Algebraic Geometry

Gao, Xu

December 5, 2015

Preface

This is a note aim to understand algebraic geometry through a category-flavor approach. My main reference is the Stacks Project [Stacks]. The framework is basically taken from it. However, some details, especially the proofs, are modified by my taste. I also take examples from [RV15].

Gao, Xu[1]

[1]Email: GauSyu@gmail.com

Contents

Conventions

Algebra

Throughout this note, **rings** are assumed to be commutative and unitary except special assumptions. For the usual notion of (unitary) rings, we call them **non-commutative rings** to emphasize. For the more general notion of non-unitary rings, we call them **associative algebras** to distinguish.

Notations

- Calligraphic letters like $\mathcal{C}, \mathcal{D}, \mathcal{E}$ will denote categories or more general structured categories. But $\mathcal{A}$ will be restricted to denote algebraic categories and $\mathcal{K}$ will denote a "cosmos" where other structures grow.
- Capital letters like F, G, H and small letters like u, v, w will denote functors, the later ones often used for functors between sites.
- Capital letters like $A, B, C; U, V, W; X, Y, Z$ will denote objects. But A, B, C often refer to structured objects such as algebraic objects, U, V, W often refer to objects will used as indexes such as objects in a site. The letter T often denotes a "test object" and S a *generator*.
- Small letters like f, g, h and the letters ϕ, ψ will denote morphisms in suitable context, the later ones often used for morphisms serving as a part of some structure.
- The script letters like $\mathscr{F}, \mathscr{G}, \mathscr{H}$ will denote presheaves and sheaves, and the letters φ, ψ will denote morphisms between them,
- The fraktur letters like $\mathfrak{U}, \mathfrak{V}, \mathfrak{W}$ will denote coverings.

- $*$ denotes the terminal object in a category, usually the singleton. We use $\varnothing$ to denote the empty. 0 denotes the zero object in a category. But we also use it to denote the trivial ring in which $0 = 1$.
- We will use $\coprod$ and $\bigoplus$ to denote coproducts. The later is particularly used in an abelian tensor category.

I

Sheaves

Sheaves are the abstract of how to glue local information into global.

§ I.1 Presheaves

In this section, we give the notion of *presheaves*: they are contravariant functors. Then, we show presheaves are colimits of the *representable ones* (Theorem 2.e). In this sense, presheaves are generalized objects. Next, we describe the *monomorphisms, epimorphisms and isomorphisms of presheaves*, defines the notions of *subpresheaves* and *image* and points out that *limits and colimits of presheaves are computed object by object.* This makes presheaves more like kind of generalized objects. Finally, we show that a functor u between categories induces a chain of adjunctions: $u_p \dashv u^p \dashv {}_pu$.

Presheaves and Yoneda lemma

1 (Presheaves are contravariant functors) A **presheaf** on a category $\mathcal{C}$ with values in another category $\mathcal{A}$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{A}$. In the case $\mathcal{A} = \mathbf{Set}$, we simply call it a presheaf. *Morphisms* between presheaves are natural transformations.

Notations:

- $\mathbf{PSh}_{\mathcal{A}}(\mathcal{C}) = [\mathcal{C}^{\mathrm{opp}}, \mathcal{A}]$: the category of presheaves on $\mathcal{C}$ with values in $\mathcal{A}$.
- $\mathbf{PSh}(\mathcal{C}) = \mathbf{PSh}_{\mathbf{Set}}(\mathcal{C})$: the category of presheaves on $\mathcal{C}$.
- An element $s \in \mathscr{F}(U)$ is called a **section** of $\mathscr{F}$ on U. For a morphism $f\colon V \to U$, we denote $\mathscr{F}(f)(s)$ by $s|_V$ or $s|_f$.

2 Example (Representable presheaves) For any object $U \in \mathcal{C}$, the **functor of points** $h_U\colon X \mapsto \mathrm{Hom}(X, U)$ is a presheaf. For any presheaf $\mathscr{F}$, a **representation** of it is a natural isomorphism from h_U to $\mathscr{F}$ for some object U. If this is the case, we say $\mathscr{F}$ is **representable** and is represented by U.

2.a Theorem (Yoneda lemma) *There is a canonical bijection*

$$\begin{aligned} \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(h_U, \mathscr{F}) &\longrightarrow \mathscr{F}(U) \\ s &\longmapsto s_U(\mathrm{id}_U) \end{aligned}$$

natural in both the object U and the presheaf $\mathscr{F}$.

2.b Corollary *The functor $\Upsilon\colon U \mapsto h_U$ is a* **full embedding**, *which means Υ is fully faithful and injective on object. This functor is called the* **Yoneda embedding**.

2.c Corollary *A representation of a presheaf $\mathscr{F}$ on $\mathcal{C}$ is precisely the terminal object in the comma category $(\Upsilon \downarrow \mathrm{const}_{\mathscr{F}})$, which means a pair (U, u) of an object $U \in \mathrm{ob}\,\mathcal{C}$ and an element $u \in \mathscr{F}(U)$ satisfies the following universal property:*

For every pair (X,x) of $X \in \operatorname{ob}\mathcal{C}$ and $x \in \mathscr{F}(X)$, there is a unique morphism $f\colon X \to U$ such that $\mathscr{F}(f)(u) = x$.

Remark The comma category $(\Upsilon \downarrow \mathrm{const}_{\mathscr{F}})$ is isomorphic to the comma category $(* \downarrow \mathscr{F})$, where $*$ denote the constant functor mapping any object to the the singleton. This is indeed another expression of the Yoneda lemma. This comma category is called **the category of sections** of $\mathscr{F}$ and is denoted by $\mathcal{C}_{\mathscr{F}}$.

2.d Lemma (The set of global sections) *Let $\mathscr{F}$ be a presheaf on a small category $\mathcal{C}$, then*

$$\varprojlim_{\mathcal{C}} \mathscr{F} = \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(*, \mathscr{F}).$$

where $*$ denote the presheaf mapping any object in $\mathcal{C}$ to the singleton. This set is called the **set of global sections** *of $\mathscr{F}$.*

Proof: Just note that a natural transformation from $*$ to $\mathscr{F}$ is the same thing as a compatible data of the system $\{\mathscr{F}(U)\}_{U \in \operatorname{ob}\mathcal{C}}$, which is an element in the limit of $\mathscr{F}$. □

2.e Theorem (Every presheaf is a colimit of representable ones) *Let $\mathscr{F}$ be a presheaf on a small category $\mathcal{C}$, then*

$$\mathscr{F} \cong \varinjlim_{h_U \to \mathscr{F}} h_U := \varinjlim_{\mathcal{C}_{\mathscr{F}}} \Upsilon(U).$$

Proof (by Urs Schreiber): Notice that for every $\mathscr{G} \in \mathbf{PSh}(\mathcal{C})$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\varinjlim_{\mathcal{C}_{\mathscr{F}}} \Upsilon(U), \mathscr{G}) &= \varprojlim_{\mathcal{C}_{\mathscr{F}}} \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\Upsilon(U), \mathscr{G}) \\
&= \varprojlim_{\mathcal{C}_{\mathscr{F}}} \mathscr{G}(U) \\
&= \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C}_{\mathscr{F}})}(*, \mathscr{G}),
\end{aligned}$$

where the last equality follows from Lemma 2.d.

Now, notice that an $\alpha \in \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C}_{\mathscr{F}})}(*, \mathscr{G})$ gives each objects $h_U \to \mathscr{F}$ in $\mathcal{C}_{\mathscr{F}}$, which is equivalent to an element of $\mathscr{F}(U)$ by the Yoneda lemma, a map $*(U) \to \mathscr{G}(U)$, i.e. an element of $\mathscr{G}(U)$. Therefore, we have

$$\mathrm{Hom}_{\mathbf{PSh}(\mathcal{C}_{\mathscr{F}})}(*, \mathscr{G}) = \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathscr{F}, \mathscr{G}).$$

Then the conclusion follows. □

3 ¶Remark All the above notions and results can be generalized to enriched categories, cf. [Kel05]. Note that, in this case, a *presheaf* should mean a contravariant $\mathcal{A}$-functor to $\mathcal{A}$; the *functor of points* h_U should mean the contravariant $\mathcal{A}$-functor $X \mapsto \underline{\mathrm{Hom}}(X, U)$, where the notation $\underline{\mathrm{Hom}}$ emphasize that this is the internal hom-object in $\mathcal{A}$ rather than a hom-set.

3.a Theorem (Weak Yoneda lemma) *There is a canonical bijection*

$$\begin{aligned}\mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(h_U, \mathscr{F}) &\longrightarrow \mathrm{Hom}_{\mathcal{A}}(I, \mathscr{F}(U)) \\ s &\longmapsto \left(I \xrightarrow{1_U} \underline{\mathrm{Hom}}(U,U) \xrightarrow{s_U} \mathscr{F}(U)\right)\end{aligned}$$

natural in both the object U and the presheaf $\mathscr{F}$.

The strong form of Yoneda lemma reqiures the completeness of $\mathcal{A}$. Then, given a small $\mathcal{A}$-enriched category $\mathcal{C}$ and $\mathcal{A}$-enriched functors $\mathscr{F}, \mathscr{G} \colon \mathcal{C} \to \mathcal{A}$, one may construct the object of $\mathcal{A}$-*natural transformations* as an *enriched end*:

$$\mathcal{A}^{\mathcal{C}}(\mathscr{F}, \mathscr{G}) := \int_{X \in \mathrm{ob}\,\mathcal{C}} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathscr{F}(X), \mathscr{G}(X)).$$

This is the hom-object in the enriched functor category $\mathcal{A}^{\mathcal{C}}$.

3.b Theorem (Strong Yoneda lemma) *There is a $\mathcal{A}$-natural isomorphism*

$$\mathcal{A}^{\mathcal{C}^{\mathrm{opp}}}(h_U, \mathscr{F}) \xrightarrow{\sim} \mathscr{F}(U).$$

$\mathcal{A}$-natural in both the object U and the presheaf $\mathscr{F}$.

3.c Corollary (Ninja Yoneda lemma) *(following T. Leinster's comment in MathOverflow) Let $\mathscr{F}$ be a presheaf, then*

$$\mathscr{F} \cong \int_{X \in \mathrm{ob}\,\mathcal{C}} \underline{\mathrm{Hom}}_{\mathcal{A}}(\underline{\mathrm{Hom}}_{\mathcal{C}}(X, -), \mathscr{F}(X)) \cong \int^{X \in \mathrm{ob}\,\mathcal{C}} \Upsilon(X) \otimes \mathscr{F}(X).$$

Category of presheaves

Recall that the morphisms between presheaves are nothing but natural transformations.

4 (Injective and surjective sheaf morphisms) A morphism $\varphi \colon \mathscr{F} \to \mathscr{G}$ of presheaves is said to be **injective** (resp. **surjective**) if $\varphi_U \colon \mathscr{F}(U) \to \mathscr{G}(U)$ is injective (resp. surjective) for every $U \in \mathrm{ob}\,\mathcal{C}$.

4.a Lemma *Let $f \colon X \to Y$ be a morphism in a category $\mathcal{C}$, then*

f is	*if and only if the induced map*
monic	$\mathrm{Hom}(U, X) \xrightarrow{f_*} \mathrm{Hom}(U, Y)$ *is injective for all* $U \in \mathrm{ob}\,\mathcal{C}$*;*
epic	$\mathrm{Hom}(X, U) \xrightarrow{f^*} \mathrm{Hom}(Y, U)$ *is injective for all* $U \in \mathrm{ob}\,\mathcal{C}$*;*
split epic	$\mathrm{Hom}(U, X) \xrightarrow{f_*} \mathrm{Hom}(U, Y)$ *is surjective for all* $U \in \mathrm{ob}\,\mathcal{C}$*;*
split monic	$\mathrm{Hom}(X, U) \xrightarrow{f^*} \mathrm{Hom}(Y, U)$ *is surjective for all* $U \in \mathrm{ob}\,\mathcal{C}$*.*

4.b Proposition (Monic/epic is pointwise) *The injective (resp. surjective) morphisms of presheaves are precisely the monomorphism (resp. epimorphism) in* $\mathbf{PSh}(\mathcal{C})$*. In particular, the isomorphisms in* $\mathbf{PSh}(\mathcal{C})$ *are those both injective and surjective.*

Remark The injective part of this statement is straightly from Theorem 2.e and lemma 4.a, while the surjective part requires the slogan "*limits in functor categories are computed pointwise*" and the fact that a morphism is epic if and only if its *cokernel pair* is trivial.

5 (Subpresheaves) We say $\mathscr{F}$ is a **subpresheaf** of $\mathscr{G}$ if for every $U \in \operatorname{ob}\mathcal{C}$, $\mathscr{F}(U) \subset \mathscr{G}(U)$ and the inclusion maps glue together to give an injective morphism of presheaves $\mathscr{F} \to \mathscr{G}$.

5.a Proposition (Image of a morphism) *For any morphism of presheaves* $\varphi\colon \mathscr{F} \to \mathscr{G}$*, there exists a unique subpresheaf* $\mathscr{G}' \subset \mathscr{G}$ *such that* φ *can be factorized into* $\mathscr{F} \to \mathscr{G}' \to \mathscr{G}$ *and that the first morphism is surjective. Such a subpresheaf* $\mathscr{G}'$ *is called the* **(presheaf) image** *of* φ*.*

6 Proposition (Limits and colimits of presheaves) *Limits and colimits exist in the category* $\mathbf{PSh}(\mathcal{C})$*. Indeed, they are computed pointwise. Moreover, for every* $U \in \operatorname{ob}\mathcal{C}$*, the* **section functors**

$$\begin{aligned} \Gamma(U, -)\colon \mathbf{PSh}(\mathcal{C}) &\longrightarrow \mathbf{Set} \\ \mathscr{F} &\longmapsto \mathscr{F}(U) \end{aligned}$$

commutes with limits and colimits.

As a result of this, statements about limits and colimits of presheaves can be deduced to the similar statements for sets. See my note *BMO* or refer [Bor94] for more details.

6.a Corollary *If* $\mathcal{C}$ *is a small category. Then the Yoneda embedding* $\Upsilon\colon \mathcal{C} \to \mathbf{PSh}(\mathcal{C})$ *commutes with limits.*

Proof: Let $\varprojlim U_i = U$ in $\mathcal{C}$ and $V \in \mathcal{C}$. Then

$$\begin{aligned} (\varprojlim \Upsilon(U_i))(V) &= \varprojlim \Upsilon(U_i)(V) \\ &= \varprojlim \operatorname{Hom}(V, U_i) \\ &= \operatorname{Hom}(V, \varprojlim U_i) \\ &= \operatorname{Hom}(V, U) = \Upsilon(U)(V). \end{aligned}$$

Thus $\varprojlim \Upsilon(U_i) = \Upsilon(\varprojlim U_i)$. $\square$

Remark However, the Yoneda embedding does not commute with colimits in general.

Changing the base space

Let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor. Let u^p denote the functor

$$\begin{aligned} u^p\colon\ \mathbf{PSh}(\mathcal{D}) &\longrightarrow \mathbf{PSh}(\mathcal{C}) \\ \mathscr{G} &\longmapsto \mathscr{G} \circ u. \end{aligned}$$

Note that this functor commutes with limits and colimits.

Now, we are going to introduce a *left adjoint* to this functor. Before we do so, we introduce a category $\mathcal{I}_V$ for every $V \in \operatorname{ob}\mathcal{D}$ as follows. The objects in $\mathcal{I}_V$ are pairs (U, ϕ) where $U \in \operatorname{ob}\mathcal{C}$ and $\phi\colon V \to u(U)$. A morphism between (U, ϕ) and (U', ϕ') is a morphism $f\colon U \to U'$ such that $u(f) \circ \phi = \phi'$. in other words, $\mathcal{I}_V$ is the *comma category* $(\mathrm{const}_V \downarrow u)$.

Before going forward, we recall the notion of *filtered colimit*.

7 (Filtered colimits) A category $\mathcal{I}$ is said to be **filtered** if it is nonempty and if every *finite diagram* in which has a *cocone*, in other words, if every functor from a finite category to $\mathcal{I}$ admits a natural transformation to a constant functor.

Like the equivalence condition of cocompleteness, we have

7.a Proposition *A category $\mathcal{I}$ is filtered, if and only if*

1. *$\mathcal{I}$ is nonempty;*
2. *For any two objects $A, B \in \operatorname{ob}\mathcal{I}$, there exists an object $C \in \operatorname{ob}\mathcal{I}$ and morphisms $A \to C$ and $B \to C$;*
3. *For any two parallel morphisms $f, g\colon A \rightrightarrows B$ in $\mathcal{I}$, there exists a morphism $h\colon B \to C$ such that $h \circ f = h \circ g$.*

We have a slogan "*in* **Set***, filtered colimits commute with finite limits*". The proof of this statement is technical, one can either refer Theorem 2.12.11 in my note *BMO*, or directly refer [Bor94].

Now, we go back to the category $\mathcal{I}_V$.

8 Lemma *Let $\mathcal{C}$ be a finite-complete category, which means it has all finite limits, and $u\colon \mathcal{C} \to \mathcal{D}$ be a functor commutes with all those finite limits, then $\mathcal{I}_V^{\mathrm{opp}}$ are filtered.*

Proof: First, we show that $\mathcal{I}_V^{\mathrm{opp}}$ is nonempty. Indeed, let X be a terminal object in $\mathcal{C}$. Then $u(X)$ is a terminal object in $\mathcal{D}$. Thus there exists a morphism $V \to u(X)$, and therefore $\mathcal{I}_V$ has at least one object.

Then we verify condition *2* in Proposition 7.a. Let $(A, \phi), (B, \psi) \in \operatorname{ob}\mathcal{I}_V$. Let C be the product of A and B in $\mathcal{C}$. Then $u(C)$ is the product of $u(A)$ and $u(B)$. Hence there exists a unique morphism $\theta\colon V \to u(C)$ compatible with ϕ and ψ. Then (C, θ) is the required object.

Finally, we verify condition *3*. Let $f, g\colon (A,\phi) \rightrightarrows (B,\psi)$ be two parallel morphisms in $\mathcal{I}_V^{\mathrm{opp}}$. Then $f, g\colon B \rightrightarrows A$ are two parallel morphisms in $\mathcal{C}$. Let $h\colon C \to B$ be the equalizer of them, then $u(h)$ is the equalizer of $u(f)$ and $u(g)$. Hence there exists a unique morphism $\theta\colon V \to u(C)$ such that $u(h) \circ \theta = \psi$. Then $u(h)$ is the required morphism. $\square$

Given a presheaf $\mathscr{F}$ on $\mathcal{C}$, we have a functor

$$\begin{aligned} \mathscr{F}_V\colon \mathcal{I}_V^{\mathrm{opp}} &\longrightarrow \mathbf{Set} \\ (U,\phi) &\longmapsto \mathscr{F}(U). \end{aligned}$$

So, we define

$$u_p\mathscr{F}(V) := \varinjlim \mathscr{F}_V.$$

Given a morphism $g\colon V' \to V$, by the functorality of comma category, we have a functor

$$\begin{aligned} \bar{g}\colon \mathcal{I}_V &\longrightarrow \mathcal{I}_{V'} \\ (U,\phi) &\longmapsto (U, \phi\circ g), \end{aligned}$$

such that $\mathscr{F}_{V'}\circ\bar{g} = \mathscr{F}_V$. Therefore, there exists a unique map $g^*\colon u_p\mathscr{F}(V) \to u_p\mathscr{F}(V')$ compatible with this relation, i.e. the following diagram commutes.

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "c(\phi)"] \arrow[d, "\mathrm{id}"'] & u_p\mathscr{F}(V) \arrow[d, "g^*"] \\ \mathscr{F}(U) \arrow[r, "c(\phi\circ g)"] & u_p\mathscr{F}(V') \end{tikzcd}$$

The uniqueness of those g^* implies that we obtain a presheaf on $\mathcal{D}$, denoted by $u_p\mathscr{F}$. Note that any morphism of presheaves $\mathscr{F} \to \mathscr{F}'$ gives rise to compatible systems of morphisms between functors $\mathscr{F}_V \to \mathscr{F}'_V$, and hence to a morphism of presheaves $u_p\mathscr{F} \to u_p\mathscr{F}'$. In this way, we have defined a functor

$$u_p\colon\ \mathbf{PSh}(\mathcal{C}) \longrightarrow \mathbf{PSh}(\mathcal{D}).$$

9 Theorem *The functor u_p is a left adjoint to the functor u^p.*

Proof: Let $\mathscr{G}$ be a presheaf on $\mathcal{D}$ and let $\mathscr{F}$ be a presheaf on $\mathcal{C}$. We need to show the following one-one corresponding:

$$\mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathscr{F}, u^p\mathscr{G}) \cong \mathrm{Hom}_{\mathbf{PSh}(\mathcal{D})}(u_p\mathscr{F}, \mathscr{G}).$$

First, given a morphism $\alpha\colon u_p\mathscr{F} \to \mathscr{G}$, we get $u^p\alpha\colon u^pu_p\mathscr{F} \to u^p\mathscr{G}$. Since there already exists a morphism $\mathscr{F} \to u^pu_p\mathscr{F}$ given by the canonical maps $c(\mathrm{id}_{u(U)})\colon \mathscr{F}(U) \to u_p\mathscr{F}(u(U))$, we find the corresponding morphism $\mathscr{F} \to u^pu_p\mathscr{F} \to u^p\mathscr{G}$.

Then, given a morphism $\beta\colon \mathscr{F} \to u^p\mathscr{G}$, we get $u_p\beta\colon u_p\mathscr{F} \to u_pu^p\mathscr{G}$. For every $V \in \mathcal{D}$, consider the set $u_pu^p\mathscr{G}(V) := \varprojlim u^p\mathscr{G}_V$. Now, for each $(U,\phi) \in \mathcal{I}_V$, its value under $u^p\mathscr{G}_V$ is $\mathscr{G}(u(U))$ which admits a map $\mathscr{G}(\phi)\colon \mathscr{G}(u(U)) \to \mathscr{G}(V)$. These maps form a natrual transformation from $u^p\mathscr{G}_V$ to the constant functor $\mathrm{const}_{\mathscr{G}(V)}$ on $\mathcal{I}_V$. Then there exists a map $u_pu^p\mathscr{G}(V) \to \mathscr{G}(V)$. These maps form a morphism of presheaves $u_pu^p\mathscr{G} \to \mathscr{G}$. Then we obtain the required morphism $u_p\mathscr{F} \to_p u^p\mathscr{G} \to \mathscr{G}$.

Finally, one can verify the above are mutually inverse. □

Remark Note that if $\mathcal{A}$ is a category such that any diagram $\mathcal{I}_V^{\mathrm{opp}} \to \mathcal{A}$ has a limit, then the functors u^p and u_p can be defined on the categories of presheaves with values in $\mathcal{A}$. Moreover, the adjointness of the pair u^p and u_p continues to hold in this setting.

9.a Corollary *Let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor between categories. Then, for any $U \in \mathrm{ob}\,\mathcal{C}$ we have $u_ph_U = h_{u(U)}$.*

Proof: By the adjointness and Yoneda lamma, we have

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{PSh}(\mathcal{D})}(u_ph_U,\mathscr{G}) \cong \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(h_U,u^p\mathscr{G}) \cong u^p\mathscr{G}(U) = \mathscr{G}(u(U)),\\ &\mathrm{Hom}_{\mathbf{PSh}(\mathcal{D})}(h_{u(U)},\mathscr{G}) \cong \mathscr{G}(u(U)).\end{aligned}$$

Therefore, $u_ph_U = h_{u(U)}$. □

Kan extensions

One can similar define a *right adjoint* ${}_pu$ of u^p as follows. First, construct the category ${}_V\mathcal{I}$ as the comma category $(u \downarrow \mathrm{const}_V)$. Then construct the functor ${}_V\mathscr{F}$ as

$$\begin{aligned}{}_V\mathscr{F}\colon {}_V\mathcal{I}^{\mathrm{opp}} &\longrightarrow \mathbf{Set}\\ (U,\phi) &\longmapsto \mathscr{F}(U).\end{aligned}$$

Finally, the functor is given by

$${}_pu\mathscr{F}(V) := \varprojlim {}_V\mathscr{F}.$$

The functor u_p (resp. ${}_pu$) is called the **left (resp. right) Kan extension operation along** u and $u_p\mathscr{F}$ (resp. ${}_pu\mathscr{F}$) is called the **left (resp. right) Kan extension of** $\mathscr{F}$ **along** u. More details can be found in §4.4 of my note *BMO* or refer [Bor94].

10 Theorem *The functor ${}_pu$ is right adjoint to u^p.*

Proof: The proof is similar to that of Theorem 9. □

The following lemma shows what happens when the change of base space already has an adjoint.

11 Lemma *Let* $u \dashv v\colon \mathcal{C} \to \mathcal{D}$ *be an* adjoint pair*, which means a pair of functors* $u\colon \mathcal{C} \to \mathcal{D}$ *and* $v\colon \mathcal{D} \to \mathcal{C}$ *such that* u *is left adjoint to* v*. Then*

1. $u^p h_V = h_{v(V)}$ *for any* $V \in \operatorname{ob}\mathcal{D}$*;*
2. *the category* $\mathcal{I}_U^v$ *has an initial object;*
3. *the category* ${}_V^u\mathcal{I}$ *has a terminal object;*
4. ${}_pu = v^p$*;*
5. $u^p = v_p$*.*

Proof: *1.* Let $V \in \operatorname{ob}\mathcal{D}$, then

$$u^p h_V(U) = h_V(u(U)) = \operatorname{Hom}(u(U), V) = \operatorname{Hom}(U, v(V)) = h_{v(V)}(U).$$

2. Let $\eta_U\colon U \to v(u(U))$ be the map adjoint to the map $\mathrm{id}_{u(U)}$. Then $(u(U), \eta_U)$ is an initial object of $\mathcal{I}_U^v$.

3. Let $\epsilon_V\colon u(v(V)) \to V$ be the map adjoint to the map $\mathrm{id}_{v(V)}$. Then $(v(V), \epsilon_V)$ is a terminal object of ${}_V^u\mathcal{I}$.

4. Indeed, for any presheaf $\mathscr{F}$ on $\mathcal{C}$, we have

$$\begin{aligned} v^p\mathscr{F}(V) &= \mathscr{F}(v(V)) \\ &= \operatorname{Hom}_{\mathbf{PSh}(\mathcal{C})}(h_{v(V)}, \mathscr{F}) \\ &= \operatorname{Hom}_{\mathbf{PSh}(\mathcal{C})}(u^p h_V, \mathscr{F}) \\ &= \operatorname{Hom}_{\mathbf{PSh}(\mathcal{D})}(h_V, {}_pu\mathscr{F}) \\ &= {}_pu\mathscr{F}(V). \end{aligned}$$

5. u^p is right adjoint to ${}_pu$, v_p is right adjoint to v^p. By the uniqueness of adjoint functor, ${}_pu = v^p$ implies $u^p = v_p$. □

The Kan extension operations u_p and ${}_pu$ above are special case of the following general notions.

12 (Kan extensions) Let $\mathscr{F}\colon \mathcal{C} \to \mathcal{D}$ and $p\colon \mathcal{C} \to \mathcal{C}'$ be two functors. The **left Kan extension** of $\mathscr{F}$ along p, if it exists, is a pair $(\mathscr{G}, \alpha)$ where

- $\mathscr{G}\colon \mathcal{C}' \to \mathcal{D}$ is a functor,
- $\alpha\colon \mathscr{F} \Rightarrow \mathscr{G} \circ p$ is a natural transformation,

satisfying the following universal property: if $(\mathscr{H}, \beta)$ is another pair with

- $\mathscr{H}\colon \mathcal{C}' \to \mathcal{D}$ is a functor,

- $\beta\colon \mathscr{F} \Rightarrow \mathscr{H} \circ p$ is a natural transformation,

then there exists a unique natural transformation $\gamma\colon \mathscr{G} \Rightarrow \mathscr{H}$ such that $(\gamma * p) \circ \alpha = \beta$.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathscr{F}"] \arrow[d, "p"'] \arrow[dr, phantom, near start, "\Downarrow\alpha"] & \mathcal{D} \\
\mathcal{C}' \arrow[ur, "\mathscr{G}"] \arrow[ur, bend right=50, "\mathscr{H}"'] \arrow[ur, phantom, bend right=25, "\Downarrow\gamma"] &
\end{tikzcd}
$$

We shall use the notation $\operatorname{Lan}_p \mathscr{F}$ to denote the **left Kan extension** of $\mathscr{F}$ along p. The notation $\operatorname{Ran}_p \mathscr{F}$ is used for the dual notion of **right Kan extension**.

13 Example (Yoneda extension) Let $\mathscr{F}\colon \mathcal{C} \to \mathcal{D}$ be a functor, then its **Yoneda extension** $\widetilde{\mathscr{F}}\colon \mathbf{PSh}(\mathcal{C}) \to \mathcal{D}$ is the left Kan extension of $\mathscr{F}$ along the Yoneda embedding, i.e. $\widetilde{\mathscr{F}} = \operatorname{Lan}_{\Upsilon} \mathscr{F}$. Note that one has $\widetilde{\mathscr{F}} \circ \Upsilon = \mathscr{F}$ and the formula

$$\widetilde{\mathscr{F}}(\mathscr{G}) = \varinjlim_{(h_U \to \mathscr{G}) \in \mathrm{ob}(\Upsilon \downarrow \mathrm{const}_{\mathscr{G}})} \mathscr{F}(U).$$

§ I.2 Sites and sheaves

In this section, we define *sheaves on a site*. One slogan about the topologies used in algebraic geometry is that "it is the covering does matter, not the open set". Therefore, a convention approach to the notion of sites is using coverings. So we list the definition of *coverings* and their *equivalence relations*. On this foundation, we define sheaves as presheaves satisfying *descent condition* respect to coverings. Then, we briefly introduce the notion sheaves with values in an algebraic category. Finally, we show equivalent sites define the same sheaves.

Sites

1 (Sites as categpries equipped with a Grothendieck pretopology) A **site** is a category $\mathcal{C}$ equipped with a **Grothendieck pretopology**, that is a collection $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target $\{U_i \to U\}$, called **coverings** on $\mathcal{C}$, satisfying the following axioms

Cov1. If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$;

Cov2. the collection of coverings is stable under pullback: if $\{U_i \to U\}_{i\in I}$ is a covering and $f\colon V \to U$ is any morphism in $\mathcal{C}$, then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I}$ is a covering;

Cov3. if $\{U_i \to U\}_{i\in I}$ is a covering and for each i, $\{U_{i,j} \to U_i\}_{j\in J_i}$ is also a covering, then the family of compositions $\{U_{i,j} \to U_i \to U\}_{i\in I, j\in J_i}$ is a covering.

Remark One may hope $\mathrm{Cov}(\mathcal{C})$ to be a set. But this may not be true even if $\mathcal{C}$ is a small category. Usually, we need to shrink the Grothendieck pretopology to make it become a set.

1.a Example (Topological space) Let X be a topological space and $\mathcal{T}_X$ the category whose objects are all the open subsets of X and morphisms are the inclusion maps. Then there is a standard Grothendieck pretopology on $\mathcal{T}_X$ given by

$$\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{T}_X) \iff \bigcup U_i = U.$$

We should point out that in this site, $U \times V = U \cap V$ and that empty covering of the empty set is a covering.

However, this Grothendieck pretopology is too big: the collection $\mathrm{Cov}(\mathcal{T}_X)$ is not a set as we allow arbitrary set as the index set I. This can be avoid if we exclude those coverings having duplicative members. But then, this set is *NOT* a Grothendieck pretopology unless we modify the axioms as follows.

Let $\mathcal{C}$ be a small category. Let $\mathcal{P}$ denote the power set and $\mathrm{Hom}(\mathcal{C})$ denote the union of all hom-sets in $\mathcal{C}$. Then a **quasi-Grothendieck pretopology** $\mathrm{Cov}(\mathcal{C})$ on $\mathcal{C}$ is given by

Cov0' $\mathrm{Cov}(\mathcal{C}) \subset \mathcal{P}(\mathrm{Hom}(\mathcal{C}))$;

Cov1' If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$;

Cov2' the collection of coverings is stable under pullback: if $\{U_i \to U\}_{i\in I}$ is a covering and $f\colon V \to U$ is any morphism in $\mathcal{C}$, then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I}$ is tautologically equivalent to an element of $\mathrm{Cov}(\mathcal{C})$;

Cov3' if $\{U_i \to U\}_{i\in I}$ is a covering and for each i, $\{U_{i,j} \to U_i\}_{j\in J_i}$ is also a covering, then the family of compositions $\{U_{i,j} \to U_i \to U\}_{i\in I, j\in J_i}$ is tautologically equivalent to an element of $\mathrm{Cov}(\mathcal{C})$.

1.b Example (G-sets) Let G be a group and $G\,\mathbf{Set}$ the category whose objects are sets X with a left G-action and whose morphisms are G-equivariant maps. Now, define

$$\{\varphi_i\colon U_i \to U\}_{i\in I} \in \mathrm{Cov}(G\,\mathbf{Set}) \iff \bigcup \varphi_i(U_i) = U.$$

One can verify this $\mathrm{Cov}(G\,\mathbf{Set})$ satisfies the axioms. However, since both $G\,\mathbf{Set}$ and $\mathrm{Cov}(G\,\mathbf{Set})$ are too big (they are proper classes), one may prefer to work with some smaller substitutes.

First, for any G-set X_0, there exists a suitable universe $\mathcal{U}$ such that the full subcategory $G\,\mathbf{Set}_{\mathcal{U}}$ of $\mathcal{U}$-small G-sets contains X_0 and, up to isomorphism, every G-sets smaller than those in this subcategory. Then replace $\mathrm{Cov}(G\,\mathbf{Set}_{\mathcal{U}})$ by a smaller one $\mathrm{Cov}(G\,\mathbf{Set}_{\mathcal{U}})_s$, which contains the coverings we care about and every covering in $\mathrm{Cov}(G\,\mathbf{Set}_{\mathcal{U}})$ is *combinatorially equivalent* to a covering in $\mathrm{Cov}(G\,\mathbf{Set}_{\mathcal{U}})_s$. This site $(G\,\mathbf{Set}_{\mathcal{U}}, \mathrm{Cov}(G\,\mathbf{Set}_{\mathcal{U}})_s)$ is denoted by $\mathcal{T}_G$.

1.c Example Any category $\mathcal{C}$ admits a canonical Grothendieck pretopology by setting $\{\mathrm{id}_U\colon U \to U\}$ as the coverings. *Sheaves* on this site are the presheaves on $\mathcal{C}$. The corresponding topology is called the **chaotic** or **indiscrete topology**.

1.d Remark (Coverages) In [Joh02], Johnstone introduced a more general concept called **coverage**, which is basically the same as a Grothendieck pretopology except the second axiom may not be satisfied. In his text, a *site* is a category equipped with a coverage, not necessary a Grothendieck pretopology. Many constructions and results still hold in this setting.

2 (Morphisms and refinements) Let $\mathfrak{U} = \{U_i \to U\}_{i\in I}$ and $\mathfrak{V} = \{V_j \to V\}_{j\in J}$ be two coverings on a category $\mathcal{C}$, a **morphism** from $\mathfrak{U}$ to $\mathfrak{V}$ consists of a morphism $f\colon U \to V$, a map $\alpha\colon I \to J$ and for each $i \in I$, a morphism

$U_i \to V_{\alpha(i)}$ making the following diagram commute.

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_{\alpha(i)} \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

When $U = V$ and $U \to V$ is the identity, we call $\mathfrak{U}$ a **refinement** of $\mathfrak{V}$.

Remark If $\mathfrak{V}$ is the empty covering, i.e. $J = \varnothing$, then no nonempty covering $\mathfrak{U}$ can refine $\mathfrak{V}$.

Now, we define the equivalence relation of coverings, so that we can shrink the Grothendieck pretopology in the case that we still have all the coverings up to equivalence.

3 (Equivalence relations of coverings) Let $\mathfrak{U} = \{\phi_i \colon U_i \to U\}_{i\in I}$ and $\mathfrak{V} = \{\psi_j \colon V_j \to U\}_{j\in J}$ be two coverings on a category $\mathcal{C}$.

1. We say $\mathfrak{U}$ and $\mathfrak{V}$ are **combinatorially equivalent** if there exist maps $\alpha \colon I \to J$ and $\beta \colon J \to I$ such that $\phi_i = \psi_{\alpha(i)}$ and $\psi_j = \phi_{\beta(j)}$.

2. We say $\mathfrak{U}$ and $\mathfrak{V}$ are **tautologically equivalent** if there exist maps $\alpha \colon I \to J$ and $\beta \colon J \to I$ such that for all $i \in I$ and $j \in J$ the following diagrams commute.

$$\begin{tikzcd} U_i \arrow[rr, "\cong"] \arrow[dr] & & V_{\alpha(i)} \arrow[dl] \\ & U & \end{tikzcd} \qquad \begin{tikzcd} V_j \arrow[rr, "\cong"] \arrow[dr] & & U_{\beta(j)} \arrow[dl] \\ & U & \end{tikzcd}$$

3.a Lemma *Let $\mathfrak{U} = \{\phi_i \colon U_i \to U\}_{i\in I}$ and $\mathfrak{V} = \{\psi_j \colon V_j \to U\}_{j\in J}$ be two coverings on a category $\mathcal{C}$.*

1. *If $\mathfrak{U}$ and $\mathfrak{V}$ are combinatorially equivalent then they are tautologically equivalent.*

2. *If $\mathfrak{U}$ and $\mathfrak{V}$ are tautologically equivalent then $\mathfrak{U}$ is a refinement of $\mathfrak{V}$ and $\mathfrak{V}$ is a refinement of $\mathfrak{U}$.*

3. *The relation "being combinatorially equivalent" is an equivalence relation.*

4. *The relation "being tautologically equivalent" is an equivalence relation.*

5. *The relation "$\mathfrak{U}$ refines $\mathfrak{V}$ and $\mathfrak{V}$ refines $\mathfrak{U}$" is an equivalence relation.*

Sheaves

4 **(Sheaves are gluing presheaves)** Let $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$ be a site. A **sheaf** on it, or a *sheaf* on $\mathcal{C}$ respect to $\mathrm{Cov}(\mathcal{C})$ is a presheaf $\mathscr{F}$ satisfying the following **gluing axiom**:

> For any covering $\{U_i \to U\}_{i\in I}$ and sections $s_i \in \mathscr{F}(U_i)$ such that
> $$s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$$
> in $\mathscr{F}(U_i \times_U U_j)$ for all $i, j \in I$, there exists a unique $s \in \mathscr{F}(U)$ such that $s_i = s|_{U_i}$.

Remark If in the above definition there is at most one such s, we say that $\mathscr{F}$ is a **separated presheaf**.

The above component-wise definition can be written into a more abstract way: A presheaf $\mathscr{F}$ is called a **sheaf** if for every covering $\{U_i \to U\}_{i\in I}$, the diagram

$$\mathscr{F}(U) \longrightarrow \prod_{i\in I} \mathscr{F}(U_i) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod_{i_0,i_1\in I} \mathscr{F}(U_{i_0} \times_U U_{i_1}) \tag{I.2.1}$$

is *exact*, which means the first arrow is an equalizer of pr_0^* and pr_1^*. This condition is also called the **descent condition**.

Remark By this definition, if there exists an empty covering $\{U_i \to U\}_{i\in I}$, which means $I = \varnothing$, then $\mathscr{F}(U)$ is a singleton, the terminal object in **Set**.

The morphisms between sheaves are the morphisms between their underlying presheaves. In this way, the category of sheaves $\mathbf{Sh}(\mathcal{C})$ is a *full subcategory* of the category of presheaves $\mathbf{PSh}(\mathcal{C})$.

4.a **Example (Sheaves on topological spaces)** Let X be a topological space and let $\mathcal{T}_X$ be the site in Example 1.a. Then the sheaves on $\mathcal{T}_X$ is called sheaves on the topological space X. Actually, this is the original notion of sheaves.

4.b **Example** Let X be a topological space and let $\mathcal{T}'_X$ be the site basically the same as $\mathcal{T}_X$ except it excludes empty coverings. The sheaves on $\mathcal{T}'_X$ are the same as sheaves on the space $X \sqcup \{\eta\}$ whose open sets are the empty set and union of open sets in X with $\{\eta\}$.

5 **(Sheaves with values in a category)** Since the *descent condition* (I.2.1) makes sense for arbitrary category, thus we can easily generalize the notion of sheaves to allow values in an arbitrary category $\mathcal{A}$.

Let $\mathscr{F}$ be a presheaf with values in $\mathcal{A}$. For any $X \in \operatorname{ob}\mathcal{A}$, We define presheaves $\mathscr{F}_X$ as

$$\mathscr{F}_X(U) := \operatorname{Hom}_{\mathcal{A}}(X, \mathscr{F}(U)).$$

Then, the Yoneda lemma tells us that $\mathscr{F}$ is a *sheaf with values in* $\mathcal{A}$ if and only if for all $X \in \operatorname{ob}\mathcal{A}$, $\mathscr{F}_X$ is a sheaf.

5.a Theorem *Presheaves (resp. sheaves) with values in the category* **Ab** *of abelian groups are precisely the abelian group objects in the category of presheaves (resp. sheaves). They are also called* **abelian presheaves** *(resp.* **abelian sheaves***). The category of abelian presheaves (resp. abelian sheaves) is also denoted by* $\mathbf{PAb}(\mathcal{C})$ *(resp.* $\mathbf{Ab}(\mathcal{C})$*).*

5.b Let $\mathcal{A}$ be a **concrete category**, which is a category equipped with a faithful functor $F\colon \mathcal{A} \to$ **Set** called the **forgetful functor**. Then a presheaf with values in $\mathcal{A}$ gives rise to a presheaf of sets $F \circ \mathscr{F}$ called the **underlying presheaf of sets** of $\mathscr{F}$.

In practice, a concrete category often appears as a category of *structured sets*. Sheaves of structured sets can be checked by their underlying presheaves of sets.

5.c Lemma *Let* $\mathcal{A}$ *be a complete concrete category with forgetful functor* F *which commutes with all limits and reflects isomorphisms. Then for any presheaf* $\mathscr{F}$ *with values in* $\mathcal{A}$*,* $\mathscr{F}$ *is a sheaf with values in* $\mathcal{A}$ *if and only if its underlying presheaf of sets is a sheaf.*

Proof: Apply F to the diagram (I.2.1) and one can check the requirements by the properties of F. $\square$

6 (Algebraic categories) A **category of algebraic structures**, or **algebraic category** is a concrete category $\mathcal{A}$ equipped with a forgetful functor F satisfying the following conditions:

1. $\mathcal{A}$ is *complete*, meaning it has limits, and F commutes with limits;
2. $\mathcal{A}$ has shifted colimits and F commutes with them;
3. F reflects isomorphisms.

Remark In fact, the correct definition requires further F admits a left adjoint, called the **free functor**. But this is equivalent to 1. under suitable assumption of sizes. See Adjoint functor theorem.

6.a (Sifted colimits) Recall that a small category $\mathcal{I}$ is said to be **sifted** if it is nonempty and the *diagonal functor* $\Delta\colon \mathcal{I} \to \mathcal{I} \times \mathcal{I}$ is *cofinal*, which means for any objects $i, j \in \operatorname{ob}\mathcal{I}$, the comma category $(\mathrm{const}_{(i,j)} \downarrow \Delta)$ is connected. Note that the objects in $(\mathrm{const}_{(i,j)} \downarrow \Delta)$ are *cospans* $i \to k \leftarrow j$ in $\mathcal{I}$. *Sifted colimits in* **Set** *commutes with finite products.* (ref. §2.12 in *BMO* or [Bor94])

6.b Example The following categories, equipped with the obvious forgetful functor, are algebraic categories:

- The category $*\,\mathbf{Set}$ of pointed sets.
- The category $\mathbf{Ab}$ of abelian groups.
- The category $\mathbf{Grp}$ of groups.
- The category $\mathbf{Mon}$ of monoids.
- The category $\mathbf{Ring}$ of (non-commutative) rings.
- The category $\mathbf{Mod}(R)$ of R-modules over a fixed ring R.
- The category of Lie algebras over a fixed field.

6.c Proposition *Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$. Then f is monic (resp. epic) if so is $F(f)$. Moreover, F reflects monomorphisms.*

Proof: Note that f is monic if and only if its *kernel pair* $X \times_Y X$ is trivial, i.e. $X \to X \times_Y X$ is an isomorphism. □

6.d Lemma *Let $f\colon X \to Y$ and $g\colon X' \to Y$ be two morphisms in $\mathcal{A}$. If $F(g)$ is injective and* $\mathrm{im}(F(f)) \subset \mathrm{im}(F(g))$, *then there exists a morphism $h\colon X \to X'$ such that $f = g \circ h$.*

Proof: Note that the assumptions imply that $F(X) \times_{F(Y)} F(X') = F(X)$. Then the conclusion follows. □

7 Theorem (Equivalent sites provide same sheaves) *Let $\mathcal{C}$ be a category and* $\mathrm{Cov}_1, \mathrm{Cov}_2$ *two Grothendieck pretopologies.*

1. *If each $\mathfrak{U} \in \mathrm{Cov}_1$ is tautologically equivalent to some $\mathfrak{V} \in \mathrm{Cov}_2$, then* $\mathbf{Sh}(\mathcal{C}, \mathrm{Cov}_2) \subset \mathbf{Sh}(\mathcal{C}, \mathrm{Cov}_1)$.
2. *If for each $\mathfrak{U} \in \mathrm{Cov}_1$, there exists a $\mathfrak{V} \in \mathrm{Cov}_2$ refining $\mathfrak{U}$, then* $\mathbf{Sh}(\mathcal{C}, \mathrm{Cov}_2) \subset \mathbf{Sh}(\mathcal{C}, \mathrm{Cov}_1)$.

Proof: Let $\mathfrak{U} = \{U_i \to U\}_{i \in I}$ be a covering in Cov_1 and $\mathfrak{V} = \{V_j \to U\}_{j \in J}$ a refinement of $\mathfrak{U}$ in Cov_2 given by the map $\alpha\colon J \to I$ and the morphisms $f_j\colon V_j \to U_{\alpha(j)}$. Let $\mathscr{F}$ be a presheaf on $\mathcal{C}$. We need to show that the *descent condition* (I.2.1) for $\mathscr{F}$ with respect to all coverings in Cov_2 implies the one with respect to $\mathfrak{U}$.

The uniqueness is easy to prove. Indeed, let $s, s' \in \mathscr{F}(U)$ such that $s|_{U_i} = s'|_{U_i}$ for all $i \in I$. Then we also have $s|_{V_j} = s'|_{V_j}$ for all V_j. Thus $s = s'$ by the descent condition respect to $\mathfrak{V}$.

Now we turn to the gluing condition. Let $s_i \in \mathscr{F}(U_i)$ be a family of sections satisfying $s_i|_{U_i \times_U U_{i'}} = s_{i'}|_{U_i \times_U U_{i'}}$ for all $i, i' \in I$. Let $s_j := \mathscr{F}(f_j)(s_{\alpha(i)}) \in \mathscr{F}(V_j)$. Then from the following Cartesian diagrams,

$$\begin{tikzcd}
V_j \times_U V_{j'} \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & V_{j'} \arrow[d] \\
\cdot \arrow[r] \arrow[d] & U_{\alpha(j)} \times_U U_{\alpha(j')} \arrow[r] \arrow[d] & U_{\alpha(j')} \arrow[d] \\
V_j \arrow[r] & U_{\alpha(j)} \arrow[r] & U
\end{tikzcd}$$

we obtain $s_j|_{V_j \times_U V_{j'}} = s_{j'}|_{V_j \times_U V_{j'}}$. By the descent condition respect to $\mathfrak{V}$, there exists a section $s \in \mathscr{F}(U)$ such that $s_j = s|_{V_j}$ for all $j \in J$. We remain to show that $s_i = s|_{U_i}$ for all $i \in I$.

Now we have to consider some other coverings. Let $i_0 \in I$, then $\mathfrak{U}' = \{U_i \times_U U_{i_0} \to U_{i_0}\}_{i \in I}$ is a covering in Cov_1 and $\mathfrak{V}' = \{V_j \times_U U_{i_0} \to U_{i_0}\}_{j \in J}$ is a covering in Cov_2 which refines $\mathfrak{U}'$ via α and $f'_j := f_j \times \mathrm{id}_{U_0}$. Then consider $s_{i_0}|_{V_j \times_U U_{i_0}}$ given by the composition

$$\mathscr{F}(U_{i_0}) \longrightarrow \mathscr{F}(U_{\alpha(j)} \times_U U_{i_0}) \longrightarrow \mathscr{F}(V_j \times_U U_{i_0}).$$

Since $s_i|_{U_i \times_U U_{i_0}} = s_{i_0}|_{U_i \times_U U_{i_0}}$ for all $i \in I$ and $s_j = \mathscr{F}(f_j)(s_{\alpha(i)})$, we have $s_{i_0}|_{V_j \times_U U_{i_0}} = s_j|_{V_j \times_U U_{i_0}}$ for all $j \in J$. Now, from the following Cartesian diagrams,

$$\begin{tikzcd}
V_j \times_U U_{i_0} \arrow[r] \arrow[d] & U_{\alpha(j)} \times_U U_{i_0} \arrow[r] \arrow[d] & U_{i_0} \arrow[d] \\
V_j \arrow[r] & U_{\alpha(j)} \arrow[r] & U
\end{tikzcd}$$

we have $s_{i_0}|_{V_j \times_U U_{i_0}} = s|_{U_{i_0}}\big|_{V_j \times_U U_{i_0}}$ for all $j \in J$. Hence $s_{i_0} = s|_{U_{i_0}}$. $\square$

§ I.3 Sheaves on topological spaces

We have seen sheaves on topological spaces in Example 2.4.a. Now we discuss something more about them.

Sheaves and stalks

1 (Sheaves) Recall that a presheaf on a topological space X is called a **sheaf** if it satisfies the *descent condition*. By the description of limits in **Set**, this means for any covering $\{U_i \subset U\}_{i\in I}$,

1. the canonical map $\mathscr{F}(U) \to \prod_{i\in I} \mathscr{F}(U_i)$ is injective,
2. the image of $\mathscr{F}(U)$ under the canonical map equals

$$\left\{(s_i) \in \mathscr{F}(U_i) \middle| \forall i_0, i_1 \in I\ s_{i_0}|_{U_{i_0}\cap U_{i_1}} = s_{i_1}|_{U_{i_0}\cap U_{i_1}}\right\}.$$

More elementarily, the descent condition equals the following two axioms:

Identity axiom For any sections $s, s' \in \mathscr{F}(U)$, if there exists a covering $\{U_i \subset U\}_{i\in I}$ such that $s|_{U_i} = s'|_{U_i}$, then $s = s'$.

Gluing axiom For any **system of compatible sections** $(s_i)_{i\in I}$ respect to a covering $\{U_i \subset U\}_{i\in I}$, namely an element $(s_i) \in \prod_{i\in I} \mathscr{F}(U_i)$ satisfying $s_{i_0}|_{U_{i_0}\cap U_{i_1}} = s_{i_1}|_{U_{i_0}\cap U_{i_1}}$ for all $i_0, i_1 \in I$.

In this case, we call the image of inclusion maps under a presheaf $\mathscr{F}$ the **restriction maps**.

1.a (global sections) Let $\mathscr{F}$ be a presheaf on a topological space X. Since X itself is the terminal object in $\mathcal{T}_X$, we have $\mathscr{F}(X) \cong \mathrm{Hom}_{\mathbf{PSh}(X)}(*, \mathscr{F})$. Therefore, we call an element $s \in \Gamma(X, \mathscr{F}) := \mathscr{F}(X)$ a **global section**.

A significant fact about a topological space is that it has points.

2 (Stalks and germs) Let X be a topological space and $x \in X$ be a point. Let $\mathscr{F}$ be a presheaf on X. The **stalk** of $\mathscr{F}$ at x is the colimit

$$\mathscr{F}_x := \varinjlim_{x\in U} \mathscr{F}(U).$$

2.a Remark (Taking stalk is exact) One can see this gives rise to a functor

$$\begin{aligned} \Gamma_x\colon \mathbf{PSh}(X) &\longrightarrow \mathbf{Set} \\ \mathscr{F} &\longmapsto \mathscr{F}_x \end{aligned}$$

called the **stalk functor**.

Moreover, note that the system of neighborhoods of x is filtered, thus the stalks are filtered colimits, thus commute with colimits and finite limits. In particular, *the stalk functor is exact.*

It is easy to describe the set $\mathscr{F}_x$. It is the quotient

$$\mathscr{F}_x = \{(U, s) | x \in U, s \in \mathscr{F}(U)\} / \sim$$

where the equivalence relation $\sim$ is given by $(U, s) \sim (U', s')$ if and only if there exists an open $U'' \subset U \cap U'$ such that $x \in U''$ and that $s|_{U''} = s'|_{U''}$. The equivalence class of (U, s) will be denoted by s_x and called the **germ** of s at x.

From this description, we get a canonical map for every open set $U \subset X$:

$$\begin{aligned} \mathscr{F}(U) &\longrightarrow \prod_{x \in U} \mathscr{F}_x \\ s &\longmapsto \prod_{x \in U} s_x. \end{aligned}$$

2.b Lemma (Sections are determined by germs) *Let X be a topological space. A presheaf $\mathscr{F}$ on X is* separated *if and only if for every open $U \subset X$ the above canonical map is injective.*

Proof: Let $\mathscr{F}$ be a separated presheaf. Let s, s' be two sections of $\mathscr{F}$ on some open set U such that they have the same germ at each $x \in U$. Then, for each $x \in U$, there exists an open neighborhood U_x of x such that $s|_{U_x} = s'|_{U_x}$. Note that $\{U_x \subset U\}_{x \in U}$ is a covering, thus $s = s'$.

Conversely, let $\mathscr{F}$ be a presheaf satisfying the condition in statement. Let $\{U_i \subset U\}_{i \in I}$ be a covering. Let s, s' be two sections of $\mathscr{F}$ on U such that $s|_{U_i} = s'|_{U_i}$ for all $i \in I$. Note that this implies that s and s' have the same germ at every point in each U_i, thus in whole U. Then, by the injectivity of the canonical map, $s = s'$. $\square$

2.c (Compatible germs) Now we turn to the image of the canonical map. The elements in the image are called **systems of compatible germs** of $\mathscr{F}$ over U. To be explicit, a *system of compatible germs* is an element $\prod_{x \in U} s_x \in \prod_{x \in U} \mathscr{F}_x$ such that for any $x \in U$, there exists some representative (U_x, s^x) of s_x with $U_x \subset U$ such that the germ of s^x at any point $y \in U_x$ is s_y.

3 Example (Sheaf of germs) Let X be a topological space. For each $x \in X$, give a set S_x. Then we have a presheaf given by $\mathscr{F}(U) = \prod_{x \in U} S_x$ with obvious restriction maps. This is a sheaf. But, usually $\mathscr{F}_x \neq S_x$. We only have a map $\mathscr{F}_x \to S_x$.

4 (Support of a section) Let $\mathscr{F}$ be an abelian sheaf on X and s be a global section. The **support** of s, denoted by $\mathrm{Supp}(s)$, is the subset of X consisting of points of X where s has nonzero germ:

$$\mathrm{Supp}(s) := \{x \in X | s_x \neq 0 \in \mathscr{F}_x\}.$$

4.a Proposition $\mathrm{Supp}(s)$ *is a closed subset of* X.

Proof: Consider any point y in the closure $\overline{\mathrm{Supp}(s)}$. Then for any neighborhood U of y, there exists a point x contained in $U \cap \mathrm{Supp}(s)$. Then $s_x \neq 0$, thus $s|_U \neq 0$. Vary U through all neighborhood of y, we get $s_y \neq 0$. □

The followings are some important examples.

5 Example (Restriction of a sheaf) Let $\mathscr{F}$ be a sheaf on X and U be an open subset of X. Then there is a natural sheaf $\mathscr{F}|_U$ on U given by $\mathscr{F}|_U(V) := \mathscr{F}(V)$ for all open subsets $V \subset U$, called the **restriction** of $\mathscr{F}$ to U.

6 Example (Skyscraper sheaves) Let X be a topological space with $x \in X$ and S a set. Let $i_x \colon x \to X$ be the inclusion. The **skyscraper sheaf** $i_{x,*}S$ is given by

$$i_{x,*}S(U) = \begin{cases} S & \text{if } x \in U; \\ * & \text{if } x \notin U. \end{cases}$$

with obvious restriction maps. Here $*$ denote a *singleton*, or more generally a *terminal object*.

Note that it may be true that there are nontrivial stalks of a skyscraper sheaf other than the one at x. Indeed, we have

$$(i_{x,*}S)_y = \begin{cases} S & \text{if } y \in \overline{\{x\}}; \\ * & \text{if } y \notin \overline{\{x\}}. \end{cases}$$

One can see that taking skyscraper sheaf at a point is a functor, moreover, we have

6.a Theorem (Stalk is left adjoint to skyscraper sheaf) *Let X be a topological space and $x \in X$. Then there exists a bijection*

$$\mathrm{Hom}_{\mathcal{A}}(\mathscr{F}_x, S) \cong \mathrm{Hom}_{\mathbf{Sh}(X)}(\mathscr{F}, i_{x,*}S)$$

natural in both the sheaf $\mathscr{F}$ of algebraic structures and the algebraic structure $S \in \mathrm{ob}\,\mathcal{A}$.

Proof: Note that stalk functor can be viewed as the *inverse image functor* for the inclusion map $i_x \colon x \to X$, thus this property is a corollary of Theorem 5.3, which will appear later. □

7 Example (Sheaf of continuous maps) Let X, Y be two topological spaces. Define $\mathscr{F}$ as follows. $\mathscr{F}(U)$ consists of all continuous maps from U to Y, and the restriction maps are the obvious ones.

Note that for S a set, regarded as a topological space with discrete topology, the continuous maps to S are precisely the locally constant maps to S. Here we define

7.a Example (Constant sheaves) Let X be a topological space and S be a set. The **constant sheaf** with value S, denoted by $\underline{S}$ or $\underline{S}_X$, is the sheaf of locally constant maps to S.

One may feel confusion about the name. Maybe a constant sheaf should be a presheaf with constant values, i.e. $\mathscr{F}(U) = S$ for all open subsets U. This presheaf is called the **constant presheaf** with value S, denoted by const_S. But this is rarely a sheaf. Even when one remember that the value of a sheaf on the empty set should be a terminal object and modify the definition of const_S, it is still far from being a sheaf.

The relationship between the constant sheaves and constant presheaves will be discussed later.

7.b Example (Representable sheaves) For any open set U of X, the representable presheaf $h_U = \mathrm{Hom}_{\mathcal{T}_X}(-, U)$ is a sheaf.

7.c Example (Sheaf of sections of a map) Let $f\colon Y \to X$ be a continuous map. Define $\mathscr{F}(U)$ to be the set of **sections** of f over U, which are continuous maps $s\colon U \to Y$ such that $f \circ s = \mathrm{id}|_U$.

When Y is further a topological group, one can see that $\mathscr{F}$ is a sheaf of groups.

8 Example (Sheaf of differential functions) Let X be a *differential manifold*. One can consider the sheaf $\mathscr{O}$ of differential functions on X similar as the sheaf of continuous maps. Since functions having same germ at a point are locally the same, it makes sense to define the **value of a germ** s_x at a point x as the value of a representative $s \in \mathscr{F}(U_x)$ at x.

Obviously, $\mathscr{O}_x$ is a ring for all $x \in X$. Moreover, it is a local ring. Let $\mathfrak{m}_x$ denote the ideal of $\mathscr{O}_x$ consisting of germs vanishing at x. Then one can check the following is an exact sequence.

$$0 \longrightarrow \mathfrak{m}_x \longrightarrow \mathscr{O}_x \xrightarrow{f \mapsto f(x)} \mathbb{R} \longrightarrow 0.$$

As any germ in $\mathscr{O}_x \setminus \mathfrak{m}_x$ is invertible, this shows that $\mathscr{O}_x$ is a local ring with the maximal ideal $\mathfrak{m}_x$.

Note that $\mathfrak{m}_x/\mathfrak{m}_x^2$ is a vector space over the residue field $\mathscr{O}_x/\mathfrak{m}_x \cong \mathbb{R}$. This vector space is called the **cotangent space** of X at x.

Sheafification

Given a presheaf $\mathscr{F}$, there is a universal way to get a sheaf. That is the notion of *sheafification*.

Let $\mathscr{F}$ be a preshaef on a topological space X. Then a morphism $\mathscr{F} \to \mathscr{F}^\#$ is called a **sheafification** of $\mathscr{F}$ if for any sheaf $\mathscr{G}$ and presheaf morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$, there exists a unique sheaf morphism $\varphi^\#\colon \mathscr{F}^\# \to \mathscr{G}$ making the following digram commute.

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[dr, "\varphi"'] & \mathscr{F}^{\#} \arrow[d, "\varphi^{\#}"] \\ & \mathscr{G} \end{tikzcd}$$

One can see that the sheafification is unique up to unique isomorphism and that sheafifications, if they exist, give rise to a functor:

$$\begin{aligned} \#\colon \mathbf{PSh}(X) &\longrightarrow \mathbf{Sh}(X) \\ \mathscr{F} &\longmapsto \mathscr{F}^{\#}. \end{aligned}$$

Now, we give the construction. First, recall we have canonical maps $\mathscr{F}(U) \to \prod_{x\in U} \mathscr{F}_x$, which induce a canonical presheaf morphism

$$\mathscr{F} \longrightarrow \Pi(\mathscr{F}),$$

here $\Pi(\mathscr{F})$ is defined as $\Pi(\mathscr{F})(U) := \prod_{x\in U} \mathscr{F}_x$. Like in Example 3, this presheaf is a sheaf but usually $\mathscr{F}_x \neq (\Pi\mathscr{F})_x$.

9 Lemma (Sheafification through stalks) *The sheafification $\mathscr{F}^{\#}$ of $\mathscr{F}$ is given by the fibre product in the following Cartesian diagram:*

$$\begin{tikzcd} \mathscr{F}^{\#} \arrow[r] \arrow[d] & \Pi(\mathscr{F}) \arrow[d] \\ \Pi(\mathscr{F}) \arrow[r] & \Pi(\Pi(\mathscr{F})) \end{tikzcd}$$

where the right vertical map is the canonical map for the sheaf $\Pi(\mathscr{F})$, while the bottom horizontal map come from the maps

$$\prod_{x\in U} \mathscr{F}_x \longrightarrow \prod_{x\in U} (\Pi(\mathscr{F}))_x,$$

which is the product of the canonical maps for the presheaf $\mathscr{F}$.

Proof: The universal property of fibre product gives us a canonical morphism $\theta\colon \mathscr{F} \to \mathscr{F}^{\#}$. Then the only things one needs to verify are *1,* $\mathscr{F}^{\#}$ is a sheaf;. *2,* $\mathscr{F}$ is a sheaf if and only if θ is an isomorphism. If so, then the functorality of Π implies the functorality of $\#$ and the universal property of sheafification follows from this and *2*.

1. By *limits commutes with limits*, we can deduce the descent condition of the fibre product $\mathscr{F}^{\#}$ from those of $\Pi(\mathscr{F})$ and $\Pi(\Pi(\mathscr{F}))$. Therefore $\mathscr{F}^{\#}$ is a sheaf.

2. We show that $\mathscr{F}^{\#}(U)$ is precisely the set of all *systems of compatible germs* of $\mathscr{F}$ over U.

For any pair (t, t') with $t, t' \in \Pi(\mathscr{F})(U)$ which have same image in $\Pi(\Pi(\mathscr{F}))$ through the bottom horizontal map and the right vertical map respectively, let $t = (s_x)_{x \in U}$, $t' = (s'_x)_{x \in U}$ and their images in $\mathscr{F}(U)$ be $(t_x)_{x \in U}$ and $(t'_x)_{x \in U}$. For each s_x, let (U_x, s^x) be a representative of it, then the germ s^x_y of s^x at each $y \in U_x$ form a representative $(s^x_y)_y$ of t_x. On the other hand, (U, t') is definitely a representative of t'_x. Then since $t_x = t'_x$, there exists a neighborhood U'_x of x such that $s^x_y = s'_y$ and thus that $s_x = s'_x$.

We have seen in 2.c that $\mathscr{F}^\#$ is precisely the image of $\mathscr{F}$ in $\Pi(\mathscr{F})$, thus θ is an isomorphism when $\mathscr{F}$ is a sheaf. □

9.a Lemma (Sheafification preserves stalks) *Let $\mathscr{F}$ be a presheaf on a topological space X, then for any $x \in X$, $\mathscr{F}_x \cong \mathscr{F}^\#_x$.*

Proof: First, let s_x, s'_x be two germs of $\mathscr{F}$ at x sharing the same image under the canonical map $\mathscr{F}_x \to \mathscr{F}^\#_x$. Then, for some representatives (U, s) (resp.(U', s')) of s_x (resp. s'_x), we have $(U, (s_y)) \sim (U', (s'_y))$ in $\mathscr{F}^\#_x$. Then there exists a neighborhood U'' of x contained in $U \cap U'$ such that $(s_y)|_{U''} = s'_y|_{U''}$, i.e. $s_y = s'_y$ for all $y \in U''$. Particularly, $s_x = s'_x$.

To show the surjectivity, consider a germ $\bar{t} \in \mathscr{F}^\#_x$. Taking any representative (U, t) of this germ, then $t = (s_y)$ is a system of compatible germs of $\mathscr{F}$ over U. Therefore, there exists a representative (V, s^x) of s_x such that $V \subset U$ and the germ of s^x at any point $y \in V$ is s_y. Thus $t|_V$ is the image of s^x under the canonical map $\mathscr{F}(V) \to \mathscr{F}^\#(V)$. Then passing to the stalks, the germ of s^x at x will be a preimage of $\bar{t}$. □

Proof: A more simple proof is this: apply Π to the Cartesian diagram in Lemma 9, then we get $\Pi(\mathscr{F}^\#) = \Pi(\mathscr{F})^\# \cong \Pi(\mathscr{F})$. □

10 Proposition (Sheafification is free) *The sheafification functor $\#$ is* left adjoint *to the forgetful functor from sheaves on X to presheaves on X.*

This is nothing but the universal property of sheafification. But its corollary is very useful.

10.a Corollary *The sheafification preserves colimits and hence is* right exact*; the forgetful functor preserves limits and hence is* left exact*. In particular, the presheaf kernel is already the sheaf kernel.*

Proof: This follows from the property of adjoint functors. One can refer my note *BMO* or [Bor94] directly. □

11 Example (Constant sheaves) Let S be a set. Then the constant sheaf $\underline{S}$ is precisely the sheafification of the constant presheaf const_S. Indeed, define maps $S \to \underline{S}(U)$ by mapping $s \in S$ to the constant map $x \mapsto s$ for all $x \in U$. Then we get a morphism $\mathrm{const}_S \to \underline{S}$, which induces a morphism $\mathrm{const}_S^\# \to \underline{S}$. One can see this is an isomorphism.

The notion of sheafification works for presheaves of algebraic structures. One can just follow the above with a slight modification. Or, one can use the following lemma.

12 Lemma (Sheafification of presheaves of algebraic structures) *Let $\mathscr{F}$ be a presheaf on a topological space X with values in an algebraic category $\mathcal{A}$. Then there exists a unique morphism $\mathscr{F} \to \mathscr{F}^{\#}$ of presheaves with values in $\mathcal{A}$ such that the corresponding morphism of underlying presheaves of sets is a sheafification and that it satisfying the universal property of a sheafification.*

Proof: The main idea is to define $\mathscr{F}^{\#}(U)$ as the fibre product:

$$\begin{tikzcd} \mathscr{F}^{\#}(U) \arrow[r] \arrow[d] & \prod(\mathscr{F})(U) \arrow[d] \\ \prod_{x\in U} \mathscr{F}_x \arrow[r] & \prod_{x\in U} \prod(\mathscr{F})_x \end{tikzcd}$$

Now, we check the conditions. First, apply the forgetful functor F to the above Cartesian diagram. Then the first statement follows. Next, let $\mathscr{G}$ be a sheaf on X with values in $\mathcal{A}$ and $\varphi\colon \mathscr{F} \to \mathscr{G}$ a presheaf morphism. Then the following diagram satisfies the assumptions in Lemma 2.6.d:

$$\begin{tikzcd} \mathscr{F}^{\#}(U) \arrow[d] & \mathscr{G}(U) \arrow[d] \\ \prod(\mathscr{F})(U) \arrow[r] & \prod(\mathscr{G})(U) \end{tikzcd}$$

the underlying map of the right vertical morphism is injective since $\mathscr{G}$ is a sheaf; the image of the composition of left and bottom morphism lies in the image of the right vertical morphism since the sections in $\mathscr{F}^{\#}(U)$ are systems of compatible germs and there already exists a morphism $\mathscr{F} \to \mathscr{G}$. Thus, by Lemma 2.6.d, there exists a morphism $\mathscr{F}^{\#}(U) \to \mathscr{G}(U)$ making the diagram commute. The uniqueness of such a morphism comes from the injectivity of the right vertical morphism. $\square$

Sheaves on bases

Let X be a topological space and $\mathcal{B}$ a *base* of it. Recall that a **base** for the topology on X is a full subcategory $\mathcal{B}$ of $\mathcal{T}_X$ such that every object of $\mathcal{T}_X$, i.e. open subset of X, is a colimit of diagrams in $\mathcal{B}$, i.e. a union of sets in $\mathcal{B}$. Moreover, $\mathcal{B}$ inherits a *coverage* (not a Grothendieck pretopology since pullbacks do not exist in $\mathcal{B}$) from $\mathcal{T}_X$, thus becomes a site. Now, we can define the notion of **(pre)sheaves on** $\mathcal{B}$ as (pre)sheaves on the site $\mathcal{B}$.

Let x be a point in X, then the **stalk** of a (pre)sheaf $\mathscr{F}$ on $\mathcal{B}$ at x is the colimit

$$\mathscr{F}_x := \varinjlim_{x\in U, U\in\mathcal{B}} \mathscr{F}(U).$$

We still call the elements in $\mathscr{F}_x$ **germs** at x. Note that the notion of *compatible germs* still works for (pre)sheaves on $\mathcal{B}$ and since neighborhoods of x in $\mathcal{B}$ are *cofinal* in the system of neighborhoods of x, one can actually define *compatible germs* for arbitrary subsets of X. Next, one can define and state the notions and facts about (pre)sheaves and stalks like on a topological space.

From the inclusion functor $\mathcal{B} \to \mathcal{T}_X$, we get two canonical functors $\mathbf{PSh}(X) \to \mathbf{PSh}(\mathcal{B})$ and $\mathbf{Sh}(X) \to \mathbf{Sh}(\mathcal{B})$. However, since we can not glue things from smaller subsets, the notion of presheaves on base and on topological space are very different. Luckily, things are good for sheaves.

On one hand, any sheaf $\mathscr{F}$ on X can be viewed as a sheaf on $\mathcal{B}$ through the canonical functor $\mathbf{Sh}(X) \to \mathbf{Sh}(\mathcal{B})$. In this case, since neighborhoods in $\mathcal{B}$ are *cofinal* in those in $\mathcal{T}_X$, the stalks of $\mathscr{F}$, viewed as a sheaf on $\mathcal{B}$ and on X respectively, are the same.

On the other hand, for any sheaf $\mathscr{F}$ on $\mathcal{B}$, its stalks induces a sheaf $\Pi(\mathscr{F})$ on X:

$$\Pi(\mathscr{F})(U) := \prod_{x \in U} \mathscr{F}_x.$$

Then the fibre product $\mathscr{F}^{\#} := \Pi(\mathscr{F}) \times_{\Pi(\Pi(\mathscr{F}))} \Pi(\mathscr{F})$ defines a functor

$$\#\colon \mathbf{Sh}(\mathcal{B}) \longrightarrow \mathbf{Sh}(X).$$

One can see this sheaf $\mathscr{F}^{\#}$ is just the *sheaf of compatible germs of* $\mathscr{F}$.

13 Theorem *The functor $\#$ is a weak inverse of the canonical functor from $\mathbf{Sh}(X)$ to $\mathbf{Sh}(\mathcal{B})$. In other words, it provides an equivalence between the two categories. Moreover, $\#$ commutes with taking stalks, i.e. there are canonical bijection*

$$\mathscr{F}_x = \mathscr{F}^{\#}_x$$

for all $x \in X$.

Remark The notions and statements also work for sheaves of algebraic structures.

§ I.4 Sheaves on topological spaces: morphisms

Now we turn to consider the category $\mathbf{Sh}(X)$ of sheaves on a topological space X.

Morphisms of sheaves

We start with the following important fact.

1 **Lemma (Morphisms are determined by stalks)** *Let φ_1, φ_2 be two morphisms from a presheaf $\mathscr{F}$ to a sheaf $\mathscr{G}$, which induce the same maps on every stalk. Then $\varphi_1 = \varphi_2$.*

Proof: Consider following commutative diagrams:

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi_i(U)"] \arrow[d] & \mathscr{G}(U) \arrow[d, hook] \\ \prod_{x\in U} \mathscr{F}_x \arrow[r] & \prod_{x\in U} \mathscr{G}_x \end{tikzcd}$$

where the vertical morphisms are the canonical maps. Then, since the right canonical map is injective, $\varphi_1 = \varphi_2$. $\square$

2 **Proposition** *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Then φ is a monomorphism (resp. epimorphism, isomorphism) if and only if for all $x \in X$, $\varphi_x\colon \mathscr{F}_x \to \mathscr{G}_x$ is injective (resp. surjective, bijective).*

Proof: The "if" part for monomorphisms and epimorphisms follow from Lemma 1, while the "only if" part follow from the exactness of the stalk functor. As for the isomorphisms, let $\psi_x\colon \mathscr{G}_x \to \mathscr{F}_x$ be the inverse of each $\varphi_x\colon \mathscr{F}_x \to \mathscr{G}_x$ induced by φ. We need to glue them into a morphism $\psi\colon \mathscr{G} \to \mathscr{F}$. To do this, consider the following diagram.

$$\begin{tikzcd} \mathscr{G}(U) \arrow[d, hook] & \mathscr{F}(U) \arrow[d, hook] \\ \prod_{x\in U} \mathscr{G}_x \arrow[r, "\prod \psi_x"] & \prod_{x\in U} \mathscr{F}_x \end{tikzcd}$$

Since the vertical canonical maps are injective and the bottom map is bijective, the assumptions of Lemma 2.6.d are satisfied. Then there exists a map $\mathscr{F}(U) \to \mathscr{G}(U)$ making the diagram commute. The commutativity also implies that those maps form a morphism $\psi\colon \mathscr{G} \to \mathscr{F}$ of sheaves. Since for all $x \in X$, $\varphi_x \circ \psi_x = \mathrm{id}, \psi_x \circ \varphi_x = \mathrm{id}$, by Lemma 1, $\varphi \circ \psi = \mathrm{id}, \psi \circ \varphi = \mathrm{id}$. This shows that φ is an isomorphism. $\square$

Proof: Note that saying for all $x \in X$, $\varphi_x \colon \mathscr{F}_x \to \mathscr{G}_x$ is bijective is equal to say $\Pi(\varphi) \colon \Pi(\mathscr{F}) \to \Pi(\mathscr{G})$ is an isomorphism. In this case, let $\overline{\psi}$ be the inverse of $\Pi(\varphi)$. Then we have the following commutative diagram

$$\begin{tikzcd}
\mathscr{G} \arrow[rr] \arrow[dr, dashed, "\exists!\psi"] \arrow[dd] & & \Pi(\mathscr{G}) \arrow[dr, "\overline{\psi}"] & & \\
& \mathscr{F} \arrow[rr] \arrow[dr, "\varphi"] \arrow[dd] & & \Pi(\mathscr{F}) \arrow[dr, "\Pi(\varphi)"] \arrow[dd] & \\
\Pi(\mathscr{G}) \arrow[dr, "\overline{\psi}"'] & & \mathscr{G} \arrow[rr] \arrow[dd] & & \Pi(\mathscr{G}) \arrow[dd] \\
& \Pi(\mathscr{F}) \arrow[rr] \arrow[dr, "\Pi(\varphi)"'] & & \Pi(\Pi(\mathscr{F})) \arrow[dr, "\Pi(\Pi(\varphi))"] & \\
& & \Pi(\mathscr{G}) \arrow[rr] & & \Pi(\Pi(\mathscr{G}))
\end{tikzcd}$$

where the front two layers are obtained by apply the canonical Cartesian diagram for sheafification to the morphism φ. Then the universal property of fibre product implies the unique existence of the inverse ψ of φ. ☐

2.a Example (Distinct sheaves may have isomorphic stalks) Note that Proposition 2 doesn't imply that sheaves having isomorphic stalks are isomorphic. This is because the isomorphisms for stalks may not be induced from a morphism of sheaves.

For instance, let X be the set $\{a, b\}$ with the topology $\{X, U = \{a\}, \varnothing\}$. Define $\mathscr{F}$ and $\mathscr{G}$ as $\mathscr{F}(X) = \mathscr{F}(U) = \mathscr{G}(X) = \mathscr{G}(U) = X$ with the restriction maps $\mathscr{F}(X) \to \mathscr{F}(U)$ the identity and $\mathscr{G}(X) \to \mathscr{G}(U)$ mapping both a and b to a. Then $\mathscr{F}$ and $\mathscr{G}$ are non-isomorphic sheaves with the same stalks.

3 Proposition (Subsheaves) *Let $\varphi \colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Then the followings are equivalent.*

1. *φ is monic;*
2. *for all $x \in X$, $\varphi_x \colon \mathscr{F}_x \to \mathscr{G}_x$ is injective;*
3. *for all open subsets $U \subset X$, $\varphi(U) \colon \mathscr{F}(U) \to \mathscr{G}(U)$ is injective.*

If this is the case, $\mathscr{F}$ is called a **subsheaf** *of $\mathscr{G}$.*

Proof: We have seen *1*⇔*2* before. *3*⇒*1* is obvious. It remains to show *2*⇒*3*. Suppose *2* and consider the following diagram.

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, hook] & \mathscr{G}(U) \arrow[d, hook] \\
\prod_{x \in U} \mathscr{F}_x \arrow[r] & \prod_{x \in U} \mathscr{G}_x
\end{tikzcd}$$

Since the vertical canonical maps and the bottom map are injective, $\varphi(U)$ must be also injective. □

4 Proposition (quotient sheaves) *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Then the followings are equivalent.*

1. *φ is epic;*
2. *for all $x \in X$, $\varphi_x\colon \mathscr{F}_x \to \mathscr{G}_x$ is surjective;*
3. *for any open subset $U \subset X$ and any section $s \in \mathscr{G}(U)$, there exists a covering $\{U_i \to U\}$ such that $s|_{U_i}$ lies in the image of $\mathscr{F}(U_i) \to \mathscr{G}(U_i)$.*

If this is the case, $\mathscr{G}$ is called a **quotient sheaf** *of $\mathscr{F}$.*

Proof: We have seen *1⇔2* before. Suppose *3* and let $\psi_1, \psi_2\colon \mathscr{G} \rightrightarrows \mathscr{T}$ be two parallel morphisms of sheaves such that $\psi_1 \circ \varphi = \psi_2 \circ \varphi$. We need to show $\psi_1 = \psi_2$. For any open subset $U \subset X$ and any section $s \in \mathscr{G}(U)$, there exists a covering $\{U_i \to U\}$ and for each i, there exists a section $t_i \in \mathscr{F}(U_i)$ such that $\varphi(U_i)(t_i) = s|_{U_i}$. Then

$$\begin{aligned}\psi_1(U)(s)|_{U_i} &= \psi_1(U_i)(s|_{U_i}) = \psi_1\varphi(U_i)(t_i) \\ &= \psi_2\varphi(U_i)(t_i) = \psi_2(U_i)(s|_{U_i}) = \psi_2(U)(s)|_{U_i}\end{aligned}$$

Since $\{U_i \to U\}$ is a covering, this shows $\psi_1(U)(s) = \psi_2(U)(s)$. Thus $\psi_1 = \psi_2$.

Now suppose *2*. For any open subset $U \subset X$ and any section $s \in \mathscr{G}(U)$, consider its germs s_x at each point $x \in U$. Since φ_x are surjective, there exists some $t_x \in \mathscr{F}_x$ such that $\varphi_x(t_x) = s_x$. Let (U_x, t^x) be a representative of t_x such that $U_x \subset U$. Then $\varphi(U_x)(t^x)$ and s have the same germ s_x at x. Thus we can shrink U_x so that $\varphi(U_x)(t^x) = s|_{U_x}$. Note that $\{U_x \subset U\}_{x\in U}$ is a covering, this shows *3*. □

Remark Recall the notions of injective and surjective presheaf morphisms, we find that a sheaf morphism is monic if and only if its underlying presheaf morphism is injective, in other word, the forgetful functor is left exact. However, the similar statement fails to be true for epimorphisms.

All the above statements can be easily generalized to *sheaves of algebraic structures*. One can also use the following lemma.

5 Lemma (Morphisms of sheaves of algebraic structures) *Let $\mathscr{F}, \mathscr{G}$ be two sheaves on a topological space X with values in an algebraic category $\mathcal{A}$. Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of underlying sheaves of sets. If for every $x \in X$, $\varphi_x\colon \mathscr{F}_x \to \mathscr{G}_x$ defines a morphism in $\mathcal{A}$, then φ defines a morphism of sheaves with values in $\mathcal{A}$.*

Proof: Consider the following commutative digram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, hook] & \mathscr{G}(U) \arrow[d, hook] \\
\prod_{x\in U}\mathscr{F}_x \arrow[r] & \prod_{x\in U}\mathscr{G}_x
\end{tikzcd}$$

in which all maps but $\varphi(U)$ define morphisms in $\mathcal{A}$. By Lemma 2.6.d and the uniqueness of $\varphi(U)$, it must also defines a morphism in $\mathcal{A}$. □

Proof: As all φ_x defines morphisms in $\mathcal{A}$, we get two morphisms of sheaves with values in $\mathcal{A}$ whose underlying morphisms of sheaves of sets are $\Pi(\varphi)$ and $\Pi(\Pi(\varphi))$. Then, consider the following commutative diagram of sheaves of sets and the corresponding one of sheaves with values in $\mathcal{A}$.

$$\begin{tikzcd}
\mathscr{F} \arrow[rr] \arrow[dr, dashed, "\varphi"] \arrow[dd] & & \Pi(\mathscr{F}) \arrow[dr, "\Pi(\varphi)"] \arrow[dd] & \\
& \mathscr{G} \arrow[rr] \arrow[dd] & & \Pi(\mathscr{G}) \arrow[dd] \\
\Pi(\mathscr{F}) \arrow[rr] \arrow[dr, "\Pi(\varphi)"'] & & \Pi(\Pi(\mathscr{F})) \arrow[dr, "\Pi(\Pi(\varphi))"] & \\
& \Pi(\mathscr{G}) \arrow[rr] & & \Pi(\Pi(\mathscr{G}))
\end{tikzcd}$$

Then, by the universal property of fibre products, there exists a unique morphism of sheaves with values in $\mathcal{A}$ whose underlying morphism of sheaves of sets is φ. □

Abelian sheaves form an abelian category

Now we turn to look at abelian sheaves.

Since presheaves are nothing but contravariant functor, the category of abelian presheaves, or more generally presheaves with values in an abelian category, form an abelian category, and subpresheaves, quotient presheaves, presheaf kernels, presheaf cokernels and presheaf images are computed open sets by open sets. In other words,

6 Theorem (Abelian presheaves form an abelian category) *Let X be a topological spaces. Then* $\mathbf{PAb}(X)$ *is an abelian category with a family of exact functors* $\{\Gamma(U,-)|U \in \operatorname{ob}\mathcal{T}_X\}$ reflecting the exactness *in the sense that a sequence of abelian presheaves is exact if and only if it is also exact after applying every functor* $\Gamma(U,-)$.

But things are not so easy for abelian sheaves. Since sheafification is left adjoint to the forgetful functor from sheaves to presheaves, it is true that for a sheaf morphism its *presheaf kernel* is already the **sheaf kernel**. The problem is the cokernel.

7 Example (Holomorphic logarithms) Let X be the complex plane, $\underline{\mathbb{Z}}$ the constant sheaf with values $\mathbb{Z}$, $\mathscr{O}_X$ the sheaf of *holomorphic functions*, and $\mathscr{F}$ the presheaf of functions admitting a *holomorphic logarithm*. Then there is an exact sequence of abelian presheaves on X:

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\times 2\pi i} \mathscr{O}_X \xrightarrow{\exp} \mathscr{F} \longrightarrow 0.$$

However, $\mathscr{F}$ is not a sheaf since there are functions that don't have a logarithm but locally have a logarithm. For instance, the function $f(z) = z$ has no logarithm in an annular region round 0, while it has logarithm in any simply connected part of this region.

So the presheaf cokernel $\operatorname{coker}^p \varphi$ is not automatically the sheaf cokernel in general. But, from the universal properties of cokernel and sheafification, the **sheaf cokernel** $\operatorname{coker}\varphi$ should be the sheafification of the *presheaf cokernel*.

7.a Example (Holomorphic logarithms) We turn back to Example 7. Now we have known that the correct cokernel of the map $\underline{\mathbb{Z}} \to \mathscr{O}_X$ should be the sheafification of $\mathscr{F}$. Now we describe it.

Let $\mathscr{O}_X^*$ denote the presheaf of *invertible* (nowhere zero) holomorphic functions. One can see it is a sheaf of abelian groups under multiplications.

Here we have a theorem

> *a holomorphic function f on a simply connected domain D is invertible if and only if f has logarithm on D.*

Indeed, the logarithm is given by the integral

$$\log f(z) := \int_\gamma \frac{df}{f},$$

where γ is a path from a fixed point z_0 to z in D. Refer [Pri03] for more details.

Now, for each germ of $\mathscr{F}$ at a point x, which is also germ of $\mathscr{F}^\#$ at x, one can always find a representative of it in a simply connected neighborhood of x. In this way, we have $\mathscr{F}_x = \mathscr{O}_x^*$. As we have seen that $\mathscr{O}_X^*$ is a sheaf, it is the sheafification of $\mathscr{F}$.

Consequently, there is an exact sequence:

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\times 2\pi i} \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^* \longrightarrow 1.$$

Now we summarize the results about abelian sheaves into the following theorem.

8 Theorem (Abelian sheaves form an abelian category) *Let X be a topological spaces. Then* $\mathbf{Ab}(X)$ *is an abelian category with a family of exact functors* $\{\Gamma_x | x \in X\}$ reflecting the exactness *in the sense that a sequence of abelian sheaves is exact if and only if it is also exact after applying every functor* Γ_x.

Proof: We have seen the statement is trivially true for presheaves. As for the statement about sheaves, we only need to show that the stalks reflect sheaf kernels and cokernels. Since the presheaf kernels are already sheaf kernels, our claim about kernels follows from the exactness of colimits (note that stalks are colimits). As for the cokernel, just note that the stalks of the sheafification of a presheaf are equal to the stalks of itself, then the argument for kernel works. $\square$

These theorems, as with as similar statements we have seen before, can be summarized into the following slogan:

> *Presheaves can be checked at the level of open sets, while sheaves at the level of stalks.*

8.a Corollary (Section functor is left exact) *The section functor* $\Gamma(U, -)$ *on* $\mathbf{Ab}(X)$ *is left exact, but is not exact.*

Limits and colimits

As in the general case, the category $\mathbf{PSh}(X)$ of *presheaves is complete and cocomplete*, and that *the limits and colimits are computed open sets by opensets* (refer Proposition 1.6). As for the stalks, one can easily check that *taking stalks commutes with all colimits and all finite limits.* But note that taking stalks in general can *NOT* commute with an arbitrary limit.

The following proposition is obvious.

9 Proposition (Limits and colimits in Sh(X)) *Let X be a topological space.*

1. $\mathbf{Sh}(X)$ *is complete and cocomplete.*
2. *The forgetful functor* $F\colon \mathbf{Sh}(X) \to \mathbf{PSh}(X)$ *commutes with all limits. In particular, the section functors* $\Gamma(U, -)$ *are left exact.*
3. *The forgetful functor* $F\colon \mathbf{Sh}(X) \to \mathbf{PSh}(X)$ *does* NOT *commute with colimits. However, we have*
$$\varinjlim \mathscr{F}_i = \left(\varinjlim F(\mathscr{F}_i)\right)^{\#}.$$
4. *The sheafification* $\#\colon \mathbf{PSh}(X) \to \mathbf{Sh}(X)$ *commutes with all colimits and all finite limits. In particular, the stalk functors* Γ_x *are exact.*

Remark Recall that in a functor between finite-complete (resp. finite-cocomplete) categories is said to be **left exact** (resp. **right exact**) if it commutes with all finite limits (resp. finite colimits). A functor is said to be **exact** if it is both left exact and right exact. Note that in an abelian category those notions coincide with the usual notion of exact functors since additive functors preserve biproducts.

Recall that a set in a topological space is said to be *quasi-compact* if its every finite covering has a finite subcovering. In category theory, an object $U \in \operatorname{ob} \mathcal{C}$ is said to be *compact* if the functor $\operatorname{Hom}_{\mathcal{C}}(U, -)$ preserves filtered colimits.

10 Lemma (Compactness) *Let X be a topological space. Let $\{\mathscr{F}_j\}_{j \in \operatorname{ob} \mathcal{J}}$ be a filtered system of sheaves of sets. Consider the canonical map*

$$\Phi\colon \varinjlim_{\mathcal{J}} \mathscr{F}_j(U) \longrightarrow (\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U).$$

1. *If all the transition morphisms are injective then Φ is injective.*
2. *If U is quasi-compact, then Φ is injective.*
3. *If U is quasi-compact and all the transition morphisms are injective then Φ is an isomorphism.*
4. *If any covering of U can be refined by some coverings $\{U_i \to U\}_{i \in I}$ with I finite and $U_i \cap U_{i'}$ quasi-compact, then Φ is bijective.*

Proof: *1.* Assume all the transition morphisms are injective. First, the canonical maps

$$\mathscr{F}_j(U) \longrightarrow \prod_{x \in U} \mathscr{F}_{j,x}$$

are injective, thus so is the colimit

$$\varinjlim_{\mathcal{J}} \mathscr{F}_j(U) \longrightarrow \varinjlim_{\mathcal{J}} \prod_{x \in U} \mathscr{F}_{j,x}.$$

It remains to show the canonical map

$$\varinjlim_{\mathcal{J}} \prod_{x \in U} \mathscr{F}_{j,x} \longrightarrow \prod_{x \in U} \varinjlim_{\mathcal{J}} \mathscr{F}_{j,x}$$

is injective.

Let $\bar{s}$ and $\bar{t}$ be two elements of $\varinjlim_{\mathcal{J}} \prod_{x \in U} \mathscr{F}_{j,x}$ having the same image in $\prod_{x \in U} \varinjlim_{\mathcal{J}} \mathscr{F}_{j,x}$ and $s = (s_x), t = (t_x)$ be their representatives in some $\prod_{x \in U} \mathscr{F}_{j,x}$. Now, the image of $\bar{s}$ and $\bar{t}$ in $\prod_{x \in U} \varinjlim_{\mathcal{J}} \mathscr{F}_{j,x}$ can be written as $(\bar{s_x})$ and $(\bar{t_x})$, where each $\bar{s_x}$ or $\bar{t_x}$ is the image of s_x or t_x in $\varinjlim_{\mathcal{J}} \mathscr{F}_{j,x}$.

Since $\bar{s_x} = \bar{t_x}$, there exists some $j_x \in \operatorname{ob} \mathcal{J}$ such that the image of s_x and t_x in $\mathscr{F}_{j_x,x}$ are the same. Then, since the transition morphism $\mathscr{F}_j \to \mathscr{F}_{j_x}$ is injective, so is the transition map $\mathscr{F}_{j,x} \to \mathscr{F}_{j_x,x}$. Therefore $s_x = t_x$. Then, we get $s = t$ and *a fortiori* $\bar{s} = \bar{t}$.

2. Assume U is quasi-compact. Let $\bar{s}$ and $\bar{t}$ be two elements of $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$ having the same image under Φ and s, t be their representatives in some $\mathscr{F}_j(U)$. Now, the image of $\bar{s}$ and $\bar{t}$ under Φ can be written as systems of compatible germs $(\bar{s}_x)$ and $(\bar{t}_x)$. For each $x \in U$, $\bar{s}_x = \bar{t}_x$ implies that there exists some open neighborhood U_x of x such that $\bar{s}|_{U_x} = \bar{t}|_{U_x}$. Then, there exists $j_x \in \operatorname{ob} \mathcal{J}$ such that the image of $s|_{U_x}$ and $t|_{U_x}$ in $\mathscr{F}_{j_x}(U_x)$ are the same. Since U is quasi-compact, the covering $\{U_x \to U\}_{x \in U}$ has a finite subcovering $\{U_i \to U\}_{i \in I}$. For this covering, we can take j_0 to be the index such that there are arrows $j_i \to j_0$. Now, the image of $s|_{U_i}$ and $t|_{U_i}$ in $\mathscr{F}_{j_0}(U_i)$ are the same. Then s and t maps to the same element in $\mathscr{F}_{j_0}(U)$, *a fortiori* in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$.

3. Assume U is quasi-compact and all the transition morphisms are injective. Then Φ is injective. It suffices to show it is surjective. Let $(\bar{s_x})$ be an element in $(\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U)$, where each $\bar{s_x}$ belongs to the stalk $\varinjlim_{\mathcal{J}} \mathscr{F}_{j,x}$. Then for each $\bar{s_x}$, let s_x be its representative in some $\mathscr{F}_{j_x,x}$ and (U_x, s^x) be the representative of s_x. Note that the image of $s^x|_{U_x \cap U_y}$ and $s^y|_{U_x \cap U_y}$ in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_x \cap U_y)$ have the same image under Φ. Thus, by *1.*, they are the same.

Since U is quasi-compact, the covering $\{U_x \to U\}_{x \in U}$ has a finite subcovering $\{U_i \to U\}_{i \in I}$. For this covering, we can take j_0 to be the index such that there are arrows $j_i \to j_0$. Then the sections $s^i \in \mathscr{F}_{j_i}(U_i)$ give rise to sections $s_i \in \mathscr{F}_{j_0}(U_i)$. Since $s_i|_{U_i \cap U_{i'}}$ and $s_{i'}|_{U_i \cap U_{i'}}$ have the same image in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \cap U_{i'})$, by the similar argument in *1.*, they are the same. Therefore (s_i) is a system of compatible sections, and thus gives a section $s \in \mathscr{F}_{j_0}(U)$ such that $s|_{U_i} = s_i$. Then, this s gives an element $\bar{s}$ of $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$ and one can see it maps to $(\bar{s_x})$ under Φ.

4. Assume the hypothesis of *4.* It is obvious that U is quasi-compact. It suffices to show Φ is surjective. Let $(\bar{s_x})$ be an element in $(\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U)$, where each $\bar{s_x}$ belongs to the stalk $\varinjlim_{\mathcal{J}} \mathscr{F}_{j,x}$. Then for each $\bar{s_x}$, let s_x be its representative in some $\mathscr{F}_{j_x,x}$ and (U_x, s^x) be the representative of s_x.

Now, the covering $\{U_x \to U\}_{x \in U}$ can be refined by a finite covering $\{U_i \to U\}_{i \in I}$ such that $U_i \cap U_{i'}$ are quasi-compact. Since the image of $s^i|_{U_i \cap U_{i'}}$ and $s^{i'}\Big|_{U_i \cap U_{i'}}$ in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \cap U_{i'})$ have the same image under Φ, by *2.*, they are the same and thus there exists $j_{ii'} \in \operatorname{ob} \mathcal{J}$ such that $s^i|_{U_i \cap U_{i'}}$ and $s^{i'}\Big|_{U_i \cap U_{i'}}$ have the same image in $\mathscr{F}_{j_{ii'}}(U_i \cap U_{i'})$.

Now, we can take j_0 to be the index such that there are arrows $j_{ii'} \to j_0$. Then the sections $s^i \in \mathscr{F}_{j_i}(U_i)$ give rise to sections $s_i \in \mathscr{F}_{j_0}(U_i)$ and furthermore, they form a system of compatible sections. Thus we get a section $s \in \mathscr{F}_{j_0}(U)$ such that $s|_{U_i} = s_i$. Then, this s gives an element $\bar{s}$ of $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$ and one can see it maps to $(\bar{s_x})$ under Φ. □

The conditions are necessary. The following is a counterexample.

10.a Example Let $X = I \cup \mathbb{N}$, where $I = \{x_1, \cdots, x_k\}$ is a finite set. Given a topology on X as following: U is an open subset if and only if it is a subset of $\mathbb{N}$ or a union of $\mathbb{N}$ with some subset of I. Write $n \in \mathbb{N}$ as ξ_n. Let $U_n = \{\xi_n, \xi_{n+1}, \cdots\}$ and $j_n\colon U_n \to X$ be the inclusions. Set $\mathscr{F}_n = j_{n,*}\underline{S}$ (refer 5.1 and 3.7.a) and transition morphisms induced by inclusions $U_n \to U_m$. This gives a filtered system of sheaves indexed by $\mathbb{N}$. Let $\mathscr{F} = \varinjlim_{\mathbb{N}} \mathscr{F}_n$.

For $m < n$, we have $\mathscr{F}_{n,\xi_m} = *$ since $\{\xi_m\}$ is an open neighborhood of ξ_m missing U_n. Therefore, passing to the colimit, we have $\mathscr{F}_{\xi_m} = *$ for all $m \in \mathbb{N}$. On the other hand, since for any open neighborhood U of x_i, we have

$$\mathscr{F}_n(U) = \mathscr{F}_n(\mathbb{N}) = \underline{S}(U_n) = \prod_{m \geqslant n} S,$$

so $\mathscr{F}_{n,x_i} = \prod_{m \geqslant n} S$ and thus $\mathscr{F}_{x_i}$ is the colimit

$$M := \varinjlim_{n \in \mathbb{N}} \prod_{m \geqslant n} S.$$

Therefore $\Pi(\mathscr{F})$ is the product of the *skyscraper sheaves* with value M at the closed points $x_1, \cdots, x_k$. Then, so is $\Pi(\Pi(\mathscr{F}))$ and thus $\mathscr{F}$

Now, $\mathscr{F}(X) = \prod_{i=1}^{k} M$, while $\varinjlim_{\mathbb{N}} \mathscr{F}_n(X) = \varinjlim_{\mathbb{N}} \underline{S}(U_n) = M$.

Gluing sheaves

11 (Sheaf Hom) Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ a sheaf on a topological space X. Define the presheaf $\mathcal{H}om(\mathscr{F}, \mathscr{G})$ by

$$\mathcal{H}om(\mathscr{F}, \mathscr{G})(U) := \mathrm{Hom}_{\mathbf{PSh}(U)}(\mathscr{F}|_U, \mathscr{G}|_U).$$

By the following Lemma 11.a, this is indeed a sheaf, called the **sheaf Hom**.

11.a Lemma (Gluing morphisms) *Let X be a topological space with a covering $\{U_i \subset X\}_{i \in I}$. Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ a sheaf on X. Suppose that there are morphisms*

$$\varphi_i\colon\ \mathscr{F}|_{U_i} \longrightarrow \mathscr{G}|_{U_i}$$

such that for all $i, j \in I$, the restrictions of φ_i and φ_j to $U_i \cap U_j$ are the same morphism $\varphi_{ij}\colon\ \mathscr{F}|_{U_i \cap U_j} \to \mathscr{G}|_{U_i \cap U_j}$. Then there exists a unique morphism

$$\varphi\colon \mathscr{F} \longrightarrow \mathscr{G}$$

whose restriction to each U_i is φ_i.

12 Lemma *There exist a canonical map* $\mathcal{H}om(\mathscr{F},\mathscr{G})_x \to \mathrm{Hom}(\mathscr{F}_x,\mathscr{G}_x)$.

Proof: By the functorality of stalks, there are canonical maps

$$\mathrm{Hom}_{\mathbf{PSh}(U)}(\mathscr{F}|_U, \mathscr{G}|_U) \to \mathrm{Hom}(\mathscr{F}_x,\mathscr{G}_x)$$

for all neighborhood U of x. Thus the existence of required canonical map follows from the universal property of colimits. $\square$

The canonical map is in general *NOT* bijective.

12.a Example (Sheaf Hom doesn't commute with taking stalks) Let X be a topological space and x a non-isolated closed point in X. Let S be a *nontrivial*, meaning neither empty or singleton, set.

First, $\mathcal{H}om(i_{x,*}S,\underline{S})_x \to \mathrm{Hom}((i_{x,*}S)_x,\underline{S}_x)$ *is not surjective.* Indeed, since any section s of $i_{x,*}S$ is trivial away from x, thus so is its image under any morphism $i_{x,*}S \to \underline{S}$. But this implies that the image of s is the trivial section. Thus $\mathcal{H}om(i_{x,*}S,\underline{S})$ is the trivial sheaf and thus $\mathcal{H}om(i_{x,*}S,\underline{S})_x$ is a singleton. On the other hand, $\mathrm{Hom}((i_{x,*}S)_x,\underline{S}_x) = \mathrm{Hom}(S,S)$, which is definitely nontrivial.

Secondly, let $V = X \setminus \{x\}$ and $\mathscr{F}$ be the sheaf satisfying $\mathscr{F}|_V = \underline{S}_V$ and $\mathscr{F}(U) = \varnothing$ if $U \not\subset V$. Then $\mathcal{H}om(\mathscr{F},\mathscr{F})_x \to \mathrm{Hom}(\mathscr{F}_x,\mathscr{F}_x)$ *is not injective.* Indeed, as $\mathscr{F}_x = \varnothing$, so $\mathrm{Hom}(\mathscr{F}_x,\mathscr{F}_x)$ is a singleton. On the other hand, $\mathrm{Hom}(\mathscr{F}|_U, \mathscr{F}|_U)$ is nontrivial, thus so is the colimit $\mathcal{H}om(\mathscr{F},\mathscr{F})_x$.

The sheaf Hom shares many properties of the Hom-set.

13 Lemma *Let $\mathscr{F}$ be a sheaf on a topological space X, then $\mathcal{H}om(\underline{*},\mathscr{F}) \cong \mathscr{F}$.*

Proof: For any $\varphi \in \mathrm{Hom}_{\mathbf{PSh}(U)}(\underline{*}_U, \mathscr{F}|_U)$, its corresponding element in $\mathscr{F}(U)$ is the image of the singleton under $\varphi(U)$. This gives rise to a morphism $\Phi\colon \mathcal{H}om(\underline{*},\mathscr{F}) \to \mathscr{F}$. To show it is an isomorphism, we check it at the level of stalks.

Let φ_x,ψ_x be two germs of $\mathcal{H}om(\underline{*},\mathscr{F})$ at x having the same image under Φ_x. Then taking representatives (U_φ,φ) and (U_ψ,ψ) of φ_x and ψ_x respectively, there exists a neighborhood U_x of x such that $U_x \subset U_\varphi \cap U_\psi$ and that $\varphi(U_\varphi)(*)|_{U_x} = \Phi(U_\varphi)(\varphi)|_{U_x} = \Phi(U_\psi)(\psi)|_{U_x} = \psi(U_\psi)(*)|_{U_x}$. Then for any open subset V of U_x, we have $\varphi(V)(*) = \varphi(U_\varphi)(*)|_V = \psi(U_\psi)(*)|_V = \psi(V)(*)$. Therefore $\varphi|_{U_x} = \psi|_{U_x}$ and thus $\varphi_x = \psi_x$.

Let s_x be any germ of $\mathscr{F}$ at x and take a representative (U,s) of it. Define $\varphi\colon \underline{*}_U \to \mathscr{F}|_U$ as $\varphi(V)(*) = s|_V$ for all open subsets $V \subset U$. Then the germ of φ at x will be mapped to s_x under Φ_x. $\square$

Proof: One can also get the lemma following general nonsense after prove Theorem 15. $\square$

14 Lemma (Sheaf Hom is left exact) *Let $\mathscr{F}$ be a sheaf on a topological space X, then $\mathcal{H}om(\mathscr{F}, -)$ is maps limits to limits and $\mathcal{H}om(-, \mathscr{F})$ is maps colimits to limits.*

Proof: Note that the section functors commute with limits. Then, we can check open set by open set, which is obvious. □

The following theorem exposes why sheaf Hom shares similar properties with the Hom-set.

15 Theorem (Sheaf Hom is the internal Hom) *For any sheaves $\mathscr{F}, \mathscr{G}$ and $\mathscr{H}$ on a topological space X, there is a canonical bijection*

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(\mathscr{F} \times \mathscr{G}, \mathscr{H}) \cong \mathrm{Hom}_{\mathbf{Sh}(X)}(\mathscr{F}, \mathcal{H}om(\mathscr{G}, \mathscr{H})).$$

Proof: Given a morphism $\varphi\colon \mathscr{F} \times \mathscr{G} \to \mathscr{H}$, one gets a family of compatible maps $\varphi(U)\colon \mathscr{F}(U) \times \mathscr{G}(U) \to \mathscr{H}(U)$, thus a compatible family of maps $\psi_U\colon \mathscr{F}(U) \to \mathrm{Hom}(\mathscr{G}(U), \mathscr{H}(U))$. For any $s \in \mathscr{F}(U)$, we have a family of maps $\psi_V(s|_V)\colon \mathscr{G}(V) \to \mathscr{H}(V)$ for all open subsets V of U. The compatibility of ψ_V guarantees the compatibility of $\psi_V(s|_V)$, thus they give rise to a morphism $\mathscr{G}|_U \to \mathscr{H}|_U$. In this way, we get a map $\psi(U)\colon \mathscr{F}(U) \to \mathrm{Hom}_{\mathbf{Sh}(U)}(\mathscr{G}|_U, \mathscr{H}|_U)$. Such kind of maps are compatible and thus forms a morphism $\mathscr{F} \to \mathcal{H}om(\mathscr{G}, \mathscr{H})$. Now, we have a map $\mathrm{Hom}_{\mathbf{Sh}(X)}(\mathscr{F} \times \mathscr{G}, \mathscr{H}) \to \mathrm{Hom}_{\mathbf{Sh}(X)}(\mathscr{F}, \mathcal{H}om(\mathscr{G}, \mathscr{H}))$, whose inverse is easy to construct. □

Remark The notion of sheaf Hom also works for sheaves of algebraic structures and the above results still hold.

16 (Gluing data) Let X be a topological space and $\mathfrak{U} = \{U_i \subset X\}_{i \in I}$ a covering on X. A **gluing data for sheaves of sets with respect to the covering** $\mathfrak{U}$ consists of the following stuff:

- For each $i \in I$, a sheaf $\mathscr{F}_i$ of sets on U_i;
- For each pair $i, j \in I$, an isomorphism $\varphi_{ij}\colon \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j|_{U_i \cap U_j}$,

satisfying the **cocycle condition**:

For any $i, j, k \in I$, the following diagram commutes.

$$\begin{tikzcd} \mathscr{F}_i|_{U_i\cap U_j\cap U_k} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & \mathscr{F}_k|_{U_i\cap U_j\cap U_k} \\ & \mathscr{F}_j|_{U_i\cap U_j\cap U_k} \arrow[ur, "\varphi_{jk}"'] & \end{tikzcd}$$

A *morphisms* between gluing data $(\mathscr{F}_i, \varphi_{ij})$ and $(\mathscr{G}_i, \psi_{ij})$ is a family of morphisms

$$f_i \colon \mathscr{F}_i \longrightarrow \mathscr{G}_i$$

such that the following diagram commutes.

$$\begin{tikzcd}
\mathscr{F}_i|_{U_i\cap U_j} \arrow[r, "f_i"] \arrow[d, "\varphi_{ij}"'] & \mathscr{G}_i|_{U_i\cap U_j} \arrow[d, "\psi_{ij}"] \\
\mathscr{F}_j|_{U_i\cap U_j} \arrow[r, "f_j"] & \mathscr{G}_j|_{U_i\cap U_j}
\end{tikzcd}$$

One can see the definitions can be easily generalized to **gluing data for sheaves of algebraic structures**.

Obviously, any sheaf $\mathscr{F}$ admits a gluing data $(\mathscr{F}_i, \varphi_{ij})$, where $\mathscr{F}_i$ is the restriction $\mathscr{F}|_{U_i}$ and φ_{ij} is the induced morphism

$$\mathscr{F}|_{U_i}\Big|_{U_i\cap U_j} \longrightarrow \mathscr{F}|_{U_j}\Big|_{U_i\cap U_j}.$$

Moreover, this construction is functoral, meaning it gives rise to a functor from $\mathbf{Sh}(X)$ to the category of gluing data.

17 Theorem (Sheaf = gluing data) *The above functor induces an equivalence of category between* $\mathbf{Sh}(X)$ *and the category of gluing data. The similar statement holds for sheaves of algebraic structures.*

Proof: The functor is fully faithful by Lemma 11.a and essentially surjective by Lemma 17.a. $\square$

17.a Lemma *Let X be a topological space and $\mathfrak{U} = \{U_i \subset X\}_{i\in I}$ a covering on X. Let $(\mathscr{F}_i, \varphi_{ij})$ be a gluing data for sheaves of sets with respect to the covering $\mathfrak{U}$. Then there exists a sheaf $\mathscr{F}$ on X together with isomorphisms*

$$\varphi_i \colon \mathscr{F}|_{U_i} \longrightarrow \mathscr{F}_i$$

such that the following diagram commutes.

$$\begin{tikzcd}
\mathscr{F}|_{U_i\cap U_j} \arrow[r, "\varphi_i"] \arrow[d, equal] & \mathscr{F}_i|_{U_i\cap U_j} \arrow[d, "\varphi_{ij}"] \\
\mathscr{F}|_{U_i\cap U_j} \arrow[r, "\varphi_j"] & \mathscr{F}_j|_{U_i\cap U_j}
\end{tikzcd}$$

The similar statement holds for sheaves of algebraic structures.

Proof: For any open subset W of X, the object $\mathscr{F}(W)$ is given as the equalizer of the morphisms:

$$\prod_{i\in I} \mathscr{F}_i(W \cap U_i) \rightrightarrows \prod_{i,j\in I} \mathscr{F}_i(W \cap U_i \cap U_j).$$

For sheaves of sets, this set can be written as

$$\mathscr{F}(W) = \left\{ (s_i) \in \prod_{i\in I} \mathscr{F}_i(W \cap U_i) \middle| \varphi_{ij}(s_i|_{W\cap U_i \cap U_j}) = s_j|_{W\cap U_i\cap U_j} \right\}.$$

By the universal property of equalizers, this this forms a presheaf. By the interchange property of limits, the descent condition for $\mathscr{F}_i$ implies that $\mathscr{F}$ is a sheaf.

As for the isomorphism, just note that a section in $\mathscr{F}|_{U_i}(W)$ is nothing but a system of compatible sections $s_j \in \mathscr{F}_i(W \cap U_i \cap U_j)$, which gives rise to a section $s \in \mathscr{F}_i(W)$. Thus the lemma follows. $\square$

§ I.5 Sheaves on topological spaces: continuous maps

In this section, we give the *direct image* and *inverse image* induced by a continuous map. later, we consider the inverse image of an open immersion. In that case, the inverse image has a left adjoint. Finally, we talk about the direct image of a closed immersion.

Direct images

1 (Direct images of sheaves) Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{F}$ a presheaf on X. Define $f_*\mathscr{F}$ by

$$f_*\mathscr{F}(V) := \mathscr{F}(f^{-1}(V))$$

with obvious restriction maps. These data form a presheaf on Y, called the **direct image** or **pushforward** of $\mathscr{F}$ by f. This construction is functoral, thus we get a functor:

$$f_*\colon \mathbf{PSh}(X) \longrightarrow \mathbf{PSh}(Y).$$

1.a Lemma *Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{F}$ a sheaf on X. Then $f_*\mathscr{F}$ is a sheaf on Y.*

Proof: Note that if $\{V_i \subset V\}$ is a covering in Y, then $\{f^{-1}(V_i) \subset f^{-1}(V)\}$ is a covering in X. Thus the descent condition for $f_*\mathscr{F}$ follows from the descent condition for $\mathscr{F}$. □

As a consequence, we get the functor

$$f_*\colon \mathbf{Sh}(X) \longrightarrow \mathbf{Sh}(Y).$$

1.b Lemma (Composition) *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous maps of topological spaces. Then the functors $(g \circ f)_*$ and $g_* \circ f_*$ are equal.*

Proof: This is because $(g \circ f)^{-1}(W) = f^{-1}g^{-1}(W)$. □

1.c Example (Skyscraper sheaves) The *skyscraper sheaf* $i_{x,*}S$ is the direct image of the constant sheaf $\underline{S}$ on an one-point space x, under the inclusion morphism $i_x\colon x \to X$.

The following lemma shows how the stalks related by direct image. This canonical map will be used later in notion of f-maps, see 6.

1.d Lemma *Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{F}$ a sheaf on X. If $f(x) = y$, then there is a canonical map $(f_*\mathscr{F})_y \to \mathscr{F}_x$.*

Proof: Note that

$$(f_*\mathscr{F})_y = \varinjlim_{y\in V} f_*\mathscr{F}(V) = \varinjlim_{y\in V} \mathscr{F}(f^{-1}(V)).$$

and that $\{f^{-1}(V)|y \in V\} \subset \{U|x \in U\}$. Then there exists a unique map $(f_*\mathscr{F})_y \to \mathscr{F}_x$ compatible with the restriction maps.

Let s_y be a germ of $f_*\mathscr{F}$ at $y \in Y$. Then its image under this canonical map can be describe as follows. Let (V, s) be a representative of s_y. Since $s \in \mathscr{F}(f^{-1}(V))$, it represents a germ s_x of $\mathscr{F}$ at x. One can see this s_x is independent of the choice of representative and is the image of s_y. $\square$

Inverse images

2 (Inverse images of presheaves) Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{G}$ a presheaf on Y. Define

$$f_p\mathscr{G}(U) := \varinjlim_{f(U)\subset V} \mathscr{G}(V)$$

with the restriction maps induces from the canonical maps between colimits. These data form a presheaf on X, called the **inverse image** or **pullback** of $\mathscr{G}$ by f.

3 Theorem (Inverse and direct images are adjoint) *Let $f\colon X \to Y$ be a continuous map of topological spaces, then $f_p \dashv f_*$ form an* adjoint pair*.*

Proof: Recall that a pair of functors L and R is called a **adjoint pair** or **adjunction** if they admit two natural transformations $\eta\colon \mathrm{id} \Rightarrow RL$ and $\epsilon\colon LR \Rightarrow \mathrm{id}$ satisfy the **triangle identities**.

$$\begin{tikzcd} & LRL \arrow[dr, "\epsilon * L"] & \\ L \arrow[ur, "L*\eta"] \arrow[rr, "\mathrm{id}"] & & L \end{tikzcd} \qquad \begin{tikzcd} & RLR \arrow[dr, "R*\epsilon"] & \\ R \arrow[ur, "\eta * R"] \arrow[rr, "\mathrm{id}"] & & R \end{tikzcd}$$

For details, refer §4.1 in my note *BMO*, or [Bor94] directly.

Let $\mathscr{F}$ be a presheaf on X and $\mathscr{G}$ a preshaef on Y.

First, note that the index system of the colimit

$$f_p\mathscr{G}(f^{-1}(V)) = \varinjlim_{V\subset V'} \mathscr{G}(V')$$

contains V itself. Thus we get a map $\mathscr{G}(V) \to f_p\mathscr{G}(f^{-1}(V))$, which induces a canonical morphism $\eta_{\mathscr{G}}\colon \mathscr{G} \to f_*f_p\mathscr{G}$.

Next, consider the colimit

$$f_pf_*\mathscr{F}(U) = \varinjlim_{f(U)\subset V} f_*\mathscr{F}(V) = \varinjlim_{f(U)\subset V} \mathscr{F}(f^{-1}(V)).$$

Since for each $V \supset f(U)$, there is a restriction map $\mathscr{F}(f^{-1}(V)) \to \mathscr{F}(U)$, we obtain a canonical map $f_p f_* \mathscr{F}(U) \to \mathscr{F}(U)$, which induces a canonical morphism $\epsilon_{\mathscr{F}} \colon f_p f_* \mathscr{F} \to \mathscr{F}$.

One can check that η and ϵ are natural transformations and that the following compositions are identities.

$$f_p \mathscr{G} \xrightarrow{f_p(\eta_{\mathscr{G}})} f_p f_* f_p \mathscr{G} \xrightarrow{\epsilon_{f_p \mathscr{G}}} f_p \mathscr{G}, \qquad f_* \mathscr{F} \xrightarrow{\eta_{f_* \mathscr{F}}} f_* f_p f_* \mathscr{F} \xrightarrow{f_*(\epsilon_{\mathscr{F}})} f_* \mathscr{F}.$$

This shows the triangle identities and thus $f_p \dashv f_*$ are an adjoint pair. $\square$

Remark One may expect to show that f_p is *left adjoint* to f_*, i.e. to prove the following natural bijection

$$\mathrm{Hom}_{\mathbf{PSh}(X)}(f_p \mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathbf{PSh}(Y)}(\mathscr{G}, f_* \mathscr{F}).$$

This follows directly from the adjoint pair: for $\phi \colon f_p \mathscr{G} \to \mathscr{F}$ a morphism of presheaves on X, the corresponding morphism is the composition

$$\mathscr{G} \xrightarrow{\eta_{\mathscr{G}}} f_* f_p \mathscr{G} \xrightarrow{f_* \phi} f_* \mathscr{F};$$

for $\psi \colon \mathscr{G} \to f_* \mathscr{F}$ a morphism of presheaves on Y, the corresponding morphism is the composition

$$f_p \mathscr{G} \xrightarrow{f_p(\psi)} f_p f_* \mathscr{F} \xrightarrow{\epsilon_{\mathscr{F}}} \mathscr{F}.$$

3.a Corollary (Composition) *Let $f \colon X \to Y$ and $g \colon Y \to Z$ be continuous maps of topological spaces. Then the functors $(g \circ f)_p$ and $g_p \circ f_p$ are equal.*

Proof: This follows from the uniqueness of adjoint functor and Lemma 1.b. $\square$

The relationship of stalks under inverse image is more imitate.

3.b Lemma *Let $f \colon X \to Y$ be a continuous map of topological spaces and $\mathscr{G}$ a sheaf on Y. Then there is a canonical bijection $(f_p \mathscr{G})_x \cong \mathscr{G}_{f(x)}$.*

Proof: This can be shown as follows.

$$\begin{aligned} (f_p \mathscr{G})_x &= \varinjlim_{x \in U} f_p \mathscr{G}(U) \\ &= \varinjlim_{x \in U} \varinjlim_{f(U) \subset V} \mathscr{G}(V) \\ &= \varinjlim_{f(x) \in V} \mathscr{G}(V) \\ &= \mathscr{G}_{f(x)} \end{aligned}$$

$\square$

4 (Inverse images of sheaves) Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{G}$ a sheaf on Y. Then we already has a presheaf $f_p\mathscr{G}$, which is called the *inverse image* of $\mathscr{G}$ by f. However, this $f_p\mathscr{G}$ is rarely a sheaf. So we define the **inverse image** or **pullback** of $\mathscr{G}$ by f as the sheafification of $f_p\mathscr{G}$, i.e.

$$f^{-1}\mathscr{G} := (f_p\mathscr{G})^{\#}.$$

5 Theorem (Inverse and direct images are adjoint) *Let $f\colon X \to Y$ be a continuous map of topological spaces, then $f^{-1} \dashv f_*$ form an* adjoint pair*.*

Proof: Consider the following commutative digram.

$$\begin{tikzcd}
\mathbf{Sh}(X) \arrow[r, shift left, "f_*"] \arrow[d, shift left, "F"] & \mathbf{Sh}(Y) \arrow[l, shift left, "f^{-1}"] \arrow[d, shift left, "F"] \\
\mathbf{PSh}(X) \arrow[u, shift left, "\#"] \arrow[r, shift left, "f_*"] & \mathbf{PSh}(Y) \arrow[u, shift left, "\#"] \arrow[l, shift left, "f_p"]
\end{tikzcd}$$

Except the upper one, all pairs are adjoint pairs, thus so is the upper one.

More precisely, this can be shown by

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}\mathscr{G}, \mathscr{F}) &= \mathrm{Hom}_{\mathbf{PSh}(X)}(f_p\mathscr{G}, \mathscr{F}) \\
&= \mathrm{Hom}_{\mathbf{PSh}(Y)}(\mathscr{G}, f_*\mathscr{F}) \\
&= \mathrm{Hom}_{\mathbf{Sh}(Y)}(\mathscr{G}, f_*\mathscr{F}).
\end{aligned}$$ □

5.a Corollary (Composition) *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous maps of topological spaces. Then the functors $(g \circ f)^{-1}$ and $g^{-1} \circ f^{-1}$ are equal.*

Proof: This follows from the uniqueness of adjoint functor and Lemma 1.b.□

Similarly to Lemma 3.b, we have

5.b Lemma *Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{G}$ a sheaf on Y. Then there is a canonical bijection $(f^{-1}\mathscr{G})_x \cong \mathscr{G}_{f(x)}$.*

Proof: This follows from Lemma 3.9.a and Lemma 3.b. □

The following notion is crucial in the definition of morphisms between ringed spaces.

6 (f-maps) Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{F}$ a sheaf on X, $\mathscr{G}$ a sheaf on Y. Combine the inverse and direct image functors, we define the **f-map** $\xi\colon \mathscr{G} \to \mathscr{F}$ as a morphism from $\mathscr{G}$ to $f_*\mathscr{F}$.

Let $f: X \to Y$ and $g: Y \to Z$ be continuous maps of topological spaces and $\mathscr{F}$ a sheaf on X, $\mathscr{G}$ a sheaf on Y, $\mathscr{H}$ a sheaf on Z. Let $\phi: \mathscr{G} \to \mathscr{F}$ be an f-map and $\psi: \mathscr{H} \to \mathscr{G}$ a g-map, then the *composition* $\psi \circ \phi$ of them is the $(g \circ f)$-map defined as the composition

$$\mathscr{H} \xrightarrow{\psi} g_*\mathscr{G} \xrightarrow{g_*\phi} g_*f_*\mathscr{F} = (g \circ f)_*\mathscr{F}.$$

Any f-map $\phi: \mathscr{G} \to \mathscr{F}$ gives rise to canonical maps at stalks:

$$\phi_x : \mathscr{G}_{f(x)} \longrightarrow \mathscr{F}_x,$$

which are given by the compositions:

$$\mathscr{G}_{f(x)} \longrightarrow (f_*\mathscr{F})_{f(x)} \longrightarrow \mathscr{F}_x.$$

7 Remark Now all the above constructions also appear in general case. Let $f: X \to Y$ be a continuous map of topological spaces and $\mathscr{F}$ a sheaf on X, $\mathscr{G}$ a sheaf on Y, both with values in an algebraic category $\mathcal{A}$. Let $\mathbf{PSh}(X, \mathcal{A})$ (resp. $\mathbf{Sh}(X, \mathcal{A})$) denote the category of presheaves (resp. sheaves) on X with values in $\mathcal{A}$. Then we have the following functors

$$\begin{aligned} f_*&: \mathbf{PSh}(X, \mathcal{A}) \longrightarrow \mathbf{PSh}(Y, \mathcal{A}) \\ f_*&: \mathbf{Sh}(X, \mathcal{A}) \longrightarrow \mathbf{Sh}(Y, \mathcal{A}) \\ f_p&: \mathbf{PSh}(Y, \mathcal{A}) \longrightarrow \mathbf{PSh}(X, \mathcal{A}) \\ f^{-1}&: \mathbf{Sh}(Y, \mathcal{A}) \longrightarrow \mathbf{Sh}(X, \mathcal{A}) \end{aligned}$$

which are compatible with the forgetful functor $F: \mathcal{A} \to \mathbf{Set}$.

We also have some formulas:

$$\begin{aligned} f_*\mathscr{F}(V) &= \mathscr{F}(f^{-1}(V)), \\ f_p\mathscr{G}(U) &= \varinjlim_{f(U) \subset V} \mathscr{G}(V), \\ f^{-1}\mathscr{G} &= (f_p\mathscr{G})^{\#}, \\ (f_p\mathscr{G})_x &= \mathscr{G}_{f(x)}, \\ (f^{-1}\mathscr{G})_x &= \mathscr{G}_{f(x)}. \end{aligned}$$

What's most important is the adjoint pairs:

$$\begin{aligned} f_p \dashv f_*&: \mathbf{PSh}(X, \mathcal{A}) \leftrightarrows \mathbf{PSh}(Y, \mathcal{A}), \\ f^{-1} \dashv f_*&: \mathbf{Sh}(X, \mathcal{A}) \leftrightarrows \mathbf{Sh}(Y, \mathcal{A}). \end{aligned}$$

Finally, the notion of *f-maps* also works.

Immersions of subspaces

Now we turn to consider a special kind of continuous maps. They are the *immersions of subspaces.*

8 Proposition (Inverse images by an open immersion) *Let X be a topological space and $j\colon U \to X$ an inclusion map of an open subset U into X.*

1. *Let $\mathscr{G}$ be a presheaf on X. Then the presheaf $j_p\mathscr{G}$ is given by $V \mapsto \mathscr{G}(V)$ for all open subsets V of U.*
2. *Let $\mathscr{G}$ be a sheaf on X. Then the sheaf $j^{-1}\mathscr{G}$ is given by $V \mapsto \mathscr{G}(V)$ for all open subsets V of U.*
3. *For any point $u \in U$ and any sheaf $\mathscr{G}$ on X we have a canonical identification of stalks*
$$j^{-1}\mathscr{G}_u = (\mathscr{G}|_U)_u = \mathscr{G}_u.$$
4. *We have $j_pj_* = \mathrm{id}$ on $\mathbf{PSh}(U)$ and $j^{-1}j_* = \mathrm{id}$ on $\mathbf{Sh}(U)$.*

The same description holds for (pre)sheaves of algebraic structures.

Proof: Note that V is *cofinal* in the system $\{W|V \subset W\}$, thus the first two results follow. Then, *3* follows from the fact that neighborhoods of u which are contained in U is *cofinal* in the system of all open neighborhoods of u in X. Finally, *4* follows from direct computing. $\square$

Remark One can see the (pre)sheaves in *1* and *2* are precisely the *restrictions* $\mathscr{G}|_U$ of $\mathscr{G}$ on an open subset U.

In the case of open immersions, there is a left adjoint functor to f^{-1}.

9 (Extension by zero) Let X be a topological space and $j\colon U \to X$ an inclusion map of an open subset U into X.

- Let $\mathscr{F}$ be a presheaf on U. Define the **extension of $\mathscr{F}$ by empty** $j_{p!}\mathscr{F}$ as the presheaf given by
$$j_{p!}\mathscr{F}(V) := \begin{cases} \mathscr{F}(V) & \text{if } V \subset U \\ \varnothing & \text{if } V \not\subset U \end{cases}$$
with obvious restriction maps.
- Let $\mathscr{F}$ be a sheaf on U. Define the **extension of $\mathscr{F}$ by empty** $j_!\mathscr{F}$ as the sheafification of the presheaf $j_{p!}\mathscr{F}$.

For sheaves of algebraic structures, there are similar notions. Let 0 denote the initial object in an algebraic category $\mathcal{A}$.

- Let $\mathscr{F}$ be a presheaf on U with values in $\mathcal{A}$. Define the **extension of $\mathscr{F}$ by zero** $j_{p!}\mathscr{F}$ as the presheaf given by

$$j_{p!}\mathscr{F}(V) := \begin{cases} \mathscr{F}(V) & \text{if } V \subset U \\ 0 & \text{if } V \not\subset U \end{cases}$$

  with obvious restriction maps.

- Let $\mathscr{F}$ be a sheaf on U with values in $\mathcal{A}$. Define the **extension of $\mathscr{F}$ by zero** $j_!\mathscr{F}$ as the sheafification of the presheaf $j_{p!}\mathscr{F}$.

Remark Although we can define the extension by zero for general sheaves of algebraic structures, but this construction depends on what the initial object is. For instance, the extension by zero of a sheaf of rings in the category of sheaves of rings is different from the one in the category of abelian sheaves. In particular, the functor $j_!$ *doesn't commute with taking underlying sheaves of sets* as other functors!

9.a Example ($j_{p!}\mathscr{F}$ is not a sheaf) Let U be the an open interval in $X = \mathbb{R}$ and $\mathscr{F}$ the sheaf of continuous functions on U. Then $j_{p!}\mathscr{F}$ is not a sheaf since one can definitely glue a nonzero function with zeros near the boundary of U with zero functions outside U to get a nonzero function on X, which does not lie in $j_{p!}\mathscr{F}(X)$.

10 Theorem (Extension by zero is left adjoint to restriction) *Let X be a topological space and $j : U \to X$ an inclusion map of an open subset $U \subset X$.*

1. *$j_{p!}$ is left adjoint to the restriction j_p.*
2. *$j_!$ is left adjoint to the restriction j^{-1}.*
3. *The stalks of the sheaf $j_!\mathscr{F}$ are described as follows*

$$j_!\mathscr{F}_x = \begin{cases} \mathscr{F}_x & \text{if } x \in U \\ \varnothing & \text{if } x \notin U. \end{cases}$$

4. *We have $j_p j_{p!} = \mathrm{id}$ on $\mathbf{PSh}(U)$ and $j^{-1}j_! = \mathrm{id}$ on $\mathbf{Sh}(U)$.*

The same results hold for (pre)sheaves of algebraic structures except that $\varnothing$ should be replaced by an initial object 0.

Proof: Let $\mathscr{F}$ be a presheaf on U and $\mathscr{G}$ a presheaf on X. First, as $j_{p!}\mathscr{F}$ vanishes outside U, a morphism from $j_{p!}\mathscr{F}$ to $\mathscr{G}$ is determined by its components on open subsets of U, which form a morphism from $\mathscr{F}$ to $\mathscr{G}|_U$. This shows the adjointness of $j_{p!} \dashv j_p$. Then the adjointness of $j_! \dashv j^{-1}$ follows from this and the adjointness of $\# \dashv F$. The rests are from direct computing. $\square$

We say that a sheaf $\mathscr{F}$ **vanishes** at a point x if $\mathscr{F}_x$ is an initial object.

11 Theorem *Let X be a topological space and $j\colon U \to X$ an inclusion map of an open subset $U \subset X$. Then the functor*

$$j_!\colon \mathbf{Sh}(U) \longrightarrow \mathbf{Sh}(X)$$

is fully faithful. Moreover, this functor induces an equivalence between $\mathbf{Sh}(U)$ and the full subcategory of $\mathbf{Sh}(X)$ consisting of sheaves vanishing outside of U. The same result holds for sheaves of algebraic structures.

Proof: $j_!$ is fully faithful since $j^{-1}j_! = \mathrm{id}$.As for the second statement, just note that the canonical morphism $\epsilon_{\mathscr{G}}\colon f_!f^{-1}\mathscr{G} \to \mathscr{G}$ is an isomorphism if $\mathscr{G}$ vanishes outside of U. $\square$

12 Proposition (Direct image by a closed immersion) *Let X be a topological space and $i\colon Z \to X$ an inclusion map of a closed subset $Z \subset X$.*

1. *Let $\mathscr{F}$ be a sheaf on Z. Then the stalks of the sheaf $i_*\mathscr{F}$ on X can be described as*
$$i_*\mathscr{F}_x = \begin{cases} \mathscr{F}_x & \text{if } x \in Z \\ * & \text{if } x \notin Z. \end{cases}$$
2. *We have $i^{-1}i_* = \mathrm{id}$ on $\mathbf{Sh}(Z)$.*

The same results hold for sheaves of algebraic structures.

Proof: Note that as a sheaf $\mathscr{F}$ should map empty set to a terminal object. Then the results follow. $\square$

13 Theorem *Let X be a topological space and $i\colon Z \to X$ an inclusion map of a closed subset $Z \subset X$. Then the functor*

$$i_*\colon \mathbf{Sh}(Z) \longrightarrow \mathbf{Sh}(X)$$

is fully faithful. Moreover, this functor induces an equivalence between $\mathbf{Sh}(Z)$ and the full subcategory of $\mathbf{Sh}(X)$ consisting of sheaves $\mathscr{G}$ satisfying $\mathscr{G}_x = *$ for all $x \in X \setminus Z$. The same result holds for sheaves of algebraic structures.

Proof: i_* is fully faithful since $i^{-1}i_* = \mathrm{id}$.As for the second statement, just note that the canonical morphism $\eta_{\mathscr{G}}\colon \mathscr{G} \to i_*i^{-1}\mathscr{G}$ is an isomorphism if $\mathscr{G}_x = *$ for all $x \in X \setminus Z$. $\square$

14 Remark (Direct image has no right adjoint) Let X be a topological space and $i\colon Z \to X$ an inclusion map of a closed subset $Z \subset X$. Let $x \in X \setminus Z$ and $\mathscr{F}$ be a sheaf on Z. Then $i_*\mathscr{F}_x = *$. Let $\mathscr{F} = \underline{*} \sqcup \underline{*}$, then $i_*\mathscr{F}_x = * \neq i_*(\underline{*})_x \sqcup i_*(\underline{*})_x$. This shows that the functor i_* is *NOT* right exact, hence can not has a right adjoint functor.

However, this is not the case for abelian sheaves. In fact, i_* on abelian sheaves is exact and does have right adjoint.

§ I.6 Sheafification

In this section, we construct the *sheafification* for presheaves on a site through the *zeroth Čech cohomology.*

Zeroth Čech cohomology

Recall that a presheaf $\mathscr{F}$ is a sheaf respect to the coverage Cov if for every covering $\mathfrak{U} = \{U_i \to U\}_{i\in I} \in \mathrm{Cov}$, the $\mathscr{F}(U)$ is the equalizer of the maps:

$$\prod_{i\in I} \mathscr{F}(U_i) \overset{\mathrm{pr}_0^*}{\underset{\mathrm{pr}_1^*}{\rightrightarrows}} \prod_{i_0,i_1\in I} \mathscr{F}(U_{i_0} \times_U U_{i_1}).$$

In general, the equalizer is not $\mathscr{F}(U)$ and its value depends on the covering $\mathfrak{U}$. This set is called the **zeroth Čech cohomology** of the presheaf $\mathscr{F}$ respect to the covering $\mathfrak{U}$ and is denoted by $\check{H}^0(\mathfrak{U}, \mathscr{F})$.

By the universal property of equalizers, there is a canonical map

$$\mathscr{F}(U) \to \check{H}^0(\mathfrak{U}, \mathscr{F}).$$

Then the *descent condition* turns out to say a presheaf $\mathscr{F}$ is a sheaf if and only if this canonical map is bijective for every covering.

Now we focus on the zeroth Čech cohomology $\check{H}^0(\mathfrak{U}, \mathscr{F})$. Let $\mathcal{J}_U$ denote the category of coverings of U.

First of all, any morphism $f \colon \mathfrak{U} \to \mathfrak{V}$ of coverings induces a map

$$f^* \colon \check{H}^0(\mathfrak{V}, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{U}, \mathscr{F}),$$

compatible with the map $\mathscr{F}(V) \to \mathscr{F}(U)$. In this way, the zeroth Čech cohomology $\check{H}^0(\mathfrak{U}, \mathscr{F})$ is a functor from $\mathcal{J}_U$ to **Set**. One may wish $\mathcal{J}_U^{\mathrm{opp}}$ to be filtered. However, this is not true in general. But luckily, we still have

1 Lemma *The diagram* $\check{H}^0(-, \mathscr{F}) \colon \mathcal{J}_U^{\mathrm{opp}} \to \mathbf{Set}$ *is filtered.*

Proof: First, since $\{\mathrm{id} \colon U \to U\}$ is a covering, the category is nonempty.

Next, for any two coverings $\mathfrak{U} = \{U_i \to U\}_{i\in I}$ and $\mathfrak{V} = \{V_j \to U\}_{j\in I}$, there is a covering

$$\mathcal{W} := \{U_i \times_U V_j \to U\}_{(i,j)\in I\times J}$$

refines both $\mathfrak{U}$ and $\mathfrak{V}$.

But now the troubles appear when we try to check the last axiom for filtered category. However, we still have the following Lemma 1.a. □

1.a Lemma *Any two morphisms* $f, g \colon \mathfrak{U} \rightrightarrows \mathfrak{V}$ *of coverings inducing the same morphism* $U \to V$ *induce the same map* $\check{H}^0(\mathfrak{V}, \mathscr{F}) \to \check{H}^0(\mathfrak{U}, \mathscr{F})$.

Proof: Let f (resp. g) is given by the map α (resp. β) and the morphisms $U_i \to V_{\alpha(i)}$ (resp. $U_i \to V_{\beta(i)}$). Then we have the following commutative diagram.

$$\begin{tikzcd}
 & V_{\alpha(i)} \arrow[dr] & \\
U_i \arrow[ur, "f_i"] \arrow[r, "\varphi"] \arrow[dr, "g_i"'] & V_{\alpha(i)} \times_V V_{\beta(i)} \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \arrow[r] & V \\
 & V_{\beta(i)} \arrow[ur] &
\end{tikzcd}$$

Then, for any $s = (s_j) \in \check{H}^0(\mathfrak{V}, \mathscr{F})$, we have $\mathrm{pr}_1^*(s_{\alpha(i)}) = \mathrm{pr}_2^*(s_{\beta(i)})$ by the definition of $\check{H}^0(\mathfrak{V}.\mathscr{F})$. Therefore, we have

$$(f^* s)_i = f_i^*(s_{\alpha(i)}) = \varphi^* \mathrm{pr}_1^*(s_{\alpha(i)}) = \varphi^* \mathrm{pr}_2^*(s_{\beta(i)}) = g_i^*(s_{\beta(i)}) = (g^* s)_i.$$

Thus $f^* = g^*$ as desired. $\square$

Now, since the diagram $\check{H}^0(-, \mathscr{F})\colon \mathcal{J}_U^{\mathrm{opp}} \to \mathbf{Set}$ is filtered, we can apply the following lemma.

2 Lemma (Description of filtered colimits) *If $D\colon \mathcal{I} \to$ **Set** is filtered, then*

$$\varinjlim_{\mathcal{I}} D = \Big(\bigsqcup_{i \in \mathrm{ob}\,\mathcal{I}} D(i)\Big)/\sim,$$

where the equivalence relation is given as following: two elements $s_i \in D(i)$ and $s_j \in D(j)$ are equivalent if there exists morphisms $f\colon i \to k$ and $g\colon j \to k$ such that $D(f)(s_i) = D(g)(s_j)$.

Now, we define

$$\check{H}^0(U, \mathscr{F}) := \varinjlim_{\mathcal{J}_U^{\mathrm{opp}}} \check{H}^0(\mathfrak{U}, \mathscr{F}),$$

which is called the **zeroth Čech cohomology** of $\mathscr{F}$ on U.

Now, let $U \to V$ be a morphism, then it induces a functor

$$\begin{aligned} \mathcal{J}_V &\longrightarrow \mathcal{J}_U \\ \{V_i \to V\} &\longmapsto \{V_i \times_V U \to U\}. \end{aligned}$$

Then, this functor induces a canonical map $\check{H}^0(V, \mathscr{F}) \to \check{H}^0(U, \mathscr{F})$. In this way, $\check{H}^0(-, \mathscr{F})$ becomes a presheaf, denoted by $\mathscr{F}^+$.

Now, notice that since $\mathfrak{U}_0 = \{\mathrm{id}\colon U \to U\}$ is a covering of U, there is a canonical map $\check{H}^0(\mathfrak{U}_0, \mathscr{F}) \to \check{H}^0(U, \mathscr{F})$. But $\check{H}^0(\mathfrak{U}_0, \mathscr{F})$ is nothing but $\mathscr{F}(U)$, thus we get a canonical map $\mathscr{F}(U) \to \mathscr{F}^+(U)$, which induces a canonical morphism

$$\mathscr{F} \longrightarrow \mathscr{F}^+.$$

Now, we claim that the corresponding $\mathscr{F} \mapsto (\mathscr{F} \to \mathscr{F}^+)$ forms a functor.

Given a presheaf morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$, it induces a map

$$\check{H}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{U}, \mathscr{G})$$

for every covering $\mathfrak{U}$, and thus also induces a map

$$\check{H}^0(U, \mathscr{F}) \longrightarrow \check{H}^0(U, \mathscr{G})$$

for every object $U \in \operatorname{ob} \mathcal{C}$. In this way, we obtain a morphism $\varphi^+\colon \mathscr{F}^+ \to \mathscr{G}^+$ and a commutative diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[d] & \mathscr{G} \arrow[d] \\ \mathscr{F}^+ \arrow[r, "\varphi^+"] & \mathscr{G}^+ \end{tikzcd}$$

and thus show that $\mathscr{F} \mapsto (\mathscr{F} \to \mathscr{F}^+)$ forms a functor.

3 Lemma *The functor $+$ is left exact, i.e. commutes with finite limits.*

Proof: First, *the functor $\check{H}^0(\mathfrak{U}, -)$ commutes with limits.* Indeed, by the definition, for any diagram $\mathcal{J} \to \mathbf{PSh}(\mathcal{C})\colon j \mapsto \mathscr{F}_j$ and any covering $\mathfrak{U} = \{U_i \to U\}_{i\in I}$, we have

$$\begin{aligned} \varprojlim_{\mathcal{J}} \check{H}^0(\mathfrak{U}, \mathscr{F}_j) &= \varprojlim_{\mathcal{J}} \ker\left(\prod_{i\in I} \mathscr{F}_j(U_i) \rightrightarrows \prod_{i_0,i_1\in I} \mathscr{F}_j(U_{i_0} \times_U U_{i_1})\right) \\ &= \ker\left(\prod_{i\in I} \varprojlim_{\mathcal{J}}(\mathscr{F}_j(U_i)) \rightrightarrows \prod_{i_0,i_1\in I} \varprojlim_{\mathcal{J}}(\mathscr{F}_j(U_{i_0} \times_U U_{i_1}))\right) \\ &= \ker\left(\prod_{i\in I} \varprojlim_{\mathcal{J}} \mathscr{F}_j(U_i) \rightrightarrows \prod_{i_0,i_1\in I} \varprojlim_{\mathcal{J}} \mathscr{F}_j(U_{i_0} \times_U U_{i_1})\right), \end{aligned}$$

where the second equality comes from the commutativity of limits and the third form the fact that limits in $\mathbf{PSh}(\mathcal{C})$ are computed pointwise.

Therefore, for any finite diagram $\mathcal{I} \to \mathbf{PSh}(\mathcal{C})\colon i \mapsto \mathscr{F}_i$ and any object $U \in \operatorname{ob}\mathcal{C}$, since filtered colimits commute with finite limits, we have

$$\begin{aligned} \varprojlim_{\mathcal{I}} \mathscr{F}_i^+(U) &= \varprojlim_{\mathcal{I}}(\mathscr{F}_i^+(U)) = \varprojlim_{\mathcal{I}} \varinjlim_{\mathcal{J}_U^{\mathrm{opp}}} \check{H}^0(\mathfrak{U}, \mathscr{F}_i) \\ &= \varinjlim_{\mathcal{J}_U^{\mathrm{opp}}} \varprojlim_{\mathcal{I}} \check{H}^0(\mathfrak{U}, \mathscr{F}_i) = \varinjlim_{\mathcal{J}_U^{\mathrm{opp}}} \check{H}^0(\mathfrak{U}, \varprojlim_{\mathcal{I}} \mathscr{F}_i) = (\varprojlim_{\mathcal{I}} \mathscr{F}_i)^+(U). \ \square \end{aligned}$$

Sheafification

4 Theorem (Sheafification) *Let $\mathscr{F}$ be a presheaf.*

1. *$\mathscr{F}^+$ is separated.*
2. *If $\mathscr{F}$ is separated, then $\mathscr{F}^+$ is a sheaf and the morphism of presheaves $\mathscr{F} \to \mathscr{F}^+$ is injective.*
3. *If $\mathscr{F}$ is a sheaf, then $\mathscr{F} \to \mathscr{F}^+$ is an isomorphism.*
4. *The presheaf $\mathscr{F}^{++}$ is always a sheaf.*

Proof: *1.* Let $\mathfrak{U} = \{U_i \to U\}_{i \in I}$ be a covering. Let $\bar{s}$ and $\bar{s}'$ be two elements of $\mathscr{F}^+(U)$ having the same image under the canonical map

$$\mathscr{F}^+(U) \to \prod_{i \in I} \mathscr{F}^+(U_i).$$

Let s (resp. s') be a representative of $\bar{s}$ (resp. $\bar{s}'$) in some $\check{H}^0(\mathfrak{V}, \mathscr{F})$, where $\mathfrak{V}$ is a covering of U. Since s and s' have the same image under the compose

$$\begin{tikzcd} \check{H}^0(\mathfrak{V}, \mathscr{F}) \arrow[r] \arrow[d] & \check{H}^0(\mathfrak{V} \times_U U_i, \mathscr{F}) \arrow[d] \\ \mathscr{F}^+(U) \arrow[r] & \mathscr{F}^+(U_i) \end{tikzcd}$$

there exists a covering $\mathfrak{W}_i$ of U_i such that s and s' have the same image under the compose

$$\check{H}^0(\mathfrak{V}, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{V} \times_U U_i, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{W}_i, \mathscr{F}).$$

But now those $\mathfrak{W}_i$ give a covering $\mathfrak{W}$ of U by compose each $\mathfrak{W}_i$ with $U_i \to U$. Then s and s' have the same image under the compose

$$\check{H}^0(\mathfrak{V}, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{V} \times_U U_i, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{W}_i, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{W}, \mathscr{F}).$$

Then as s and s' have the same image in the component $\check{H}^0(\mathfrak{W}, \mathscr{F})$, *a fortiori* the colimit $\check{H}^0(U, \mathscr{F})$. Thus $\bar{s} = \bar{s}'$.

2. Now for every covering $\mathfrak{U} = \{U_i \to U\}_{i \in I}$, the canonical map

$$\mathscr{F}(U) \longrightarrow \prod_{i \in I} \mathscr{F}(U_i)$$

is injective, thus so is

$$\mathscr{F}(U) \longrightarrow \check{H}^0(\mathfrak{U}, \mathscr{F}),$$

As the diagram $\check{H}^0(-, \mathscr{F}) \colon \mathcal{J}_U^{\mathrm{opp}} \to \mathbf{Set}$ is filtered, we further obtain the injectivity of the canonical map

$$\mathscr{F}(U) \longrightarrow \mathscr{F}^+(U).$$

Therefore, $\mathscr{F} \to \mathscr{F}^+$ *is injective.*

Now, let's prove $\mathscr{F}^+$ is a sheaf by checking the canonical map

$$\mathscr{F}^+(U) \longrightarrow \check{H}^0(\mathfrak{U}, \mathscr{F}^+)$$

is bijective for all coverings $\mathfrak{U}$. By *1*, it's already injective. Therefore we only need to show it's *surjective.* Let $\bar{s} = (\bar{s}_i)$ be an element of $\check{H}^0(\mathfrak{U}, \mathscr{F}^+)$. For each $\bar{s}_i \in \mathscr{F}^+(U_i)$, chose a representative $s_i = (s_{i\alpha}) \in \check{H}^0(\mathfrak{U}_i, \mathscr{F})$, where $\mathfrak{U}_i$ is a covering of U_i. Now, compose those coverings with $\mathfrak{U}$, we get another covering $\mathfrak{W} = \{U_{i\alpha} \to U\}$. Then $(s_{i\alpha})$ forms an element of $\prod \mathscr{F}(U_{i\alpha})$. Now, we wish *it lies in* $\check{H}^0(\mathfrak{W}, \mathscr{F})$. In other words, for any $i, j \in I$ and $\alpha \in I_i, \beta \in I_j$, we need to show $s_{i\alpha} \in \mathscr{F}(U_{i\alpha})$ *and* $s_{j\beta} \in \mathscr{F}(U_{j\beta})$ *have the same image under the maps*

$$\mathscr{F}(U_{i\alpha}) \longrightarrow \mathscr{F}(U_{i\alpha} \times_U U_{j\beta}) \longleftarrow \mathscr{F}(U_{j\beta}).$$

To do this, consider the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{U}_i \times_U \mathfrak{U}_j \arrow[r] \arrow[d] & U_i \times_U \mathfrak{U}_j \arrow[r] \arrow[d] & \mathfrak{U}_j \arrow[d] \\
\mathfrak{U}_i \times_U U_j \arrow[r] \arrow[d] & U_i \times_U U_j \arrow[r] \arrow[d] & U_j \arrow[d] \\
\mathfrak{U}_i \arrow[r] & U_i \arrow[r] & U
\end{tikzcd}$$

Let s_{ij}^1 and s_{ij}^2 denote the images of s_i and s_j on $\mathfrak{U}_i \times_U U_j$ and $U_i \times_U \mathfrak{U}_j$ respectively. Since $(\bar{s}_i) \in \check{H}^0(\mathfrak{U}, \mathscr{F}^+)$, s_{ij}^1 and s_{ij}^2 have the same image in $\mathscr{F}^+(U_i \times_U U_j)$. Then there exists a covering $\mathfrak{V}$ refining both $\mathfrak{U}_i \times_U U_j$ and $U_i \times_U \mathfrak{U}_j$ such that s_{ij}^1 and s_{ij}^2 have the same image in $\check{H}^0(\mathfrak{V}, \mathscr{F})$. Now, let

$$\mathfrak{U}_{ij} = \mathfrak{U}_i \times_U \mathfrak{U}_j,$$

which is a common refinement of $\mathfrak{U}_i \times_U U_j$ and $U_i \times_U \mathfrak{U}_j$. Then, by Lemma 4.a below, s_{ij}^1 and s_{ij}^2 have the same image in $\check{H}^0(\mathfrak{U}_{ij}, \mathscr{F})$. Thus s_i and s_j have the same image in $\check{H}^0(\mathfrak{U}_{ij}, \mathscr{F})$. In particular, $s_{i\alpha}$ and $s_{j\beta}$ have the same image in $\mathscr{F}(U_{i\alpha} \times_U U_{j\beta})$.

Now $(s_{i\alpha}) \in \check{H}^0(\mathfrak{W}, \mathfrak{F})$, so it represents an element $\bar{s}'$ of $\mathscr{F}^+(U)$. Since $\bar{s}'|_{U_i}$ and $\bar{s}_i$ have the same representative $s_i \in \check{H}^0(\mathfrak{U}_i, \mathfrak{F})$, they have to be the same. In this way, we see that the canonical map $\mathscr{F}^+(U) \to \check{H}^0(\mathfrak{U}, \mathscr{F}^+)$ maps $\bar{s}'$ to $\bar{s}$ as desired.

3. Now, assume $\mathscr{F}$ is a sheaf. Since for every covering $\mathfrak{U}$, the canonical map $\mathscr{F}(U) \to \check{H}^0(\mathfrak{U}, \mathscr{F})$ is bijective, the diagram $\check{H}^0(-, \mathscr{F}) \colon \mathscr{J}_U^{\mathrm{opp}} \to \mathbf{Set}$ is constant. Thus passing to the colimit, $\mathscr{F}(U) \to \mathscr{F}^+(U)$ is also bijective. Therefore, $\mathscr{F} \to \mathscr{F}^+$ is an isomorphism.

4. It is obvious now. $\square$

4.a Lemma *Let $\mathscr{F}$ be a separated presheaf.*

1. *If there is a refinement $f\colon \mathfrak{V} \to \mathfrak{U}$, then the map*
$$f^*\colon \check{H}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{V}, \mathscr{F})$$
is injective.

2. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be two coverings of U and $s_{\mathfrak{U}} \in \check{H}^0(\mathfrak{U}, \mathscr{F})$, $s_{\mathfrak{V}} \in \check{H}^0(\mathfrak{V}, \mathscr{F})$. If there exists a common refinement $\mathfrak{W}_0$ of them such that $s_{\mathfrak{U}}$ and $s_{\mathfrak{V}}$ have the same image in $\check{H}^0(\mathfrak{W}_0, \mathscr{F})$, then for any common refinement $\mathfrak{W}$ of them, $s_{\mathfrak{U}}$ and $s_{\mathfrak{V}}$ have the same image in $\check{H}^0(\mathfrak{W}, \mathscr{F})$.*

Proof: *1.* Let $\mathfrak{W}$ denote the covering $\mathfrak{U} \times_U \mathfrak{V}$ obtained by fibre products. $\mathfrak{W}$ admits two morphisms $\mathrm{pr}_1\colon \mathfrak{W} \to \mathfrak{U}$ and $\mathrm{pr}_2\colon \mathfrak{W} \to \mathfrak{V}$ via projections. Now, for each $U_i \to U$,

$$\mathscr{F}(U_i) \longrightarrow \prod_{j\in J} \mathscr{F}(U_i \times_U V_j)$$

is injective. Thus so is the product

$$\prod_{i\in I} \mathscr{F}(U_i) \longrightarrow \prod_{(i,j)\in I\times J} \mathscr{F}(U_i \times_U V_j).$$

Then, by the definition of zeroth Čech cohomology,

$$\mathrm{pr}_1^*\colon \check{H}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{W}, \mathscr{F})$$

is injective. Now, note that since there is a refinement $f\colon \mathfrak{V} \to \mathfrak{U}$, thus $\mathrm{pr}_1 = f \circ \mathrm{pr}_2$. Then since $\mathrm{pr}_2^* \circ f^* = \mathrm{pr}_1^*$ is injective, so is

$$f^*\colon \check{H}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{V}, \mathscr{F}).$$

2. Assumptions as in *2*, let $\mathfrak{W}$ be another common refinement of $\mathfrak{U}$ and $\mathfrak{V}$. Then there is a common refinement $\mathfrak{W}'$ of $\mathfrak{W}_0$ and $\mathfrak{W}$. Now, $s_{\mathfrak{U}}$ and $s_{\mathfrak{V}}$ have the same image in $\check{H}^0(\mathfrak{W}_0, \mathscr{F})$, *a fortiori* in $\check{H}^0(\mathfrak{W}', \mathscr{F})$. But since $\mathfrak{W}$ is a refinement of $\mathfrak{U}$, the map $\check{H}^0(\mathfrak{U}, \mathscr{F}) \to \check{H}^0(\mathfrak{W}_0, \mathscr{F})$ factors through $\check{H}^0(\mathfrak{W}, \mathscr{F})$. For $\mathfrak{V}$, the story is similar. Now, $s_{\mathfrak{U}}$ and $s_{\mathfrak{V}}$ have the same image under the following composes

$$\begin{tikzcd}
\check{H}^0(\mathfrak{U}, \mathscr{F}) \arrow[dr] & & \\
& \check{H}^0(\mathfrak{W}, \mathscr{F}) \arrow[r] & \check{H}^0(\mathfrak{W}', \mathscr{F}) \\
\check{H}^0(\mathfrak{V}, \mathscr{F}) \arrow[ur] & &
\end{tikzcd}$$

where the last map is injective, thus have the same image in $\check{H}^0(\mathfrak{W}, \mathscr{F})$. □

5 (Sheafification) Let $\mathscr{F}$ be a presheaf on a site $\mathcal{C}$. Then the sheaf $\mathscr{F}^\# := \mathscr{F}^{++}$ together with the canonical morphism $\mathscr{F} \to \mathscr{F}^\#$ is called the **sheafification** of $\mathscr{F}$.

The sheafification has the following universal property:

> For any sheaf $\mathscr{G}$ and presheaf morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$, there exists a unique sheaf morphism $\varphi^\#\colon \mathscr{F}^\# \to \mathscr{G}$ making the following digram commute.

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[dr, "\varphi"'] & \mathscr{F}^\# \arrow[d, "\varphi^\#"] \\ & \mathscr{G} \end{tikzcd}$$

In other words,

5.a Proposition (Sheafification is free) *The sheafification functor $\#$ is* left adjoint *to the forgetful functor from sheaves on $\mathcal{C}$ to presheaves on $\mathcal{C}$.*

Proof: Indeed, for any presheaf morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$ with $\mathscr{G}$ a sheaf, the unique sheaf morphism factors it is its sheafification $\varphi^\#$. □

5.b Corollary *The sheafification preserves colimits and hence is* right exact*; the forgetful functor preserves limits and hence is* left exact. *In particular, the presheaf kernel is already the sheaf kernel.*

Proof: This follows from the property of adjoint functors. One can refer my note *BMO* or [Bor94] directly. □

6 Proposition *The sheafification is exact.*

Proof: The right exactness comes from the freeness. As for the left exactness, just note that the colimit used to construct the functor $+$ is filtered, thus commutes with finite limits.

More precisely, since Corollary 5.b, the limits in $\mathbf{Sh}(\mathcal{C})$ are computed in the category $\mathbf{PSh}(\mathcal{C})$. Then, for any finite diagram $\mathcal{I} \to \mathbf{PSh}(\mathcal{C})\colon i \mapsto \mathscr{F}_i$, by Lemma 3, we have

$$\varprojlim_{\mathcal{I}} \mathscr{F}_i^\# = \varprojlim_{\mathcal{I}} \mathscr{F}_i^{++} = (\varprojlim_{\mathcal{I}} \mathscr{F}_i^{+})^+ = (\varprojlim_{\mathcal{I}} \mathscr{F}_i)^{++} = (\varprojlim_{\mathcal{I}} \mathscr{F}_i)^\#.$$ □

Now, we give a concrete description of the sheafification.

7 (Compatible sections) The sheafification can also be described by *compatible sections*. Let $\mathscr{F}$ be a presheaf on a site $\mathcal{C}$ and $\mathfrak{U} = \{U_i \to U\}_{i\in I}$ a covering. A **system of compatible sections** respect to $\mathfrak{U}$ is an element $(s_i) \in \prod \mathscr{F}(U_i)$ satisfying the following property:

For every $i, j \in I$, there exists a covering $\{U_{ijk} \to U_i \times_U U_j\}$ such that the pullbacks of s_i and s_j to each U_{ijk} agrees.

One can verify that given an element $s \in \mathscr{F}^{\#}(U)$ is equivalent to giving a system of compatible sections (s_i) for every coverings $\mathfrak{U}$ of U such that $s|_{U_i}$ is the image of s_i under the canonical map $\mathscr{F}(U_i) \to \mathscr{F}^{\#}(U_i)$.

§ I.7 The category of sheaves

In this section, we study the category $\mathbf{Sh}(\mathcal{C})$ of sheaves on a site $\mathcal{C}$. Such kind of category are called *topoi*. First, we consider monomorphisms, epimorphisms and isomorphisms in $\mathbf{Sh}(\mathcal{C})$. Then, we give some words on *images* and *local images*. The we discuss the relation between *quasi-compact object* in a site and *compact object* in its topos. Finally, we talk about canonical topology and presentable sheaves.

Morphisms of sheaves

Before going forward, we briefly recall the morphisms in $\mathbf{PSh}(\mathcal{C})$.

1. The monomorphisms (resp. epimorphisms, isomorphisms) in $\mathbf{PSh}(\mathcal{C})$ are precisely the *injective* (resp. *surjective*, *bijective*) morphisms, meaning injective (resp. surjective, bijective) on each object $U \in \operatorname{ob}\mathcal{C}$.

2. As a corollary, the category $\mathbf{PSh}(\mathcal{C})$ is *balanced*, meaning isomorphism = monomorphism + epimorphism.

3. Moreover, since the limits and colimits in $\mathbf{PSh}(\mathcal{C})$ are computed pointwise, $\mathbf{PSh}(\mathcal{C})$ inherits many wonderful properties of **Set**, such as

3.1. The monomorphisms are *regular*, meaning they are equalizers of some parallel morphisms. Indeed, in **Set**, a map $f\colon X \to Y$ is monic if and only if it is the equalizer of the *characteristic function*

$$\chi_f(y) = \begin{cases} 1 & \text{if } y \in \operatorname{im}(f) \\ 0 & \text{others} \end{cases}$$

and the constant function $1(y) = 1$. In $\mathbf{PSh}(\mathcal{C})$, the story is the same: just replace sets by presheaves and $\{0,1\}$ by the constant presheaf $\Delta_{\{0,1\}}$ and notice that the equalizer is computed pointwise.

Note that *a regular monomorphism is an isomorphism if and only if it is also epic.* Therefore this leads to another proof of *iso = monic + epic.*

3.2. The epimorphisms are *effective*, meaning they are *coequalizers* of their *kernel pair*. Indeed, in **Set**, a map $f\colon X \to Y$ is epic if and only if it is the coequalizer of its kernel pair, i.e. the two projections $\mathrm{pr}_1\,\mathrm{pr}_2$ in the following Cartesian diagram.

$$\begin{tikzcd} \ker(f) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

In $\mathbf{PSh}(\mathcal{C})$, the story is the same: just replace sets by presheaves and notice that the coequalizer and kernel pair are computed pointwise.

Note that *an effective epimorphism is an isomorphism if and only if it is also monic.* Therefore this leads to another proof of *iso = monic + epic.*

1 Proposition (Monomorphisms) *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a site $\mathcal{C}$. Then, the following are equivalent.*

1. *φ is a monomorphism.*

2. *φ is an injective presheaf morphism, i.e. for any object $U \in \operatorname{ob}\mathcal{C}$ the map $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is injective.*

3. *φ is a regular monomorphism.*

Proof: *2⇒1* and *3⇒1* are obvious. *1⇒2* comes form the left exactness of the forgetful functor $\mathbf{Sh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{C})$ and the exactness of the section functors $\Gamma(U, -)$.

As for *2⇒3*, consider the presheaf morphisms $\chi_\varphi, 1\colon \mathscr{G} \rightrightarrows \Delta_{\{0,1\}}$. Then φ, viewed as a presheaf morphism, is the equalizer of them. Apply sheafification to them, by Proposition 6.6, we see that the sheafification of φ, i.e. itself, is the equalizer of the sheafification of χ_φ and 1. □

2 Proposition (Isomorphisms) *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a site $\mathcal{C}$. Then, the following are equivalent.*

1. *φ is an isomorphism.*

2. *φ is an bijective presheaf morphism, i.e. for any object $U \in \operatorname{ob}\mathcal{C}$ the map $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is bijective.*

3. *φ is both monic and epic.*

Proof: *1⇒3* is trivial. Recall that $\mathbf{Sh}(\mathcal{C})$ is a full subcategory of $\mathbf{PSh}(\mathcal{C})$. Thus a sheaf morphism is an isomorphism if and only if it is an isomorphism in $\mathbf{PSh}(\mathcal{C})$. Therefore *1⇔2*.

As for *3⇒1*, just notice that *iso = regular monic + epic* is always true and then use Proposition 1. □

3 Proposition (Epimorphisms) *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a site $\mathcal{C}$. Then, the following are equivalent.*

1. *φ is an epimorphism.*

2. *φ is a* **locally surjective** *presheaf morphism, which means that for any object $U \in \operatorname{ob}\mathcal{C}$ and any section $s \in \mathscr{G}(U)$, there exists a covering $\{U_i \to U\}$ such that $s|_{U_i}$ lies in the image of $\varphi(U_i)\colon \mathscr{F}(U_i) \to \mathscr{G}(U_i)$.*

3. *φ is an effective epimorphism.*

Proof: *3⇒1* is trivial.

2⇒1: Let $\psi_1, \psi_2\colon \mathscr{G} \rightrightarrows \mathscr{T}$ be two parallel morphisms of sheaves such that $\psi_1 \circ \varphi = \psi_2 \circ \varphi$. We need to show $\psi_1 = \psi_2$. For any object $U \in \operatorname{ob}\mathcal{C}$

and any section $s \in \mathscr{G}(U)$, there exists a covering $\{U_i \to U\}$ such that for each i, there exists a section $t_i \in \mathscr{F}(U_i)$ such that $\varphi(U_i)(t_i) = s|_{U_i}$. Then

$$\begin{aligned}\psi_1(U)(s)|_{U_i} = \psi_1(U_i)(s|_{U_i}) = \psi_1\varphi(U_i)(t_i)& \\ = \psi_2\varphi(U_i)(t_i) = \psi_2(U_i)(s|_{U_i}) &= \psi_2(U)(s)|_{U_i}\end{aligned}$$

Since $\{U_i \to U\}$ is a covering, this shows $\psi_1(U)(s) = \psi_2(U)(s)$. Thus $\psi_1 = \psi_2$.

1⇒2: Define a subpresheaf $\mathscr{G}'$ of $\mathscr{G}$ as follows:

> $s \in \mathscr{G}'(U)$ if and only if there exists a covering $\{U_i \to U\}$ such that $s|_{U_i}$ lies in the image of $\varphi(U_i)\colon \mathscr{F}(U_i) \to \mathscr{G}(U_i)$.

It remains to show $\mathscr{G}' = \mathscr{G}$.

First of all, this $\mathscr{G}'$ is actually a sheaf. Indeed, we only need to verify the gluing condition. Let $(s_i) \in \prod \mathscr{G}'(U_i)$ be a system of compatible sections respect to a covering $\{U_i \to U\}$. Then, it corresponds to a section $s \in \mathscr{G}(U)$. It remains to show $s \in \mathscr{G}'(U)$. Indeed, for each s_i, there exists a covering $\{U_{ij} \to U_i\}$ such that $s_i|_{U_{ij}}$ lies in the image of $\varphi(U_{ij})\colon \mathscr{F}(U_{ij}) \to \mathscr{G}(U_{ij})$. Now, combine all those coverings, we obtain a covering $\{U_{ij} \to U\}$ such that $s|_{U_{ij}} = s_i|_{U_{ij}}$ lies in the image of $\varphi(U_{ij})\colon \mathscr{F}(U_{ij}) \to \mathscr{G}(U_{ij})$. Therefore $s \in \mathscr{G}'(U)$.

Remark This sheaf $\mathscr{G}'$ is called the **sheaf image** of φ.

Now, we have sheaf morphisms

$$\mathscr{F} \longrightarrow \mathscr{G}' \xrightarrow{i} \mathscr{G}.$$

Since the compose φ is epic, so is $i\colon \mathscr{G}' \to \mathscr{G}$. But this i is the inclusion morphism from the subsheaf $\mathscr{G}'$ to $\mathscr{G}$, thus is monic. Now, i is both monic and epic in $\mathbf{Sh}(\mathcal{C})$, thus is an isomorphism by Proposition 2. This shows $\mathscr{G}' = \mathscr{G}$.

1⇒3: Let $\mathscr{T}$ be a sheaf and $\psi\colon \mathscr{F} \to \mathscr{T}$ a morphism coequalize the *kernel pair* of φ:

$$\mathscr{F} \times_{\mathscr{G}} \mathscr{F} \rightrightarrows \mathscr{F}.$$

Let $\mathscr{G}' \subset \mathscr{G}$ be the presheaf image of φ. Since the sheafification is exact, it maps $\mathscr{G}'$ to the sheaf image of φ, here which is $\mathscr{G}$ itself. Therefore, the above kernel pair can be obtained by applying sheafification to the kernel pair of $\varphi'\colon \mathscr{F} \to \mathscr{G}'$ in $\mathbf{PSh}(\mathcal{C})$. But since sheafification is exact, the two kernel pairs are the same.

Now, ψ coequalize the kernel pair of $\varphi'\colon \mathscr{F} \to \mathscr{G}'$ in $\mathbf{PSh}(\mathcal{C})$. Since φ' is epic, hence effective epic in $\mathbf{PSh}(\mathcal{C})$, there exists a unique presheaf morphism $\tau'\colon \mathscr{G} \to \mathscr{T}$ such that $\tau' \circ \varphi' = \psi$. Apply sheafification to them, we obtain a unique sheaf morphism $\tau\colon \mathscr{G} \to \mathscr{T}$ such that $\tau \circ \varphi = \psi$. □

Recall that the presheaf image $\operatorname{im}^p \varphi$ of a presheaf morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$ is the unique subpresheaf of the codomain $\mathscr{G}$ such that the morphism factors through it and that $\mathscr{F} \to \operatorname{im}^p \varphi$ is an epimorphism. Then one can verify that this epimorphism is nothing but the *coequalizer of the kernel pair* of φ, i.e. the **category-theoretic image** in $\mathbf{PSh}(\mathcal{C})$.

Then, from the above proof, we see

4 Lemma *The sheaf image is the sheafification of the presheaf image and is the category-theoretic image in* $\mathbf{Sh}(\mathcal{C})$.

The locally surjectivity can also be defined for presheaf morphisms. Obviously, surjective morphisms are locally surjective and the converse is false. Likely, the sheaf image can be defined for presheaves. But now, it may not be a sheaf and thus should have another name, the **local image**.

5 Proposition *A presheaf morphism is locally surjective if and only if its sheafification is an epimorphism.*

Proof: First, assume $\varphi\colon \mathscr{F} \to \mathscr{G}$ is locally surjective. For any $\bar{s} \in \mathscr{G}^{\#}(U)$, let $(s_i) \in \prod \mathscr{G}(U_i)$ be a family of compatible sections respect to the a covering $\{U_i \to U\}$. For each s_i, there exists a covering $\{U_{ij} \to U_i\}$ such that $s_i|_{U_{ij}}$ has a preimage $t_{ij} \in \mathscr{F}(U_{ij})$. Let $\bar{t_{ij}}$ be the image of t_{ij} in $\mathscr{F}^{\#}(U_{ij})$, then $\varphi^{\#}(U_{ij})(\bar{t_{ij}}) = \bar{s}|_{U_{ij}}$. This shows $\varphi^{\#}$ is locally surjective, hence epic.

Next, assume the sheafification of $\varphi\colon \mathscr{F} \to \mathscr{G}$ is an epimorphism. For any section $s \in \mathscr{G}(U)$ of $\mathscr{G}$, consider its image under the canonical map $\mathscr{G}(U) \to \mathscr{G}^{\#}(U)$, saying $\bar{s}$. Since $\varphi^{\#}$ is epic, by Proposition 3, there exists a covering $\mathfrak{U} = \{U_i \to U\}$ such that each $\bar{s}|_{U_i}$ lies in the image under $\varphi^{\#}(U_i)$. Let $\bar{t_i} \in \mathscr{F}^{\#}(U_i)$ be the preimage of $\bar{s}|_{U_i}$. Now, by 6.7, each $\bar{t_i}$ is equivalent to a system of compatible sections $(t_{ij}) \in \mathscr{F}(U_{ij})$ for each covering $\{U_{ij} \to U_i\}$. Now, consider the covering $\{U_{ij} \to U\}$, one can verify that $\varphi(U_{ij})(t_{ij}) = s|_{U_{ij}}$. This shows φ is locally surjective. $\square$

6 Proposition *The sheafification preserves local images.*

Proof: Indeed, let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves and $\operatorname{im}^p \varphi$ the local image of it. Apply sheafification to them, we have the following commutative diagram.

$$\begin{tikzcd}
\mathscr{F} \arrow[r] \arrow[d] & \operatorname{im}^p \varphi \arrow[r] \arrow[d] & \mathscr{G} \arrow[d] \\
\mathscr{F}^{\#} \arrow[r] & (\operatorname{im}^p \varphi)^{\#} \arrow[r] & \mathscr{G}^{\#}
\end{tikzcd}$$

By the exactness of $\#$, $(\operatorname{im}^p \varphi)^{\#}$ is a subsheaf of $\mathscr{G}^{\#}$. By Proposition 5, $\mathscr{F}^{\#} \to (\operatorname{im}^p \varphi)^{\#}$ is epic. Since the sheaf image $\operatorname{im} \varphi^{\#}$ of $\varphi^{\#}$ is the unique subsheaf of the codomain $\mathscr{G}^{\#}$ such that $\varphi^{\#}$ factors through it and $\mathscr{F}^{\#} \to \operatorname{im} \varphi^{\#}$ is epic, we have $(\operatorname{im}^p \varphi)^{\#} = \operatorname{im} \varphi^{\#}$. $\square$

Compactness

7 (Quasi-compactness) Let $\mathcal{C}$ be a site. An object U of $\mathcal{C}$ is said to be **quasi-compact** if every covering of U can be refined by a finite covering.

Recall that, in a category $\mathcal{C}$, an object U is said to be **compact** if the functor $\mathrm{Hom}_{\mathcal{C}}(U, -)$ preserves filtered limits. What does the following lemma say?

8 Lemma (Compactness) *Let $\mathcal{C}$ be a site. Let $\mathcal{J} \to \mathbf{Sh}(\mathcal{C})$ be a filtered diagram of sheaves of sets. Let $U \in \mathrm{ob}\,\mathcal{C}$. Consider the canonical map*

$$\Phi\colon \varinjlim_{\mathcal{J}} \mathscr{F}_j(U) \longrightarrow (\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U).$$

1. *If all the transition morphisms are injective then Φ is injective.*
2. *If U is quasi-compact, then Φ is injective.*
3. *If U is quasi-compact and all the transition morphisms are injective then Φ is an isomorphism.*
4. *If any covering of U can be refined by some coverings $\{U_i \to U\}_{i \in I}$ with I finite and $U_i \times_U U_{i'}$ quasi-compact, then Φ is bijective.*

Proof: *1.* Assume all the transition morphisms are injective. First of all, we show the presheaf $\mathscr{F}_{\mathcal{J}}\colon U \mapsto \varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$ is *separated.* Indeed the canonical maps

$$\mathscr{F}_j(U) \longrightarrow \prod \mathscr{F}_j(U_i)$$

are injective, thus so is the colimit

$$\varinjlim_{\mathcal{J}} \mathscr{F}_j(U) \longrightarrow \varinjlim_{\mathcal{J}} \prod \mathscr{F}_j(U_i).$$

It remains to show the canonical map

$$\varinjlim_{\mathcal{J}} \prod \mathscr{F}_j(U_i) \longrightarrow \prod \varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i)$$

is injective.

Let $\bar{s}$ and $\bar{t}$ be two elements of $\varinjlim_{\mathcal{J}} \prod \mathscr{F}_j(U_i)$ having the same image in $\prod \varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i)$ and $s = (s_i), t = (t_i)$ be their representatives in some $\prod \mathscr{F}_j(U_i)$. Now, the image of $\bar{s}$ and $\bar{t}$ in $\prod \varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i)$ can be written as $(\bar{s_i})$ and $(\bar{t_i})$, where each $\bar{s_i}$ or $\bar{t_i}$ is the image of s_i or t_i in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i)$. Since $\bar{s_i} = \bar{t_i}$, there exists some $j_i \in \mathrm{ob}\,\mathcal{J}$ such that the image of s_i and t_i in $\mathscr{F}_{j_i}(U_i)$ are the same. Then, since the transition morphism $\mathscr{F}_j \to \mathscr{F}_{j_i}$ is injective, we have $s_i = t_i$. Then, we get $s = t$ and *a fortiori* $\bar{s} = \bar{t}$.

By Proposition 6.6, $\varinjlim_{\mathcal{J}} \mathscr{F}_j$ is the sheafification of the presheaf $\mathscr{F}_{\mathcal{J}}$. Then, by Theorem 6.4, Φ is injective.

2. Assume U is quasi-compact. Let $\bar{s}$ and $\bar{t}$ be two elements of $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$ having the same image under Φ and s, t be their representatives in some $\mathscr{F}_j(U)$. Now, for any covering $\mathfrak{U} = \{U_i \to U\}_{i\in I}$, the image of $\bar{s}$ and $\bar{t}$ under Φ can be written as systems of compatible sections $(\bar{s}_i)$ and $(\bar{t}_i)$ of the presheaf$\mathscr{F}_{\mathcal{J}}$. Then, there exists $j_i \in \operatorname{ob}\mathcal{J}$ such that the image of $s|_{U_i}$ and $t|_{U_i}$ in $\mathscr{F}_{j_i}(U_i)$ are the same. Since U is quasi-compact, the covering $\mathfrak{U}$ can be refined by a finite covering $\mathfrak{V} = \{V_i \to U\}_{i\in I'}$ with the index transformation $\alpha \colon I' \to I$. For this covering, we can take j_0 to be the index such that there are arrows $j_{\alpha(i)} \to j_0$. Now, the image of $s|_{V_i}$ and $t|_{V_i}$ in $\mathscr{F}_{j_0}(V_i)$ are the same. Then s and t maps to the same element in $\mathscr{F}_{j_0}(U)$, *a fortiori* in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$.

3. Assume U is quasi-compact and all the transition morphisms are injective. Then Φ is injective. It suffices to show it is surjective. Let $\bar{s}$ be a section in $(\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U)$. For any covering $\mathfrak{U} = \{U_i \to U\}_{i\in I}$, $\bar{s}$ can be written as a system of compatible sections $(\bar{s}_i)$ of the presheaf $\mathscr{F}_{\mathcal{J}}$. Let s^i be the representative of $\bar{s}_i$ in some $\mathscr{F}_{j_i}$. Then the images of $s^i|_{U_i\times_U U_{i'}}$ and $s^{i'}\Big|_{U_i\times_U U_{i'}}$ in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i\times_U U_{i'})$ have the same image under the composition

$$\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \times_U U_{i'}) \xrightarrow{\Phi} (\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U_i \times_U U_{i'}) \longrightarrow (\varinjlim_{\mathcal{J}} \mathscr{F}_j)(V_{ii'})$$

for each $V_{ii'k} \to U_i \times_U U_{i'}$ in some covering $\mathfrak{V}_{ii'}$. Thus, by *1.*, the images of $s^i|_{V_{ii'}}$ and $s^{i'}\Big|_{V_{ii'}}$ in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(V_{ii'})$ are the same.

Since U is quasi-compact, the covering $\mathfrak{U}$ can be refined by a finite covering. Thus we may assume $\mathfrak{U}$ is finite. Now, we can take j_0 to be the index such that there are arrows $j_i \to j_0$. Then the sections $s^i \in \mathscr{F}_{j_i}(U_i)$ give rise to sections $s_i \in \mathscr{F}_{j_0}(U_i)$. For any $V_{ii'}$, $s_i|_{V_{ii'}}$ and $s_{i'}|_{V_{ii'}}$ have the same image in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(V_{ii'})$, by the similar argument in *1.*, they are the same. Therefore (s_i) is a system of compatible sections of $\mathscr{F}_{j_0}$, and thus gives a section $s \in \mathscr{F}_{j_0}(U)$ such that $s|_{U_i} = s_i$. Then, this s gives an element of $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$ which maps to $\bar{s}$ under Φ.

4. Assume the hypothesis of *4.* It is obvious that U is quasi-compact and thus Φ is injective. It suffices to show Φ is surjective. Let $\bar{s}$ be an element in $(\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U)$. For any covering $\mathfrak{U} = \{U_i \to U\}_{i\in I}$, $\bar{s}$ can be written as a system of compatible sections $(\bar{s}_i)$ of the presheaf $\mathscr{F}_{\mathcal{J}}$. We may assume that the covering $\mathfrak{U}$ is finite and that $U_i \times_U U_{i'}$ are quasi-compact. Let s^i be the representative of $\bar{s}_i$ in some $\mathscr{F}_{j_i}$. Then the images of $s^i|_{U_i\times_U U_{i'}}$ and

$s^{i'}\Big|_{U_i\times_U U_{i'}}$ in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \times_U U_{i'})$ have the same image under the composition

$$\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \times_U U_{i'}) \xrightarrow{\Phi} (\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U_i \times_U U_{i'}) \longrightarrow (\varinjlim_{\mathcal{J}} \mathscr{F}_j)(V_{ii'})$$

for each $V_{ii'k} \to U_i \times_U U_{i'}$ in some covering $\mathfrak{V}_{ii'}$. Since $\varinjlim_{\mathcal{J}} \mathscr{F}_j$ is a sheaf, this shows the images of $s^i|_{U_i\times_U U_{i'}}$ and $s^{i'}\Big|_{U_i\times_U U_{i'}}$ in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \times_U U_{i'})$ have the same image under Φ. Since $U_i \times_U U_{i'}$ is quasi-compact, by *2.*, this shows $s^i|_{U_i\times_U U_{i'}}$ and $s^{i'}\Big|_{U_i\times_U U_{i'}}$ have the same image in $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \times_U U_{i'})$, and thus in $\mathscr{F}_{j_{ii'}}(U_i \times_U U_{i'})$ for some $j_{ii'}$.

Now, we can take j_0 to be the index such that there are arrows $j_{ii'} \to j_0$. Then the sections $s^i \in \mathscr{F}_{j_i}(U_i)$ give rise to sections $s_i \in \mathscr{F}_{j_0}(U_i)$ and furthermore, they form a system of compatible sections. Thus we get a section $s \in \mathscr{F}_{j_0}(U)$ such that $s|_{U_i} = s_i$. Then, this s gives an element of $\varinjlim_{\mathcal{J}} \mathscr{F}_j(U)$ which maps to $\bar{s}$ under Φ. $\square$

We also give another proof.

Proof: *2.* Assume U is quasi-compact. First of all, for a finite covering $\mathfrak{U}$ of U, since filtered colimits commute with finite limits, the canonical map $\mathscr{F}_J(U) \to \check{H}^0(\mathfrak{U}, \mathscr{F}_J)$ is injective. As for an arbitrary covering $\mathfrak{V}$ of U, let $\mathfrak{U}$ be a finite covering refining $\mathfrak{V}$, then we have the following commutative diagram

$$\begin{tikzcd} \mathscr{F}_J(U) \arrow[r] \arrow[d, equal] & \check{H}^0(\mathfrak{V}, \mathscr{F}_J) \arrow[d] \\ \mathscr{F}_J(U) \arrow[r] & \check{H}^0(\mathfrak{U}, \mathscr{F}_J) \end{tikzcd}$$

where the bottom $\mathscr{F}_J(U) \to \check{H}^0(\mathfrak{U}, \mathscr{F}_J)$ is injective, thus so is the upper $\mathscr{F}_J(U) \to \check{H}^0(\mathfrak{V}, \mathscr{F}_J)$. Therefore the canonical map $\mathscr{F}_J(U) \to \mathscr{F}_J^+(U)$ is injective. By Theorem 6.4, $\mathscr{F}_J^+$ is separated, thus $\mathscr{F}_J^+(U) \to \mathscr{F}_J^{++}(U)$ is injective. Since Φ is the canonical map $\mathscr{F}_J(U) \to \mathscr{F}_J^{\#}(U)$, which equals the composition $\mathscr{F}_J(U) \to \mathscr{F}_J^+(U) \to \mathscr{F}_J^{++}(U)$, it is also injective.

3. Assume U is quasi-compact and all the transition morphisms are injective. First of all, by the proof of *1.*, $\mathscr{F}_J$ is separated. Thus $\varinjlim_{\mathcal{J}} \mathscr{F}_j = \mathscr{F}_J^+$ and the canonical map Φ is just the map $\mathscr{F}_J(U) \to \mathscr{F}_J^+(U)$. Since U is quasi-compact, the finite coverings are cofinal in all coverings of U. Thus

$$(\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U) = \mathscr{F}_J^+(U) = \varinjlim_{\mathfrak{U} \text{ is a finite covering of } U} \check{H}^0(\mathfrak{U}, \mathscr{F}_J).$$

For any finite covering $\mathfrak{U} = \{U_i \to U\}$ of U, since filtered colimits commute with finite limits, we have

$$\begin{aligned}
\check{H}^0(\mathfrak{U}, \mathscr{F}_J) &= \ker\left(\prod \varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i) \rightrightarrows \prod \varinjlim_{\mathcal{J}} \mathscr{F}_j(U_i \times_U U_{i'})\right) \\
&= \varinjlim_{\mathcal{J}} \ker\left(\prod \mathscr{F}_j(U_i) \rightrightarrows \prod \mathscr{F}_j(U_i \times_U U_{i'})\right) \\
&= \varinjlim_{\mathcal{J}} \mathscr{F}_j(U).
\end{aligned}$$

Therefore

$$\mathscr{F}_J^+(U) = \varinjlim_{\mathfrak{U} \text{ is a finite covering of } U} \mathscr{F}_J(U)$$

and thus the map $\mathscr{F}_J(U) \to \mathscr{F}_J^+(U)$ is an isomorphism as desired.

4. Assume the hypothesis of *4.* It is obvious that U is quasi-compact. By the proof of *3.*, we have $\mathscr{F}_J(U) \cong \mathscr{F}_J^+(U)$. Let $\mathfrak{U} = \{U_i \to U\}$ be a covering satisfying the assumption in *4.*, we claim that each U_i is quasi-compact. Indeed, consider any covering $\mathfrak{U}_i = \{U_{ij} \to U_i\}$ of U_i. Then $\mathfrak{U}_i \times_U U_{i'}$ is a covering of $U_i \times_U U_{i'}$ for all i'. Since $U_i \times_U U_{i'}$ is quasi-compact, $\mathfrak{U}_i \times_U U_{i'}$ has a finite refinement $\mathfrak{V}_{i'}$. Combining all those $\mathfrak{V}_{i'}$, we get a finite covering $\{V_{i'k} \to U_i \times_U U_{i'} \to U_i\}$ which refines $\mathfrak{U}_i$.

The coverings satisfying the assumption in *4.* are cofinal in all coverings of U. Thus

$$(\varinjlim_{\mathcal{J}} \mathscr{F}_j)(U) = \mathscr{F}_J^{++}(U) = \varinjlim_{\mathfrak{U} \text{ satisfies the assumption in } 4.} \check{H}^0(\mathfrak{U}, \mathscr{F}_J^+).$$

But for such a covering $\mathfrak{U} = \{U_i \to U\}$, since all U_i and $U_i \times_U U_{i'}$ are quasi-compact, we have

$$\begin{aligned}
\check{H}^0(\mathfrak{U}, \mathscr{F}_J^+) &= \ker\left(\prod \mathscr{F}_J^+(U_i) \rightrightarrows \prod \mathscr{F}_J^+(U_i \times_U U_{i'})\right) \\
&\cong \ker\left(\prod \mathscr{F}_J(U_i) \rightrightarrows \prod \mathscr{F}_J(U_i \times_U U_{i'})\right) \\
&= \mathscr{F}_J(U).
\end{aligned}$$

Therefore

$$\mathscr{F}_J^{++}(U) \cong \varinjlim_{\mathfrak{U} \text{ satisfies the assumption in } 4.} \mathscr{F}_J(U)$$

and thus the map $\mathscr{F}_J(U) \to \mathscr{F}_J^{++}(U)$ is an isomorphism as desired. □

Representable sheaves

9 **(Canonical topology)** A site is said to have **subcanonical topology** if all representable presheaves on this site are sheaves. A pretopology defines such a site is called a **subcanonical pretopology**. The largest subcanonical pretopology is called the **canonical pretopology**.

But what does a subcanonical topology look like?

10 **(Effective epimorphisms)** In a category $\mathcal{C}$, a family of morphisms with fixed target $\mathfrak{U} = \{U_i \to U\}$ is **effective-epic** if all the morphisms $U_i \to U$ have pullbacks and for any $X \in \operatorname{ob}\mathcal{C}$, the presheaf h_X satisfies the descent condition respect to $\mathfrak{U}$ (refer (I.2.1)). We say $\mathfrak{U}$ is **universal effective-epic** if its pullback along any morphism $V \to U$ is effective-epic.

Remark In the case $\mathcal{C}$ has coproducts, saying $\mathfrak{U}$ is effective-epic is equivalent to say the morphism $\coprod_i U_i \to U$ is an effective epimorphism.

Obviously, we have

11 **Proposition (Subcanonical and universal effective-epic)** *A site $\mathcal{C}$ has subcanonical if and only if its coverings are all universal effective-epic. Moreover, if $\mathcal{C}$ has fibre products, then the canonical pretopology contains precisely all the universal effective-epic families.*

12 **Example (Canonical topology on Set)** The canonical topology on **Set** is given by the coverage Cov as follows.

$$\{\varphi_i \colon U_i \to U\}_{i\in I} \in \mathrm{Cov} \iff \bigcup_{i\in I} \varphi_i(U_i) = U.$$

To show this, it suffices to show that for any set S, $h_S(\bigcup_{i\in I} \varphi_i(U_i))$ is the equalizer of

$$\prod h_S(U_i) \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} \prod h_S(U_i \times_U U_{i'}).$$

Indeed, for any $f \in \operatorname{Hom}(\bigcup_{i\in I} \varphi_i(U_i), S)$, the compositions $f \circ \varphi_i$ give rise to an element in $\prod h_S(U_i)$, which has same image under pr_1 and pr_2. Conversely, for any element $(g_i) \in \prod h_S(U_i)$ having same image under pr_1 and pr_2, we have the following commutative diagram.

$$\begin{tikzcd}
U_i \times_U U_{i'} \arrow[r] \arrow[d] & U_{i'} \arrow[d, "g_{i'}"] \\
U_i \arrow[r, "g_i"] & S
\end{tikzcd}$$

For any element of $s \in \bigcup_{i\in I} \varphi_i(U_i)$, taking any preimage of it, say $u_i \in U_i$, we define $f(s) = g_i(u_i)$. The above commutative diagram guarantees the map $f \colon \bigcup_{i\in I} \varphi_i(U_i) \to S$ is well-defined.

13 **Example (Canonical topology on a topological space)** For X a topological space, the canonical topology on $\mathcal{T}_X$ is the original one.

It suffices to show for any family of open sets $U_{ii\in I}$ and an open set V of X, $\mathrm{Hom}_{\mathcal{T}_X}(\bigcup_{i\in I} U_i, V)$ is the equalizer of

$$\prod_{i\in I} \mathrm{Hom}_{\mathcal{T}_X}(U_i, V) \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \prod_{i,i'\in I} \mathrm{Hom}_{\mathcal{T}_X}(U_i \cap U_{i'}, V).$$

But this is obvious since $\mathrm{Hom}_{\mathcal{T}_X}(U, V)$ is either a singleton, when $U \subset V$, or the empty.

14 (Representable sheaves) Let $\mathcal{C}$ be a site having subcanonical topology. Then a **representable sheaf** is a sheaf of the form h_U for some $U \in \mathrm{ob}\,\mathcal{C}$. In this case, h_U is also denoted by $\widetilde{U}$.

Note that for any sheaf $\mathscr{F}$, we have

$$\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(h_U^{\#}, \mathscr{F}) \cong \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(h_U, \mathscr{F}) \cong \mathscr{F}(U).$$

In the case that $\mathcal{C}$ has subcanonical topology, $h_U^{\#} = h_U$ and then the Yoneda embedding identifies $\mathcal{C}$ with the full subcategory of $\mathbf{Sh}(\mathcal{C})$ consisting of representable sheaves.

15 Lemma *Let $\mathcal{C}$ be a site. If $\{U_i \to U\}_{i\in I}$ is a covering, then the morphism of presheaves of setas*

$$\coprod_{i\in I} h_{U_i} \longrightarrow h_U$$

becomes surjective after sheafification.

Proof: We need to show $\coprod_{i\in I} h_{U_i}^{\#} \to h_U^{\#}$ is epic, which is equivalent to that for any sheaf $\mathscr{F}$,

$$\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(h_U^{\#}, \mathscr{F}) \to \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\coprod_{i\in I} h_{U_i}^{\#}, \mathscr{F})$$

is injective. But $\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(h_U^{\#}, \mathscr{F}) = \mathscr{F}(U)$ and $\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\coprod_{i\in I} h_{U_i}^{\#}, \mathscr{F}) = \prod_{i\in I} \mathscr{F}(U_i)$. Thus this is just the descent condition of $\mathscr{F}$. $\square$

The following construct will be used later.

16 Lemma *Let $\mathcal{C}$ be a site and $\mathcal{E}$ be a set of objects in $\mathcal{C}$ such that any object U in $\mathcal{C}$ has a covering by members of $\mathcal{E}$. Then for any sheaf of sets $\mathscr{F}$, there exists a diagram of sheaves of sets*

$$\mathscr{F}_1 \rightrightarrows \mathscr{F}_0 \longrightarrow \mathscr{F}$$

Where $\mathscr{F}_0$ and $\mathscr{F}_1$ are coproducts of sheaves of the form $h_U^{\#}$ with $U \in \mathcal{E}$ and $\mathscr{F}$ is the coequalizer.

Proof: First, for any $U \in \mathcal{E}$, a section $s \in \mathscr{F}(U)$ corresponds to a morphism $h_U^\# \to \mathscr{F}$. Taking the coproduct of them, we get a morphism

$$\mathscr{F}_0 = \coprod_{U \in \mathcal{E}, s \in \mathscr{F}(U)} h_U^\# \longrightarrow \mathscr{F}.$$

This is an epimorphism since for any section $s \in \mathscr{F}(V)$, choosing a covering $\{U_i \to V\}$ with $U_i \in \mathcal{E}$, we see each $s|_{U_i}$ has preimage $\mathrm{id}_{U_i} \in h_{U_i}^\#(U_i)$.

Let $\mathscr{G} = \mathscr{F}_0 \times_{\mathscr{F}} \mathscr{F}_0$, then $\mathscr{F}$ is the coequalizer of the kernel pair $\mathscr{G} \rightrightarrows \mathscr{F}_0$ of the above epimorphism.

Now, construct an epimorphism $\mathscr{F}_1 = \mathscr{G}_0 \to \mathscr{G}$ as above. Then, since compositing with an epimorphism does not change the coequalizer, this $\mathscr{F}_1$ is the required one. $\square$

§ I.8 Topoi and geometric morphisms

In this section, we discuss *geometric morphisms*. They are morphisms between topoi. Later, we focus on special types of functors between sites, which induce geometric morphisms. Such kinds of functor should be viewed as *morphisms of sites*. However, there are many distinct kinds of functors all induce the geometric morphisms. That's fine, since what crucial is the topoi not the sites.

Topoi and geometric morphisms

1 **(Topoi)** A **(Grothendieck) topos** is a category $\mathbf{Sh}(\mathcal{C})$ of sheaves on a site $\mathcal{C}$. A **geometric moprhism** $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ between topoi is an *adjunction*:

$$f^{-1} \dashv f_*\colon \mathbf{Sh}(\mathcal{D}) \leftrightarrows \mathbf{Sh}(\mathcal{C}),$$

in which the left adjoint f^{-1} is left exact. The left adjoint f^{-1} is called the **inverse image** and the right adjoint is called the **direct image**.

The *composition of geometric morphisms* are just the composition of adjunctions. More precisely, if $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ and $g\colon \mathbf{Sh}(\mathcal{D}) \to \mathbf{Sh}(\mathcal{E})$ are two geometric morphisms, then their composition $g \circ f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{E})$ is given by $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ and $(g \circ f)_* = g_* \circ f_*$.

1.a **(2-morphisms of topoi)** Let $f, g\colon \mathbf{Sh}(\mathcal{C}) \rightrightarrows \mathbf{Sh}(\mathcal{D})$ be two geometric morphisms, a 2**-morphism** from f to g is given by a natural transformation $\alpha\colon f_* \Rightarrow g_*$. Usually we denote it as

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{Sh}(\mathcal{D}). \end{tikzcd}$$

Note that since f^{-1} is left adjoint to f_* and g^{-1} is left adjoint to g_*, α also induces a natural transformation from g^{-1} to f^{-1} uniquely characterized by the commutative diagram.

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(f^{-1}\mathscr{F}, \mathscr{G}) \arrow[r, equal] \arrow[d, "\alpha\circ-"'] & \mathrm{Hom}_{\mathbf{Sh}(\mathcal{D})}(\mathscr{F}, f_*\mathscr{G}) \arrow[d, "-\circ\alpha"] \\ \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(g^{-1}\mathscr{F}, \mathscr{G}) \arrow[r, equal] & \mathrm{Hom}_{\mathbf{Sh}(\mathcal{D})}(\mathscr{F}, g_*\mathscr{G}) \end{tikzcd}$$

The *vertical and horizontal compositions of of 2-morphisms* are just the vertical and horizontal compositions of natural transformations. In this way, the topoi together with geometric morphisms between them and 2-morphisms between geometric morphisms from a *strict* 2*-category*. Note that this is a big category since the geometric morphisms between two topoi may not form a set.

1.b Remark Obviously the notion of geometric morphisms does not only work for topoi. For instance, let $F\colon \mathcal{C} \to \mathcal{D}$ be an equivalence of categories with weak inverse G, then $G \dashv F$ is a geometric morphism. Moreover, if both $\mathcal{C}$ and $\mathcal{D}$ are topoi, then this geometric morphism is an equivalence in the 2-category of topoi. Therefore, it makes sense to extend our definition by saying a category is a topos when it is equivalent to a category of sheaves on a site.

2 Example (Empty topos) Let $*$ denote the category consisting of only one object $*$ with only one morphism id_*. It has two possible site structure on it. One is to treat it as the category of open sets of $\varnothing$, so this site is denoted as $\varnothing$. Then, since any sheaf on it must sent the initial object $*$ to a singleton, $\mathbf{Sh}(\varnothing)$ consists only one sheaf with only one morphism. In this way, $\mathbf{Sh}(\varnothing)$ is equivalent to $*$ itself.

The following example is very important in topos theory.

3 Example (Punctual topos) Another coverage on $*$ consists only one covering, namely the identity covering $\{\mathrm{id}_*\}$. Denote this site by pt. Then, it is clearly that $\mathbf{Sh}(\mathrm{pt}) = \mathbf{PSh}(\mathrm{pt}) = \mathbf{Set}$.

But $*$ is not the only site giving the punctual topos. For instance, let $\mathcal{S}$ be a small full subcategory of **Set** which contains at least one nonempty set and has fibre products. Define coverings on $\mathcal{S}$ as surjective families of maps and shrink this coverage so that it defines a site. Now, we claim that $\mathbf{Sh}(\mathcal{S})$ is equivalent to **Set**.

First, if $\mathcal{S}$ contains a singleton, then the equivalence $i\colon \mathbf{Sh}(\mathrm{pt}) \to \mathbf{Sh}(\mathcal{S})$ is given by

$$i_* S = \mathrm{Hom}_{\mathbf{Set}}(-, S) \qquad \text{and} \qquad i^{-1}\mathscr{F} = \mathscr{F}(*).$$

Indeed, if $\mathscr{F}$ is a sheaf on $\mathcal{S}$, then for any $U \in \mathrm{ob}\,\mathcal{S}$, there is a covering $\{\varphi_u\colon * \to U\}_{u\in U}$. The descent condition for this covering implies that

$$\mathscr{F}(U) \cong \prod_{u\in U} \mathscr{F}(*) \cong \mathrm{Hom}_{\mathbf{Set}}(U, \mathscr{F}(*)).$$

Moreover, this equality is compatible with restriction maps. Therefore the geometric morphism i is an equivalence of topoi.

Next, for general $\mathcal{S}$, we still have i_*. Let $\widetilde{*}$ be a nonempty set in $\mathcal{S}$ and $\varphi\colon \widetilde{*} \to \widetilde{*}$ is a map whose image is a singleton. Let

$$i^{-1}\mathscr{F} = \mathrm{im}\,\mathscr{F}(\varphi).$$

Then this i^{-1} is also a weak inverse of i_*.

4 Example (Geometric morphisms between presheaf topoi) Any functor $u\colon \mathcal{C} \to \mathcal{D}$ induces a geometric morphism

$$u^p \dashv {}_pu\colon\ \mathbf{PSh}(\mathcal{D}) \leftrightarrows \mathbf{PSh}(\mathcal{C}),$$

where $u^p\mathscr{F} = \mathscr{F} \circ u$ for any $\mathscr{F} \in \operatorname{ob}\mathbf{PSh}(\mathcal{D})$ and ${}_pu$ is the *right Kan extension operator along* u. Moreover, if u itself is left exact, then there is a second geometric morphism

$$u_p \dashv u^p\colon\ \mathbf{PSh}(\mathcal{C}) \leftrightarrows \mathbf{PSh}(\mathcal{D}),$$

where u_p is the *left Kan extension operator along* u.

Proof: This first statement follows from Theorem 1.9 and Theorem 1.10. To show the second one, it suffices to show u_p is left exact when u is left exact. This follows from Lemma 1.8 and the fact that filtered colimit commutes with finite limits. □

5 (Geometric embeddings) A geometric morphism $f = (f^{-1} \dashv f_*)$ is called a **geometric embedding** if f_* is fully faithful.

5.a Example Let $\mathcal{C}$ be a site. The sheafification $\#$ and the forgetful functor F from $\mathbf{Sh}(\mathcal{C})$ to $\mathbf{PSh}(\mathcal{C})$ forms a geometric embedding $\# \dashv F$.

Morphisms of sites

One approach to geometric morphisms is induced by the morphisms of sites. But there are many candidates. For the classical one, we introduce the notions of *continuous functors*.

6 (Continuous functors) Let $\mathcal{C}, \mathcal{D}$ be two sites. A functor $u\colon \mathcal{C} \to \mathcal{D}$ is called **continuous** if for every $\mathfrak{U} = \{U_i \to U\}_{i\in I} \in \operatorname{Cov}(\mathcal{C})$ we have

1. $u(\mathfrak{U}) := \{u(U_i) \to u(U)\}_{i\in I} \in \operatorname{Cov}(\mathcal{D})$,
2. For any morphism $T \to U$ in $\mathcal{C}$, $u(T \times_U U_i) \to u(T) \times_{u(U)} u(U_i)$ is an isomorphism.

Remark Do *NOT* confuse this "continuous functor" with the one used in category theory, which means a functor commuting with all limits.

6.a Lemma *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a continuous functor. If $\mathscr{F}$ is a sheaf on $\mathcal{D}$, then $u^p\mathscr{F}$ is a sheaf on $\mathcal{C}$.*

Proof: The descent condition for $u^p\mathscr{F}$ respect to a covering $\mathfrak{U}$ is the same as the descent condition for $\mathscr{F}$ respect to the covering $u(\mathfrak{U})$. □

Therefore, if $u\colon \mathcal{C} \to \mathcal{D}$ is continuous, then the restriction of u^p on $\mathbf{Sh}(\mathcal{D})$ gives a functor

$$u^s\colon\ \mathbf{Sh}(\mathcal{D}) \longrightarrow \mathbf{Sh}(\mathcal{C}).$$

6.b Lemma (The left adjoint of u^s) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a continuous functor. Then*

1. *$u_s := \# \circ u_p \circ F$ is a left adjoint to u^s;*
2. *For any presheaf $\mathscr{G}$ on $\mathcal{C}$, we have $(u_p\mathscr{G})^{\#} = (u_p(\mathscr{G}^{\#}))^{\#}$;*
3. *For any object U of $\mathcal{C}$, we have $u_s h_U^{\#} = h_{u(U)}^{\#}$.*

Proof: *1.* By Theorem 1.9 and Proposition 6.5.a.

2. For any sheaf $\mathscr{F}$ on $\mathcal{D}$ we have

$$\begin{aligned}
\mathrm{Hom}_{Sh(\mathcal{D})}((u_p\mathscr{G})^{\#}, \mathscr{F}) &\cong \mathrm{Hom}_{\mathbf{PSh}(\mathcal{D})}(u_p\mathscr{G}, \mathscr{F}) \\
&\cong \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathscr{G}, u^p\mathscr{F}) \\
&\cong \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathscr{G}^{\#}, u^s\mathscr{F}) \\
&\cong \mathrm{Hom}_{\mathbf{Sh}(\mathcal{D})}(u_s(\mathscr{G}^{\#}), \mathscr{F}).
\end{aligned}$$

3. By Corollary 1.9.a and *2.* □

7 (Morphisms of sites) Let $\mathcal{C}, \mathcal{D}$ be two sites. A **morphism of sites** $f\colon \mathcal{D} \to \mathcal{C}$ is given by a continuous functor $u\colon \mathcal{C} \to \mathcal{D}$ (*NOT* $\mathcal{D} \to \mathcal{C}$!) such that the functor u_s is exact. A morphism of sites $f\colon \mathcal{D} \to \mathcal{C}$ gives a geometric morphism $f = (f^{-1}, f_*)$ as

$$f^{-1} := u_s\colon\ \mathbf{Sh}(\mathcal{C}) \longrightarrow \mathbf{Sh}(\mathcal{D}), \qquad f_* := u^s\colon\ \mathbf{Sh}(\mathcal{D}) \longrightarrow \mathbf{Sh}(\mathcal{C}).$$

7.a Example (Continuous maps) Let $f\colon X \to Y$ be a continuous map between topological spaces. Recall that in Example 2.1.a, we defines the sites $\mathcal{T}_X$ and $\mathcal{T}_Y$. Then one can see the functor

$$\begin{aligned}
u\colon \mathcal{T}_Y &\longrightarrow \mathcal{T}_X \\
V &\longmapsto f^{-1}(V)
\end{aligned}$$

is a continuous functor. Moreover, u^s equals the *direct image functor* f_* defined in 5.1 and thus its left adjoint u_s is isomorphic to the *inverse image functor* f^{-1} defined in 5.4. Since the inverse image functor is exact, so is u_s. Thus u induces a morphism of sites $f\colon \mathcal{T}_X \to \mathcal{T}_Y$.

7.b Lemma (Composition of continuous functors) *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be sites and $u\colon \mathcal{D} \to \mathcal{C}$ and $v\colon \mathcal{E} \to \mathcal{D}$ be continuous functors which induce morphisms of sites. Then the functor $u \circ v\colon \mathcal{E} \to \mathcal{C}$ is continuous and defines a morphism of sites $\mathcal{C} \to \mathcal{E}$.*

7.c **(Composition of morphisms of sites)** Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be three sites and $f: \mathcal{C} \to \mathcal{D}$ and $g: \mathcal{D} \to \mathcal{E}$ be morphisms of sites given by continuous functors $u: \mathcal{D} \to \mathcal{C}$ and $v: \mathcal{E} \to \mathcal{D}$. The *composition* $g \circ f$ is the morphism of sites induced by the continuous functor $u \circ v$.

8 **(Conditions defining morphisms of sites)** The following lemmas give some conditions for a continuous functor to define a morphism of sites.

8.a **Lemma** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites and $u: \mathcal{C} \to \mathcal{D}$ be continuous functor. Assume all the categories $\mathcal{I}_V^{\mathrm{opp}}$ are filtered. Then u defines a morphism of sites $\mathcal{D} \to \mathcal{C}$, in other words u_s is exact.*

Proof: It suffices to show u_s is left exact, which follows from the fact that filtered colimit commutes with finite limits plus Proposition 6.5.a. $\square$

8.b **Lemma** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites and $u: \mathcal{C} \to \mathcal{D}$ be continuous functor. Assume $\mathcal{C}$ is finite-complete and u is left exact. Then u defines a morphism of sites $\mathcal{D} \to \mathcal{C}$, in other words u_s is exact.*

Proof: It suffices to show u_s is left exact, which follows from Lemma 1.8 and Lemma 8.a. $\square$

8.c **Lemma** *A continuous functor between sites which has a continuous left adjoint defines a morphism of sites.*

Proof: Let $u: \mathcal{D} \to \mathcal{C}$ be a continuous functor and $v: \mathcal{C} \to \mathcal{D}$ its continuous left adjoint. By Lemma 1.11, $u_p = v^p$, and hence $u_s = v^s$ has both left and right adjoint, whence is exact. $\square$

Cocontinuous functors

Another way to get geometric morphisms is via cocontinuous functors.

9 **(Cocontinuous functors)** Let $\mathcal{C}, \mathcal{D}$ be two sites. A functor $u: \mathcal{C} \to \mathcal{D}$ is called **cocontinuous** if for every $U \in \operatorname{ob} \mathcal{C}$ and every covering $\mathfrak{V} = \{V_j \to u(U)\}_{j \in J} \in \mathrm{Cov}(\mathcal{D})$, there exists a covering $\mathfrak{U} = \{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ such that $u(\mathfrak{U})$ refines the covering $\mathfrak{V}$.

Remark Note that this $u(\mathfrak{U})$ is in general *NOT* a covering.

Remark Do *NOT* confuse this "cocontinuous functor" with the one used in category theory, which means a functor commuting with all colimits.

9.a **Lemma** *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u: \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. If $\mathscr{F}$ is a sheaf on $\mathcal{C}$, then ${}_p u \mathscr{F}$ is a sheaf on $\mathcal{D}$.*

Proof: Let $\mathfrak{V} = \{V_j \to V\}$ be a covering in $\mathcal{D}$. We need to show the sequence

$$ {}_p u\mathscr{F}(V) \longrightarrow \prod {}_p u\mathscr{F}(V_j) \rightrightarrows \prod {}_p u\mathscr{F}(V_{j_0} \times_V V_{j_1}) $$

is exact. But ${}_p u\mathscr{F}$ is right adjoint to u^p, thus

$$ \begin{aligned} {}_p u\mathscr{F}(V) &= \mathrm{Hom}_{\mathbf{PSh}(\mathcal{D})}(h_V, {}_p u\mathscr{F}) \\ &= \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(u^p h_V, \mathscr{F}) = \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}((u^p h_V)^{\#}, \mathscr{F}). \end{aligned} $$

Therefore, it suffices to show that the sheafification of

$$ \coprod u^p h_{V_{j_0} \times_V V_{j_1}} \rightrightarrows \coprod u^p h_{V_j} \longrightarrow u^p h_V $$

is exact.

First, we show the sheafification of $\coprod u^p h_{V_j} \to u^p h_V$ is an epimorphism. To do this, we use Proposition 7.5. Thus it suffices to show it is locally surjective. Let $s \in u^p h_V(U)$ be a section, which is a morphism $u(U) \to V$. Then $\mathfrak{V} \times_V u(U) = \{V_j \times_V u(U) \to u(U)\}$ is a covering in $\mathcal{D}$. As u is cocontinuous, there is a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that $\{u(U_i) \to u(U)\}$ refines $\mathfrak{V} \times_V u(U)$. This means that each restriction $s|_{U_i} : u(U_i) \to V$ factors through a morphism $s_i \colon u(U_i) \to V_j$ for some j. Thus $s|_{U_i}$ lies in the image of $u^p h_{V_j} \to u^p h_V$ as desired.

$$ \begin{tikzcd} u(U_i) \arrow[r] \arrow[dr] & V_j \times_V u(U) \arrow[r] \arrow[d] & V_j \arrow[d] \\ & u(U) \arrow[r] & V \end{tikzcd} $$

Now we show the exactness. To do this, we use Proposition 7.3. Thus, note that sheafification is exact, it suffices to show $\coprod u^p h_{V_{j_0} \times_V V_{j_1}} \rightrightarrows \coprod u^p h_{V_j}$ is the kernel pair of $\coprod u^p h_{V_j} \to u^p h_V$. Let $s \colon u(U) \to V_j$ and $s' \colon u(U) \to V_{j'}$ be two sections of $\coprod u^p h_{V_j}$ having the same image in $u^p h_V$. Then we have the following commutative diagram

$$ \begin{tikzcd} u(U) \arrow[r, "s'"] \arrow[d, "s"'] & V_{j'} \arrow[d] \\ V_j \arrow[r] & V \end{tikzcd} $$

and thus get a morphism $t = (s, s') \colon u(U) \to V_j \times_V V_{j'}$. This is a section of $\coprod u^p h_{V_{j_0} \times_V V_{j_1}}$ mapping to s and s' as desired. $\square$

Therefore, if $u \colon \mathcal{C} \to \mathcal{D}$ is cocontinuous, then the restriction of ${}_p u$ on $\mathbf{Sh}(\mathcal{C})$ gives a functor

$$ {}_s u \colon \mathbf{Sh}(\mathcal{C}) \longrightarrow \mathbf{Sh}(\mathcal{D}). $$

9.b Theorem (The left adjoint of $_s u$) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Then*

1. $\# \circ u^p \circ F$ *is a left adjoint to $_s u$ and is exact;*

2. *For any presheaf $\mathscr{G}$ on $\mathcal{C}$, we have $(u^p \mathscr{G})^\# = (u^p(\mathscr{G}^\#))^\#$.*

Proof: *1.* The adjunction follows from Theorem 1.10 and Proposition 6.5.a. The left exactness follows from the exactness of $\#$, u^p and the left exactness of F.

2.: For any sheaves $\mathscr{F}$ on $\mathcal{C}$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}((u^p\mathscr{G})^\#, \mathscr{F}) &= \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(u^p\mathscr{G}, \mathscr{F}) \\ &= \mathrm{Hom}_{\mathbf{PSh}(\mathcal{D})}(\mathscr{G}, {}_p u\mathscr{F}) \\ &= \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathscr{G}^\#, {}_s u\mathscr{F}) \\ &= \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}((u^p(\mathscr{G}^\#))^\#, \mathscr{F}).\end{aligned}$$

□

9.c Corollary *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Then $g_* = {}_s u$ and $g^{-1} = (u^p)^\#$ define a geometric morphism $g\colon$ $\mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$.*

9.d Lemma (Composition of cocontinuous functors) *Let $u\colon \mathcal{C} \to \mathcal{D}$ and $v\colon \mathcal{D} \to \mathcal{E}$ be two cocontinuous functors inducing geometric morphisms $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ and $g\colon \mathbf{Sh}(\mathcal{D}) \to \mathbf{Sh}(\mathcal{E})$. Then $v \circ u$ is a cocontinuous functor and induces the geometric morphism $g \circ f$.*

Proof: Let $U \in \mathrm{ob}\,\mathcal{C}$. Let $\mathfrak{W} = \{W_i \to v(u(U))\}$ be a covering in $\mathcal{E}$. As v is cocontinuous, there exists a covering $\mathfrak{V} = \{V_j \to u(U)\}$ in $\mathcal{D}$ such that $v(\mathfrak{V})$ refines $\mathfrak{W}$. As u is cocontinuous, there exists a covering $\mathfrak{U} = \{U_k \to U\}$ in $\mathcal{C}$ such that $u(\mathfrak{U})$ refines $\mathfrak{V}$. Then $v(u(\mathfrak{U}))$ refines $\mathfrak{W}$. This shows $v \circ u$ is cocontinuous. As for the last assertion, it suffices to show ${}_s v \circ {}_s u = {}_s v \circ u$, which suffices to show ${}_p v \circ {}_p u = {}_p v \circ u$. By Theorem 1.10, it suffices to show $u^p \circ v^p = (v \circ u)^p$, which is obvious. □

10 Example (Open immersion) Let X be a topological space and $j\colon U \to X$ an inclusion map of an open subset U into X. Recall that we have sites $\mathcal{T}_X$ and $\mathcal{T}_U$ and continuous functor $u\colon \mathcal{T}_X \to \mathcal{T}_U$ as in Example 7.a. Next, consider the functor

$$\begin{aligned} v\colon \mathcal{T}_U &\longrightarrow \mathcal{T}_X \\ V &\longmapsto V. \end{aligned}$$

It is a cocontinuous functor, thus it induces a geometric morphism $(v^p)^\# \dashv {}_s v$. One can see $(v^p)^\# = j^{-1}$ and ${}_s v = j_*$ by Proposition 5.8. In other words, the cocontinuous functor v induces the same geometric morphism as the continuous functor u.

11 Example (Open map) Let $f\colon X \to Y$ be an open map between topological spaces. Then we have sites $\mathcal{T}_X$ and $\mathcal{T}_Y$ and continuous functor $u\colon \mathcal{T}_Y \to \mathcal{T}_X$ as in Example 7.a. Next, consider the functor

$$\begin{aligned} v\colon \mathcal{T}_X &\longrightarrow \mathcal{T}_Y \\ U &\longmapsto f(U). \end{aligned}$$

It is a cocontinuous functor, thus it induces a geometric morphism $(v^p)^{\#} \dashv {}_sv$. One can see $(v^p)^{\#} = f^{-1}$ and ${}_sv = f_*$. Indeed, for any sheaf $\mathscr{G}$ on Y, we have

$$v^p\mathscr{G}(U) = \mathscr{G}(f(U)) \qquad \text{and} \qquad u_p\mathscr{G}(U) = \varinjlim_{f(U)\subset V} \mathscr{G}(V) = \mathscr{G}(f(U)).$$

Therefore, the cocontinuous functor v induces the same geometric morphism as the continuous functor u.

Remark Note that the functor v in Example 10 is both cocontinuous and continuous, while the functor v in Example 11 is not continuous in general.

Extension by empty

12 Lemma (Extension by empty) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a functor which is both continuous and cocontinuous. Let $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be the geometric morphism induced by u, then*

1. *$f^{-1} = u^p$;*
2. *f^{-1} has a left adjoint $f_! := (u_p)^{\#}$.*

In this case, we have a sequence of adjunctions:

$$f_! \dashv f^{-1} \dashv f_*.$$

For any sheaf $\mathscr{F}$ on $\mathcal{C}$, the sheaf $f_!\mathscr{F}$ is called the **extension of $\mathscr{F}$ by empty**. In algebraic value case, it is called **extension of $\mathscr{F}$ by zero**. Note that the functor $f_!$ is *NOT* exact in general.

The following lemmas describe properties of $f_!$ under certain assumptions.

13 Lemma (Commutativity with finite connected limits) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a functor which is both continuous and cocontinuous. Let $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be the geometric morphism induced by u. Assume $\mathcal{C}$ has fibre products and equalizers and u commutes with them. Then $f_!$ commutes with finite connected limits.*

Proof: By Lemma 1.8 and the fact that *coproducts commute with connected limits* (refer §2.12 in *BMO*) It suffices to show the opposite of $\mathcal{I}_V^u$ is a disjoint union of filtered categories. To do this, it suffices to show

1. For any $f\colon (A,\phi) \to (C,\chi)$ and $g\colon (B,\psi) \to (C,\chi)$, there exists some $f'\colon (D,\theta) \to (A,\phi)$ and $g'\colon (D,\theta) \to (B,\psi)$.

2. For any $f,g\colon (A,\phi) \rightrightarrows (B,\psi)$, there exists a $h\colon (C,\theta) \to (A,\phi)$ such that $f \circ h = g \circ h$.

For *1.*, let D be the fibre product of A and B over C in $\mathcal{C}$, then $u(D) = u(A) \times_{u(C)} u(B)$. Thus there exists a unique morphism $\theta\colon V \to u(D)$ compatible with ϕ and ψ. Then the two projections $(D,\theta) \to (A,\phi)$ and $(D,\theta) \to (B,\psi)$ are the required ones.

The prove for *2.* is the same in Lemma 1.8. $\square$

14 Lemma (Fully faithful) *Let $\mathcal{C},\mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a functor which is both continuous and cocontinuous. Let $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be the geometric morphism induced by u. If u is fully faithful, then $\eta\colon \mathrm{id}_{\mathbf{Sh}(\mathcal{C})} \Rightarrow f^{-1}f_!$ and $\epsilon\colon f^{-1}f_* \Rightarrow \mathrm{id}_{\mathbf{Sh}(\mathcal{C})}$ are natural isomorphisms.*

Proof: Let $U \in \operatorname{ob}\mathcal{C}$. We have

$$f^{-1}f_*\mathscr{F}(U) = f_*\mathscr{F}(u(U)) = \varprojlim_{{}_{u(U)}\mathcal{I}^{\mathrm{opp}}} \mathscr{F}(V).$$

For any $(V,\phi) \in \operatorname{ob}{}_{u(U)}\mathcal{I}$, since u is fully faithful, there exists a morphism $\psi\colon V \to U$ such that $u(\psi) = \phi$. Therefore, ${}_{u(U)}\mathcal{I}$ has a terminal object $(U, \mathrm{id}_{u(U)})$. Thus $f^{-1}f_*\mathscr{F}(U) = \mathscr{F}(U)$.

On the other hand, we have

$$f^{-1}f_!\mathscr{F}(U) = f_!\mathscr{F}(u(U)) = (u_p\mathscr{F})^{\#}(u(U)),$$

and

$$u_p\mathscr{F}(u(U)) = \varinjlim_{\mathcal{I}_{u(U)}^{\mathrm{opp}}} \mathscr{F}(V).$$

For any $(V,\phi) \in \operatorname{ob}\mathcal{I}_{u(U)}$, since u is fully faithful, there exists a morphism $\psi\colon U \to V$ such that $u(\psi) = \phi$. Therefore, $\mathcal{I}_{u(U)}$ has an initial object $(U, \mathrm{id}_{u(U)})$. Thus $u_p\mathscr{F}(u(U)) = \mathscr{F}(U)$. Since u is both continuous and cocontinuous, any covering of $u(U)$ in $\mathcal{D}$ can be refined by a covering $\{u(U_i) \to u(U)\}$ in $\mathcal{D}$, where $\{U_i \to U\}$ is a covering in $\mathcal{C}$. Therefore $(u_p\mathscr{F})^+(u(U)) = \mathscr{F}(U)$. Thus $(u_p\mathscr{F})^{\#}(u(U)) = \mathscr{F}(U)$ as desired. $\square$

15 Lemma (Inducing morphisms of sites) *Let $\mathcal{C},\mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a functor which is both continuous and cocontinuous. Let $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be the geometric morphism induced by u. Assume*

- *u is fully faithful;*

- *$\mathcal{C}$ is finite-complete and u is left exact.*

Then u induces a morphism of sites $g\colon \mathcal{D} \to \mathcal{C}$ such that

1. *$g^{-1} = f_!$ and $g_* = f^{-1}$;*
2. *the composition $g \circ f$ of f and g is isomorphic to the identity geometric morphism $\mathrm{id}_{\mathbf{Sh}(\mathcal{C})}$;*
3. *g^{-1} is fully faithful.*

Proof: *1.* follows from Lemma 8.b, *2.* from Lemma 14. As for *3.*, consider any sheaves $\mathscr{F}$ and $\mathscr{G}$ on $\mathcal{C}$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathscr{F}, \mathscr{G}) &\cong \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(g_* g^{-1}\mathscr{F}, \mathscr{G}) \\ &\cong \mathrm{Hom}_{\mathbf{Sh}(\mathcal{D})}(g^{-1}\mathscr{F}, g^{-1}\mathscr{G}).\end{aligned}$$ □

16 Lemma (Cocontinuous functors with a right adjoint) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Let $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be the geometric morphism induced by u. Assume u has a right adjoint v. Then $f_* = v^s$, $f^{-1} = v_s := (v_p)^{\#}$. Therefore, when v is continuous, it defines a morphism of sites which induces the same geometric morphism with u.*

Proof: Let $\mathscr{F}$ be a sheaf on $\mathcal{C}$, it suffices to show $f_*\mathscr{F}(V) = \mathscr{F}(v(V))$ for any object $V \in \mathrm{ob}\,\mathcal{D}$. First, we have $u^p h_V = h_{v(V)}$ by Lemma 1.11. Then, by Theorem 9.b, we have

$$f^{-1}(h_V^{\#}) = (u^p h_V)^{\#} = h_{v(V)}^{\#}.$$

Therefore

$$\begin{aligned}f_*\mathscr{F}(V) &= \mathrm{Hom}_{\mathbf{Sh}(\mathcal{D})}(h_V^{\#}, f_*\mathscr{F}) \\ &= \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(f^{-1}h_V^{\#}, \mathscr{F}) \\ &= \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(h_{v(V)}^{\#}, \mathscr{F}) \\ &= \mathscr{F}(v(V)).\end{aligned}$$ □

16.a Corollary (Cocontinuous functors with a left adjoint) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a functor which is both continuous and cocontinuous. Let $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be the geometric morphism induced by u. Assume u has a left adjoint w.*

1. *$f_! = (w^p)^{\#}$ and it is exact.*
2. *If w is continuous, then $f_!$ has a left adjoint.*
3. *If w is cocontinuous and induces a geometric morphism $g\colon \mathbf{Sh}(\mathcal{D}) \to \mathbf{Sh}(\mathcal{C})$, then $g^{-1} = f_!$ and $g_* = f^{-1}$.*

Proof: *1.* follows from Lemma 1.11. If w is continuous, then $f_! = w^p = w^s$, which has a left adjoint w_s as in Lemma 6.b. If w is cocontinuous, then the statement is nothing but Lemma 16. $\square$

17 Lemma (Existence of lower shriek) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be a geometric morphism. Assume $\mathcal{C}$ has a subcategory $\mathcal{E}$ such that*

- *for every $U \in \operatorname{ob}\mathcal{E}$, there exists a sheaf $\mathscr{G}_U$ on $\mathcal{D}$ such that $g^{-1}\mathscr{F}(U) = \operatorname{Hom}_{\mathbf{Sh}(\mathcal{D})}(\mathscr{G}_U, \mathscr{F})$ functorally for $\mathscr{F} \in \operatorname{ob}\mathbf{Sh}(\mathcal{D})$;*
- *every object in $\mathcal{C}$ has a covering by objects in $\mathcal{E}$.*

Then f^{-1} has a left adjoint $f_!$.

Proof: Since $f^{-1}\mathscr{F}(U) = \operatorname{Hom}_{\mathbf{Sh}(\mathcal{C})}(h_U^{\#}, f^{-1}\mathscr{F})$, we have the bijection

$$\operatorname{Hom}_{\mathbf{Sh}(\mathcal{D})}(\mathscr{G}_U, \mathscr{F}) \cong \operatorname{Hom}_{\mathbf{Sh}(\mathcal{C})}(h_U^{\#}, f^{-1}\mathscr{F})$$

natural on $U \in \operatorname{ob}\mathcal{E}$ and $\mathscr{F} \in \operatorname{ob}\mathbf{Sh}(\mathcal{D})$.

Therefore, we already have a functor $f_!\colon h_U^{\#} \mapsto \mathscr{G}_U$ from the full subcategory of $\mathbf{Sh}(\mathcal{C})$ consisting of sheaves of the form $h_U^{\#}$ with $U \in \operatorname{ob}\mathcal{E}$ to $\mathbf{Sh}(\mathcal{D})$. It remains to extend it to a functor from $\mathbf{Sh}(\mathcal{C})$ to $\mathbf{Sh}(\mathcal{D})$.

First, since $\prod \operatorname{Hom}(-,-) = \operatorname{Hom}(\coprod, -)$, we have natural bijection

$$\operatorname{Hom}_{\mathbf{Sh}(\mathcal{D})}(\coprod \mathscr{G}_{U_j}, \mathscr{F}) \cong \operatorname{Hom}_{\mathbf{Sh}(\mathcal{C})}(\coprod h_{U_j}^{\#}, f^{-1}\mathscr{F}).$$

This extends $f_!$ to coproducts of some $h_U^{\#}$ with each $U \in \operatorname{ob}\mathcal{E}$.

Now, for any sheaf $\mathscr{H}$ on $\mathcal{C}$, by Lemma 7.16, we have a coequalizer diagram

$$\mathscr{H}_1 \rightrightarrows \mathscr{H}_0 \longrightarrow \mathscr{H}.$$

where $\mathscr{H}_1$ and $\mathscr{H}_0$ are coproducts of some $h_U^{\#}$ with each $U \in \operatorname{ob}\mathcal{E}$. Apply $f_!$ to $\mathscr{H}_1 \rightrightarrows \mathscr{H}_0$, then the coequalizer of it gives $f_!\mathscr{H}$. One can see this functor $f_!\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ is left adjoint to f^{-1}. $\square$

§ I.9 Localization

In this section, we discuss *localizations* at the level of sites. First, we discuss localization of sites, then morphisms of sites, then combination of them. Finally, as applications, we defines the notions of *sheaf Hom* and *gluing data.*

Localization of sites

1 **(Localization)** Let $\mathcal{C}$ be a site and $X \in \operatorname{ob}\mathcal{C}$. Then the *slice category* $\mathcal{C}/X$ (whose objects are morphisms $U \to X$ in $\mathcal{C}$ with fixed target X, usually denoted as U/X and viewed as *object related to* X, and morphisms are morphisms in $\mathcal{C}$ compatible with them) inherits a coverage from $\mathcal{C}$: a family $\mathfrak{U}/X = \{U_i/X \to U/X\}$ in $\mathcal{C}/X$ is a covering when $\mathfrak{U} = \{U_i \to U\}$ is a covering in $\mathcal{C}$. This site $\mathcal{C}/X$ is called the **localization** of $\mathcal{C}$ at the **base object** X.

Consider the forgetful functor $j_X : \mathcal{C}/X \to \mathcal{C}$. It is both continuous and cocontinuous. Therefore, by Lemma 8.12, it induces a geometric morphism

$$j_X \colon \mathbf{Sh}(\mathcal{C}/X) \longrightarrow \mathbf{Sh}(\mathcal{C}),$$

called the **localization morphism**. This geometric morphism is given by the *direct image functor* j_{X*} and the *inverse image functor* j_X^{-1}. For any sheaf $\mathscr{F}$ on $\mathcal{C}$, we call the sheaf $j_X^{-1}\mathscr{F}$ the **restriction of** $\mathscr{F}$ **on** X and denoted by $\mathscr{F}|_X$. Obviously, we have $\mathscr{F}|_X(U/X) = \mathscr{F}(U)$. Moreover, j_X^{-1} has a left adjoint $j_{X!}$. For any sheaf $\mathscr{G}$ on $\mathcal{C}/X$, we call the sheaf $j_{X!}\mathscr{G}$ the **extension of** $\mathscr{G}$ **by empty**.

2 **Lemma (Description of $j_{X!}$)** *Let $\mathcal{C}$ be a site and $X \in \operatorname{ob}\mathcal{C}$. Let $\mathscr{G}$ be a presheaf on $\mathcal{C}/X$. Then $j_{X!}(\mathscr{G}^{\#})$ is the sheafification of the presheaf*

$$V \mapsto \coprod_{\phi \in \operatorname{Hom}_{\mathcal{C}}(U,X)} \mathscr{G}(U \xrightarrow{\phi} X).$$

Proof: First of all, by definition,

$$j_{Xp}\mathscr{G}(U) = \varinjlim_{(V/X, U\to V) \in \operatorname{ob}\mathcal{I}_U^{\mathrm{opp}}} \mathscr{G}(V).$$

By the family $\{(U/X, \mathrm{id}_U) | U/X \in \operatorname{Hom}_{\mathcal{C}}(U,X)\}$ is final in $\mathcal{I}_U^{\mathrm{opp}}$, thus

$$j_{Xp}\mathscr{G}(U) = \coprod_{\phi \in \operatorname{Hom}_{\mathcal{C}}(U,X)} \mathscr{G}(U \xrightarrow{\phi} X).$$

By Lemma 8.12, $j_{X!}(\mathscr{G}^{\#}) = (j_{Xp}(\mathscr{G}^{\#}))^{\#}$, which equals to $(j_{Xp}\mathscr{G})^{\#}$ by Lemma 8.6.b. $\square$

2.a Corollary *Let $\mathcal{C}$ be a site and $X \in \operatorname{ob}\mathcal{C}$. Let $U/X \in \operatorname{ob}\mathcal{C}/X$. Then we have $j_{X!}(h^\#_{U/X}) = h^\#_U$.*

Proof: Since $h_{U/X}(V/X) = \operatorname{Hom}_{\mathcal{C}/X}(V/X, U/X)$ and

$$\operatorname{Hom}_{\mathcal{C}}(V,U) = \coprod_{V/X \in \operatorname{Hom}_{\mathcal{C}}(V,X)} \operatorname{Hom}_{\mathcal{C}/X}(V/X, U/X),$$

the statement follows from Lemma 2. □

2.b Corollary *Let $\mathcal{C}$ be a site and $X \in \operatorname{ob}\mathcal{C}$. For any sheaf $\mathscr{F}$ on $\mathcal{C}$, we have $j_{X!}j_X^{-1}\mathscr{F} = \mathscr{F} \times h^\#_X$.*

Proof: By Lemma 2, $j_{X!}j_X^{-1}\mathscr{F}$ is the sheafification of $U \mapsto \mathscr{F}(U) \times h_X(U)$. Thus $j_{X!}j_X^{-1}\mathscr{F} = \mathscr{F} \times h^\#_X$. □

3 Theorem *Let $\mathcal{C}$ be a site and $X \in \operatorname{ob}\mathcal{C}$. The functor $j_{X!}$ gives an equivalence of categories*

$$\mathbf{Sh}(\mathcal{C}/X) \longrightarrow \mathbf{Sh}(\mathcal{C})/h^\#_X.$$

Proof: First, any topos has a terminal object, namely the constant sheaf $\underline{*}$ of singleton. But in $\mathbf{Sh}(\mathcal{C}/X)$, we have $\underline{*} = h^\#_{X/X}$. Thus for any sheaf $\mathscr{G}$ on $\mathcal{C}/X$, we have a canonical morphism

$$j_{X!}(\mathscr{G}) \longrightarrow j_{X!}(\underline{*}) = j_{X!}(h^\#_{X/X}) = h^\#_X.$$

This gives a functor from $\mathbf{Sh}(\mathcal{C}/X)$ to $\mathbf{Sh}(\mathcal{C})/h^\#_X$.

Conversely, for any sheaf $\mathscr{F}$ on $\mathcal{C}$ and morphism $\varphi\colon \mathscr{F} \to h^\#_X$, we define $\mathscr{F}_\varphi(U/X)$ to be the fiber of $U/X \in h_X(U) \to h^\#_X(U)$ along $\mathscr{F}(U) \to h^\#_X(U)$.

$$\begin{tikzcd}
\mathscr{F}_\varphi(U/X) \arrow[rr, hook] \arrow[d] & & \mathscr{F}(U) \arrow[d, "\varphi"] \\
U/X \quad \in & h_X(U) \arrow[r] & h^\#_X(U)
\end{tikzcd}$$

Then one can see this $\mathscr{F}_\varphi$ is a sheaf on $\mathcal{C}/X$ and further the functor $\varphi \mapsto \mathscr{F}_\varphi$ gives a weak inverse of the functor above. □

Note that this lemma factors $j_{X!}$ as

$$\mathbf{Sh}(\mathcal{C}/X) \longrightarrow \mathbf{Sh}(\mathcal{C})/h^\#_X \longrightarrow \mathbf{Sh}(\mathcal{C}),$$

where the first functor is an equivalence of categories and the second is a localization again! We sometimes also denote the equivalence as $j_{X!}$.

3.a Corollary *Let $\mathcal{C}$ be a site and $X \in \operatorname{ob}\mathcal{C}$. The functor $j_{X!}$ commutes with finite connected limits. In particular, if $\mathscr{F} \subset \mathscr{F}'$ in $\mathbf{Sh}(\mathcal{C}/X)$, then we have $j_{X!}\mathscr{F} \subset j_{X!}\mathscr{F}'$.*

Proof: One can see the forgetful functor $\mathbf{Sh}(\mathcal{C})/h_X^{\#} \to \mathbf{Sh}(\mathcal{C})$ commutes with fibre products and equalizers. Therefore it commutes with finite connected limits and so does $j_{X!}$. $\square$

Proof: Note that coproducts commute with finite connected limites, then by Lemma 2, so is $j_{X!}$. $\square$

Now, we discuss properties of localization under certain assumptions.

First, if the site $\mathcal{C}$ has subcanonical topology, then the description of $j_{X!}$ would be more simply.

4 Lemma *Let $\mathcal{C}$ be a site and $X \in \operatorname{ob}\mathcal{C}$. Assume $\mathcal{C}$ has subcanonical topology. Then for any sheaf $\mathscr{G}$ on $\mathcal{C}/X$, we have*

$$j_{X!}\mathscr{G}(U) = \coprod_{\phi \in \operatorname{Hom}_{\mathcal{C}}(U,X)} \mathscr{G}(U \xrightarrow{\phi} X).$$

Proof: One needs to show that $U \mapsto \coprod_{\phi \in \operatorname{Hom}_{\mathcal{C}}(U,X)} \mathscr{G}(U \xrightarrow{\phi} X)$ defines a sheaf $\mathscr{H}$ on $\mathcal{C}$. Let $\mathfrak{U} = \{U_i \to U\}_{i \in I}$ be a covering in $\mathcal{C}$. We need to show $\mathscr{H}(U) \cong \check{H}^0(\mathfrak{U}, \mathscr{H})$. Note that there is a canonical morphism $\mathscr{H} \to h_X$ given by $(s, \phi) \mapsto \phi$, where $\phi \colon U \to X$ is an object in $\mathcal{C}/X$ and $s \in \mathscr{G}(\phi)$.

Let $(s_i, \phi_i)_{i \in I} \in \check{H}^0(\mathfrak{U}, \mathscr{H})$. Then $(\phi_i)_{i \in I} \in \check{H}^0(\mathfrak{U}, h_X)$. Since $\mathcal{C}$ has subcanonical topology, h_X is a sheaf and thus $\check{H}^0(\mathfrak{U}, h_X) = h_X(U)$. Therefore there exists a unique $\phi \colon U \to X$ such that ϕ_i are compositions of ϕ with $U_i \to U$. Then, $\mathfrak{U}/X$ is a covering of ϕ in $\mathcal{C}/X$. In this case, $(s_i)_{i \in I}$ lies in $\check{H}^0(\mathfrak{U}/X, \mathscr{G})$ and thus defines a section $s \in \mathscr{G}(\phi)$. One can see the pair $(s, \phi) \in \mathscr{H}(U)$ is unique the preimage of $(s_i, \phi_i)_{i \in I}$ under the canonical map $\mathscr{H}(U) \to \check{H}^0(\mathfrak{U}, \mathscr{H})$. This shows the desired isomorphism. $\square$

Next, under suitable assumption, the forgetful functor j_X can also be viewed as a morphism of sites. The following lemma describes this assumption and gives a description under it.

5 Lemma *Let $\mathcal{C}$ be a site and $X \in \operatorname{ob}\mathcal{C}$. Assume $\mathcal{C}$ has finite products. Then*

1. *j_X has a continuous right adjoint v given by $v(U) = U \times X/X$;*
2. *the functor v defines a morphism of sites $\mathcal{C}/X \to \mathcal{C}$ which induces the same geometric morphisms as j_X;*
3. *$j_{X*}\mathscr{F}(U) = \mathscr{F}(U \times X/X)$.*

Proof: *1.* To show the adjunction, we only need to verify that for any $U/X \in \mathcal{C}/X$ and $V \in \mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(U, V) \cong \mathrm{Hom}_{\mathcal{C}/X}(U/X, V \times X/X),$$

which is clear. To see v is continuous, let $\mathfrak{U} = \{U_i \to U\}$ be a covering of $\mathcal{C}$, then $\mathfrak{U} \times_U (U \times X)$ is a covering of $U \times X$ in $\mathcal{C}$ and therefore $\mathfrak{U} \times_U (U \times X)/X$ is a covering in $\mathcal{C}/X$.

2. and *3.* follow from Lemma 8.16. $\square$

The following lemma generalizes Proposition 5.8 and Theorem 5.10.

6 Lemma *Let $\mathcal{C}$ be a site and $X \in \mathrm{ob}\,\mathcal{C}$. Assume every $U \in \mathrm{ob}\,\mathcal{C}$ has at most one morphism to X. Then $\eta\colon \mathrm{id}_{\mathbf{Sh}(\mathcal{C}/X)} \Rightarrow j_X^{-1} j_{X!}$ and $\epsilon\colon j_X^{-1} j_{X*} \Rightarrow \mathrm{id}_{\mathbf{Sh}(\mathcal{C}/X)}$ are natural isomorphisms.*

Proof: Note that the assumption implies j_X is fully faithful. Then the statement follows from Lemma 8.14. $\square$

Now, ewe turn to consider what happens when change the base object of localization.

7 Lemma (Changing base object) *Let $\mathcal{C}$ be a site and $f\colon Y \to X$ be a morphism of $\mathcal{C}$. Then there exists a commutative diagram*

$$\begin{tikzcd} \mathcal{C}/Y \arrow[rr, "j"] \arrow[dr, "j_Y"'] & & \mathcal{C}/X \arrow[dl, "j_X"] \\ & \mathcal{C} & \end{tikzcd}$$

of cocontinuous functors. Here $j\colon \mathcal{C}/Y \to \mathcal{C}/X$ is identified with the functor $j_{Y/X}\colon (\mathcal{C}/X)/(Y/X) \to \mathcal{C}/X$ via the identification $(\mathcal{C}/X)/(Y/X) = \mathcal{C}/Y$. Moreover, we have $j_{Y!} = j_{X!} \circ j_!$, $j_Y^{-1} = j^{-1} \circ j_X^{-1}$, and $j_{Y*} = j_{X*} \circ j_*$.

Proof: The identification is obvious and the statements then follow from Lemma 8.9.d. $\square$

7.a Lemma *Notations as in Lemma 7. Through the identifications $\mathbf{Sh}(\mathcal{C}/Y) = \mathbf{Sh}(\mathcal{C})/h_Y^{\#}$ and $\mathbf{Sh}(\mathcal{C}/X) = \mathbf{Sh}(\mathcal{C})/h_X^{\#}$ of Theorem 3, the functor j^{-1} has the following description*

$$j^{-1}(\mathscr{H} \xrightarrow{\varphi} h_X^{\#}) = (\mathscr{H} \times_{h_X^{\#}} h_Y^{\#} \longrightarrow h_Y^{\#}).$$

Proof: Let $\varphi\colon \mathscr{H} \to h_X^{\#}$ be an object in $\mathbf{Sh}(\mathcal{C})/h_X^{\#}$. By Theorem 3, it corresponds to the sheaf $\mathscr{H}_\varphi$ on $\mathcal{C}/X$. Let $\mathscr{H}'_{\varphi'}$ denote the sheaf on $\mathcal{C}/Y$ corresponding to

$$\mathscr{H}' = \mathscr{H} \times_{h_X^{\#}} h_Y^{\#} \xrightarrow{\varphi'} h_Y^{\#}.$$

For any $U/Y \in \mathrm{ob}\,\mathcal{C}/Y$, on one hand, we have

$$j^{-1}\mathscr{H}_\varphi(U/Y) = \mathscr{H}_\varphi(f \circ U/Y),$$

which is the fiber of $f \circ U/Y \in h_X(U) \to h_X^\#(U)$ along $\mathscr{H}(U) \xrightarrow{\varphi} h_X^\#(U)$. On the other hand, $\mathscr{H}'_{\varphi'}(U/Y)$ is the fiber of $U/Y \in h_Y(U) \to h_Y^\#(U)$ along $\mathscr{H}'(U) \xrightarrow{\varphi'} h_Y^\#(U)$. But we have the following Cartesian diagram

$$\begin{tikzcd}
\mathscr{H}'(U) \arrow[r] \arrow[d, "\varphi'"'] & \mathscr{H}(U) \arrow[d, "\varphi"] \\
h_Y^\#(U) \arrow[r] & h_X^\#(U)
\end{tikzcd}$$

thus $\mathscr{H}'_{\varphi'}(U/Y)$ is also the fiber of $U/Y \in h_Y(U) \to h_Y^\#(U) \to h_X^\#(U)$ along $\mathscr{H}(U) \xrightarrow{\varphi} h_X^\#(U)$. But the image of $U/Y \in h_Y(U) \to h_Y^\#(U) \to h_X^\#(U)$ coincide with that of $f \circ U/Y \in h_X(U) \to h_X^\#(U)$, thus so do their fibers. Therefore $\mathscr{H}'_{\varphi'} = j^{-1}\mathscr{H}_\varphi$ as desired. □

The following is the same proof in more concrete statements.

Proof: Let $\varphi\colon \mathscr{H} \to h_X^\#$ be an object in $\mathbf{Sh}(\mathcal{C})/h_X^\#$. Let θ denote the canonical morphism induced by sheafification. By Theorem 3, it corresponds to the sheaf $\mathscr{H}_\varphi$ on $\mathcal{C}/X$ and we have

$$\mathscr{H}_\varphi(U/X) = \{s \in \mathscr{H}(U) | \varphi(s) = \theta(U/X)\}.$$

Write f as Y/X, we have

$$\begin{aligned}
j^{-1}\mathscr{H}_\varphi(U/Y) &= j^{-1}_{Y/X}\mathscr{H}_\varphi((Y/X \circ U/Y)/(Y/X)) \\
&= \mathscr{H}_\varphi(Y/X \circ U/Y) \\
&= \{s \in \mathscr{H}(U) | \varphi(s) = \theta(Y/X \circ U/Y)\}.
\end{aligned}$$

On the other hand, the sheaf on $\mathcal{C}/Y$ corresponding to

$$\mathscr{H}' = \mathscr{H} \times_{h_X^\#} h_Y^\# \xrightarrow{\varphi'} h_Y^\#,$$

is given by

$$\begin{aligned}
\mathscr{H}'_{\varphi'}(U/Y) &= \left\{s' \in \mathscr{H}'(U) \middle| \varphi'(s') = \theta(U/Y)\right\} \\
&= \left\{(s,a) \in \mathscr{H}(U) \times h_Y^\#(U) \middle| a = \theta(U/Y), \varphi(s) = \theta(Y/X \circ U/Y)\right\} \\
&= \{s \in \mathscr{H}(U) | \varphi(s) = \theta(Y/X \circ U/Y)\}.
\end{aligned}$$

Therefore $j^{-1}\mathscr{H}_\varphi = \mathscr{H}'_{\varphi'}$. □

Under suitable assumption, the forgetful functor j can also be viewed as a morphism of sites.

8 Lemma *Let $\mathcal{C}$ be a site and $f\colon Y \to X$ be a morphism of $\mathcal{C}$. Assume $\mathcal{C}$ has fibre products. Then*

1. *j has a continuous right adjoint v given by $v(U/X) = U \times_X Y/Y$;*
2. *the functor v defines a morphism of sites $\mathcal{C}/Y \to \mathcal{C}/X$ which induces the same geometric morphisms as j;*
3. *$j_*\mathscr{F}(U) = \mathscr{F}(U \times_X Y/Y)$.*

Proof: Follows from the identification $j = j_{Y/X}$ and Lemma 5 . □

A typical way to obtain cocontinuous functors is checking the following *Property P*.

9 (Property P) Let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor between categories. Given morphisms $g\colon u(U) \to V$ and $f\colon W \to V$ in $\mathcal{D}$ we can define a presheaf on $\mathcal{C}$ by

$$T \longmapsto \mathrm{Hom}_{\mathcal{C}}(T, U) \times_{\mathrm{Hom}_{\mathcal{D}}(u(T),V)} \mathrm{Hom}_{\mathcal{D}}(u(T), W).$$

If this presheaf is representable, then we denote the representative object by $U \times_{g,V,f} W$ or simply $U \times_V W$ or $g \times_V f$. Obviously this is a generalization of fibre product and we have

$$u(U \times_V W) = u(U) \times_V W.$$

Assume $\mathcal{C}$ and $\mathcal{D}$ are sites. A functor $u\colon \mathcal{C} \to \mathcal{D}$ is said to have **property P** if for every covering $\mathfrak{V}$ and any morphism $g\colon u(U) \to V$ in $\mathcal{D}$, $U \times_V \mathfrak{V}$ exists and is a covering in $\mathcal{C}$. One can see this property is similar to the definition of continuous functor.

Note that if this is the case, then

$$u(U \times_V \mathfrak{V}) = u(U) \times_V \mathfrak{V}$$

is a covering and refines $\mathfrak{V}$. Therefore u have property P implies that u is cocontinuous but the converse fails to be true.

10 Lemma *The localization j satisfies property P.*

Proof: Let $\mathfrak{U}/X$ be a covering and $g\colon j(V/Y) \to (U/X)$ be a morphism in $\mathcal{C}/X$. Then $(V/Y) \times_{(U/X)} (\mathfrak{U}/X) = (V \times_U \mathfrak{U}/X)$ and is a covering in $\mathcal{C}/Y$.□

Localization of morphisms of sites

Now, we talk about localization of a morphism of sites. That is a functorial morphism of sites connecting the localizations of sites. Here are the details:

11 **Lemma (Localization of morphisms of sites)** *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a morphism of sites defined by a continuous functor $u\colon \mathcal{D} \to \mathcal{C}$. Let $Y \in \operatorname{ob} \mathcal{D}$ and $X = u(Y)$. Then the functor*

$$\begin{aligned} u'\colon \mathcal{D}/Y &\longrightarrow \mathcal{C}/X \\ V/Y &\longmapsto u(V)/X \end{aligned}$$

defines a morphism of sites $f'\colon \mathcal{C}/X \to \mathcal{D}/Y$, which satisfies a commutative diagram of topoi

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}/X) \arrow[r, "j_X"] \arrow[d, "f'"'] & \mathbf{Sh}(\mathcal{C}) \arrow[d, "f"] \\ \mathbf{Sh}(\mathcal{D}/Y) \arrow[r, "j_Y"] & \mathbf{Sh}(\mathcal{D}) \end{tikzcd}$$

Through the identifications $\mathbf{Sh}(\mathcal{C}/X) = \mathbf{Sh}(\mathcal{C})/h_X^\#$ and $\mathbf{Sh}(\mathcal{D}/Y) = \mathbf{Sh}(\mathcal{D})/h_Y^\#$ of Theorem 3, the functor f'^{-1} has the following description

$$f'^{-1}(\mathscr{H} \xrightarrow{\varphi} h_Y^\#) = (f^{-1}\mathscr{H} \xrightarrow{f^{-1}\varphi} h_X^\#).$$

Finally, we have $f'_* \circ j_X^{-1} = j_Y^{-1} \circ f_*$.

Proof: It is clear that u' is also continuous. Then u' induces an adjunction

$$f'^* \dashv f'_*\colon\ \mathbf{Sh}(\mathcal{C}/X) \leftrightarrows \mathbf{Sh}(\mathcal{C}/Y)$$

by $f'_* = u'^s$ and $f'^{-1} = u'_s$. Then for any sheaf $\mathscr{F}$ on $\mathcal{C}$, we have

$$f'_* j_X^{-1}\mathscr{F}(V/Y) = j_X^{-1}\mathscr{F}(u(V)/X) = \mathscr{F}(u(V)) = f_*\mathscr{F}(V) = j_Y^{-1} f_*\mathscr{F}(V/Y).$$

Thus $f'_* \circ j_X^{-1} = j_Y^{-1} \circ f_*$. It remains to show u' is exact, the diagram of topoi is commutative and the description of f'^{-1} is as given.

First, to show the escription of f'^{-1} is as given, it suffices to show the following diagram is commutative:

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}/X) \arrow[r, "j_{X!}"] & \mathbf{Sh}(\mathcal{C})/h_X^\# \arrow[r] & \mathbf{Sh}(\mathcal{C}) \\ \mathbf{Sh}(\mathcal{D}/Y) \arrow[r, "j_{Y!}"] \arrow[u, "f'^{-1}"] & \mathbf{Sh}(\mathcal{D})/h_Y^\# \arrow[r] \arrow[u, "f^{-1}"] & \mathbf{Sh}(\mathcal{D}) \arrow[u, "f^{-1}"] \end{tikzcd}$$

By Lemma 8.6.b, we have $f^{-1}h_Y^{\#} = h_{u(Y)}^{\#} = h_X^{\#}$. Thus the right square is commutative. For any sheaf $\mathscr{H}$ on $\mathcal{D}/Y$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Sh}(\mathcal{D}/Y)}(j_{X!}f'^{-1}\mathscr{H}, \mathscr{F}) &= \mathrm{Hom}_{\mathbf{Sh}\,\mathcal{C}}(\mathscr{H}, f'_* j_X^{-1}\mathscr{F}) \\
&= \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathscr{H}, j_Y^{-1} f_*\mathscr{F}) \\
&= \mathrm{Hom}_{\mathbf{Sh}(\mathcal{D}/Y)}(f^{-1} j_{Y!}\mathscr{H}, \mathscr{F}).
\end{aligned}$$

Thus $j_{X!} \circ f'^{-1} = f^{-1} \circ j_{Y!}$. Therefore the diagram above is commutative.

Since f^{-1} is exact, the equivalences shows so is f'^{-1}.

Finally, we show $f'^{-1} \circ j_Y^{-1} = j_X^{-1} \circ f^{-1}$. Indeed, for any sheaf $\mathscr{G}$ on $\mathcal{D}$, by Corollary 2.b, we have

$$j_{X!}f'^{-1}j_Y^{-1}\mathscr{G} = f^{-1}j_{Y!}j_Y^{-1}\mathscr{G} = f^{-1}(\mathscr{G} \times h_Y^{\#}) = f^{-1}\mathscr{G} \times h_X^{\#} = j_{X!}j_X^{-1}f^{-1}\mathscr{G}.$$

Since $j_{X!}$ is an equivalence, we have $f'^{-1} \circ j_Y^{-1} = j_X^{-1} \circ f^{-1}$. The commutativity of the diagram of topoi then follows. □

11.a Lemma (Localization of morphisms of sites) *Let $u\colon \mathcal{D} \to \mathcal{C}$ be a continuous functor. Let $Y \in \operatorname{ob}\mathcal{D}$, $X = u(Y)$ and u' be*

$$\begin{aligned}
u'\colon \mathcal{D}/Y &\longrightarrow \mathcal{C}/X \\
V/Y &\longmapsto u(V)/X.
\end{aligned}$$

Assume $\mathcal{C}$ and $\mathcal{D}$ are finite-complete and u is left exact. Then there exists a commutative diagram of morphisms of sites

$$\begin{tikzcd}
\mathcal{C}/X \arrow[r, "j_X"] \arrow[d, "f'"'] & \mathcal{C} \arrow[d, "f"] \\
\mathcal{D}/Y \arrow[r, "j_Y"] & \mathcal{D}
\end{tikzcd}$$

where the horizontal morphisms are defined as in Lemma 8 and the vertical morphisms are defined by u and u'. Moreover, this commutative diagram induces the same commutative diagram of topoi as in Lemma 11. In particular, $f'_* \circ j_X^{-1} = j_Y^{-1} \circ f_*$.

Proof: By Lemma 8.8.b, u defines a morphism of sites $f\colon \mathcal{C} \to \mathcal{D}$. Then by Lemma 11, u' defines a morphism of sites $f'\colon \mathcal{C}/X \to \mathcal{D}/Y$. It remains to show $f \circ j_X = j_Y \circ f'$ as morphisms of sites, which follows from Lemma 8 and that

$$u(V) \times X = u(V) \times u(Y) = u(V \times Y).$$

□

Combining Lemma 7 and Lemma 11, we have

12 Lemma (General localization of morphisms of sites) *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a morphism of sites defined by a continuous functor $u\colon \mathcal{D} \to \mathcal{C}$. Let $Y \in \operatorname{ob}\mathcal{D}$, $X \in \operatorname{ob}\mathcal{C}$ and $c\colon X \to u(Y)$ a morphism in $\mathcal{C}$. Then there exists a commutative diagram of topoi*

$$\begin{tikzcd}
\mathbf{Sh}(\mathcal{C}/X) \arrow[r, "j_X"] \arrow[d, "f_c"'] & \mathbf{Sh}(\mathcal{C}) \arrow[d, "f"] \\
\mathbf{Sh}(\mathcal{D}/Y) \arrow[r, "j_Y"] & \mathbf{Sh}(\mathcal{D})
\end{tikzcd}$$

where $f_c = f' \circ j_c$, $f'\colon \mathbf{Sh}(\mathcal{C}/u(Y)) \to \mathbf{Sh}(\mathcal{D}/Y)$ is defined in Lemma 11, $j_c\colon \mathbf{Sh}(\mathcal{C}/X) \to \mathbf{Sh}(\mathcal{C}/u(Y))$ is defined in Lemma 7. Through the identifications $\mathbf{Sh}(\mathcal{C}/X) = \mathbf{Sh}(\mathcal{C})/h_X^{\#}$ and $\mathbf{Sh}(\mathcal{D}/Y) = \mathbf{Sh}(\mathcal{D})/h_Y^{\#}$ of Theorem 3, the functor f_c^{-1} has the following description

$$f_c^{-1}(\mathscr{H} \xrightarrow{\varphi} h_Y^{\#}) = (f^{-1}\mathscr{H} \times_{h^{\#}_{u(Y)}} h_X^{\#} \to h_X^{\#}).$$

Finally, for any morphisms $a\colon X \to X'$, $b\colon Y \to Y'$ and $c'\colon X' \to u(Y')$ making the diagram

$$\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "a"'] & u(Y) \arrow[d, "u(b)"] \\
X \arrow[r, "c'"] & u(Y')
\end{tikzcd}$$

the diagram

$$\begin{tikzcd}
\mathbf{Sh}(\mathcal{C}/X) \arrow[r, "j_a"] \arrow[d, "f_c"'] & \mathbf{Sh}(\mathcal{C}/X') \arrow[d, "f_{c'}"] \\
\mathbf{Sh}(\mathcal{D}/Y) \arrow[r, "j_b"] & \mathbf{Sh}(\mathcal{D}/Y')
\end{tikzcd}$$

commutes.

Proof: The first commutative diagram follows from those in Lemma 7 and Lemma 11. The description follows from those in Lemma 7.a and Lemma 11. The last statement follows from that all but the front square in the following diagram commutes thus so does the front:

$$\begin{tikzcd}
 & \mathbf{Sh}(\mathcal{C}) \arrow[dd, "f"] & \\
\mathbf{Sh}(\mathcal{C}/X) \arrow[ur, "j_X"] \arrow[rr, "j_a"] \arrow[dd, "f_c"'] & & \mathbf{Sh}(\mathcal{C}/X') \arrow[ul, "j_{X'}"'] \arrow[dd, "f_{c'}"] \\
 & \mathbf{Sh}(\mathcal{D}) & \\
\mathbf{Sh}(\mathcal{D}/Y) \arrow[ur, "j_Y"] \arrow[rr, "j_b"] & & \mathbf{Sh}(\mathcal{D}/Y') \arrow[ul, "j_{Y'}"']
\end{tikzcd}$$

Or, one can use the identifications and the descriptions. Let $\mathscr{H}$ be a sheaf on $\mathcal{D}$, then

$$\begin{aligned}
j_a^{-1} f_{c'}^{-1}(\mathscr{H} \to h_{Y'}^{\#}) &= j_a^{-1}(f^{-1}\mathscr{H} \times_{h_{u(Y')}^{\#}} h_{X'}^{\#} \to h_{X'}^{\#}) \\
&= (f^{-1}\mathscr{H} \times_{h_{u(Y')}^{\#}} h_{X'}^{\#} \times_{h_{X'}^{\#}} h_X^{\#} \to h_X^{\#}) \\
&= (f^{-1}\mathscr{H} \times_{h_{u(Y')}^{\#}} h_X^{\#} \to h_X^{\#}) \\
&= (f^{-1}\mathscr{H} \times_{h_{u(Y')}^{\#}} h_{u(Y)}^{\#} \times_{h_{u(Y)}^{\#}} h_X^{\#} \to h_X^{\#}) \\
&= (f^{-1}(\mathscr{H} \times_{h_{Y'}^{\#}} h_Y^{\#}) \times_{h_{u(Y)}^{\#}} h_X^{\#} \to h_X^{\#}) \\
&= f_c^{-1}(\mathscr{H} \times_{h_{Y'}^{\#}} h_Y^{\#} \to h_Y^{\#}) \\
&= f_c^{-1} j_b^{-1}(\mathscr{H} \to h_{Y'}^{\#}).
\end{aligned}$$

This shows the required commutativity. □

Unsurprised, there is a *cocontinuous* version of Lemma 11:

13 Lemma (Localization of a cocontinuous functor) *Let $u\colon \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Let $X \in \operatorname{ob}\mathcal{C}$ and $Y = u(X)$. Then the functor*

$$\begin{aligned}
u'\colon \mathcal{C}/X &\longrightarrow \mathcal{D}/Y \\
U/X &\longmapsto u(U)/Y
\end{aligned}$$

satisfies the commutative digram

$$\begin{tikzcd}
\mathcal{C}/X \arrow[r, "j_X"] \arrow[d, "u'"'] & \mathcal{C} \arrow[d, "u"] \\
\mathcal{D}/Y \arrow[r, "j_Y"] & \mathcal{D}
\end{tikzcd}$$

Moreover, u' is cocontinuous and we have a commutative diagram

$$\begin{tikzcd}
\mathbf{Sh}(\mathcal{C}/X) \arrow[r, "j_X"] \arrow[d, "f'"'] & \mathbf{Sh}(\mathcal{C}) \arrow[d, "f"] \\
\mathbf{Sh}(\mathcal{D}/Y) \arrow[r, "j_Y"] & \mathbf{Sh}(\mathcal{D})
\end{tikzcd}$$

where f (resp. f') is induced by u (resp. u').

Proof: The commutativity of the first diagram is clear. Once we proved u' is cocontinuous, the second commutative diagram follows.

To show u' is cocontinuous, let $\mathfrak{V}/Y = \{V_j/Y \to u(U)/Y\}$ be a covering in $\mathcal{D}/Y$. Since u is cocontinuous, there is a covering $\mathfrak{U} = \{U_i \to U\}$ in $\mathcal{C}$ such that $u(\mathfrak{U})$ refines $\mathfrak{V} = \{V_j \to u(U)\}$. Then $\mathfrak{U}/X = \{U_i/X \to U/X\}$ is a covering in $\mathcal{C}/X$ refining $\mathfrak{V}/Y$. Hence u' is cocontinuous. □

Remark In general, the equality $f'_* \circ j_X^{-1} = j_Y^{-1} \circ f_*$ does not hold in the case of Lemma 13.

Application: gluing sheaves

14 **(Sheaf Hom)** Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ a sheaf on a site $\mathcal{C}$. Define the presheaf $\mathcal{H}om(\mathscr{F},\mathscr{G})$ by

$$\mathcal{H}om(\mathscr{F},\mathscr{G})(U) := \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C}/U)}(\mathscr{F}|_U, \mathscr{G}|_U).$$

By the following Lemma 14.a, this is indeed a sheaf, called the **sheaf Hom**.

14.a **Lemma (Gluing morphisms)** *Let $\mathcal{C}$ be a site and $\{U_i \to X\}_{i\in I}$ a covering. Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ a sheaf on $\mathcal{C}$. Suppose that there are morphisms*

$$\varphi_i \colon \mathscr{F}|_{U_i} \longrightarrow \mathscr{G}|_{U_i}$$

such that for all $i, j \in I$, the restrictions of φ_i and φ_j to $U_i \times_X U_j$ are the same morphism $\varphi_{ij} \colon \mathscr{F}|_{U_i\times_X U_j} \to \mathscr{G}|_{U_i\times_X U_j}$. Then there exists a unique morphism

$$\varphi \colon \mathscr{F} \longrightarrow \mathscr{G}$$

whose restriction to each U_i is φ_i.

The sheaf Hom shares many properties of the Hom-set.

15 **Lemma** *Let $\mathscr{F}$ be a sheaf on a site $\mathcal{C}$, then $\mathcal{H}om(\underline{*}, \mathscr{F}) \cong \mathscr{F}$.*

Proof: Let $U \in \operatorname{ob}\mathcal{C}$. For any $\varphi \in \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C}/U)}(\underline{*}|_U, \mathscr{F}|_U)$, its corresponding section in $\mathscr{F}(U)$ is the image of the singleton under $\varphi(U)$. This gives rise to a morphism $\Phi \colon \mathcal{H}om(\underline{*}, \mathscr{F}) \to \mathscr{F}$. Conversely, any section $s \in \mathscr{F}(U)$ gives a map $* \to \mathscr{F}(U)$, and furthermore a map $* \to \mathscr{F}(U) \to \mathscr{F}(V)$ for any $V/U \in \mathcal{C}/U$. We can see this gives a morphism $\underline{*} \to \mathscr{F}|_U$. In this way we find the inverse of Φ, thus it's an isomorphism. □

The following two lemmas are easy to verify.

16 **Lemma (Sheaf Hom is left exact)** *Let $\mathscr{F}$ be a sheaf on a site $\mathcal{C}$, then $\mathcal{H}om(\mathscr{F}, -)$ is a left exact covariant functor and $\mathcal{H}om(-, \mathscr{F})$ is a left exact contravariant functor.*

The following theorem exposes why sheaf Hom shares similar properties with the Hom-set.

17 **Theorem (Sheaf Hom is the internal Hom)** *For any sheaves $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ on a site $\mathcal{C}$, there is a canonical bijection*

$$\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathscr{F} \times \mathscr{G}, \mathscr{H}) \cong \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathscr{F}, \mathcal{H}om(\mathscr{G}, \mathscr{H})).$$

Proof: Given a morphism $\varphi\colon \mathscr{F} \times \mathscr{G} \to \mathscr{H}$, one gets a family of compatible maps $\varphi(U)\colon \mathscr{F}(U) \times \mathscr{G}(U) \to \mathscr{H}(U)$, thus a compatible family of maps $\psi_U\colon \mathscr{F}(U) \to \mathrm{Hom}(\mathscr{G}(U), \mathscr{H}(U))$. For any $s \in \mathscr{F}(U)$, we have a family of maps $\psi_V(s|_V)\colon \mathscr{G}(V) \to \mathscr{H}(V)$ for all objects V/U of $\mathcal{C}/U$. The compatibility of ψ_V guarantees the compatibility of $\psi_V(s|_V)$, thus they give rise to a morphism $\mathscr{G}|_U \to \mathscr{H}|_U$. In this way, we get a map $\psi(U)\colon \mathscr{F}(U) \to \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C}/U)}(\mathscr{G}|_U, \mathscr{H}|_U)$. Such kind of maps are compatible and thus forms a morphism $\mathscr{F} \to \mathcal{H}om(\mathscr{G}, \mathscr{H})$. Now, we have a map $\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathscr{F} \times \mathscr{G}, \mathscr{H}) \to \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathscr{F}, \mathcal{H}om(\mathscr{G}, \mathscr{H}))$, whose inverse is easy to construct. $\square$

Remark The notion of sheaf Hom also works for sheaves of algebraic structures and the above results still hold.

18 (Gluing data) Let $\mathcal{C}$ be a site and $\mathfrak{U} = \{U_i \to U\}_{i\in I}$ a covering. A **gluing data for sheaves of sets with respect to the covering** $\mathfrak{U}$ consists of the following stuff:

- For each $i \in I$, a sheaf $\mathscr{F}_i$ of sets on $\mathcal{C}/U_i$;
- For each pair $i, j \in I$, an isomorphism $\varphi_{ij}\colon\ \mathscr{F}_i|_{U_i\times_U U_j} \to \mathscr{F}_j|_{U_i\times_U U_j}$,

satisfying the **cocycle condition**:

For any $i, j, k \in I$, the following diagram commutes.

$$\begin{tikzcd}
\mathscr{F}_i|_{U_i\times_U U_j\times_U U_k} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & \mathscr{F}_k|_{U_i\times_U U_j\times_U U_k} \\
& \mathscr{F}_j|_{U_i\times_U U_j\times_U U_k} \arrow[ur, "\varphi_{jk}"'] &
\end{tikzcd}$$

A *morphisms* between gluing data $(\mathscr{F}_i, \varphi_{ij})$ and $(\mathscr{G}_i, \psi_{ij})$ is a family of morphisms

$$f_i\colon \mathscr{F}_i \longrightarrow \mathscr{G}_i$$

such that the following diagram commutes.

$$\begin{tikzcd}
\mathscr{F}_i|_{U_i\times_U U_j} \arrow[r, "f_i"] \arrow[d, "\varphi_{ij}"'] & \mathscr{G}_i|_{U_i\times_U U_j} \arrow[d, "\psi_{ij}"] \\
\mathscr{F}_j|_{U_i\times_U U_j} \arrow[r, "f_j"] & \mathscr{G}_j|_{U_i\times_U U_j}
\end{tikzcd}$$

One can see this definition can be easily generalized to **gluing data for sheaves of algebraic structures**.

Let $\mathcal{C}$ be a site and $\mathfrak{U} = \{U_i \to U\}_{i\in I}$ a covering. Any sheaf $\mathscr{F}$ on $\mathcal{C}/U$ admits a canonical gluing data $(\mathscr{F}_i, \varphi_{ij})$, where $\mathscr{F}_i$ is the restriction $\mathscr{F}|_{U_i}$ and φ_{ij} is the induced morphism

$$\mathscr{F}|_{U_i}\Big|_{U_i\times_U U_j} \longrightarrow \mathscr{F}|_{U_j}\Big|_{U_i\times_U U_j}.$$

Moreover, this construction is functoral, meaning it gives rise to a functor from $\mathbf{Sh}(\mathcal{C}/U)$ to the category of gluing data.

19 Theorem (Sheaf = gluing data) *The above functor induces an equivalence of category between* $\mathbf{Sh}(\mathcal{C}/U)$ *and the category of gluing data. The similar statement holds for sheaves of algebraic structures.*

Proof: The functor is fully faithful by Lemma 14.a and essentially surjective by Lemma 19.a. □

19.a Lemma *Let $\mathcal{C}$ be a site and $\mathfrak{U} = \{U_i \to U\}_{i\in I}$ a covering. Let $(\mathscr{F}_i, \varphi_{ij})$ be a gluing data for sheaves of sets with respect to the covering $\mathfrak{U}$. Then there exists a sheaf $\mathscr{F}$ on $\mathcal{C}/U$ together with isomorphisms*

$$\varphi_i\colon\ \mathscr{F}|_{U_i} \longrightarrow \mathscr{F}_i$$

such that the following diagram commutes.

$$\begin{tikzcd}
\mathscr{F}|_{U_i\times_U U_j} \arrow[r, "\varphi_i"] \arrow[d, equal] & \mathscr{F}_i|_{U_i\times_U U_j} \arrow[d, "\varphi_{ij}"] \\
\mathscr{F}|_{U_i\times_U U_j} \arrow[r, "\varphi_j"] & \mathscr{F}_j|_{U_i\times_U U_j}
\end{tikzcd}$$

The similar statement holds for sheaves of algebraic structures.

Proof: For any object $V/U \in \operatorname{ob}\mathcal{C}/U$, the object $\mathscr{F}(V/U)$ is given as the equalizer of the morphisms:

$$\textstyle\prod_{i\in I}\mathscr{F}_i(V\times_U U_i/U_i) \rightrightarrows \prod_{i,j\in I}\mathscr{F}_i(V\times_U U_i\times_U U_j/U_i).$$

For sheaves of sets, this set can be written as

$$\mathscr{F}(V/U) = \left\{(s_i)\in\prod_{i\in I}\mathscr{F}_i(V\times_U U_i/U_i)\,\middle|\,\varphi_{ij}(s_i|_{V\times_U U_i\times_U U_j}) = s_j|_{V\times_U U_i\times_U U_j}\right\}.$$

As for the isomorphism, just note that a section in $\mathscr{F}|_{U_i}(V/U_i)$ is nothing but a system of compatible sections $(s_j)\in\prod_{j\in I}\mathscr{F}_i(V\times_U U_i\times_U U_j/U_i)$, which gives rise to a section $s\in\mathscr{F}_i(V/U_i)$. Thus the lemma follows. □

§ I.10 Special cocontinuous functors

In this section we show that any geometric morphism is equivalent to one comes from a morphism of sites.

Geometric morphisms are induced by morphisms of sites

The key tool is the notion of *special cocontinuous functors.*

1 (Special cocontinuous functors) Let $\mathcal{C}, \mathcal{D}$ be two sites. A **special cocontinuous functor** $u\colon \mathcal{C} \to \mathcal{D}$ is a functor which is both continuous and cocontinuous and satisfies the followings.

SC1. (*locally faithful*) For any $a, b\colon U \rightrightarrows U'$ in $\mathcal{C}$ such that $u(a) = u(b)$, there exists a covering $\{\phi_i\colon U_i \to U\}$ in $\mathcal{C}$ such that $a \circ \phi_i = b \circ \phi_i$.

SC2. (*locally full*) For any $U, U' \in \operatorname{ob}\mathcal{C}$ and a morphism $c\colon u(U) \to u(U')$ in $\mathcal{D}$, there exists a covering $\{\phi_i\colon U_i \to U\}$ and morphisms $c_i\colon U_i \to U'$ in $\mathcal{C}$ such that $u(c_i) = c \circ u(\phi_i)$.

SC3. (*locally essentially surjective*) Any $V \in \operatorname{ob}\mathcal{D}$ admits a covering of the form $\{u(U_i) \to V\}$.

2 Lemma (Special cocontinuous functors induce equivalences of topoi) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a special cocontinuous functors. Let $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be the geometric morphism induced by u. Then f is an equivalence of topoi.*

Proof: By Lemma 8.12, the cocontinuous functor u induces a geometric morphism $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ as

$$f_*\mathscr{F}(V) := \varprojlim_{{}_V\mathcal{I}^{\mathrm{opp}}} \mathscr{F}(U) \qquad \text{and} \qquad f^{-1}\mathscr{G}(U) := \mathscr{G}(u(U)).$$

We need to show $\eta\colon \mathrm{id} \Rightarrow f_* \circ f^{-1}$ and $\epsilon\colon f^{-1} \circ f_* \Rightarrow \mathrm{id}$ are isomorphisms.

To begin with, for any sheaf $\mathscr{F}$ on $\mathcal{C}$, we have

$$f^{-1}f_*\mathscr{F}(U) = \varprojlim_{{}_{u(U)}\mathcal{I}^{\mathrm{opp}}} \mathscr{F}(U').$$

As (U, id) is an object in ${}_{u(U)}\mathcal{I}$, there is a canonical map from this limit to $\mathscr{F}(U)$. We need to show it is bijective.

One one hand, *any element $(s_{(U',\psi)})$ in the limit is uniquely determined by $s_{(U,\mathrm{id})}$*. Indeed, for any $\psi\colon u(U') \to u(U)$, by SC2, there is a covering $\mathfrak{U}' = \{\phi_i\colon U'_i \to U'\}$ and morphisms $\psi_i\colon U'_i \to U$ in $\mathcal{C}$ such that $u(\psi_i) = \psi \circ u(\phi_i)$. Therefore $s_{(U',\psi)}\big|_{U'_i} = s_{(U,\mathrm{id})}\big|_{U'_i}$. As $\mathfrak{U}'$ is a covering, this shows that $(s_{(U',\psi)})$ is uniquely determined by $s_{(U,\mathrm{id})}$.

On the other hand, *given any $s \in \mathscr{F}(U)$, there exists an element $(s_{(U',\psi)})$ in the limit such that $s_{(U,\mathrm{id})} = s$.* For any $\psi\colon u(U') \to u(U)$, let U'_i, ϕ_i, ψ_i be as before. Then we only need to show any $s|_{U'_i}$ and $s|_{U'_j}$ agree on $U'_i \times_{U'} U'_j$. Consider the diagram

$$\begin{tikzcd}
U'_i \times_{U'} U'_j \arrow[r, "p_2"] \arrow[d, "p_1"'] & U'_j \arrow[d, "\phi_j"] \\
U'_i \arrow[r, "\phi_i"] & U
\end{tikzcd}$$

which needs not to be commutative but becomes commutative after apply u to it. By SC1, there exists coverings $\{\phi_{ijk}\colon U'_{ijk} \to U'_i \times_{U'} U'_j\}$ such that $\phi_i \circ p_1 \circ \phi_{ijk} = \phi_j \circ p_2 \circ \phi_{ijk}$. Therefore

$$s|_{U'_i}\Big|_{U'_i \times_{U'} U'_j}\Big|_{U'_{ijk}} = s|_{U'_j}\Big|_{U'_i \times_{U'} U'_j}\Big|_{U'_{ijk}}.$$

Hence $s|_{U'_i}$ and $s|_{U'_j}$ agree on $U'_i \times_{U'} U'_j$ as desired and thus we get a section $s_{(U',\psi)} \in \mathscr{F}(U')$. Those sections form an element $(s_{(U',\phi)})$ in the limit and satisfies $s_{(U,\mathrm{id})} = s$.

Next, for any sheaf $\mathscr{G}$ on $\mathcal{D}$, by the *triangle identity* $f^{-1} \circ f_* \circ f^{-1} = f^{-1}$, we see the canonical morphism

$$\mathscr{G} \to f_* f^{-1} \mathscr{G}$$

is bijective on objects of the form $u(U)$. By SC3, any object V has a covering of the form $\{u(U_i) \to V\}$, thus $\mathscr{G}(V) \to f_* f^{-1}\mathscr{G}(V)$ is also bijective. Therefore, $\mathscr{G} \to f_* f^{-1}\mathscr{G}$ is an isomorphism. $\square$

The localization of a special cocontinuous functor is again a special cocontinuous functor.

3 **Lemma (Localization of a special cocontinuous functor)** *Let $\mathcal{C}, \mathcal{D}$ be two sites and $u\colon \mathcal{C} \to \mathcal{D}$ be a special cocontinuous functor. Then we have a commutative diagram*

$$\begin{tikzcd}
\mathcal{C}/X \arrow[r, "j_X"] \arrow[d, "u'"'] & \mathcal{C} \arrow[d, "u"] \\
\mathcal{D}/u(X) \arrow[r, "j_{u(X)}"] & \mathcal{D}
\end{tikzcd}$$

as in Lemma 9.13 and the vertical arrows are special cocontinuous functors. Hence in the commutative diagram of topoi,

$$\begin{tikzcd}
\mathbf{Sh}(\mathcal{C}/X) \arrow[r, "j_X"] \arrow[d, "f'"'] & \mathbf{Sh}(\mathcal{C}) \arrow[d, "f"] \\
\mathbf{Sh}(\mathcal{D}/u(X)) \arrow[r, "j_{u(X)}"] & \mathbf{Sh}(\mathcal{D})
\end{tikzcd}$$

vertical arrows are equivalences of topoi.

Proof: What we have to show is that the induced functor u' is a special cocontinuous functor.

u' is cocontinuous. This is Lemma 9.13.

u' is continuous. Let $\mathfrak{U}/X = \{U_i/X \to U/X\}$ be a covering in $\mathcal{C}/X$, then $\mathfrak{U} = \{U_i \to U\}$ is a covering in $\mathcal{C}$. Since u is continuous, $u(\mathfrak{U})$ is a covering in $\mathcal{D}$. Thus $u'(\mathfrak{U}/X) = u(\mathfrak{U})/u(X)$ is a covering in $\mathcal{D}/X$.

u' satisfies SC1. Let $a, b\colon U/X \rightrightarrows U'/X$ be two morphisms such that $u'(a) = u'(b)$. Let a, b also denote the corresponding morphisms $U \rightrightarrows U'$. Then, since u satisfies SC1, there is a covering $\mathfrak{U} = \{\phi_i\colon U_i \to U\}$ in $\mathcal{C}$ such that $a \circ \phi_i = b \circ \phi_i$. This gives a covering $\mathfrak{U}/X$ in $\mathcal{C}/X$ such that $a \circ \phi_i = b \circ \phi_i$.

u' satisfies SC2. Let $U/X, U/X \in \operatorname{ob} \mathcal{C}/X$ and $c\colon u'(U/X) \to u'(U'/X)$ a morphism in $\mathcal{D}/X$. Then, since u satisfies SC2, there is a covering $\mathfrak{U} = \{\phi_i\colon U_i \to U\}$ and morphisms $c_i\colon U_i \to U'$ in $\mathcal{C}$ such that $u(c_i) = c \circ u(\phi_i)$. This gives a covering $\mathfrak{U}/X$ and morphisms $c_i\colon U_i/X \to U'/X$ such that $u(c_i) = c \circ u(\phi_i)$.

u' satisfies SC3. Let $V/u(X)$ be an object in $\mathcal{D}/u(X)$. Then, since u satisfies SC3, there is a covering of V of the form $\{u(U_i) \to V\}$. This gives $V/u(X)$ a covering of the form $\{u'(U_i/X) \to V/u(X)\}$. $\square$

Up to equivalence of topoi, any small site can be enlarged to a subcanonical one.

4 **Lemma (Enlarging a site)** *Let $\mathcal{C}$ be a small site. Let $\mathcal{C}' \subset \mathbf{Sh}(\mathcal{C})$ be a small full subcategory such that*

- $h_U^{\#} \in \operatorname{ob} \mathcal{C}'$ *for all* $U \in \operatorname{ob} \mathcal{C}$,
- $\mathcal{C}'$ *is preserved under pullbacks.*

Define a covering of $\mathcal{C}'$ as a family $\{\mathscr{F}_i \to \mathscr{F}\}$ such that $\coprod \mathscr{F}_i \to \mathscr{F}$ is an epimorphism. Then

1. *$\mathcal{C}'$ is a site;*
2. *$\mathcal{C}'$ has subcanonical topology;*
3. *$u\colon U \mapsto h_U^{\#}$ is a special cocontinuous functor, hence induces an equivalence of topoi $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C}')$;*
4. *for any $\mathscr{F} \in \operatorname{ob} \mathcal{C}'$, we have $f^{-1} h_{\mathscr{F}} = \mathscr{F}$;*
5. *for any $U \in \operatorname{ob} \mathcal{C}$, we have $f_* h_U^{\#} = h_{u(U)} = h_{h_U^{\#}}$.*

Proof: *1.* is obvious.

2. Let $\{\mathscr{F}_i \to \mathscr{F}\}$ be a covering in $\mathcal{C}'$, i.e. a surjective family of morphisms of sheaves, and let $\mathscr{G}$ an object in $\mathcal{C}'$, i.e. a sheaf on $\mathcal{C}$. We need to show $\{\mathscr{F}_i \to \mathscr{F}\}$ is effective-epic, which follows from Proposition 7.3, saying $\coprod_i \mathscr{F}_i \to \mathscr{F}$ is an epimorphism if and only if it is an effective epimorphism.

3. u is cocontinuous: Let $\mathfrak{F} = \{\mathscr{F}_j \to h_U^{\#}\}$ bw a covering in $\mathcal{C}'$, then $\coprod_j \mathscr{F}_j \to h_U^{\#}$ is an epimorphism. By Proposition 7.3, it is locally surjective, which means there is a covering $\mathfrak{U} = \{U_i \to U\}$ in $\mathcal{C}$ such that each $\mathrm{id}_U|_{U_i} \in h_U^{\#}(U_i)$ has a preimage $s_i \in \coprod_j \mathscr{F}_j(U_i)$. Assume $s_i \in \mathscr{F}_{j_i}(U_i)$, then they give rise to a refinement $u(\mathfrak{U}) \to \mathfrak{F}$ given by $h_{U_i}^{\#} \to \mathscr{F}_{j_i}$.

u is continuous: Follows from that Yoneda embedding and sheafification commutes with limits and Lemma 7.15.

u satisfies SC1: Let $a, b\colon U \rightrightarrows U'$ be two morphisms in $\mathcal{C}$ such that $u(a) = u(b)$, which means, as sections of $h_{U'}(U)$, a and b have the same image in the sheafification. By the construction of $\#$, this means there is a covering $\mathfrak{U} = \{\phi_i\colon U_i \to U\}$ in $\mathcal{C}$ such that a and b agree on each U_i. Then $a \circ \phi_i = b \circ \phi_i$.

u' satisfies SC2. Let $U, U \in \mathrm{ob}\,\mathcal{C}$ and $c\colon h_U^{\#} \to h_{U'}^{\#}$ a morphism in $\mathcal{C}'$. Then c is also a section in the seafification of $h_{U'}$ on U, thus there is a covering $\{\phi_i\colon U_i \to U\}$ in $\mathcal{C}$ such that each $c|_{U_i}$ lies in the image of the canonical map, thus has a preimage $c_i \in h_{U'}(U_i)$. That means, $u(c_i) = c \circ u(\phi_i)$.

u' satisfies SC3. Let $\mathscr{F}$ be an object in $\mathcal{C}'$. We need to find an epimorphism $\mathscr{F}_0 \to \mathscr{F}$, where $\mathscr{F}_0$ is a coproduct of sheaves of the form $h_U^{\#}$ with $U \in \mathrm{ob}\,\mathcal{C}$. This follows from Lemma 7.16.

4. For any $\mathscr{F} \in \mathrm{ob}\,\mathcal{C}'$ and $U \in \mathrm{ob}\,\mathcal{C}$, we have

$$f^{-1}h_{\mathscr{F}}(U) = h_{\mathscr{F}}(u(U)) = \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(h_U^{\#}, \mathscr{F}) = \mathscr{F}(U).$$

5. For any $\mathscr{F} \in \mathrm{ob}\,\mathcal{C}'$ and $U \in \mathrm{ob}\,\mathcal{C}$, we have

$$\begin{aligned} f_* h_U^{\#}(\mathscr{F}) &= \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C}')}(h_{\mathscr{F}}, f_* h_U^{\#}) \\ &= \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(f^{-1}h_{\mathscr{F}}, h_U^{\#}) \\ &= \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathscr{F}, h_U^{\#}) = h_{u(U)}(\mathscr{F}). \end{aligned}$$ □

Moreover, we can enlarge a site to a finite-complete one.

5 Lemma (Enlarging a site) *Let $\mathcal{C}$ be a small site. Let $\{\mathscr{F}_i\}$ be a set of sheaves on $\mathcal{C}$. Then there exists an equivalence of topoi $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C}')$ induced by a special cocontinuous functor $u\colon \mathcal{C} \to \mathcal{C}'$ of sites such that*

1. *$\mathcal{C}'$ has subcanonical topology;*
2. *a family $\{V_i \to V\}$ of morphisms of $\mathcal{C}'$ is (combinatorially equivalent to) a covering of $\mathcal{C}'$ if and only if $\coprod h_{V_i} \to h_V$ is epic;*
3. *$\mathcal{C}'$ is finite-complete;*

4. *every subsheaf of a representable sheaf on $\mathcal{C}'$ is representable;*

5. *each $f_*\mathscr{F}_i$ is a representable sheaf.*

Proof: Let $\mathcal{C}_1$ be a full subcategory of $\mathbf{Sh}(\mathcal{C})$ consisting of all $h_U^\#$ with $U \in \mathcal{C}$, the given sheaves $\mathscr{F}_i$, the terminal sheaf $\underline{*}$ and all their subsheaves. If $\mathcal{C}$ is already finite-complete, then setting $\mathcal{C}' = \mathcal{C}_1$ with the topology inheriting from $\mathbf{Sh}(\mathcal{C})$, $u\colon \mathcal{C} \to \mathcal{C}'$ given by $U \mapsto h_U^\#$ and everything is done.

If $\mathcal{C}$ is not finite-complete, then let $\mathcal{C}_{n+1}$ be the full subcategory of $\mathbf{Sh}(\mathcal{C})$ consisting of fibre products and subsheaves of objects in $\mathcal{C}_n$. In this way, we get:

$$\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_3 \subset \cdots \subset \mathbf{Sh}(\mathcal{C}).$$

Let $\mathcal{C}'$ be the union of those $\mathcal{C}_i$, then everything follows. □

Now, we face the title of this subsection.

6 Theorem (Geometric morphisms are induced by morphisms of sites) *Let $\mathcal{C}, \mathcal{D}$ be two sites and $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ be a geometric morphism. Then there exists a site $\mathcal{C}'$ and a diagram of functors*

$$\mathcal{C} \xrightarrow{v} \mathcal{C}' \xleftarrow{u} \mathcal{D}$$

such that

1. *v is a special cocontinuous functor;*

2. *u commutes with fibre products, is continuous and defines a morphism of sites $\mathcal{C}' \to \mathcal{D}$;*

3. *the geometric morphism f agree with the composition*

$$\mathbf{Sh}(\mathcal{C}) \longrightarrow \mathbf{Sh}(\mathcal{C}') \longrightarrow \mathbf{Sh}(\mathcal{D}).$$

   which is induced by v and u.

Proof: By Lemma 5, we have a special cocontinuous functor

$$\begin{aligned} v\colon \mathcal{C} &\longrightarrow \mathcal{C}' \\ U &\longmapsto h_U^\#, \end{aligned}$$

where $\mathcal{C}'$ is finite-complete, has subcanonical topology compatible with the one of $\mathbf{Sh}(\mathcal{C})$, guarantees that every subsheaf of a representable sheaf is representable and contains all $f^{-1}h_V^\#$. Let g denotes the geometric morphism induced by v.

Now, define the functor u by $V \mapsto f^{-1}h_V^\#$. Since f^{-1} is exact and commutes with all colimits, the functor u transforms a covering into a effective-epic family of morphisms of sheaves. Hence u is continuous. Since both $h^\#$

and f^{-1} are left exact, so is u and thus u defines a morphism of sites by Lemma 1.8.

To show the last statement, it suffices to show $u^s \circ g_* = f_*$. Indeed, for any $U \in \operatorname{ob} \mathcal{C}$ and $V \in \operatorname{ob} \mathcal{D}$, we have

$$\begin{aligned}(u^s \circ g_* h_U^\#)(V) &= g_* h_U^\#(f^{-1} h_V^\#) \\ &= \operatorname{Hom}_{\mathbf{Sh}(\mathcal{C})}(f^{-1} h_V^\#, h_U^\#) \\ &= \operatorname{Hom}_{\mathbf{Sh}(\mathcal{D})}(h_V^\#, f_* h_U^\#) \\ &= f_* h_U^\#(V).\end{aligned}$$

By Lemma 7.16, this shows $u^s \circ g_* = f_*$. □

6.a Remark Notation as in Theorem 6. Assume $\mathcal{D}$ is finite-complete. Then, by the construction of v, it is left exact. Then, by Lemma 8.8.b, v defines a morphism of sites. Apply Lemma 5 to $\mathcal{D}$ to get $\mathcal{D}'$, then v' induces a morphism of site $v' \colon \mathcal{C}' \to \mathcal{D}'$. let f' be the geometric morphisms induced by v'. Next, apply Theorem 6 to the geometric morphism $\mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D}')$. We finally have the following statement:

6.b Corollary *Notation and assumptions as above. Then the geometric morphism satisfies a commutative diagram of topoi.*

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}) \arrow[r, "f"] \arrow[d] & \mathbf{Sh}(\mathcal{D}) \arrow[d] \\ \mathbf{Sh}(\mathcal{C}') \arrow[r, "f'"] & \mathbf{Sh}(\mathcal{D}') \end{tikzcd}$$

Moreover, we have

1. *the vertical arrows are equivalence of topoi induced by the special co-continuous functors $\mathcal{C} \to \mathcal{C}'$ and $\mathcal{D} \to \mathcal{D}'$;*

2. *the sites $\mathcal{C}'$ and $\mathcal{D}'$ are finite-complete and have subcanonical topologies;*

3. *given any set of sheaves $\{\mathscr{F}_i\}$ (resp. $\{\mathscr{G}_i\}$) on $\mathcal{C}$ (resp. $\mathcal{D}$), we may assume each of them is a representable sheaf on $\mathcal{C}'$ (resp. $\mathcal{D}'$).*

Localizations of topoi

Since any geometric morphism can be identified with one induced by morphism of sites, results in §9 holds for general case.

§ I.11 Points

1 **(Points of a site)** Let $\mathcal{C}$ be a site. A **point** p **of** $\mathcal{C}$ is a continuous functor $u\colon \mathcal{C} \to \mathbf{Set}$ which defines a morphism of sites $\mathbf{Set} \to \mathcal{C}$. To avoid the size issue, we should restrict on a small subcategory $\mathbf{Set}$ containing images of p. For convention, we write out the conditions straightforward.

By Example 8.3, the sheaves on (small) $\mathbf{Set}$ is equivalent to the sets obtained by apply them to the singleton. For any sheaf $\mathscr{F}$ on $\mathcal{C}$, we have

$$u_s\mathscr{F}(*) = \varinjlim_{\mathcal{I}_*^{\text{opp}}} \mathscr{F}(U).$$

Recall that an object in $\mathcal{I}_*$ is is a pair (U, x) with $U \in \operatorname{ob}\mathcal{C}$ and $x\colon * \to u(U)$, which is equivalent to an element of $u(U)$. So we define a **neighborhood** of p to be a pair (U, x) with $U \in \operatorname{ob}\mathcal{C}$ and $x \in u(U)$ and a *morphism of neighborhoods* $(U, x) \to (V, y)$ to be a morphism $\phi\colon U \to V$ such that $u(\phi)(x) = y$. In this setting, $u_s\mathscr{F}(*)$ is called the **stalk** of $\mathscr{F}$ at p and is denoted by $\mathscr{F}_p$. Note that the definition

$$\mathscr{F}_p := \varinjlim_{(U,x)} \mathscr{F}(U)$$

also apply to presheaves. In this way, we get a *stalk functor*:

$$\begin{aligned} \mathbf{PSh}(\mathcal{C}) &\longrightarrow \mathbf{Set} \\ \mathscr{F} &\longmapsto \mathscr{F}_p. \end{aligned}$$

So, here are the conditions:

Pt1. For every overing $\{U_i \to U\}$, the map $\coprod u(U_i) \to u(U)$ is surjective;

Pt2. For every overing $\{U_i \to U\}$ and every morphism $V \to U$, the maps $u(U_i \times_U V) \to u(U_i) \times_{u(U)} u(V)$ are bijective;

Pt3. The stalk functor $\mathscr{F} \mapsto \mathscr{F}_p$ is left exact.

Recall that under the equivalence of (small) $\mathbf{Set}$ and sheaves on it, a set S can be viewed as a sheaf $\operatorname{Hom}_{\mathbf{Set}}(-, S)$. The functor u^p then apply to this sheaf and then defines a functor

$$\begin{aligned} u^p\colon\ \mathbf{Set} &\longrightarrow \mathbf{PSh}(\mathcal{C}) \\ S &\longmapsto \operatorname{Hom}_{\mathbf{Set}}(u(-), S). \end{aligned}$$

1.a Lemma *Let $\mathcal{C}$ be a site and $p = u\colon \mathcal{C} \to \mathbf{Set}$ a functor. Then*

1. *u^p is right adjoint to the stalk functor on presheaves;*
2. *there are isomorphisms $(h_U)_p = u(U)$ natural in $U \in \operatorname{ob}\mathcal{C}$.*

Proof: *1* follows from Theorem 1.9. By Corollary 1.9.a, $u_p h_U = h_{u(U)}$, thus $(h_U)_p = u(U)$. $\square$

1.b Lemma *Let $\mathcal{C}$ be a site and $p = u\colon \mathcal{C} \to \mathbf{Set}$ a functor such that for every overing $\mathfrak{U} = \{U_i \to U\}$*

1. *the map $\coprod u(U_i) \to u(U)$ is surjective;*
2. *the maps $u(U_i \times_U U_j) \to u(U_i) \times_{u(U)} u(U_j)$ are surjective.*

Then

1. *the presheaf $u^p S$ is a sheaf for all set S, denoted by $u^s S$;*
2. *u^s is right adjoint to the stalk functor on sheaves;*
3. *for every presheaf $\mathscr{F}$ on $\mathcal{C}$, $\mathscr{F}_p = \mathscr{F}^{\#}_p$.*

Proof: The first two are obvious. For any set S, we have

$$\mathrm{Hom}(\mathscr{F}_p, S) = \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathscr{F}, u^p S) = \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathscr{F}^{\#}, u^s S) = \mathrm{Hom}(\mathscr{F}^{\#}_p, S).$$

Then the third assertion follows. $\square$

1.c (Skyscraper sheaves) Let p be a point of a site $\mathcal{C}$ defined by a functor $u\colon \mathcal{C} \to \mathbf{Set}$. For any set S, the sheaf $u^s S$ is denoted by $p_* S$ and is called a **skyscraper sheaf**.

Denote the stalk functor by p^{-1}, then we get a geometric morphism

$$p^{-1} \dashv p_*\colon\ \mathbf{Set}\,\mathbf{Sh}(\mathcal{C}).$$

2 (Points of a topos) Let $\mathbf{Sh}(\mathcal{C})$ be a topos, then a **point** p **of it** is a geometric morphism $p\colon \mathbf{Sh}(\mathrm{pt}) \to \mathbf{Sh}(\mathcal{C})$.

We have seen that a point of a site $\mathcal{C}$ induces a point of the topos $\mathbf{Sh}(\mathcal{C})$. Conversely,

2.a Lemma *Let $\mathcal{C}$ be a site and $p = (p^{-1} \dashv p_*)$ a point of $\mathbf{Sh}(\mathcal{C})$. Then the functor $u(U) = p^{-1}(h^{\#}_U)$ gives rise to a point of $\mathcal{C}$ which induces a point equal to p.*

Proof: First, we show u gives rise to a point of $\mathcal{C}$.

Pt1. Let $\{U_i \to U\}$ be a covering, then by Lemma 7.15, the morphism $\coprod h^{\#}_{U_i} \to h^{\#}_U$ is an epimorphism. Since p^{-1} commutes with colimits, the map $\coprod u(U_i) \to u(U)$ is surjective.

Since Yoneda embedding, sheafification and p^{-1} are left exact, so is u. Then *Pt2.* and *Pt3.* follows.

Let p' denote the point of topos induced by u. It remains to show that $p'^{-1} = p^{-1}$. By Lemma 1.a and Lemma 1.b, this is true for $h^{\#}_U$ with $U \in \mathrm{ob}\,\mathcal{C}$. By Lemma 7.16, any sheaf $\mathscr{F}$ on $\mathcal{C}$ is a colimits over a family of sheaves of the form $h^{\#}_U$. Since both p'^{-1} and p^{-1} commute with colimits, we see $p'^{-1}\mathscr{F} = p^{-1}\mathscr{F}$. $\square$

Apply Theorem 10.6, we have

2.b Lemma *Let p be a point of a site $\mathcal{C}$ defined by a functor $u\colon \mathcal{C} \to \mathbf{Set}$. Let S_0 be an infinite set such that all $u(U)$ lies in S_0 and $\mathcal{S}$ be the full subcategory of* **Set** *whose objects are subsets of S_0. Then*

1. *there is an equivalence $i\colon\ \mathbf{Sh}(\mathrm{pt}) \to \mathbf{Sh}(\mathcal{S})$;*
2. *the functor $u\colon \mathcal{C} \to \mathcal{S}$ defines a morphism of sites $f\colon \mathcal{S} \to \mathcal{C}$;*
3. *the composition*
$$\mathbf{Sh}(\mathrm{pt}) \longrightarrow \mathbf{Sh}(\mathcal{S}) \longrightarrow \mathbf{Sh}(\mathcal{C})$$
is the point $(p^{-1} \dashv p_*)$ induced by p..

Proof: It remains to show $f_*i_* = p_*$. Indeed, for any set S in $\mathcal{S}$, we have

$$f_*i_*S(U) = i_*(u(U)) = \mathrm{Hom}(u(U), S),$$

which shows $f_*i_*S = p_*S$. □

In general $(p_*S)_p$ may not be S, but we have

2.c Lemma *Let $\mathcal{C}$ be a site and p a point of $\mathbf{Sh}(\mathcal{C})$. For any set S, there are canonical maps*
$$S \longrightarrow p^{-1}p_*S \longrightarrow S$$
whose composition is id_S.

Proof: There already is a canonical map $p^{-1}p_*S \to S$. Since both p_* and p^{-1} are left exact, $p^{-1}p_*(*) \cong *$. But then this defines a map $S \to p^{-1}p_*S$ by mapping a $s \in S$, viewed as a map $s\colon * \to S$, to the element corresponding to the map $p^{-1}p_*(s)\colon p^{-1}p_*(*) \cong * \to p^{-1}p_*S$. □

2.d Lemma *Let $\mathcal{C}$ be a site and p a point of $\mathbf{Sh}(\mathcal{C})$. Then p_* has the following properties:*

1. *It commutes with limits,*
2. *it is faithful,*
3. *it reflects monomorphisms, epimorphisms and isomorphisms,*
4. *it transforms epimorphisms into epimorphisms and commutes with coequalizers.*

Proof: Since p_* has a left adjoint, it commutes with limits. Since the canonical map $p^{-1}p_*S \to S$ is epic, p_* is faithful. Indeed if f and g are two parallel maps transformed into same morphism by p_*, then the fit into the following commutative diagram.

$$\begin{tikzcd} p^{-1}p_*S \arrow[r] \arrow[d] & S \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] \\ p^{-1}p_*T \arrow[r] & T \end{tikzcd}$$

Since the horizontal maps are epic, we have $f = g$.

Since the canonical map $p^{-1}p_*S \to S$ is split epic, p_* reflects monomorphisms, epimorphisms and isomorphisms. Indeed in the following commutative diagram,

$$\begin{tikzcd} p^{-1}p_*S \arrow[r] \arrow[d, "f'"'] & S \arrow[d, "f"] \\ p^{-1}p_*T \arrow[r] & T \end{tikzcd}$$

since the horizontal maps are split epic, the vertical maps are monomorphisms, epimorphisms or isomorphisms at the same time.

By Lemma 2.a, we may assume p is induced by a point $u\colon \mathcal{C} \to \mathbf{Set}$ of $\mathcal{C}$. Then, for any set S, we have

$$p_*S(U) = \mathrm{Hom}(u(U), S).$$

Since epimorphisms in **Set** are split epimorphisms, p_* transforms epimorphisms into epimorphisms. Since coequalizers in **Set** are split coequalizers (ref. §2.10 in *BMO*), they commute with any functor, *a fortiori* p_*. $\square$

3 Theorem (Constructing points) *Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ is finite-complete. Then a point p of $\mathcal{C}$ is given by a left exact functor $u\colon \mathcal{C} \to \mathbf{Set}$ such that for every overing $\{U_i \to U\}$, the map $\coprod u(U_i) \to u(U)$ is surjective.*

Proof: If p is a point given by a functor u, then u satisfies the later condition. By Lemma 2.a, we u equals the composition $p^{-1} \circ \# \circ h$, which are all left exact, thus so is u.

Conversely, if u is a functor satisfying the conditions, then it remains to show the stalk functor associated to u is left exact. Indeed, it suffices to show the opposite of category of neighborhoods is filtered, which follows from Lemma 1.8. $\square$

3.a Example Let $\mathcal{C}$ be a site and u a functor as in Lemma 1.b. Assume $\mathcal{C}$ is finite-complete and u maps terminal object to the singleton. Then, in general u is *NOT* a point!

Let $\mathcal{C}$ be the poset $\{1 \to 2\}$, viewed as a site with *chaotic topology* (ref. Example 2.1.c). Note that 2 is the terminal object and $1 \times_2 1 = 1$, thus $\mathcal{C}$ is finite-complete. Consider the functor u: $u(1) = \{a, b\}$ and $u(2) = \{c\}$.

One can see this u satisfies all conditions, but the stalk functor is not exact. Indeed, let $\mathscr{F}$ and $\mathscr{G}$ be the sheaf $\mathscr{F}(1) = *$, $\mathscr{F}(2) = \varnothing$ and $\mathscr{G}(1) = \mathscr{G}(2) = *$ respectively. Then there is an obvious monomorphism $\mathscr{F} \to \mathscr{G}$. However, since their stalks are

$$\mathscr{F}_p = * \coprod *, \qquad \mathscr{G}_p = *,$$

the stalk of this monomorphism is not injective.

3.b Example (Points of a topological space) Let X be a topological space. Consider the functor $u\colon \mathcal{T}_X \to \mathbf{Set}$ given by

$$u(U) = \begin{cases} \varnothing & \text{if } x \notin U \\ * & \text{if } x \in U. \end{cases}$$

Then this u defines a point of $\mathcal{T}_X$ which induces a stalk functor coinciding with stalk at x.

But not all points come from the really existed points in X.

Let p be a point of $\mathcal{T}_X$. By Theorem 3, we have $u(\varnothing) = \varnothing$ and $u(U \cap V) = u(U) \times u(V)$. Then, $u(U) = u(U) \times u(U)$ and thus is either empty or a singleton. Moreover, for any covering $\{U_i \subset U\}$, we have

$$u(U) = \varnothing \iff \forall i, u(U_i) = \varnothing.$$

Therefore, there is a largest open set W in X with $u(W) = \varnothing$, which is the union of all such open sets. Let $Z = X \setminus W$.

This Z *is irreducible*. Indeed, if $Z = Z_1 \cup Z_2$ with Z_i closed, then $W = (X \setminus Z_1) \cap (X \setminus Z_2)$ and thus $u(X \setminus Z_1) \times u(X \setminus Z_2) = u(W) = \varnothing$, which implies either $X \setminus Z_1 = W$ or $X \setminus Z_2 = W$.

Now, we can see

$$u(U) = \begin{cases} \varnothing & \text{if } Z \cap U = \varnothing \\ * & \text{if } Z \cap U \neq \varnothing. \end{cases}$$

Conversely, for any irreducible closed set Z of X, the above formula defines a point of $\mathcal{T}_X$. Therefore:

> *Points of the site $\mathcal{T}_X$ are one-to-one correspondence to irreducible closed sets of X.*

Recall that a topological space is called a *sober space* if it is T_0 and every irreducible closed set is the closure of a point. Thus if X is a sober space, then points of $\mathcal{T}_X$ are one-to-one correspondence to points of X.

3.c Example (Points of a chaotic site) Let $\mathcal{C}$ be a category, viewed as a site with chaotic topology. Then for any object U of $\mathcal{C}$, the functor $\operatorname{Hom}_{\mathcal{C}}(U, -)$ defines a point of $\mathcal{C}$.

§ I.12 Flatness

For the first, we follows [Kar04].

1 (Flatness respect to a site) Let $\mathcal{C}$ be a site and $\mathcal{D}$ a category. A functor $u\colon \mathcal{D} \to \mathcal{C}$ is said to be **flat respect to** $\mathcal{C}$ if for any finite diagram $\mathscr{D}\colon \mathcal{I} \to \mathcal{D}$ and any cone $\alpha\colon \Delta_U \Rightarrow u \circ \mathscr{D}$ over $u \circ \mathscr{D}$ in $\mathcal{C}$, the family

$$S_\alpha := \{\varphi\colon V \to U | \exists \beta\colon \Delta_W \Rightarrow \mathscr{D} \text{ s.t. } \alpha \circ \varphi \text{ factors through } u * \beta\}$$

is a covering in $\mathcal{C}$.

Remark This family is in fact a **sieve**, which means a family of morphisms with fixed target S which is *closed under precomposition*: if $\varphi\colon V \to U \in S$ and $\psi\colon W \to V$ is a morphism, then $\varphi \circ \psi \in S$. Thus any family of morphisms with fixed target $\mathfrak{U}$ can be extended into a sieve $S_{\mathfrak{U}}$ by added up with precompositions. The notion of sieves will be used in the definition of *Grothendieck topology.*

To see why this property is called flatness, we introduce the notion of tensor products of **Set**-valued functors. Who doesn't care this can skip the following few pages.

1.a Lemma (Tensor products) *Let $\mathcal{C}$ be a small category. Let $\mathscr{F}\colon \mathcal{C}^{\mathrm{opp}} \to$ **Set** and $\mathscr{G}\colon \mathcal{C} \to$ **Set** be a pair of a presheaf and a functor. Let Υ' denote the presheaf*

$$\begin{aligned} \Upsilon'\colon \mathcal{C}^{\mathrm{opp}} &\longrightarrow [\mathcal{C}, \mathbf{Set}] \\ U &\longmapsto \mathrm{Hom}_{\mathcal{C}}(U, -). \end{aligned}$$

Then, there is a natural bijection

$$(\mathrm{Lan}_{\Upsilon'} \mathscr{F})(\mathscr{G}) \cong (\mathrm{Lan}_{\Upsilon} \mathscr{G})(\mathscr{F}).$$

Proof: Since (refer Theorem 1.2.e and Example 1.13)

$$\begin{aligned} (\mathrm{Lan}_{\Upsilon'} \mathscr{F})(\mathscr{G}) &\cong \varinjlim_{(\Upsilon'(U)\to\mathscr{G})\in\mathrm{ob}(\Upsilon'\downarrow\mathrm{const}_{\mathscr{G}})} \mathscr{F}(U), \\ (\mathrm{Lan}_{\Upsilon} \mathscr{G})(\mathscr{F}) &\cong \varinjlim_{(\Upsilon(U)\to\mathscr{F})\in\mathrm{ob}(\Upsilon\downarrow\mathrm{const}_{\mathscr{F}})} \mathscr{G}(U), \\ \mathscr{G} &\cong \varinjlim_{(\Upsilon'(U)\to\mathscr{G})\in\mathrm{ob}(\Upsilon'\downarrow\mathrm{const}_{\mathscr{G}})} \Upsilon'(U), \\ \mathscr{F} &\cong \varinjlim_{(\Upsilon(U)\to\mathscr{F})\in\mathrm{ob}(\Upsilon\downarrow\mathrm{const}_{\mathscr{F}})} \Upsilon(U), \end{aligned}$$

we have

$$\begin{aligned}
(\operatorname{Lan}_{\Upsilon'}\mathscr{F})(\mathscr{G}) &\cong \varinjlim_{(\Upsilon'(U)\to\mathscr{G})\in\operatorname{ob}(\Upsilon'\downarrow\mathrm{const}_{\mathscr{G}})} \mathscr{F}(U)\\
&\cong \varinjlim_{(\Upsilon'(U)\to\mathscr{G})\in\operatorname{ob}(\Upsilon'\downarrow\mathrm{const}_{\mathscr{G}})} \varinjlim_{(\Upsilon(V)\to\mathscr{F})\in\operatorname{ob}(\Upsilon\downarrow\mathrm{const}_{\mathscr{F}})} \operatorname{Hom}_{\mathcal{C}}(U,V)\\
&\cong \varinjlim_{(\Upsilon(V)\to\mathscr{F})\in\operatorname{ob}(\Upsilon\downarrow\mathrm{const}_{\mathscr{F}})} \varinjlim_{(\Upsilon'(U)\to\mathscr{G})\in\operatorname{ob}(\Upsilon'\downarrow\mathrm{const}_{\mathscr{G}})} \operatorname{Hom}_{\mathcal{C}}(U,V)\\
&\cong \varinjlim_{(\Upsilon(U)\to\mathscr{F})\in\operatorname{ob}(\Upsilon\downarrow\mathrm{const}_{\mathscr{F}})} \mathscr{G}(U)\\
&\cong (\operatorname{Lan}_{\Upsilon}\mathscr{G})(\mathscr{F}). \qquad \square
\end{aligned}$$

1.b (Tensor products) The construction in the lemma is called the **tensor product** of $\mathscr{F}$ and $\mathscr{G}$, and is denoted by $\mathscr{F}\otimes_{\mathcal{C}}\mathscr{G}$ or $\mathscr{F}\otimes\mathscr{G}$.

Recall Example 1.13, we see that $-\otimes_{\mathcal{C}}\mathscr{F}$ is nothing but the *Yoneda extension* $\widetilde{\mathscr{F}}$. This functor has a right adjoint, which is $\Upsilon_{\mathbf{Set}}(-)\circ\mathscr{F}$. Indeed, we have a bijection

$$\operatorname{Hom}_{\mathbf{Set}}(\mathscr{G}\otimes_{\mathcal{C}}\mathscr{F},S)\cong\operatorname{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathscr{G},\operatorname{Hom}_{\mathbf{Set}}(\mathscr{F}(-),S))$$

natural in $\mathscr{G}\in\operatorname{ob}\mathbf{PSh}(\mathcal{C})$ and $S\in\operatorname{ob}\mathbf{Set}$.

1.c ¶Example Let $\mathcal{C}$ be a small category and $\mathscr{F}\colon\mathcal{C}\to\mathbf{Set}$ a functor. View **Set** as a site via its canonical topology, then the followings are equivalent.

1. $\mathscr{F}$ is flat respect to **Set**;
2. the opposite of its **category of elements** $\mathbf{El}(\mathscr{F}) := (*\downarrow\mathscr{F})$, is filtered;
3. its *Yoneda extension* $\widetilde{\mathscr{F}}\colon\ \mathbf{PSh}(\mathcal{C})\to\mathbf{Set}$ is left exact.

Proof: *1.⇒2.*: For any finite diagram $\mathscr{D}\colon\mathcal{I}\to\mathcal{C}$, consider the limit cone $\lambda\colon\Delta_{\varprojlim\mathscr{F}\circ\mathscr{D}}\Rightarrow\mathscr{F}\circ\mathscr{D}$ of $\mathscr{F}\circ\mathscr{D}$, then the sieve S_λ is generated by the family

$$\mathfrak{U}_\lambda := \left\{\varphi\colon\mathscr{F}(V)\to\varprojlim\mathscr{F}\circ\mathscr{D}\,\middle|\,\exists\beta\colon\Delta_V\Rightarrow\mathscr{D}\text{ s.t. }\mathscr{F}*\beta=\lambda\circ\varphi\right\}.$$

Then *1.* implies that such kind of families are surjective. Next, we show that the opposite of $\mathbf{El}(\mathscr{F})$ is filtered in this case. Recall that the opposite of a category is filtered if and only if any finite diagram in that category has a cone.

Let $\mathscr{D}'\colon\mathcal{I}\to\mathbf{El}(\mathscr{F})$ be a finite diagram. Then, itself gives a cone $\alpha\colon\Delta_*\Rightarrow\mathscr{F}\circ\mathscr{D}$ over the image of a diagram $\mathscr{D}\colon\mathcal{I}\to\mathcal{C}$ under $\mathscr{F}$. This cone uniquely factors through the limit cone $\lambda\colon\Delta_{\varprojlim\mathscr{F}\circ\mathscr{D}}\Rightarrow\mathscr{F}\circ\mathscr{D}$ and thus gives an element $a \in \varprojlim \mathscr{F} \circ \mathscr{D}$. Therefore, by the surjectivity of the family above, there exists a map $\varphi \colon \mathscr{F}(V) \to \varprojlim \mathscr{F} \circ \mathscr{D}$ and a cone $\beta \colon \Delta_V \Rightarrow \mathscr{D}$ such that $\mathscr{F} * \beta = \lambda \circ \varphi$ and that $a \in \operatorname{im} \varphi$. Now taking any preimage s of a in $\mathscr{F}(V)$, viewing it as a map from $*$ to $\mathscr{F}(V)$, we have $\varphi \circ s = a$. Therefore, we have a commutative diagram

$$\begin{tikzcd} \Delta_* \arrow[r, "s"] \arrow[dr, "\alpha"'] & \Delta_{\mathscr{F}(V)} \arrow[d, "\mathscr{F}*\beta"] \\ & \mathscr{F} \circ \mathscr{D} \end{tikzcd}$$

which induces a cone $\Delta_{s\colon * \to \mathscr{F}(V)} \Rightarrow \mathscr{D}'$ over $\mathscr{D}'$.

2.⇒3.: Let $\mathscr{G}$ be any presheaf on $\mathcal{C}$, then

$$\begin{aligned} \widetilde{\mathscr{F}}(\mathscr{G}) &\cong \varinjlim_{(\Upsilon'(U) \to \mathscr{F}) \in \operatorname{ob}(\Upsilon' \downarrow \operatorname{const}_{\mathscr{F}})} \mathscr{G}(U) \\ &\cong \left(\varinjlim_{(\Upsilon'(U) \to \mathscr{F}) \in \operatorname{ob}(\Upsilon' \downarrow \operatorname{const}_{\mathscr{F}})} \Gamma(U, -) \right)(\mathscr{G}). \end{aligned}$$

Since $(\Upsilon' \downarrow \operatorname{const}_{\mathscr{F}})$ is the opposite of $\mathbf{El}(\mathscr{F})$, it is filtered. Note that the functor $\Gamma(U, -)$ is exact. Now $\mathscr{F}$ is a filtered colimit of exact functors, thus is left exact.

3.⇒1.: For any finite diagram $\mathscr{D} \colon \mathcal{I} \to \mathcal{C}$, let $\lambda \colon \Delta_{\varprojlim \mathscr{F} \circ \mathscr{D}} \Rightarrow \mathscr{F} \circ \mathscr{D}$ be the limit cone of $\mathscr{F} \circ \mathscr{D}$. Then it suffices to show the family $\mathfrak{U}_\lambda$ is surjective. Indeed, if so, then the sieve S_λ is surjective, thus a covering. Now, for an arbitrary cone $\alpha \colon \Delta_U \Rightarrow \mathscr{F} \circ \mathscr{D}$, pullback S_λ along the canonical map $U \to \varprojlim \mathscr{F} \circ \mathscr{D}$, we obtain a covering of U which is contained in the sieve S_α, thus S_α is also surjective.

Now, note that $\varprojlim \mathscr{F} \circ \mathscr{D} = \varprojlim \widetilde{\mathscr{F}} \circ \Upsilon \circ \mathscr{D} = \widetilde{\mathscr{F}}(\varprojlim \Upsilon \circ \mathscr{D})$ and that a section $s \in (\varprojlim \Upsilon \circ \mathscr{D})(U)$ is a cone $s \colon \Delta_U \Rightarrow \mathscr{D}$. Denote $\varprojlim \Upsilon \circ \mathscr{D}$ by $\mathscr{L}$. For any object $t \colon h_V \to \mathscr{L}$ in $(\Upsilon \downarrow \operatorname{const}_{\mathscr{L}})$, denote the image of id_V by β, which is a cone $\beta \colon \Delta_V \Rightarrow \mathscr{D}$. Now $\Upsilon * \beta \colon \Delta_{h_V} \Rightarrow \Upsilon \circ \mathscr{D}$ is nothing but the composition of t with λ. Therefore $\mathscr{F} * \beta = \lambda \circ \widetilde{\mathscr{F}}(t)$. Therefore

$$\left\{ \widetilde{\mathscr{F}}(t) \colon \mathscr{F}(V) \to \varprojlim \mathscr{F} \circ \mathscr{D} \,\middle|\, t \colon h_V \to \varprojlim \Upsilon \circ \mathscr{D} \right\} \subset \mathfrak{U}_\lambda$$

but the left family is already surjective, *a fortiori* $\mathfrak{U}_\lambda$. □

Remark Recall that $- \otimes_{\mathcal{C}} \mathscr{F}$ has a right adjoint and thus is already right exact. Therefore $\mathscr{F}$ is flat respect to **Set** if and only if the tensor product functor $- \otimes_{\mathcal{C}} \mathscr{F}$ is exact.

1.d Corollary *If $\mathcal{C}$ is finite complete, then $\mathscr{F}$ is flat respect to* **Set** *if and only if it is left exact.*

Proof: *only if*: Note that $\mathscr{F} = \widetilde{\mathscr{F}} \circ \Upsilon$ and that Υ commutes with limits.

if: similarly to Lemma 1.8, one can show that $\mathbf{El}(\mathscr{F})^{\mathrm{opp}}$ is filtered if $\mathscr{F}$ is left exact. $\square$

II

Ringed spaces and $\mathscr{O}$-modules

Basically this chapter is talking about the commutative algebra theory in a topos.

§ II.1 ¶Algebraic structures

In this section, we discuss the notion of algebraic structures in a categorical flavor. We also show that the notion of algebraic category is worthy of its name.

Algebraic theories

Recall that a *sheaf* with value in an algebraic category $\mathcal{A}$ is a presheaf satisfying suitable descent condition (ref. I.2.5 and I.2.6). Of course one can give another description (Theorem 2) of this notion by introducing the notion of *algebraic structures*. Here we do a bit.

1 (Lawvere theory) We begin with declare some notions about algebraic objects in an elementary way. Let $\mathcal{C}$ be a category. Let $*$ denote the (possibly existed) terminal object. For $X \in \operatorname{ob}\mathcal{C}$, let X^n denote the n-fold product of X. In particular, $X^0 = *$.

- A n**-ary operation** on X is a morphism from X^n to X. Conversely, a n**-ary cooperation** on X is a morphism from X to X^n. More generally, a (m,n)**-operation** on X is a morphism from X^n to X^m.

In this way, the notion of operations on a set has been generalized into an arbitrary category. The relations of operations on sets are described by equations. Now, they become communities of diagrams.

1.a Example (group object) A **group object** in a category $\mathcal{C}$ is an object G with a 0-ary operation e called *unit*, a 1-ary operation $\imath$ called *inverse* and a 2-ary operation m called *multiplication* satisfying the following commutative diagrams.

$$\begin{tikzcd} G^3 \arrow[r, "\mathrm{id}\times m"] \arrow[d, "m\times\mathrm{id}"'] & G^2 \arrow[d, "m"] \\ G^2 \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(e,\mathrm{id})"] \arrow[d, "(\mathrm{id},e)"'] \arrow[dr, equal] & G^2 \arrow[d, "m"] \\ G^2 \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(\imath,\mathrm{id})"] \arrow[d, "(\mathrm{id},\imath)"'] \arrow[dr, "e"] & G^2 \arrow[d, "m"] \\ G^2 \arrow[r, "m"] & G \end{tikzcd}$$

where $e\colon G \to G$ is the composition $G \to * \xrightarrow{e} G$.

The followings are common examples:

- A group object in **Set** is a *group*.
- A group object in **Top** is a *topological group*.
- A group object in **Diff** is a *Lie group*.
- A group object in **Grp** is an *abelian group*.
- A group object in **Ab** is an *abelian group* again.

- A group object in **Cat** is a *strict* 2-*group*.
- A group object in **Grpd** is a *strict* 2-*group* again.
- A group object in $\mathbf{CRing}^{\mathrm{opp}}$ is a *commutative Hopf algebra*.
- A group object in a functor category is a *group functor*.
- A group object in the category of schemes is a *group scheme*.

The data of operations and relations of them can be encoded into a category $\mathbb{T}$ which has finite products and all its objects are finite products of a distinguished object x called the **generic object**. Such a category is called a **Lawvere theory**. A *morphism* of Lawvere theories is a functor preserving finite products.

Use this notion, we define

- An **algebraic object of type** $\mathbb{T}$ in $\mathcal{C}$ is an object X with a functor, also denoted by X, from $\mathbb{T}$ to $\mathcal{C}$ which maps x to X and preserves finite products. A **homomorphism** between algebraic objects X and Y is a morphism $f\colon X \to Y$ inducing a natural transformation $X \Rightarrow Y$.

1.b **($\mathbb{T}$-algebras)** Specially, we call an algebraic object of type $\mathbb{T}$ in **Set** a $\mathbb{T}$-algebra. Let $\mathbf{Alg}(\mathbb{T})$ denote the category of $\mathbb{T}$-algebras.

The following theorem describes the algebraic objects in the category of presheaves.

2 Theorem (Presheaves of $\mathbb{T}$-algebras) *The algebraic objects of type $\mathbb{T}$ in $\mathbf{PSh}(\mathcal{C})$ for some category $\mathcal{C}$ having finite products are equivalent to the presheaves of $\mathbb{T}$-algebras. In particular, algebraic objects of type $\mathbb{T}$ in $\mathcal{C}$ are equivalent to the representable presheaves of $\mathbb{T}$-algebras.*

Proof: Given an algebraic object $(\mathscr{X}, X)$ of type $\mathbb{T}$ in $\mathbf{PSh}(\mathcal{C})$, since the section functor $\Gamma(U, -)$ is exact, we obtain $\mathbb{T}$-algebras $(\mathscr{X}(U), \Gamma(U, -) \circ X)$. For any morphism $V \to U$ in $\mathcal{C}$, the transition map $\mathscr{X}(U) \to \mathscr{X}(V)$ is a homomorphism by the functorality of $\mathscr{X}^n$. This shows $\mathscr{X}$ is a presheaf of $\mathbb{T}$-algebras. Conversely, let $\mathscr{X}$ be a presheaf of $\mathbb{T}$-algebras, then $X\colon x^n \mapsto \mathscr{X}^n$ defines an algebraic object $(\mathscr{X}, X)$ of type $\mathbb{T}$ in $\mathbf{PSh}(\mathcal{C})$. □

3 Proposition (Limits and colimits in Alg($\mathbb{T}$)) $\mathbf{Alg}(\mathbb{T})$ *has all limits and sifted colimits, and they are computed pointwise.*

Proof: Since $\mathbf{Alg}(\mathbb{T})$ can be viewed as a full sub category of $[\mathbb{T}, \mathbf{Set}]$, it suffices to show the limits and sifted colimits of finite-products-preserving functors also preserve finite products. This comes from that limits and sifted colimits commute with finite products. □

4 Proposition **Alg**$(\mathbb{T})$ *is an algebraic category.*

Proof: For any $\mathbb{T}$-algebra X, the set X is called the **underlying set** of X. Taking underlying set forms a functor $F\colon \mathbf{Alg}(\mathbb{T}) \to \mathbf{Set}$. This functor is obviously faithful and reflects isomorphisms. By Proposition 3, $\mathbf{Alg}(\mathbb{T})$ has limits and sifted colimits, and they are computed pointwise. Thus F commutes with them. □

But the correct definition of algebraic category further requires F has a left adjoint.

4.a Proposition *The forgetful functor* $F\colon \mathbf{Alg}(\mathbb{T}) \to \mathbf{Set}$ *has a left adjoint.*

Proof: Let $[n]$ denote a finite set with n elements. For any $\mathbb{T}$-algebra X, we have

$$\mathrm{Hom}_{\mathbf{Set}}([n], F(X)) = X^n = X(x^n).$$

Let $\widetilde{n}$ denote the functor $\mathrm{Hom}_{\mathbb{T}}(x^n, -)$. One can see it defines a $\mathbb{T}$-algebra. By the Yoneda lemma, we have

$$\mathrm{Hom}_{\mathbf{Alg}(\mathbb{T})}(\widetilde{n}, X) = X(x^n).$$

Therefore we have

$$\mathrm{Hom}_{\mathbf{Alg}(\mathbb{T})}(\widetilde{n}, X) = \mathrm{Hom}_{\mathbf{Set}}([n], F(X)).$$

For any set S, let $\mathbf{Sub}(S)$ be the category of finite subsets of S and their inclusions. Then we have

$$S = \varinjlim_{U \in \mathbf{Sub}(S)} U,$$

and thus

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Set}}(S, F(X)) &= \mathrm{Hom}_{\mathbf{Set}}(\varinjlim_{U \in \mathbf{Sub}(S)} U, F(X)) \\
&= \varprojlim_{U \in \mathbf{Sub}(S)} \mathrm{Hom}_{\mathbf{Set}}(U, F(X)) \\
&= \varprojlim_{U \in \mathbf{Sub}(S)} \mathrm{Hom}_{\mathbf{Alg}(\mathbb{T})}(\widetilde{\mathrm{Card}(U)}, X) \\
&= \mathrm{Hom}_{\mathbf{Alg}(\mathbb{T})}(\varinjlim_{U \in \mathbf{Sub}(S)} \widetilde{\mathrm{Card}(U)}, X).
\end{aligned}$$

In this way we get a functor $T\colon S \mapsto \varinjlim_{U \in \mathbf{Sub}(S)} \widetilde{\mathrm{Card}(U)}$ which is left adjoint to F. This $T(S)$ is called the **free $\mathbb{T}$-algebra generated by** S. One can further write down the underlying set of $T(S)$, which is

$$\{f(s_1, \cdots, s_n) | n \in \mathbb{N}, f \in \mathrm{Hom}_{\mathbb{T}}(x^n, x), s_1, \cdots, s_n \in S\}.$$ □

We have seen $\mathbb{T}$-algebras form an algebraic category. What about the converse?

Before going forward, we take a glance at a generalization.

5 ¶Remark (PROPs) Any category with finite products can be considered as a *cartesian monoidal category*. So we may generalize the above into *monoidal categories*. Let $\mathcal{C}$ be a monoidal category with tensor product $\otimes$ and unit $\mathbf{1}$. For any $X \in \operatorname{ob}\mathcal{C}$, let $X^{\otimes n}$ denote the n-fold tensor product of X. In particular, $X^{\otimes 0} = \mathbf{1}$.

- A **n-ary operation** on X is a morphism from $X^{\otimes n}$ to X. Conversely, a **n-ary cooperation** on X is a morphism from X to $X^{\otimes n}$. More generally, a (m,n)**-operation** on X is a morphism from $X^{\otimes n}$ to $X^{\otimes m}$.

5.a Example (Monoids) A **monoid** in a monoidal category $\mathcal{C}$ is an object M with a 0-ary operation e called *unit* and a 2-ary operation m called *multiplication* satisfying the following commutative diagrams.

$$\begin{tikzcd}
M^{\otimes 3} \arrow[r, "\mathrm{id}\otimes m"] \arrow[d, "m\otimes\mathrm{id}"'] & M^{\otimes 2} \arrow[d, "m"] \\
M^{\otimes 2} \arrow[r, "m"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
M^{\otimes 2} \arrow[dr, "m"] & \mathbf{1}\otimes M \arrow[l, "e\otimes\mathrm{id}"'] \arrow[d, "\cong"] \\
M\otimes\mathbf{1} \arrow[u, "\mathrm{id}\otimes e"] \arrow[r, "\cong"] & M
\end{tikzcd}$$

The followings are common examples:

- A monoid in **Set** is a *monoid*.
- A monoid in **Top** is a *topological monoid*.
- A monoid in **Mon** is an *abelian monoid*.
- A monoid in **Ab** (with the tensor products of abelian groups) is a *ring*.
- A monoid in **Mod**(A) (with the tensor products of A-modules) for some commutative algebra A is an *A-algebra*.
- A monoid in $\operatorname{End}(\mathcal{C})$ for some category $\mathcal{C}$ is a *monad* on $\mathcal{C}$.

A *homomorphism* of monoids is then a morphism commutes with unit and multiplication. The category of monoid is denoted by $\mathbf{Mon}(\mathcal{C})$.

5.b Example (Commutative monoid) Let $\mathcal{C}$ be a symmetric monoidal category, then there is a natural transformation $\gamma_{X,Y}\colon X\otimes Y \to Y\otimes X$ called *braiding*. Then any monoid (M,m,e) in $\mathcal{C}$ admits another monoid structure $(M, m\circ\gamma_{M,M}, e)$, called the **opposite monoid** of M. If the two monoid structures coincide, we call it a **commutative monoid**. The category of commutative monoids is denoted by $\mathbf{CMon}(\mathcal{C})$.

The data of operations and relations of them can be encoded into a *monoidal category* $\mathbb{T}$ in which all objects are finite tensor products of a distinguished object x called the **generic object**. Such a monoidal category is called a **PROP** (abbreviating of "products and permutations category"). A *morphism* of PROP is a *monoidal functor*.

Use this notion, we define

- An **algebraic object of type** $\mathbb{T}$ in a monoidal category $\mathcal{C}$ is an object X with a monoidal functor, also denoted by X, from $\mathbb{T}$ to $\mathcal{C}$ which maps x to X. A **homomorphism** between algebraic objects X and Y is a *monoidal natural transformation* $X \Rightarrow Y$.

Monads

6 **(Monads)** We now turn to another way to encode an algebraic theory. Recall that a **monad** on a category $\mathcal{C}$ is a monoid in the category $\mathrm{End}(\mathcal{C})$ of endofunctors. More precisely, a **monad** on $\mathcal{C}$ is an endofunctor $\mathfrak{T}\colon \mathcal{C} \to \mathcal{C}$ with two natural transformations $\eta\colon\ \mathrm{id}_{\mathcal{C}} \Rightarrow \mathfrak{T}$ (called *unit*) and $\mu\colon \mathfrak{T}\circ\mathfrak{T} \Rightarrow \mathfrak{T}$ (called *multiplication*) satisfying the following commutative diagrams.

$$\begin{tikzcd}
\mathfrak{T}\circ\mathfrak{T}\circ\mathfrak{T} \arrow[r, "\mathrm{id}*\mu"] \arrow[d, "\mu*\mathrm{id}"'] & \mathfrak{T}\circ\mathfrak{T} \arrow[d, "\mu"] \\
\mathfrak{T}\circ\mathfrak{T} \arrow[r, "\mu"] & \mathfrak{T}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{T}\circ\mathfrak{T} \arrow[dr, "\mu"] & \mathrm{id}_{\mathcal{C}}\circ\mathfrak{T} \arrow[l, "\eta*\mathrm{id}"'] \arrow[d, "\cong"] \\
\mathfrak{T}\circ\mathrm{id}_{\mathcal{C}} \arrow[u, "\mathrm{id}*\eta"] \arrow[r, "\cong"] & \mathfrak{T}
\end{tikzcd}$$

7 **(Modules over a monad and $\mathfrak{T}$-algebras)** Let $\mathfrak{T}$ be a monad on a category $\mathcal{C}$, a **left $\mathfrak{T}$-module** is a functor $M\colon \mathcal{D} \to \mathcal{C}$ with a natural transformation $\lambda\colon \mathfrak{T}\circ M \Rightarrow M$ satisfying the following commutative diagrams.

$$\begin{tikzcd}
\mathfrak{T}\circ\mathfrak{T}\circ M \arrow[r, "\mathfrak{T}*\lambda"] \arrow[d, "\mu*M"'] & \mathfrak{T}\circ M \arrow[d, "\lambda"] \\
\mathfrak{T}\circ M \arrow[r, "\lambda"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{id}_{\mathcal{C}}\circ M \arrow[r, "\eta*M"] \arrow[dr, "\cong"'] & \mathfrak{T}\circ M \arrow[d, "\lambda"] \\
 & M
\end{tikzcd}$$

A **homomorphism** of left $\mathfrak{T}$-modules is a natural transformation preserving λ. Similarly, one can define the notion of **right $\mathfrak{T}$-modules**. In the case M is a constant functor, we call it a **$\mathfrak{T}$-algebra**. The category of $\mathfrak{T}$-algebras is called the **Eilenberg-Moore category** of $\mathfrak{T}$ and denoted by $\mathbf{Alg}(\mathfrak{T})$.

For any object S in $\mathcal{C}$, it is easy to see that $(\mathfrak{T}(S), \mu_S)$ is a $\mathfrak{T}$-algebra, called the **free $\mathfrak{T}$-algebra generated by** S. This factors $\mathfrak{T}$ itself as $F\circ\overline{\mathfrak{T}}$ with $F\colon\ \mathbf{Alg}(\mathfrak{T}) \to \mathcal{C}$ the forgetful functor. The full subcategory $\langle\mathfrak{T}\rangle$ of $\mathbf{Alg}(\mathfrak{T})$ consisting of free $\mathfrak{T}$-algebras is called the **Kleisli category**.

8 **Lemma (Free algebras)** *$\overline{\mathfrak{T}}\colon \mathcal{C} \to \mathbf{Alg}(\mathfrak{T})$ is left adjoint to the forgetful functor $F\colon\ \mathbf{Alg}(\mathfrak{T}) \to \mathcal{C}$.*

Proof: By the definition of monad, $\overline{\mathfrak{T}} \dashv F$ is an adjunction with the unit $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow F \circ \overline{\mathfrak{T}}$ and the counit $\epsilon\colon \overline{\mathfrak{T}} \circ F \Rightarrow \mathrm{id}_{\mathbf{Alg}(\mathfrak{T})}$ given by setting $\epsilon_{(M,\lambda)}$ as λ for every $\mathfrak{T}$-algebra (M, λ). $\square$

A typical way to get a monad is through an adjunction.

9 Lemma *Any adjunction induces a monad.*

Proof: Let $L \dashv R\colon \mathcal{D} \to \mathcal{C}$ be an adjunction. Let $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow R \circ L$ be the unit and $\epsilon\colon L \circ R \Rightarrow \mathrm{id}_{\mathcal{C}}$ the counit. Then the counit induces a natural transformation $\mu\colon R \circ L \circ R \circ L \Rightarrow R \circ L$. Let $\mathfrak{T} = R \circ L$, then the commutative diagram comes from the triangle identities for adjunction. $\square$

10 Example (Monadic adjunction) We have seen that for a monad $\mathfrak{T}$, the free algebra functor and the forgetful functor form an adjunction. The monad associated to this adjunction is nothing but $\mathfrak{T}$ itself.

More generally, let $L \dashv R\colon \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Let $\mathfrak{T} = R \circ L$. Then the natural transformation $\mathfrak{T} * \epsilon\colon \mathfrak{T} \circ R \Rightarrow R$ defines a canonical $\mathfrak{T}$-algebra structure on every $R(S)$ with $S \in \operatorname{ob} \mathcal{D}$. This makes R become a functor $\kappa = \overline{R}\colon \mathcal{D} \to \mathbf{Alg}(\mathfrak{T})$, called the **comparison functor**. An adjunction $L \dashv R$ is said to be **monadic** if this comparison functor is an equivalence of categories. A functor R is said to be **monadic** if it admits a monadic adjunction.

Recall that a Lawvere theory $\mathbb{T}$ admits a forgetful functor $F\colon \mathbf{Alg}(\mathbb{T}) \to \mathbf{Set}$ and its left adjoint T. Then the adjunction $T \dashv F$ provides a monad $\mathfrak{T}$ on $\mathbf{Set}$. The following theorem shows the relation of them.

11 Theorem *Let $\mathfrak{T}$ be a monad associate to a Lawvere theory $\mathbb{T}$. Then the $\mathfrak{T}$-algebras are equivalent to the $\mathbb{T}$-algebras.*

Proof: Given a $\mathbb{T}$-algebra X. Define $M = X(x)$ and $\lambda = F * \epsilon$. Then this (M, λ) is a $\mathfrak{T}$-algebra by the triangle identities. A homomorphism of $\mathbb{T}$-algebras induces a homomorphism of $\mathfrak{T}$-algebras by the functorality of F and ϵ. Therefore, we have a functor Φ from $\mathbf{Alg}(\mathbb{T})$ to $\mathbf{Alg}(\mathfrak{T})$. In particular, each $\widetilde{n}$ induces a $\mathfrak{T}$-algebra, also denoted by $\widetilde{n}$.

Conversely, any $\mathfrak{T}$-algebra (M, λ) admits a functor

$$\begin{aligned} X\colon \mathbb{T} &\longrightarrow \mathbf{Set} \\ x^n &\longmapsto \mathrm{Hom}_{\mathbf{Alg}(\mathfrak{T})}(\widetilde{n}, M). \end{aligned}$$

By Lemma 8, we have

$$\mathrm{Hom}_{\mathbf{Alg}(\mathfrak{T})}(\mathfrak{T}([n]), M) \cong \mathrm{Hom}_{\mathbf{Set}}([n], M).$$

Note that $\mathrm{Hom}_{\mathbf{Set}}([n], M) = M^n$. Therefore $X(x^n) = M^n$ and thus X is a finite-product preserving functor. In this way, we get a functor

$$\begin{aligned} \Psi\colon\ \mathbf{Alg}(\mathfrak{T}) &\longrightarrow \mathbf{Alg}(\mathbb{T}) \\ (M, \lambda) &\longmapsto X. \end{aligned}$$

Obviously, Φ and Ψ are weakly inverse to each other and thus $\mathbf{Alg}(\mathfrak{T})$ is equivalent to $\mathbf{Alg}(\mathbb{T})$. $\square$

Given a monad $\mathfrak{T}$ on **Set**, one can see from the above proof that $\mathfrak{T}([n])$ is the n-fold coproduct of $\mathfrak{T}([1])$ in $\mathbf{Alg}(\mathfrak{T})$. There fore the opposite of the full subcategory $\mathbb{T}$ of $\mathbf{Alg}(\mathfrak{T})$ consisting of object of the form $\mathfrak{T}([n])$ is a Lawvere theory which is isomorphic to the original one if $\mathfrak{T}$ is associated to some. Let $\mathfrak{T}'$ be the monad associated to $\mathbb{T}$. What's the relation of $\mathfrak{T}'$ and $\mathfrak{T}$?

12 **Proposition** $\mathfrak{T}' \cong \mathfrak{T}$.

Proof: First, for $[n]$, we have

$$\begin{aligned} \mathfrak{T}'([n]) &= T([n])(\mathfrak{T}([1])) \\ &= \mathrm{Hom}_{\mathbf{Alg}(\mathfrak{T})}(\mathfrak{T}([1]), \mathfrak{T}([n])) \\ &\cong \mathrm{Hom}_{\mathbf{Set}}([1], \mathfrak{T}([n])) \cong \mathfrak{T}([n]). \end{aligned}$$

For any set S, write it as the colimit $\varinjlim U$ of its finite subsets. Then, since $\mathfrak{T}$ has a right adjoint, it commutes with colimits and therefore $\mathfrak{T}'(S) = \varinjlim \mathfrak{T}'(U) = \varinjlim \mathfrak{T}(U) = \mathfrak{T}(S)$. $\square$

For now, we have algebras over a Lawvere theory or a monad, and those notions are equivalent. So, what's the relation of the category of algebras over a Lawvere theory or a monad with the notion of *algebraic category*?

It turns out that they are equivalent. This is a corollary of the following monadicity theorem of Beck.

13 **Theorem (Monadicity theorem)** *A functor $F\colon : \mathcal{D} \to \mathcal{C}$ is monadic if and only if*

1. *F admits a left adjoint;*
2. *F reflects isomorphisms;*
3. *For any pair of parallel morphisms (f, g) in $\mathcal{D}$, if $(F(f), F(g))$ admits a split coequalizer, then (f, g) has a coequalizer preserved by F.*

Proof: If F is monadic with the associated monad $\mathfrak{T}$, then it is equivalent to the forgetful functor $F\colon\ \mathbf{Alg}(\mathfrak{T}) \to \mathcal{C}$. For the forgetful functor, *1.* and

2. are obvious. As for *3.*, consider any pair of parallel morphisms (f, g) of $\mathfrak{T}$-algebras A, B. Let

$$F(A) \underset{F(g)}{\overset{F(f)}{\rightrightarrows}} F(B) \xrightarrow{c} C,$$

be the split coequalizer of $(F(f), F(g))$. Since such coequalizer is absolute, it is preserved by $\mathfrak{T}$. This gives C a $\mathfrak{T}$-algebra structure and making c a homomorphism of $\mathfrak{T}$-algebras.

Conversely, assume F satisfying *1.* to *3.*, and let L be its left adjoint. Let $\mathfrak{T}$ be the associated monad and $\kappa\colon \mathcal{D} \to \mathbf{Alg}(\mathfrak{T})$ the comparison functor. It remains to show κ is an equivalence. By Lemma 13.b, there is a unique comparison functor $K\colon \mathbf{Alg}(\mathfrak{T}) \to \mathcal{D}$. One can see that the compositions $K \circ \kappa$ and $\kappa \circ K$ are also comparison functors. Therefore, by the uniqueness, $K \circ \kappa \cong \mathrm{id}_{\mathcal{D}}$, $\kappa \circ K \cong \mathrm{id}_{\mathbf{Alg}(\mathfrak{T})}$. □

13.a (Beck coequalizers) Recall (ref. §2.10 in *BMO* or [Bor94]) that a colimit is said to be **absolute** if it is preserved by any functors. A **fork** is a diagram of the form

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{c} C,$$

where $c \circ f = c \circ g$. A fork is called a **split coequalizer** if the morphism $(f, c)\colon g \to c$ admits a section in the *arrow category*. A split coequalizer is an absolute coequalizer.

Every $\mathfrak{T}$-algebra (M, λ) is the coequalizer of the first stage of its **bar resolution**:

$$(\mathfrak{T}^2(M), \mu_{\mathfrak{T}(M)}) \underset{\mathfrak{T}(\lambda)}{\overset{\mu_M}{\rightrightarrows}} (\mathfrak{T}(M), \mu_M) \xrightarrow{\lambda} (M, \lambda).$$

Moreover, the underlying fork in $\mathcal{C}$

$$\mathfrak{T}^2(M) \underset{\mathfrak{T}(\lambda)}{\overset{\mu_M}{\rightrightarrows}} \mathfrak{T}(M) \xrightarrow{\lambda} M.$$

is a split coequalizer with the splitting

$$\mathfrak{T}^2(M) \overset{\eta_{\mathfrak{T}(M)}}{\longleftarrow} \mathfrak{T}(M) \overset{\eta_M}{\longleftarrow} M.$$

Moreover, those conditions determines the $\mathfrak{T}$-algebra structure (M, λ).

Note that for (M, λ) a $\mathfrak{T}$-algebra, $\mathfrak{T}(\eta_M)$ is a common section of both μ_M and λ_M. Therefore the first stage of bar resolution is actually a sifted colimit diagram.

13.b Lemma *Let $F\colon \mathcal{D} \to \mathcal{C}$ be a functor satisfying condition 1 and 3 of Theorem 13. Let $L' \dashv R'\colon \mathcal{D}' \to \mathcal{C}$ be another adjunction defining the same monad* *with F. Then there exists a unique (up to unique isomorphism)* **comparison functor** $K\colon \mathcal{D}' \to \mathcal{D}$ *such that the following diagrams commute.*

$$\begin{tikzcd} \mathcal{D}' \arrow[r, "K"] \arrow[dr, "R'"'] & \mathcal{D} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D}' \arrow[r, "K"] & \mathcal{D} \\ & \mathcal{C} \arrow[ul, "L'"] \arrow[u, "L"'] \end{tikzcd}$$

Proof: If such a functor K exists, then we have $LFK = KL'R'$ and $\epsilon * K = K * \epsilon'$. Note that any object $S' \in \operatorname{ob}\mathcal{D}'$ admits a fork

$$L'R'L'R'(S') \underset{L'R'*\epsilon'}{\overset{\epsilon'*L'R'}{\rightrightarrows}} L'R'(S') \xrightarrow{\epsilon'_{S'}} S'.$$

Apply K to it, we obtain a fork

$$LFLR'(S') \underset{LR'*\epsilon'}{\overset{\epsilon*LR'}{\rightrightarrows}} LR'(S') \xrightarrow{K*\epsilon'} K(S').$$

Apply F to it, we obtain

$$\mathfrak{T}^2R'(S') \underset{\mathfrak{T}R'\epsilon'}{\overset{F\epsilon LR'}{\rightrightarrows}} \mathfrak{T}R'(S') \xrightarrow{R'\epsilon'} R'(S'),$$

which is of the form as in 13.a. Note that $(R'(S'), R'\epsilon'_{S'})$ is a $\mathfrak{T}$-algebra, thus the above fork is a split coequalizer. Since F satisfies condition *3.*, the former fork is a coequalizer. Then. the uniqueness of coequalizer ensure the uniqueness of K.

Moreover, the above discussion also shows the existence of K: $K(S')$ is the coequalizer of $(\epsilon_{LR'(S')}, LR'(\epsilon'_{S'}))$. $\square$

14 Theorem (Universal algebra theorem) *Let $\mathcal{C}$ be an algebraic category, then it is equivalent to the category of algebras over a monad.*

Proof: It suffices to show the forgetful functor is monadic. By its definition, it remains to show the condition *3.* in Theorem 13. Let L be the left adjoint of the forgetful functor $F\colon \mathcal{C} \to \mathbf{Set}$ and $\mathfrak{T}$ be the monad associated to them.

Let $f, g\colon A \rightrightarrows B$ be two morphisms in $\mathcal{C}$ such that the following diagram is a split coequalizer diagram of sets.

$$F(A) \underset{F(g)}{\overset{F(f)}{\rightrightarrows}} F(B) \xrightarrow{c} C$$

Then, since split coequalizers are absolute, the horizontal arrows in the following diagram are coequalizers.

$$\begin{tikzcd} \mathfrak{T}F(A) \arrow[r, shift left, "\mathfrak{T}F(f)"] \arrow[r, shift right, "\mathfrak{T}F(g)"'] \arrow[d] & \mathfrak{T}F(B) \arrow[r, "\mathfrak{T}(c)"] \arrow[d] & \mathfrak{T}(C) \arrow[d, dashed, "\lambda"] \\ F(A) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & F(B) \arrow[r, "c"] & C \end{tikzcd}$$

The first two vertical arrows are just $F * \epsilon$, they give the canonical $\mathfrak{T}$-algebraic structure on $F(A)$ and $F(B)$. Then, the universal property of coequalizers guarantees the third vertical arrow and further gives a $\mathfrak{T}$-algebraic structure on C and making c becoming a homomorphism.

Now consider the diagram

$$L\mathfrak{T}(C) \underset{L(\lambda)}{\overset{\epsilon_{L(C)}}{\rightrightarrows}} L(C).$$

One can see $L(\eta_C)$ is a common section of them. Therefore the coeqalizer $\Lambda\colon L(C) \to D$ of this diagram is a sifted colimit, thus exists in $\mathcal{C}$ and is preserved by F. But we already have a coeqalizer diagram

$$\mathfrak{T}^2(C) \underset{\mathfrak{T}(\lambda)}{\overset{\mu_C}{\rightrightarrows}} \mathfrak{T}(C) \xrightarrow{\lambda} C,$$

where the left pair is obtained by apply F to the previous diagram. Therefore $F(\Lambda) = \lambda$.

Now, consider the following diagram.

$$\begin{tikzcd}
L\mathfrak{T}F(A) \arrow[r, shift left, "L\mathfrak{T}F(f)"] \arrow[r, shift right, "L\mathfrak{T}F(g)"'] \arrow[d, shift left] \arrow[d, shift right] & L\mathfrak{T}F(B) \arrow[r, "L\mathfrak{T}(c)"] \arrow[d, shift left] \arrow[d, shift right] & L\mathfrak{T}(C) \arrow[d, shift right, "L(\lambda)"'] \arrow[d, shift left, "\epsilon_{L(C)}"] \\
LF(A) \arrow[r, shift left, "LF(f)"] \arrow[r, shift right, "LF(g)"'] \arrow[d] & LF(B) \arrow[r, "L(c)"] \arrow[d] & L(C) \arrow[d, "\Lambda"] \\
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, dashed, "d"] & D
\end{tikzcd}$$

Here, we have seen the first two rows and the last column are coequalizer. Apply F to the firs two columns, we obtained the first stage of the bar resolutions of $F(A)$ and $F(B)$. By Lemma 14.b, F reflects sifted colimits, thus the firs two columns are coequalizers. Then the dotted arrow exists by the universal property. By the 9-lemma, the bottom is a coequalizer diagram. Apply F to the whole diagram and use 9-lemma again, we see $F(d)$ is the coequalizer of $F(f)$ and $F(g)$, thus $F(d) = c$ as desired. □

14.a (Regular and effective epimorphisms) Recall that a morphism is called a **regular epimorphism** if it is a coequalizer, called an **effective epimorphism** if it is the coequalizer of its kernel pair. Since *if a coequalizer has a kernel pair, then it is the coequalizer of its kernel pair*, we see the two notions coincides in an algebraic category.

14.b Lemma *Let $\mathcal{C}$ be an algebraic category with the forgetful functor F. Then F reflects limits, sifted colimits, isomorphisms, monomorphisms and regular epimorphisms.*

Proof: By the definition and Proposition I.2.6.c, F reflects isomorphisms and monomorphisms and commutes with limits and sifted colimits. Since F reflects isomorphisms, by the universal property, F reflects a limit or colimit when commutes with it. Thus it remains to show F reflects regular epimorphisms.

Note that for $k_1, k_2 \colon \bullet \rightrightarrows \bullet$ a kernel pair, there is a unique section s of both k_1 and k_2 via the universal property of kernel pair. In this way, the coequalizer of a kernel pair is the same as the colimit of the diagram

$$\begin{tikzcd} \bullet \arrow[r, shift left=2, "k_1"] \arrow[r, shift right=2, "k_1"'] & \bullet \arrow[l, "s" description] \end{tikzcd}$$

which is, however, sifted. Therefore, since F reflects sifted colimits, it reflects regular epimorphisms. $\square$

§ II.2 Linear cosmoses

In this section, we give the notion of linear cosmoses and discuss some basic algebraic notions in them.

1 **(Linear cosmoses)** Recall that an *pre-abelian category* is a finite-complete and finite-cocomplete **Ab**-enriched category. Then we have the notion of *biproducts*, *kernels*, *cokernels*, *images* and *coimages*. Any morphism in a pre-abelian category can be uniquely factored as a composition of an effective epimorphism (its coimage), a canonical morphism and a regular monomorphism (its image). A pre-abelian category is an **abelian category** if for each morphism, the canonical morphism is an isomorphism.

An **abelian tensor category**, or **finitary linear cosmos**, is

- an abelian category $\mathcal{K}$
- equipped with a symmetric monoidal structure on it,

such that

- for any $X \in \operatorname{ob}\mathcal{K}$, $X \otimes -$ gives rise to an additive functor which is right exact.

The *morphisms* between them, called the **abelian lax functors**, or **changes of linear cosmoses**, are functors additive, symmetric lax and commute with colimits. Similar, we have **abelian tensor functors**.

Remark Note that the tensor product functor is *NOT* a lax functor, *a fortiori* a change of linear cosmoses.

A **cocomplete abelian tensor category**[1], or a **linear cosmos**, is an abelian tensor category $\mathcal{K}$ which is also a **cocomplete tensor category**, meaning

- $\mathcal{K}$ is cocomplete and for any $X \in \operatorname{ob}\mathcal{K}$, $X \otimes -$ commutes with colimits (this is equivalent to say $\mathcal{K}$ is a cocomplete closed monoidal category under suitable size assumptions).

Finally, a **Grothendieck cosmos**[2] is a linear cosmos $\mathcal{K}$ which is a **Grothendieck abelian category**, meaning

1. $\mathcal{K}$ has a *generator*, which is an object S such that $\operatorname{Hom}_{\mathcal{K}}(S, -)$ is faithful.
2. Grothendieck's axiom *AB5*:

   $\mathcal{K}$ is cocomplete and filtered colimits of exact sequences are exact.

[1] The terminology is different with Lurie's in [Lur04].

[2] The terminologies "linear cosmos" and "Grothendieck cosmos" are not standard.

1.a Example The category **Ab** of abelian groups is definitely a Grothendieck cosmos. More general, the category $\mathbf{Mod}(R)$ of modules over a ring R is a Grothendieck cosmos. For $f\colon R \to S$ a homomorphism of rings, the induced functor $S \otimes_R -\colon \mathbf{Mod}(R) \to \mathbf{Mod}(S)$ is an abelian tensor functor. However, the forgetful functor $\mathbf{Mod}(S) \to \mathbf{Mod}(R)$ taking S-modules to their restrictions to R is merely an abelian lax functor.

1.b Example Let $\mathcal{I}$ be a small category and $\mathcal{K}$ a (cocomplete) abelian tensor category. Then the category $[\mathcal{I}, \mathcal{K}]$ of functors $\mathcal{I} \to \mathcal{K}$ is a (cocomplete) abelian tensor category, denoted by ${}^{\mathcal{I}}\mathcal{K}$. In particular, Let G be a group. Viewed G as a category with one object, we get the category $\mathbf{Rep}_{\mathcal{K}}(G)$ of representations of G on $\mathcal{K}$. Note that this category is rarely a Grothendieck cosmos even if $\mathcal{K}$ is since $S \in \operatorname{ob}\mathcal{K}$ is a generator does *NOT* imply that the constant functor Δ_S is a generator.

1.c Example The category $\mathbf{Vect}(R)$ of free modules over a ring R is *NOT* a linear cosmos since it has no cokernels. In fact, $\mathbf{Mod}(R)$ is the cocompletion of $\mathbf{Vect}(R)$. So, $\mathbf{Vect}(R)$ is a linear cosmos if and only if R is a field.

1.d Example The category of vector bundles on a nontrivial manifold is *NOT* a linear cosmos since it has no kernels and cokernels. One way to make up this is to enalrge the category, for instance, the category of *quasi-coherent sheaves.*

1.e (Elements) In a category $\mathcal{C}$, one can use the **element notation** is help reasoning. In this notation system, an **element** of an object is actually an arbitrary morphism to it. The Yoneda lemma guarantees that we can verify commutative diagrams by chasing an arbitrary element.

If $\mathcal{C}$ has a generator, then we may limit elements to refer to those from the generator, saying an **element** of an object X in a category with a generator S is a morphism from S to X.

In a monoidal category, we call a morphism from the tensor unit $\mathbf{1}$ to an object X a **global element** of X. Note that the global elements may *NOT* be enough to determine X.

The following basic result will be used very often.

1.f Lemma (Tensor product of presentations) *Let $\mathcal{K}$ be an abelian tensor category. Then*

1. *The tensor product of two epimorphisms is again an epimorphism.*
2. *The tensor product of two coequalizer diagrams is again a coequalizer diagram. In particular, the tensor product of two regular epimorphisms is again a regular epimorphism.*

Monoids

2 (Algebras) A monoid in an abelian tensor category is called an **algebra**. Similarly, a commutative monoid in an abelian tensor category is called a **commutative algeba**. We have seen the definition and basic examples of monoids and commutative monoids in Example 1.5.a and Example 1.5.b.

Before going forward to deal with algebras, we give some further discussion on general monoids. To distinguish, we use $\mathcal{C}$ to refer to an arbitrary symmetric monoidal category and $\mathcal{K}$ to refer in particular to an abelian tensor category.

The following lemmas are obvious.

2.a Lemma (Monoids form a monoidal category) *Let $\mathcal{C}$ be a symmetric monoidal category, then the category $\mathbf{Mon}(\mathcal{C})$ of monoids in $\mathcal{C}$ is again a symmetric monoidal category, in which the tensor product of two monoids (M_1, m_1, e_1) and (M_2, m_2, e_2) is given by $(M_1 \otimes M_2, m, e)$ where m is the composition*

$$(M_1 \otimes M_2) \otimes (M_1 \otimes M_2) \xrightarrow{\gamma} (M_1 \otimes M_1) \otimes (M_2 \otimes M_2) \xrightarrow{m_1 \otimes m_2} M_1 \otimes M_2,$$

and e is the composition

$$\mathbf{1} \cong \mathbf{1} \otimes \mathbf{1} \xrightarrow{e_1 \otimes e_2} M_1 \otimes M_2.$$

As a full subcategory of $\mathbf{Mon}(\mathcal{C})$, the category $\mathbf{CMon}(\mathcal{C})$ of commutative monoids in $\mathcal{C}$ inherits this symmetric monoidal category structure. Moreover, the coproducts in $\mathbf{CMon}(\mathcal{C})$ are given by the tensor products.

2.b Lemma (Eckmann-Hilton Principle) $\mathbf{Mon}(\mathbf{Mon}(\mathcal{C}))$ *is isomorphic to* $\mathbf{CMon}(\mathcal{C})$*. In precise, if an object M equipped two monoid structure such that one of them makes M an object in $\mathbf{Mon}(\mathcal{C})$ and the other gives also a monoid structure on it, then the two structures coincide and become a commutative monoid structure.*

Proof: Follows from the Eckmann-Hilton argument. □

2.c Lemma (Lifting adjunctions) *Any monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a monoidal functor $\widehat{F}\colon \mathbf{Mon}(\mathcal{C}) \to \mathbf{Mon}(\mathcal{D})$. Moreover, if F admits a right adjoint G, then so does $\widehat{G}$ and the two adjunctions are compatible with the forgetful functors $\mathbf{Mon}(\mathcal{C}) \to \mathcal{C}$ and $\mathbf{Mon}(\mathcal{D}) \to \mathcal{D}$.*

2.d Example (Initial monoid) The category $\mathbf{Mon}(\mathcal{C})$ has an initial object, that is $\mathbf{1}$ with the obvious monoid structure. Furthermore, $\mathbf{1}$ is commutative.

2.e Example (Endomorphism monoid) Let $\mathcal{C}$ be a closed symmetric monoidal category. Then any object $M \in \operatorname{ob}\mathcal{C}$ admit a monoid $\underline{\operatorname{End}}(M)$ whose underlying object is $\underline{\operatorname{Hom}}(M, M)$ and whose unit is the *identity operation* $\mathbf{1} \to \underline{\operatorname{Hom}}(M, M)$ and whose multiplication is the *composition operation*

$$\underline{\operatorname{Hom}}(M, M) \otimes \underline{\operatorname{Hom}}(M, M) \longrightarrow \underline{\operatorname{Hom}}(M, M).$$

This monoid is called the **endomorphism monoid** of M.

2.f Example (Center) Let $\mathcal{C}$ be a closed symmetric monoidal category and A a monoid in it. Then corresponding to the multiplications of A and its opposite monoid, we have two morphisms $A \to \underline{\operatorname{Hom}}(A, A)$. The equalizer of them is called the **center** of A, denoted by $Z(A)$. It turns out to be a submonoid of A.

2.g Lemma (Algebraicness) *Let $\mathcal{C}$ be a closed symmetric monoidal category and $F\colon \mathbf{Mon}(\mathcal{C}) \to \mathcal{C}$ the forgetful functor, then*

1. *F is faithful;*
2. *F commutes with limits;*
3. *F commutes with filtered colimits;*
4. *F reflects isomorphisms.*

Proof: Note that the unit and multiplication of limits are the dotted morphisms obtained by the universal property in the following diagrams.

$$\begin{tikzcd}
\mathbf{1} \arrow[r, dashed, "e"] \arrow[dr, "e_i"'] & \varprojlim M_i \arrow[d] \\
& M_i
\end{tikzcd}
\qquad
\begin{tikzcd}
(\varprojlim M_i) \otimes (\varprojlim M_j) \arrow[r, dashed, "m"] \arrow[d] & \varprojlim M_i \arrow[d] \\
M_i \otimes M_i \arrow[r, "m_i"] & M_i
\end{tikzcd}$$

Let $M = \varinjlim M_i$ be a filtered colimit of underlying objects of monoids in $\mathcal{C}$. Then the composition $\mathbf{1} \to M_i \to \varinjlim M_i$ gives M a unit, which is well-defined since the diagram is filtered. Since $\mathcal{C}$ is closed, we have

$$(\varinjlim M_i) \otimes (\varinjlim M_j) \cong \varinjlim (M_i \otimes M_j)$$

and this filtered diagram has a final family $M_i \otimes M_i$. Therefore

$$M \otimes M \cong \varinjlim (M_i \otimes M_i).$$

Then the multiplications $m_i\colon M_i \otimes M_i \to M_i$ induces a multiplication

$$\varinjlim (M_i \otimes M_i) \xrightarrow{m} \varinjlim M_i.$$

The rests are easy to verify. $\square$

2.h Example (Tensor algebras) Let $\mathcal{C}$ be a closed symmetric monoidal category having denumerable coproducts. Then, for any object M in $\mathcal{C}$, there is a canonical monoid structure on the coproduct

$$T(M) := \bigoplus_{n=0}^{\infty} M^{\otimes n},$$

where the unit is the inclusion $M^{\otimes 0} \hookrightarrow T(M)$ and the multiplication is the isomorphism $M^{\otimes p} \otimes M^{\otimes q} \to M^{\otimes p+q}$. This monoid is called the **tensor algebra** generated by M.

Note that the tensor algebras give rise to a symmetric monoidal functor $T\colon \mathcal{C} \to \mathbf{Mon}(\mathcal{C})$, which is left adjoint to the forgetful functor $\mathbf{Mon}(\mathcal{C}) \to \mathcal{C}$. In particular, T maps the initial object 0 of $\mathcal{C}$ to the initial monoid $\mathbf{1}$.

3 ¶ (Bimonoids) A **comonoid** in a monoidal category $\mathcal{C}$ is an object which is a monoid in $\mathcal{C}^{\mathrm{opp}}$. Note that the category $\mathbf{CoMon}(\mathcal{C})$ of comonoids is $(\mathbf{Mon}(\mathcal{C}^{\mathrm{opp}}))^{\mathrm{opp}}$, not $\mathbf{Mon}(\mathcal{C}^{\mathrm{opp}})$.

A **bimonoid** is then an object which is both a monoid and a comonoid in a compatible way. In the case $\mathcal{C}$ is symmetric monoidal, this compatibility is easy to describe: a **bimonoid** is an object B with a 0-ary operation e called *unit*, a 0-ary cooperation ϵ called *counit*, a 2-ary operation m called *multiplication* and a 2-ary cooperation δ called *comultiplication* such that

- (B, m, e) is a monoid;
- (B, δ, ϵ) is a comonoid;

and satisfying one of the following equivalent compatible conditions:

1. δ and ϵ are homomorphisms of monoids;
2. m and e are homomorphisms of comonoids.

A *homomorphism* of bimonoids $f\colon (B, m, e, \delta, \epsilon) \to (B', m', e', \delta', \epsilon')$ is then a morphism $f\colon B \to B'$ which is both a homomorphism of monoids $f\colon (B, m, e) \to (B', m', e')$ and a homomorphism of comonoids $f\colon (B, \delta, \epsilon) \to (B', \delta', \epsilon')$. The category of bimonoids is denoted by $\mathbf{BiMon}(\mathcal{C})$.

A bimonoid $(B, m, e, \delta, \epsilon)$ is said to be **commutative** (resp. **cocommutative**) if (B, m, e) is commutative (resp. (B, δ, ϵ) is cocommutative).

3.a Proposition *The following three symmetric monoidal categories are equivalent:* $\mathbf{Mon}(\mathbf{CoMon}(\mathcal{C}))$, $\mathbf{CoMon}(\mathbf{Mon}(\mathcal{C}))$ *and* $\mathbf{BiMon}(\mathcal{C})$.

4 ¶ (Hopf monoids) A **Hopf monoid** H is a bimonoid with an extra 1-ary operation called **antipode** such that the following diagram commutes.

$$\begin{tikzcd}
H \arrow[r, "e"] \arrow[d, "\delta"'] & \mathbf{1} \arrow[r, "\epsilon"] & H \\
H \otimes H \arrow[rr, shift left, "H\otimes s"] \arrow[rr, shift right, "s\otimes H"'] & & H \otimes H \arrow[u, "m"']
\end{tikzcd}$$

4.a Example (Groups) In a cartesian monoidal category, for example **Set**, every monoid object is a bimonoid in a unique way. Such a bimonoid is a Hopf monoid if and only if it is a group object. In this way *Hopf monoid is a generalization of group objects.*

Remark We will meet Hopf algebras many times. They are actually a very important kind of quantum groups.

4.b Proposition *A commutative Hopf monoid in $\mathcal{C}$ is the same thing as a group object in* $\mathbf{CMon}(\mathcal{C})$

Modules

Next, we consider modules.

5 (Modules over a monoid) Recall that monads on a category $\mathcal{C}$ are the same as monoids in $\mathrm{End}(\mathcal{C})$. The definition in 1.7 can be easily generalized. Let A be a monoid in a monoidal category $\mathcal{C}$, a **left A-module**, or simply called **A-module**, is an object $M \in \mathrm{ob}\,\mathcal{C}$ with a morphism $l\colon A \otimes M \to M$, called *action*, satisfying the following commutative diagrams.

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "A\otimes l"] \arrow[d, "m\otimes M"'] & A \otimes M \arrow[d, "l"] \\ A \otimes M \arrow[r, "l"] & M \end{tikzcd} \qquad \begin{tikzcd} \mathbf{1} \otimes M \arrow[r, "e\otimes M"] \arrow[dr, "\cong"'] & A \otimes M \arrow[d, "l"] \\ & M \end{tikzcd}$$

A **homomorphism** of A-modules, or an **A-homomorphism**, is a morphism commuting with the A-module actions. The category of A-modules is denoted by $\mathbf{Mod}(A)$. Similarly, one can define the notion of **right A-modules**. Let A, B be two monoids, then an **(A, B)-bimodule** is an object equipped both a left A-module structure and a right B-module structure. An (A, A)-bimodule is called an **A-bimodule**.

Remark This definition actually generalize nothing. Indeed, one can see that a monoid in $\mathcal{C}$ admits a monad $A \otimes -$ and that the modules over a monoid A are precisely the algebras over the monad $A \otimes -$.

Remark If $\mathcal{C}$ is closed. Then a left A-module structure on M can be identified with a homomorphism of monoids $A \to \underline{\mathrm{Hom}}(M, M)$.

5.a Remark (Category of modules over a commutative monoid) Let $\mathcal{C}$ be a symmetric monoidal category and A a commutative monoid in it, then any left A-module is also a right A-module and thus an A-bimodule. Then, just like in commutative algebras, we can define the **tensor product** of two A-modules M and N as follows.

First, the object $M \otimes_A N$ is the coequalizer of

$$M \otimes A \otimes N \underset{M\otimes l}{\overset{l\otimes N}{\rightrightarrows}} M \otimes N$$

in $\mathcal{C}$. Then action $A \otimes (M \otimes_A N) \to M \otimes_A N$ is unique morphism making the following commutative diagram whose uniqueness and existence are guaranteed by the universal property of coequalizer.

$$\begin{tikzcd}
A \otimes M \otimes A \otimes N \arrow[r, shift left, "l\otimes N"] \arrow[r, shift right, "M\otimes l"'] \arrow[d, "l\otimes A\otimes N"'] & A \otimes M \otimes N \arrow[r] \arrow[d, "l\otimes N"'] & A \otimes (M \otimes_A N) \arrow[d, dashed] \\
M \otimes A \otimes N \arrow[r, shift left, "l\otimes N"] \arrow[r, shift right, "M\otimes l"'] & M \otimes N \arrow[r] & M \otimes_A N
\end{tikzcd}$$

In this way, $\mathbf{Mod}(A)$ becomes a symmetric monoidal category with tensor product $\otimes_A$, unit A and braiding $M \otimes_A N \to N \otimes_A M$ induced by the braiding γ of $\mathcal{C}$.

If $\mathcal{C}$ is further closed and has equalizers. Then so does $\mathbf{Mod}(A)$. The internal Hom $\underline{\mathrm{Hom}}_A(M, N)$ is given by the equalizer of the morphism

$$\underline{\mathrm{Hom}}(M, N) \longrightarrow \underline{\mathrm{Hom}}(M, \underline{\mathrm{Hom}}(A, N)),$$

which is induced by the action of A on N, and the morphism

$$\underline{\mathrm{Hom}}(M, N) \longrightarrow \underline{\mathrm{Hom}}(M \otimes A, N) \cong \underline{\mathrm{Hom}}(M, \underline{\mathrm{Hom}}(A, N)),$$

which is induced by the action of A on M.

Recall (Example A.2.5) that the A-homomorphism

$$\mathrm{ev}\colon M \otimes_A \underline{\mathrm{Hom}}_A(M, N) \longrightarrow N$$

corresponding to $\mathrm{id}_{\underline{\mathrm{Hom}}_A(M,N)}$ is called the **evaluation**.

5.b Remark (Internal Tensor-Hom adjunction) Recall (Example A.2.17) that in a right closed monoidal category $\mathcal{C}$, the *Tensor-Hom adjunction*

$$- \otimes A \dashv \underline{\mathrm{Hom}}(A, -)$$

further induce natural isomorphisms

$$\underline{\mathrm{Hom}}(X \otimes A, Y) \cong \underline{\mathrm{Hom}}(X, \underline{\mathrm{Hom}}(A, Y)).$$

5.c Lemma (Limits and colimits in a module category) *Let $\mathcal{C}$ be a closed symmetric monoidal category and A a monoid in it. Then the limits and colimits in* $\mathbf{Mod}(A)$ *are computed in $\mathcal{C}$.*

Proof: The A-module structure of a limit of A-modules is given by the dotted morphism in the following diagram.

$$\begin{tikzcd} A \otimes (\varprojlim M_i) \arrow[r, dashed] \arrow[d] & \varprojlim M_i \arrow[d] \\ A \otimes M_i \arrow[r] & M_i \end{tikzcd}$$

The A-module structure of a colimit of A-modules is given by the dotted morphism in the following diagram.

$$\begin{tikzcd} & A \otimes M_i \arrow[r] \arrow[d] & M_i \arrow[d] \\ A \otimes (\varinjlim M_i) \arrow[r, phantom, "\cong"] & \varinjlim (A \otimes M_i) \arrow[r, dashed] & \varinjlim M_i \end{tikzcd}$$

$\square$

Therefore the forgetful functor $\mathbf{Mod}(A) \to \mathcal{C}$ is

1. faithful,
2. reflects limits, colimits and isomorphisms.

It is further a *symmetric lax functor* as we have canonical morphisms:

$$M \otimes N \longrightarrow M \otimes_A N \qquad \text{and} \qquad \mathbf{1} \xrightarrow{e_A} A.$$

However, it is *NOT* monoidal since it doesn't preserve the tensor unit.

5.d Example (Free modules) Let $\mathcal{C}$ be a symmetric monoidal category and A a commutative monoid in it. For any object X in $\mathcal{C}$, $A \otimes X$ is an A-module with the action

$$A \otimes A \otimes X \xrightarrow{m_A} A \otimes X.$$

This module is called the **free module** on X.

The free modules give rise to the left adjoint functor of the forgetful functor $\mathbf{Mod}(A) \to \mathcal{C}$. The unit $- \to A \otimes -$ is given by the unit of A and the counit $A \otimes - \to -$ is given by the module actions.

Moreover the free module functor is a symmetric monoidal functor since we have the natural isomorphisms

$$A \otimes (X \otimes Y) \cong (A \otimes X) \otimes_A (A \otimes Y) \qquad \text{and} \qquad A \otimes \mathbf{1} \cong A.$$

In the case $A \otimes -$ commutes with coproducts, the n-fold coproducts $A^{\oplus n}$ of A is a free module, called the **free module of rank** n.

5.e Example (Cofree modules) Let $\mathcal{C}$ be a closed symmetric monoidal category and A a commutative monoid in it. For any object X in $\mathcal{C}$, we first have a morphism

$$\underline{\mathrm{Hom}}(A, X) \xrightarrow{m_A} \underline{\mathrm{Hom}}(A \otimes A, X)$$

induced by the multiplication of A. Apply the *Tensor-Hom* adjunction to it and note that (by Remark 5.b)

$$\underline{\mathrm{Hom}}(A \otimes A, X) \cong \underline{\mathrm{Hom}}(A, \underline{\mathrm{Hom}}(A, X))$$

we get a morphism

$$A \otimes \underline{\mathrm{Hom}}(A, X) \longrightarrow \underline{\mathrm{Hom}}(A, X).$$

Furthermore, this morphism gives an A-module structure on $\underline{\mathrm{Hom}}(A, X)$. This module is called the **cofree module** on X.

The cofree modules give rise to the right adjoint functor of the forgetful functor $\mathbf{Mod}(A) \to \mathcal{C}$. The unit is given by the A-homomorphisms

$$M \longrightarrow \underline{\mathrm{Hom}}(A, M)$$

corresponding to the module actions $M \otimes A \to M$. The counit is given by the composition of the unit of A and the *evaluations* (ref. Example A.2.5)

$$\underline{\mathrm{Hom}}(A, X) \xrightarrow{e_A} A \otimes \underline{\mathrm{Hom}}(A, X) \xrightarrow{\mathrm{ev}} X.$$

Note that the cofree module functor is *NOT* a symmetric lax functor. For instance, $\underline{\mathrm{Hom}}(A, \mathbf{1})$ is rarely isomorphic to A as A-modules, actually there may be even no A-homomorphisms.

5.f Example (Dual modules) Let M be a module over a commutative monoid A in a closed symmetric monoidal category $\mathcal{C}$. The module $\underline{\mathrm{Hom}}_A(M, A)$ is called the **weak dual** of M, denoted by $M^\vee$.

6 ¶ (Modules over Hopf monoids) A **module** over a bimonoid B is a module over the monoid B. We have seen $\mathbf{Mod}(B)$ is a symmetric monoidal category. However, we give $\mathbf{Mod}(B)$ another monoidal structure which directly inherits from $\mathcal{C}$. Indeed, if B is a bimonoid, then for any two B-modules M and N, their tensor product $M \otimes N$ admits an action as the composition

$$B \otimes (M \otimes N) \xrightarrow{\delta} (B \otimes B) \otimes (M \otimes N) \xrightarrow{\gamma} (B \otimes M) \otimes (B \otimes N) \xrightarrow{l \otimes l} M \otimes N.$$

Similarly, the counit $\epsilon \colon B \to \mathbf{1}$ induces a B-module structure on $\mathbf{1}$. In this way, we see the forgetful functor $\mathbf{Mod}(B) \to \mathcal{C}$ is a *strong monoidal functor*, meaning it preserves the monoidal structures.

6.a Theorem (Closed monoidal structure on modules) *If H is a Hopf monoid in a closed symmetric monoidal category $\mathcal{C}$, then the category $H\,\mathbf{Mod}$ is also closed and the forgetful functor is strong closed, meaning it preserves internal hom.*

Proof: It suffices to give an H-module structure on $\underline{\mathrm{Hom}}(M,N)$ for any H-modules M,N. Indeed, since

$$\mathrm{Hom}_{\mathcal{C}}(H\otimes \underline{\mathrm{Hom}}(M,N),\underline{\mathrm{Hom}}(M,N)) = \mathrm{Hom}_{\mathcal{C}}(H\otimes \underline{\mathrm{Hom}}(M,N)\otimes M,N),$$

it suffices to give a morphism $H\otimes \underline{\mathrm{Hom}}(M,N)\otimes M\to N$. Here it is

$$\begin{aligned} H\otimes \underline{\mathrm{Hom}}(M,N)\otimes M &\xrightarrow{\delta} (H\otimes H)\otimes \underline{\mathrm{Hom}}(M,N)\otimes M \\ &\xrightarrow{\gamma} (H\otimes \underline{\mathrm{Hom}}(M,N))\otimes (H\otimes M) \\ &\xrightarrow{l\otimes l} \underline{\mathrm{Hom}}(M,N)\otimes M \\ &\xrightarrow{\mathrm{ev}} N, \end{aligned}$$

where ev is the **evaluation**. □

Algebras and modules in a linear cosmos

We now, turn back to consider modules in an abelian tensor category.

7 **Theorem (Abelian tensor category of modules)** *Let A be a monoid in an abelian tensor category $\mathcal{K}$.*

1. *The category* $\mathbf{Mod}(A)$ *of A-modules is naturally an abelian category.*
2. *If A is further commutative, then* $\mathbf{Mod}(A)$ *is also an abelian tensor category. In this case, the forgetful functor* $\mathbf{Mod}(A)\to\mathcal{K}$ *reflects colimits and had a left adjoint $A\otimes -$, which is an abelian tensor functor.*
3. *If $\mathcal{K}$ is further a linear cosmos, then so is* $\mathbf{Mod}(A)$.
4. *If $\mathcal{K}$ is further a Grothendieck cosmos, then so is* $\mathbf{Mod}(A)$.

Proof: First, $\mathbf{Mod}(A)$ is $\mathbf{Ab}$-enriched. Indeed, for any two A-modules M and N, $\mathrm{Hom}_A(M,N)$ is a subset of $\mathrm{Hom}_{\mathcal{K}}(M,N)$. It is further a subgroup: let f,g be two A-morphisms from M to N, then so is $f-g$ since $A\otimes -$ is an additive functor.

Next, $\mathbf{Mod}(A)$ is finite-complete and finite-cocomplete. This follows from a similar argument for Lemma 5.c except we only consider finite limits and finite colimits.

Now, any A-morphism can be uniquely factored into a triple which gives the its factorization as $\mathcal{K}$-morphisms. Thus the canonical morphism must be an isomorphism. This shows $\mathbf{Mod}(A)$ is an abelian category.

Assume A is commutative, then as shown in Remark 5.a, $\mathbf{Mod}(A)$ has a tensor product $\otimes_A$ with the tensor unit A making it a symmetric monoidal category. Moreover, by the construction of $\otimes_A$, we see that for any A-module M, $M\otimes_A -$ is additive and right exact. The adjunction of the free module functor $A\otimes -$ and the forgetful functor $\mathbf{Mod}(A) \to \mathcal{K}$ is given by the unit of A and the A-module structures. We have seen in Example 5.d that $A\otimes -$ is a symmetric monoidal functor. Therefore $A\otimes -$ is an abelian tensor functor and actually furthermore commutes with colimits.

Now assume $\mathcal{K}$ is a linear cosmos. Obviously $\mathbf{Mod}(A)$ is cocomplete, colimits and exact sequences in $\mathbf{Mod}(A)$ are computed in $\mathcal{K}$ and $M \otimes_A -$ has a right adjoint $\underline{\mathrm{Hom}}_A(M, -)$ for every A-module M. If S is a generator of $\mathcal{K}$, then $A \otimes S$ is an A-module and is a generator of $\mathbf{Mod}(A)$. $\square$

8 Lemma (Uniqueness of rank) *Let A be a commutative monoid in an abelian tensor category $\mathcal{K}$ and assume $A \neq 0$. If $A^{\oplus n} \cong A^{\oplus m}$ as A-modules, then $n = m$.*

Proof: Let $R = \mathrm{Hom}_A(A, A)$. This is a nontrivial ring since $A \neq 0$. Then, apply the functor $\mathrm{Hom}_A(A, -)\colon \mathbf{Mod}(A) \to \mathbf{Mod}(R)$ to the isomorphism $A^{\oplus n} \cong A^{\oplus m}$, we get $R^n \cong R^m$, which implies $n = m$. $\square$

Remark The similar statement fails to be true in general.

9 (Algebras over a commutative monoid) Let $\mathcal{K}$ be an abelian tensor category $\mathcal{K}$ and A a commutative monoid in it. Then, a monoid in $\mathbf{Mod}(A)$ is called an **A-algebra**.

Note that an A-algebra (B, n, f) is also a monoid in $\mathcal{K}$, whose unit e is the composition

$$\mathbf{1} \to A \xrightarrow{f} B$$

and whose multiplication m is the composition

$$B \otimes B \to B \otimes_A B \xrightarrow{n} B.$$

Then, the action $l\colon A \otimes B \to B$ can be factored as the composition

$$A \otimes B \xrightarrow{f\otimes B} B \otimes B \xrightarrow{m} B.$$

Moreover, the unit $A \to B$ is a homomorphism of monoids in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd} A \otimes B \arrow[r, "f\otimes B"] \arrow[d, "f\otimes B"'] & B \otimes B \arrow[r, "m"] & B \arrow[d, equal] \\ B \otimes B \arrow[r, "\gamma_{B,B}"] & B \otimes B \arrow[r, "m"] & B \end{tikzcd}$$

Conversely, given any homomorphism $f\colon A \to B$ of monoids satisfying the above commutative diagram, then B is an A-algebra. The A-module structure is given by

$$A \otimes B \xrightarrow{f} B \otimes B \xrightarrow{m} B,$$

the unit of B is f, and the multiplication of B is the dotted morphism determined by the universal property in the following diagram.

$$\begin{tikzcd} B \otimes A \otimes B \arrow[r, shift left] \arrow[r, shift right] & B \otimes B \arrow[r] \arrow[dr, "m"'] & B \otimes_A B \arrow[d, dashed] \\ & & B \end{tikzcd}$$

In particular, we have the following equivalence of symmetric monoidal categories:

$$\mathbf{CMon}(A) \simeq \mathbf{CMon}(\mathcal{K})/A.$$

10 **(Restrictions and extensions of scalars)** Let $f\colon A \to B$ be a homomorphism of commutative monoids in an abelian tensor category $\mathcal{K}$. On one hand, any B-module N admits an A-module structure given by the composition

$$A \otimes N \xrightarrow{f} B \otimes N \longrightarrow N.$$

This A-module is called the **restriction** of N to A, denoted by $N|_A$. The restrictions give rise to a functor, which is actually the forgetful functor from $\mathbf{Mod}(B)$ to $\mathbf{Mod}(A)$. On the other hand, for any A-module M, there is a natural B-module structure on the tensor product $B \otimes_A M$ given by the multiplication m of B:

$$B \otimes B \otimes_A M \xrightarrow{m} B \otimes_A M.$$

This gives rise to the **extension of scalars** functor $\mathbf{Mod}(A) \to \mathbf{Mod}(B)$.

It turns out to be that the two functors form an adjunction. In fact, this is just a special case of Example 5.d.

The *restriction functor* also has a right adjoint, given by mapping each A-module M to the B-module $\underline{\mathrm{Hom}}_A(B, M)$. The B-module structure on $\underline{\mathrm{Hom}}_A(B, M)$ is given by the morphism

$$B \otimes_A \underline{\mathrm{Hom}}_A(B, M) \longrightarrow \underline{\mathrm{Hom}}_A(B, M)$$

corresponding to

$$\underline{\mathrm{Hom}}_A(B, M) \xrightarrow{m_A} \underline{\mathrm{Hom}}_A(B \otimes_A B, M) \cong \underline{\mathrm{Hom}}_A(B, \underline{\mathrm{Hom}}_A(B, M))$$

under the *Tensor-Hom adjunction*. Actually, this is just a special case of Example 5.e.

§ II.3 Ringed spaces

Ringed spaces and $\mathscr{O}_X$-modules

In this subsection, we introduce ringed spaces and their modules. For now, we temporarily not assume rings to be commutative.

1 (Ringed spaces) A **ringed space** is a pair $(X, \mathscr{O}_X)$ of a topological space X and a sheaf of rings on X. A *morphism of ringed spaces* is is a pair $(f, f^\sharp)$ of a continuous map $f\colon X \to Y$ and an *f-map* (ref. I.5.6) of sheaves of rings $f^\sharp\colon \mathscr{O}_Y \to \mathscr{O}_X$.

Let $(f, f^\sharp)\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ and $(g, g^\sharp)\colon (Y, \mathscr{O}_Y) \to (Z, \mathscr{O}_Z)$ be two morphisms of ringed spaces. The *composition of morphisms of ringed spaces* is given by

$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$

Note that here the composition $f^\sharp \circ g^\sharp$ follows I.5.6.

2 ($\mathscr{O}$-modules) Let $\mathscr{F}$ and $\mathscr{G}$ be two presheaves of abelian groups, then their **tensor product** $\mathscr{F} \otimes \mathscr{G}$ is given by

$$(\mathscr{F} \otimes \mathscr{G})(U) := \mathscr{F}(U) \otimes \mathscr{G}(U).$$

In this way, $\mathbf{PAb}(X)$ becomes a symmetric monoidal category. By Theorem 1.2 or direct check, *a presheaf of rings $\mathscr{O}$ is the same as a monoid in* $\mathbf{PAb}(X)$. A **presheaf of $\mathscr{O}$-modules** is then a module over such a monoid. In other words, an $\mathscr{O}$-module is an abelian presheaf $\mathscr{F}$ with an abelian presheaf morphism $\mathscr{O} \otimes \mathscr{F} \to \mathscr{F}$, called the *action*, such that each $\mathscr{O}(U) \otimes \mathscr{F}(U) \to \mathscr{F}(U)$ gives $\mathscr{F}(U)$ an $\mathscr{O}(U)$-module structure. A *morphism between presheaves of $\mathscr{O}$-modules* is then a morphism between such abelian presheaves preserving the actions. The category of presheaves of $\mathscr{O}$-modules is denoted by $\mathbf{PMod}(\mathscr{O})$.

2.a Lemma *Let $\mathscr{F}$ and $\mathscr{G}$ be two abelian presheaves, then*

1. $(\mathscr{F} \otimes \mathscr{G})_x = \mathscr{F}_x \otimes \mathscr{G}_x$ *for all* $x \in X$;
2. $(\mathscr{F} \otimes \mathscr{G})^\# = \mathscr{F}^\# \otimes \mathscr{G}^\#$.

Proof: *1.* For any $x \in X$, we have

$$\begin{aligned} \mathscr{F}_x \otimes \mathscr{G}_x &= (\varinjlim_{x \in U} \mathscr{F}(U)) \otimes (\varinjlim_{x \in V} \mathscr{G}(V)) \\ &= \varinjlim_{x \in U} \varinjlim_{x \in V} (\mathscr{F}(U) \otimes \mathscr{G}(V)). \end{aligned}$$

Note that pairs of the form (U, U) with $x \in U$ are cofinal in the category of pairs (U, V) of neighborhoods U, V of x. Therefore $\mathscr{F}_x \otimes \mathscr{G}_x = (\mathscr{F} \otimes \mathscr{G})_x$.

2. We now have $\Pi(\mathscr{F} \otimes \mathscr{G}) = \Pi(\mathscr{F}) \otimes \Pi(\mathscr{G})$ and further $\Pi(\Pi(\mathscr{F} \otimes \mathscr{G})) = \Pi(\Pi(\mathscr{F})) \otimes \Pi(\Pi(\mathscr{G}))$. Then, by Lemma I.3.9, $(\mathscr{F} \otimes \mathscr{G})^\# = \mathscr{F}^\# \otimes \mathscr{G}^\#$. $\square$

This lemma shows that $\mathbf{Ab}(X)$ inherits a symmetric monoidal structure form $\mathbf{PAb}(X)$. In this symmetric monoidal category, *a sheaf of rings is the same as a monoid.* We call a module over such a monoid $\mathscr{O}$ as a **sheaf of $\mathscr{O}$-modules** or simply **$\mathscr{O}$-module**. A *morphism of $\mathscr{O}$-modules* is then a morphism of such abelian sheaves preserving actions. The category of $\mathscr{O}$-modules is denoted by $\mathbf{Mod}(\mathscr{O})$.

2.b Corollary *Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}$ a presheaf of $\mathscr{O}_X$-modules. Then $\mathscr{F}^\#$ is an $\mathscr{O}_X$-module and for each $x \in X$, $\mathscr{F}_x$ is an $\mathscr{O}_{X,x}$-module.*

We have seen $\mathbf{Ab}(X)$ is an abelian category in Theorem I.4.8, and we also have seen $\mathbf{Ab}(X)$ is a symmetric monoidal category. What's the relation?

3 Lemma $\mathbf{Ab}(X)$ *is an abelian tensor category and a cocomplete tensor category.*

Proof: It is easy to see that the category of abelian presheaves form an abelian category. Furthermore, we have seen in Theorem I.4.8 that abelian sheaves also form an abelian category. Then, Proposition I.4.9 implies that $\mathbf{Ab}(X)$ is complete and cocomplete and colimits in it are sheafifications of colimits in $\mathbf{PAb}(X)$. Next, it is easy to see that the constant sheaf $\underline{\mathbb{Z}}$ is a generator. Finally, since colimits are computed open sets by open sets in $\mathbf{PAb}(X)$ and filtered colimits in $\mathbf{Ab}$ are exact, so are filtered colimits in $\mathbf{PAb}(X)$. Since colimits in $\mathbf{Ab}(X)$ are sheafifications of colimits in $\mathbf{PAb}(X)$ and the sheafification is exact, so are filtered colimits in $\mathbf{Ab}(X)$. $\square$

4 Theorem (Linear cosmoses for sheaf theory) *Let $(X, \mathscr{O}_X)$ be a ringed space. Then $\mathbf{PMod}(\mathscr{O}_X)$ and $\mathbf{Mod}(\mathscr{O}_X)$ are Grothendieck categories and $\# \dashv F$ is an adjunction of additive functors. If $\mathscr{O}_X$ is further commutative, then $\mathbf{PMod}(\mathscr{O}_X)$ and $\mathbf{Mod}(\mathscr{O}_X)$ are linear cosmoses and $\# \dashv F$ is an adjunction of abelian tensor functors.*

Proof: As the category of modules over a monoid in a linear cosmos, $\mathbf{Mod}(\mathscr{O}_X)$ is a Grothendieck category. If further $\mathscr{O}_X$ is commutative, then $\mathbf{Mod}(\mathscr{O}_X)$ is again a linear cosmos by Theorem 2.7. $\square$

4.a Remark In the case $\mathscr{O}_X$ is commutative, we denote the tensor product in $\mathbf{PMod}(\mathscr{O}_X)$ and $\mathbf{Mod}(\mathscr{O}_X)$ by $\otimes^{\mathrm{pre}}_{\mathscr{O}_X}$ and $\otimes_{\mathscr{O}_X}$ respectively. For any presheaves of $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, obviously we have

$$(\mathscr{F} \otimes^{\mathrm{pre}}_{\mathscr{O}_X} \mathscr{G})(U) = \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U).$$

But the similar formula doesn't hold in $\mathbf{Mod}(\mathscr{O}_X)$ since there $\otimes_{\mathscr{O}_X}$ is obtained by sheafification of $\otimes^{\mathrm{pre}}_{\mathscr{O}_X}$. However, since sheafification preserves stalks, for any $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, we still have

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_x.$$

As we have get the linear cosmos, from now on, we assume rings to be commutative.

4.b (Internal Hom) Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}, \mathscr{G}$ two $\mathscr{O}_X$-modules. Then the sheaf $\mathcal{H}om(\mathscr{F}, \mathscr{G})$ (ref. I.4.11) is an $\mathscr{O}_X$-module whose action is given by

$$\begin{aligned} \mathscr{O}_X(U) \otimes \operatorname{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U) &\longrightarrow \operatorname{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U) \\ (f, \varphi) &\longmapsto f\varphi, \end{aligned}$$

here $f\varphi$ is the composition of φ and multiplication by f. This $\mathscr{O}_X$-module is denoted by $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$.

One can verify the followings (similar to those in I.4.11).

1. There are canonical homomorphisms of $\mathscr{O}_{X,x}$-modules
$$\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x \to \operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x).$$
2. There are canonical isomorphisms of $\mathscr{O}_{X,x}$-modules
$$\mathcal{H}om_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{F}) \cong \mathscr{F}.$$
3. The covariant functor $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, -)$ maps limits to limits and the contravariant functor $\mathcal{H}om_{\mathscr{O}_X}(-, \mathscr{F})$ maps colimits to limits.
4. There is an adjunction of additive functors on $\mathbf{Mod}(\mathscr{O}_X)$:
$$\mathscr{F} \otimes_{\mathscr{O}_X} - \dashv \mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, -).$$
5. There are canonical homomorphisms of $\mathscr{O}_{X,x}$-modules
$$\mathscr{F} \otimes_{\mathscr{O}_X} \mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{G},$$
called the **evaluation**.

4.c Proposition *If there is a homomorphism of sheaves of rings $\mathscr{O}_1 \to \mathscr{O}_2$, then we have the following adjunctions of additive functors.*

$$\begin{aligned} \mathscr{O}_2 \otimes_{\mathscr{O}_1} - \dashv |_{\mathscr{O}_1} &: \mathbf{Mod}(\mathscr{O}_2) \leftrightarrows \mathbf{Mod}(\mathscr{O}_1), \\ |_{\mathscr{O}_1} \dashv \mathcal{H}om_{\mathscr{O}_1}(\mathscr{O}_2, -) &: \mathbf{Mod}(\mathscr{O}_1) \leftrightarrows \mathbf{Mod}(\mathscr{O}_2), \end{aligned}$$

In particular,

1. $\mathscr{O}_2 \otimes_{\mathscr{O}_1} - \colon \mathbf{Mod}(\mathscr{O}_1) \to \mathbf{Mod}(\mathscr{O}_2)$ *is an abelian tensor functor;*
2. $|_{\mathscr{O}_1} : \mathbf{Mod}(\mathscr{O}_2) \to \mathbf{Mod}(\mathscr{O}_1)$ *is a faithful, symmetric lax and reflects limits, colimits and isomorphisms.*

Proof: From Theorem 4, one can see this is just a special case of 2.10 and further of Lemma 2.5.c, Example 2.5.d and Example 2.5.e. $\square$

5 **(Generated by global sections)** We have defined the notion of *global sections* in I.3.1.a. For $(X, \mathscr{O}_X)$ a ringed space and $\mathscr{F}$ an $\mathscr{O}_X$-module, we say $\mathscr{F}$ is **generated by global sections** if there is a family of global sections $s_i \in \Gamma(X, \mathscr{F})$ indexed by a set I such that the homomorphism

$$\bigoplus_{i \in I} \mathscr{O}_X \longrightarrow \mathscr{F},$$

whose summand is induced by s_i, is an epimorphism. If this is the case, we say $\{s_i\}$ **generates** $\mathscr{F}$. If this homomorphism is further an isomorphism, we call $\{s_i\}$ is a **global basis** of $\mathscr{F}$.

Remark One can the this is a generalization of the notion of a module over a ring generated by elements.

The following lemmas are obvious.

5.a **Lemma** *Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}$ an $\mathscr{O}_X$-module. Let $\{s_i\}_{i \in I}$ be a family of global sections indexed by a set I. Then $\{s_i\}$ generates $\mathscr{F}$ if and only if for all $x \in X$, the family of their germs $\{s_{i,x}\}$ generates the $\mathscr{O}_{X,x}$-modules $\mathscr{F}_x$.*

5.b **Lemma** *Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}, \mathscr{G}$ two $\mathscr{O}_X$-modules. If both $\mathscr{F}$ and $\mathscr{G}$ are generated by global sections, then so is $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$.*

Pushforwards and pullbacks

Our goal in this section is the following adjunction.

6 **Theorem (Pushforwards and pullbacks)** *Let $(f, f^\sharp)\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces, then there is an adjunction*

$$f^* \dashv f_* \colon \mathbf{Mod}(\mathscr{O}_X) \leftrightarrows \mathbf{Mod}(\mathscr{O}_Y).$$

The meaning of the notations will be explained in the follows.

6.a **Lemma** *Let $f\colon X \to Y$ be a continuous map. Then we have adjunctions of abelian tensor functors:*

$$\begin{aligned} f_p \dashv f_* &\colon \mathbf{PAb}(X) \leftrightarrows \mathbf{PAb}(Y), \\ f^{-1} \dashv f_* &\colon \mathbf{Ab}(X) \leftrightarrows \mathbf{Ab}(Y). \end{aligned}$$

Proof: It is obvious that they are additive. It suffices to show that for any abelian presheaves $\mathscr{F}$ and $\mathscr{G}$ on X and $\mathscr{F}'$ and $\mathscr{G}'$ on Y. We have

$$\begin{aligned} f_*(\mathscr{F} \otimes \mathscr{G}) &= f_*\mathscr{F} \otimes f_*\mathscr{G} \\ f_p(\mathscr{F}' \otimes \mathscr{G}') &= f_p\mathscr{F}' \otimes f_p\mathscr{G}', \end{aligned}$$

which are straightforward. $\square$

Let $f\colon X \to Y$ be a continuous map. Let $\mathscr{O}_X$ be a sheaf of rings on X and let $\mathscr{O}_Y$ be a sheaf of rings on Y. Then we have the following abelian tensor functors:

$$\begin{aligned} f_*\colon \mathbf{PMod}(\mathscr{O}_X) &\longrightarrow \mathbf{PMod}(f_*\mathscr{O}_X), \\ f_*\colon \mathbf{Mod}(\mathscr{O}_X) &\longrightarrow \mathbf{Mod}(f_*\mathscr{O}_X), \\ f_p\colon \mathbf{PMod}(\mathscr{O}_Y) &\longrightarrow \mathbf{PMod}(f_p\mathscr{O}_Y), \\ f^{-1}\colon \mathbf{Mod}(\mathscr{O}_Y) &\longrightarrow \mathbf{Mod}(f^{-1}\mathscr{O}_Y). \end{aligned}$$

So, what's their relation?

6.b Lemma *Let $f\colon X \to Y$ be a continuous map. Let $\mathscr{O}$ be a sheaf of rings on Y. Then we have adjunctions of abelian tensor functors:*

$$\begin{aligned} f_p \dashv f_*\colon \mathbf{PMod}(f_p\mathscr{O}) &\leftrightarrows \mathbf{PMod}(\mathscr{O}), \\ f^{-1} \dashv f_*\colon \mathbf{Mod}(f^{-1}\mathscr{O}) &\leftrightarrows \mathbf{Mod}(\mathscr{O}). \end{aligned}$$

Proof: One can verify that if $\mathscr{F}$ is a presheaf of $f_p\mathscr{O}$-modules, then $f_*\mathscr{F}$ is a presheaf of $\mathscr{O}$-modules whose modules action is given by the composition

$$\mathscr{O} \otimes f_*\mathscr{F} \xrightarrow{\eta} f_*f_p\mathscr{O} \otimes f_*\mathscr{F} = f_*(f_p\mathscr{O} \otimes \mathscr{F}) \longrightarrow f_*\mathscr{F}.$$

One can verify that the unit $\eta\colon \mathscr{O} \to f_*f_p\mathscr{O}$, the counit $\epsilon\colon f_pf_*\mathscr{O} \to \mathscr{O}$ and the forgetful functors are compatible with the module actions. Therefore the required adjunction follows. $\square$

6.c Lemma *Let $f\colon X \to Y$ be a continuous map. Let $\mathscr{O}$ be a sheaf of rings on X. Then we have adjunctions of abelian tensor functors:*

$$\begin{aligned} \mathscr{O} \otimes^{\mathrm{pre}}_{f_pf_*\mathscr{O}} f_p \dashv f_*\colon \mathbf{PMod}(\mathscr{O}) &\leftrightarrows \mathbf{PMod}(f_*\mathscr{O}), \\ \mathscr{O} \otimes_{f^{-1}f_*\mathscr{O}} f^{-1} \dashv f_*\colon \mathbf{Mod}(\mathscr{O}) &\leftrightarrows \mathbf{Mod}(f_*\mathscr{O}). \end{aligned}$$

Proof: Apply Proposition 4.c to $\epsilon\colon f_pf_*\mathscr{O} \to \mathscr{O}$, we have adjunctions

$$\begin{aligned} \mathscr{O} \otimes^{\mathrm{pre}}_{f_pf_*\mathscr{O}} - \dashv |_{f_pf_*\mathscr{O}}\colon \mathbf{PMod}(\mathscr{O}) &\leftrightarrows \mathbf{PMod}(f_pf_*\mathscr{O}), \\ \mathscr{O} \otimes_{f^{-1}f_*\mathscr{O}} - \dashv |_{f^{-1}f_*\mathscr{O}}\colon \mathbf{Mod}(\mathscr{O}) &\leftrightarrows \mathbf{Mod}(f^{-1}f_*\mathscr{O}). \end{aligned}$$

Apply Lemma 6.b to $f_*\mathscr{O}$, we have adjunctions

$$\begin{aligned} f_p \dashv f_* \colon \mathbf{PMod}(f_p f_*\mathscr{O}) &\leftrightarrows \mathbf{PMod}(f_*\mathscr{O}), \\ f^{-1} \dashv f_* \colon \mathbf{Mod}(f^{-1} f_*\mathscr{O}) &\leftrightarrows \mathbf{Mod}(f_*\mathscr{O}). \end{aligned}$$

Combining them, we get the required adjunctions. □

Let $(f, f^\sharp)\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces.

- For any $\mathscr{O}_X$-module $\mathscr{F}$, we define the **pushforward** of $\mathscr{F}$ as the $\mathscr{O}_Y$-module obtained by restricting the $f_*\mathscr{O}_X$-module $f_*\mathscr{F}$ to $\mathscr{O}_Y$ via $f^\sharp\colon \mathscr{O}_Y \to f_*\mathscr{O}_X$. We still denote it by $f_*\mathscr{F}$.
- For any $\mathscr{O}_Y$-module $\mathscr{G}$, we define the **pullback** of $\mathscr{G}$ as the $\mathscr{O}_X$-module obtained by the extension of scalars:

$$f^*\mathscr{G} := \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{G},$$

where the homomorphism $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ is the one corresponding to $f^\sharp$ via the adjunction $f^{-1} \dashv f_*$.

Proof (Proof of Theorem 6): This proof is just an analogy of Lemma 6.c. Apply Proposition 4.c to $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$, we have the adjunction

$$\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} - \dashv |_{f^{-1}\mathscr{O}_Y} \colon \mathbf{Mod}(\mathscr{O}_X) \leftrightarrows \mathbf{Mod}(f^{-1}\mathscr{O}_Y).$$

Apply Lemma 6.b to $\mathscr{O}_Y$, we have the adjunction

$$f^{-1} \dashv f_* \colon \mathbf{Mod}(f^{-1}\mathscr{O}_Y) \leftrightarrows \mathbf{Mod}(\mathscr{O}_Y).$$

Combining them, we get the required adjunctions. □

7 (f-maps) Let $(f, f^\sharp)\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces. Given an $\mathscr{O}_X$-module F and an $\mathscr{O}_Y$-module $\mathscr{G}$, an **f-map** $\varphi\colon \mathscr{G} \to \mathscr{F}$ **of modules** is an f-map of abelian sheaves compatible with the module actions. The canonical map at stalks

$$\varphi_x \colon \mathscr{G}_{f(x)} \longrightarrow \mathscr{F}_x$$

is now a homomorphism of $\mathscr{O}_{Y,f(x)}$-modules. Here $\mathscr{F}_x$ is viewed as an $\mathscr{O}_{Y,f(x)}$-module by restricting it via the homomorphism $f_x^\sharp\colon \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$.

Here is a related lemma.

7.a Lemma *Let $(f, f^\sharp)\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces and $\mathscr{G}$ an $\mathscr{O}_Y$-module. Then for any $x \in X$, we have an isomorphism*

$$(f^*\mathscr{G})_x = \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{G}_{f(x)}$$

of $\mathscr{O}_{X,x}$-modules. Here the extension of scalars is induced by the homomorphism $f_x^\sharp\colon \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$.

8 (Open immersion of topological spaces) Recall that for open immersion $j\colon U \hookrightarrow X$ of topological spaces, there is a left adjoint $j_!$ of j^{-1}. Usually, this functor is not left exact. But we have

8.a Lemma *Let $j\colon U \hookrightarrow X$ be an open immersion of topological spaces. Then the functor $j_!\colon \mathbf{Ab}(U) \to \mathbf{Ab}(X)$ is exact.*

Proof: It remains to show $j_!$ is left exact, which is obvious sine j maps the zero sheaf on U to the zero sheaf on X. $\square$

Let $(X, \mathscr{O})$ be a ringed space. Then for any open subset U of X, $(U, \mathscr{O}|_U)$ is also a ringed space. The open immersion $j\colon U \hookrightarrow X$ gives rise to a morphism of ringed spaces $j\colon (U, \mathscr{O}|_U) \to (X, \mathscr{O})$.

8.b Proposition *Let $j\colon U \hookrightarrow X$ be an open immersion of topological spaces.*

1. *We have adjunctions*

$$j_{p!} \dashv j_p\colon \mathbf{PMod}(\mathscr{O}) \leftrightarrows \mathbf{PMod}(\mathscr{O}|_U),$$
$$j_! \dashv j^{-1}\colon \mathbf{Mod}(\mathscr{O}) \leftrightarrows \mathbf{Mod}(\mathscr{O}|_U).$$

2. *Let $\mathscr{F}$ be an $\mathscr{O}|_U$-module. Then the stalks of the sheaf $j_!\mathscr{F}$ are described as follows*

$$j_!\mathscr{F}_x = \begin{cases} \mathscr{F}_x & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases}$$

3. *We have $j_p j_{p!} = \mathrm{id}$ on $\mathbf{PMod}(\mathscr{O}|_U)$ and $j^{-1}j_! = \mathrm{id}$ on $\mathbf{Mod}(\mathscr{O}|_U)$.*

4. *The functor*

$$j_!\colon \mathbf{Mod}(\mathscr{O}|_U) \longrightarrow \mathbf{Mod}(\mathscr{O})$$

*is fully faithful and induces an equivalence between $\mathbf{Mod}(\mathscr{O}|_U)$ and the full subcategory of $\mathbf{Mod}(\mathscr{O})$ consisting of $\mathscr{O}$-modules vanishing outside of U. Those modules can be called $\mathscr{O}$-*modules on U.

Proof: Similar to Theorem I.5.10 and Theorem I.5.11. $\square$

8.c (Closed immersions of topological spaces) content

§ II.4 Manifolds, bundles and étalé spaces

1 (Manifolds) Recall that a manifold M is a topological space locally like $\mathbb{R}^n$. Usually, there are some extra requirements such as *second countability* and *Hausdorff property.* More precisely, for every point $x \in M$, there exists a neighborhood U of x equipped with an embedding $\phi\colon \mathbb{R}^n$, called a *chart*, and those charts are *compatible.* Here two charts (U, ϕ) and (V, ψ) are said to be *compatible* if the *transition function* $\psi \circ \phi^{-1}$ is a continuous (or k-differential, smooth, etc. depending on what kind of manifold is considered) map in $\mathbb{R}^n$.

1.a (Structure sheaf) Any manifold M admits a canonical sheaf $\mathscr{O}_M$, called its **structure sheaef**. For a topological manifold, it is just the sheaf $\mathscr{C}_M$ of continuous maps to $\mathbb{R}$.

Next, we consider the differential manifolds. But before that, let's recall that there are many subsheaves of $\mathscr{C}$ on the Euclidean space $\mathbb{R}^n$ such as $\mathscr{C}^k$, the sheaf of k-differential functions, $\mathscr{C}^\infty$, the sheaf of smooth functions, $\mathscr{C}^\omega$, the sheaf of real analytic functions, et cetera.

Anyhow, let $\mathscr{O}$ denotes one of those subsheaf, for instance $\mathscr{C}^\infty$. Let's see how the definition of a manifold translate them to the manifold M. Recall a chart is nothing but a embedding $\phi\colon U \to \mathbb{R}^n$, this embedding translates the sheaf $\mathscr{O}$ to U via inverse image ϕ^{-1}. Then the compatible condition of charts (U, ϕ) and (V, ψ) require the transition functions $\psi \circ \phi^{-1}$ being smooth. In this way the transition functions provides isomorphisms between $\phi^{-1}\mathscr{O}\big|_{U\cap V}$ and $\psi^{-1}\mathscr{O}\big|_{U\cap V}$ via

$$\begin{aligned} \psi^{-1}\mathscr{O}\big|_{U\cap V}(W) = \mathscr{O}(\psi(W)) &\longrightarrow \phi^{-1}\mathscr{O}\big|_{U\cap V}(W) = \mathscr{O}(\phi(W)) \\ f &\longmapsto f \circ \psi \circ \phi^{-1}. \end{aligned}$$

Now, we have a system of sheaves on open sets of M together with isomorphisms on their overlaps. Obviously, this system can be extended into a gluing data. Then, by Theorem I.4.17, we obtain a sheaf $\mathscr{O}_M$ on M. This sheaf is called the **structure sheaef** of M and in the smooth case it is called the **sheaf of smooth functions** on M and denoted by $\mathscr{C}^\infty_M$.

Conversely, we have

1.b Theorem *A manifold M is equivalent to a locally ringed space $(M, \mathscr{O}_M)$, which is locally isomorphic to an open subset of $\mathbb{R}^n$.*

1.c Lemma *Let $(f, \psi)\colon X \to Y$ be a morphism of locally ringed spaces, where X and Y are smooth manifolds with their sheaves of smooth functions. If $\psi\colon \mathscr{C}^\infty_Y \to f_*\mathscr{C}^\infty_X$ is a morphism of sheaves of $\mathbb{R}$-algebras, then f is smooth and $\psi = f^\sharp$.*

Now, we give another definition of manifolds using the language of sheaves. To begin with, we need a standard model consisting of a space like $\mathbb{R}^n$, $\mathbb{C}^n$, etc. and a sheaf of special type of maps on it.

1.d Let X be a topological space. A **transformation group** G on X is a subsheaf of the sheaf of continuous functions.

2 (Espace étalé) Let **Top** denote the category of topological spaces and continuous maps. Define a Grothendieck pretopology Cov on **Top** given by

$$\{\phi_i\colon U_i \to U\}_{i\in I} \in \mathrm{Cov} \Longleftrightarrow \bigcup_{i\in I} \phi_i(U_i) = U \text{ and } \phi_i \text{ are injective and open.}$$

One can see the representable presheaves on **Top** are sheaves. In this way, *sheaves can be thought as generalized spaces.*

Let X be a topological space. Then the category $\mathbf{Top}/X$ of continuous maps to X has an inherited Grothendieck pretopology and form a site $\mathbf{Top}_X$. Now, one can see that $\mathcal{T}_X$ forms a *subsite* of $\mathbf{Top}_X$. So there is a functor $\mathbf{Sh}(\mathbf{Top}_X) \to \mathbf{Sh}(X)$. Compositing it with the Yoneda embedding, we get the following functor

$$\mathbf{Top}/X \xrightarrow{\Upsilon} \mathbf{Sh}(\mathbf{Top}_X) \longrightarrow \mathbf{Sh}(X).$$

To simplify notations, we still use Υ denote this functor. Now, what surprising is this functor has a right adjoint, meaning for every sheaf $\mathscr{F}$ on X, there is a canonical topological space $E_{\mathscr{F}}$ with a canonical map $\pi_{\mathscr{F}}\colon E_{\mathscr{F}} \to X$ such that

$$\mathrm{Hom}(Y, E_{\mathscr{F}}) = \mathrm{Hom}(\Upsilon(Y), \mathscr{F}).$$

One may wants to conversely extend every sheaf $\mathscr{F}$ on X to a sheaf on $\mathbf{Top}_X$. Let $f\colon Y \to X$ be an arbitrary object in $\mathbf{Top}/X$, one attempt is to define $\mathscr{F}(f)$ as the same with $\mathscr{F}(f(Y))$. However, $f(Y)$ is in general not an open set, thus $\mathscr{F}(f(Y))$ is still non-defined. So, one may try to restrict to a suitable subsite of $\mathbf{Top}_X$. The first candidate is the subcategory of $\mathbf{Top}/X$ consisting of only open maps.

More precisely, let

III

Schemes

§ III.1 Schemes

1 (Affine schemes) content

A

Some generalities

§ A.1 Monoidal categories and coherence theorems

In this section, we give basic notions on monoidal categories. Especially, we give a proof of the coherence theorems.

Monoidal categories, functors and natural transformations

1 **(Monoidal categories) monoidal category** is a category $\mathcal{C}$ equipped with a bifunctor

$$\otimes\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C},$$

called the **tensor product**, an object

$$\mathbf{1} \in \operatorname{ob}\mathcal{C},$$

called the **unit object** or **tensor unit**, a natural isomorphism

$$\alpha_{A,B,C}\colon (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C),$$

called the **associator**, a natural isomorphism

$$\lambda_A\colon \ \mathbf{1} \otimes A \xrightarrow{\cong} A,$$

called the **left unitor**, and a natural isomorphism

$$\rho_A\colon A \otimes \mathbf{1} \xrightarrow{\cong} A,$$

called the **right unitor**, such that the *pentagon identity* hold for every $A, B, C, D \in \operatorname{ob}\mathcal{C}$,

$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C\otimes D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A\otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C}\otimes D"'] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "A\otimes\alpha_{B,C,D}"']
\end{tikzcd}$$

and the *triangle identity* holds for every $A, B \in \operatorname{ob}\mathcal{C}$.

$$\begin{tikzcd}
(A \otimes \mathbf{1}) \otimes B \arrow[rr, "\alpha_{A,\mathbf{1},B}"] \arrow[dr, "\rho_A\otimes B"'] & & A \otimes (\mathbf{1} \otimes B) \arrow[dl, "A\otimes\lambda_B"] \\
 & A \otimes B &
\end{tikzcd}$$

Moreover, a **symmetric monoidal category** is a monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ equipped a natural isomorphism

$$\gamma_{A,B}\colon A \otimes B \xrightarrow{\cong} B \otimes A,$$

called the **symmetric braiding**, such that the *hexagon identity* holds for every $A, B, C \in \operatorname{ob}\mathcal{C}$,

$$\begin{tikzcd}[column sep=small]
 & A \otimes (B \otimes C) \arrow[dr, "\gamma_{A,B\otimes C}"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha_{A,B,C}"] \arrow[d, "\gamma_{A,B}\otimes C"'] & & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[dr, "\alpha_{B,A,C}"'] & & B \otimes (C \otimes A) \\
 & B \otimes (A \otimes C) \arrow[ur, "B\otimes\gamma_{A,C}"'] &
\end{tikzcd}$$

and the *involution law* holds for every $A, B \in \operatorname{ob}\mathcal{C}$.

$$\begin{tikzcd}
A \otimes B \arrow[rr, "1_{A\otimes B}"] \arrow[dr, "\gamma_{A,B}"'] & & A \otimes B \\
 & B \otimes A \arrow[ur, "\gamma_{B,A}"'] &
\end{tikzcd}$$

Remark The symmetric monoidal categories are specail cases of *braiding monoidal categories*, which are defined like symmetric monoidal categories except the involution law fails and another hexagon identity is required.

1.a Example (Linear cosmoses) There is a standard symmetric monoidal category structure on the category **Ab** of abelian groups, or more general, the category $\mathbf{Mod}(R)$ of modules over a ring R. Those symmetric monoidal categories are the usual *linear cosmoses*.

1.b Example (Cartesian categories) There are other familiar examples of monoidal categories: The cartesian products provide a symmetric monoidal category structure on **Set**, whose tansor unit is the singleton. The cartesian products also provide another symmetric monoidal category structure on **Ab**, whose tensor unit is the trivial group. Such kind of monoidal categories are so-called **cartesian** ones.

2 (Monoidal functors and natural transformations) As monoidal categories are categories equipped with extra structures, the correct morphisms between them should be functors preserving those extra structures.

But this requirement may be too strong. So, we define a **lax functor** to be a functor $F\colon \mathcal{C} \to \mathcal{C}'$ between monoidal categories, equipped with a natural transformation

$$\Phi_{A,B}\colon F(A) \otimes F(B) \longrightarrow F(A \otimes B),$$

and a morphism

$$\phi\colon\ \mathbf{1} \longrightarrow F(\mathbf{1}),$$

such that the following diagram commutes for every $A, B, C \in \operatorname{ob}\mathcal{C}$,

$$\begin{tikzcd}
& F(A\otimes B)\otimes F(C) \arrow[dr, "\Phi_{A\otimes B,C}"] & \\
(F(A)\otimes F(B))\otimes F(C) \arrow[ur, "\Phi_{A,B}\otimes F(C)"] \arrow[d, "\alpha_{F(A),F(B),F(C)}"'] & & F((A\otimes B)\otimes C) \arrow[d, "F(\alpha_{A,B,C})"] \\
F(A)\otimes (F(B)\otimes F(C)) \arrow[dr, "F(A)\otimes\Phi_{B,C}"'] & & F(A\otimes (B\otimes C)) \\
& F(A)\otimes F(B\otimes C) \arrow[ur, "\Phi_{A,B\otimes C}"'] &
\end{tikzcd}$$

and the following diagrams commute for every $A \in \operatorname{ob}\mathcal{C}$.

$$\begin{tikzcd}
\mathbf{1}\otimes F(A) \arrow[r, "\lambda_{F(A)}"] \arrow[d, "\phi\otimes F(A)"'] & F(A) \\
F(\mathbf{1})\otimes F(A) \arrow[r, "\Phi_{\mathbf{1},A}"] & F(\mathbf{1}\otimes A) \arrow[u, "F(\lambda_A)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A)\otimes \mathbf{1} \arrow[r, "\rho_{F(A)}"] \arrow[d, "F(A)\otimes\phi"'] & F(A) \\
F(A)\otimes F(\mathbf{1}) \arrow[r, "\Phi_{A,\mathbf{1}}"] & F(A\otimes \mathbf{1}) \arrow[u, "F(\rho_A)"']
\end{tikzcd}$$

Remark The morphism ϕ is uniquely determined by the above diagrams.

Like functors, lax functors can be *composed*. More precisely, if F and G are two lax functors and can be composed as functors. Then the composition $G \circ F$ has a natural lax functor structure given by

$$G(F(A)) \otimes G(F(B)) \longrightarrow G(F(A) \otimes F(B)) \longrightarrow G(F(A \otimes B)).$$

A **symmetric lax functor** is simply a lax functor $F\colon \mathcal{C} \to \mathcal{C}'$ making the following diagram commute for every $A, B \in \operatorname{ob}\mathcal{C}$.

$$\begin{tikzcd}
F(A)\otimes F(B) \arrow[r, "\gamma_{F(A),F(B)}"] \arrow[d, "\Phi_{A,B}"'] & F(B)\otimes F(A) \arrow[d, "\Phi_{B,A}"] \\
F(A\otimes B) \arrow[r, "F(\gamma_{A,B})"] & F(B\otimes A)
\end{tikzcd}$$

Dually, we have the notion of **oplax functors**, which are similar to lax functors except reversing the canonical morphisms Φ and ϕ.

Let (F, Φ, ϕ) be a lax functor. If Φ is further a natural isomorphism and ϕ is an isomorphism, we call F a **monoidal functor**. If they are identities, we call F a **strict monoidal functor**. For example, the identity functor of a monoidal category has a trivial obvious strict monoidal functor structure.

2.a Example Any (symmetric) monoidal category $\mathcal{C}$ admits a (symmetric) lax functor to **Set**, given by

$$|-| := \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, -).$$

However, this functor may *NOT* be faithful in general. But if it is, then I is a *generator* in $\mathcal{K}$. In this case, we call a morphism from I to an object $A \in \mathrm{ob}\,\mathcal{K}$ an **element** of A.

A monoidal functor is said to be an **equivalence** of monoidal categories if it is an equivalence of categories. This definition seems not correct since the right concept of "equivalence" of monoidal categories should be stronger than the concept of equivalence of categories.

To explain this in detail, we should define the *monoidal natural transformations*. They are the natural transformations respect the monoidal structure in an obvious way.

More precisely, a **monoidal natural transformation** is a natural transformation $\eta\colon F \Rightarrow G$ between two monoidal functors (or, symmetric monoidal functors) (F, Φ, ϕ) and (G, Ψ, ψ) making the following diagrams commute for every $A, B \in \mathrm{ob}\,\mathcal{C}$.

$$\begin{tikzcd} F(A) \otimes F(B) \arrow[r, "\eta_A \otimes \eta_B"] \arrow[d, "\Phi_{A,B}"'] & G(A) \otimes G(B) \arrow[d, "\Psi_{A,B}"] \\ F(A \otimes B) \arrow[r, "\eta_{A \otimes B}"] & G(A \otimes B) \end{tikzcd} \qquad \begin{tikzcd} & \mathbf{1} \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ F(\mathbf{1}) \arrow[rr, "\eta_{\mathbf{1}}"] & & G(\mathbf{1}) \end{tikzcd}$$

A monoidal functor $F\colon \mathcal{C} \to \mathcal{C}'$ between two monoidal categories is said to be an **monoidal equivalence** and thus $\mathcal{C}$ and $\mathcal{C}'$ are said to be **monoidally equivalent** if it has a **monoidal weak inverse**, that is a monoidal functor $G\colon \mathcal{C}' \to \mathcal{C}$ such that there exist monoidal natural isomorphisms $F \circ G \cong \mathrm{id}_{\mathcal{C}'}$ and $\mathrm{id}_{\mathcal{C}} \cong G \circ F$. The **symmetric monoidal equivalence** and **symmetric monoidal weak inverse** are defined likewise.

However, we have

3 Lemma *If a monoidal functor is an equivalence of monoidal categories, then it is a monoidal equivalence.*

Proof: Let $F\colon \mathcal{C} \to \mathcal{C}'$ be an equivalence of monoidal categories and G be its weak inverse. Then we have a natural isomorphism:

$$G(A \otimes B) \cong G(F(G(A)) \otimes F(G(B))) \cong G(F(G(A) \otimes G(B))) \cong G(A) \otimes G(B).$$

It is straightforward to see that this gives G a monoidal functor structure. Moreover, it is not difficult to verify that the natural isomorphisms $F \circ G \cong \mathrm{id}_{\mathcal{C}'}$ and $\mathrm{id}_{\mathcal{C}} \cong G \circ F$ are monoidal. $\square$

Coherence theorems

Now we should prove the coherence theorems for monoidal categories and symmetric monoidal categories. But before going further, we should clarify that the theorems should talk about the commutativities of "formal" diagrams instead of individual diagrams because in a particular monoidal category, a same object can accidentally become tensor products defined from different sorts, thus provides accidental identities and brings some diagrams beyond our expectation and may be unfortunately not commutative. A suitable approach is to consider the diagrams of functors and natural transformations instead of objects and morphisms.

So, the coherence theorem states

4 Theorem (Coherence theorem for monoidal categories) [ML63] *In a monoidal category, any two chains of associators and unit isomorphisms connecting two functors built from tensor products and identities will provide the same identification.*

Remark The coherence theorem is first proved by Mac Lane and a well-known version can be found in his famous textbook *Categories for the Working Mathematician*, where the statement is a little different. Our statement basically follows a course note of [Eti00].

Other coherence conditions

To prove the coherence theorem, we need some lemmas, they are actually the extra axioms given by [ML63] and showed by [Kel64] later that they could be implied from the *pentagon* and *triangle* identities.

5 Lemma *In a monoidal category, the following diagrams commute.*

$$\begin{tikzcd}
(\mathbf{1}\otimes A)\otimes B \arrow[rr, "\alpha_{\mathbf{1},A,B}"] \arrow[dr, "\lambda_A\otimes B"'] & & \mathbf{1}\otimes(A\otimes B) \arrow[dl, "\lambda_{A\otimes B}"] \\
& A\otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes(B\otimes\mathbf{1}) \arrow[rr, "\alpha_{A,B,\mathbf{1}}"] \arrow[dr, "A\otimes\rho_B"'] & & (A\otimes B)\otimes\mathbf{1} \arrow[dl, "\rho_{A\otimes B}"] \\
& A\otimes B &
\end{tikzcd}$$

Proof: We only prove the first. Note that the functor $\mathbf{1}\otimes -$ is isomorphic to the identity, thus an equivalence. Therefore it suffices to check the commutativity of the triangle below.

$$\begin{tikzcd}
\mathbf{1}\otimes((\mathbf{1}\otimes A)\otimes B) \arrow[rr, "\mathbf{1}\otimes\alpha_{\mathbf{1},A,B}"] \arrow[dr, "\mathbf{1}\otimes(\lambda_A\otimes B)"'] & & \mathbf{1}\otimes(\mathbf{1}\otimes(A\otimes B)) \arrow[dl, "\mathbf{1}\otimes\lambda_{A\otimes B}"] \\
& \mathbf{1}\otimes(A\otimes B) &
\end{tikzcd}$$

To do this, we consider the following diagram,

$$\begin{tikzcd}
\mathbf{1}\otimes((\mathbf{1}\otimes A)\otimes B) \arrow[rr, "\mathbf{1}\otimes\alpha_{\mathbf{1},A,B}"] \arrow[dr, "\mathbf{1}\otimes(\lambda_A\otimes B)"'] \arrow[r, phantom, "\mathrm{I}"] & & \mathbf{1}\otimes(\mathbf{1}\otimes(A\otimes B)) \arrow[dl, "\mathbf{1}\otimes\lambda_{A\otimes B}"] \\
 & \mathbf{1}\otimes(A\otimes B) \arrow[r, phantom, "\mathrm{II}"] \arrow[dl, phantom, "\mathrm{V}"] & \\
 & & (\mathbf{1}\otimes\mathbf{1})\otimes(A\otimes B) \arrow[uu, "\alpha_{\mathbf{1},\mathbf{1},A\otimes B}"'] \arrow[ul, "\rho_{\mathbf{1}}\otimes(A\otimes B)"] \\
 & (\mathbf{1}\otimes A)\otimes B \arrow[uu, "\alpha_{\mathbf{1},A,B}"] \arrow[ur, phantom, "\mathrm{III}"] \arrow[d, phantom, "\mathrm{IV}"] & \\
(\mathbf{1}\otimes(\mathbf{1}\otimes A))\otimes B \arrow[uuuu, "\alpha_{\mathbf{1},\mathbf{1}\otimes A,B}"] \arrow[ur, "(\mathbf{1}\otimes\lambda_A)\otimes B"] & & ((\mathbf{1}\otimes\mathbf{1})\otimes A)\otimes B \arrow[ll, "\alpha_{\mathbf{1},\mathbf{1},A}\otimes B"'] \arrow[uu, "\alpha_{\mathbf{1}\otimes\mathbf{1},A,B}"'] \arrow[ul, "(\rho_{\mathbf{1}}\otimes A)\otimes B"']
\end{tikzcd}$$

Here, the outside commutes by the *pentagon identity*, regions II and IV by the *triangle identity*, and regions III and V by the naturality of α. It follows that region I commutes as desired. □

6 Lemma *In a monoidal category,* $\lambda_{\mathbf{1}} = \rho_{\mathbf{1}}$.

Proof: From the *triangle identity*, we have $(\mathbf{1}\otimes\lambda_{\mathbf{1}})\circ\alpha_{\mathbf{1},\mathbf{1},\mathbf{1}} = \rho_{\mathbf{1}}\otimes\mathbf{1}$. On the other hand, by Lemma 5, we have $(\mathbf{1}\otimes\lambda_{\mathbf{1}})\circ\alpha_{\mathbf{1},\mathbf{1},\mathbf{1}} = \lambda_{\mathbf{1}}\otimes\mathbf{1}$. Therefore $\rho_{\mathbf{1}}\otimes\mathbf{1} = \lambda_{\mathbf{1}}\otimes\mathbf{1}$. Then $\lambda_{\mathbf{1}} = \rho_{\mathbf{1}}$ since $-\otimes\mathbf{1}$ is an equivalence. □

The strictness theorem

It is obvious that if the associator α and the two unitors λ and ρ are all identities, then the coherence theorem for monoidal categories trivially holds. Such kind of monoidal categories are said to be **strict**.

Then, the main step of proving the coherence theorem is to prove the following strictness theorem.

7 Theorem (Strictness theorem for monoidal categories) [JS93] *Every monoidal category is monoidally equivalent to a strict monoidal category.*

Proof: Let $\mathcal{C}$ be a monoidal category, we now construct a strict monoidal category $\mathcal{C}_s$ as follows. The objects of $\mathcal{C}_s$ are pairs (F,ζ), where $F\colon\mathcal{C}\to\mathcal{C}$ is a functor and

$$\zeta_{A,B}\colon F(A)\otimes B \xrightarrow{\cong} F(A\otimes B)$$

is a natural isomorphism, such that the following diagram commutes for any

$A, B, C \in \operatorname{ob}\mathcal{C}$.

$$\begin{tikzcd}
& F(A) \otimes (B \otimes C) \arrow[dr, "\zeta_{A,B\otimes C}"] & \\
(F(A) \otimes B) \otimes C \arrow[ur, "\alpha_{F(A),B,C}"] \arrow[d, "\zeta_{A,B}\otimes C"'] & & F(A \otimes (B \otimes C)) \\
F(A \otimes B) \otimes C \arrow[rr, "\zeta_{A\otimes B,C}"] & & F((A \otimes B) \otimes C) \arrow[u, "F(\alpha_{A,B,C})"']
\end{tikzcd}$$

A morphism $\theta\colon (F,\zeta) \to (F',\zeta')$ in $\mathcal{C}_s$ is a natural transformation $\theta\colon F \Rightarrow F'$ such that the following square commutes for any $A, B \in \operatorname{ob}\mathcal{C}$.

$$\begin{tikzcd}
F(A) \otimes B \arrow[r, "\theta_A \otimes B"] \arrow[d, "\zeta_{A,B}"'] & F'(A) \otimes B \arrow[d, "\zeta'_{A,B}"] \\
F(A \otimes B) \arrow[r, "\theta_{A\otimes B}"] & F'(A \otimes B)
\end{tikzcd}$$

Composition of morphisms is the *vertical composition* of natural transformations. The tensor product of objects is given by

$$(F,\zeta) \otimes (F',\zeta') := (F \circ F', \tilde{\zeta}),$$

where $\tilde{\zeta}$ is given by the composition

$$F(F'(A)) \otimes B \stackrel{\zeta_{F'(A),B}}{\longrightarrow} F(F'(A) \otimes B) \stackrel{F(\zeta'_{A,B})}{\longrightarrow} F(F'(A \otimes B)),$$

and the tensor product of morphisms is the *horizontal composition* of natural transformations. Then one can verify that $\mathcal{C}_s$ is a strict monoidal category with identity functor id as the tensor unit.

Next, we define a functor $L\colon \mathcal{C} \to \mathcal{C}_s$ as follows:

$$L(A) = (A \otimes -, \alpha_{A,-,-}), \qquad L(f) = f \otimes -.$$

Now, we claim that this functor L is a monoidal equivalence.

First, it is easy to verify that for any $(F,\zeta) \in \operatorname{ob}\mathcal{C}_s$, (F,ζ) is isomorphic to $L(F(\mathbf{1}))$ by check the commutativity of the following square. Here the natural transformation θ is defined by $\theta_A := F(\lambda_A) \circ \zeta_{\mathbf{1},A}$.

$$\begin{tikzcd}
(F(\mathbf{1}) \otimes A) \otimes B \arrow[r, "\theta_A \otimes B"] \arrow[d, "\alpha_{\mathbf{1},A,B}"'] & F(A) \otimes B \arrow[d, "\zeta_{A,B}"] \\
F(\mathbf{1}) \otimes (A \otimes B) \arrow[r, "\theta_{A\otimes B}"] & F(A \otimes B)
\end{tikzcd}$$

Thus L is essentially surjective.

Then us should show that L is full. Let $\theta\colon L(A) \to L(B)$ be a morphism in $\mathcal{C}_s$, we define $f\colon A \to B$ to be the composite

$$A \xrightarrow{\rho_A^{-1}} A \otimes \mathbf{1} \xrightarrow{\theta_{\mathbf{1}}} B \otimes \mathbf{1} \xrightarrow{\rho_B} B.$$

To show $\theta = L(f)$, it suffices to show that for any $C \in \operatorname{ob}\mathcal{C}$, $\theta_C = f \otimes C$. Indeed, this follows from the commutativity of the diagram,

$$\begin{tikzcd}
A \otimes C \arrow[r, "\rho_A^{-1}\otimes C"] \arrow[d, "f\otimes C"'] & (A \otimes \mathbf{1}) \otimes C \arrow[r, "\alpha_{A,\mathbf{1},C}"] \arrow[d, "\theta_{\mathbf{1}}\otimes C"'] & A \otimes (\mathbf{1} \otimes C) \arrow[r, "A\otimes\lambda_C"] \arrow[d, "\theta_{\mathbf{1}\otimes C}"'] & A \otimes C \arrow[d, "\theta_C"'] \\
B \otimes C \arrow[r, "\rho_B^{-1}\otimes C"] & (B \otimes \mathbf{1}) \otimes C \arrow[r, "\alpha_{B,\mathbf{1},C}"] & B \otimes (\mathbf{1} \otimes C) \arrow[r, "B\otimes\lambda_C"] & B \otimes C
\end{tikzcd}$$

where the rows are the identities by the *triangle identity*, the left square commutes by the definition of f, the right square commutes by naturality of θ, and the central square commutes since θ is a morphism in $\mathcal{C}_s$.

Next, if $L(f) = L(g)$ for some morphisms f, g in $\mathcal{C}$ then, in particular $f \otimes \mathbf{1} = g \otimes \mathbf{1}$ so that $f = g$. Thus L is faithful.

Finally, we define a monoidal functor structure

$$\Psi_{A,B}\colon L(A) \otimes L(B) \xrightarrow{\cong} L(A \otimes B), \qquad \psi\colon \mathrm{id} \xrightarrow{\cong} L(\mathbf{1})$$

on L as follows. First,

$$\begin{aligned} L(A) \otimes L(B) &= (A \otimes (B \otimes -), \zeta)\,, \\ L(A \otimes B) &= \big((A \otimes B) \otimes -, \zeta'\big)\,. \end{aligned}$$

(Here we omit the explicit formulas for ζ and ζ').

Thus we can define $\Psi_{A,B}$ to be $\alpha^{-1}_{A,B,-}$. To show it is a morphism in $\mathcal{C}_s$, we should check the commutativity of the square for any $C, D \in \operatorname{ob}\mathcal{C}$.

$$\begin{tikzcd}
L(A) \circ L(B)(C) \otimes D \arrow[r, "\alpha^{-1}_{A,B,C}\otimes D"] \arrow[d, "\zeta_{C,D}"'] & L(A \otimes B)(C) \otimes D \arrow[d, "\zeta'_{C,D}"] \\
L(A) \circ L(B)(C \otimes D) \arrow[r, "\alpha^{-1}_{A,B,C\otimes D}"] & L(A \otimes B)(C \otimes D)
\end{tikzcd}$$

But $\zeta_{C,D}$ is the composite

$$(A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha_{A,B\otimes C,D}} A \otimes ((B \otimes C) \otimes D) \xrightarrow{A\otimes\alpha_{B,C,D}} A \otimes (B \otimes (C \otimes D)),$$

and $\zeta'_{C,D}$ is $\alpha_{A\otimes B,C,D}$. Thus the square becomes the *pentagon identity.*

The tensor unit of $\mathcal{C}_s$ is the identity functor id, so the isomorphism ϕ can be defined as $\phi_A = \lambda_A^{-1}\colon A \xrightarrow{\cong} \mathbf{1} \otimes A$. To show it is a morphism in $\mathcal{C}_s$, we need to verify the commutativity of the square for any $A, B \in \mathrm{ob}\,\mathcal{C}$.

$$\begin{tikzcd}
A \otimes B \arrow[r, "\lambda_A^{-1} \otimes B"] \arrow[d, equal] & (\mathbf{1} \otimes A) \otimes B \arrow[d, "\alpha_{\mathbf{1},A,B}"] \\
A \otimes B \arrow[r, "\lambda_{A\otimes B}^{-1}"] & \mathbf{1} \otimes (A \otimes B)
\end{tikzcd}$$

It follows directly from Lemma 5.

Now we check that (Φ, ϕ) give a monoidal functor structure on L. To check the hexagon condition of (L, Φ, ϕ), i.e.

$$\begin{tikzcd}
& L(A \otimes B) \otimes L(C) \arrow[dr, "\Phi_{A\otimes B,C}"] & \\
(L(A) \otimes L(B)) \otimes L(C) \arrow[ur, "\Phi_{A,B}\otimes L(C)"] \arrow[d, equal] & & L((A \otimes B) \otimes C) \arrow[d, "L(\alpha_{A,B,C})"] \\
L(A) \otimes (L(B) \otimes L(C)) \arrow[dr, "L(A)\otimes \Phi_{B,C}"'] & & L(A \otimes (B \otimes C)) \\
& L(A) \otimes L(B \otimes C) \arrow[ur, "\Phi_{A,B\otimes C}"'] &
\end{tikzcd}$$

we apply it on an object D. Then it becomes the *pentagon identity*, thus holds. The rest two conditions are

$$\begin{tikzcd}
\mathbf{1} \otimes L(A) \arrow[r, equal] \arrow[d, "\phi\otimes L(A)"'] & L(A) \\
L(\mathbf{1}) \otimes L(A) \arrow[r, "\Phi_{\mathbf{1},A}"] & L(\mathbf{1} \otimes A) \arrow[u, "L(\lambda_A)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
L(A) \otimes \mathbf{1} \arrow[r, equal] \arrow[d, "L(A)\otimes\phi"'] & L(A) \\
L(A) \otimes L(\mathbf{1}) \arrow[r, "\Phi_{A,\mathbf{1}}"] & L(A \otimes \mathbf{1}) \arrow[u, "L(\rho_A)"']
\end{tikzcd}$$

we apply them on an object B, then they become

$$\begin{tikzcd}
& A \otimes B \arrow[dl, "\lambda_{A\otimes B}^{-1}"'] & \\
\mathbf{1} \otimes (A \otimes B) \arrow[rr, "\alpha_{\mathbf{1},A,B}"] & & (\mathbf{1} \otimes A) \otimes B \arrow[ul, "\lambda_A \otimes B"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \otimes B \arrow[dl, "A\otimes\lambda_B^{-1}"'] & \\
A \otimes (\mathbf{1} \otimes B) \arrow[rr, "\alpha_{A,\mathbf{1},B}"] & & (A \otimes \mathbf{1}) \otimes B \arrow[ul, "\rho_A \otimes B"']
\end{tikzcd}$$

where the left triangle commute by Lemma 5 and the right by the *triangle identity*. □

The coherence theorems

Once we have the strictness theorem, the coherence theorem follows almost directly.

Proof (Proof for Theorem 4): Let $F\colon \mathcal{C} \to \mathcal{C}_s$ be a monoidal equivalence from a monoidal category to a strict monoidal category. Then, for two chains θ and θ' connected two expressions f and g, we have the following commutative diagram by the definition of monoidal functors.

$$\begin{tikzcd}
f(F(A_1), F(A_2), \cdots, F(A_n)) \arrow[r, "\theta_s", "\theta'_s"'] \arrow[d, "\Phi_f"'] & g(F(A_1), F(A_2), \cdots, F(A_n)) \arrow[d, "\Phi_g"] \\
F(f(A_1, A_2, \cdots, A_n)) \arrow[r, "F(\theta)", "F(\theta')"'] & F(g(A_1, A_2, \cdots, A_n))
\end{tikzcd}$$

Here, the columns are natural isomorphisms and rows are chains of associators and unit isomorphisms. Since $\mathcal{C}_s$ is strict, both θ_s and θ'_s are just chains of identities, thus equal. Then we have

$$F(\theta) \circ \Phi_f = \Phi_g = F(\theta') \circ \Phi_f.$$

Since Φ_f is an isomorphism and F is an equivalence, $\theta = \theta'$ as desired. $\square$

Now we give the coherence theorem for symmetric monoidal categories and prove it.

8 **Theorem (Coherence theorem for symmetric monoidal categories)** *In a symmetric monoidal category, any two chains of associators, unit isomorphisms and braiding isomorphisms connecting two functors built from tensor products, identities and permutations will provide the same identification.*

Proof: Let $\mathcal{C}$ be a symmetric monoidal category. Then, by Theorem 4, we can safely assume that it is strict, thus all $n-$ary functors built from tensor products and identities are equal. Let's call it the $n-$**ary tensor product functor**. Then we only need to show that any two chains of braiding isomorphisms connecting two permutations of the $n-$ary tensor product functor will provide the same identification. Moreover, we only need to check this for the *closed chains*, that is a chain whose start equals to the end.

Note that, the *hexagon identity* in the strict case shows that we can replace any braiding isomorphism by a chain of *adjacent transposition braiding isomorphisms*, that is a braiding isomorphism whose domain and codomain differ by an adjacent transposition. For example, the braiding isomorphism $\gamma_{A,B\otimes C}\colon A \otimes B \otimes C \to B \otimes C \otimes A$ is equal to the composite

$$A \otimes B \otimes C \overset{\gamma_{A,B}\otimes C}{\longrightarrow} B \otimes A \otimes C \overset{B\otimes\gamma_{A,C}}{\longrightarrow} B \otimes C \otimes A.$$

Then, we can label every adjacent transposition braiding isomorphism by the corresponding adjacent transposition. We define the label of a chain to be the *formal* product of the labels of its components. That means we do not take products in the nth symmetric group $\mathfrak{S}_n$, instead, we should take products in the free group $\mathfrak{F}_n$ generated by the adjacent transpositions. Thus a chain is closed if and only if its label belongs to the kernel of the canonical map $\mathfrak{F}_n \to \mathfrak{S}_n$.

Recall that this kernel is generated by the following elements

$$\begin{aligned} {\sigma_i}^2, &\quad 1 \leqslant i \leqslant n-1; \\ (\sigma_i\sigma_j)^2, &\quad 1 \leqslant i < j-1 \leqslant n-2; \\ (\sigma_i\sigma_{i+1})^3, &\quad 1 \leqslant i \leqslant n-2. \end{aligned}$$

Then, we can decompose a closed chain as a composite of some *elementary* ones, each of them is labelled by one of the above elements.

Thus, it suffices to verify that every elementary closed chain provides the trivial identification. For chains labelled by ${\sigma_i}^2$, this follows by the *involution law*; for chains labelled by $(\sigma_i\sigma_j)^2$, this follows from the naturality of γ; for chains labelled by $(\sigma_i\sigma_{i+1})^3$, this follows from the commutativity of the diagram below.

$$\begin{tikzcd}
& A\otimes B\otimes C \arrow[dl, "\gamma_{A,B}\otimes C"'] \arrow[dr, "A\otimes\gamma_{B,C}"] & \\
B\otimes A\otimes C \arrow[d, "B\otimes\gamma_{A,C}"'] & & A\otimes C\otimes B \arrow[d, "\gamma_{A,C}\otimes B"] \\
B\otimes C\otimes A \arrow[dr, "\gamma_{B,C}\otimes A"'] & & C\otimes A\otimes B \arrow[dl, "C\otimes\gamma_{A,B}"] \\
& C\otimes B\otimes A &
\end{tikzcd}$$

This is the **Yang-Baxter identity**, one can verify it directly from the *hexagon identity*. □

Reference

[ML63] Saunders Mac Lane, *Natural associativity and commutativity*, Rice Univ. Studies **49** (1963), no. 4, 28–46.

[Kel64] Gregory Maxwell Kelly, *On MacLane's conditions for coherence of natural associativities, commutativities, etc*, J. Algebra **1** (1964), 397–402.

[Eti00] Pavel Etingof, *Tensor categories* (2000), http://www-math.mit.edu/~etingof/tenscat.pdf.

[JS93] André Joyal and Ross Street, *Braided tensor categories*, Adv. Math. **102** (1993), no. 1, 20–78.

[Lei04] Tom Leinster, *Higher operads, higher categories*, London Mathematical Society Lecture Note Series, vol. 298, Cambridge University Press, Cambridge, 2004.

§ A.2 Enriched categories

In this section, we give basic notions on enriched categories.

Enriched categories

1 **(Enriched categories)** Let $\mathcal{K}$ be a monoidal category. Then a **$\mathcal{K}$-enriched category**, or simply a **$\mathcal{K}$-category** $\mathcal{C}$ consists of the following data:

- a collection $\operatorname{ob}\mathcal{C}$ of objects.
- for every two objects A and B in $\mathcal{C}$, an object $\underline{\operatorname{Hom}}_{\mathcal{C}}(A,B) \in \operatorname{ob}\mathcal{K}$ called the **hom-object**.
- for every three objects A, B and C in $\mathcal{C}$, a morphism

$$\bullet_{A,B,C} \colon \underline{\operatorname{Hom}}_{\mathcal{C}}(B,C) \otimes \underline{\operatorname{Hom}}_{\mathcal{C}}(A,B) \longrightarrow \underline{\operatorname{Hom}}_{\mathcal{C}}(A,C)$$

  called the **composition**.
- for every object A in $\mathcal{C}$, a $\mathcal{K}$-morphism $\operatorname{id}_A \colon \mathbf{1} \to \underline{\operatorname{Hom}}_{\mathcal{C}}(A,A)$, called the **identity**.

subject to the following axioms:

1. the composition is *associative*, which means for every $A, B, C, D \in \operatorname{ob}\mathcal{C}$, the following diagram commutes;

$$\begin{tikzcd}
 & \underline{\operatorname{Hom}}_{\mathcal{C}}(C,D) \otimes \underline{\operatorname{Hom}}_{\mathcal{C}}(B,C) \otimes \underline{\operatorname{Hom}}_{\mathcal{C}}(A,B) \arrow[dl] \arrow[dr] & \\
\underline{\operatorname{Hom}}_{\mathcal{C}}(C,D) \otimes \underline{\operatorname{Hom}}_{\mathcal{C}}(A,C) \arrow[dr] & & \underline{\operatorname{Hom}}_{\mathcal{C}}(B,D) \otimes \underline{\operatorname{Hom}}_{\mathcal{C}}(A,B) \arrow[dl] \\
 & \underline{\operatorname{Hom}}_{\mathcal{C}}(A,D) &
\end{tikzcd}$$

2. composition is *unital*, which means for every $A, B \in \operatorname{ob}\mathcal{C}$, the following diagram commutes.

$$\begin{tikzcd}
\mathbf{1} \otimes \underline{\operatorname{Hom}}_{\mathcal{C}}(A,B) \arrow[r, "\operatorname{id}_B"] \arrow[dr, "\cong"'] & \underline{\operatorname{Hom}}_{\mathcal{C}}(B,B) \otimes \underline{\operatorname{Hom}}_{\mathcal{C}}(A,B) \arrow[d] \\
 & \underline{\operatorname{Hom}}_{\mathcal{C}}(A,B) \\
\underline{\operatorname{Hom}}_{\mathcal{C}}(A,B) \otimes \mathbf{1} \arrow[r, "\operatorname{id}_A"] \arrow[ur, "\cong"] & \underline{\operatorname{Hom}}_{\mathcal{C}}(A,B) \otimes \underline{\operatorname{Hom}}_{\mathcal{C}}(A,A) \arrow[u]
\end{tikzcd}$$

2 **Example** **Set** is a monoidal category with Cartesian products as tensor products. In this sense, a **Set**-category is merely a category.

3 Example (Change of bases) Let $u\colon \mathcal{K} \to \mathcal{K}'$ be a *lax functor*. Let $\mathcal{C}$ be a $\mathcal{K}$-category. Then we may define a $\mathcal{K}'$-category $u\mathcal{C}$ as follows:

- objects of $u\mathcal{C}$ are objects of $\mathcal{C}$;
- for every two objects A and B in $\mathcal{C}$, we set

$$\underline{\mathrm{Hom}}_{u\mathcal{C}}(A,B) = u(\underline{\mathrm{Hom}}_{\mathcal{C}}(A,B));$$

- for every three objects A, B and C in $\mathcal{C}$, the *composition* is give by the composition of $u(\bullet)$ with the comparison of monoidal structures:

$$\begin{aligned} u(\underline{\mathrm{Hom}}_{\mathcal{C}}(B,C)) \otimes u(\underline{\mathrm{Hom}}_{\mathcal{C}}(A,B)) &\longrightarrow u(\underline{\mathrm{Hom}}_{\mathcal{C}}(B,C) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(A,B)) \\ &\xrightarrow{u(\bullet)} u(\underline{\mathrm{Hom}}_{\mathcal{C}}(A,C)); \end{aligned}$$

- for every object A in $\mathcal{C}$, the identity of A in $u\mathcal{C}$ is the composition of $u(\mathrm{id}_A)$ with the comparison of monoidal structures:

$$\mathbf{1} \longrightarrow u(\mathbf{1}) \xrightarrow{u(\mathrm{id}_A)} u(\underline{\mathrm{Hom}}_{\mathcal{C}}(A,A)).$$

Especially, for $\mathcal{K}' = \mathbf{Set}$, we have

4 (Underlying category) Let $\mathcal{K}$ be a monoidal category and $\mathcal{C}$ a $\mathcal{K}$-category. Then we may define a category via the monoidal functor $K := \mathrm{Hom}_{\mathcal{K}}(\mathbf{1}, -)$ from $\mathcal{K}$ to **Set**. This category is called the **underlying ccategory** of $\mathcal{C}$. For any $A, B \in \mathrm{ob}\,\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{K}}(\mathbf{1}, \underline{\mathrm{Hom}}_{\mathcal{C}}(A,B))$ is called the **underlying Hom-set** and denoted by $\mathrm{Hom}_{\mathcal{C}}(A,B)$. Usually, we do not distinguish $\mathcal{C}$ with its underlying category and thus we may call an element f of $\mathrm{Hom}_{\mathcal{C}}(A,B)$ as a morphism from A to B and denote it as $f\colon A \to B$.

Let $f\colon A \to B$ be a morphism. Then it induces two $\mathcal{K}$-morphisms

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathcal{C}}(B,C) &\longrightarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(B,C) \otimes \mathbf{1} \\ &\xrightarrow{f} \underline{\mathrm{Hom}}_{\mathcal{C}}(B,C) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(A,B) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(A,C); \\ \underline{\mathrm{Hom}}_{\mathcal{C}}(C,A) &\longrightarrow \mathbf{1} \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(C,A) \\ &\xrightarrow{f} \underline{\mathrm{Hom}}_{\mathcal{C}}(A,B) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(C,A) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(C,B). \end{aligned}$$

In this way, $\underline{\mathrm{Hom}}_{\mathcal{C}}(-,-)$ becomes a bifunctor from $\mathcal{C}$ to $\mathcal{K}$.

5 Example (Tensor-Hom adjunction) A monoidal category $\mathcal{C}$ is said to be **right closed** if every functor $-\otimes M$ with $M \in \mathrm{ob}\,\mathcal{C}$ has a right adjoint, denoted by $\underline{\mathrm{Hom}}(M,-)$.

If this is the case, we get a bifunctor $\underline{\mathrm{Hom}}(-,-)\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the **internal Hom** in $\mathcal{C}$. If this is the case, then $\mathcal{C}$ is enriched over itself by setting $\underline{\mathrm{Hom}}_{\mathcal{C}}(A,B)$ to be the internal-Hom $\underline{\mathrm{Hom}}(A,B)$. Then, there is a underlying category of the $\mathcal{C}$-category $\mathcal{C}$. But $\mathcal{C}$ itself is *a priori* a category. So, what's the relation?

By the *Tensor-Hom adjunction*, we have

$$\mathrm{Hom}_{\mathcal{C}}(A,B) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbf{1} \otimes A, B) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \underline{\mathrm{Hom}}(A,B)).$$

Therefore, the two category structures on $\mathcal{C}$ coincides.

In general, for any object A of $\mathcal{C}$, we call the set $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A)$ the set of **global element** of A.

For any objects A and B in a right closed monoidal category, there is a canonical morphism

$$\mathrm{ev}\colon\ \underline{\mathrm{Hom}}(A,B) \otimes A \longrightarrow B$$

corresponding to $\mathrm{id}_{\underline{\mathrm{Hom}}(A,B)}$ under the *Tensor-Hom adjunction.* It is called the **evaluation**.

6 (Enriched functors and enriched natural transformations) Given two $\mathcal{K}-$categories $\mathcal{C}$ and $\mathcal{D}$, a **$\mathcal{K}$-enriched functor**, or simply a **$\mathcal{K}$-functor** $F\colon \mathcal{C} \to \mathcal{D}$ consists of following data:

- A mapping between the collection of objects;
- A collection of morphisms $\underline{\mathrm{Hom}}_{\mathcal{C}}(A,B) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(FA,FB)$ which are natural in $A, B \in \mathrm{ob}\,\mathcal{C}$.

subject to the following axioms:

1. the functor is compatible with enriched compositions, which means for every $A, B, C \in \mathrm{ob}\,\mathcal{C}$, the following diagram commutes;

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{C}}(B,C) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(A,B) \arrow[r] \arrow[d, "F"'] & \underline{\mathrm{Hom}}_{\mathcal{C}}(A,C) \arrow[d, "F"] \\
\underline{\mathrm{Hom}}_{\mathcal{D}}(FB,FC) \otimes \underline{\mathrm{Hom}}_{\mathcal{D}}(FA,FB) \arrow[r] & \underline{\mathrm{Hom}}_{\mathcal{D}}(FA,FC)
\end{tikzcd}$$

2. the functor is compatible with enriched identity, which means for every $A \in \mathrm{ob}\,\underline{\mathrm{Hom}}_{\mathcal{C}}$, the following diagram commutes.

$$\begin{tikzcd}
 & \mathbf{1} \arrow[dl, "\mathrm{id}_A"'] \arrow[dr, "\mathrm{id}_{FA}"] & \\
\underline{\mathrm{Hom}}_{\mathcal{C}}(A,A) \arrow[rr, "F"] & & \underline{\mathrm{Hom}}_{\mathcal{D}}(FA,FA)
\end{tikzcd}$$

A $\mathcal{K}$-functor F is said to be **full** (resp. **faithful**, **fully faithful**) if each morphism $\underline{\mathrm{Hom}}_{\mathcal{C}}(A,B) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(FA,FB)$ is a monomorphism (resp. epimorphism, isomorphism).

For two $\mathcal{K}-$enriched functors $F, G\colon \mathcal{C} \to \mathcal{D}$, a **$\mathcal{K}$-enriched natural transformation**, or simply a **$\mathcal{K}$-natural transformation** $\alpha\colon F \Rightarrow G$ is a natural transformation $\alpha_A\colon \mathbf{1} \to \underline{\mathrm{Hom}}_{\mathcal{D}}(FA, GA)$ of functors from $\mathcal{C}$ to $\mathcal{K}$ such that the following diagram commutes for every $A, B \in \mathrm{ob}\,\mathcal{C}$.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{C}}(A,B) \arrow[r, "F"] \arrow[d, "G"'] & \underline{\mathrm{Hom}}_{\mathcal{D}}(FA,FB) \arrow[d, "\alpha_B"] \\
\underline{\mathrm{Hom}}_{\mathcal{D}}(GA,GB) \arrow[r, "\alpha_A"] & \underline{\mathrm{Hom}}_{\mathcal{D}}(FA,GB)
\end{tikzcd}$$

The **vertical composition** $\beta \circ \alpha$ of two $\mathcal{K}$-enriched natural transformations $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$ is obtained by the composition:

$$\mathbf{1} \cong \mathbf{1} \otimes \mathbf{1} \xrightarrow{\beta\otimes\alpha} \underline{\mathrm{Hom}}_{\mathcal{D}}(GA, HA) \otimes \underline{\mathrm{Hom}}_{\mathcal{D}}(FA, GA) \xrightarrow{\bullet} \underline{\mathrm{Hom}}_{\mathcal{D}}(FA, HA).$$

As for the **horizontal composition** $\beta * \alpha$ of two $\mathcal{K}$-enriched natural transformations:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E}
\end{tikzcd}$$

there are two possible compositions:

$$\begin{tikzcd}
 & \mathbf{1}\otimes\underline{\mathrm{Hom}}_{\mathcal{D}}(FA,GA) \arrow[r, "\beta"] & \underline{\mathrm{Hom}}_{\mathcal{E}}(F'GA,G'GA)\otimes\underline{\mathrm{Hom}}_{\mathcal{E}}(F'FA,F'GA) \arrow[d] \\
\mathbf{1} \arrow[r, "\cong"] & \mathbf{1}\otimes\mathbf{1} \arrow[u, "\alpha"] \arrow[d, "\alpha"'] & \underline{\mathrm{Hom}}_{\mathcal{E}}(F'FA,G'GA) \\
 & \underline{\mathrm{Hom}}_{\mathcal{D}}(FA,GA)\otimes\mathbf{1} \arrow[r, "\beta"] & \underline{\mathrm{Hom}}_{\mathcal{E}}(G'FA,G'GA)\otimes\underline{\mathrm{Hom}}_{\mathcal{E}}(F'FA,G'FA) \arrow[u]
\end{tikzcd}$$

one can verify that they are the same.

6.a Remark (Underlying functors and natural transformations) Every $\mathcal{K}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a functor between their underlying categories: for any $A, B \in \mathrm{ob}\,\mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(FA, GA)$$

is given by mapping any $f\colon A \to B$ to the composition

$$\mathbf{1} \xrightarrow{f} \underline{\mathrm{Hom}}_{\mathcal{C}}(A, B) \xrightarrow{F} \underline{\mathrm{Hom}}_{\mathcal{D}}(FA, FB).$$

A $\mathcal{K}$-natural transformation $\alpha\colon F \Rightarrow G$, by its definition, is already a natural transformation between their underlying functors. So there is no need to distinguish them. In this way all notions about natural transformations apply to $\mathcal{K}$-natural transformations. In particular, we have the notion of *natural isomorphisms*, *vertical composition*, *horizontal composition* and the *interchange law*.

Tensor products of $\mathcal{K}$-categories

We have seen that $\mathcal{K}$-categories form a 2-category $\mathcal{K}\,\mathbf{Cat}$. In the case $\mathcal{K}$ is symmetric monoidal, $\mathcal{K}\,\mathbf{Cat}$ is future a *symmetric monoidal* 2*-category.* Throughout this subsection, $\mathcal{K}$ is assumed to be symmetric monoidal.

7 **(Tensor products of $\mathcal{K}$-categories)** Let $\mathcal{K}$ be a symmetric monoidal category and $\mathcal{C}, \mathcal{D}$ two $\mathcal{K}$-categories. We define the $\mathcal{K}$-category $\mathcal{C} \otimes \mathcal{D}$ as follows.

- objects of $\mathcal{C} \otimes \mathcal{D}$ is $\operatorname{ob}\mathcal{C} \times \operatorname{ob}\mathcal{D}$;
- for every two objects A and B in $\mathcal{C}$ and two objects A' and B' in $\mathcal{D}$, the *hom-object* is

$$\underline{\mathrm{Hom}}_{\mathcal{C}\otimes\mathcal{D}}((A, A'), (B, B') := \underline{\mathrm{Hom}}_{\mathcal{C}}(A, B) \otimes \underline{\mathrm{Hom}}_{\mathcal{D}}(A', B');$$

- for every three objects A, B and C in $\mathcal{C}$ and ,three objects A', B' and C' in $\mathcal{D}$, the *composition* is given by the composition

$$\begin{aligned} &\underline{\mathrm{Hom}}_{\mathcal{C}\otimes\mathcal{D}}((B, B'), (C, C')) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}\otimes\mathcal{D}}((A, A'), (B, B')) \\ \longrightarrow\, &\underline{\mathrm{Hom}}_{\mathcal{C}}(B, C) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(A, B) \otimes \underline{\mathrm{Hom}}_{\mathcal{D}}(B', C') \otimes \underline{\mathrm{Hom}}_{\mathcal{D}}(A', B') \\ \longrightarrow\, &\underline{\mathrm{Hom}}_{\mathcal{C}\otimes\mathcal{D}}((A, A'), (C, C')); \end{aligned}$$

- for every object A in $\mathcal{C}$ and A' in $\mathcal{D}$, the *identity* is given by the composition

$$\mathbf{1} \longrightarrow \mathbf{1} \otimes \mathbf{1} \xrightarrow{\mathrm{id}_A \otimes \mathrm{id}_{A'}} \underline{\mathrm{Hom}}_{\mathcal{C}}(A, A) \otimes \underline{\mathrm{Hom}}_{\mathcal{D}}(A', A').$$

Then, one can define the tensor products of $\mathcal{K}$-functors and $\mathcal{K}$-natural transformations. In this way, $\mathcal{K}$-categories form a *symmetric monoidal* 2*-category* $\mathcal{K}\,\mathbf{Cat}$. Here the unit is the $\mathcal{K}$-category $\mathcal{I}$ with one object $*$ and $\underline{\mathrm{Hom}}_{\mathcal{I}}(*, *) = \mathbf{1}$.

Note that the underlying category of $\mathcal{C} \otimes \mathcal{D}$ is rarely $\mathcal{C} \times \mathcal{D}$ and the underlying category of $\mathcal{I}$ is usually not $*$. Instead, there are canonical functors $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ and $* \to \mathcal{I}$.

8 **(Partial functors)** Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be three $\mathcal{K}$-categories. Then a $\mathcal{K}$-functor $F\colon \mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ gives rise to **partial functors** $F(A, -)\colon \mathcal{D} \to \mathcal{E}$ and $F(-, A')\colon \mathcal{C} \to \mathcal{E}$ by

$$\mathcal{D} \cong \mathcal{I} \otimes \mathcal{D} \xrightarrow{A \otimes \mathrm{id}} \mathcal{C} \otimes \mathcal{D} \xrightarrow{F} \mathcal{E} \quad \text{and} \quad \mathcal{C} \cong \mathcal{C} \otimes \mathcal{I} \xrightarrow{\mathrm{id} \otimes A'} \mathcal{C} \otimes \mathcal{D} \xrightarrow{F} \mathcal{E}.$$

The partial functors of the composition

$$\mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{C} \otimes \mathcal{D} \xrightarrow{F} \mathcal{E}$$

are just the underlying functors of the partial functors of F.

Conversely, given a family of $\mathcal{K}$-functors $G_A\colon \mathcal{D} \to \mathcal{E}$ indexed by objects of $\mathcal{C}$ and a family of $\mathcal{K}$-functors $H_{A'}\colon \mathcal{C} \to \mathcal{E}$ indexed by objects of $\mathcal{D}$ such that $G_A(A') = H_{A'}(A)$ for all $A \in \operatorname{ob}\mathcal{C}$ and $A' \in \operatorname{ob}\mathcal{D}$. Let $T(A, A')$ denote the object $G_A(A')$. Then *there is a $\mathcal{K}$-functor $T\colon \mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ whose partial functors are G_A and $H_{A'}$ if and only if they fit into the following commutative diagram.*

$$\begin{tikzcd}
 & \underline{\mathrm{Hom}}_{\mathcal{E}}(H_{B'}(A), H_{B'}(B)) \otimes \underline{\mathrm{Hom}}_{\mathcal{E}}(G_A(A'), G_A(B')) \arrow[dd] \\
\underline{\mathrm{Hom}}_{\mathcal{C}}(A, B) \otimes \underline{\mathrm{Hom}}_{\mathcal{D}}(A', B') \arrow[ur, "H_{B'} \otimes G_A"] \arrow[dd, "\gamma"'] & \\
 & \underline{\mathrm{Hom}}_{\mathcal{E}}(T(A, A'), T(B, B')) \\
\underline{\mathrm{Hom}}_{\mathcal{D}}(A', B') \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(A, B) \arrow[dr, "H_{B'} \otimes G_A"'] & \\
 & \underline{\mathrm{Hom}}_{\mathcal{E}}(H_{B'}(A), H_{B'}(B)) \otimes \underline{\mathrm{Hom}}_{\mathcal{E}}(G_A(A'), G_A(B')) \arrow[uu]
\end{tikzcd}$$

Now, we take a look at natural transformations. Let $T, S\colon \mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ be two $\mathcal{K}$-functors and $\alpha\colon T \Rightarrow S$ a natural transformation between the underlying functors of T and S. Then *the following two statements are equivalent.*

1. *α is a natural transformation between $\mathcal{K}$-functors;*
2. *for each $A \in \operatorname{ob}\mathcal{C}$ and $A' \in \operatorname{ob}\mathcal{D}$, $\alpha_{A,-}\colon T(A,-) \Rightarrow S(A,-)$ and $\alpha_{-,A'}\colon T(-,A') \Rightarrow S(-,A')$ are natural transformations between $\mathcal{K}$-functors.*

9 (Opposites and presheaves) For any $\mathcal{K}$-category $\mathcal{C}$, there is a $\mathcal{K}$-category $\mathcal{C}^{\mathrm{opp}}$ whose underlying category is the opposite of $\mathcal{C}$. Indeed, for every two objects A, B in $\mathcal{C}$, the hom-object $\underline{\mathrm{Hom}}_{\mathcal{C}^{\mathrm{opp}}}(A, B)$ is defined as the object $\underline{\mathrm{Hom}}_{\mathcal{C}}(B, A)$. One can see $(\mathcal{C} \otimes \mathcal{D})^{\mathrm{opp}} = \mathcal{C}^{\mathrm{opp}} \otimes \mathcal{D}^{\mathrm{opp}}$.

A $\mathcal{K}$-functor from $\mathcal{C}^{\mathrm{opp}}$ is called a **$\mathcal{K}$-presheaf** on $\mathcal{C}$.

10 Example (Hom bifunctor) Let $\mathcal{C}$ be a $\mathcal{K}$-category, then $\underline{\mathrm{Hom}}_{\mathcal{C}}(-,-)$ is a $\mathcal{K}$-functor $\mathcal{C}^{\mathrm{opp}} \otimes \mathcal{C} \to \mathcal{K}$. Indeed, fixing $A \in \operatorname{ob}\mathcal{C}$, then $\underline{\mathrm{Hom}}_{\mathcal{C}}(A,-)$ is a $\mathcal{K}$-functor and $\underline{\mathrm{Hom}}_{\mathcal{C}}(-,A)$ is a $\mathcal{K}$-presheaf. Furthermore, one can see they give rise to a $\mathcal{K}$-functor $\underline{\mathrm{Hom}}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{opp}} \otimes \mathcal{C} \to \mathcal{K}$.

Now, we consider the composition

$$\mathcal{C}^{\mathrm{opp}} \times \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{opp}} \otimes \mathcal{C} \xrightarrow{\underline{\mathrm{Hom}}_{\mathcal{C}}} \mathcal{K} \xrightarrow{K} \mathbf{Set}\,.$$

Since the partial functors of the composition of the first two are precisely the partial functors of $\underline{\mathrm{Hom}}_{\mathcal{C}}$ and that

$$K \circ \underline{\mathrm{Hom}}_{\mathcal{C}}(A,-) = \mathrm{Hom}_{\mathcal{C}}(A,-), \qquad K \circ \underline{\mathrm{Hom}}_{\mathcal{C}}(-,A) = \mathrm{Hom}_{\mathcal{C}}(-,A),$$

we see the resulted composition is $\mathrm{Hom}_{\mathcal{C}}(-,-)$.

11 Example (Tensor product functor) The tensor product in $\mathcal{K}$ gives rise to a $\mathcal{K}$-functor $\mathrm{Ten}\colon \mathcal{K}\otimes\mathcal{K} \to \mathcal{K}$. Here $\mathrm{Ten}(A,B) := A\otimes B$ and the morphism

$$\underline{\mathrm{Hom}}(A,A') \otimes \underline{\mathrm{Hom}}(B,B') \longrightarrow \underline{\mathrm{Hom}}(A\otimes B, A'\otimes B')$$

is given as the one corresponding under the *Tensor-Hom adjunction* to the composition

$$\begin{aligned} &(\underline{\mathrm{Hom}}(A,A') \otimes \underline{\mathrm{Hom}}(B,B')) \otimes (A\otimes B) \\ \longrightarrow &(\underline{\mathrm{Hom}}(A,A') \otimes A) \otimes (\underline{\mathrm{Hom}}(B,B') \otimes B) \\ \longrightarrow &A' \otimes B'. \end{aligned}$$

One can see the composition

$$\mathcal{K}\times\mathcal{K} \longrightarrow \mathcal{K}\otimes\mathcal{K} \xrightarrow{\mathrm{Ten}} \mathcal{K}$$

is precisely the bifunctor $-\otimes-$. Note that the natural transformations in the symmetric monoidal structure on $\mathcal{K}$ are thus the $\mathcal{K}$-natural transformations in its $\mathcal{K}$-enriched symmetric monoidal structure.

12 Lemma (Hom of tensor products) *Let $\mathcal{C},\mathcal{D}$ be two $\mathcal{K}$-categories, then the composition*

$$\mathcal{C}^{\mathrm{opp}} \otimes \mathcal{C} \otimes \mathcal{D}^{\mathrm{opp}} \otimes \mathcal{D} \xrightarrow{\underline{\mathrm{Hom}}_{\mathcal{C}} \otimes \underline{\mathrm{Hom}}_{\mathcal{D}}} \mathcal{K}\otimes\mathcal{K} \xrightarrow{\mathrm{Ten}} \mathcal{K}$$

is $\underline{\mathrm{Hom}}_{\mathcal{C}\otimes\mathcal{D}}$.

Representability

13 Theorem (Weak Yoneda lemma) *Let $\mathcal{K}$ be a right closed monoidal category and $\mathcal{C}$ be a $\mathcal{K}$-category. There is a canonical bijection*

$$\begin{aligned} \mathrm{Nat}(\underline{\mathrm{Hom}}_{\mathcal{C}}(A,-),F) &\longrightarrow \mathrm{Hom}_{\mathcal{K}}(\mathbf{1},F(A)) \\ \alpha &\longmapsto \Big(\mathbf{1} \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(A,A) \xrightarrow{\alpha_A} F(A),\Big) \end{aligned}$$

which is natural in both the object $A\in \mathrm{ob}\,\mathcal{C}$ and the $\mathcal{K}$-functor $F\colon \mathcal{C}\to\mathcal{K}$.

Proof: Let $x\colon \mathbf{1} \to F(A)$ be an element of $F(A)$, then the following composition

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(A,B) \xrightarrow{F} \underline{\mathrm{Hom}}(F(A),F(B)) \xrightarrow{x} \underline{\mathrm{Hom}}(\mathbf{1},F(B)) \xrightarrow{\mathrm{ev}} F(B)$$

gives rise to a natural transformation α from $\underline{\mathrm{Hom}}_{\mathcal{C}}(A,-)$ to F.

This corresponding gives the inverse of the map described in the statement. Indeed, $\alpha_A \circ \mathrm{id}_A$ equals x by the is the following commutative diagrams.

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "F"] \arrow[d] & \mathbf{1} \arrow[r, "x"] \arrow[d] & F(A) \arrow[d] \arrow[dr, equal] & \\
\underline{\mathrm{Hom}}_{\mathcal{C}}(A,A) \arrow[r, "F"] & \underline{\mathrm{Hom}}(F(A),F(A)) \arrow[r, "x"] & \underline{\mathrm{Hom}}(\mathbf{1},F(A)) \arrow[r, "\mathrm{ev}"] & F(A)
\end{tikzcd}$$

Conversely, given a natural transformation α from $\underline{\mathrm{Hom}}_{\mathcal{C}}(A,-)$ to F, if it corresponds to $x\colon \mathbf{1} \to F(A)$. Then, the natural transformation corresponding to x is the whole composition of the following commutative diagrams.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{C}}(A,B) \arrow[r, "F"] \arrow[d, "\underline{\mathrm{Hom}}_{\mathcal{C}}(A,-)"'] & \underline{\mathrm{Hom}}(F(A),F(B)) \arrow[d, "\alpha_A"] \\
\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}_{\mathcal{C}}(A,A),\underline{\mathrm{Hom}}_{\mathcal{C}}(A.B)) \arrow[r, "\alpha_B"] \arrow[d, "\mathrm{id}_A"'] & \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}_{\mathcal{C}}(A,A),F(B)) \arrow[d, "\mathrm{id}_A"] \\
\underline{\mathrm{Hom}}(\mathbf{1},\underline{\mathrm{Hom}}_{\mathcal{C}}(A.B)) \arrow[r, "\alpha_B"] \arrow[d, "\mathrm{ev}"'] & \underline{\mathrm{Hom}}(\mathbf{1},F(B)) \arrow[d, "\mathrm{ev}"] \\
\underline{\mathrm{Hom}}_{\mathcal{C}}(A,B) \arrow[r, "\alpha_B"] & F(B)
\end{tikzcd}$$

Since the composition of the left verticals is the identity, the whole composition is α_B as desired. ☐

14 **(Representable functors)** A $\mathcal{K}$-functor (resp. $\mathcal{K}$-presheaf) F is said to be **representable** if there is a natural isomorphism from $\underline{\mathrm{Hom}}_{\mathcal{C}}(A,-)$ (resp. $\underline{\mathrm{Hom}}_{\mathcal{C}}(-,A)$) to F. Such a natural isomorphism is called a **representation** and we say F is *represented by A*. By the weak Yoneda lemma, a representation can also be written as a pair (A,x) of an object $A \in \operatorname{ob}\mathcal{C}$ and a morphism $x\colon \mathbf{1} \to F(A)$ such that the corresponding natural transformation is invertible,

15 **Lemma** *Let $\mathcal{K}$ be a right closed monoidal category, $\mathcal{C}$ be a $\mathcal{K}$-category and $F\colon \mathcal{C} \to \mathcal{K}$ be a $\mathcal{K}$-functor. Given two representations (A,x) and (B,y) of F, there exists a unique morphism $f\colon A \to B$ in the underlying category of $\mathcal{C}$ such that $F(f) \circ x = y$.*

16 **(Adjunctions)** A pair of $\mathcal{K}$-functors $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ is called a **adjoint pair** or **adjunction** if they admit two natural transformations $\eta\colon \mathrm{id} \Rightarrow RL$ and $\epsilon\colon LR \Rightarrow \mathrm{id}$ satisfy the **triangle identities**.

$$\begin{tikzcd}
& LRL \arrow[dr, "\epsilon * L"] & \\
L \arrow[ur, "L * \eta"] \arrow[rr, "\mathrm{id}"] & & L
\end{tikzcd}
\qquad
\begin{tikzcd}
& RLR \arrow[dr, "R * \epsilon"] & \\
R \arrow[ur, "\eta * R"] \arrow[rr, "\mathrm{id}"] & & R
\end{tikzcd}$$

If this is the case, then the compositions

$$\underline{\mathrm{Hom}}_{\mathcal{D}}(L(A),B) \xrightarrow{R} \underline{\mathrm{Hom}}_{\mathcal{C}}(RL(A),R(B) \xrightarrow{\underline{\mathrm{Hom}}_{\mathcal{C}}(\eta_A,R(B))} \underline{\mathrm{Hom}}_{\mathcal{C}}(A,R(B))$$

and

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(A,R(B)) \xrightarrow{L} \underline{\mathrm{Hom}}_{\mathcal{D}}(L(A),LR(B) \xrightarrow{\underline{\mathrm{Hom}}_{\mathcal{C}}(L(A),\epsilon_B)} \underline{\mathrm{Hom}}_{\mathcal{D}}(L(A),B)$$

give rise to a mutually inverse pair of natural transformations:

$$\underline{\mathrm{Hom}}_{\mathcal{D}}(L(-),-) \cong \underline{\mathrm{Hom}}_{\mathcal{C}}(-,R(-)).$$

Conversely, if there are such a mutually inverse pair of natural transformations

$$\underline{\mathrm{Hom}}_{\mathcal{D}}(L(-),-) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \underline{\mathrm{Hom}}_{\mathcal{C}}(-,R(-)),$$

then the compositions

$$I \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{D}}(L(A),L(A)) \xrightarrow{\alpha_{A,L(A)}} \underline{\mathrm{Hom}}_{\mathcal{C}}(A,RL(A))$$

and

$$I \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(R(B),R(B)) \xrightarrow{\beta R(B),B} \underline{\mathrm{Hom}}_{\mathcal{D}}(LR(B),B)$$

give rise to a pair of natural transformations $\eta\colon \mathrm{id} \Rightarrow RL$ and $\epsilon\colon LR \Rightarrow \mathrm{id}$ satisfy the **triangle identities**.

17 Example (Enriched Tensor-Hom adjunction) Recall (Example 5) that in a right closed monoidal category $\mathcal{K}$, there are adjunction

$$-\otimes A \dashv \underline{\mathrm{Hom}}(A,-)$$

of functors. But each of them induces a $\mathcal{K}$-functor.

For $-\otimes A$, the morphism

$$\underline{\mathrm{Hom}}(B,C) \longrightarrow \underline{\mathrm{Hom}}(B\otimes A, C\otimes A)$$

is the one corresponding to the evaluation

$$\underline{\mathrm{Hom}}(B,C)\otimes B\otimes A \xrightarrow{\mathrm{ev}\otimes A} C\otimes A$$

under the *Tensor-Hom adjunction*.

For $\underline{\mathrm{Hom}}(A,-)$, the morphism

$$\underline{\mathrm{Hom}}(B,C) \longrightarrow \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(A,B),\underline{\mathrm{Hom}}(A,C))$$

is the one corresponding to the composition operation

$$\underline{\mathrm{Hom}}(B,C)\otimes\underline{\mathrm{Hom}}(A,B) \xrightarrow{\bullet} \underline{\mathrm{Hom}}(A,C)$$

under the *Tensor-Hom adjunction*.

Furthermore, the above two $\mathcal{K}$-functors form an adjunction.

18 Lemma (Equivalence of $\mathcal{K}$-categories) *Let $\mathcal{K}$ be a monoidal category and $F : \mathcal{C} \to \mathcal{D}$ a $\mathcal{K}$-functor. Then the followings are equivalent:*

1. *F is fully faithful and essentially surjective;*
2. *F admits a left or right adjoint $\mathcal{K}$-functor such that the unit and the counit are natural transformations;*
3. *there exist an adjunction $L \dashv R$ of $\mathcal{K}$-functors and natural isomorphisms $\mathrm{id}_{\mathcal{C}} \cong LF$ and $FR \cong \mathrm{id}_{\mathcal{D}}$.*

Actions

Index

Bibliography

[Stacks] The Stacks Project Authors, *Stacks Project*, 2015.

[RV15] Ravi Vakil, *THE RISING SEA: Foundations of Algebraic Geometry*, 2015.

[Milne] James S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, 1980.

[Pri03] H. A. Priestley, *Introduction to complex analysis*, 2nd ed., Oxford University Press, Oxford, 2003.

[Kel05] G. M. Kelly, *Basic concepts of enriched category theory*, Repr. Theory Appl. Categ. **10** (2005), vi+137. Reprint of the 1982 original.

[Bor94] Francis Borceux, *Handbook of categorical algebra. 1*, Encyclopedia of Mathematics and its Applications, vol. 50, Cambridge University Press, Cambridge, 1994. Basic category theory.

[Joh02] Peter T. Johnstone, *Sketches of an elephant: a topos theory compendium. Vol. 2*, Oxford Logic Guides, vol. 44, The Clarendon Press, Oxford University Press, Oxford, 2002.

[Kar04] Panagis Karazeris, *Notions of flatness relative to a Grothendieck topology*, Theory Appl. Categ. **12** (2004), No. 5, 225–236.

[Mar09] Florian Marty, *Des ouverts Zariski et des morphismes lisses en géométrie relative*, Ph.D. Thesis, Université Toulouse III-Paul Sabatier, 2009.

[Lur04] J. Lurie, *Tannaka Duality for Geometric Stacks*, ArXiv Mathematics e-prints (2004), available at `math/0412266`.