Note on

Chiral algebras

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§ 1 Categories

Multicategories and pseudo-tensor categories

Pseudo-tensor category = symmetric multicategory.

1.1 (Motivation from monoidal categories) In a monoidal category, as the string diagram language suggests, we can view a morphism from a tensor product as a *multimorphism*. This motivates the notion of multicategories.

Note that

1. The coherence theorem for monoidal category guarantees that the (ordered) tensor product $X_1 \otimes \cdots X_n$ is well-defined up to a unique isomorphism.

2. The coherence theorem for symmetric monoidal category guarantees that the (unordered) tensor product $\bigotimes_{i \in I} X_i$ is well-defined up to a unique isomorphism.

Therefore we consider the following two *suitable categories of indexes*:

$\mathcal{S}1$ The category of natural numbers[1] and maps between them.

$\mathcal{S}2$ The category of finite sets and maps between them.

Remark To simply notation, we use the following alternative notation for natural numbers:

$$\underline{n} := \{1, 2, \cdots, n\}.$$

If there is no ambiguity, we simply write n instead of $\underline{n}$. More precisely, we will never write $i \in n$.

1.2 (Indexes) Recall that a family of sets indexed by an set I is actually a mapping which associating each $i \in I$ a set. In general, *a family of objects* in a category $\mathcal{C}$ indexed by an other category I is a functor from I to $\mathcal{C}$. The category of functors from I to $\mathcal{C}$ is denoted by $\mathcal{C}^I$.

1.2.1 Example Let n be a natural number then a *n-indexed family of objects* of $\mathcal{C}$ is a list $X_1, X_2, \cdots, X_n$ of objects of $\mathcal{C}$.

1.2.2 Example Let I be a finite set, then a *I-indexed family of objects* of $\mathcal{C}$ is a mapping which associating each $i \in I$ an object $X_i \in \mathcal{C}$.

[1] We follow *von Neumann's construction*:

$$0 := \emptyset, 1 := \{0\}, \cdots, n := \{0, 1, \cdots, n-1\}.$$

Note that the natural numbers are naturally ordinals under this construction.

1.3 **(Category of indexes)** From now on, we fix a *suitable category of indexes*, naming either $\mathcal{S}1$ or $\mathcal{S}2$. Note that in both case we have: for $I \in \mathcal{S}$, if $|I| = n$, then $\mathcal{S}(I, I) = \mathbb{S}_n$, the n-th symmetric group.

Then critical difference between $\mathcal{S}1$ and $\mathcal{S}2$ is that a natural number is an ordinal, i.e. it carries a canonical linear order on it, while a finite set doesn't. Another difference is that $\mathcal{S}2$ is closed under usual set-theoretic constructions while $\mathcal{S}1$ is not. However, we still have corresponding constructions.

1.3.1 **Example** The Cartesian product of two finite sets is again a finite set, while of two natural numbers m, n are not a natural numbers. However, by putting the lexicography order on $m \times n$, we obtain a canonical isomorphism from $m \times n$ to the natural number mn.

1.3.2 **Example** Let $\{m_i\}_{i \in \underline{n}}$ be a n-indexed family of natural numbers. Then their n-indexed Cartesian product $\prod_{i \in \underline{n}} m_i = \{(k_1, \cdots, k_n) | k_i \in \underline{m_i}\}$ is not a natural number. By putting the lexicography order on it, we obtain a canonical isomorphism from it to the natural number $m_1 m_2 \cdots m_n$.

1.3.3 **Example** The disjoint union $\coprod_{i \in \underline{n}} m_i = \{(i, k) | i \in \underline{n}, k \in \underline{m_i}\}$ is not a natural number. By putting lexicography order on it, we obtain a canonical isomorphism from it to the natural number $m_1 + m_2 + \cdots + m_n$.

1.3.4 **Example** Given a map $\pi \colon m \to n$, the preimage set $\pi^{-1}(i)$ of $i \in \underline{n}$ is not a natural number. However, since it is a subset of a natural number, it has a canonical linear order, which gives us an isomorphism to the natural number $|\pi^{-1}(i)|$.

1.3.5 **Remark** The reason why there are such constructions is follows. Since any natural number is a finite set, we have the following inclusion of categories

$$\mathcal{S}1 \longrightarrow \mathcal{S}2.$$

Note that it is an equivalence of category. Then the usual set-theoretic constructions in $\mathcal{S}2$ are actually certain limits/colimits in it and the corresponding constructions in $\mathcal{S}1$ are corresponding limits/colimits in it.

1.3.6 **Remark** In both $\mathcal{S}1$ and $\mathcal{S}2$, $\underline{1}$ is a terminal object. We fix it as *the* terminal object. In particular, we identify $\mathcal{C}^1$ with $\mathcal{C}$ for any category $\mathcal{C}$. Note that I haven't identify $\mathcal{C}$ with every $\mathcal{C}^I$ where I is a singleton yet. However, since the isomorphism between a singleton and $\underline{1}$ is unique, we can always do so.

1.4 Recall that a category $\mathcal{C}$ consists of

- A class of *objects* $\mathcal{C}_0$.
- For each pair of objects $X, Y \in \mathcal{C}_0$, a *Hom set* $\mathcal{C}(X, Y)$. In other words, a mapping $\mathcal{C}_0 \times \mathcal{C}_0 \longrightarrow \mathbf{Set}_0$. Note that the Hom set construction can be extended to a bifunctor $\mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow \mathbf{Set}$ through composition.

- For each triple of objects $X, Y, Z \in \mathcal{C}_0$, a map

$$\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X, Z),$$

called the *composition*. Note that when we extend the Hom sets to a bifunctor, then the composition become a *dinatural transformation*

$$\mathrm{Hom} \times \mathrm{Hom} \Longrightarrow \mathrm{Hom}\,.$$

Remark For $F\colon \mathcal{A} \times \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{D}, G\colon \mathcal{A} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ two functors, a **dinatural** transformation (or more precisely natural in $\mathcal{A}$ and dinatural in $\mathcal{B}$ and $\mathcal{C}$) is a collection of $\mathcal{E}$-morphisms

$$\alpha_{a,b,c}\colon F(a, b, b) \longrightarrow G(a, c, c),$$

such that for any $f\colon a \to a'$ in $\mathcal{A}$, $g\colon b \to b'$ in $\mathcal{B}$ and $h\colon c \to c'$ in $\mathcal{C}$, we have the following commutative diagrams

$$\begin{aligned} \alpha_{a',b,c} \circ F(f, 1, 1) &= G(f, 1, 1) \circ \alpha_{a,b,c}, \\ \alpha_{a,b',c} \circ F(1, g, 1) &= \alpha_{a,b,c} \circ F(1, 1, g), \\ G(1, 1, h) \circ \alpha_{a,b,c'} &= G(1, h, 1) \circ \alpha_{a,b,c}. \end{aligned}$$

Remark Mappings are functors: indeed, a mapping to the class of objects of a category $\mathcal{V}$ can be viewed as a functor from a discrete category to $\mathcal{V}$. Then it make sense to talk about the morphisms between mappings, which are called *transformations* as there is no need of naturalness.

- For each object $X \in \mathcal{C}_0$, an *identity* $\mathrm{id}_X \in \mathcal{C}(X, X)$. In other words, a transformation $\underline{1} \Rightarrow \mathrm{Hom} \circ \Delta$, where Δ is the diagonal mapping $\Delta\colon \mathcal{C}_0 \to \mathcal{C}_0 \times \mathcal{C}_0$ and $\underline{1}$ is the constant mapping which maps each $X \in \mathcal{C}_0$ to the terminal object $\underline{1} \in \mathbf{Set}$.

and satisfies the associativity and unity.

1.5 A **$\mathcal{S}$-multicategory**[2] $\mathcal{M}$ consists of the following data:

- A class of *objects* $\mathcal{M}_0$.
- For any $I \in \mathcal{S}$, a mapping $P_I = P_I^{\mathcal{M}}\colon (\mathcal{M}_0^I)^{\mathrm{op}} \times \mathcal{M}_0 \to \mathbf{Set}$, called the set of ***I*-indexed multimorphisms**.

Remark For any $\mathcal{S}$-morphism $\pi\colon J \to I$, we can define a mapping $P_\pi\colon (\mathcal{M}_0^J)^{\mathrm{op}} \times \mathcal{M}_0^I \to \mathbf{Set}^I$ as follows. First, put $J_i := \pi^{-1}(i)$. Then putting the mappings P_{J_i} together, we get the desired mapping.

[2]This is not a standard name.

Remark For $f \in P_I(\{M_i\}_{i\in I}, N)$, we write $f\colon \{M_i\}_{i\in I} \to N$. For $f \in P_n((M_1, \cdots, M_n), N)$, we write $f\colon M_1, \cdots, M_n \to N$.

- For any $\mathcal{S}$-morphism $\pi\colon J \to I$, a dinatural transformation (between mappings), called the **composition**

$$\circ_\pi\colon P_I \times P_\pi \Longrightarrow P_J,$$

where the mapping $P_I \times P_\pi$ is the composition

$$(\mathcal{M}_0^I)^{\mathrm{op}} \times \mathcal{M}_0 \times (\mathcal{M}_0^J)^{\mathrm{op}} \times \mathcal{M}_0^I \xrightarrow{P_I \times P_\pi} \mathbf{Set} \times \mathbf{Set}^I \xrightarrow{\prod} \mathbf{Set}.$$

Remark For $f_i \in P_{J_i}(\{L_j\}_{j\in J_i}, M_i)$ and $g \in P_I(\{M_i\}_{i\in I}, N)$, we write $g \circ \{f_i\}_{i\in I}$ instead of $\circ_\pi(g, \{f_i\}_{\in I})$.

Remark For any two $\mathcal{S}$-morphisms $\pi\colon J \to I$ and $\rho\colon H \to J$, we can define a transformation $\circ_{\pi,\rho}\colon P_\pi \times P_\rho \Rightarrow P_{\pi\circ\rho}$ as follows. First, separate ρ as collection of $\mathcal{S}$-morphisms $\rho_i\colon H_i \to J_i$, where $H_i = \rho^{-1}(J_i) = (\pi \circ \rho)^{-1}(i)$. Then putting the transformations $\circ_{\rho_i}$ together, we get the desired transformation.

- For any one element index set $I \in \mathcal{S}$, a dinatural transformation, called the ***I*-indexed identity**

$$\mathbf{1}_I\colon \underline{1} \Longrightarrow P_I,$$

where $\underline{1}$ is the trivial mapping and P_I is viewed as a mapping from $\mathcal{M}_0^{\mathrm{op}} \times \mathcal{M}_0$ via the identification $\mathcal{M}_0^I \cong \mathcal{M}_0$.

Remark Hence for each $|I| = 1$ and $M \in \mathcal{M}_0$, the I-indexed identity specifies an element in $P_I(\{M\}_{i\in I}, M)$, denoted by 1_M^I.

Those data are subject to the following axioms:

1. The composition is *associative*, i.e. if $\pi\colon J \to I$, $\rho\colon H \to J$ are two $\mathcal{S}$-morphisms, then the following diagram commutes:

$$\begin{tikzcd}
P_I \times P_\pi \times P_\rho \arrow[r, "\circ_\pi"] \arrow[d, "\circ_{\pi,\rho}"'] & P_J \times P_\rho \arrow[d, "\circ_\rho"] \\
P_I \times P_{\pi\circ\rho} \arrow[r, "\circ_{\pi\circ\rho}"] & P_H
\end{tikzcd}$$

2. The identity works as *left identity* of composition, i.e. if $|I| = 1$, $J \in \mathcal{S}$ and $\pi\colon J \to I$ is the unique map, then the following diagram commutes.

$$\begin{tikzcd}
\underline{1} \times P_J \arrow[r, "\cong"] \arrow[d, "\mathbf{1}_I"'] & P_J \\
P_I \times P_J \arrow[r, equal] & P_I \times P_\pi \arrow[u, "\circ_\pi"']
\end{tikzcd}$$

3. The identity works as *right identity* of composition, i.e. if $I \in \mathcal{S}$ and $\pi\colon I \to I$ is the identity, then the following diagram commutes.

$$\begin{tikzcd}
P_I \times \underline{1}^I \arrow[r, "\cong"] \arrow[d, "\mathbf{1}_I"'] & P_I \\
P_I \times P_\pi \arrow[ur, "\circ_\pi"'] &
\end{tikzcd}$$

A **map of $\mathcal{S}$-multicategories** $F\colon \mathcal{M} \to \mathcal{N}$ consists of the following data:

- A mapping $F\colon \mathcal{M}_0 \to \mathcal{N}_0$.
- For any $I \in \mathcal{S}$, a F-transformation[3] $F_I\colon P_I^{\mathcal{M}} \Rightarrow P_I^{\mathcal{N}}$, which is the same as a transformation from $P_I^{\mathcal{M}}$ to the composition of $P_I^{\mathcal{N}}$ with the following mapping

$$F^I \times F\colon \mathcal{M}_0^I \times \mathcal{M}_0 \longrightarrow \mathcal{N}_0^I \times \mathcal{N}_0.$$

Remark One can define a F-transformation F_π from F_Is.

Those data are subject to the axiom that for any $\mathcal{S}$-morphism $\pi\colon J \to I$, the following diagram commutes.

$$\begin{tikzcd}
P_I^{\mathcal{M}} \times P_\pi^{\mathcal{M}} \arrow[r, "\circ_\pi"] \arrow[d, "F_I \times F_\pi"'] & P_J^{\mathcal{M}} \arrow[d, "F_J"] \\
P_I^{\mathcal{N}} \times P_\pi^{\mathcal{N}} \arrow[r, "\circ_\pi"] & P_J^{\mathcal{N}}
\end{tikzcd}$$

A **transformation** between maps of $\mathcal{S}$-multicategories is a transformation $\alpha\colon F \Rightarrow G$ such that the following diagram commutes.

$$\begin{tikzcd}
P_I^{\mathcal{M}} \arrow[r, "F"] \arrow[d, "G"'] & P_J^{\mathcal{N}} \circ ((F^I)^{\mathrm{op}} \times F) \arrow[d] \\
P_I^{\mathcal{N}} \circ ((G^I)^{\mathrm{op}} \times G) \arrow[r] & P_J^{\mathcal{N}} \circ ((F^I)^{\mathrm{op}} \times G)
\end{tikzcd}$$

where the right is the transformation obtained by apply the mapping $P_J^{\mathcal{N}} \circ (F^I \times -)$ to components of α, and the bottom is the transformation obtained by apply the mapping $P_J^{\mathcal{N}} \circ (-^I \times G)$ to components of α.

The transformations compose in obvious ways just as usual horizontal and vertical compositions of natural transformations. This makes the category $\mathcal{S}$**MC** of $\mathcal{S}$-multicategory a strict 2-category.

[3]Recall that, if we have functors $F\colon \mathcal{A} \to \mathcal{B}$, $P\colon \mathcal{A} \to \mathcal{C}$ and $Q\colon \mathcal{B} \to \mathcal{C}$, then a natural F-transformation from P to Q is a natural transformation from P to $Q \circ F$.

When we choose $\mathcal{S}$ to be $\mathcal{S}1$, a $\mathcal{S}$-multicategory is called a **multicategory/non-symmetric pseudo-tensor category**. The 2-category $\mathcal{S}\mathbf{MC}$ is simply denoted by $\mathbf{MC}$ in this case. When we choose $\mathcal{S}$ to be $\mathcal{S}2$, a $\mathcal{S}$-multicategory is called a **symmetric multicategory/pseudo-tensor category**. The 2-category $\mathcal{S}\mathbf{MC}$ is denoted by $\mathbb{S}\mathbf{MC}$ in this case.

1.5.1 Recall that for $\mathcal{C}$ a category and $\mathcal{V}$ another category, a $\mathcal{V}$*-valued presheaf* on $\mathcal{C}$ is a contravariant functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{V}$. The category of $\mathcal{V}$-valued presheaves on $\mathcal{C}$ is denoted by $\mathcal{V}_{\mathcal{C}}$, comparing with the notation of the category $\mathcal{V}^{\mathcal{C}}$ of functors from $\mathcal{C}$ to $\mathcal{V}$. For instance, P_I can be viewed as a $\mathbf{Set}^{\mathcal{M}_0}$-valued presheaf on $\mathcal{M}_0^I$.

Given a category I and an I-indexed family of categories $\mathcal{C}_i$, the **union** $\bigsqcup_I \mathcal{C}_i$ of this family is the category

- whose objects are pairs (i, X) where $i \in I$ and $X \in \mathcal{C}_i$.
- whose morphisms are pairs $(\pi, f)\colon (i, X) \to (j, Y)$ where $\pi\colon j \to i$ is a morphism in I and f is a morphism from X to $\pi^*(Y)$ (here π^* is the functor $\mathcal{C}_j \to \mathcal{C}_i$ induced by π).

This category carries a forgetful functor mapping each (i, X) to $i \in I$. We simply denote (i, X) by X and call i the *index* of X.

Let $\mathcal{M}_0$ be a class. In the category $\mathcal{S}P^{\mathcal{M}_0} := \bigsqcup_{\mathcal{S}} \mathbf{Set}^{\mathcal{M}_0}_{\mathcal{M}_0^I}$, an object is just a mapping like P_I. Therefore we can be call the *category of multimorphism structures* on $\mathcal{M}_0$.

1.5.2 Recall that a representation of a groupoid $\mathcal{G}$ in a category $\mathcal{C}$ is just a functor $\mathcal{G} \to \mathcal{C}$.

Let $\mathcal{S}^c$ be the *core* of $\mathcal{S}$, i.e. the maximal subcategory of $\mathcal{S}$ consisting only isomorphisms. The following lemma is critical, especially when we choose $\mathcal{S}$ to be $\mathcal{S}2$.

1.5.3 Lemma *Let $\mathcal{M}$ be a $\mathcal{S}$-multicategory, then there is a canonical representation of $\mathcal{S}^c$ in the category of multimorphism structures on $\mathcal{M}_0$. In precise, for $I \in \mathcal{S}$, we have the object P_I. For a $\mathcal{S}$-morphism $\pi\colon J \to I$, we have a transformation $\mu_\pi\colon P_I \Rightarrow \pi^*(P_J)$, where π^* is the obvious one.*

1.5.4 It is clear that if $\mathcal{M}$ is a multicategory, then given a fixed one-element set $I \in \mathcal{S}$, $(\mathcal{M}_0, P_I, \mathbf{1}_I)$ is a category. Moreover, from the lemma, we can see that although different choices of I give different category, but all of them are isomorphic. Hence it make sense to talk about *the* **underlying category** of a $\mathcal{S}$-multicategory.

1.5.5 From the lemma, we can replace $\mathcal{S}2$ by its *skeleton*, i.e. the category obtained by identify isomorphic objects in $\mathcal{S}2$. Note that this category is isomorphic to $\mathcal{S}1$. In this way, a pseudo-tensor category is a multicategory equipped with extra structures coming from the linear orders on natural numbers.

1.5.6 Using the above fact, we have the following alternative definition:

A **pre-$\mathcal{S}$-multicategory structure** on a category $\mathcal{M}$ consists of the following data

- For any $I \in \mathcal{S}$, a bifunctor $P_I = P_I^{\mathcal{M}} \colon (\mathcal{M}^I)^{\mathrm{op}} \times \mathcal{M} \to \mathbf{Set}$, called the set of ***I*-indexed multimorphisms**.
- For any $\mathcal{S}$-morphism $\pi \colon J \to I$, a dinatural transformation, called the **composition**
$$\circ_\pi \colon P_I \times P_\pi \Longrightarrow P_J,$$
where the functor $P_I \times P_\pi$ is the composition
$$(\mathcal{M}^I)^{\mathrm{op}} \times \mathcal{M} \times (\mathcal{M}^J)^{\mathrm{op}} \times \mathcal{M}^I \xrightarrow{P_I \times P_\pi} \mathbf{Set} \times \mathbf{Set}^I \xrightarrow{\prod} \mathbf{Set}.$$
- For any one element index set $I \in \mathcal{S}$, a dinatural transformation, called the ***I*-indexed identity**
$$\mathbf{1}_I \colon \underline{1} \Longrightarrow P_I,$$
where P_I is viewed as a functor from $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$ via the identification $\mathcal{M}^I \cong \mathcal{M}$.

Those data are subject to the following axioms:

1. The composition is *associative*, i.e. if $\pi \colon J \to I$, $\rho \colon H \to J$ are two $\mathcal{S}$-morphisms, then the following diagram commutes:
$$\begin{tikzcd} P_I \times P_\pi \times P_\rho \arrow[r, "\circ_\pi"] \arrow[d, "\circ_{\pi,\rho}"'] & P_J \times P_\rho \arrow[d, "\circ_\rho"] \\ P_I \times P_{\pi\circ\rho} \arrow[r, "\circ_{\pi\circ\rho}"] & P_H \end{tikzcd}$$
2. The identity works as *left identity* of composition, i.e. if $|I| = 1$, $J \in \mathcal{S}$ and $\pi \colon J \to I$ is the unique map, then the following diagram commutes
$$\begin{tikzcd} \underline{1} \times P_J \arrow[r, "\cong"] \arrow[d, "\mathbf{1}_I"'] & P_J \\ P_I \times P_J \arrow[r, equal] & P_I \times P_\pi \arrow[u, "\circ_\pi"'] \end{tikzcd}$$
3. The identity works as *right identity* of composition, i.e. if $I \in \mathcal{S}$ and $\pi \colon I \to I$ is the identity, then the following diagram commutes.
$$\begin{tikzcd} P_I \times \underline{1}^I \arrow[r, "\cong"] \arrow[d, "\mathbf{1}_I"'] & P_I \\ P_I \times P_\pi \arrow[ur, "\circ_\pi"'] & \end{tikzcd}$$

Note that a pre-$\mathcal{S}$-multicategory structure on $\mathcal{M}$ defines a $\mathcal{S}$-multicategory $\widetilde{\mathcal{M}}$ with objects the same as $\mathcal{M}$. Moreover, a pre-$\mathcal{S}$-multicategory structure is a **$\mathcal{S}$-multicategory structure** if it satisfying the following *compatible axiom*:

> The underlying category of $\widetilde{\mathcal{M}}$ is isomorphic to $\mathcal{M}$.

Then, we have

Proposition *a $\mathcal{S}$-multicategory is equivalent to a category equipped with a $\mathcal{S}$-multicategory structure on it.*

1.5.7 Using above equivalence, a **map of $\mathcal{S}$-multicategories** $F\colon \mathcal{M} \to \mathcal{N}$ can be reformulated as the following data

- A functor $F\colon \mathcal{M} \to \mathcal{N}$.
- For any $I \in \mathcal{S}$, a natural F-transformation $F_I\colon P_I^{\mathcal{M}} \Rightarrow P_I^{\mathcal{N}}$, which is the same as a natural transformation from $P_I^{\mathcal{M}}$ to the composition of $P_I^{\mathcal{N}}$ with the following functor

$$(F^I)^{\mathrm{op}} \times F\colon (\mathcal{M}^I)^{\mathrm{op}} \times \mathcal{M} \longrightarrow (\mathcal{N}^I)^{\mathrm{op}} \times \mathcal{N}.$$

Those data are subject to the axiom that for any $\mathcal{S}$-morphism $\pi\colon J \to I$, the following diagram (where the horizontal ones are dinatural transformations and the vertical ones are natural F-transformations) commutes.

$$\begin{tikzcd}
P_I^{\mathcal{M}} \times P_\pi^{\mathcal{M}} \arrow[r, "\circ_\pi"] \arrow[d, "F_I \times F_\pi"'] & P_J^{\mathcal{M}} \arrow[d, "F_J"] \\
P_I^{\mathcal{N}} \times P_\pi^{\mathcal{N}} \arrow[r, "\circ_\pi"] & P_J^{\mathcal{N}}
\end{tikzcd}$$

Similarly, a **transformation** between maps of $\mathcal{S}$-multicategories can be reformulated as a natural transformation $\alpha\colon F \Rightarrow G$ such that the following diagram commutes.

$$\begin{tikzcd}
P_I^{\mathcal{M}} \arrow[r, "F"] \arrow[d, "G"'] & P_J^{\mathcal{N}} \circ ((F^I)^{\mathrm{op}} \times F) \arrow[d, "\alpha"] \\
P_I^{\mathcal{N}} \circ ((G^I)^{\mathrm{op}} \times G) \arrow[r, "\alpha"] & P_J^{\mathcal{N}} \circ ((F^I)^{\mathrm{op}} \times G)
\end{tikzcd}$$

where the right α denote the natural transformation obtained by apply the functor $P_J^{\mathcal{N}} \circ ((F^I)^{\mathrm{op}} \times -)$ to components of α, while the bottom α denote the natural transformation obtained by apply the functor $P_J^{\mathcal{N}} \circ ((-^I)^{\mathrm{op}} \times G)$ to components of α.

1.5.8 On a category $\mathcal{M}$, there maybe many different $\mathcal{S}$-multicategory structures. We can define a map between them as a map between $\mathcal{S}$-multicategories such that its restriction on the underlying category $\mathcal{M}$ is the identity functor. In this sense, we can talk about the *universal $\mathcal{S}$-multicategory structure* on $\mathcal{M}$. It is clear that we have an adjunction

$$\mathbf{Cat} \rightleftarrows \mathcal{S}\mathbf{MC}.$$

1.6 We have the following typical examples of multicategories/pseudo-tensor categories.

1.6.1 Example Any category $\mathcal{M}$ can be viewed as a $\mathcal{S}$-multicategory by putting P_I to empty when $|I| > 1$.

1.6.2 Example Any monoidal category $(\mathcal{M}, \otimes)$ has an *underlying multicategory*, which has the same objects with $\mathcal{M}$ and the functor P_n is given by

$$P_n(\{M_i\}_{i\in\underline{n}}, N) = \mathcal{M}(\otimes_{i\in\underline{n}} M_i, N).$$

Here the tensor product $\otimes_{i\in\underline{n}} M_i$ is defined as $(\cdots(M_1 \otimes M_2) \otimes \cdots \otimes M_n)$. Different associating settings give different multicategories. However, the coherence theorem guarantees they are equivalent.

1.6.3 Example Any symmetric monoidal category $\mathcal{M}$ has an *underlying pseudo-tensor category*, which has the same objects with $\mathcal{M}$ and the functor P_I is given by

$$P_I(\{M_i\}_{i\in I}, N) = \mathcal{M}(\otimes_{i\in I} M_i, N).$$

Here the tensor product $\otimes_{i\in I} M_i$ is defined as $M_1 \otimes M_2 \otimes \cdots \otimes M_n$ if we identify I with $\underline{n} = |I|$. Note that such an identification is equivalent to a linear order on I. Different identifications give different pseudo-tensor categories. However, the coherence theorem guarantees they are equivalent.

1.6.4 Example More generally, for $\mathcal{M}$ a monoidal category and C a collection of its objects (not necessarily closed under tensor product), there is a multicategory whose class of objects is C and sets of multimorphisms are constructed as above. This example shows that not any multicategory is an underlying multicategory of a monoidal category.

1.7 A notion related to previous examples is *representation* of multimorphism structures. For $\mathcal{M}$ a $\mathcal{S}$-multicategory, and $\{M_i\}_{i\in I}$ a I-indexed family of objects of $\mathcal{M}$, $P_I(\{M_i\}_{i\in I}, -)$ is a functor from $\mathcal{M}$ to $\mathbf{Set}$. If this functor is *represented* by an object, then that object is called the **pseudo-tensor product** of $\{M_i\}_{i\in I}$, denoted by $\bigotimes_{i\in I} M_i$. If all such kind of functors are *representable*, then we say the $\mathcal{S}$-multicategory $\mathcal{M}$ is **pre-representable**. Note that if so, for any $\mathcal{S}$-morphism $\pi\colon J \to I$, we have the morphisms

$$\epsilon_\pi\colon \bigotimes_{j\in J} M_j \longrightarrow \bigotimes_{i\in I}(\bigotimes_{j\in J_i} M_j)$$

naturally in $\{M_j\}_{j\in J}$. In particular, we have the natural morphisms

$$(M_1 \otimes M_2) \otimes M_3 \longrightarrow M_1 \otimes (M_2 \otimes M_3).$$

However, there is no guarantee that the above is a natural isomorphism, see Leinster's book.

1.7.1 Let $\{M_i\}_{i\in I}$ be an object in $\mathcal{M}^I$, we say a $\mathcal{S}$-multimorphism $u\colon \{M_i\}_{i\in I} \to N$ is *pre-universal* if for any multimorphism

$$f\colon \{M_i\}_{i\in I} \to N'$$

there is a unique morphism

$$\tilde{f}\colon N \to N'$$

such that $f = \tilde{f} \circ u$.

It is clear that for any family $\{M_i\}_{i\in I}$ of objects of $\mathcal{M}$, the pseudo-tensor product $\bigotimes_{i\in I} M_i$, if it exists, gives a pre-universal multimorphism $u\colon \{M_i\}_{i\in I} \to \bigotimes_{i\in I} M_i$ and conversely, if $u\colon \{M_i\}_{i\in I} \to N$ is a pre-universal multimorphism, then N represents the functor $P_I(\{M_i\}_{i\in I}, -)$.

1.7.2 We say a $\mathcal{S}$-multicategory $\mathcal{M}$ is **representable** if it is pre-representable and composition of pre-universal multimorphisms is again pre-universal.

1.7.3 Let $I, J \in \mathcal{S}$ and $p\colon I \to J$ a $\mathcal{S}$-morphism factoring through $\underline{1}$. Define $J \vee_p I$ as the disjoint union of J-indexed $\mathcal{S}$-objects U_j where $U_j = \underline{1}$ unless $j \in p(I)$, denote the element of J which p maps I to, define $U_p = I$. Let $\pi\colon J \vee_p I \to J$ be the projection and $\rho\colon I \to J \vee_p I$ the inclusion. We denote $g \circ \{f_j\}_{j\in J}$ by $g \circ_p f_p$ if $f_j = 1$ for $j \notin p(I)$.

In particular, for $\mathcal{S} = \mathcal{S}2$, $J \vee_p I$ is obtained by replacing $p \in J$ by elements of I; for $\mathcal{S} = \mathcal{S}1$, $J \vee_p I$ is obtained by inset I into $p \in J$ and reordering to make it a natural number.

1.7.4 Let $\{M_i\}_{i\in I}$ be an object in $\mathcal{M}$, we say a multimorphism $u\colon \{M_i\}_{i\in I} \to N$ is *universal* if for any $p\colon I \to J$ a $\mathcal{S}$-morphism factoring through $\underline{1}$ and any multimorphism

$$f\colon \{M'_j\}_{j\in J\vee_p I} \to N'$$

with $M'_{\rho(i)} = M_i$, there is a unique multimorphism

$$\tilde{f}\colon \{M'_j\}_{j\in J} \to N'$$

where $M'_p = N$, such that $f = \tilde{f} \circ_p u$.

Now, we have a theorem

Theorem *For a multicategory $\mathcal{M}$, the following conditions are equivalent:*

1. *$\mathcal{M}$ is the underlying multicategory of some monoidal category.*
2. *$\mathcal{M}$ is representable.*
3. *Any family of objects of $\mathcal{M}$ is a domain of some universal multimorphism.*

Similar equivalence holds for pseudo-tensor categories.

1.7.5 It is not difficult to extend the underlying $\mathcal{S}$-multicategory construction to a faithful functor V from the category of monoidal categories/symmetric monoidal categories to the category of multicategories/pseudo-tensor categories.

Conversely, if $\mathcal{M}, \mathcal{N}$ are two pre-representable $\mathcal{S}$-multicategories, then a map between them $F\colon \mathcal{M} \to \mathcal{N}$ is precisely a functor $F\colon \mathcal{M} \to \mathcal{N}$ together with morphisms

$$\nu_I\colon \bigotimes_{i\in I} F(M_i) \longrightarrow F(\bigotimes_{i\in I} M_i)$$

naturally in $\{M_j\}_{j\in J}$ for each $I \in \mathcal{S}$ such that for any $\mathcal{S}$-morphism $\pi\colon J \to I$, the following diagram commutes.

$$\begin{tikzcd}
\bigotimes_{j\in J} F(M_j) \arrow[r, "\nu_J"] \arrow[d, "\epsilon_\pi"'] & F(\bigotimes_{j\in J} M_j) \arrow[dd, "F(\epsilon_\pi)"] \\
\bigotimes_{i\in I}(\bigotimes_{j\in J_i} F(M_j)) \arrow[d, "\otimes_{i\in I}\nu_{J_i}"'] & \\
\bigotimes_{i\in I} F(\bigotimes_{j\in J_i} M_j) \arrow[r, "\nu_I"] & F(\bigotimes_{i\in I}(\bigotimes_{j\in J_i} M_j))
\end{tikzcd}$$

Therefore any map of representable $\mathcal{S}$-multicategories is obtained from a (symmetric) lax functor between the corresponding (symmetric) monoidal categories. This maybe the reason why the maps of $\mathcal{S}$-multicategories is also called **pseudo-tensor functors**. Moreover, we have

Theorem *For a map of representable multicategories $F\colon \mathcal{M} \to \mathcal{N}$, the following conditions are equivalent:*

1. *$F = V(P)$ with P a monoidal functor.*
2. *F preserves universal multimorphisms.*

Likewise, for a map of representable pseudo-tensor categories $F\colon \mathcal{M} \to \mathcal{N}$, the following conditions are equivalent:

1. *$F = V(P)$ with P a symmetric monoidal functor.*

2. *F preserves universal multimorphisms.*

1.7.6 The functor V from the category of monoidal categories to the category of multicategories has a left adjoint defined as follows. For $\mathcal{M}$ a multicategory. The **strict monoidal category generated by $\mathcal{M}$** is the strict monoidal category $\mathcal{M}^{\otimes}$ whose objects are lists of objects of $\mathcal{M}$, whose morphisms are obtained from the multimorphisms in $\mathcal{M}$ in the obvious way and whose monoidal structure is given by concatenation of lists.

Similar construction, called the **symmetric monoidal category generated by $\mathcal{M}$**, exists for a pseudo-tensor category $\mathcal{M}$.

1.8 Recall that if $\{\mathcal{C}_i\}_{i\in I}$ is a discrete family of categories, then we have the following two constructions.

1. The **product** $\prod_{i\in I}\mathcal{C}_i$ of them whose objects are families $(X_i)_{i\in I}$ with $X_i \in \mathcal{C}_i$ and whose morphisms are families $(f_i)_{i\in I}$ with f_i in $\mathcal{C}_i$. The composition and identity are obvious.
2. The **coproduct** $\coprod_{i\in I}\mathcal{C}_i$ of them whose objects are disjoint union of objects of $\mathcal{C}_i$s and whose morphisms are morphisms in $\mathcal{C}_i$s.

Similarly, for $\{\mathcal{M}_i\}_{i\in I}$ a family of $\mathcal{S}$-multicategories, their products and coproducts are defined in an obvious way.

1.9 The category **Set** is a Cartesian monoidal category. Its underlying multicategory is also denoted by **Set**.

1.9.1 Let $\mathcal{M}$ be a $\mathcal{S}$-multicategory. Then a **$\mathcal{M}$-algebra** is a map from $\mathcal{M}$ to **Set**. A morphism between $\mathcal{M}$-algebras is then a transformation between them. More generally, if $\mathcal{V}$ is another $\mathcal{S}$-multicategory, one can consider $\mathcal{V}$-*valued* $\mathcal{M}$-*algebras*, which are maps from $\mathcal{M}$ to $\mathcal{V}$.

1.10 A multicategory having only one object is called an **(planar) operad**. Note that in this case the functor P_n is actually a set. Hence an *operad* $\mathcal{P}$ is a collection of sets $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ (whose elements called **n-ary operations**) together with functions $\mathcal{P}_n \times \mathcal{P}_\pi \to \mathcal{P}_m$ satisfying certain properties.

1.10.1 Example The terminal operad $\mathbf{1}$ has exactly one n-ary operation for each n. Then a $\mathbf{1}$-algebra is the same as a *monoid*. Let $\mathcal{M}$ be a multicategory, then a **monoid** in $\mathcal{M}$ is a $\mathcal{M}$-valued $\mathbf{1}$-algebra.

1.10.2 Example The operad for *semigroups* is defined like $\mathbf{1}$ except $\mathcal{P}_0 = \emptyset$.

1.10.3 Example If we put $\mathcal{P}_1 = \underline{1}$ and $\mathcal{P}_n = \emptyset$ for other n. Then the $\mathcal{P}$-algebras are just *sets*.

1.10.4 Example If we put $\mathcal{P}_0 = \mathcal{P}_1 = \underline{1}$ and $\mathcal{P}_n = \emptyset$ for other n. Then the $\mathcal{P}$-algebras are *pointed sets*.

1.10.5 Example Let M be a monoid. Define $\mathcal{P}_1 = M$ and $\mathcal{P}_n =$ for other n. The composition and identity are the multiplication and unit of M. Then, a $\mathcal{P}$-algebra is a M*-set*, i.e. a set with left M-action.

1.10.6 Example Any commutative monoid $(A, +, 0)$ has a underlying operad obtained by viewing A as a one-object strict monoidal category. More precisely, we have $\mathcal{P}_n = A$ for all n and the composition is

$$\begin{aligned} \mathcal{P}_n \times \mathcal{P}_{m_1} \times \cdots \times \mathcal{P}_{m_n} &\longrightarrow \mathcal{P}_{m_1+\cdots+m_n} \\ (a, a_1, \cdots, a_n) &\longmapsto a + a_1 + \cdots + a_n, \end{aligned}$$

and the identity is $0 \in \mathcal{P}_1$. Then a $\mathcal{P}$-algebra is an A*-set*.

1.10.7 Example For M an object in a monoidal category $\mathcal{M}$, the underlying multicategory construction of $\mathcal{M}$ defines an operad, called the **endomorphism operad** of M, denoted by $\mathrm{End}(M)$. Note that for $\mathcal{P}$ an operad, a $\mathcal{M}$*-valued* $\mathcal{P}$*-algebra* is equivalent to an object $M \in \mathcal{M}$ with a morphism of operads $\mathcal{P} \to \mathrm{End}(M)$.

1.10.8 Example We define an operad $\mathbb{S}$, called the **operad of symmetries**, as follows. For any $n \in \mathcal{S}$, the set $\mathbb{S}_n$ is the n-th symmetric group. For any $\mathcal{S}$-morphism $\pi \colon m \to n$, the composition is defined as follows. For any $\sigma \in \mathbb{S}_n$, $\tau_i \in \mathbb{S}_{m_i}, (i \in n)$, the composition of them is $\sigma(\tau_i | i \in n)$ (or denoted by $\sigma(\tau_1, \cdots, \tau_n)$) which works as follows. For any $i \in n$ and $j \in m_i$,

$$\sigma(\tau_i | i \in n)(m_1 + \cdots + m_{i-1} + j) = m_{\sigma^{-1}(1)} + \cdots + m_{\sigma^{-1}(\sigma(i)-1)} + \tau_i(j).$$

Remark Note that $\mathbb{S}^{\otimes} \cong \mathcal{S}^c$ for $\mathcal{S} = \mathcal{S}1$. Note that $\mathbb{S}^{\otimes}$ is automatically a strict symmetric monoidal category.

1.11 A **symmetric multicategory** is a multicategory $\mathcal{M}$ with a representation of the groupoid $\mathbb{S}^{\otimes}$ in the category of multimorphism structures which extends the mapping $n \mapsto P_n$ and is compatible with the composition in the sense that for any $\pi \colon m \to n$ in $\mathcal{S}$ and $\sigma \in \mathbb{S}_n$, $\tau_i \in \mathbb{S}_{m_i}, (i \in n)$, the following diagram commutes

$$\begin{tikzcd} P_n \times P_\pi \arrow[r, "\circ_\pi"] \arrow[d, "\mu_\sigma \cdot \prod_{i \in n} \mu_{\tau_i}"'] & P_m \arrow[d, "\mu_{\sigma(\tau_i | i \in n)}"] \\ P_n \times P_\pi \arrow[r, "\circ_\pi"] & P_m \end{tikzcd}$$

where $\mu_\sigma \cdot \prod_{i \in n} \mu_{\tau_i}$ denote the product of the natural isomorphisms μ_σ and $\mu_{\tau_i}(i \in n)$.

Remark By Lemma 1.5.3, a symmetric multicategory is equivalent to a pseudo-tensor category.

A **map of symmetric multicategories** is a map of multicategories preserving the symmetric structure. The transformations between such maps are just transformations between them as maps of multicategories.

A **symmetric operad** is a one-object symmetric multicategory.

1.11.1 Example $\mathbb{S}$ become a symmetric operad by multiplication in the symmetric groups. Treat is as a pseudo-tensor category, we have $\mathbb{S}^{\otimes} \cong \mathcal{S}^c$ where $\mathcal{S} = \mathcal{S}2$.

1.12 We have a forgetful functor from $\mathbb{S}\mathbf{MC}$ to $\mathbf{MC}$, which forgets the symmetric structure. This functor has a left adjoint

$$\mathrm{Sym}\colon \mathbf{MC} \longrightarrow \mathbb{S}\mathbf{MC}$$

sending a multicategory to its **symmetrization**. More precisely, it replaces each $\mathcal{P}_n$ by the free $\mathbb{S}_n$-set generated by $\mathcal{P}_n$, and getting a symmetric operad by the obvious symmetric structure.

1.12.1 Remark By the adjunction, an algebra for an operad $\mathcal{P}$ is equivalent to an algebra for its symmetrization $\mathrm{Sym}(\mathcal{P})$.

1.12.2 Remark Some operad have natural symmetric structure making it a symmetric operad. However, note that by definition, the algebras for it, viewed as operad and symmetric operad are different.

1.12.3 Example $\mathrm{Sym}(\mathbf{1})$ is just $\mathbb{S}$, which is not the terminal object in the category of symmetric operads. Let $\mathcal{M}$ be a pseudo-tensor category, then a **monoid** in $\mathcal{M}$ is a $\mathcal{M}$-valued $\mathbb{S}$-algebra.

1.12.4 Example For the terminal symmetric operad $\mathbf{1}$, its algebras are *commutative monoids*. Let $\mathcal{M}$ be a pseudo-tensor category, then a **commutative monoid** in $\mathcal{M}$ is a $\mathcal{M}$-valued $\mathbf{1}$-algebra.

1.13 Let $\mathcal{A}$ be a symmetric monoidal category. Note that the tensor product $\otimes\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a pseudo-tensor functor. Recall that a **(left) module category** over $\mathcal{A}$ is a category $\mathcal{M}$ together with an **$\mathcal{A}$-action** on $\mathcal{M}$, which is a monoidal functor from $\mathcal{A}$ to $\mathrm{End}(\mathcal{M})$, the monoidal category of endofunctors on $\mathcal{M}$. Such an action can be reformulated as a pseudo-tensor functor $\mathcal{A} \times \mathcal{M} \to \mathcal{M}$ satisfies some compatibilities.

Likewise, for $\mathcal{M}$ a pseudo-tensor category, an **$\mathcal{A}$-action** on $\mathcal{M}$ consists of the following data:

- A pseudo-tensor functor $\otimes^{\mathcal{M}}\colon \mathcal{A} \times \mathcal{M} \to \mathcal{M}$.
- A transformation $\alpha\colon (- \otimes -) \otimes^{\mathcal{M}} - \Rightarrow - \otimes (- \otimes^{\mathcal{M}} -)$.

Those data should satisfy certain compatibilities analogous to the compatibilities for an $\mathcal{A}$-action on a category.

In concrete terms, such an action is given by an action on the underlying category of $\mathcal{M}$ with natural transformations whose components are

$$\mathcal{A}(\bigotimes_I A_i, B) \times P_I(\{M_i\}_{i\in I}, N) \longrightarrow P_I(\{A_i \otimes^{\mathcal{M}} M_i\}_{i\in I}, B \otimes^{\mathcal{M}} N).$$

Note that they are determined by the natural maps

$$P_I(\{M_i\}_{i\in I}, N) \longrightarrow P_I(\{A_i \otimes^{\mathcal{M}} M_i\}_{i\in I}, (\bigotimes_I A_i) \otimes^{\mathcal{M}} N).$$

1.13.1 Remark Give $\mathcal{A}$, $\mathcal{M}$ as above. For P a commutative monoid in $\mathcal{A}$, it is clear that $M \mapsto (P, M)$ defines a pseudo-tensor functor from $\mathcal{M}$ to $\mathcal{A} \times \mathcal{M}$. Hence we have the pseudo-tensor functor $\mathcal{M} \to \mathcal{M}$ by $M \mapsto P \otimes^{\mathcal{M}} M$. Similarly, if F is a commutative monoid in $\mathcal{M}$, then $A \mapsto A \otimes^{\mathcal{M}} F$ defines a pseudo-tensor functor $\mathcal{A} \to \mathcal{M}$. So if a A is a $\mathcal{P}$-algebra in $\mathcal{A}$ and M is a $\mathcal{P}$-algebra in $\mathcal{M}$, then $P \otimes^{\mathcal{M}} M$, $A \otimes^{\mathcal{M}} F$ are also $\mathcal{P}$-algebras in $\mathcal{M}$.

Enrichment over k

1.14 Now we want to do the aboves in the enrichment contexts.

THE FOLLOWING IS WAITING TO FILL!!!

1.15 Let $\{\mathcal{M}_j\}_{j\in J}$ be a finite family of pseudo-tensor categories. Their **tensor product** $\bigotimes_J \mathcal{M}_j$ is the pseudo-tensor category defined as follows.

- Objects the same as $\prod_J \mathcal{M}_j$. We denote an object $\{M_j\}_{j\in J}$ in $\bigotimes_J \mathcal{M}_j$ as $\bigotimes_J M_j$
- For each I, define

$$P_I = \bigotimes$$