(V).
\end{aligned}$$

in this way, $\mathcal{L}\mathscr{F}$ is a presheaf.

Since $\mathrm{id}\colon h_U \to h_U$ is an object in $\mathcal{LI}_U$, we get a canonical map

$$\mathscr{F}(U) = \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(h_U, \mathscr{F}) \longrightarrow \varinjlim_{\mathcal{LI}_U} \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathscr{U}, \mathscr{F}) = \mathcal{L}\mathscr{F}(U).$$

In this way, we get a presheaf morphism $\mathscr{F} \to \mathcal{L}\mathscr{F}$.

Now, one can verify that $\mathscr{F} \mapsto (\mathscr{F} \to \mathcal{L}\mathscr{F})$ gives a functor.

2.c Theorem (Sheafification) *Let $\mathscr{F}$ be a presheaf on a site $\mathcal{C}$. Then $\mathcal{L}\mathscr{F}$ is a sheaf. Moreover, if $\mathscr{F}$ is already a sheaf, then $\mathscr{F} \to \mathcal{L}\mathscr{F}$ is an isomorphism.*

Proof: First, we prove that for any covering $\{U_i \to U\}$, the canonical map

$$\mathcal{L}\mathscr{F}(U) \longrightarrow \prod \mathcal{L}\mathscr{F}(U_i)$$

is injective. Indeed, $\square$

2.d (Deloopings and duals) Just as any monoid in **Set** can be viewed as a one-object category, any monoidal category $\mathcal{C}$ can be viewed as a one-object **bicategory**, which is similar to a *strict* 2*-category* except the associativity and unity laws hold only up to coherent isomorphism. This bicategory is called the **delooping** of $\mathcal{C}$ and is denoted by $\mathbf{B}\mathcal{C}$.

Now, an object in $\mathcal{C}$ is a morphism in $\mathbf{B}\mathcal{C}$, thus it make sense to define its adjoint. Indeed, An **adjunction** $L \dashv R$ in a bicategory is a pair of morphisms with two 2-morphisms $\eta\colon \mathrm{id} \to R \circ L$ and $\epsilon\colon L \circ R \to \mathrm{id}$ satisfying the following *triangle identities.*

$$\begin{tikzcd}
 & LRL \arrow[dr, "\epsilon * L"] & \\
L \arrow[ur, "L * \eta"] \arrow[rr, "\mathrm{id}"] & & L
\end{tikzcd}
\qquad
\begin{tikzcd}
 & RLR \arrow[dr, "R * \epsilon"] & \\
R \arrow[ur, "\eta * R"] \arrow[rr, "\mathrm{id}"] & & R
\end{tikzcd}$$

Let $M \in \mathrm{ob}\,\mathcal{C}$, a *left* adjoint of M is called its **(right) dual**. Down to earth, we define an object M is **(right) dualizable** if there exists an object M^* and two morphisms $\eta\colon I \to M \otimes M^*$ (called *unit*) and $\epsilon\colon M^* \otimes M \to I$ (called *counit*). Note that when $\mathcal{C}$ is a symmetric monoidal category, there is no difference between right and left duals. For a morphism $f\colon M \to N$, its *dual* $f^*\colon N^* \to M^*$ is the composition

$$N^* \xrightarrow{\eta} N^* \otimes M \otimes M^* \xrightarrow{f} N^* \otimes N \otimes M^* \xrightarrow{\epsilon} M^*.$$

For M a dualizable object in $\mathcal{C}$, if its unit and counit are isomorphisms, then we call its dual the **inverse** of M.

If all objects in $\mathcal{C}$ is dualizable on both sides, then we say $\mathcal{C}$ is **rigid**. If $\mathcal{C}$ is also symmetric, then we call it a **compact closed category**. Here the adjective "closed" appears since in this case $\mathcal{C}$ is closed. Indeed, the internal Hom is given by

$$\hom(M, N) := M^* \otimes N.$$

Conversely, assume $\mathcal{C}$ is a closed symmetric monoidal category, then there is another notion of dual. For any $M \in \mathcal{C}$, its **weak dual** is

$$M^\vee := \hom(M, I).$$

However, the weak duals are usually *NOT* the monoidal duals. Nevertheless, there is always a morphism $\epsilon\colon M^\vee \otimes M \to I$, i.e. the evaluation. Braiding it and through the tensor-Hom adjunction, we get a morphism from M to $M^{\vee\vee}$, which is usually *NOT* an isomorphism. But if this is the case, we call this weak dual is **reflexive**. If moreover $M \otimes M^\vee$ is canonically dual to itself, then we call $M^\vee$ is a **strong dual**. Note that