

Note on
Chiral algebras

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§ 1 Categories

Multicategories and pseudo-tensor categories

Pseudo-tensor category = symmetric multicategory.

1.1 (Motivation from monoidal categories) In a monoidal category, as the string diagram language suggests, we can view a morphism from a tensor product as a *multimorphism*. This motivates the notion of multicategories.

Note that

1. The coherence theorem for monoidal category guarantees that the (ordered) tensor product $X_1 \otimes \cdots X_n$ is well-defined up to a unique isomorphism.
2. The coherence theorem for symmetric monoidal category guarantees that the (unordered) tensor product $\bigotimes_{i \in I} X_i$ is well-defined up to a unique isomorphism.

Therefore we consider the following two *suitable categories of indexes*:

S1 The category of natural numbers¹ and maps between them.

S2 The category of finite sets and maps between them.

Remark To simplify notation, we use the following alternative notation for natural numbers:

$$\underline{n} := \{1, 2, \dots, n\}.$$

If there is no ambiguity, we simply write n instead of \underline{n} . More precisely, we will never write $i \in n$.

1.2 (Indexes) Recall that a family of sets indexed by an set I is actually a mapping which associating each $i \in I$ a set. In general, a *family of objects* in a category \mathcal{C} indexed by an other category I is a functor from I to \mathcal{C} . The category of functors from I to \mathcal{C} is denoted by \mathcal{C}^I .

1.2.1 Example Let n be a natural number then a *n -indexed family of objects* of \mathcal{C} is a list X_1, X_2, \dots, X_n of objects of \mathcal{C} .

1.2.2 Example Let I be a finite set, then a *I -indexed family of objects* of \mathcal{C} is a mapping which associating each $i \in I$ an object $X_i \in \mathcal{C}$.

¹We follow *von Neumann's construction*:

$$0 := \emptyset, 1 := \{0\}, \dots, n := \{0, 1, \dots, n-1\}.$$

Note that the natural numbers are naturally ordinals under this construction.

1.3 (Category of indexes) From now on, we fix a *suitable category of indexes*, naming either $\mathcal{S}1$ or $\mathcal{S}2$. Note that in both case we have: for $I \in \mathcal{S}$, if $|I| = n$, then $\mathcal{S}(I, I) = \mathfrak{S}_n$, the n -th symmetric group.

Then critical difference between $\mathcal{S}1$ and $\mathcal{S}2$ is that a natural number is an ordinal, i.e. it carries a canonical linear order on it, while a finite set doesn't. Another difference is that $\mathcal{S}2$ is closed under usual set-theoretic constructions while $\mathcal{S}1$ is not. However, we still have corresponding constructions.

1.3.1 Example The Cartesian product of two finite sets is again a finite set, while of two natural numbers m, n are not a natural numbers. However, by putting the lexicography order on $m \times n$, we obtain a canonical isomorphism from $m \times n$ to the natural number mn .

1.3.2 Example Let $\{m_i\}_{i \in \underline{n}}$ be a n -indexed family of natural numbers. Then their n -indexed Cartesian product $\prod_{i \in \underline{n}} m_i = \{(k_1, \dots, k_n) | k_i \in \underline{m_i}\}$ is not a natural number. By putting the lexicography order on it, we obtain a canonical isomorphism from it to the natural number $m_1 m_2 \dots m_n$.

1.3.3 Example The disjoint union $\bigsqcup_{i \in \underline{n}} m_i = \{(i, k) | i \in \underline{n}, k \in \underline{m_i}\}$ is not a natural number. By putting lexicography order on it, we obtain a canonical isomorphism from it to the natural number $m_1 + m_2 + \dots + m_n$.

1.3.4 Example Given a map $\pi: m \rightarrow n$, the preimage set $\pi^{-1}(i)$ of $i \in \underline{n}$ is not a natural number. However, since it is a subset of a natural number, it has a canonical linear order, which gives us an isomorphism to the natural number $|\pi^{-1}(i)|$.

1.3.5 Remark The reason why there are such constructions is follows. Since any natural number is a finite set, we have the following inclusion of categories

$$\mathcal{S}1 \longrightarrow \mathcal{S}2.$$

Note that it is an equivalence of category. Then the usual set-theoretic constructions in $\mathcal{S}2$ are actually certain limits/colimits in it and the corresponding constructions in $\mathcal{S}1$ are corresponding limits/colimits in it.

1.3.6 Remark In both $\mathcal{S}1$ and $\mathcal{S}2$, $\underline{1}$ is a terminal object. We fix it as *the* terminal object. In particular, we identify \mathcal{C}^1 with \mathcal{C} for any category \mathcal{C} . Note that I haven't identify \mathcal{C} with every \mathcal{C}^I where I is a singleton yet. However, since the isomorphism between a singleton and $\underline{1}$ is unique, we can always do so.

1.4 Recall that a category \mathcal{C} consists of

- A class of *objects* \mathcal{C}_0 .
- For each pair of objects $X, Y \in \mathcal{C}_0$, a *Hom set* $\mathcal{C}(X, Y)$. In other words, a mapping $\mathcal{C}_0 \times \mathcal{C}_0 \longrightarrow \mathbf{Set}_0$. Note that the Hom set construction can be extended to a bifunctor $\mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow \mathbf{Set}$ through composition.

- For each triple of objects $X, Y, Z \in \mathcal{C}_0$, a map

$$\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X, Z),$$

called the *composition*. Note that when we extend the Hom sets to a bifunctor, then the composition become a *dinatural transformation*

$$\text{Hom} \times \text{Hom} \Longrightarrow \text{Hom}.$$

Remark For $F: \mathcal{A} \times \mathcal{B}^{\text{op}} \times \mathcal{B} \rightarrow \mathcal{D}, G: \mathcal{A} \times \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ two functors, a **dinatural** transformation (or more precisely natural in \mathcal{A} and dinatural in \mathcal{B} and \mathcal{C}) is a collection of \mathcal{E} -morphisms

$$\alpha_{a,b,c}: F(a, b, b) \longrightarrow G(a, c, c),$$

such that for any $f: a \rightarrow a'$ in \mathcal{A} , $g: b \rightarrow b'$ in \mathcal{B} and $h: c \rightarrow c'$ in \mathcal{C} , we have the following commutative diagrams

$$\begin{aligned} \alpha_{a',b,c} \circ F(f, 1, 1) &= G(f, 1, 1) \circ \alpha_{a,b,c}, \\ \alpha_{a,b',c} \circ F(1, g, 1) &= \alpha_{a,b,c} \circ F(1, 1, g), \\ G(1, 1, h) \circ \alpha_{a,b,c'} &= G(1, h, 1) \circ \alpha_{a,b,c}. \end{aligned}$$

Remark Mappings are functors: indeed, a mapping to the class of objects of a category \mathcal{V} can be viewed as a functor from a discrete category to \mathcal{V} . Then it make sense to talk about the morphisms between mappings, which are called *transformations* as there is no need of naturalness.

- For each object $X \in \mathcal{C}_0$, an *identity* $\text{id}_X \in \mathcal{C}(X, X)$. In other words, a transformation $\underline{1} \Rightarrow \text{Hom} \circ \Delta$, where Δ is the diagonal mapping $\Delta: \mathcal{C}_0 \rightarrow \mathcal{C}_0 \times \mathcal{C}_0$ and $\underline{1}$ is the constant mapping which maps each $X \in \mathcal{C}_0$ to the terminal object $\underline{1} \in \mathbf{Set}$.

and satisfies the associativity and unity.

1.5 A **S-multicategory**² \mathcal{M} consists of the following data:

- A class of *objects* \mathcal{M}_0 .
- For any $I \in \mathcal{S}$, a mapping $P_I = P_I^{\mathcal{M}}: (\mathcal{M}_0^I)^{\text{op}} \times \mathcal{M}_0 \rightarrow \mathbf{Set}$, called the set of **I-indexed multimorphisms**.

Remark For any \mathcal{S} -morphism $\pi: J \rightarrow I$, we can define a mapping $P_\pi: (\mathcal{M}_0^J)^{\text{op}} \times \mathcal{M}_0^I \rightarrow \mathbf{Set}^I$ as follows. First, put $J_i := \pi^{-1}(i)$. Then putting the mappings P_{J_i} together, we get the desired mapping.

²This is not a standard name.

Remark For $f \in P_I(\{M_i\}_{i \in I}, N)$, we write $f: \{M_i\}_{i \in I} \rightarrow N$. For $f \in P_n((M_1, \dots, M_n), N)$, we write $f: M_1, \dots, M_n \rightarrow N$.

- For any \mathcal{S} -morphism $\pi: J \rightarrow I$, a dinatural transformation (between mappings), called the **composition**

$$\circ_\pi: P_I \times P_\pi \Longrightarrow P_J,$$

where the mapping $P_I \times P_\pi$ is the composition

$$(\mathcal{M}_0^I)^{\text{op}} \times \mathcal{M}_0 \times (\mathcal{M}_0^J)^{\text{op}} \times \mathcal{M}_0^I \xrightarrow{P_I \times P_\pi} \mathbf{Set} \times \mathbf{Set}^I \xrightarrow{\Pi} \mathbf{Set}.$$

Remark For $f_i \in P_{J_i}(\{L_j\}_{j \in J_i}, M_i)$ and $g \in P_I(\{M_i\}_{i \in I}, N)$, we write $g \circ \{f_i\}_{i \in I}$ instead of $\circ_\pi(g, \{f_i\}_{i \in I})$.

Remark For any two \mathcal{S} -morphisms $\pi: J \rightarrow I$ and $\rho: H \rightarrow J$, we can define a transformation $\circ_{\pi, \rho}: P_\pi \times P_\rho \Rightarrow P_{\pi \circ \rho}$ as follows. First, separate ρ as collection of \mathcal{S} -morphisms $\rho_i: H_i \rightarrow J_i$, where $H_i = \rho^{-1}(J_i) = (\pi \circ \rho)^{-1}(i)$. Then putting the transformations \circ_{ρ_i} together, we get the desired transformation.

- For any one element index set $I \in \mathcal{S}$, a dinatural transformation, called the **I -indexed identity**

$$\mathbf{1}_I: \underline{1} \Longrightarrow P_I,$$

where $\underline{1}$ is the trivial mapping and P_I is viewed as a mapping from $\mathcal{M}_0^{\text{op}} \times \mathcal{M}_0$ via the identification $\mathcal{M}_0^I \cong \mathcal{M}_0$.

Remark Hence for each $|I| = 1$ and $M \in \mathcal{M}_0$, the I -indexed identity specifies an element in $P_I(\{M\}_{i \in I}, M)$, denoted by 1_M^I .

Those data are subject to the following axioms:

1. The composition is *associative*, i.e. if $\pi: J \rightarrow I$, $\rho: H \rightarrow J$ are two \mathcal{S} -morphisms, then the following diagram commutes:

$$\begin{array}{ccc} P_I \times P_\pi \times P_\rho & \xrightarrow{\circ_\pi} & P_J \times P_\rho \\ \circ_{\pi, \rho} \downarrow & & \downarrow \circ_\rho \\ P_I \times P_{\pi \circ \rho} & \xrightarrow{\circ_{\pi \circ \rho}} & P_H \end{array}$$

2. The identity works as *left identity* of composition, i.e. if $|I| = 1$, $J \in \mathcal{S}$ and $\pi: J \rightarrow I$ is the unique map, then the following diagram commutes.

$$\begin{array}{ccc} \underline{1} \times P_J & \xrightarrow{\cong} & P_J \\ \mathbf{1}_I \downarrow & & \uparrow \circ_\pi \\ P_I \times P_J & \xlongequal{\quad} & P_I \times P_\pi \end{array}$$

3. The identity works as *right identity* of composition, i.e. if $I \in \mathcal{S}$ and $\pi: I \rightarrow I$ is the identity, then the following diagram commutes.

$$\begin{array}{ccc} P_I \times \underline{1}^I & \xrightarrow{\cong} & P_I \\ \mathbf{1}_I \downarrow & \nearrow \circ_\pi & \\ P_I \times P_\pi & & \end{array}$$

A **map of \mathcal{S} -multicategories** $F: \mathcal{M} \rightarrow \mathcal{N}$ consists of the following data:

- A mapping $F: \mathcal{M}_0 \rightarrow \mathcal{N}_0$.
- For any $I \in \mathcal{S}$, a F -transformation³ $F_I: P_I^{\mathcal{M}} \Rightarrow P_I^{\mathcal{N}}$, which is the same as a transformation from $P_I^{\mathcal{M}}$ to the composition of $P_I^{\mathcal{N}}$ with the following mapping

$$F^I \times F: \mathcal{M}_0^I \times \mathcal{M}_0 \longrightarrow \mathcal{N}_0^I \times \mathcal{N}_0.$$

Remark One can define a F -transformation F_π from F_I s.

Those data are subject to the axiom that for any \mathcal{S} -morphism $\pi: J \rightarrow I$, the following diagram commutes.

$$\begin{array}{ccc} P_I^{\mathcal{M}} \times P_\pi^{\mathcal{M}} & \xrightarrow{\circ_\pi} & P_J^{\mathcal{M}} \\ F_I \times F_\pi \downarrow & & \downarrow F_J \\ P_I^{\mathcal{N}} \times P_\pi^{\mathcal{N}} & \xrightarrow{\circ_\pi} & P_J^{\mathcal{N}} \end{array}$$

A **transformation** between maps of \mathcal{S} -multicategories is a transformation $\alpha: F \Rightarrow G$ such that the following diagram commutes.

$$\begin{array}{ccc} P_I^{\mathcal{M}} & \xrightarrow{F} & P_J^{\mathcal{N}} \circ ((F^I)^{\text{op}} \times F) \\ G \downarrow & & \downarrow \\ P_I^{\mathcal{N}} \circ ((G^I)^{\text{op}} \times G) & \longrightarrow & P_J^{\mathcal{N}} \circ ((F^I)^{\text{op}} \times G) \end{array}$$

where the right is the transformation obtained by apply the mapping $P_J^{\mathcal{N}} \circ (F^I \times -)$ to components of α , and the bottom is the transformation obtained by apply the mapping $P_J^{\mathcal{N}} \circ (-^I \times G)$ to components of α .

The transformations compose in obvious ways just as usual horizontal and vertical compositions of natural transformations. This makes the category **SMC** of \mathcal{S} -multicategory a strict 2-category.

³Recall that, if we have functors $F: \mathcal{A} \rightarrow \mathcal{B}$, $P: \mathcal{A} \rightarrow \mathcal{C}$ and $Q: \mathcal{B} \rightarrow \mathcal{C}$, then a natural F -transformation from P to Q is a natural transformation from P to $Q \circ F$.

When we choose \mathcal{S} to be $\mathcal{S}1$, a \mathcal{S} -multicategory is called a **multicategory/non-symmetric pseudo-tensor category**. The 2-category \mathcal{SMC} is simply denoted by \mathbf{MC} in this case. When we choose \mathcal{S} to be $\mathcal{S}2$, a \mathcal{S} -multicategory is called a **symmetric multicategory/pseudo-tensor category**.

1.5.1 Recall that for \mathcal{C} a category and \mathcal{V} another category, a \mathcal{V} -valued presheaf on \mathcal{C} is a contravariant functor $\mathcal{C}^{\text{op}} \rightarrow \mathcal{V}$. The category of \mathcal{V} -valued presheaves on \mathcal{C} is denoted by $\mathcal{V}_{\mathcal{C}}$, comparing with the notation of the category $\mathcal{V}^{\mathcal{C}}$ of functors from \mathcal{C} to \mathcal{V} . For instance, P_I can be viewed as a $\mathbf{Set}^{\mathcal{M}_0}$ -valued presheaf on \mathcal{M}_0^I .

Given a category I and an I -indexed family of categories \mathcal{C}_i , the **union** $\bigsqcup_I \mathcal{C}_i$ of this family is the category

- whose objects are pairs (i, X) where $i \in I$ and $X \in \mathcal{C}_i$.
- whose morphisms are pairs $(\pi, f): (i, X) \rightarrow (j, Y)$ where $\pi: j \rightarrow i$ is a morphism in I and f is a morphism from X to $\pi^*(Y)$ (here π^* is the functor $\mathcal{C}_j \rightarrow \mathcal{C}_i$ induced by π).

This category carries a forgetful functor mapping each (i, X) to $i \in I$. We simply denote (i, X) by X and call i the *index* of X .

Let \mathcal{M}_0 be a class. In the category $\mathcal{SP}^{\mathcal{M}_0} := \bigsqcup_{\mathcal{S}} \mathbf{Set}_{\mathcal{M}_0^I}^{\mathcal{M}_0}$, an object is just a mapping like P_I . Therefore we can call the *category of multimorphism structures* on \mathcal{M}_0 .

1.5.2 Recall that a representation of a groupoid \mathcal{G} in a category \mathcal{C} is just a functor $\mathcal{G} \rightarrow \mathcal{C}$.

Let \mathcal{S}^c be the *core* of \mathcal{S} , i.e. the maximal subcategory of \mathcal{S} consisting only of isomorphisms. The following lemma is critical, especially when we choose \mathcal{S} to be $\mathcal{S}2$.

1.5.3 Lemma *Let \mathcal{M} be a \mathcal{S} -multicategory, then there is a canonical representation of \mathcal{S}^c in the category of multimorphism structures on \mathcal{M}_0 . In precise, for $I \in \mathcal{S}$, we have the object P_I . For a \mathcal{S} -morphism $\pi: J \rightarrow I$, we have a transformation $\mu_{\pi}: P_I \Rightarrow \pi^*(P_J)$, where π^* is the obvious one.*

1.5.4 It is clear that if \mathcal{M} is a multicategory, then given a fixed one-element set $I \in \mathcal{S}$, $(\mathcal{M}_0, P_I, \mathbf{1}_I)$ is a category. Moreover, from the lemma, we can see that although different choices of I give different category, but all of them are isomorphic. Hence it make sense to talk about *the underlying category* of a \mathcal{S} -multicategory.

1.5.5 From the lemma, we can replace $\mathcal{S}2$ by its *skeleton*, i.e. the category obtained by identify isomorphic objects in $\mathcal{S}2$. Note that this category is isomorphic to $\mathcal{S}1$. In this way, a pseudo-tensor category is a multicategory equipped with extra structures coming from the linear orders on natural numbers.

1.5.6 Using the above fact, we have the following alternative definition:

A **pre- \mathcal{S} -multicategory structure** on a category \mathcal{M} consists of the following data

- For any $I \in \mathcal{S}$, a bifunctor $P_I = P_I^{\mathcal{M}}: (\mathcal{M}^I)^{\text{op}} \times \mathcal{M} \rightarrow \mathbf{Set}$, called the set of **I -indexed multimorphisms**.
- For any \mathcal{S} -morphism $\pi: J \rightarrow I$, a dinatural transformation, called the **composition**

$$\circ_{\pi}: P_I \times P_{\pi} \Longrightarrow P_J,$$

where the functor $P_I \times P_{\pi}$ is the composition

$$(\mathcal{M}^I)^{\text{op}} \times \mathcal{M} \times (\mathcal{M}^J)^{\text{op}} \times \mathcal{M} \xrightarrow{P_I \times P_{\pi}} \mathbf{Set} \times \mathbf{Set} \xrightarrow{\Pi} \mathbf{Set}.$$

- For any one element index set $I \in \mathcal{S}$, a dinatural transformation, called the **I -indexed identity**

$$\mathbf{1}_I: \underline{1} \Longrightarrow P_I,$$

where P_I is viewed as a functor from $\mathcal{M}^{\text{op}} \times \mathcal{M}$ via the identification $\mathcal{M}^I \cong \mathcal{M}$.

Those data are subject to the following axioms:

1. The composition is *associative*, i.e. if $\pi: J \rightarrow I$, $\rho: H \rightarrow J$ are two \mathcal{S} -morphisms, then the following diagram commutes:

$$\begin{array}{ccc} P_I \times P_{\pi} \times P_{\rho} & \xrightarrow{\circ_{\pi}} & P_J \times P_{\rho} \\ \circ_{\pi, \rho} \downarrow & & \downarrow \circ_{\rho} \\ P_I \times P_{\pi \circ \rho} & \xrightarrow{\circ_{\pi \circ \rho}} & P_H \end{array}$$

2. The identity works as *left identity* of composition, i.e. if $|I| = 1$, $J \in \mathcal{S}$ and $\pi: J \rightarrow I$ is the unique map, then the following diagram commutes

$$\begin{array}{ccc} \underline{1} \times P_J & \xrightarrow{\cong} & P_J \\ \mathbf{1}_I \downarrow & & \uparrow \circ_{\pi} \\ P_I \times P_J & \xlongequal{\quad} & P_I \times P_{\pi} \end{array}$$

3. The identity works as *right identity* of composition, i.e. if $I \in \mathcal{S}$ and $\pi: I \rightarrow I$ is the identity, then the following diagram commutes.

$$\begin{array}{ccc} P_I \times \underline{1}^I & \xrightarrow{\cong} & P_I \\ \mathbf{1}_I \downarrow & \nearrow \circ_{\pi} & \\ P_I \times P_{\pi} & & \end{array}$$

Note that a pre- \mathcal{S} -multicategory structure on \mathcal{M} defines a \mathcal{S} -multicategory $\tilde{\mathcal{M}}$ with objects the same as \mathcal{M} . Moreover, a pre- \mathcal{S} -multicategory structure is a **\mathcal{S} -multicategory structure** if it satisfying the following *compatible axiom*:

The underlying category of $\tilde{\mathcal{M}}$ is isomorphic to \mathcal{M} .

Then, we have

Proposition *a \mathcal{S} -multicategory is equivalent to a category equipped with a \mathcal{S} -multicategory structure on it.*

1.5.7 Using above equivalence, a **map of \mathcal{S} -multicategories** $F: \mathcal{M} \rightarrow \mathcal{N}$ can be reformulated as the following data

- A functor $F: \mathcal{M} \rightarrow \mathcal{N}$.
- For any $I \in \mathcal{S}$, a natural F -transformation $F_I: P_I^{\mathcal{M}} \Rightarrow P_I^{\mathcal{N}}$, which is the same as a natural transformation from $P_I^{\mathcal{M}}$ to the composition of $P_I^{\mathcal{N}}$ with the following functor

$$(F^I)^{\text{op}} \times F: (\mathcal{M}^I)^{\text{op}} \times \mathcal{M} \longrightarrow (\mathcal{N}^I)^{\text{op}} \times \mathcal{N}.$$

Those data are subject to the axiom that for any \mathcal{S} -morphism $\pi: J \rightarrow I$, the following diagram (where the horizontal ones are dinatural transformations and the vertical ones are natural F -transformations) commutes.

$$\begin{array}{ccc} P_I^{\mathcal{M}} \times P_{\pi}^{\mathcal{M}} & \xrightarrow{\circ_{\pi}} & P_J^{\mathcal{M}} \\ F_I \times F_{\pi} \downarrow & & \downarrow F_J \\ P_I^{\mathcal{N}} \times P_{\pi}^{\mathcal{N}} & \xrightarrow{\circ_{\pi}} & P_J^{\mathcal{N}} \end{array}$$

Similarly, a **transformation** between maps of \mathcal{S} -multicategories can be reformulated as a natural transformation $\alpha: F \Rightarrow G$ such that the following diagram commutes.

$$\begin{array}{ccc} P_I^{\mathcal{M}} & \xrightarrow{F} & P_J^{\mathcal{N}} \circ ((F^I)^{\text{op}} \times F) \\ G \downarrow & & \downarrow \alpha \\ P_I^{\mathcal{N}} \circ ((G^I)^{\text{op}} \times G) & \xrightarrow{\alpha} & P_J^{\mathcal{N}} \circ ((F^I)^{\text{op}} \times G) \end{array}$$

where the right α denote the natural transformation obtained by apply the functor $P_J^{\mathcal{N}} \circ ((F^I)^{\text{op}} \times -)$ to components of α , while the bottom α denote the natural transformation obtained by apply the functor $P_J^{\mathcal{N}} \circ ((-)^{\text{op}} \times G)$ to components of α .

1.5.8 On a category \mathcal{M} , there maybe many different \mathcal{S} -multicategory structures. We can define a map between them as a map between \mathcal{S} -multicategories such that its restriction on the underlying category \mathcal{M} is the identity functor. In this sense, we can talk about the *universal \mathcal{S} -multicategory structure* on \mathcal{M} . It is clear that we have an adjunction

$$\mathbf{Cat} \rightleftarrows \mathbf{SMC}.$$

1.6 We have the following typical examples of multicategories/pseudo-tensor categories.

1.6.1 Example Any category \mathcal{M} can be viewed as a \mathcal{S} -multicategory by putting P_I to empty when $|I| > 1$.

1.6.2 Example Any monoidal category (\mathcal{M}, \otimes) has an *underlying multicategory*, which has the same objects with \mathcal{M} and the functor P_n is given by

$$P_n(\{M_i\}_{i \in \underline{n}}, N) = \mathcal{M}(\otimes_{i \in \underline{n}} M_i, N).$$

Here the tensor product $\otimes_{i \in \underline{n}} M_i$ is defined as $(\cdots (M_1 \otimes M_2) \otimes \cdots \otimes M_n)$. Different associating settings give different multicategories. However, the coherence theorem guarantees they are equivalent.

1.6.3 Example Any symmetric monoidal category \mathcal{M} has an *underlying pseudo-tensor category*, which has the same objects with \mathcal{M} and the functor P_I is given by

$$P_I(\{M_i\}_{i \in I}, N) = \mathcal{M}(\otimes_{i \in I} M_i, N).$$

Here the tensor product $\otimes_{i \in I} M_i$ is defined as $M_1 \otimes M_2 \otimes \cdots \otimes M_n$ if we identify I with $\underline{n} = |I|$. Note that such an identification is equivalent to a linear order on I . Different identifications give different pseudo-tensor categories. However, the coherence theorem guarantees they are equivalent.

1.6.4 Example More generally, for \mathcal{M} a monoidal category and C a collection of its objects (not necessarily closed under tensor product), there is a multicategory whose class of objects is C and sets of multimorphisms are constructed as above. This example shows that not any multicategory is an underlying multicategory of a monoidal category.

1.7 A notion related to previous examples is *representation* of multimorphism structures. For \mathcal{M} a \mathcal{S} -multicategory, and $\{M_i\}_{i \in I}$ a I -indexed family of objects of \mathcal{M} , $P_I(\{M_i\}_{i \in I}, -)$ is a functor from \mathcal{M} to **Set**. If this functor is *represented* by an object, then that object is called the **pseudo-tensor product** of $\{M_i\}_{i \in I}$, denoted by $\bigotimes_{i \in I} M_i$. If all such kind of functors are *representable*, then we say the \mathcal{S} -multicategory \mathcal{M} is **pre-representable**. Note that if so, for any \mathcal{S} -morphism $\pi: J \rightarrow I$, we have the morphisms

$$\epsilon_\pi: \bigotimes_{j \in J} M_j \longrightarrow \bigotimes_{i \in I} (\bigotimes_{j \in J_i} M_j)$$

naturally in $\{M_j\}_{j \in J}$. In particular, we have the natural morphisms

$$(M_1 \otimes M_2) \otimes M_3 \longrightarrow M_1 \otimes (M_2 \otimes M_3).$$

However, there is no guarantee that the above is a natural isomorphism, see Leinster's book.

- 1.7.1** Let $\{M_i\}_{i \in I}$ be an object in \mathcal{M}^I , we say a \mathcal{S} -multimorphism $u: \{M_i\}_{i \in I} \rightarrow N$ is *pre-universal* if for any multimorphism

$$f: \{M_i\}_{i \in I} \rightarrow N'$$

there is a unique morphism

$$\tilde{f}: N \rightarrow N'$$

such that $f = \tilde{f} \circ u$.

It is clear that for any family $\{M_i\}_{i \in I}$ of objects of \mathcal{M} , the pseudo-tensor product $\bigotimes_{i \in I} M_i$, if it exists, gives a pre-universal multimorphism $u: \{M_i\}_{i \in I} \rightarrow \bigotimes_{i \in I} M_i$ and conversely, if $u: \{M_i\}_{i \in I} \rightarrow N$ is a pre-universal multimorphism, then N represents the functor $P_I(\{M_i\}_{i \in I}, -)$.

- 1.7.2** We say a \mathcal{S} -multicategory \mathcal{M} is **representable** if it is pre-representable and composition of pre-universal multimorphisms is again pre-universal.

- 1.7.3** Let $I, J \in \mathcal{S}$ and $p: I \rightarrow J$ a \mathcal{S} -morphism factoring through $\underline{1}$. Define $J \vee_p I$ as the disjoint union of J -indexed \mathcal{S} -objects U_j where $U_j = \underline{1}$ unless $j \in p(I)$, denote the element of J which p maps I to, define $U_p = I$. Let $\pi: J \vee_p I \rightarrow J$ be the projection and $\rho: I \rightarrow J \vee_p I$ the inclusion. We denote $g \circ \{f_j\}_{j \in J}$ by $g \circ_p f_p$ if $f_j = 1$ for $j \notin p(I)$.

In particular, for $\mathcal{S} = \mathcal{S}2$, $J \vee_p I$ is obtained by replacing $p \in J$ by elements of I ; for $\mathcal{S} = \mathcal{S}1$, $J \vee_p I$ is obtained by inset I into $p \in J$ and reordering to make it a natural number.

- 1.7.4** Let $\{M_i\}_{i \in I}$ be an object in \mathcal{M} , we say a multimorphism $u: \{M_i\}_{i \in I} \rightarrow N$ is *universal* if for any $p: I \rightarrow J$ a \mathcal{S} -morphism factoring through $\underline{1}$ and any multimorphism

$$f: \{M'_j\}_{j \in J \vee_p I} \rightarrow N'$$

with $M'_{\rho(i)} = M_i$, there is a unique multimorphism

$$\tilde{f}: \{M'_j\}_{j \in J} \rightarrow N'$$

where $M'_p = N$, such that $f = \tilde{f} \circ_p u$.

Now, we have a theorem

Theorem For a multicategory \mathcal{M} , the following conditions are equivalent:

1. \mathcal{M} is the underlying multicategory of some monoidal category.
2. \mathcal{M} is representable.
3. Any family of objects of \mathcal{M} is a domain of some universal multimorphism.

Similar equivalence holds for pseudo-tensor categories.

1.7.5 It is not difficult to extend the underlying \mathcal{S} -multicategory construction to a faithful functor V from the category of monoidal categories/symmetric monoidal categories to the category of multicategories/pseudo-tensor categories.

Conversely, if \mathcal{M}, \mathcal{N} are two pre-representable \mathcal{S} -multicategories, then a map between them $F: \mathcal{M} \rightarrow \mathcal{N}$ is precisely a functor $F: \mathcal{M} \rightarrow \mathcal{N}$ together with morphisms

$$\nu_I: \bigotimes_{i \in I} F(M_i) \longrightarrow F(\bigotimes_{i \in I} M_i)$$

naturally in $\{M_j\}_{j \in J}$ for each $I \in \mathcal{S}$ such that for any \mathcal{S} -morphism $\pi: J \rightarrow I$, the following diagram commutes.

$$\begin{array}{ccc} \bigotimes_{j \in J} F(M_j) & \xrightarrow{\nu_J} & F(\bigotimes_{j \in J} M_j) \\ \epsilon_\pi \downarrow & & \downarrow F(\epsilon_\pi) \\ \bigotimes_{i \in I} (\bigotimes_{j \in J_i} F(M_j)) & & \\ \otimes_{i \in I} \nu_{J_i} \downarrow & & \\ \bigotimes_{i \in I} F(\bigotimes_{j \in J_i} M_j) & \xrightarrow{\nu_I} & F(\bigotimes_{i \in I} (\bigotimes_{j \in J_i} M_j)) \end{array}$$

Therefore any map of representable \mathcal{S} -multicategories is obtained from a lax functor between the corresponding monoidal categories/symmetric monoidal categories. Moreover, we have

Theorem For a map of representable \mathcal{S} -multicategories $F: \mathcal{M} \rightarrow \mathcal{N}$, the following conditions are equivalent:

1. $F = V(P)$ with P a monoidal functor.
2. F preserves universal multimorphisms.

1.7.6 The functor V from the category of monoidal categories to the category of multicategories has a left adjoint defined as follows. For \mathcal{M} a multicategory. The **strict monoidal category generated by \mathcal{M}** is the strict monoidal category \mathcal{M}^\otimes whose objects are lists of objects of \mathcal{M} , whose morphisms are obtained from the multimorphisms in \mathcal{M} in the obvious way and whose monoidal structure is given by concatenation of lists.

Similar construction exists for pseudo-tensor categories.

1.8 Recall that if $\{\mathcal{C}_i\}_{i \in I}$ is a discrete family of categories, then we have the following two constructions.

1. The **product** $\prod_{i \in I} \mathcal{C}_i$ of them whose objects are families $(X_i)_{i \in I}$ with $X_i \in \mathcal{C}_i$ and whose morphisms are families $(f_i)_{i \in I}$ with f_i in \mathcal{C}_i . The composition and identity are obvious.
2. The **coproduct** $\coprod_{i \in I} \mathcal{C}_i$ of them whose objects are disjoint union of objects of \mathcal{C}_i s and whose morphisms are morphisms in \mathcal{C}_i s.

Similarly, for $\{\mathcal{M}_i\}_{i \in I}$ a family of \mathcal{S} -multicategories, their products and co-products are defined in an obvious way.

1.9 The category **Set** is a Cartesian monoidal category. Its underlying multicategory is also denoted by **Set**.

1.9.1 Let \mathcal{M} be a \mathcal{S} -multicategory. Then a **\mathcal{M} -algebra** is a map from \mathcal{M} to **Set**. A morphism between \mathcal{M} -algebras is then a transformation between them. More generally, if \mathcal{V} is another \mathcal{S} -multicategory, one can consider \mathcal{V} -valued \mathcal{M} -algebras, which are maps from \mathcal{M} to \mathcal{V} .

1.10 A multicategory having only one object is called an **operad**. Note that in this case the functor P_n is actually a set. Hence an *operad* \mathcal{P} is a collection of sets $\{\mathcal{P}_n\}_{n \in \mathbb{N}}$ (whose elements called **n -ary operations**) together with functions $\mathcal{P}_n \times \mathcal{P}_\pi \rightarrow \mathcal{P}_m$ satisfying certain properties.

1.10.1 Example The terminal operad **1** has exactly one n -ary operation for each n . Then a **1**-algebra is the same as a *monoid*.

1.10.2 Example The operad for *semigroups* is defined like **1** except $\mathcal{P}_0 = \emptyset$.

1.10.3 Example If we put $\mathcal{P}_1 = \underline{1}$ and $\mathcal{P}_n = \emptyset$ for other n . Then the \mathcal{P} -algebras are just *sets*.

1.10.4 Example If we put $\mathcal{P}_0 = \mathcal{P}_1 = \underline{1}$ and $\mathcal{P}_n = \emptyset$ for other n . Then the \mathcal{P} -algebras are *pointed sets*.

1.10.5 Example Let M be a monoid. Define $\mathcal{P}_1 = M$ and $\mathcal{P}_n = \emptyset$ for other n . The composition and identity are the multiplication and unit of M . Then, a \mathcal{P} -algebra is a *M -set*, i.e. a set with left M -action.

1.10.6 Example Any commutative monoid $(A, +, 0)$ has a underlying operad obtained by viewing A as a one-object strict monoidal category. More precisely, we have $\mathcal{P}_n = A$ for all n and the composition is

$$\begin{aligned} \mathcal{P}_n \times \mathcal{P}_{m_1} \times \cdots \times \mathcal{P}_{m_n} &\longrightarrow \mathcal{P}_{m_1 + \cdots + m_n} \\ (a, a_1, \cdots, a_n) &\longmapsto a + a_1 + \cdots + a_n, \end{aligned}$$

and the identity is $0 \in \mathcal{P}_1$. Then a \mathcal{P} -algebra is an *A -set*.

1.10.7 Example For M an object in a monoidal category \mathcal{M} , the underlying multicategory construction of \mathcal{M} defines an operad, called the **endomorphism operad** of M , denoted by $\text{End}(M)$. Note that for \mathcal{P} an operad, a \mathcal{M} -valued \mathcal{P} -algebra is equivalent to an object $M \in \mathcal{M}$ with a morphism of operads $\mathcal{P} \rightarrow \text{End}(M)$.

1.10.8 Example We define an operad \mathfrak{S} , called the **operad of symmetries**, as follows. For any $n \in \mathcal{S}$, the set \mathfrak{S}_n is the n -th symmetric group. For any \mathcal{S} -morphism $\pi: m \rightarrow n$, the composition is defined as follows. For any $\sigma \in \mathfrak{S}_n$, $\tau_i \in \mathfrak{S}_{m_i}$, ($i \in n$), the composition of them is $\sigma(\tau_i|i \in n)$ (or denoted by $\sigma(\tau_1, \dots, \tau_n)$) which works as follows. For any $i \in n$ and $j \in m_i$,

$$\sigma(\tau_i|i \in n)(m_1 + \dots + m_{i-1} + j) = m_{\sigma^{-1}(1)} + \dots + m_{\sigma^{-1}(\sigma(i)-1)} + \tau_i(j).$$

Remark Note that $\mathfrak{S}^\otimes \cong \mathcal{S}^c$ for $\mathcal{S} = \mathcal{S}1$. Note that \mathfrak{S}^\otimes is automatically a strict symmetric monoidal category.

1.11 A **symmetric multicategory** is a multicategory \mathcal{M} with a representation of the groupoid \mathfrak{S}^\otimes in the category of multimorphism structures which extends the mapping $n \mapsto P_n$ and is compatible with the composition in the sense that for any $\pi: m \rightarrow n$ in \mathcal{S} and $\sigma \in \mathfrak{S}_n$, $\tau_i \in \mathfrak{S}_{m_i}$, ($i \in n$), the following diagram commutes

$$\begin{array}{ccc} P_n \times P_\pi & \xrightarrow{\circ_\pi} & P_m \\ \mu_\sigma \cdot \prod_{i \in n} \mu_{\tau_i} \downarrow & & \downarrow \mu_{\sigma(\tau_i|i \in n)} \\ P_n \times P_\pi & \xrightarrow{\circ_\pi} & P_m \end{array}$$

where $\mu_\sigma \cdot \prod_{i \in n} \mu_{\tau_i}$ denote the product of the natural isomorphisms μ_σ and μ_{τ_i} ($i \in n$).

Remark By Lemma 1.5.3, a symmetric multicategory is equivalent to a pseudo-tensor category.

A **map of symmetric multicategories** is a map of multicategories preserving the symmetric structure. The transformations between such maps are just transformations between them as maps of multicategories.

A **symmetric operad** is a one-object symmetric multicategory.

1.11.1 Example \mathfrak{S} become a symmetric operad by multiplication in the symmetric groups. Treat it as a pseudo-tensor category, we have $\mathfrak{S}^\otimes \cong \mathcal{S}^c$ where $\mathcal{S} = \mathcal{S}2$.

THE FOLLOWING IS WAITING TO FILL!!!

1.12

1.13 Let \mathcal{V} be a monoidal category. Then a **(left) module category** over \mathcal{V} is a category \mathcal{M} together with a **\mathcal{V} -action** on \mathcal{M} , which is a monoidal functor from \mathcal{V} to $\text{End}(\mathcal{M})$, the monoidal category of endofunctors on \mathcal{M} .