Note on
Homological Algebra

Xu Gao

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Abstract

This note is on homological algebra with a homotopy-theoretical perspective and aims to introduce a framework for homotopy theory based on the notion of dg-categories. Such a framework, as I know, is a special case of the full general machinery of infinite-category theory and thus should be thought as well-known fact or even common sense.

Contents

§ 1 Homotopy theory for topological spaces

Before going to the main topics of this note, let's take a glance to the homotopy theory. One can refer to either a standard textbook on algebraic topology like [1], or a homotopy-first textbook like [2], or the wonderful textbook [3]. For further reading, refer [4].

1.1 Let $f, g\colon X \to Y$ be two (continues) maps between topological spaces, a **(left) homotopy** $\Phi\colon f \Rightarrow g$ is a commutative diagram (in the category of topological spaces) of the form

$$\begin{tikzcd} X \arrow[d, "{(\mathrm{id},\delta_0)}"'] \arrow[dr, "f"] & \\ X \times I \arrow[r, dashed, "\Phi"] & Y \\ X \arrow[u, "{(\mathrm{id},\delta_1)}"] \arrow[ur, "g"'] & \end{tikzcd}$$

where I is the unit interval $[0,1]$ and δ_0 (resp. δ_1) is the inclusion $\{0\} \hookrightarrow I$ (resp. $\{1\} \hookrightarrow I$). If such a homotopy exists, then we say f and g are **homotopic**, denoted by $f \simeq g$. Let $x_0 \in X$ and $y_0 \in Y$ be base points and suppose f and g preserve the base point. Then Φ is called a **based homotopy** if $\Phi(x_0, t) = y_0$ for all $t \in I$. More generally, let $A \subset X$ and $B \subset Y$ be subspaces and $f|_A = g|_A$ and $f(A) \subset B$. Then Φ is called a **relative homotopy** or **homotopy rel** A if $\Phi(x,t) = f(x)$ for all $x \in A$. To emphasize the base point x_0, or the subspace A, we use the notations $f \simeq_{x_0} g$ or $f \simeq_A g$ to denote that f and g are **based homotopic** or **homotopic rel** A.

The set $\mathrm{Map}(X,Y)$ of all continues maps from X to Y, equipped with the compact-open topology, is called the **mapping space** from X to Y. If X is a good topological space, for instant a locally compact Hausdorff space, then there is a natural bijection

$$\mathrm{Map}(Z \times X, Y) \cong \mathrm{Map}\big(Z, \mathrm{Map}(X,Y)\big),$$

where $Z \times X$ carries the product topology. If this is the case, then the exponential law implies that there is a natural bijection between the set of homotopy classes of maps $X \to Y$ and the set of path-components of $\mathrm{Map}(X,Y)$. This set will be denoted by $[X,Y]$, called the **free homotopy class set**.

Let $A \subset X$ and $B \subset Y$ be subspaces. The **product** of the pairs (X,A) and (Y,B) is the pair $(X \times Y, X \times B \cup A \times Y)$. The subspace $\mathrm{Map}(X,A;Y,B)$ of $\mathrm{Map}(X,Y)$ consists of those maps $f\colon X \to Y$ satisfying $f(A) \subset B$. It is called the **(relative) mapping space** from (X,A) to (Y,B). There is a special subspace of it, which consists of those factoring through B, thus can be identified to $\mathrm{Map}(X, B)$. Again, if (X, A) is good enough, then there is a natural bijection

$$\mathrm{Map}\left(Z\times X, Z\times A\cup C\times X; Y, B\right) \cong \mathrm{Map}\left(Z, C; \mathrm{Map}(X, A; Y, B), \mathrm{Map}(X, B)\right).$$

Let $(Z.C)$ be $(I, \varnothing)$, then we see that if (X, A) is good enough, then there is a natural bijection between the set of relative homotopy classes of maps $(X, A) \to (Y, B)$ and the set of path-components of $\mathrm{Map}(X, A; Y, B)$. This set is denoted by $[X, A; Y, B]$, called the **relative homotopy class set**.

Let (X, x_0) and (Y, y_0) are pointed spaces, i.e. topological spaces with a base point. The subspace $\mathrm{Map}(X, x_0; Y, y_0)$ is simply denoted by $\mathrm{Map}_*(X, Y)$, called the **(based) mapping space**. (In many case, the base point is clear or irrelevant to the discussion, we should simplify our notation by just write X instead of (X, x_0).) If X is good enough, from the previous paragraph, there is a natural bijection between the set of based homotopy classes of based maps $X \to Y$ and the set of path-components of $\mathrm{Map}_*(X, Y)$. This set will be denoted by $[X, Y]_*$, or $\langle X, Y\rangle$, called the **based homotopy class set**. Beside the Cartesian product, there is another *tensor product* of pointed spaces, which is the **smash product** $X \wedge Y$: it is precisely the pointed space obtained from the pair $(X \times Y, X \vee Y)$ by modulo the later, where $X \vee Y$ is the wedge sum. There is a natural base point of $\mathrm{Map}_*(X, Y)$, that is the map $\widetilde{y_0}\colon X \to \{y_0\}$. In the case X is good enough, there is a natural bijection

$$\mathrm{Map}_*(Z \wedge X, Y) \cong \mathrm{Map}_*(Z, \mathrm{Map}_*(X, Y)).$$

1.2 Before going further, notice that the natural objection

$$\mathrm{Map}(X \times I, Y) \cong \mathrm{Map}\left(X, \mathrm{Map}(I, Y)\right)$$

gives another equivalent definition of homotopy: let $f, g\colon X \to Y$ be two maps between topological spaces, a **right homotopy** $\Phi\colon f \Rightarrow g$ is a commutative diagram of the form

$$\begin{tikzcd} & Y \\ X \arrow[ur, "f"] \arrow[r, dashed, "\Phi"] \arrow[dr, "g"'] & \mathrm{Map}(I, Y) \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & Y \end{tikzcd}$$

where ev_0 (resp. ev_1) is the evaluation at $0 \in I$ (resp. $1 \in I$).

1.3 One can also define the notion of based homotopy using pure diagrammatic language. Write Y_+ for the pointed space obtained as the union of Y and a disjoint base point $*$. Note that if X is a pointed space, then $X \wedge Y_+$

can be identified with the one obtained from the pair $(X \times Y, \{*\} \times Y)$ and $\mathrm{Map}_*(Y_+, X)$ can be identified as $\mathrm{Map}(Y, X)$ specified the base point to be the map collapsing to the base point of X. Let $f, g\colon X \to Y$ be two based maps between pointed spaces. A **based homotopy** $\Phi\colon f \Rightarrow g$ can be defined as a commutative diagram of the form

$$\begin{tikzcd} X \arrow[d, "\mathrm{id} \wedge \delta_0"'] \arrow[dr, "f"] & \\ X \wedge I_+ \arrow[r, dashed, "\Phi"] & Y \\ X \arrow[u, "\mathrm{id} \wedge \delta_1"] \arrow[ur, "g"'] & \end{tikzcd}$$

where inclusions δ_i are viewed as $\{i\}_+ \hookrightarrow I_+$. Using the natural bijection for pointed spaces, a **based right homotopy** can be defined as the same commutative diagram for right homotopy with additional requirement that all maps involved must be based.

Remark The functor $Y \mapsto Y_+$ is in fact the left adjoint of the forgetful functor from **Top** to **Top**$_*$, the category of pointed spaces with based maps.

1.4 The topological spaces with continuous maps form a category **Top**. However, this category lost informations since it ignores the topologies on the mapping spaces. A better category is the one obtained by replacing every mapping space by the corresponding homotopy class set[1]. This can be done since homotopy respect the composition of maps. The result category is called **the homotopy category** $\mathcal{H}$. Two topological spaces are said to be **(strong) homotopy equivalent** if they are isomorphic in $\mathcal{H}$.

Similar discussion apply to relative and pointed spaces.

1.5 Let (X, x_0) be a pointed space. Then a **(based) loop** on (X, x_0) is a base-point-preserving map from $(S^1, *)$, where $*$ is a fixed base point of S^1, to it. $\mathrm{Map}_*(S^1, X)$ is called the **loop space** on it, denoted by $\Omega(X, x_0)$ or simply ΩX. There is a natural "multiplication" on this space: any two such loops can be concatenated to obtain a third loop. Although this "multiplication" is not associative, it does induce an associative multiplication on the quotient set $\pi_1(X, x_0)$ of it by modulo the based homotopies. The set $\pi_1(X, x_0)$ then carries a group structure and is called the **fundamental group** of (X, x_0).

Similarly, one can define the **n-th homotopy group** as $\pi_n(X, x_0) = [S^n, X]_*$ with the addition induced by $c\colon S^n \to S^n \vee S^n$ where c collapses a

[1] There is an issue that the notion of homotopy class sets, although can be defined for arbitrary topological spaces, does not behave well unless the topological space is good enough. Therefore, it is better to work on a subcategory of **Top** consisting of *good topological spaces*, or on a *convenient category of topological spaces* instead of **Top**. For the purpose of this note, we ignore this issue.

equator S^{n-1} (containing the base point) in S^n to the base point. As the notation suggests, $\pi_0(X, x_0)$ should be $[S^0, X]_*$, where S^0 is the 0-sphere, i.e. the set of two points with one of them being the base point. Note that there is no natural group structure on it anymore. Since S^0 is merely a set of two points and one of them must be mapped to x_0, the space $\mathrm{Map}_*(S^0, X)$ is homeomorphic to $\mathrm{Map}(\mathrm{pt}, X)$ and hence X itself. Thus $\pi_0(X, x_0)$ actually has nothing to do with x_0 and is precisely the set of path-components of X.

Note that for (X, A) a pair of space and subspace and (Y, y_0) a pointed space, there is a canonical bijection $[X, A; Y, y_0] \cong [X/A, [A]; Y, y_0]$. Thus the n-the homotopy group can also be defined as $[I^n, \partial I^n; X, x_0]$ with the addition induced by concatenation (there are n different ways to do this, but by the *Eckmann-Hilton argument*, they all give the same commutative binary operation on the homotopy class set). This characterization is easier to compute.

1.6 Note that we have a natural bijection

$$\mathrm{Map}_*(X \wedge S^1, Y) \cong \mathrm{Map}_*(X, \Omega Y)$$

for any pointed spaces X and Y. Let ΣX denote the pointed space $X \wedge S^1$. It is called the **suspension** of X. From this we get

$$\pi_n(X) = [\Sigma^n S^0, X]_* = \pi_0(\Omega^n X).$$

1.7 We can always view the loop space $\Omega(X, x_0)$ as a subspace of $\mathrm{Map}(I, X)$ by identify it as $\mathrm{Map}(I, \partial I; X, x_0)$. Note that there are two canonical maps from $\mathrm{Map}(I, X)$ to X: one maps $f\colon I \to X$ to $f(0)$, another to $f(1)$. If we ignore the issue that concatenation is not strict associative, those data defines a *topological groupoid*. To fix this issue, we can consider $[I, X]$ instead of $\mathrm{Map}(I, X)$. Then the result construction is a *groupoid*, called the **fundamental groupoid** of X and denoted by $\Pi_1(X)$. If X is good enough (locally path-connected and locally simply-connected), then $[I, X]$ has a natural topology on it and $\Pi_1(X)$ becomes a *topological groupoid*.

In any case, using those two maps, we obtain a bundle $[I, X] \to X \times X$ whose fiber at any point (x_0, x_0) in the diagonal is precisely $\pi_1(X, x_0)$. Thus, if we pullback it along the diagonal map $\Delta\colon X \to X \times X$, we obtain a bundle above X, or equivalently a sheaf on X. This is another realization of the notion of *fundamental groupoid*.

It is clear that the fundamental groupoid $\Pi_1(X)$ encodes the information of homotopies between points, i.e. paths connecting them, and is essentially (up to equivalences of categories) determined by $\pi_0(X)$ and $\pi_1(X, x_0)$ with x_0 go through a presenting system of $\pi_0(X)$.

1.8 Then one may try to obtain a higher analogy of fundamental groupoids. That is a *functorial* construction $\Pi(X)$ for each topological space X, which encodes the information of not only homotopies between points, but homotopies between homotopies, homotopies between those between homotopies and so on. Moreover, $\Pi(X)$ must be essentially determined by $\pi_0(X)$ and $\pi_n(X, x_0)$ for all n with x_0 go through a presenting system of $\pi_0(X)$. This object is called the **homotopy type** or **fundamental ∞-groupoid** of X.

The later terminology suggests it should be an ∞-groupoid, i.e. an ∞-category with all morphisms invertible. Ideally, for a given topological space X, its points should be the objects of $\Pi(X)$, homotopies between them should be 1-morphisms of $\Pi(X)$, homotopies between 1-morphisms should be 2-morphisms and so on. Conversely, there is a requirement of ∞-category theory called the **homotopy hypothesis**, which states that the ∞-category of ∞-groupoid is equivalent (in the sense of ∞-category theory) to the ∞-category of homotopy types.

A naïve approach is just define an ∞-groupoid as a topological space and an ∞-category as a category enriched over $\mathcal{H}$. However, this does not work due to the reason below.

1.9 Let $f\colon X \to Y$ be a map between topological spaces. We can view it as a based map by choosing a base point x_0 of X. Then, by composing with f, we obtain natural maps $\mathrm{Map}_*(S^n, X) \to \mathrm{Map}_*(S^n, Y)$ and hence homomorphisms $f_*\colon \pi_n(X) \to \pi_n(Y)$ for all n. f is called a **weak homotopy equivalence** if f_* is an isomorphism for all n and all choices of base point. Two topological spaces are said to be **weak homotopy equivalent**, or have the same **(weak) homotopy type** if there is a zigzag of weak homotopy equivalences between them. By the homotopy hypothesis, if $f\colon X \to Y$ is weak homotopy equivalence, then the induces morphism $f_*\colon \Pi(X) \to \Pi(Y)$ must be an equivalent of ∞-groupoid, or an isomorphism in $\mathcal{H}$.

It is not difficult to show that homotopy equivalences are weak homotopy equivalences. However, the converse is not true. Therefore to get the correct ∞-category theory, the homotopy category $\mathcal{H}$ should be modified such that two topological spaces are weak homotopy equivalent if and only if they are isomorphic in $\mathcal{H}$.

One way to do this is to restrict $\mathcal{H}$ to a suitable subcategory such that:

1) in this subcategory, every weak homotopy equivalence becomes an isomorphism;

2) every topological space is weak homotopy equivalent to an object in this subcategory.

1.10 There is a special class of topological spaces called **CW complexes**. For which we have

Whitehead theorem Every weak homotopy equivalence between CW complexes is a strong homotopy equivalence.

CW approximation Every topological space admits a weak homotopy equivalence from a CW complex to it.

Cellular approximation Every maps of CW complexes is homotopic to a cellular map, i.e. preserving the skeletons.

Thus, a good modification of $\mathcal{H}$ is to restrict it to the subcategory of CW complexes.

With this modification, we can built an ∞-category theory satisfying the homotopy hypothesis. To summary[2]:

1. **The homotopy category** $\mathcal{H}$ is the category of CW complexes whose morphisms are homotopy classes of maps between CW complexes. Furthermore, such a morphism can be presented by a cellular map.

2. Hence, an **∞-groupoid** is a CW complex and an **∞-category** is a category enriched over $\mathcal{H}$. This definition gives naturally a notion of **homotopy category** of an ∞-category, which is the plain category obtained by apply the *change of base categories* $\pi_0 \colon \mathcal{H} \to \mathbf{Set}$.

3. The **fundamental ∞-groupoid** of a topological space is then the CW approximation of it.

The above version of ∞-category theory provided a good framework to study homotopy theory and has the advantage that it is pretty geometric. However, it also has some disadvantages: it is not algebraic enough for general application and the constructions in CW complex theory involves cumbersome and irrelevant choices. Another well-developed ∞-category theory can be find in [5]. An axiomatic approach to ∞-category theory can be find in a book in progress [6].

1.11 Leaving the general ∞-category theory aside, let's return to the homotopy theory of topological spaces. First of first, the category **Top** of topological spaces now can be viewed as an ∞-category. Note that, in our setting, the **Hom space** from X to Y is not $\mathrm{Map}(X, Y)$, but its CW approximation. Let's denote it by $\mathcal{Hom}(X, Y)$.

1.12 A significant feature of ∞-category theory is it admits **homotopy limits** and **homotopy colimits**. To see the difference between those notions and *limits/colimits*, let's consider a simple diagram: $\bullet \to \bullet$. A digram of this shape in **Top** is just a continuous map $f \colon X \to Y$. It is easy to see that the limit (resp. colimit) of it is just X (resp. Y).

However, when consider homotopy limit of it, one looks at the category of *homotopy triangle above* f. An object of this category is a space T (called

[2] But there is still some pathological issue in this framework. A really workable definition needs to replace **Top** by a convenient category of topological spaces.

the *vertex*) together with a triangle

$$\begin{tikzcd} T \arrow[d] \arrow[dr] & \\ X \arrow[r, "f"'] \arrow[ur, phantom, "\Leftarrow"] & Y \end{tikzcd}$$

where the bold arrow denoted a homotopy. If $S \to T$ is a continuous map, then by composing it with a homotopy triangle above f with vertex T, we obtain a homotopy triangle above f with vertex S. A *morphism* between homotopy triangles is such a continuous map. Then the *homotopy limit* of the diagram $X \xrightarrow{f} Y$ is the terminal object in this category.

To spell out the homotopy limit, we translate the homotopy triangles into usual commutative diagrams

$$\begin{tikzcd} & Y \\ T \arrow[ur] \arrow[r] \arrow[d] & \mathrm{Map}(I,Y) \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

which is equivalent to the following diagram.

$$\begin{tikzcd} T \arrow[r] \arrow[d] & \mathrm{Map}(I,Y) \arrow[d, "\mathrm{ev}_1"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Therefore, the homotopy limit of the diagram $X \xrightarrow{f} Y$ is the pullback of $\mathrm{ev}_1 \colon \mathrm{Map}(I,Y) \to Y$ along f. More concretely, it is the space

$$Nf := \big\{(x,\gamma) \in X \times \mathrm{Map}(I,Y) : f(x) = \gamma(1)\big\}$$

equipped with the subspace topology. This space is called the **mapping path space** of f. It is clear that Nf is not homeomorphic to X in general. However, they are homotopy equivalent.

The similar story happens to the dual situation, where the homotopy triangle is eventually translated into the following diagram.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\delta_0"'] & Y \arrow[d] \\ X \times I \arrow[r] & T \end{tikzcd}$$

Therefore, the homotopy colimit of the diagram $X \xrightarrow{f} Y$ is the pushout of $\delta_0 \colon X \to X \times I$ along f. More concretely, it is the quotient space

$$\mathrm{Cly}(f) := X \times I \coprod Y/\sim,$$

where $\sim$ is generated by $(x, 0) \sim f(x)$. This space is called the **mapping cylinder** of f. It is clear that $\mathrm{Cly}(f)$ is not homeomorphic to X in general. However, they are homotopy equivalent.

Remark Note that in the above diagrams, one can invert the orientation of I, i.e. switch ev_0 and ev_1 (resp. δ_0 and δ_1), while the resulting space is homeomorphic to the one defined there.

1.13 However, the *homotopy limits/colimits* are even not limits/colimits in the homotopy category. To see this, let's consider the diagram $\bullet \to \bullet \leftarrow \bullet$. A diagram of this shape in **Top** is a pair of continuous maps $X \xrightarrow{f} Y \xleftarrow{g} Z$. Then a *homotopy square to* it is such a diagram

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & Z \arrow[d, "g"] \arrow[dl, dashed, Rightarrow] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

where the dashed arrow denote a homotopy. Such a homotopy diagram is equivalent to the following commutative diagram.

$$\begin{tikzcd}
Z \arrow[r, "g"] & Y \\
T \arrow[u] \arrow[r] \arrow[d] & \mathrm{Map}(I, Y) \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Hence, the homotopy limit of the diagram $X \xrightarrow{f} Y \xleftarrow{g} Z$ is the fiber product of Nf and Ng over $\mathrm{Map}(I, Y)$, that is the space

$$X \times^h_Y Z := \big\{(x, \gamma, z) \in X \times \mathrm{Map}(I, Y) \times Z : f(x) = \gamma(1), g(z) = \gamma(0)\big\}.$$

This space is called the **homotopy fiber product**, or the **homotopy pullback** of g along f.

Dually, one can consider the digram $X \xleftarrow{f} Y \xrightarrow{g} Z$ and the homotopy colimit of it is the fiber coproduct of $\mathrm{Cly}(f)$ and $\mathrm{Cly}(g)$ under $Y \times I$, which is the quotient space

$$X \amalg^h_Y Z := X \coprod (Y \times I) \coprod Z/\sim,$$

where $\sim$ is generated by $f(y) \sim (y, 0)$ and $(y, 1) \sim g(y)$. This is called the **homotopy fiber coproduct**. the **homotopy pushout** of f along g.

1.14 Now, let's consider a special case of previous constructions: where Z is the singleton pt. In this case, we can identify $X \times_Y^h \mathrm{pt}$ with the space

$$\mathrm{Fib}(f) := \big\{(x,\gamma) \in X \times \mathrm{Map}(I,Y) : f(x) = \gamma(1), \gamma(0) = *\big\},$$

where $*$ is the image of pt in Y. This space is called the **homotopy fiber** of f at the point $* \in Y$. We can identify $X \amalg_Y^h \mathrm{pt}$ as the quotient space

$$\mathrm{Cofib}(f) := X \coprod (Y \times I)/\sim,$$

where $\sim$ is generated by $f(y) \sim (y,0)$ and $(y,1) \sim (y',1)$. This space is called the **homotopy cofiber** of f, or the **mapping cone** of f with notation Cf.

1.15 Let's consider a even more special case: both X and Z are singleton pt and mapping to the same point $*$ of Y. In this case, we can surprisingly identify $\mathrm{pt} \times_Y^h \mathrm{pt}$ with the loop space ΩY by viewing Y as the pointed space with the base point $*$. It is clear that the loop space of a topological space is in general not contractible.

Besides, we can identify $\mathrm{pt} \amalg_Y^h \mathrm{pt}$ as the quotient space

$$SY := Y \times I/\sim,$$

where $\sim$ is generated by $(y,i) \sim (y',i)$ for $i = 0,1$. This space is called the **unreduced suspension** of Y. Let $Y = S^1$, it is clear that $SS^1 = S^2$, which is not contractible. Note that if Y is pointed as a base point $*$, then SY admits a distinguish subspace $\{*\} \times I$ and the quotient by modulo this subspace is the pointed space ΣY.

1.16 Recall that if $D\colon \mathcal{I} \to \mathcal{C}$ is a diagram in a category $\mathcal{C}$, then there are natural isomorphisms of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(-, \lim D) &\cong \lim \mathrm{Hom}_{\mathcal{C}}(-, D), \\ \mathrm{Hom}_{\mathcal{C}}(\operatorname{colim} D, -) &\cong \lim \mathrm{Hom}_{\mathcal{C}}(D, -). \end{aligned}$$

Analogously, if $D\colon \mathcal{I} \to \mathcal{C}$ is a diagram in a ∞-category $\mathcal{C}$, then there should be natural equivalences of (functors to) ∞-groupoids[3]

$$\begin{aligned} \mathcal{H}om_{\mathcal{C}}(-, \operatorname{holim} D) &\simeq \operatorname{holim} \mathcal{H}om_{\mathcal{C}}(-, D), \\ \mathcal{H}om_{\mathcal{C}}(\operatorname{hocolim} D, -) &\simeq \operatorname{holim} \mathcal{H}om_{\mathcal{C}}(D, -). \end{aligned}$$

Therefore, since we have worked out the homotopy limits/colimits of previous diagrams, we can make the following definitions in an arbitrary ∞-category $\mathcal{C}$.

[3] However, the right-hand side is not a CW complex in general. Hence one needs to replace it by its CW approximation and makes the statements meaningful only for weak homotopy equivalences. Consequently, the notions of homotopy limits/colimits make sense only up to weak homotopy equivalences.

(i) Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. Then a **mapping path object** is an object Nf inducing a natural equivalence of ∞-groupoids

$$\mathcal{H}om_{\mathcal{C}}(T, Nf) \simeq N\,\mathcal{H}om_{\mathcal{C}}(T, f),$$

where the continuous map $\mathcal{H}om_{\mathcal{C}}(T, f)\colon \mathcal{H}om_{\mathcal{C}}(T, X) \to \mathcal{H}om_{\mathcal{C}}(T, Y)$ is given by composing with f, for each object T of $\mathcal{C}$. Dually, a **mapping cylinder object** is an object $\mathrm{Cly}(f)$ inducing a natural equivalence of ∞-groupoids

$$\mathcal{H}om_{\mathcal{C}}\big(\mathrm{Cly}(f), T\big) \simeq P\,\mathcal{H}om_{\mathcal{C}}(f, T)$$

for each object T of $\mathcal{C}$.

(ii) Let $X \xrightarrow{f} Y \xleftarrow{g} Z$ be two morphisms in $\mathcal{C}$. Then a **homotopy fiber product** is a object $X \times^h_Y Z$ inducing a natural equivalence of ∞-groupoids

$$\mathcal{H}om_{\mathcal{C}}(T, X \times^h_Y Z) \simeq \mathcal{H}om_{\mathcal{C}}(T, X) \times^h_{\mathcal{H}om_{\mathcal{C}}(T,Y)} \mathcal{H}om_{\mathcal{C}}(T, Z),$$

where the right-hand side is obtained from the diagram

$$\mathcal{H}om_{\mathcal{C}}(T, X) \xrightarrow{f_*} \mathcal{H}om_{\mathcal{C}}(T, Y) \xleftarrow{g_*} \mathcal{H}om_{\mathcal{C}}(T, Z),$$

for each object T of $\mathcal{C}$.

(iii) As special cases of previous, we have the notions of **homotopy fiber** and **loop space object** (also called **looping**) in $\mathcal{C}$.

(iv) Let $X \xleftarrow{f} Y \xrightarrow{g} Z$ be two morphisms in $\mathcal{C}$. Then a **homotopy fiber coproduct** is a object $X \amalg^h_Y Z$ inducing a natural equivalence of ∞-groupoids

$$\mathcal{H}om_{\mathcal{C}}(X \amalg^h_Y Z, T) \simeq \mathcal{H}om_{\mathcal{C}}(X, T) \times^h_{\mathcal{H}om_{\mathcal{C}}(Y,T)} \mathcal{H}om_{\mathcal{C}}(Z, T),$$

where the right-hand side is obtained from the diagram

$$\mathcal{H}om_{\mathcal{C}}(X, T) \xrightarrow{f^*} \mathcal{H}om_{\mathcal{C}}(Y, T) \xleftarrow{g^*} \mathcal{H}om_{\mathcal{C}}(Z, T),$$

for each object T of $\mathcal{C}$.

(v) As special cases of previous, we have the notions of **homotopy cofiber** and **suspension object** in $\mathcal{C}$.

1.17 Apply the previous to the ∞-category $\mathbf{Top}_*$, we obtain the following constructions.

(i) The **mapping path space** of a based map $f\colon (X, x_0) \to (Y, y_0)$ is the same space as Nf with the base point $(x_0, \widetilde{y_0})$, where $\widetilde{y_0}$ is the constant path at y_0.

(ii) The **(reduced) mapping cylinder** of a based map $f\colon (X, x_0) \to (Y, y_0)$ is the quotient space

$$\mathrm{Cly}(f) := X \times I \coprod Y/\sim,$$

where $\sim$ is generated by $(x, 0) \sim f(x)$ and $(x_0, t) \sim (x_0, t')$, with the base point the class of $(x_0, 0)$.

(iii) The **homotopy fiber product** of a pair of based maps $(X, x_0) \xrightarrow{f} (Y, y_0) \xleftarrow{g} (Z, z_0)$ is the same space as $X \times_Y^h Z$ with the base point $(x_0, \widetilde{y_0}, z_0)$.

(iv) In particular, the **homotopy fiber** of a based map $f\colon (X, x_0) \to (Y, y_0)$ is the same space as $\mathrm{Fib}(f)$ with the base point $(x_0, \widetilde{y_0})$.

(v) In particular, the **looping** of pointed space (X, x_0) is the loop space ΩX with the based point the constant loop at x_0.

(vi) The **(reduced) homotopy fiber coproduct** of based maps $(X, x_0) \xleftarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$ is the quotient space

$$X \amalg_Y^h Z := X \coprod (Y \times I) \coprod Z/\sim,$$

where $\sim$ is generated by $f(y) \sim (y, 0)$, $(y, 1) \sim g(y)$ and $(y_0, t) \sim (y_0, t')$, with the base point the class of (y_0, t).

(vii) In particular, the **(reduced) homotopy cofiber** of a based map $f\colon (X, x_0) \to (Y, y_0)$ is the quotient space

$$\mathrm{Cofib}(f) := X \coprod (Y \times I)/\sim,$$

where $\sim$ is generated by $f(y) \sim (y, 0)$, $(y, 1) \sim (y', 1)$ and $(y_0, t) \sim (y_0, t')$, with the base point the class of (y_0, t).

(viii) In particular, the **(reduced) suspension** of pointed space (X, x_0) is the suspension ΣX.

1.18 Let $f\colon X \to Y$ be a map between topological spaces. The preimage $f^{-1}(y_0)$ of $y_0 \in Y$ is called the **fiber** of X at the point y_0. Viewing f as a based map by specifying y_0 as the base point of Y, the notion of fiber is similar to the notion of kernel: let $f\colon A \to B$ be a homomorphism between abelian groups, then the kernel is the preimage $f^{-1}(0)$.

Note that in the category of pointed spaces, the singleton pt is both an initial and terminal object, hence is a *zero object*. Let $\mathcal{C}$ be a category having pullbacks and a zero object $\mathbf{0}$. For $f\colon A \to B$ a morphism in $\mathcal{C}$, its **kernel** is the pullback of the zero morphism $\mathbf{0} \to B$ along f.

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r] \arrow[d] & \mathbf{0} \arrow[d] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Dually, if $\mathcal{C}$ has pushouts, the **cokernel** of f is the pushout of the zero morphism $A \to \mathbf{0}$ along f.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ \mathbf{0} \arrow[r] & \mathrm{Coker}(f) \end{tikzcd}$$

A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is called a **left exact sequence** if A is the kernel of g, a **right exact sequence** if C is the cokernel of f and a **short exact sequence** if both of previous are true.

In the category of pointed sets, or pointed spaces, we further have the notion of *exact sequence*: a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is said to be **exact** at Y if $\mathrm{im}(f) = \ker(g)$.

1.19 Let $\mathcal{C}$ be a category with terminal object pt. Then the category under pt has a zero object $\mathrm{pt} \to \mathrm{pt}$. This category is denoted by $\mathcal{C}_*$. An object $x_0 \colon \mathrm{pt} \to X$ in $\mathcal{C}_*$ is called a **pointed object** in $\mathcal{C}$, viewed as an object X in $\mathcal{C}$ with the **base point** x_0. A morphism in $\mathcal{C}_*$ is called a **based morphism**.

Suppose $\mathcal{C}$ has limits and colimits. Then we have the followings.

(i) The forgetful functor sending each pointed object (X, x_0) to X has a left adjoint $+\colon \mathcal{C} \to \mathcal{C}_*$ sending each object X to the pointed object $(X_+, *)$, where X_+ is the coproduct of X and pt and $*$ is the morphism $\mathrm{pt} \to X \amalg \mathrm{pt}$.

(ii) Therefore the limits of pointed objects can be computed in the category $\mathcal{C}$: it is precisely the limit together with the unique morphism obtained from the base points by the universal property.

(iii) Secondly, the colimits of pointed objects are obtained by apply the functor $+$ to the colimits of their underlying objects.

(iv) The coproduct of two pointed objects X, Y is called the **wedge sum** of them, denoted by $X \vee Y$. Clearly, there is canonical morphism $X \times Y \to X \vee Y$. The cokernel of this morphism is called the **smash product** and denoted by $X \wedge Y$.

Suppose $\mathcal{C}$ is further *Cartesian closed*, i.e. the functor $X \times -$ has a right adjoint $[X, -]$.

(v) Then the smash product gives $\mathcal{C}_*$ a closed symmetric monoidal structure: the unit is pt_+ and the internal Hom object $[X, Y]_*$ is obtained as the pullback of the morphism $\mathrm{pt} \to [\mathrm{pt}, Y]$ along $[X, Y] \to [\mathrm{pt}, Y]$ with the base point obtained from the morphism $\mathrm{pt} \to [X, Y]$ whose adjunct is the composition $\mathrm{pt} \times X \to \mathrm{pt} \to Y$.

1.20 Now, let $\mathcal{C}$ be a ∞-category having terminal object pt. Then we can define the ∞-category $\mathcal{C}_*$ of pointed objects as previous. Suppose $\mathcal{C}$ has homotopy pullbacks and homotopy pushouts. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of is called a **fibration sequence** if X is a homotopy fiber of g and a **cofibration sequence** if Z is a homotopy cofiber of f. Unlike left/right exact sequences, fibration/cofibration sequences are automatically long.

Indeed, let $f\colon X \to Y$ be a based morphism of pointed objects in $\mathcal{C}$. Then, we have the fibration sequence

$$\mathrm{Fib}(f) \xrightarrow{i} X \xrightarrow{f} Y.$$

Consider the *reversed* homotopy fiber $\bar{\mathrm{Fib}}(i)$ of i. To see what does this means and why we need this, look at the following diagram

$$\begin{tikzcd}
\bar{\mathrm{Fib}}(i) \arrow[r] \arrow[d] & \mathrm{Fib}(f) \arrow[r] \arrow[d, "i"'] \arrow[dl, Rightarrow, dashed] & \mathrm{pt} \arrow[d] \arrow[dl, Rightarrow, dashed] \\
\mathrm{pt} \arrow[r] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

where the right square exhibits $\mathrm{Fib}(f)$ as the homotopy fiber of f while the left square, instead of exhibiting $\bar{\mathrm{Fib}}(i)$ as the homotopy fiber of i which is the homotopy pullback of $\mathrm{pt} \to X$ along i, exhibits $\bar{\mathrm{Fib}}(i)$ as the homotopy pullback of i along $\mathrm{pt} \to X$. Note that, by pasting the two squares, the rectangle becomes a homotopy square and exhibits $\bar{\mathrm{Fib}}(i)$ as the homotopy pullback of $\mathrm{pt} \to Y$ along itself, i.e. the *loop space object* ΩY. Note that, by our construction, the reversed homotopy fiber and the homotopy fiber are canonically isomorphic[4]. Therefore we have anther fibration sequence

$$\Omega Y \longrightarrow \mathrm{Fib}(f) \xrightarrow{i} X.$$

If we keep going, obtaining the following diagram

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d, "-\Omega f"'] & \mathrm{pt} \arrow[d] \arrow[dl, Rightarrow, dashed] & \\
\Omega Y \arrow[r] \arrow[d] & \mathrm{Fib}(f) \arrow[r] \arrow[d, "i"'] \arrow[dl, Rightarrow, dashed] & \mathrm{pt} \arrow[d] \arrow[dl, Rightarrow, dashed] \\
\mathrm{pt} \arrow[r] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

where the $-\Omega f$ denotes the *reversed* loop morphism. The reversion appear due to the reversed homotopy in the left-below square.

[4] In fact, since the notions of homotopy limits only make sense up to weak homotopy equivalences, the statement here is literally wrong. However, it is true that the constructions of reversed homotopy fiber (which is given by just invert I in the construction of the homotopy fiber) and the homotopy fiber given in **Top** and $\mathbf{Top}_*$ are canonically homeomorphic.

Therefore, if we have a fibration sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$, then we have a *long fibration sequence*

$$\cdots \longrightarrow \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z.$$

The similar story applies to cofibration sequences. If we have a cofibration sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$, then we have a *long cofibration sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma g} \Sigma Z \longrightarrow \cdots .$$

1.21 Let $f\colon X \to Y$ be a morphism in $\mathcal{C}_*$. The adjunction of Σ and Ω gives rise to the following commutative diagram.

$$\begin{tikzcd}
 & \Sigma\Omega\mathrm{Fib}(f) \arrow[r] \arrow[d] & \Sigma\Omega X \arrow[r] \arrow[d] & \Sigma\Omega Y \arrow[r] \arrow[d] & \Sigma\Omega\mathrm{Cofib}(f) \arrow[r] \arrow[d] & \Sigma\Omega\Sigma X \arrow[d] \\
\Omega Y \arrow[r] \arrow[d] & \mathrm{Fib}(f) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & \mathrm{Cofib}(f) \arrow[r] \arrow[d] & \Sigma X \\
\Omega\Sigma\Omega Y \arrow[r] & \Omega\Sigma\mathrm{Fib}(f) \arrow[r] & \Omega\Sigma X \arrow[r] & \Omega\Sigma Y \arrow[r] & \Omega\Sigma\mathrm{Cofib}(f) &
\end{tikzcd}$$

Considering the following homotopy commutative diagram:

$$\begin{tikzcd}
\mathrm{Fib}(f) \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \arrow[dl, Rightarrow, dashed] \\
X \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dl, Rightarrow, dashed] \\
\mathrm{pt} \arrow[r] & \mathrm{Cofib}(f)
\end{tikzcd}$$

one see that there are homotopy equivalence:

$$\mathrm{Fib}(f) \xrightarrow{\sim} \Omega\mathrm{Cofib}(f), \qquad \Sigma\mathrm{Fib}(f) \xrightarrow{\sim} \mathrm{Cofib}(f).$$

Together with the triangle identities for the $\Sigma \dashv \Omega$, we obtain the following commutative diagram

$$\begin{tikzcd}
 & \Sigma\Omega\mathrm{Fib}(f) \arrow[r] \arrow[d] & \Sigma\Omega X \arrow[r] \arrow[d] & \Sigma\Omega Y \arrow[r] \arrow[d] & \Sigma\mathrm{Fib}(f) \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
\Omega Y \arrow[r] \arrow[d, equal] & \mathrm{Fib}(f) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & \mathrm{Cofib}(f) \arrow[r] \arrow[d] & \Sigma X \\
\Omega Y \arrow[r] & \Omega\mathrm{Cofib}(f) \arrow[r] & \Omega\Sigma X \arrow[r] & \Omega\Sigma Y \arrow[r] & \Omega\Sigma\mathrm{Cofib}(f) &
\end{tikzcd}$$

where the top row is the suspension of a fiber sequence and the bottom row is the looping of a cofiber sequence.

1.22 It turns out that the functor $[Z,-]_*\colon \mathbf{Top}_* \to \mathbf{Set}_*$ is left exact for any pointed space Z. In particular, π_0 is left exact. So, if we have a fiber sequence of pointed spaces

$$\cdots \longrightarrow \Omega^2 Z \longrightarrow \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z.$$

Notice that $\pi_0(\Omega^n X) = \pi_n(X)$. Then we get a long exact sequence of pointed sets

$$\begin{aligned}\cdots \longrightarrow \pi_2(X) \xrightarrow{f_*} \pi_2(Y) \xrightarrow{g_*} \pi_2(Z) \longrightarrow& \\ \pi_1(X) \xrightarrow{f_*} \pi_1(Y) \xrightarrow{g_*} \pi_1(Z) \longrightarrow& \pi_0(X) \xrightarrow{f_*} \pi_0(Y) \xrightarrow{g_*} \pi_0(Z).\end{aligned}$$

Moreover, since π_0 is left exact, the above maps preserve group structures if there exists one.

For $\mathcal{C}$ an ∞-category and C any object in $\mathcal{C}_*$, the functor $\mathcal{H}om_{\mathcal{C}_*}(C,-)$ is left exact, i.e. preserves homotopy limits. Hence, if we have a fiber sequence of pointed objects

$$\cdots \longrightarrow \Omega^2 Z \longrightarrow \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z.$$

Then we get a fiber sequence of pointed spaces

$$\begin{aligned}&\cdots \longrightarrow \mathcal{H}om_{\mathcal{C}_*}(C,\Omega^2 Z) \longrightarrow \mathcal{H}om_{\mathcal{C}_*}(C,\Omega X) \xrightarrow{f_*} \mathcal{H}om_{\mathcal{C}_*}(C,\Omega Y) \xrightarrow{g_*} \\ &\mathcal{H}om_{\mathcal{C}_*}(C,\Omega Z) \longrightarrow \mathcal{H}om_{\mathcal{C}_*}(C,X) \xrightarrow{f_*} \mathcal{H}om_{\mathcal{C}_*}(C,Y) \xrightarrow{g_*} \mathcal{H}om_{\mathcal{C}_*}(C,Z),\end{aligned}$$

and thus a long exact sequence of pointed sets

$$\begin{aligned}&\cdots \longrightarrow \pi_0\,\mathcal{H}om_{\mathcal{C}_*}(C,\Omega^2 Z) \longrightarrow \\ &\quad \pi_0\,\mathcal{H}om_{\mathcal{C}_*}(C,\Omega X) \xrightarrow{f_*} \pi_0\,\mathcal{H}om_{\mathcal{C}_*}(C,\Omega Y) \xrightarrow{g_*} \pi_0\,\mathcal{H}om_{\mathcal{C}_*}(C,\Omega Z) \\ &\qquad \longrightarrow \pi_0\,\mathcal{H}om_{\mathcal{C}_*}(C,X) \xrightarrow{f_*} \pi_0\,\mathcal{H}om_{\mathcal{C}_*}(C,Y) \xrightarrow{g_*} \pi_0\,\mathcal{H}om_{\mathcal{C}_*}(C,Z),\end{aligned}$$

where the maps preserve (possibly exist) group structures. To simplify notation, denote $\pi_0\,\mathcal{H}om_{\mathcal{C}_*}(-,-)$ by $\langle -,-\rangle$ if there is no ambiguity. Dually, the functor $\mathcal{H}om_{\mathcal{C}_*}(-,C)$ sends homotopy colimits to homotopy limits. Hence, if we have a cofiber sequence of pointed objects

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma g} \Sigma Z \longrightarrow \Sigma^2 X \longrightarrow \cdots.$$

Then we get a fiber sequence of pointed spaces

$$\begin{aligned}&\cdots \longrightarrow \mathcal{H}om_{\mathcal{C}_*}(\Sigma^2 X,C) \longrightarrow \mathcal{H}om_{\mathcal{C}_*}(\Sigma Z,C) \xrightarrow{g^*} \mathcal{H}om_{\mathcal{C}_*}(\Sigma Y,C) \xrightarrow{f^*} \\ &\mathcal{H}om_{\mathcal{C}_*}(\Sigma X,C) \longrightarrow \mathcal{H}om_{\mathcal{C}_*}(Z,C) \xrightarrow{g^*} \mathcal{H}om_{\mathcal{C}_*}(Y,C) \xrightarrow{f^*} \mathcal{H}om_{\mathcal{C}_*}(X,C),\end{aligned}$$

and thus a long exact sequence of pointed sets

$$\cdots \longrightarrow \langle \Sigma^2 X, C\rangle \longrightarrow \langle \Sigma Z, C\rangle \xrightarrow{g^*} \langle \Sigma Y, C\rangle \xrightarrow{f^*}$$
$$\langle \Sigma X, C\rangle \longrightarrow \langle Z, C\rangle \xrightarrow{g^*} \langle Y, C\rangle \xrightarrow{f^*} \langle X, C\rangle,$$

where the maps preserve (possibly exist) group structures. The above two exact sequences are related by the following identification of pointed sets

$$\pi_0 \mathcal{H}om_{\mathcal{C}_*}(\Sigma^n X, Y) = \pi_0 \mathcal{H}om_{\mathcal{C}_*}(X, \Omega^n Y) = \pi_n \mathcal{H}om_{\mathcal{C}_*}(X, Y).$$

1.23 Let $\mathcal{C}$ be an ∞-category. An **Ω-spectrum** $\mathbb{E}$ is a sequence of pointed objects $\{E_n\}_{n\in\mathbb{N}}$ together with weak equivalences $f_n\colon E_n \to \Omega E_{n+1}$. Here f_n is a weak equivalence means $f_{n*}\colon \pi_n \mathcal{H}om_{\mathcal{C}_*}(C, E_n) \to \pi_n \mathcal{H}om_{\mathcal{C}_*}(C, \Omega E_{n+1})$ are isomorphisms for all n and pointed object C.

Then, once we have a cofibration sequence $X \to Y \to Z$, we have long exact sequences of pointed sets

$$\cdots \longrightarrow \langle \Sigma Z, E_n\rangle \longrightarrow \langle \Sigma Y, E_n\rangle \longrightarrow \langle \Sigma X, E_n\rangle$$
$$\longrightarrow \langle Z, E_n\rangle \longrightarrow \langle Y, E_n\rangle \longrightarrow \langle X, E_n\rangle,$$

for all n. By the adjunction $\Sigma \dashv \Omega$ and the definition of Ω-spectrum, we deduce a long exact sequence of abelian groups

$$\cdots \longrightarrow \langle Z, E_{n-1}\rangle \longrightarrow \langle Y, E_{n-1}\rangle \longrightarrow \langle X, E_{n-1}\rangle$$
$$\longrightarrow \langle Z, E_n\rangle \longrightarrow \langle Y, E_n\rangle \longrightarrow \langle X, E_n\rangle \longrightarrow$$
$$\langle Z, E_{n+1}\rangle \longrightarrow \langle Y, E_{n+1}\rangle \longrightarrow \langle X, E_{n+1}\rangle \longrightarrow \cdots .$$

Let $H^n(X, \mathbb{E})$ denote $\langle X, E_n\rangle$. Then using above discussion, it is easy to show that $H^n(-, \mathbb{E})$ defines a generalized cohomology theory, i.e it satisfies analogy of Eilenberg-Steenrod axioms. This functor is called the **intrinsic cohomology** with coefficient $\mathbb{E}$.

Note that at the beginning of the long exact sequence we have

$$H^0(Z, \mathbb{E}) \longrightarrow H^0(Y, \mathbb{E}) \longrightarrow H^0(X, \mathbb{E}) \longrightarrow \cdots$$

but that not all, we further have

$$\cdots \longrightarrow \langle \Sigma Z, E_0\rangle \longrightarrow \langle \Sigma Y, E_0\rangle \longrightarrow \langle \Sigma X, E_0\rangle$$
$$\longrightarrow H^0(Z, \mathbb{E}) \longrightarrow H^0(Y, \mathbb{E}) \longrightarrow H^0(X, \mathbb{E}).$$

Note that

$$\langle \Sigma^n(-), E_0\rangle = \pi_n \mathcal{H}om_{\mathcal{C}_*}(-, E_0).$$

Hence we conclude that if we want to extend the intrinsic cohomology to negative degrees so that we have long exact sequence tending to both directions, then we have to put

$$H^{-n}(-, \mathbb{E}) = \pi_n \mathcal{H}om_{\mathcal{C}_*}(-, E_0).$$

In other words, *negative cohomology groups are homotopy groups.*

§ 2 Chain complexes

2.1 Let I be a set and $\mathcal{C}$ a category. An ***I*-graded object** in $\mathcal{C}$ is a functor from I, viewed as a discrete category, to $\mathcal{C}$. Hence the category of I-graded objects is denoted by $\mathcal{C}^I$. In plain words, an I-graded object is a family of objects $\{X_i\}_{i\in I}$ in $\mathcal{C}$ indexed by I. We denote it by $X_\bullet$ or simply X if there is no ambiguity. A $\mathbb{Z}$-graded object is simply called a **graded object** and the category $\mathcal{C}^{\mathbb{Z}}$ will be denoted by $\mathrm{Gr}(\mathcal{C})$. A **morphism** between I-graded objects $f\colon X \to Y$ is thus a family of morphisms $\{f\colon X_i \to Y_i\}_{i\in I}$ in $\mathcal{C}$ indexed by I. In other words,

$$\mathrm{Hom}_{\mathcal{C}^I}(X,Y) = \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_i).$$

Let $\iota\colon \mathcal{C} \to \mathcal{C}^I$ be the functor sending each object Y to the I-graded object $\underline{Y}$ whose each degree is Y. Then we have a functor

$$\mathrm{Hom}_{\mathcal{C}^I}(X,\iota)\colon \mathcal{C} \longrightarrow \mathbf{Set}.$$

Suppose $\mathcal{C}$ has direct sums, then the above functor can be represented by the direct sum

$$\bigoplus_{i\in I} X_i.$$

We call it the representative of X and denoted also by X.

2.2 Now, suppose G is a commutative monoid. Let X be a G-graded object and g an element of G. The ***g*-twisted object** of X is the G-graded object $X(g)$ defined as

$$X(g)_u := X_{g+u}, \quad \forall u \in G.$$

Let X, Y be two G-graded objects. A morphism from X to $Y(g)$ is called a ***g*-twisted morphism** from X to Y. The 0-twisted morphisms are the usual morphisms can called **homogeneous morphisms**. The G-graded set defined by

$$\mathrm{Hom}(X,Y)_g := \mathrm{Hom}_{\mathcal{C}^G}(X, Y(g))$$

is called the ***G*-graded Hom**.

2.3 Now, suppose $\mathcal{A}$ is an *abelian tensor category*. For A, B two G-graded objects in $\mathcal{A}$, their **tensor product** is defined by

$$(A\otimes B)_g := \bigoplus_{u+v=g} (A_u \otimes B_v), \quad \forall g \in G.$$

In this way, $\mathcal{A}^G$ becomes an abelian tensor category. If furthermore $\mathcal{A}$ is *closed*, admitting internal Hom bifunctor $[-,-]\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{A}$. Then $\mathcal{A}^G$ can be viewed as a $\mathcal{A}$-enriched category by setting the *Hom-object* as

$$\underline{\mathrm{Hom}}_{\mathcal{A}^G}(A,B) := \prod_{g\in G} [A_g, B_g].$$

Moreover, we define the **internal G-graded Hom-object** by

$$[A, B]_g := \underline{\mathrm{Hom}}_{\mathcal{A}^G}(A, B(g)).$$

The internal G-graded Hom-objects turn to be the *internal Hom-objects* in $\mathcal{A}^G$ and we have the following (enriched) adjunctions:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}^G}(A \otimes B, C) &\cong \mathrm{Hom}_{\mathcal{C}^G}(A, [B, C]),\\
\underline{\mathrm{Hom}}_{\mathcal{C}^G}(A \otimes B, C) &\cong \underline{\mathrm{Hom}}_{\mathcal{C}^G}(A, [B, C]),\\
[A \otimes B, C] &\cong [A, [B, C]].
\end{aligned}$$

(However, to prove the above statements, one needs to deal with $\mathcal{A}^G$-enrichment first and then apply the obverse *change of base categories* $\mathcal{A}^G \to \mathcal{A}$.)

2.4 Let $\mathcal{C}$ be a category admitting a *zero object* 0.

(i) A **chain complex** in $\mathcal{C}$ is a graded object endowed with a (-1)-twisted endomorphism ∂, called the **boundary operator** or **codifferential**, such that $\partial \circ \partial = 0$. We use the notation $X_\bullet$ to indicate it is a chain complex.

(ii) Dually, a **cochain complex** in $\mathcal{C}$ is a graded object endowed with a 1-twisted endomorphism d, called the **differential** or **coboundary operator**, such that $\mathrm{d} \circ \mathrm{d} = 0$. We use the notation $X^\bullet$ to indicate it is a cochain complex.

(iii) Let $X_\bullet, Y_\bullet$ be two chain complexes. A **chain morphism** $f\colon X_\bullet \to Y_\bullet$ between them is a homogeneous morphism such that the following diagrams commute.

$$\begin{tikzcd}
\cdots \arrow[r] & X_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & X_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & Y_n \arrow[r, "\partial_n"] & Y_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

(iv) Dually, let $X^\bullet, Y^\bullet$ be two cochain complexes. A **cochain morphism** $f\colon X^\bullet \to Y^\bullet$ between them is a homogeneous morphism such that the following diagrams commute.

$$\begin{tikzcd}
\cdots \arrow[r] & X^n \arrow[r, "\mathrm{d}^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & Y^n \arrow[r, "\mathrm{d}^n"] & Y^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

The category of chain complexes (resp. cochain complexes) in $\mathcal{C}$ with chain morphisms (resp. cochain morphisms) between them is denoted by $\mathbf{Ch}_*(\mathcal{C})$ (resp. $\mathbf{Ch}^*(\mathcal{C})$). Note that this category also has a zero object $\underline{0}$ whose each degree is 0.

2.5 A chain complex $X_\bullet$ is said to be

- **connective** if $X_n = 0$ for all $n < 0$;
- **coconnective** if $X_n = 0$ for all $n > 0$;
- **bounded above** if $X_n = 0$ for sufficiently large n;
- **bounded below** if $X_n = 0$ for sufficiently small n;
- **bounded** if it is both bounded above and bounded below.

The full subcategory of $\mathbf{Ch}_*(\mathcal{C})$ spanned by connective (resp. coconnective, bounded above, bounded below, bounded) chain complexes is denoted by $\mathbf{Ch}_c(\mathcal{C})$ or $\mathbf{Ch}_{\geqslant 0}(\mathcal{C})$ (resp. $\mathbf{Ch}_{\leqslant 0}(\mathcal{C})$, $\mathbf{Ch}_-(\mathcal{C})$, $\mathbf{Ch}_+(\mathcal{C})$, $\mathbf{Ch}_b(\mathcal{C})$).

Dually, a cochain complex $X^\bullet$ is said to be

- **coconnective** if $X^n = 0$ for all $n < 0$;
- **connective** if $X^n = 0$ for all $n > 0$;
- **bounded above** if $X^n = 0$ for sufficiently large n;
- **bounded below** if $X^n = 0$ for sufficiently small n;
- **bounded** if it is both bounded above and bounded below.

The full subcategory of $\mathbf{Ch}^*(\mathcal{C})$ spanned by connective (resp. coconnective, bounded above, bounded below, bounded) chain complexes is denoted by $\mathbf{Ch}^c(\mathcal{C})$ or $\mathbf{Ch}^{\leqslant 0}(\mathcal{C})$ (resp. $\mathbf{Ch}^{\geqslant 0}(\mathcal{C})$, $\mathbf{Ch}^-(\mathcal{C})$, $\mathbf{Ch}^+(\mathcal{C})$, $\mathbf{Ch}^b(\mathcal{C})$).

2.6 Any chain complex $X_\bullet$ can be transformed into a cochain complex by

$$X^n := X_{-n}, \qquad \mathrm{d}^n := \partial_{-n}$$

and *vice versa*. Thus we can identify the following two categories

$$\mathbf{Ch}_*(\mathcal{C}) \cong \mathbf{Ch}^*(\mathcal{C})$$

and safely use the notation $\mathbf{Ch}(\mathcal{C})$ instead of $\mathbf{Ch}_*(\mathcal{C})$ or $\mathbf{Ch}^*(\mathcal{C})$ to denote those categories. In this sense, we can safely use the terminology **complex** to indicate both chain complexes and cochain complexes, and **morphism of complexes** to indicate both chain morphisms and cochain morphisms.

On the other hand, one can see that chain complexes in $\mathcal{C}$ are the same as cochain complexes in $\mathcal{C}^{\mathrm{op}}$, hence

$$\mathbf{Ch}_*(\mathcal{C})^{\mathrm{op}} = \mathbf{Ch}^*(\mathcal{C}^{\mathrm{op}}).$$

So we can canonically identify $\mathbf{Ch}(\mathcal{C}^{\mathrm{op}})$ and $\mathbf{Ch}(\mathcal{C})^{\mathrm{op}}$.

Restricting the full subcategories mentioned before, we have the following natural isomorphisms

$$\mathbf{Ch}_{\geqslant 0}(\mathcal{C})^{\mathrm{op}} = \mathbf{Ch}^{\geqslant 0}(\mathcal{C}^{\mathrm{op}}) \cong \mathbf{Ch}_{\leqslant 0}(\mathcal{C}^{\mathrm{op}}),$$
$$\mathbf{Ch}_{\leqslant 0}(\mathcal{C})^{\mathrm{op}} = \mathbf{Ch}^{\leqslant 0}(\mathcal{C}^{\mathrm{op}}) \cong \mathbf{Ch}^{\geqslant 0}(\mathcal{C}^{\mathrm{op}}).$$

Therefore, we can identify connective (resp. coconnective) chain complexes with connective (resp. coconnective) cochain complexes and call simply call them *connective (resp. coconnective) complexes*. In practice, the terminology **connective complexes** often refers to connective chain complexes while **coconnective complexes** to coconnective cochain complexes.

We also have the following natural isomorphisms

$$\mathbf{Ch}_{-}(\mathcal{C})^{\mathrm{op}} = \mathbf{Ch}^{-}(\mathcal{C}^{\mathrm{op}}) \cong \mathbf{Ch}_{+}(\mathcal{C}^{\mathrm{op}}),$$
$$\mathbf{Ch}_{+}(\mathcal{C})^{\mathrm{op}} = \mathbf{Ch}^{+}(\mathcal{C}^{\mathrm{op}}) \cong \mathbf{Ch}_{-}(\mathcal{C}^{\mathrm{op}}),$$
$$\mathbf{Ch}_{b}(\mathcal{C})^{\mathrm{op}} = \mathbf{Ch}^{b}(\mathcal{C}^{\mathrm{op}}) \cong \mathbf{Ch}_{b}(\mathcal{C}^{\mathrm{op}}).$$

Hence, we can identify bounded above (resp. bounded below) chain complexes with bounded below (resp. bounded above) cochain complexes. In this sense bounded above and bounded below chain complexes are dual notions while the notion of bounded complexes is self-dual.

We say a complex $X_\bullet$ is **concentrated** at degree $n_1, \cdots, n_k$ if $X_i = 0$ unless $i = n_1, \cdots, n_k$. It is clear that concentrated complexes are bounded complexes and *vice versa*.

2.7 There are many ways to embed $\mathcal{C}$ into the category $\mathbf{Ch}(\mathcal{C})$. Let X be an object in $\mathcal{C}$.

(i) The complex $\underline{X}_\bullet$ has X at its every degree and 0 as its boundary operator.
(ii) The complex $X[n]$ concentrated at degree $-n$ with component X.
(iii) We simply denote $X[0]$ by X if there is no ambiguity.

The notation $X[n]$ suggests that this complex is obtained by apply a **translation of degree n** functor to the complex X.

In the case $\mathcal{C}$ is an additive category, the functor $[n]$ is defined as follows. Let $X_\bullet$ be a complex. Then the complex $X[n]_\bullet$ is defined by

$$X[n]_i := X_{n+i}, \qquad \partial_{X[n]} := (-1)^n \partial_X, \qquad \forall i \in \mathbb{Z}.$$

Let f be a chain morphism. Then the chain morphism $f[n]$ is defined by $f[n]_i = f_{n+i}$ for all $i \in \mathbb{Z}$.

Remark Note that the functor $[n]$ on $\mathbf{Ch}^*(\mathcal{C})$ is usually defined by

$$X[n]^i := X^{n+i}, \qquad \mathrm{d}^{X[n]} := (-1)^n \mathrm{d}^X, \qquad \forall i \in \mathbb{Z}.$$

Under this setting, we encounter that

$$X[n]^{\bullet} \neq X[n]_{-\bullet},$$

which goes against our identification!

One should rather think the functor $[n]$ as an extension of (n) from the category of graded objects (which can be viewed as complexes with zero differentials) to the category of complexes. So $X[n]$ is not a complex unless we specify it is a chain complex or cochain complex.

2.8 When $\mathcal{C} = \mathbf{Ab}$, the category of abelian groups, we simply denote $\mathbf{Ch}(\mathbf{Ab})$ by $\mathbf{Ch}$. More generally, let k be a ring and $\mathcal{C} = k\mathbf{Mod}$, the category of k-modules, we simply denote $\mathbf{Ch}(k\mathbf{Mod})$ by $\mathbf{Ch}(k)$. The notations for subcategories $\mathbf{Ch}_?$ and $\mathbf{Ch}^?$ ($?$ equals $c, \geqslant 0, \leqslant 0, +, -, b$) are similar.

2.9 From now on, let $\mathcal{A}$ be an abelian category. When $\mathcal{A}$ is $\mathbf{Ab}$ or $k\mathbf{Mod}$, we can talk about *elements* of an object. For general abelian tensor category, a **global element** of an object refers to a morphism from the unit to it, and a **(general) element** refers to a morphism from arbitrary object.

Let $(C_\bullet, \partial)$ be a chain complex in $\mathcal{A}$.

(i) The n-th **cycle object** of $C_\bullet$ is $Z_n(C) := \operatorname{Ker} \partial_n$, whose elements are called **n-cycles**.

(ii) The n-th **boundary object** of $C_\bullet$ is $B_n(C) := \operatorname{Im} \partial_{n+1}$, whose elements are called **n-boundaries**.

Since $\partial \circ \partial = 0$, the inclusion $B_n(C) \rightarrowtail C_n$ factors through $Z_n(C)$.

(iii) The cokernel of the resulted inclusion $B_n(C) \rightarrowtail Z_n(C)$ is called the n-th **homology object** of $C_\bullet$ and denoted by $H_n(C)$. The elements of $H_n(C)$ are called **homology classes**.

Dually, let $(C^\bullet, \mathrm{d})$ be a cochain complex in $\mathcal{A}$.

(iv) The n-th **cocycle object** of $C^\bullet$ is $Z^n(C) := \operatorname{Ker} \mathrm{d}_n$, whose elements are called **n-cocycles**.

(v) The n-th **coboundary object** of $C^\bullet$ is $B^n(C) := \operatorname{Im} \mathrm{d}_{n-1}$, whose elements are called **n-coboundaries**.

Since $\mathrm{d} \circ \mathrm{d} = 0$, the inclusion $B^n(C) \rightarrowtail C^n$ factors through $Z^n(C)$.

(vi) The cokernel of the resulted inclusion $B^n(C) \rightarrowtail Z^n(C)$ is called the n-th **cohomology object** of $C^\bullet$ and denoted by $H^n(C)$. The elements of $H^n(C)$ are called **cohomology classes**.

The above constructions extend to the following additive functors

$$Z_\bullet, B_\bullet, H_\bullet \colon \mathbf{Ch}_*(\mathcal{A}) \longrightarrow \mathcal{A}^{\mathbb{Z}},$$
$$Z^\bullet, B^\bullet, H^\bullet \colon \mathbf{Ch}^*(\mathcal{A}) \longrightarrow \mathcal{A}^{\mathbb{Z}}.$$

In particular, any chain morphism $f\colon C_\bullet \to D_\bullet$ (resp. cochain morphism $f\colon C^\bullet \to D^\bullet$) induces a homogeneous morphism

$$H(f)\colon H_\bullet(C) \to H_\bullet(D). \qquad \text{(resp. } H(f)\colon H^\bullet(C) \to H^\bullet(D))$$

Obviously, if f is an isomorphism, then so is $H(f)$. But the converse may not be true. A chain morphism (resp. cochain morphism) f is called a **quasi-isomorphism** if $H(f)$ is an isomorphism. A chain complex $C_\bullet$ (resp. cochain complex $C^\bullet$) is said to be **acyclic** if it is *quasi-isomorphic* to 0.

2.10 Since complexes is a special kind of diagrams, the limits and colimits in $\mathbf{Ch}(\mathcal{A})$ are computed degree-wisely. Note that filtered colimits commute with finite limits and all colimits, hence by the construction of the functors $B_\bullet, Z_\bullet$ and $H_\bullet$ (resp. $B^\bullet, Z^\bullet$ and $H^\bullet$), they preserve filtered colimits.

2.11 Suppose $\mathcal{A}$ is an abelian tensor category. Let $C_\bullet$, $D_\bullet$ be two complexes. Then there exists a natural boundary operator ∂ on the tensor product $(C \otimes D)_\bullet$ of their underlying graded objects. The resulted complex is called the **Koszul product** of $C_\bullet$ and $D_\bullet$. By its construction, we only need to define the following morphisms

$$C_p \otimes D_q \xrightarrow{\partial^{(1)}_{p,q}} C_{p-1} \otimes D_q, \qquad C_p \otimes D_q \xrightarrow{\partial^{(2)}_{p,q}} C_p \otimes D_{q-1}.$$

Note that the condition $\partial \circ \partial = 0$ requires that the following two morphisms must be negative to each other.

$$C_p \otimes D_q \xrightarrow{\partial^{(2)}_{p,q}} C_p \otimes D_{q-1} \xrightarrow{\partial^{(1)}_{p,q-1}} C_{p-1} \otimes D_{q-1},$$
$$C_p \otimes D_q \xrightarrow{\partial^{(1)}_{p,q}} C_{p-1} \otimes D_q \xrightarrow{\partial^{(2)}_{p-1,q}} C_{p-1} \otimes D_{q-1}.$$

The common convention is

$$\partial^{(1)}_{p,q} := \partial_p \otimes \mathrm{id}_{D_q}, \qquad \partial^{(2)}_{p,q} := (-1)^p \,\mathrm{id}_{C_p} \otimes \partial_q.$$

In element notation, it reads

$$\partial(x \otimes y) = \partial x \otimes y + (-1)^{|x|} x \otimes \partial y,$$

where $|x|$ denotes the degree of x. Then one can verify that the above construction makes $\mathbf{Ch}(\mathcal{A})$ into an abelian tensor category with the unit $\mathbf{1}$, which is $\mathbf{1}[0]_\bullet$ with $\mathbf{1}$ the unit of $\mathcal{A}$, and with the non-trivial braiding $\gamma(C, D)_\bullet\colon (C \otimes D)_\bullet \to (D \otimes C)_\bullet$ whose component in each degree is

$$(-1)^{pq} \gamma(C_p, D_q)\colon C_p \otimes D_q \longrightarrow D_q \otimes C_p,$$

where γ is the braiding in $\mathcal{A}$.

Remark One can see that $C[n]_\bullet$ is precisely $(\mathbf{1}[n] \otimes C)_\bullet$. This could be a reason why one may dislike the common convention. However, if we use $(C \otimes D)_\bullet$ to denote what usually means $(D \otimes C)_\bullet$, then (using the element notation) the boundary operator reads as

$$\partial(x \otimes y) = (-1)^{|y|} \partial x \otimes y + x \otimes \partial y.$$

In a middle way, we use the notation $(C \otimes^\gamma D)_\bullet$ to denote $(D \otimes C)_\bullet$. To illustrate how the braiding $(C \otimes D)_\bullet \to (C \otimes^\gamma D)_\bullet$ works, let's accept the following formal rule for element notation

$$x \otimes^\gamma y := \gamma(x \otimes y) = (-1)^{|x||y|} y \otimes x,$$

It is often the case that elements of $C \otimes D$ are written as xy. If this is that case, elements of $C \otimes^\gamma D$ can be written as $x^\gamma y$ and the rule above reads

$$x^\gamma y = (-1)^{|x||y|} yx.$$

Since the two tensor structures $\otimes$ and $\otimes^\gamma$ are isomorphic, it doesn't matter which we use as long as we don't mix them. The $\otimes$-convention is intuitive when you do algebraic calculation while the $\otimes^\gamma$-convention is convenient to spell out formulas in homotopy theory.

Note that, under $\otimes^\gamma$-convention, we have $C[n]_\bullet = (C \otimes^\gamma \mathbf{1}[n])_\bullet$.

Remark The **Koszul product** of two cochain complexes $C^\bullet$ and $D^\bullet$ is

$$(C \otimes D)^n = \bigoplus_{p+q=n} C^p \otimes D^q,$$
$$\mathrm{d}(x \otimes y) = \mathrm{d}x \otimes y + (-1)^{|x|} x \otimes \mathrm{d}y.$$

We have $C[n]^\bullet = (\mathbf{1}[n] \otimes C)^\bullet$, where $\mathbf{1}[n]$ is a cochain complex version.

2.12 Suppose further $\mathcal{A}$ is a closed abelian tensor category. Let $C_\bullet$, $D_\bullet$ be two complexes. Then there exists a natural boundary operator ∂ on the internal Hom $[C, D]_\bullet$ of their underlying graded objects. The resulted complex is called the **Koszul Hom complex** of $C_\bullet$ and $D_\bullet$. By its construction, we only need to define the following morphisms

$$[C_p, D_q] \xrightarrow{\partial^{(1)}_{-p,q}} [C_{p+1}, D_q], \qquad [C_p, D_q] \xrightarrow{\partial^{(2)}_{-p,q}} [C_p, D_{q-1}].$$

Note that the condition $\partial \circ \partial = 0$ requires that the following two morphisms must be negative to each other.

$$[C_p, D_q] \xrightarrow{\partial^{(2)}_{-p,q}} [C_p, D_{q-1}] \xrightarrow{\partial^{(1)}_{-p,q-1}} [C_{p+1}, D_{q-1}],$$
$$[C_p, D_q] \xrightarrow{\partial^{(1)}_{-p,q}} [C_{p+1}, D_q] \xrightarrow{\partial^{(2)}_{-p-1,q}} [C_{p+1}, D_{q-1}].$$

The common convention is

$$\partial^{(1)}_{-p,q} := -(-1)^{-p+q}[\partial_{p+1}, D_q], \qquad \partial^{(2)}_{-p,q} := [C_p, \partial_q].$$

In element notation, it reads

$$(\partial f)(x) = \partial f(x) - (-1)^{|f|} f(\partial x).$$

Then one can verify that this construction together with previous ones makes $\mathbf{Ch}(\mathcal{A})$ a closed abelian tensor category.

Remark The functor $-\otimes^{\gamma} C$, i.e. $C \otimes -$ admits a right adjoint $\langle C, -\rangle$ which gives another, although equivalent to the above one, closed abelian tensor category structure. The complex $\langle C, D\rangle_\bullet$ (called the **left Koszul Hom complex**) is defined as follows. Its components are the same as $[C, D]_\bullet$ and the boundary operator reads

$$(\partial f)(x) = (-1)^{|x|}\big(\partial f(x) - f(\partial x)\big).$$

The Koszul Hom complex has the advantages that the signature is independent on where the "function" f acts on, hence is conventional if one only focus on those elements. On the other hand, the left Koszul Hom complex has the advantages that the signature only depends on where "function" f acts on, hence is conventional if one only focus on how those functions act. Note that those two complexes are canonically isomorphic since we have the braiding isomorphism γ. Indeed, the isomorphism reads

$$f \longmapsto \big(x \mapsto (-1)^{|f||x|} f(x)\big).$$

So we are free to use one of them for best fulfill our purpose.

Remark The **Koszul Hom complex** $[C, D]^\bullet$ of two cochain complexes $C_\bullet$ and $D_\bullet$ is

$$[C, D]^n = \prod_{-p+q=n} [C^p, D^q]$$
$$(\mathrm{d} f)(x) = \mathrm{d} f(x) - (-1)^{|f|} f(\mathrm{d} x),$$

and the differential for **left Koszul Hom complex** $\langle C, D\rangle^\bullet$ is

$$(\mathrm{d} f)(x) = (-1)^{|x|}(\mathrm{d} f(x) - f(\mathrm{d} x)).$$

2.13 Let $\mathcal{A}$ be an abelian tensor category. We have seen that so is $\mathbf{Ch}(\mathcal{A})$. Moreover, since the full subcategories $\mathbf{Ch}_?(\mathcal{A})$ and $\mathbf{Ch}^?(\mathcal{A})$ with ? equals $+, -, b$ are closed under finite limits and colimits and Koszul product, they are also abelian tensor categories. As for the full subcategories $\mathbf{Ch}_?(\mathcal{A})$ and $\mathbf{Ch}^?(\mathcal{A})$ with ? equals $\geqslant 0, \leqslant 0$, we use the following proposition.

2.14 Proposition *Let $\mathcal{A}$ be an abelian category. Then*

(i) the inclusion $\mathbf{Ch}_{\geqslant 0}(\mathcal{A}) \hookrightarrow \mathbf{Ch}(\mathcal{A})$ admits a left adjoint $\mathrm{sk}_{\geqslant 0}$ and a right adjoint $\tau_{\geqslant 0}$ and hence is exact;

(ii) the inclusion $\mathbf{Ch}_{\leqslant 0}(\mathcal{A}) \hookrightarrow \mathbf{Ch}(\mathcal{A})$ admits a right adjoint $\mathrm{sk}_{\leqslant 0}$ and a left adjoint $\tau_{\leqslant 0}$ and hence is exact.

In particular, $\mathbf{Ch}_{\geqslant 0}(\mathcal{A})$ and $\mathbf{Ch}_{\leqslant 0}(\mathcal{A})$ are abelian categories.

Proof: The functors $\mathrm{sk}_{\geqslant 0}$ and $\tau_{\geqslant 0}$ are defined as follows.

$$\mathrm{sk}_{\geqslant 0}(C)_n = \begin{cases} C_n & n \geqslant 0, \\ 0 & n < 0; \end{cases}$$

$$\tau_{\geqslant 0}(C)_n = \begin{cases} C_n & n > 0, \\ Z_0(C) & n = 0, \\ 0 & n < 0. \end{cases}$$

Notice that, for any chain complex $C_\bullet$, we have canonical chain morphisms

$$\pi'_C \colon C_\bullet \longrightarrow \mathrm{sk}_{\geqslant 0}(C)_\bullet, \qquad i_C \colon \tau_{\geqslant 0}(C)_\bullet \longrightarrow C_\bullet.$$

On the other hand, for any connective chain complex $C_\bullet$, we have $Z_0(C) = C_0$. Therefore

$$\mathrm{sk}_{\geqslant 0}(C)_\bullet = \tau_{\geqslant 0}(C)_\bullet = C_\bullet.$$

Those identities and previous chain morphisms give rise to the units and counits for the adjunctions.

The functors $\mathrm{sk}_{\leqslant 0}$ and $\tau_{\leqslant 0}$ are defined as follows.

$$\mathrm{sk}_{\leqslant 0}(C)_n = \begin{cases} C_n & n \leqslant 0, \\ 0 & n > 0; \end{cases}$$

$$\tau_{\leqslant 0}(C)_n = \begin{cases} C_n & n < 0, \\ C_0/B_0(C) & n = 0, \\ 0 & n > 0. \end{cases}$$

Notice that, for any chain complex $C_\bullet$, we have canonical chain morphisms

$$i'_C \colon \mathrm{sk}_{\leqslant 0}(C)_\bullet \longrightarrow C_\bullet, \qquad \pi_C \colon C_\bullet \longrightarrow \tau_{\leqslant 0}(C)_\bullet.$$

On the other hand, for any connective chain complex $C_\bullet$, we have $B_0(C) = 0$. Therefore

$$\mathrm{sk}_{\leqslant 0}(C)_\bullet = \tau_{\leqslant 0}(C)_\bullet = C_\bullet.$$

Those identities and previous chain morphisms give rise to the units and counits for the adjunctions. □

Remark The complex $\tau_{\geqslant 0}(C)_\bullet$ (resp. $\tau_{\geqslant 0}(C)_\bullet$) is called the **0-th truncation from below** (resp. **0-th truncation from above**) of $C_\bullet$. One can similarly define n-th truncation functors $\tau_{\geqslant n}$ and $\tau_{\leqslant n}$.

2.15 Let $\mathbf{1}$ be the unit of $\mathcal{A}$. Consider the chain complex $I_\bullet$ defined as

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{1} \xrightarrow{(-\operatorname{id},\operatorname{id})} \mathbf{1}\oplus\mathbf{1} \longrightarrow 0 \longrightarrow \cdots$$

where $\mathbf{1}\oplus\mathbf{1}$ is of degree 0. This complex is called the **standard interval complex**. To justify this terminology and give an intuition, consider that the topological interval $[0,1]$ admits the following cellular decomposition: it has a 1-cell *the interior* $e=(0,1)$ and two 0-cells *the endpoints* $v_0=0$ and $v_1=1$. Then the associated cellular chain complex is the connective complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}e \xrightarrow{\partial} \mathbb{Z}v_0\oplus\mathbb{Z}v_1,$$

where $\partial(e)=v_1-v_0$. To illustrate, we formally write the complex $I_\bullet$ as

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{1}e \xrightarrow{\partial^I} \mathbf{1}v_0\oplus\mathbf{1}v_1 \longrightarrow 0 \longrightarrow \cdots .$$

Let $C_\bullet$ be a complex. Let's spell out the complex $(I\otimes C)_\bullet$. First,

$$(I\otimes C)_n = C_{n-1}e\oplus C_n v_0\oplus C_n v_1.$$

To illustrate, an element (f,x,y) of this object is written as $f\colon x\rightsquigarrow y$, called a **copath** in C_n. Then the boundary operator $\partial_n^{I\otimes C}$ is induced by

$$\partial^I\otimes\operatorname{id}_C,\quad \operatorname{id}_{I_0}\otimes\partial_n^C,\quad\text{and}\quad -\operatorname{id}_{I_1}\otimes\partial_{n-1}^C.$$

To spell out this boundary operator more concretely, let's use the following notation. Let A_j, B_i $(1\leqslant j\leqslant n, 1\leqslant i\leqslant m)$ be objects in $\mathcal{A}$, then the matrix

$$\begin{pmatrix} f_{11} & \cdots & f_{1n}\\ \vdots & \ddots & \vdots\\ f_{m1} & \cdots & f_{mn}\end{pmatrix}$$

denotes the morphism $\bigoplus_{1\leqslant j\leqslant n} A_j \to \bigoplus_{1\leqslant i\leqslant m} B_i$ induced by the following morphisms

$$f_{ij}\colon A_j\to B_i,\qquad 1\leqslant j\leqslant n, 1\leqslant i\leqslant m.$$

Using this notation, the boundary operators can be written as

$$\partial_n^{I\otimes C} = \begin{pmatrix} -\partial_{n-1}^C & 0 & 0\\ -1 & \partial_n^C & 0\\ 1 & 0 & \partial_n^C\end{pmatrix}.$$

Withing element notation, this reads

$$\partial(f\colon x\rightsquigarrow y) = \big(-\partial f\colon -f+\partial x\rightsquigarrow f+\partial y\big).$$

On the other hand, let's spell out the complex $\langle I, C\rangle_\bullet$. First,

$$\langle I, C\rangle_n = [\mathbf{1}e, C_{n+1}] \oplus [\mathbf{1}v_0, C_n] \oplus [\mathbf{1}v_1, C_n] =: C_{n+1}e^* \oplus C_n v_0^* \oplus C_n v_1^*.$$

To illustrate, an element (f, x, y) of this object is written as $f\colon x \rightsquigarrow y$, called a **path** in C_n. Then the boundary operator $\partial_n^{\langle I,C\rangle}$ is induced by

$$-[I_1, \partial_{n+1}^C], \quad [I_0, \partial_n^C], \quad \text{and} \quad [\partial^I, C_n].$$

Using matrix notation, the boundary operators can be written as

$$\partial_n^{\langle I,C\rangle} = \begin{pmatrix} -\partial_{n+1}^C & -1 & 1 \\ 0 & \partial_n^C & 0 \\ 0 & 0 & \partial_n^C \end{pmatrix}.$$

Withing element notation, this reads

$$\partial(f\colon x \rightsquigarrow y) = \big(-\partial f - x + y\colon \partial x \rightsquigarrow \partial y\big).$$

2.16 Dually, one can consider the **co-interval complex** $\hat{I}^\bullet$. It is actually motivated by the cellular cochain complex of the interval $[0,1]$:

$$\mathbb{Z}v_0^* \oplus \mathbb{Z}v_1^* \xrightarrow{\mathrm{d}} \mathbb{Z}e^* \longrightarrow 0 \longrightarrow \cdots,$$

where d is the morphism $(-\mathrm{id}, \mathrm{id})$. To illustrate, we formally write the complex $\hat{I}^\bullet$ as

$$\mathbf{1}v_0^* \oplus \mathbf{1}v_1^* \xrightarrow{\mathrm{d}_I} \mathbf{1}e^* \longrightarrow 0 \longrightarrow \cdots,$$

Apply the equivalence between cochain complexes and chain complexes, one can see that this complex is precisely $\langle I, \mathbf{1}\rangle^\bullet$, i.e. it is the *weak dual* of $I^\bullet$.

Moreover, let $C^\bullet$ be a complex. Then the complex $(\hat{I} \otimes C)^\bullet$ is

$$(\hat{I} \otimes C)^n = C^{n-1}e^* \oplus C^n v_0^* \oplus C^n v_1^*$$

with differential

$$\mathrm{d}^n_{\hat{I}\otimes C} = \begin{pmatrix} -\mathrm{d}_C^{n-1} & -1 & 1 \\ 0 & \mathrm{d}_C^n & 0 \\ 0 & 0 & \mathrm{d}_C^n \end{pmatrix}.$$

Withing element notation, this reads

$$\mathrm{d}(f\colon x \rightsquigarrow y) = \big(-\mathrm{d}f - x + y\colon \mathrm{d}x \rightsquigarrow \mathrm{d}y\big).$$

Apply the equivalence between cochain complexes and chain complexes, one can see that this complex is precisely $\langle I, C\rangle^\bullet$.

On the other hand, the complex $\langle \hat{I}, C\rangle^\bullet$ is

$$\langle \hat{I}, C\rangle^n = C^{n+1}e^{**} \oplus C^n v_0^{**} \oplus C^n v_1^{**}$$

with differential

$$\mathrm{d}^n_{\langle \hat{I}, C\rangle} = \begin{pmatrix} -\mathrm{d}_C^{n+1} & 0 & 0 \\ -1 & \mathrm{d}_C^n & 0 \\ 1 & 0 & \mathrm{d}_C^n \end{pmatrix}.$$

Withing element notation, this reads

$$\mathrm{d}(f\colon x \rightsquigarrow y) = \big(-\mathrm{d}f\colon\ -f + \mathrm{d}x \rightsquigarrow f + \mathrm{d}y\big).$$

Apply the equivalence between cochain complexes and chain complexes, one can see that this complex is precisely $(I \otimes C)^\bullet$.

The reason for the aboves is that $\hat{I}^\bullet$ is indeed the *strong dual* of $I_\bullet$. To see this, let's translate $\hat{I}^\bullet$ into a chain complex. Then the chain complex $(\hat{I} \otimes I)_\bullet$ is concentrated at degree $1, 0, -1$ with components

$$\begin{aligned} (\hat{I} \otimes I)_1 &= \mathbf{1}v_0^* e \oplus \mathbf{1}v_1^* e, \\ (\hat{I} \otimes I)_0 &= \mathbf{1}e^* e \oplus \mathbf{1}v_0^* v_0 \oplus \mathbf{1}v_1^* v_0 \oplus \mathbf{1}v_0^* v_1 \oplus \mathbf{1}v_1^* v_1, \\ (\hat{I} \otimes I)_{-1} &= \mathbf{1}e^* v_0 \oplus \mathbf{1}e^* v_1. \end{aligned}$$

The boundary operators are presented by matrices as

$$\partial_1 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \partial_0 = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Then the **evaluation** $\mathrm{ev}\colon \hat{I} \otimes I \to \mathbf{1}$ is the chain morphism given by

$$\mathrm{ev}_0 = \begin{pmatrix} -1 & 1 & 0 & 0 & 1 \end{pmatrix},$$

which can be illustrated by the rule

$$\mathrm{ev}(x^* y) = (-1)^{|x||y|}\delta_{x,y} := \begin{cases} (-1)^{|x||y|} & x = y, \\ 0 & x \neq y. \end{cases}$$

The **unit morphism** $\iota'\colon \mathbf{1} \to I \otimes^\gamma \hat{I}$ is the chain morphism given by

$$\iota'_0 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \end{pmatrix}^{\mathrm{t}},$$

where t denotes the transpose of a matrix.

On the other hand, since braiding $\hat{I} \otimes I \to I \otimes \hat{I}$ can be illustrated by the following rule

$$\gamma(x^* y) = (-1)^{|x||y|} y x^*,$$

it follows that the chain complex $(I\otimes\hat{I})_\bullet$ is concentrated at degree $1,0,-1$ with components

$$\begin{aligned}(I\otimes\hat{I})_1 &= \mathbf{1}ev_0^*\oplus\mathbf{1}ev_1^*,\\ (I\otimes\hat{I})_0 &= \mathbf{1}ee^*\oplus\mathbf{1}v_0v_0^*\oplus\mathbf{1}v_1v_0^*\oplus\mathbf{1}v_0v_1^*\oplus\mathbf{1}v_1v_1^*,\\ (I\otimes\hat{I})_{-1} &= \mathbf{1}v_0e^*\oplus\mathbf{1}v_1e^*,\end{aligned}$$

and the boundary operators are presented by matrices as

$$\partial_1=\begin{pmatrix}1 & -1\\ -1 & 0\\ 1 & 0\\ 0 & -1\\ 0 & 1\end{pmatrix},\qquad \partial_0=\begin{pmatrix}-1 & -1 & 0 & 1 & 0\\ 1 & 0 & -1 & 0 & 1\end{pmatrix}.$$

Then the **evaluation** $\mathrm{ev}'\colon \hat{I}\otimes^\gamma I\to\mathbf{1}$ is the chain morphism given by

$$\mathrm{ev}'_0=\begin{pmatrix}1 & 1 & 0 & 0 & 1\end{pmatrix},$$

which can be illustrated by the rule

$$\mathrm{ev}'(xy^*)=\delta_{x,y}:=\begin{cases}1 & x=y,\\ 0 & x\neq y.\end{cases}$$

The **unit morphism** $\iota\colon\mathbf{1}\to I\otimes\hat{I}$ is the chain morphism given by

$$\iota_0=\begin{pmatrix}-1 & 1 & 0 & 0 & 1\end{pmatrix}^{\mathrm{t}},$$

where t denotes the transpose of a matrix. Then one can verify that the data (ev,ι) *exhibits* $\hat{I}$ *as a strong dual of* I while (ev',ι') *exhibits* I *as a strong dual of* $\hat{I}$.

Remark In a tensor category $\mathcal{C}$, an object X is **dualizable** if it has a **strong dual** X^*, which is another object in $\mathcal{C}$, and a **(strong) duality**, which is a pair of morphisms $\mathrm{ev}\colon X^*\otimes X\to\mathbf{1}$ (called the **evaluation**) and $\iota\colon\mathbf{1}\to X\otimes X^*$ satisfying the following commutative diagrams

$$\begin{array}{ccc}X^*\otimes(X\otimes X^*) & \xrightarrow{\cong} & (X^*\otimes X)\otimes X^*\\ {\scriptstyle\mathrm{id}\otimes\iota}\uparrow & & \downarrow{\scriptstyle\mathrm{ev}\otimes\mathrm{id}}\\ X^*\otimes\mathbf{1} & \xrightarrow{\cong} & \mathbf{1}\otimes X^*\end{array}\qquad\begin{array}{ccc}(X\otimes X^*)\otimes X & \xrightarrow{\cong} & X\otimes(X^*\otimes X)\\ {\scriptstyle\iota\otimes\mathrm{id}}\uparrow & & \downarrow{\scriptstyle\mathrm{id}\otimes\mathrm{ev}}\\ \mathbf{1}\otimes X & \xrightarrow{\cong} & X\otimes\mathbf{1}\end{array}$$

where the horizontal isomorphisms are the canonical ones.

Suppose $\mathcal{C}$ is further closed. Then the **weak dual** of an object X is precisely the object $[X,\mathbf{1}]$. If X is dualizable, then the weak dual is also the strong dual X^*. If this is the case, then for any object Y, we have a canonical isomorphism

$$Y\otimes X^*\xrightarrow{\sim}[X,Y].$$

2.17 There are two natural chain morphisms from $\mathbf{1}$ to I: s_i $(i = 0, 1)$ sends $\mathbf{1}$ to the factor $\mathbf{1}v_i$ in the 0-th degree of I. Then for any complex C, we have canonical morphisms of complexes

$$\begin{aligned} \iota_i\colon C &\longrightarrow (I \otimes C) \quad (i = 0, 1), \\ \mathrm{ev}_i\colon \langle I, C\rangle &\longrightarrow \langle \mathbf{1}, C\rangle \cong C \quad (i = 0, 1). \end{aligned}$$

To illustrate, let's spell out them by element notation:

$$\begin{aligned} \iota_0(x) &= (0\colon x \rightsquigarrow 0), & \iota_1(y) &= (0\colon 0 \rightsquigarrow y), \\ \mathrm{ev}_0(f\colon x \rightsquigarrow y) &= x, & \mathrm{ev}_1(f\colon x \rightsquigarrow y) &= y. \end{aligned}$$

2.18 Let $f, g\colon C \to D$ be two morphisms of complexes. As in algebraic topology, a **(left) homotopy** $\Phi\colon f \Rightarrow g$ between them is a commutative diagram of complexes as left hand side and a **right homotopy** is a commutative diagram as right hand side.

$$\begin{tikzcd} C \arrow[d, "\iota_0"'] \arrow[dr, "f"] & \\ (I \otimes C) \arrow[r, dashed, "\Phi"] & D \\ C \arrow[u, "\iota_1"] \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} & D \\ C \arrow[ur, "f"] \arrow[r, dashed, "\Phi"] \arrow[dr, "g"'] & \langle I, D\rangle \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & D \end{tikzcd}$$

Applying the previous conventions to chain complexes, a left homotopy is of the form

$$\Phi_n = \begin{pmatrix} \phi_{n-1} & f_n & g_n \end{pmatrix},$$

and the fact Φ is a chain morphism is then equivalent to that the 1-twisted morphism $\phi_\bullet$ satisfies the following equality:

$$g_n - f_n = \partial_n \circ \phi_n + \phi_{n-1} \circ \partial_n.$$

Dually, a right homotopy $\Phi\colon f \Rightarrow g$ is of the form

$$\Phi_n = \begin{pmatrix} \phi_n & f_n & g_n \end{pmatrix}^{\mathrm{t}},$$

and the fact that Φ is a chain morphism is then equivalent to that the 1-twisted morphism $\phi_\bullet$ satisfies the same equality as above. This equality can be illustrated as the following diagram.

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[dl, "\phi_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[r] \arrow[dl, "\phi_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & \cdots \\ \cdots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}"] & D_n \arrow[r, "\partial_n"] & D_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

A 1-twisted morphism $\phi_\bullet$ as above is called a **chain homotopy** from f to g, also denoted by $\phi\colon f \Rightarrow g$.

Similarly, applying to cochain complexes, a left homotopy is of the form

$$\Phi^n = \begin{pmatrix} \phi^{n+1} & f^n & g^n \end{pmatrix},$$

and the fact Φ is a chain morphism is then equivalent to that the (-1)-twisted morphism $\phi^\bullet$ satisfies the following equality:

$$g^n - f^n = \mathrm{d}^n \circ \phi^n + \phi^{n+1} \circ \mathrm{d}^n.$$

Dually, a right homotopy $\Phi\colon f \Rightarrow g$ is of the form

$$\Phi^n = \begin{pmatrix} \phi^n & f^n & g^n \end{pmatrix}^{\mathrm{t}},$$

and the fact that Φ is a chain morphism is then equivalent to that the (-1)-twisted morphism $\phi^\bullet$ satisfies the same equality as above. This equality can be illustrated as the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n-1} \arrow[r, "\mathrm{d}^{n-1}"] \arrow[d, shift right, "f^{n-1}"'] \arrow[d, shift left, "g^{n-1}"] & C^n \arrow[r, "\mathrm{d}^n"] \arrow[dl, "\phi^n"'] \arrow[d, shift right, "f^n"'] \arrow[d, shift left, "g^n"] & C^{n+1} \arrow[r] \arrow[dl, "\phi^{n+1}"'] \arrow[d, shift right, "f^{n+1}"'] \arrow[d, shift left, "g^{n+1}"] & \cdots \\
\cdots \arrow[r] & D^{n-1} \arrow[r, "\mathrm{d}^{n-1}"] & D^n \arrow[r, "\mathrm{d}^n"] & D^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

A (-1)-twisted morphism $\phi^\bullet$ as above is called a **cochain homotopy** from f to g, also denoted by $\phi\colon f \Rightarrow g$.

Note that these notions are equivalent and we'll not distinguish them if no necessary.

Remark The above definitions form the basic blocks of the machinery of homotopy theory. Obviously, if we replace the above $\otimes^\gamma$-version of closed tensor structure by $\otimes$-version, we can still obtained an equivalent theory. However, the concrete formulas would become cumbersome and looks far from the those in usual text of homological algebra.

2.19 Two (co)chain maps $f, g\colon C \rightrightarrows D$ are said to be **homotopic**, denoted by $f \simeq g$, if there exists a (co)chain homotopy $\Phi\colon f \Rightarrow g$. A (co)chain morphism $f\colon C \to D$ is called a **homotopy equivalence** if there exists another (co)chain morphism $g\colon D \to C$ such that $g \circ f \simeq \mathrm{id}_C$ and $f \circ g \simeq \mathrm{id}_D$. Two (co)chain complexes C and D are said to be **homotopy equivalent** if there exists a (co)chain homotopy equivalence $f\colon C \to D$.

In this way, we can form a new category $K(\mathcal{A})$ as follows:

- the objects of $K(\mathcal{A})$ are as of $\mathbf{Ch}(\mathcal{A})$,
- the Hom set $\mathrm{Hom}_{K(\mathcal{A})}(C, D)$ is the quotient set of $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{A})}(C, D)$ modulo homotopies.

This category is called the **homotopy category** of $\mathbf{Ch}(\mathcal{A})$ or $\mathcal{A}$ if there are no ambiguities. In the same way, we have subcategories $K_?(\mathcal{A})$ and $K^?(\mathcal{A})$ with ? equals $c, \geqslant 0, \leqslant 0, +, -, b$.

Given two homotopies $\Phi\colon f \Rightarrow g$ and $\Psi\colon g \Rightarrow h$, then the **vertical composition** of them is more or less the sum of them:

$$\Psi \dotplus \Phi := (\phi + \psi, f, h).$$

Note that $\Psi \dotplus \Phi \neq \Phi \dotplus \Psi$, the later even doesn't make sense. Under this composition rule, the inverse of a homotopy $\Phi\colon f \Rightarrow g$ is the homotopy $-\Phi\colon g \Rightarrow f$ defined as

$$-\Phi := (-\phi, g, f).$$

Given two homotopies Φ, Ψ as below:

$$\begin{tikzcd} C \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=L}] \arrow[r, phantom, "\Downarrow\Phi"] & D \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\Psi"] & E \end{tikzcd}$$

the **horizontal composition** is defined as

$$\Psi * \Phi := \Psi \circ g \dotplus f' \circ \Phi,$$

where the composition $f' \circ \Phi$ should be consider as given by

$$(I \otimes C)_\bullet \xrightarrow{\Phi} D_\bullet \xrightarrow{f'} E_\bullet \quad \text{or} \quad (\hat{I} \otimes C)^\bullet \xrightarrow{\Phi} D^\bullet \xrightarrow{f'} E^\bullet,$$

while the composition $\Psi \circ g$ given by

$$C_\bullet \xrightarrow{g} D_\bullet \xrightarrow{\Psi} \langle I, E\rangle_\bullet \quad \text{or} \quad C^\bullet \xrightarrow{g} D^\bullet \xrightarrow{\Psi} \langle \hat{I}, E\rangle^\bullet.$$

Therefore, the definition can be reads as

$$\Psi * \Phi := \big(f' \circ \phi + \psi \circ g, f' \circ f, g' \circ g\big).$$

Treat homotopies between chain morphisms as 2-morphisms, we obtain a 2-category structure on $\mathbf{Ch}(\mathcal{A})$. Further, we can involves composition rules of homotopies between 2-morphisms, and homotopies between those homotopies, etc. Conceptually, we should obtain an ∞-category structure.

However, this structure is, if it exists, at least not strict. To see this, consider the following diagram.

$$\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\Phi"] \arrow[r, phantom, shift right=3, "\Downarrow\Psi"] & D \arrow[r, bend left=60, "f'"] \arrow[r, "g'" description] \arrow[r, bend right=60, "h'"'] \arrow[r, phantom, shift left=3, "\Downarrow\Phi'"] \arrow[r, phantom, shift right=3, "\Downarrow\Psi'"] & E \end{tikzcd}$$

There are two ways to compose them:

$$(\Psi' \dotplus \Phi') * (\Psi \dotplus \Phi) \qquad \text{and} \qquad \Psi' * \Psi \dotplus \Phi' * \Phi.$$

The **interchange law** in the axioms of 2-category says that the above two compositions are the same. However, they do not equal. In fact, there is a homotopy Θ between them (viewed as chain morphisms) given by the 1-twisted morphism

$$\theta = (\phi' \circ \psi, 0, 0) \colon I \otimes C \longrightarrow E,$$

or equivalently the 2-twisted morphism

$$\phi' \circ \psi \colon C \longrightarrow E.$$

2.20 Passing to the homotopy category $K(\mathcal{A})$, one may expect the *interchange law* as well as more *coherence law* holds strictly. However, even the notion of homotopies itself is lack of sense. Two chain morphisms present the same morphism in $K(\mathcal{A})$ if and only if there is a homotopy between them. But such a homotopy is not unique, even up to homotopy! Indeed there are non-homotopic 2-morphisms between chain morphisms. Consequently, the notion of homotopies between morphisms in $K(\mathcal{A})$ is not well-defined!

2.21 Recall that in the homotopy theory for topological spaces, the key step to build a workable framework is to define a suitable notion of ∞-groupoids as well as the category of them. In particular, we choose CW complexes as such a model in § 1.

Let $C_\bullet, D_\bullet$ be two chain complexes. First note that

(i) A chain morphism $f \colon C_\bullet \to D_\bullet$ is a homogeneous morphism between the underlying graded objects satisfying certain properties, hence an element in $\mathrm{Hom}_{\mathrm{Gr}(\mathcal{A})}(C, D)$.

(ii) A homotopy is determined by a 1-twisted morphism, hence an element in $\mathrm{Hom}_{\mathrm{Gr}(\mathcal{A})}(C, D(1))$.

(iii) A homotopy between homotopies is determined by a 2-twisted morphism, hence an element in $\mathrm{Hom}_{\mathrm{Gr}(\mathcal{A})}(C, D(2))$.

Invested by the above, one may expect the *Hom-space*, i.e. the ∞-groupoid encoding the higher homotopies of chain morphisms from $C_\bullet$ to $D_\bullet$ is the complex $\mathcal{H}om_{\mathbf{Ch}_*(\mathcal{A})}(C, D)_\bullet$ whose underlying graded abelian group is precisely $\mathrm{Hom}_{\mathrm{Gr}(\mathcal{A})}(C, D)_\bullet$ and the boundary operator reads

$$\partial(f) = \partial^D \circ f - (-1)^{|f|} f \circ \partial^C.$$

This complex is called the **Hom-complex**.

Similarly, let $C^\bullet, D^\bullet$ be two chain complexes. Then we also have the **Hom-complex** $\mathcal{H}om_{\mathbf{Ch}^*(\mathcal{A})}(C,D)^\bullet$ with differential

$$\mathrm{d}(f) = \mathrm{d}^D \circ f - (-1)^{|f|} f \circ \mathrm{d}^C.$$

It is natural to ask: after identify chain complexes and cochain complexes, what's the relation between the above two complexes? Let's spell out their components first:

$$\begin{aligned}
\mathcal{H}om_{\mathbf{Ch}_*(\mathcal{A})}(C,D)_n &= \prod_{-p+q=n} \mathrm{Hom}_{\mathcal{A}}(C_p, D_q),\\
\mathcal{H}om_{\mathbf{Ch}^*(\mathcal{A})}(C,D)^n &= \prod_{-p+q=n} \mathrm{Hom}_{\mathcal{A}}(C^p, D^q).
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathcal{H}om_{\mathbf{Ch}^*(\mathcal{A})}(C,D)_n &= \mathcal{H}om_{\mathbf{Ch}^*(\mathcal{A})}(C,D)^{-n}\\
&= \prod_{-p+q=-n} \mathrm{Hom}_{\mathcal{A}}(C^p, D^q)\\
&= \prod_{-p+q=-n} \mathrm{Hom}_{\mathcal{A}}(C_{-p}, D_{-q})\\
&= \prod_{-p+q=n} \mathrm{Hom}_{\mathcal{A}}(C_p, D_q) = \mathcal{H}om_{\mathbf{Ch}_*(\mathcal{A})}(C,D)_n.
\end{aligned}$$

Then one can verify that they are the same complex, hence can be simply denoted by $\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,D)$.

2.22 Recall that, for a pointed topological space (X, x_0), its n-th homotopy group $\pi_n(X, x_0)$ is defined as either the set of homotopy classes of based maps $S^n \to X$ or the set of homotopy classes of maps $(I^n, \partial I^n) \to (X, x_0)$.

The complex corresponding to the n-*cube* I^n is $I_\bullet^{\otimes n}$, the n-fold Koszul product of $I_\bullet$. Let's spell out this complex concretely. To do this, let's introduce the following notion:

- An object M in $\mathcal{A}$ is **free** if it is isomorphic to a direct sum of copies of $\mathbf{1}$. A **basis** of a free object M is an isomorphism from a direct sum of copies of $\mathbf{1}$ to it. In particular, an **member of the basis** is a component $\mathbf{1} \to M$ of this isomorphism. In this way, we can always present a basis as the collection of its members.

The complex $I_\bullet$ has the basis $\{v_0, v_1\}$ at degree 0 and the basis $\{e\}$ at degree 1. Using this *basis notation*, the boundary operator can be written as

$$\partial(e) = v_1 - v_0.$$

Let α be a $\{v_0, v_1, e\}$**-string**, i.e a sequence of letters consisting of v_0, v_1 and e. Then the **length** of α is the number of letters in it and the **total degree** $|\alpha|$ is the sum of degrees of the letters (where v_0, v_1 are of degree 0 and e is of degree 1). Therefore

- $I_i^{\otimes n}$ has basis consisting of $\{v_0, v_1, e\}$-strings of length n and degree i;
- the boundary operator reads

$$\partial(\alpha\beta) = (\partial\alpha)\beta + (-1)^{|\alpha|}\alpha(\partial\beta).$$

Since ∂I^n is n-cube without its unique n-cell, the corresponding complex $\partial I_\bullet^{\otimes n}$ should be the complex $I_\bullet^{\otimes n}$ without its top degree $I_n^{\otimes n} = \mathbf{1}ee\cdots e$.

Note that the n-sphere S^n has a cellular decomposition: the 0-cell is its base point and the n-cell is all outside that point. Using this cellular decomposition, the complex corresponding to S^n is the complex $\mathbf{1} \oplus \mathbf{1}[-n]$, where the first factor presents the base point.

Let $C_\bullet$ be a complex. A **(cubic) n-loop** in $C_\bullet$ is a chain morphism $\gamma\colon I_\bullet^{\otimes n} \to C_\bullet$ such that the composition of it with the canonical inclusion $\partial I_\bullet^{\otimes n} \hookrightarrow I_\bullet^{\otimes n}$ is 0. Likewise, a **(spheric) n-loop** in $C_\bullet$ is a chain morphism $\gamma\colon \mathbf{1} \oplus \mathbf{1}[-n] \to C_\bullet$ such that the composition of it with the canonical inclusion $\mathbf{1} \hookrightarrow \mathbf{1} \oplus \mathbf{1}[-n]$ is 0. It is clear that both of them are equivalent to a morphism $\gamma_n\colon \mathbf{1} \to C_n$ such that $\partial \circ \gamma_n = 0$. In other words,

$$\gamma_n \in Z_n\, \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(\mathbf{1}, C).$$

On the other hand, a homotopy $H\colon \gamma \Rightarrow \eta$ between two n-loops is determined by a morphism $h\colon \mathbf{1} \to C_{n+1}$ such that $\partial \circ h = \eta_n - \gamma_n$, i.e.

$$\eta_n - \gamma_n \in B_n\, \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(\mathbf{1}, C).$$

Therefore we have canonical isomorphisms

$$\pi_n(C) := \{\text{homotopy classes of } n\text{-loops in } C\ \} \cong H_n\, \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(\mathbf{1}, C).$$

This abelian group is called the **n-th homotopy group** of $C_\bullet$.

Remark Be aware that $\pi_n(C)$ is in general not the underlying abelian group of $H_n(C)$, i.e. $\pi_n(C) \neq \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, H_n(C))$. The reason is that the functor $\mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, -)$ is in general not exact. As an example, consider the category of abelian sheaves on a general topological space.

2.23 The above procession works for general homotopies:

(i) By a **boundary condition** of $C_\bullet$, we mean a chain morphism from $\partial I_\bullet^{\otimes n}$ to $C_\bullet$. By a **n-cell** attaching to $C_\bullet$ via a boundary condition δ, we mean a chain morphism from $I_\bullet^{\otimes n}$ to $C_\bullet$ whose restriction to $\partial I_\bullet^{\otimes n}$ is δ. Then it is clear that the set of all n-cells attaching to $C_\bullet$ via a boundary condition δ equals to the coset

$$\{n\text{-loop in } C_\bullet\} + \delta.$$

(ii) Let $\delta\colon \partial I_\bullet^{\otimes n} \to C_\bullet$ be a boundary condition. By a **homotopy rel δ**, we mean a homotopy whose restriction to $\partial I_\bullet^{\otimes n}$ is $I \otimes \delta$. Then it is clear that the quotient set of all n-cells attaching to $C_\bullet$ via a boundary condition δ up to homotopy rel δ equals to the coset

$$\pi_n(C) + \delta.$$

(iii) With above notions, a chain morphism from $C_\bullet$ to $D_\bullet$ can be viewed as a 0-cell attaching to $[C, D]_\bullet$ via the empty boundary condition and a homotopy is a homotopy rel nothing. Hence

$$\begin{aligned}\operatorname{Hom}_{\mathbf{Ch}(\mathcal{A})}(C, D) &\cong Z_0\, \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C, D),\\ \operatorname{Hom}_{K(\mathcal{A})}(C, D) &\cong H_0\, \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C, D).\end{aligned}$$

(iv) A given pair of chain morphisms from $C_\bullet$ to $D_\bullet$ can be viewed as a boundary condition $\partial I_\bullet \to [C, D]_\bullet$. Then a homotopy between them is a 1-cell attaching to $[C, D]_\bullet$ via that boundary condition and a 2-homotopy between such homotopies is a homotopy rel that boundary condition. Hence

$$\begin{aligned}\{\text{homotopy between given chain morphisms}\} &\cong Z_1\, \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C, D),\\ \{\text{homotopy class of above}\} &\cong H_1\, \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C, D).\end{aligned}$$

(v) A given pair of homotopies can be viewed as a boundary condition $\partial I_\bullet^{\otimes 2} \to [C, D]_\bullet$, where the 1-degree encodes the two homotopies and 0-degree the domain and codomains of them. Then a 2-homotopy between them is a 2-cell attaching to $[C, D]_\bullet$ via that boundary condition and a 3-homotopy between such 2-homotopies is a homotopy rel that boundary condition. Hence

$$\begin{aligned}\{\text{2-homotopy between given homotopies}\} &\cong Z_2\, \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C, D),\\ \{\text{homotopy class of above}\} &\cong H_2\, \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C, D).\end{aligned}$$

(vi) In general, a given pair of $(n-1)$-homotopies can be viewed as a boundary condition $\partial I_\bullet^{\otimes n} \to [C, D]_\bullet$, where the components with basis consisting of strings starting with v_0 (resp. v_1) comes from the first (resp. second) $(n-1)$-homotopy. Then a n-homotopy between them is a n-cell attaching to $[C, D]_\bullet$ via that boundary condition and a $(n+1)$-homotopy between such n-homotopies is a homotopy rel that boundary condition. Hence

$$\begin{aligned}\{n\text{-homotopy between given homotopies}\} &\cong Z_n\, \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C, D),\\ \{\text{homotopy class of above}\} &\cong H_n\, \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C, D).\end{aligned}$$

$$
\begin{tikzcd}
& I^{\otimes n-1} \otimes C \arrow[d] \arrow[dr] & & I^{\otimes n} \otimes C \arrow[d] \arrow[dr] & & \\
\cdots & I^{\otimes n} \otimes C \arrow[r] & D & I^{\otimes n+1} \otimes C \arrow[r] & D & \cdots \\
& I^{\otimes n-1} \otimes C \arrow[u] \arrow[ur] & & I^{\otimes n} \otimes C \arrow[u] \arrow[ur] & &
\end{tikzcd}
$$

2.24 It is straightforward to show that both the *evaluation* $\hat{I} \otimes I \to \mathbf{1}$ and the *unit moprhism* $\mathbf{1} \to I \otimes \hat{I}$ are quasi-isomorphisms. In this way, we may think $\hat{I}_\bullet$ as $I_\bullet^{\otimes -1}$ and more generally $\hat{I}_\bullet^{\otimes n}$ as $I_\bullet^{\otimes -n}$ for any natural number n. Then the previous discussion still works.

In details. The complex $\hat{I}_\bullet$ has the basis $\{v_0^*, v_1^*\}$ at degree 0 and the basis $\{e^*\}$ at degree -1. The boundary operator of $I_\bullet^{\otimes -1}$ reads

$$\partial(v_0^*) = e^*, \qquad \partial(v_1^*) = e^*.$$

Then, the complex $I_\bullet^{\otimes -n}$ can be described as follows.

- $I_i^{\otimes -n}$ has a basis of $\{v_0^*, v_1^*, e^*\}$-strings of length n and degree i;
- the boundary operator reads

$$\partial(\alpha\beta) = (\partial\alpha)\beta + (-1)^{|\alpha|}\alpha(\partial\beta).$$

Then the complex $\partial I_\bullet^{\otimes -n}$ is the complex $I_\bullet^{\otimes -n}$ without its bottom degree $I_{-n}^{\otimes -n} = \mathbf{1}e^*e^* \cdots e^*$.

We can also define the complex corresponding to S^{-n} as the complex $\mathbf{1} \oplus \mathbf{1}[n]$, where the first factor presents the base point.

Then, one can define the notions of **cubic** and **spheric** **$(-n)$-loops** as before and verify the similar statements:

(i) a $(-n)$-loop in $C_\bullet$ is equivalent to an element in $Z_{-n}\,\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(\mathbf{1}, C)$;
(ii) two $(-n)$-loops are homotopic if they are different by an element in $B_{-n}\,\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(\mathbf{1}, C)$;
(iii) $\pi_{-n}(C) = H_{-n}\,\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(\mathbf{1}, C)$.

2.25 Through the identification of cochain complexes and chain complexes, the above statements can be translated as:

(i) a n-loop in $C^\bullet$ is equivalent to an element in $Z^{-n}\,\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(\mathbf{1}, C)$;
(ii) two n-loops of $C^\bullet$ are homotopic if they are different by an element in $B^{-n}\,\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(\mathbf{1}, C)$.

Then we can have similar statements for general homotoies, hence

$$\{n\text{-homotopy between given homotopies}\} \cong Z^{-n}\,\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C, D),$$
$$\{\text{homotopy class of above}\} \cong H^{-n}\,\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C, D).$$

In this way, we can think the Hom-complex $\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,D)$ *encodes homotopy informations into its connective truncation.* If one remember how homotopy groups can be viewed as negative-degree intrinsic cohomology groups. Then one would agree that it is more natural to view the complex $\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,D)$ as a cochain complex.

2.26 From previous observation, we can encode the ∞-category structure on $\mathbf{Ch}(\mathcal{A})$ into the Hom-complexes. To summarize, we have the followings.

(i) A **n-morphism** from C to D is a $(1-n)$-cocycle, i.e. $(n-1)$-cycle, of $\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,D)$.

(ii) A **n-homotopy** between n-morphisms $\phi\colon f \Rightarrow g$ is an element of $\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,D)^{-n}$ such that $\mathrm{d}\phi = g - f$.

(iii) The composition rules are encoded into the bilinear map

$$\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(D,E) \otimes \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,D) \longrightarrow \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,E)$$

induced from the bilinear maps

$$\mathrm{Hom}\big(D(q), E(p+q)\big) \otimes \mathrm{Hom}\big(C, D(q)\big) \longrightarrow \mathrm{Hom}\big(C, E(p+q)\big)$$

given by $g \otimes f \mapsto g \circ f$.

(iv) The identity morphism is encoded into a homomorphism from $\mathbb{Z}$ to $\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,C)$ defined by $1 \mapsto \mathrm{id}_C$.

(v) The coherent axioms are encoded into the commutative diagrams

$$\begin{tikzcd}
\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(E,F) \otimes \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(D,E) \otimes \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,D) \arrow[r] \arrow[d] & \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(E,F) \otimes \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,E) \arrow[d] \\
\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(E,D) \otimes \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,D) \arrow[r] & \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,F)
\end{tikzcd}$$

(which encodes the associativities) and

$$\begin{tikzcd}
\mathbb{Z} \otimes \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,D) \arrow[r] \arrow[dr, "\cong"'] & \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(D,D) \otimes \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,D) \arrow[d] \\
 & \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,D) \\
\mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,D) \otimes \mathbb{Z} \arrow[r] \arrow[ur, "\cong"] & \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,D) \otimes \mathcal{H}om_{\mathbf{Ch}(\mathcal{A})}(C,C) \arrow[u]
\end{tikzcd}$$

(which encodes the identity laws).

§ 3 Dg-category theory

3.1 Inspirited by previous section, the following definition arises.

A **dg-category** is precisely a **Ch**-enriched category. (Of course, one can slightly generalize this notion by replacing **Ch** with $\mathbf{Ch}(k)$). More precisely, a dg-category $\mathcal{C}$ consists of

- a collection of *objects* $\operatorname{ob}\mathcal{C}$;
- for any two objects C and D, a **Hom-complex** $\mathcal{H}om_{\mathcal{C}}(C, D) \in \mathbf{Ch}$;
  - an element of $\mathcal{H}om_{\mathcal{C}}(C, D)^n$ is called a **(general) morphism of (cohomological) degree n**;
  - a closed morphism of degree $1 - n$ is called a **n-morphism** from C to D, denoted by $f\colon C \to D$;
  - a **n-homotopy** $\phi\colon f \Rightarrow g$ is a morphism of degree $-n$ such that $\mathrm{d}\phi = g - f$;
- for any three objects C, D and E, a cochain map

$$\mathcal{H}om_{\mathcal{C}}(D, E) \otimes \mathcal{H}om_{\mathcal{C}}(C, D) \longrightarrow \mathcal{H}om_{\mathcal{C}}(C, E)$$

  called the *composition rule*;
- for any object C, a cochain map $\mathbb{Z} \to \mathcal{H}om_{\mathcal{C}}(C, C)$ called the *identity*.

Those data must satisfies the following axioms:

1. for any objects C, D, E, F, the following diagram commutes;

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{C}}(E, F) \otimes \mathcal{H}om_{\mathcal{C}}(D, E) \otimes \mathcal{H}om_{\mathcal{C}}(C, D) \arrow[r] \arrow[d] & \mathcal{H}om_{\mathcal{C}}(E, F) \otimes \mathcal{H}om_{\mathcal{C}}(C, E) \arrow[d] \\
\mathcal{H}om_{\mathcal{C}}(E, D) \otimes \mathcal{H}om_{\mathcal{C}}(C, D) \arrow[r] & \mathcal{H}om_{\mathcal{C}}(C, F)
\end{tikzcd}$$

2. for any objects C, D, the following diagram commutes.

$$\begin{tikzcd}
\mathbb{Z} \otimes \mathcal{H}om_{\mathcal{C}}(C, D) \arrow[r] \arrow[dr, "\cong"'] & \mathcal{H}om_{\mathcal{C}}(D, D) \otimes \mathcal{H}om_{\mathcal{C}}(C, D) \arrow[d] \\
 & \mathcal{H}om_{\mathcal{C}}(C, D) \\
\mathcal{H}om_{\mathcal{C}}(C, D) \otimes \mathbb{Z} \arrow[r] \arrow[ur, "\cong"] & \mathcal{H}om_{\mathcal{C}}(C, D) \otimes \mathcal{H}om_{\mathcal{C}}(C, C) \arrow[u]
\end{tikzcd}$$

Any dg-category $\mathcal{C}$ admits a (pre-additive) category $\mathcal{C}_0$ (its **underlying category**) obtained by applying the *change of base categories* $Z^0\colon \mathbf{Ch} \to \mathbf{Ab}$ (or $Z^0\colon \mathbf{Ch} \to \mathbf{Ab} \to \mathbf{Set}$ if one insists on an ordinary category) and another $\mathrm{h}\mathcal{C}$ (its **homotopy category**) obtained by applying the *change of base categories* $H^0\colon \mathbf{Ch} \to \mathbf{Ab}$ (or $H^0\colon \mathbf{Ch} \to \mathbf{Ab} \to \mathbf{Set}$).

Example Let $\mathcal{A}$ be an additive category. Then $\mathbf{Ch}(\mathcal{A})$ is automatically a dg-category. The underlying category of $\mathbf{Ch}(\mathcal{A})$ is the ordinary category of complexes. The homotopy category $\mathrm{h}\mathbf{Ch}(\mathcal{A})$ is precisely $\mathcal{K}(\mathcal{A})$. The similar conventions apply to the subcategories $\mathbf{Ch}_?(\mathcal{A})$ and $\mathbf{Ch}^?(\mathcal{A})$ with ? equals $c, \geqslant 0, \leqslant 0, +, -, b$.

3.2 A **dg-functor** between dg-categories $F\colon \mathcal{C} \to \mathcal{D}$ is an enriched functor. Equivalently, a dg-functor F consists of

- a mapping between objects $F_0\colon \operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{D}$,
- a family of cochain maps $F_{C,D}\colon \mathcal{H}om_{\mathcal{C}}(C,D) \to \mathcal{H}om_{\mathcal{D}}(F(C),F(D))$, indexed by $C,D \in \operatorname{ob}\mathcal{C}$,

satisfying the following associative and unitary laws:

1. for any objects C, D, E, the following diagram commutes;

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{C}}(D,E)\otimes\mathcal{H}om_{\mathcal{C}}(C,D) \arrow[r] \arrow[d, "F"] & \mathcal{H}om_{\mathcal{C}}(C,E) \arrow[d, "F"] \\
\mathcal{H}om_{\mathcal{D}}(F(D),F(E))\otimes\mathcal{H}om_{\mathcal{C}}(F(C),F(D)) \arrow[r] & \mathcal{H}om_{\mathcal{D}}(F(C),F(E))
\end{tikzcd}$$

2. for any object C, the following diagram commutes.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[dr] & \mathcal{H}om_{\mathcal{C}}(C,C) \arrow[d, "F"] \\
 & \mathcal{H}om_{\mathcal{D}}(F(C),F(C))
\end{tikzcd}$$

Given two dg-functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$, their **composition** $G \circ F$ is given as follows:

- the mapping $(G\circ F)_0\colon \operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{E}$ is the composition $G_0 \circ F_0$;
- the cochain maps $(G\circ F)_{C,D}\colon \mathcal{H}om_{\mathcal{C}}(C,D) \to \mathcal{H}om_{\mathcal{E}}(GF(C),GF(D))$ is given by the composition of $F_{C,D}\colon \mathcal{H}om_{\mathcal{C}}(C,D) \to \mathcal{H}om_{\mathcal{D}}(F(C),F(D))$ and $G_{F(C),F(D)}\colon \mathcal{H}om_{\mathcal{D}}(F(C),F(D)) \to \mathcal{H}om_{\mathcal{E}}(GF(C),GF(D))$.

Then the unity of the composition is the **identity dg-functor** Id which is identity on objects and each cochain map $\mathrm{Id}_{C,D}$ is just the identity map.

One can then define the **isomorphisms** of dg-categories as those dg-functors admits an inverse. It is clear that this condition is equivalent to say that the functor F is *surjective on objects* and the cochain maps $F_{C,D}$ are chain *isomorphisms*.

Any dg-functor $F\colon \mathcal{C} \to \mathcal{D}$ admits a **underlying functor** $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ obtained by applying the *change of base categories* $Z^0\colon \mathbf{Ch} \to \mathbf{Ab}$ and a **homotopy functor** $\mathrm{h}F\colon \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ obtained by applying the *change of base categories* $H^0\colon \mathbf{Ch} \to \mathbf{Ab}$.

3.3 Given two dg-functors $F, G\colon \mathcal{C} \to \mathcal{D}$. A **dg-transformation** $\alpha\colon F \Rightarrow G$ consists of a family of cochain maps $\alpha_C\colon \mathbb{Z} \to \mathcal{H}om_{\mathcal{D}}(F(C), G(C))$ indexed by objects of $\mathcal{C}$, satisfying that for any objects C and D, the following diagram commutes.

$$\begin{tikzcd}
\mathbb{Z} \otimes \mathcal{H}om_{\mathcal{C}}(C, D) \arrow[r, "\alpha_D \otimes F"] & \mathcal{H}om_{\mathcal{D}}(F(D), G(D)) \otimes \mathcal{H}om_{\mathcal{D}}(F(C), F(D)) \arrow[d] \\
\mathcal{H}om_{\mathcal{C}}(C, D) \arrow[u, "\cong"] \arrow[d, "\cong"'] & \mathcal{H}om_{\mathcal{D}}(F(C), G(D)) \\
\mathcal{H}om_{\mathcal{C}}(C, D) \otimes \mathbb{Z} \arrow[r, "G \otimes \alpha_C"] & \mathcal{H}om_{\mathcal{D}}(G(C), G(D)) \otimes \mathcal{H}om_{\mathcal{D}}(F(D), G(D)) \arrow[u]
\end{tikzcd}$$

Given two dg-transformations $\alpha\colon F \Rightarrow G$, $\beta\colon G \Rightarrow H$, their **vertical composition** $\beta \cdot \alpha$ is given by

$$\begin{tikzcd}
\begin{matrix} \mathbb{Z} \\ \otimes \\ \mathbb{Z} \end{matrix} \arrow[r, "\begin{matrix} \alpha_C \\ \otimes \\ \beta_C \end{matrix}"] & \begin{matrix} \mathcal{H}om_{\mathcal{D}}(F(C), G(C)) \\ \otimes \\ \mathcal{H}om_{\mathcal{D}}(G(C), H(C)) \end{matrix} \arrow[d] \\
\mathbb{Z} \arrow[u, "\cong"] \arrow[r, dashed, "(\beta\cdot\alpha)_C"] & \mathcal{H}om_{\mathcal{D}}(F(C), H(C)).
\end{tikzcd}$$

The unity of this composition is the **identity dg-transformation** id which gives the identity for each object. A dg-transformation α is called a **natural isomorphism** if it admits an inverse β, i.e. $\alpha \cdot \beta = \mathrm{id}$, $\beta \cdot \alpha = \mathrm{id}$. It is called a **natural equivalence** if it admits an weak inverse β, i.e. $\alpha \cdot \beta \simeq \mathrm{id}$, $\beta \cdot \alpha \simeq \mathrm{id}$.

Given two dg-transformations

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E}
\end{tikzcd}$$

their **horizontal composition** $\beta * \alpha$ is given by the following two equivalent compositions.

$$\begin{tikzcd}
\begin{matrix} \mathbb{Z} \\ \otimes \\ \mathbb{Z} \end{matrix} \arrow[r, "\begin{matrix} F'(\alpha_C) \\ \otimes \\ \beta_{G(C)} \end{matrix}"] & \begin{matrix} \mathcal{H}om_{\mathcal{E}}(F'F(C), F'G(C)) \\ \otimes \\ \mathcal{H}om_{\mathcal{E}}(F'G(C), G'G(C)) \end{matrix} \arrow[d] \\
\mathbb{Z} \arrow[u, "\cong"] \arrow[d, "\cong"'] \arrow[r, dashed, "(\beta*\alpha)_C"] & \mathcal{H}om_{\mathcal{E}}(F'F(C), G'G(C)) \\
\begin{matrix} \mathbb{Z} \\ \otimes \\ \mathbb{Z} \end{matrix} \arrow[r, "\begin{matrix} \beta_{F(C)} \\ \otimes \\ G'(\alpha_C) \end{matrix}"] & \begin{matrix} \mathcal{H}om_{\mathcal{E}}(F'F(C), G'F(C)) \\ \otimes \\ \mathcal{H}om_{\mathcal{E}}(G'F(C), G'G(C)) \end{matrix} \arrow[u]
\end{tikzcd}$$

3.4 The previous abstract definition can be spelled out elementary as follows.

(i) A cochain map from $\mathbb{Z}$ to a complex $C^\bullet$ is the same as a 0-cocycle of $C^\bullet$. Hence a **dg-transformation** $\alpha\colon F \Rightarrow G$ is the same as a family of 1-morphisms $\alpha_C\colon F(C) \to F(D)$ in $\mathcal{D}$, (hence morphisms in $\mathcal{D}_0$), satisfying that for any element $f \in \mathcal{H}om_{\mathcal{C}}(C, D)$, the following diagram commutes.

$$\begin{tikzcd} F(C) \arrow[r, "\alpha_C"] \arrow[d, "F(f)"'] & G(C) \arrow[d, "G(f)"] \\ F(D) \arrow[r, "\alpha_D"] & G(D) \end{tikzcd}$$

Be aware that a natural transformation $\alpha\colon F_0 \Rightarrow G_0$ requires merely above commutative diagrams for 1-morphisms $f\colon C \to D$.

(ii) Given two dg-transformations $\alpha\colon F \Rightarrow G$, $\beta\colon G \Rightarrow H$, their **vertical composition** $\beta \cdot \alpha$ is given by the family $(\beta \cdot \alpha)_C := \beta_C \circ \alpha_C$ viewed as compositions of 1-morphisms.

(iii) The **identity dg-transformation** id is the same as the family of identity morphisms $\mathrm{id}_{F(C)}\colon F(C) \to F(C)$. Hence a dg-transformation α is a **natural isomorphism** if and only if its each component α_C is an isomorphism in $\mathcal{D}_0$, and a **natural equivalence** if and only if its each component α_C is an isomorphism in $\mathrm{h}\mathcal{D}$.

(iv) Given two dg-transformations

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd}$$

their **horizontal composition** $\beta * \alpha$ is the family $(\beta * \alpha)_C$ given by two equivalent compositions which can be encoded into the following commutative diagram of 1-morphisms.

$$\begin{tikzcd} F'F(C) \arrow[r, "F'(\alpha_C)"] \arrow[d, "\beta_{F(C)}"'] & F'G(C) \arrow[d, "\beta_{G(C)}"] \\ G'F(C) \arrow[r, "G'(\alpha_C)"] & G'G(C) \end{tikzcd}$$

(v) Then one can verify that the **interchange law** holds: whenever we have dg-transformations

$$\begin{tikzcd}[column sep=large] \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathcal{D} \arrow[r, bend left=60, "F'"] \arrow[r, "G'" description] \arrow[r, bend right=60, "H'"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha'"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & \mathcal{E} \end{tikzcd}$$

the following two compositions are the same:

$$(\beta' \cdot \alpha') * (\beta \cdot \alpha) \quad = \quad (\beta' * \beta) \cdot (\alpha' * \alpha).$$

(vi) Note that, as in ordinary category theory, the identity transformation id_F in a formula of dg-transformations is usually denoted as F. For example, $F' * \alpha$ means $\mathrm{id}_{F'} * \alpha$, whose components are $F'(\alpha_C)$, and $\beta * G$ means $\beta * \mathrm{id}_G$, whose components are $\beta_{G(C)}$. Then the interchange law tells us

$$(\beta * G) \cdot (F' * \alpha) = \beta * \alpha.$$

Likewise, we also have

$$(G' * \alpha) \cdot (\beta * F) = \beta * \alpha.$$

Hence the fact that the two ways of horizontal composition agree is a special case of the interchange law.

3.5 Let $\mathcal{C}$ and $\mathcal{D}$ be two dg-categories. The category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ consists of

- dg-functors from $\mathcal{C}$ to $\mathcal{D}$ as its objects;
- dg-transformations between those dg-functors as its morphisms.

The natural isomorphisms are precisely isomorphisms in this category. Two dg-functors are said to be **isomorphic** if there is a natural isomorphism between them. Note that Two dg-functors $F, G \colon \mathcal{C} \to \mathcal{D}$ are isomorphic if and only if they are isomorphic in the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Two dg-transformations $\alpha, \beta \colon F \Rightarrow G$ are said to be **homotopic** if its components α_C and β_C are homotopic (as cochain maps, using the abstract definition, or equivalently, as 1-morphisms in $\mathcal{D}$, using the elementary description). Then the category $\mathrm{hFun}(\mathcal{C}, \mathcal{D})$ consists of

- dg-functors from $\mathcal{C}$ to $\mathcal{D}$ as its objects;
- homotopy classes of dg-transformations as its morphisms.

The natural equivalences are precisely isomorphisms in this category. Two dg-functors are said to be **equivalent** if there is a natural equivalence between them. Note that Two dg-functors $F, G \colon \mathcal{C} \to \mathcal{D}$ are **equivalent** if and only if they are isomorphic in the category $\mathrm{hFun}(\mathcal{C}, \mathcal{D})$.

As the notations suggest, the above ordinary categories $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and $\mathrm{hFun}(\mathcal{C}, \mathcal{D})$ should be viewed as the underlying category and the homotopy category of the dg-category of dg-functors $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})$ respectively. This dg-category will be constructed later.

3.6 An **adjunction of dg-functors** is a quadruple (F, G, η, ϵ), where

- $F\colon \mathcal{C} \to \mathcal{D}$ (the **left adjoint**) and $G\colon \mathcal{D} \to \mathcal{C}$ (the **right adjoint**) are two dg-functors,
- $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F$ (the **unit**) and $\epsilon\colon F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$ (the **counit**) are two dg-transformations,

satisfying the following two commutative diagram (the **triangle identities**) of dg-transformations.

$$\begin{tikzcd} & F\circ G\circ F \arrow[dr, "\epsilon * F"] & \\ F \arrow[ur, "F*\eta"] \arrow[rr, "\mathrm{id}_F"] & & F \end{tikzcd} \qquad \begin{tikzcd} & G\circ F\circ G \arrow[dr, "G * \epsilon"] & \\ G \arrow[ur, "\eta*G"] \arrow[rr, "\mathrm{id}_G"] & & G \end{tikzcd}$$

If this is the case, then the compositions (of cochain maps)

$$\begin{tikzcd} \begin{matrix} \mathbb{Z} \\ \otimes \\ \mathcal{H}om_{\mathcal{D}}(F(C), D) \end{matrix} \arrow[r, "\eta_C \otimes G"] & \begin{matrix} \mathcal{H}om_{\mathcal{C}}(C, GF(C)) \\ \otimes \\ \mathcal{H}om_{\mathcal{C}}(GF(C), G(D)) \end{matrix} \arrow[d] \\ \mathcal{H}om_{\mathcal{D}}(F(C), D) \arrow[u, "\cong"] \arrow[r, dashed, "\alpha_{C,D}"] & \mathcal{H}om_{\mathcal{C}}(C, G(D)) \end{tikzcd}$$

and

$$\begin{tikzcd} \begin{matrix} \mathcal{H}om_{\mathcal{C}}(C, G(D)) \\ \otimes \\ \mathbb{Z} \end{matrix} \arrow[r, "F \otimes \epsilon_D"] & \begin{matrix} \mathcal{H}om_{\mathcal{D}}(F(C), FG(D)) \\ \otimes \\ \mathcal{H}om_{\mathcal{D}}(FG(D), D) \end{matrix} \arrow[d] \\ \mathcal{H}om_{\mathcal{C}}(C, G(D)) \arrow[u, "\cong"] \arrow[r, dashed, "\beta_{C,D}"] & \mathcal{H}om_{\mathcal{D}}(F(C), D) \end{tikzcd}$$

give rise to a pair of natural isomorphisms

$$\mathcal{H}om_{\mathcal{D}}(F(-), -) \cong \mathcal{H}om_{\mathcal{C}}(-, G(-)).$$

Conversely, if there is a pair of natural isomorphisms

$$\mathcal{H}om_{\mathcal{D}}(F(-), -) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathcal{H}om_{\mathcal{C}}(-, G(-)),$$

then the compositions

$$\mathbb{Z} \to \mathcal{H}om_{\mathcal{D}}(F(C), F(C)) \xrightarrow{\alpha_{C,F(C)}} \mathcal{H}om_{\mathcal{C}}(C, GF(C))$$

and

$$\mathbb{Z} \longrightarrow \mathcal{H}om_{\mathcal{C}}(G(D), G(D)) \xrightarrow{\beta_{G(D),D}} \mathcal{H}om_{\mathcal{D}}(FG(D), D)$$

give rise to the unit η and the counit η making the quadruple (F, G, η, ϵ) an adjunction of dg-functors.

3.7 A dg-functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be

- **fully faithful**, if for any two objects C, D of $\mathcal{C}$, the cochain map

$$F_{C,D}\colon \mathcal{H}om_{\mathcal{C}}(C, D) \to \mathcal{H}om_{\mathcal{D}}(F(C), F(D))$$

  is an isomorphism;

- **homotopically fully faithful**, if for any two objects C, D of $\mathcal{C}$, the cochain map

$$F_{C,D}\colon \mathcal{H}om_{\mathcal{C}}(C, D) \to \mathcal{H}om_{\mathcal{D}}(F(C), F(D))$$

  is a homotopy equivalence;

- **essentially surjective**, if the underlying functor of it $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ is essentially surjective;

- **homotopically essentially surjective**, if the homotopy functor of it $\mathrm{h}F\colon \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is essentially surjective;

- an **equivalence**, if there is another dg-functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G \cong \mathrm{Id}_{\mathcal{D}}$ and $\mathrm{Id}_{\mathcal{C}} \cong G \circ F$ ($\cong$ denotes natural isomorphism);

- a **homotopically equivalence**, if there exists another dg-functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G \simeq \mathrm{Id}_{\mathcal{D}}$ and $\mathrm{Id}_{\mathcal{C}} \simeq G \circ F$ ($\simeq$ denotes natural equivalence);

- an **adjoint equivalence**, if it admits a right adjoint G such that the unit $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F$ and the counit $\epsilon\colon F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$ are natural isomorphisms;

- an **adjoint homotopically equivalence**, if it admits a right adjoint G such that the unit $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F$ and the counit $\epsilon\colon F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$ are natural equivalence.

3.8 Proposition *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a dg-functor. Then the followings are equivalent.*

(i) F is fully faithful and essentially surjective.
(ii) F is an equivalence.
(iii) F is an adjoint equivalence.

Moreover, the followings are equivalent.

(iv) F is homotopically fully faithful and homotopically essentially surjective.
(v) F is an homotopically equivalence.
(vi) F is an adjoint homotopically equivalence.

Proof: Suppose (ii), let's prove (iii). To do this, we need a lemma.

3.8.1 Lemma *Let $\alpha\colon F \Rightarrow \mathrm{Id}$ be a natural isomorphism between dg-endofunctors. Then we have*

$$(F * \alpha) \cdot (\alpha^{-1} * F) = (\alpha * F) \cdot (F * \alpha^{-1}) = \mathrm{id}_F \,.$$

Proof: The result follows from the following commutative diagrams.

$$\begin{tikzcd}
F(-) \arrow[r, "\alpha^{-1}_{F(-)}"] \arrow[d, "\alpha_{(-)}"'] & FF(-) \arrow[d, "F(\alpha_{(-)})"] \\
(-) \arrow[r, "\alpha^{-1}_{(-)}"] & F(-)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(-) \arrow[r, "\alpha_{(-)}"] \arrow[d, "F(\alpha^{-1}_{(-)})"'] & (-) \arrow[d, "\alpha^{-1}_{(-)}"] \\
FF(-) \arrow[r, "\alpha_{F(-)}"] & F(-)
\end{tikzcd}$$

□

Let $G\colon \mathcal{D} \to \mathcal{C}$ be the inverse of F with natural isomorphisms $\eta\colon \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} G \circ F$ and $\varepsilon\colon F \circ G \xrightarrow{\sim} \mathrm{Id}_{\mathcal{D}}$. Then, let $\epsilon\colon F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$ be the composition

$$F \circ G \xrightarrow{F \circ G * \varepsilon^{-1}} F \circ G \circ F \circ G \xrightarrow{F * \eta^{-1} * G} F \circ G \xrightarrow{\varepsilon} \mathrm{Id}_{\mathcal{D}} \,.$$

Then ϵ is a natural isomorphism and by the following commutative diagrams

$$\begin{tikzcd}
F(-) \arrow[r, "\varepsilon^{-1}_{F(-)}"] \arrow[d, "F(\eta_{(-)})"'] & FGF(-) \arrow[d, "FGF(\eta_{(-)})"'] \arrow[dr, dashed, equal] & F(-) \\
FGF(-) \arrow[dr, "FG(\varepsilon^{-1}_{F(-)})"'] & FGFGF(-) \arrow[l, "\varepsilon_{FGF(-)}"'] \arrow[d, dashed, equal] & FGF(-) \arrow[u, "\varepsilon_{F(-)}"'] \\
& FGFGF(-) \arrow[ur, "F(\eta^{-1}_{GF(-)})"'] &
\end{tikzcd}$$

(where the dashed identity transformations come from the lemma) and

$$\begin{tikzcd}
G(-) \arrow[r, "\eta_{G(-)}"] \arrow[d, "G(\varepsilon^{-1}_{(-)})"'] & GFG(-) \arrow[d, "GFG(\varepsilon^{-1}_{(-)})"] \\
GFG(-) \arrow[r, "\eta_{GFG(-)}"] \arrow[dr, dashed, equal] & GFGFG(-) \arrow[d, "GF(\eta^{-1}_{G(-)})"] \\
G(-) & GFG(-) \arrow[l, "G(\varepsilon_{(-)})"]
\end{tikzcd}$$

(where the dashed identity transformation comes from the lemma), the quadruple (F, G, η, ϵ) is an adjunction of dg-functors.

Now, suppose (iii), let's prove (i). First, F is essentially surjective. Indeed, for each object D of $\mathcal{D}$, the 1-morphism $\epsilon_D \colon FG(D) \to D$ gives the desired isomorphism.

To show F is fully faithful, consider the following composition which gives the inverse of the cochain map $F_{C,D}$.

$$\begin{array}{ccccc}
\mathbb{Z} & & \eta_C & & \mathcal{H}om_{\mathcal{C}}(C, GF(C)) \\
\otimes & & \otimes & & \otimes \\
\mathcal{H}om_{\mathcal{D}}(F(C), F(D)) & \longrightarrow & G & \longrightarrow & \mathcal{H}om_{\mathcal{C}}(GF(C), GF(D)) \\
\otimes & & \otimes & & \otimes \\
\mathbb{Z} & & \eta_D^{-1} & & \mathcal{H}om_{\mathcal{C}}(GF(D), D) \\
\cong\big\uparrow & & & & \big\downarrow \\
\mathcal{H}om_{\mathcal{D}}(F(C), F(D)) & & \overset{\alpha_{C,D}}{\dashrightarrow} & & \mathcal{H}om_{\mathcal{C}}(C, D)
\end{array}$$

Indeed, $\alpha_{C,D} \circ F_{C,D} = \mathrm{id}$ follows from the following commutative diagram

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{D}}(F(C), F(D)) \arrow[r, "G"] & \mathcal{H}om_{\mathcal{C}}(GF(C), GF(D)) \arrow[d, "\circ\eta_C"] \\
\mathcal{H}om_{\mathcal{C}}(C, D) \arrow[u, "F"] \arrow[r, "\eta_D\circ"] & \mathcal{H}om_{\mathcal{C}}(C, GF(D))
\end{tikzcd}$$

and $F_{C,D} \circ \alpha_{C,D} = \mathrm{id}$ follows from the following commutative diagram

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{D}}(F(C), F(D)) \arrow[d, "G"] \arrow[ddd, dotted, bend right, "\circ\epsilon_{F(C)}"'] & & \\
\mathcal{H}om_{\mathcal{C}}(GF(C), GF(D)) \arrow[r, "\circ\eta_C"] \arrow[d, "F"] & \mathcal{H}om_{\mathcal{C}}(C, GF(D)) \arrow[r, "\eta_D^{-1}\circ"] \arrow[d, "F"] & \mathcal{H}om_{\mathcal{C}}(C, D) \arrow[d, "F"] \\
\mathcal{H}om_{\mathcal{D}}(FGF(C), FGF(D)) \arrow[r, "\circ F(\eta_C)"] \arrow[d, "\epsilon_{F(D)}\circ"] & \mathcal{H}om_{\mathcal{D}}(F(C), FGF(D)) \arrow[r, "F(\eta_D^{-1})\circ"] \arrow[d, "\epsilon_{F(D)}\circ"] & \mathcal{H}om_{\mathcal{D}}(F(C), F(D)) \arrow[dl, dashed, equal] \\
\mathcal{H}om_{\mathcal{D}}(FGF(C), F(D)) \arrow[r, dotted, "\circ F(\eta_C)"] & \mathcal{H}om_{\mathcal{D}}(F(C), F(D)) &
\end{tikzcd}$$

where the dashed identity as well as that the composition of dotted arrow is identity follows from the triangle identities, and the blue arrows emphasize the desired composition.

Next, suppose (i), let's prove (ii). First, let's construct the dg-functor $G \colon \mathcal{D} \to \mathcal{C}$.

1. For any object D of $\mathcal{D}$, CHOOSE an object C of $\mathcal{C}$ such that $F(C) \cong D$. Then put $G(D) = C$ and denote this isomorphism by ϵ_D.

2. For any pair of objects D, D' of $\mathcal{D}$, the cochain map $G_{D,D'}$ is given by the composition:

$$\mathcal{H}om_{\mathcal{D}}(D, D') \xrightarrow[\epsilon_{D'}^{-1}\circ]{\circ\epsilon_D} \mathcal{H}om_{\mathcal{D}}(FG(D), FG(D')) \xrightarrow{F_{C,D}^{-1}} \mathcal{H}om_{\mathcal{C}}(G(D), G(D')).$$

3. Then G is a dg-functor by straightforward verification using elements.
4. Now ϵ form a dg-transformation by the construction of G and it is clear a natural isomorphism.
5. For each object C of $\mathcal{C}$, define $\eta_C\colon C \to GF(C)$ as the preimage of $\epsilon^{-1}_{F(C)}\colon F(C) \to FGF(C)$ under the cochain map

$$F_{C,GF(C)}\colon \mathcal{H}om_{\mathcal{C}}(C, GF(C)) \longrightarrow \mathcal{H}om_{\mathcal{D}}(F(C), FGF(C)).$$

6. Now η form another dg-transformation since ϵ^{-1} is a dg-transformation and F is fully faithful. It is clear that η is a natural isomorphism.

Finally, the proofs of (iv) implies (v) implies (vi) implies (iv) are similar as above, but:

1. instead of working in the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, one works in the category $\mathrm{hFun}(\mathcal{C}, \mathcal{D})$;
2. instead of using inverse dg-transformations, one needs to use weak inverse;
3. instead of CHOOSE isomorphisms, one has to CHOOSE homotopy equivalences. ☐

Remark It would be helpful if one is familiar with the proof of similar statements in ordinary category theory. One can also try to use elements to drop above proof down to earth.

3.9 Let $\mathcal{C}$ and $\mathcal{D}$ be two dg-categories, then their **(tensor) product** is the dg-category $\mathcal{C} \otimes \mathcal{D}$

- whose collection of objects is the product $\operatorname{ob}\mathcal{C} \times \operatorname{ob}\mathcal{D}$,
- each Hom object $\mathcal{H}om_{\mathcal{C}\otimes\mathcal{D}}((C, D), (C', D'))$ is the tensor product

$$\mathcal{H}om_{\mathcal{C}}(C, C') \otimes \mathcal{H}om_{\mathcal{D}}(D, D'),$$

- the composition rule and the unity is given by obverse constructions.

One should think $\mathcal{C} \otimes \mathcal{D}$ as a dg-version of $\mathcal{C} \times \mathcal{D}$. This tensor product has a unit $\mathbf{1}$, which is the dg-category having one object $*$ with the Hom-complex $\mathcal{H}om_{\mathbf{1}}(*, *) = \mathbb{Z}$, and is symmetric with the braiding given by the braiding γ of **Ch**. In this way, the 2-category dg**Cat** becomes a tensor 2-category.

The **opposite dg-category** of a dg-category $\mathcal{C}$, denoted by $\mathcal{C}^{\mathrm{op}}$, is the dg-category with the same objects as $\mathcal{C}$ and Hom-complex

$$\mathcal{H}om_{\mathcal{C}^{\mathrm{op}}}(C, D) := \mathcal{H}om_{\mathcal{C}}(D, C).$$

Clearly, $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$ and $(\mathcal{C} \otimes \mathcal{D})^{\mathrm{op}} = \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D}^{\mathrm{op}}$. One should think $\mathcal{C}^{\mathrm{op}}$ as a dg-version of opposite category $\mathcal{C}^{\mathrm{op}}$. In this way, dg**Cat** becomes a tensor 2-category with an involution $(-)^{\mathrm{op}}$.

3.10 The change of base categories Z^0 and H^0 gives rise to two 2-functors between 2-categories dg**Cat** $\to$ **Cat** (more precisely, they land in the 2-category of pre-additive categories). It is clear that $(\mathcal{C}^{\mathrm{op}})_0 = \mathcal{C}_0^{\mathrm{op}}$ and that $\mathrm{h}(\mathcal{C}^{\mathrm{op}}) = (\mathrm{h}\mathcal{C})^{\mathrm{op}}$. So they respect the involution structures. However, in general, $(\mathcal{C} \otimes \mathcal{D})_0$ is not isomorphic to $\mathcal{C}_0 \times \mathcal{D}_0$ (as product of categories) or $\mathcal{C}_0 \otimes \mathcal{D}_0$ (as product of pre-additive categories). Similarly for $\mathrm{h}(\mathcal{C} \otimes \mathcal{D})$.

However, there is a canonical additive functor

$$\mathcal{C}_0 \otimes \mathcal{D}_0 \longrightarrow (\mathcal{C} \otimes \mathcal{D})_0,$$
$$\mathrm{h}\mathcal{C} \otimes \mathrm{h}\mathcal{D} \longrightarrow \mathrm{h}(\mathcal{C} \otimes \mathcal{D}).$$

Hence both Z^0 and H^0 induce lax 2-functors between tensor 2-categories. This is because Z^0 and H^0 are lax functors, i.e. for any complexes $C^\bullet$ and $D^\bullet$, there are homomorphisms

$$Z^0(C) \otimes Z^0(D) \longrightarrow Z^0(C \otimes D), \qquad H^0(C) \otimes H^0(D) \longrightarrow H^0(C \otimes D).$$

Indeed, they are just the homomorphisms induced by the inclusion

$$C^0 \otimes D^0 \longrightarrow (C \otimes D)^0.$$

3.11 A **dg-bifunctor** is simply a dg-functor from a product $\mathcal{C} \otimes \mathcal{D}$. One should think it as the dg-version of bifunctor.

As in ordinary case, given two dg-functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C}' \to \mathcal{D}'$, we always have the dg-functor

$$F \otimes G\colon \mathcal{C} \otimes \mathcal{C}' \longrightarrow \mathcal{D} \otimes \mathcal{D}'.$$

One should think it as the dg-version of $F \times G$.

As in ordinary case, any dg-bifunctor $T\colon \mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ induces **partial functors** $T(C, -)\colon \mathcal{D} \to \mathcal{E}$ and $T(-, D)\colon \mathcal{C} \to \mathcal{E}$ by evaluating T at objects C of $\mathcal{C}$ and D of $\mathcal{D}$ respectively. In other words, $T(C, -)$ is precisely the composition

$$\mathcal{D} \cong \mathbf{1} \otimes \mathcal{D} \xrightarrow{C \otimes \mathrm{Id}_{\mathcal{D}}} \mathcal{C} \otimes \mathcal{D} \xrightarrow{T} \mathcal{E},$$

and $T(-, D)$ is the composition

$$\mathcal{C} \cong \mathcal{C} \otimes \mathbf{1} \xrightarrow{\mathrm{Id}_{\mathcal{C}} \otimes D} \mathcal{C} \otimes \mathcal{D} \xrightarrow{T} \mathcal{E}.$$

Conversely, if we have two families of dg-functors $\{F_C\colon \mathcal{D} \to \mathcal{E}\}_{C \in \mathrm{ob}\,\mathcal{C}}$ and $\{G_D\colon \mathcal{C} \to \mathcal{E}\}_{D \in \mathrm{ob}\,\mathcal{D}}$ such that

$$F_C(D) = G_D(C)$$

for any $C \in \mathrm{ob}\,\mathcal{C}$, $D \in \mathrm{ob}\,\mathcal{D}$. Then we can put $T(C, D) = F_C(D) = G_D(C)$. To make this a dg-functor, it remains to define cochain maps

$$\mathcal{H}om_{\mathcal{C}}(C, C') \otimes \mathcal{H}om_{\mathcal{D}}(D, D') \longrightarrow \mathcal{H}om_{\mathcal{E}}(T(C, D), T(C', D')).$$

There are two way to define it:

$$
\begin{tikzcd}
\begin{matrix}\mathcal{H}om_{\mathcal{D}}(D,D') \\ \otimes \\ \mathcal{H}om_{\mathcal{C}}(C,C')\end{matrix} \arrow[r, "F_C \otimes G_{D'}"] \arrow[d, leftrightarrow, "\cong"] & \begin{matrix}\mathcal{H}om_{\mathcal{E}}(T(C,D),T(C,D')) \\ \otimes \\ \mathcal{H}om_{\mathcal{E}}(T(C,D'),T(C',D'))\end{matrix} \arrow[d] \\
 & \mathcal{H}om_{\mathcal{E}}(T(C,D),T(C',D')) \\
\begin{matrix}\mathcal{H}om_{\mathcal{C}}(C,C') \\ \otimes \\ \mathcal{H}om_{\mathcal{D}}(D,D')\end{matrix} \arrow[r, "G_D \otimes F_{C'}"] & \begin{matrix}\mathcal{H}om_{\mathcal{E}}(T(C,D),T(C',D)) \\ \otimes \\ \mathcal{H}om_{\mathcal{E}}(T(C',D),T(C',D'))\end{matrix} \arrow[u]
\end{tikzcd}
$$

Hence the two families defines a dg-bifunctor T such that $T(C,-) = F_C$ and $T(-,D) = G_D$ if and only if the above diagram commutes.

Given two dg-bifunctors $T, S\colon \mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$. Using above characterization of dg-bifunctors, it is easy to show that a family of 1-morphisms $\{\alpha_{C,D}\}_{(C,D)\in \mathrm{ob}(\mathcal{C}\otimes\mathcal{D})}$ forms a dg-transformation from T to S if and only if

1. for any $C \in \mathcal{C}$, the family $\{\alpha_{C,D}\}_{D\in \mathrm{ob}\,\mathcal{D}}$ forms a dg-transformation from $T(C,-)$ to $S(C,-)$, and
2. for any $D \in \mathcal{D}$, the family $\{\alpha_{C,D}\}_{C\in \mathrm{ob}\,\mathcal{C}}$ forms a dg-transformation from $T(-,D)$ to $S(-,D)$.

In other words, dg-transformation can be verified variable by variable.

Remark On can verify that: if $T\colon \mathcal{C}\otimes\mathcal{D} \longrightarrow \mathcal{E}$ is a dg-bifunctor, then the partial functors of the bifunctor

$$\mathcal{C}_0 \times \mathcal{D}_0 \longrightarrow (\mathcal{C}\otimes\mathcal{D})_0 \xrightarrow{T_0} \mathcal{E}_0$$

are precisely the underlying functors of the partial functors of T.

3.12 Example For any dg-category $\mathcal{C}$, there is a natural *dg-bifunctor*

$$\mathcal{H}om_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \longrightarrow \mathbf{Ch}.$$

To see this, it suffices to give the canonical cochain maps

$$\mathcal{H}om_{\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}}((C,D),(C',D')) \longrightarrow [\mathcal{H}om_{\mathcal{C}}(C,D), \mathcal{H}om_{\mathcal{C}}(C',D')].$$

But this is just the adjunct of

$$\mathcal{H}om_{\mathcal{C}}(C',C)\otimes \mathcal{H}om_{\mathcal{C}}(D,D')\otimes \mathcal{H}om_{\mathcal{C}}(C,D) \longrightarrow \mathcal{H}om_{\mathcal{C}}(C',D').$$

In this way, any dg-functor $F\colon \mathcal{C}\to\mathcal{D}$ gives a dg-transformation

$$F_{-,-}\colon \mathcal{H}om_{\mathcal{C}}(-,-) \longrightarrow \mathcal{H}om_{\mathcal{D}}(F(-),F(-)).$$

3.13 Consider the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of dg-functors from a small dg-category $\mathcal{C}$ to another dg-category $\mathcal{D}$. To enhance it into a dg-category, notice that for any two dg-functors F and G and any pair of objects (C, D) in $\mathcal{C}$, there is a Hom-complex

$$\mathcal{H}om_{\mathcal{D}}(F(C), G(D)).$$

Hence the Hom-complex $\mathcal{H}om_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G)$ has to be certain universal construction from them.

Note that, the condition for a family $\{\alpha_C\}_{C\in \mathrm{ob}\,\mathcal{C}}$ from a dg-transformation from F to G can be translated into the following diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\alpha_D"] \arrow[d, "\alpha_C"'] & \mathcal{H}om_{\mathcal{D}}(F(D), G(D)) \arrow[d, "\rho_{C,D}"] \\
\mathcal{H}om_{\mathcal{D}}(F(C), G(C)) \arrow[r, "\lambda_{C,D}"] & {[\mathcal{H}om_{\mathcal{C}}(C, D), \mathcal{H}om_{\mathcal{D}}(F(C), G(D))]}
\end{tikzcd}$$

where $\lambda_{C,D}$ is given by the adjunct of

$$\mathcal{H}om_{\mathcal{C}}(C, D) \otimes \mathcal{H}om_{\mathcal{D}}(F(C), G(C)) \longrightarrow \mathcal{H}om_{\mathcal{D}}(F(C), G(D)),$$

which is the adjunct of

$$\mathcal{H}om_{\mathcal{C}}(C, D) \overset{\mathcal{H}om_{\mathcal{D}}(F(C),G(-))}{\longrightarrow} [\mathcal{H}om_{\mathcal{D}}(F(C), G(C)), \mathcal{H}om_{\mathcal{D}}(F(C), G(D))];$$

similarly, $\rho_{C,D}$ is given by the adjunct of

$$\mathcal{H}om_{\mathcal{D}}(F(D), G(D)) \otimes \mathcal{H}om_{\mathcal{C}}(C, D) \longrightarrow \mathcal{H}om_{\mathcal{D}}(F(C), G(D)),$$

which is the adjunct of

$$\mathcal{H}om_{\mathcal{C}}(C, D) \overset{\mathcal{H}om_{\mathcal{D}}(F(-),G(D))}{\longrightarrow} [\mathcal{H}om_{\mathcal{D}}(F(D), G(D)), \mathcal{H}om_{\mathcal{D}}(F(C), G(D))].$$

Inspired by this, for $T(-,-)\colon \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathbf{Ch}$ a bifunctor (for instant, $T(-,-) = \mathcal{H}om_{\mathcal{D}}(F(-), G(-))$), an **extraordinary naturality** of T is a family of cochain maps $\{\alpha_C\}_{C\in \mathrm{ob}\,\mathcal{C}}$ fitting the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha_D"] \arrow[d, "\alpha_C"'] & T(D, D) \arrow[d, "\rho_{C,D}"] \\
T(C, C) \arrow[r, "\lambda_{C,D}"] & {[\mathcal{H}om_{\mathcal{C}}(C, D), T(C, D)]}
\end{tikzcd}$$

where $\lambda_{C,D}$ is given by the functor $T(C,-)$ and $\rho_{C,D}$ by $T(-,D)$. Then the **end** of T is the universal extraordinary naturality of T. The complex representing the end is denoted by $\int_{C\in\mathcal{C}} T(C, C)$ and the canonical cochain maps $\pi_C\colon \int_{C\in\mathcal{C}} T(C, C) \to T(C, C)$ is called the **counit** at C.

One should notice that an extraordinary naturality is nothing than a cone over the diagram consisting of λ and ρ. Hence, the universal extraordinary naturality is the limit of this diagram. Therefore we have the following equalizer diagram.

$$\int_{C\in\mathcal{C}} T(C,C) \xrightarrow{\alpha} \prod_{C\in\mathcal{C}} T(C,C) \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \prod_{C,D\in\mathcal{C}} [\mathcal{H}om_{\mathcal{C}}(C,D), T(C,D)].$$

With the notion of ends, we can enhance $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ into a dg-category $\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})$ as follows.

(i) The Hom-complex is

$$\mathcal{H}om_{\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})}(F,G) := \int_{C\in\mathcal{C}} \mathcal{H}om_{\mathcal{D}}(F(C),G(C)).$$

(ii) The composition rule is given by the dashed arrow in the following commutative diagrams (with C goes through all the objects of $\mathcal{C}$) uniquely determined by the universal property of end.

$$\begin{tikzcd}
\begin{array}{c}\mathcal{H}om_{\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})}(F,G)\\ \otimes \\ \mathcal{H}om_{\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})}(G,H)\end{array} \arrow[r, "\begin{array}{c}\pi_C\\ \otimes\\ \pi_C\end{array}"] \arrow[d, dashed] & \begin{array}{c}\mathcal{H}om_{\mathcal{D}}(F(C),G(C))\\ \otimes \\ \mathcal{H}om_{\mathcal{D}}(G(C),H(C))\end{array} \arrow[d] \\
\mathcal{H}om_{\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})}(F,H) \arrow[r, "\pi_C"] & \mathcal{H}om_{\mathcal{D}}(F(C),H(C))
\end{tikzcd}$$

(iii) The identity is determined similarly, which turns out to be the *identity dg-transformation.*

3.14 Proposition *The underlying category of* $\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})$ *is* $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ *while the homotopy category is* $\mathrm{hFun}(\mathcal{C},\mathcal{D})$*.*

Proof: First we have

$$\begin{aligned}
Z^0\,\mathcal{H}om_{\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})}(F,G) &= \mathrm{Hom}_{\mathbf{Ch}}\big(\mathbb{Z}, \mathcal{H}om_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F,G)\big),\\
H^0\,\mathcal{H}om_{\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})}(F,G) &= \mathrm{Hom}_{K(\mathbf{Ab})}\big(\mathbb{Z}, \mathcal{H}om_{\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})}(F,G)\big).
\end{aligned}$$

By the universal property of $\int_{C\in\mathcal{C}} \mathcal{H}om_{\mathcal{D}}(F(C),G(C))$, a cochain map from $\mathbb{Z}$ to $\mathcal{H}om_{\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})}(F,G)$ is equivalent to a family of cochain maps $\{\alpha_C\}$ fitting the commutative diagrams

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\alpha_D"] \arrow[d, "\alpha_C"'] & \mathcal{H}om_{\mathcal{D}}(F(D),G(D)) \arrow[d, "\rho_{C,D}"] \\
\mathcal{H}om_{\mathcal{D}}(F(C),G(C)) \arrow[r, "\lambda_{C,D}"] & {[\mathcal{H}om_{\mathcal{C}}(C,D), \mathcal{H}om_{\mathcal{D}}(F(C),G(D))]}
\end{tikzcd}$$

hence a dg-transformation from F to G. Moreover, a cochain homotopy between such cochain maps is equivalent to a family of cochain homotopies between the components of the two families, hence a homotopy between dg-transformations. □

3.15 Proposition *We have natural equivalences of categories*

$$\mathrm{Fun}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \cong \mathrm{Fun}(\mathcal{C}, \mathcal{F}\mathrm{un}(\mathcal{D}, \mathcal{E}))$$

for any dg-categories $\mathcal{C}$*,* $\mathcal{D}$ *and* $\mathcal{E}$ *with* $\mathcal{D}$ *small. Moreover, if both* $\mathcal{C}$ *and* $\mathcal{D}$ *are small, we have natural equivalences of dg-categories*

$$\mathcal{F}\mathrm{un}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \cong \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{F}\mathrm{un}(\mathcal{D}, \mathcal{E})).$$

Proof: We'll construct the an adjunction of transformations as follows.

1. The unit $\eta\colon \mathrm{Id} \Rightarrow \mathcal{F}\mathrm{un}(\mathcal{D}, - \otimes \mathcal{D})$ and
2. the evaluation $\mathrm{ev}\colon \mathcal{F}\mathrm{un}(\mathcal{D}, -) \otimes \mathcal{D} \Rightarrow \mathrm{Id}$.

Then natural equivalence are given by the pair

$$\begin{aligned} \alpha\colon \mathcal{F}\mathrm{un}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) &\longrightarrow \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{F}\mathrm{un}(\mathcal{D}, \mathcal{E})) \\ F &\longmapsto \widehat{F}, \\ \beta\colon \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{F}\mathrm{un}(\mathcal{D}, \mathcal{E})) &\longrightarrow \mathcal{F}\mathrm{un}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \\ G &\longmapsto \widetilde{G}. \end{aligned}$$

Where $\widehat{F}$ is the composition

$$\mathcal{C} \xrightarrow{\eta_{\mathcal{C}}} \mathcal{F}\mathrm{un}(\mathcal{D}, \mathcal{C} \otimes \mathcal{D}) \xrightarrow{\mathcal{F}\mathrm{un}(\mathcal{D}, F)} \mathcal{F}\mathrm{un}(\mathcal{D}, \mathcal{E})$$

and $\widetilde{G}$ is the composition

$$\mathcal{C} \otimes \mathcal{D} \xrightarrow{G \otimes \mathcal{D}} \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \otimes \mathcal{D} \xrightarrow{\mathrm{ev}_{\mathcal{E}}} .\mathcal{E}$$

The components of η are dg-functors

$$\eta_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \mathcal{F}\mathrm{un}(\mathcal{D}, \mathcal{C} \otimes \mathcal{D})$$

which takes an object C of $\mathcal{C}$ to the dg-functor

$$\widehat{C}\colon \mathcal{D} \longrightarrow \mathcal{C} \otimes \mathcal{D}$$

which takes an object D of $\mathcal{D}$ to the object

$$(C, D) \in \mathrm{ob}(\mathcal{C} \otimes \mathcal{D}).$$

For any $D, D' \in \mathrm{ob}\,\mathcal{D}$, the map

$$\widehat{C}_{D,D'}\colon \mathcal{H}om_{\mathcal{D}}(D, D') \longrightarrow \mathcal{H}om_{\mathcal{C}}(C, C) \otimes \mathcal{H}om_{\mathcal{D}}(D, D')$$

is given by $g \mapsto \mathrm{id}_C \otimes g$. For any $C, C' \in \mathrm{ob}\,\mathcal{C}$, the map

$$\eta_{\mathcal{C},C,C'}\colon \mathcal{H}om_{\mathcal{C}}(C, C') \longrightarrow \int_{D \in \mathcal{D}} \mathcal{H}om_{\mathcal{C}}(C, C') \otimes \mathcal{H}om_{\mathcal{D}}(D, D)$$

is induced by $f \mapsto f \otimes \mathrm{id}_D$.

The components of ev are dg-functors

$$\mathrm{ev}_{\mathcal{C}} \colon \mathcal{F}\mathrm{un}(\mathcal{D}, \mathcal{C}) \otimes \mathcal{D} \longrightarrow \mathcal{C}$$

which takes a pair of a dg-functor $F \colon \mathcal{D} \to \mathcal{C}$ and an object D of $\mathcal{D}$ to the object $F(D)$ of $\mathcal{C}$. For any pairs (F_1, D_1) and (F_2, D_2) the map

$$\int_{D \in \mathcal{D}} \mathcal{H}om_{\mathcal{C}}(F_1(D), F_2(D)) \otimes \mathcal{H}om_{\mathcal{D}}(D_1, D_2) \xrightarrow{\mathrm{ev}_{\mathcal{C}}} \mathcal{H}om_{\mathcal{C}}(F_1(D_1), F_2(D_2))$$

is induced by the map

$$\int_{D \in \mathcal{D}} \mathcal{H}om_{\mathcal{C}}(F_1(D), F_2(D)) \longrightarrow [\mathcal{H}om_{\mathcal{D}}(D_1, D_2), \mathcal{H}om_{\mathcal{C}}(F_1(D_1), F_2(D_2))]$$

which is precisely the commutative diagram in the definition of the end $\int_{D \in \mathcal{D}} \mathcal{H}om_{\mathcal{C}}(F_1(D), F_2(D))$.

Once finish above setup, it is straightforward to verify that the pair (η, ev) exhibits the adjunction $- \otimes \mathcal{D} \dashv \mathcal{F}\mathrm{un}(\mathcal{D}, -)$ and then the fact that (α, β) form an equivalence of dg-categories follows. $\square$

3.16 Let $\mathcal{C}$ be a dg-category. A **dg-module** over $\mathcal{C}$ is a dg-functor $M \colon \mathcal{C}^{\mathrm{op}} \to$ **Ch**. In other words, it consists of

- complexes $M(C)$ for each $C \in \mathrm{ob}\,\mathcal{C}$,
- cochain maps

$$M_{C,D} \colon \mathcal{H}om_{\mathcal{C}}(C, D) \otimes M(D) \longrightarrow M(C),$$

and satisfies certain axioms looks like those for modules.

Let's denote the category of dg-modules over $\mathcal{C}$ by $\mathrm{dg}(\mathcal{C})$. For any object C of $\mathcal{C}$, the construction $\mathcal{H}om_{\mathcal{C}}(-, C)$ can be made into a dg-module $\Upsilon(C)$ by letting the cochain map $\Upsilon(C)_{C_1, C_2}$ be

$$\mathcal{H}om_{\mathcal{C}}(C_1, C_2) \otimes \mathcal{H}om_{\mathcal{C}}(C_2, C) \longrightarrow \mathcal{H}om_{\mathcal{C}}(C_1, C).$$

This construction can be made into a dg-functor by letting

$$\Upsilon_{C,D} \colon \mathcal{H}om_{\mathcal{C}}(C, D) \longrightarrow \mathcal{H}om_{\mathrm{dg}(\mathcal{C})}(\Upsilon(C), \Upsilon(D))$$

be induced by the natural cochain maps (the adjunct of composition rule)

$$\mathcal{H}om_{\mathcal{C}}(C, D) \longrightarrow [\mathcal{H}om_{\mathcal{C}}(X, C), \mathcal{H}om_{\mathcal{C}}(X, D)].$$

We have the following Yoneda lemmas.

3.17 Theorem (Yoneda Lemma) *Let $\mathcal{C}$ be a dg-category. For any object C of $\mathcal{C}$ and any dg module M over $\mathcal{C}$, we have an isomorphism*

$$\mathcal{H}om_{\mathrm{dg}(\mathcal{C})}(\Upsilon(C), M) \cong M(C)$$

both natural in C and M.

Proof: For any object D of $\mathcal{C}$, we have a canonical cochain map

$$M_{D,C} \colon \mathcal{H}om_{\mathcal{C}}(D, C) \otimes M(C) \longrightarrow M(D)$$

hence a canonical cochain map

$$M(C) \longrightarrow [\mathcal{H}om_{\mathcal{C}}(D, C), M(D)].$$

It is easy to verify that these cochain maps give rise to an extraordinary naturality of $[\mathcal{H}om_{\mathcal{C}}(-, C), M(-)]$. Hence, it remains to show that this extraordinary naturality satisfies the universal property of $\mathcal{H}om_{\mathrm{dg}(\mathcal{C})}(\Upsilon(C), M)$.

Let $\alpha_{(-)} \colon X^\bullet \to [\mathcal{H}om_{\mathcal{C}}(-, C), M(-)]$ be any extraordinary naturality. Then taking the composition of $\widehat{\alpha_C}$, the adjunct of α_C, with the identity of C, we get a cochain map

$$X^\bullet \cong X^\bullet \otimes \mathbb{Z} \longrightarrow X^\bullet \otimes \mathcal{H}om_{\mathcal{C}}(C, C) \xrightarrow{\widehat{\alpha_C}} M(C).$$

Then it is not difficult to show this is the desired unique cochain map in the universal property of $\mathcal{H}om_{\mathrm{dg}(\mathcal{C})}(\Upsilon(C), M)$. □

3.18 Corollary *The dg-functor*

$$\Upsilon \colon \mathcal{C} \longrightarrow \mathrm{dg}(\mathcal{C})$$

is fully faithful.

Remark This dg-functor is called the **dg-Yoneda embedding**.

3.19 Corollary *For any objects C and D of $\mathcal{C}$ the followings are equivalent*

(i) $\Upsilon(C)$ *and* $\Upsilon(D)$ *are isomorphic;*
(ii) $\Upsilon(C)_0$ *and* $\Upsilon(D)_0$ *are isomorphic;*
(iii) C *and* D *are isomorphic.*

Moreover, if this is the case, any natural isomorphism between $\Upsilon(C)$ and $\Upsilon(D)$ is induced by an isomorphism between C and D.

Proof: Combine the dg-version of Yoneda lemma with the ordinary Yoneda lemma, this is clear. □

3.20 A dg-transformation α between dg-functors to $\mathbf{Ch}(\mathcal{A})$ is called a **natural quasi-isomorphism** if its each component α_C is a quasi-isomorphism. Two dg-functors are said to be **quasi-isomorphic** if there is a zigzag of quasi-isomorphisms between them.

Apply this notion to the dg-transformation given by a dg-functor, we have the following notions:

- a dg-functor $F\colon \mathcal{C} \to \mathcal{D}$ is **quasi-fully faithful**, if for any two objects C, D of $\mathcal{C}$, the cochain map

$$F_{C,D}\colon \mathcal{H}om_{\mathcal{C}}(C, D) \longrightarrow \mathcal{H}om_{\mathcal{D}}(F(C), F(D))$$

  is a quasi-isomorphism;

- a quasi-fully faithful and homotopically essentially surjective dg-functor is called a **quasi-equivalence**.

Note that since natural quasi-isomorphism are not invertible, even up to homotopy, in the dg-category of dg-functors, the notion of quasi-equivalences doesn't have a similar characterization as in Proposition 3.8. This drawback suggests that $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})$ is not a good model for the full higher category of functors.

3.21 Proposition *For any objects C and D of $\mathcal{C}$ the followings are equivalent*

(i) $\Upsilon(C)$ *and* $\Upsilon(D)$ *are equivalent;*
(ii) $\Upsilon(C)$ *and* $\Upsilon(D)$ *are quasi-isomorphic;*
(iii) $\mathrm{h}\Upsilon(C)$ *and* $\mathrm{h}\Upsilon(D)$ *are isomorphic;*
(iv) C *and* D *are equivalent.*

Moreover, if this is the case, any natural equivalence between $\Upsilon(C)$ *and* $\Upsilon(D)$ *is induced by an equivalence between* C *and* D*.*

Proof: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is clear.

(iii) $\Leftrightarrow$ (iv) follows by apply the ordinary Yoneda to the category $\mathrm{h}\mathcal{C}$.

(i) $\Leftrightarrow$ (iv) follows by apply the dg-Yoneda to the category $\mathrm{dg}(\mathcal{C})$, and then apply the lax functor H^0. □

3.22 A dg-module is said to be **representable** (resp. **weak representable**, **quasi-representable**) if it is isomorphic (resp. equivalent, quasi-isomorphic) to some $\Upsilon(C)$.

§ 4 Homotopy limits

4.1 Let $\mathcal{C}$ be a dg-category. A **diagram** is a functor D from a small category $\mathcal{I}$ to the underlying category of $\mathcal{C}$. We can simply denote it by $D\colon \mathcal{I} \to \mathcal{C}$. One can then talk about the notions of limits/colimits in $\mathcal{C}$. However, this doesn't involve the higher structures. The more natural notions should be *homotopy limits/colimits*. Note that, if these notions are defined, then we should have natural quasi-isomorphisms

$$\mathcal{H}om_{\mathcal{C}}(-, \operatorname{holim} D) \xrightarrow{qis} \operatorname{holim} \mathcal{H}om_{\mathcal{C}}(-, D),$$
$$\mathcal{H}om_{\mathcal{C}}(\operatorname{hocolim} D, -) \xrightarrow{qis} \operatorname{holim} \mathcal{H}om_{\mathcal{C}}(D, -).$$

Therefore, to define the general notions of homotopy limits/colimits, it is sufficient to define them in **Ch**.

Let D be a diagram in **Ch**, the *homotopy limits/colimit* of D, is expected to be the *universal homotopy cone/cocone* of the diagram D, where a *homotopy cone/cocone* of D is expected to be the homotopy analogy of cone/cocone of a diagram in ordinary category. More precisely, there should be a notion of **homotopy cone** from an object X to D and a complex $\operatorname{HoCone}(X, D)$ of homotopy cones from X to D. Then the **homotopy limit** of D is characterized by the quasi-isomorphism

$$\mathcal{H}om_{\mathcal{C}}(-, \operatorname{holim} D) \xrightarrow{qis} \operatorname{HoCone}(-, D).$$

Similarly, there should be a notion of **homotopy cocone** from D to an object X and a complex $\operatorname{HoCocone}(D, X)$ of homotopy cocones from D to X. Then the **homotopy colimit** of D is characterized by the quasi-isomorphism

$$\mathcal{H}om_{\mathcal{C}}(\operatorname{hocolim} D, -) \xrightarrow{qis} \operatorname{HoCocone}(D, -).$$

Remark Of course D is merely a functor not a dg-functor. So how can one get a dg-functor from it? This comes from the fact that taking underlying category admits a left adjoint: taking the free dg-category of an ordinary/pre-additive category. This operation can be easily built as long as one knows the following adjunctions:

$$(\iota \dashv Z_0)\colon \mathbf{Ab} \rightleftarrows \mathbf{Ch},$$
$$(\text{Free abelian group} \dashv \text{Forgetful})\colon \mathbf{Set} \rightleftarrows \mathbf{Ab}.$$

It may be ambiguous as we write the dg-functor induced by the functor $D\colon \mathcal{I} \to \mathcal{C}_0$ also by $D\colon \mathcal{I} \to \mathcal{C}$ since the free dg-category of $\mathcal{I}$ usually has different underlying category than $\mathcal{I}$. However, this notation is meaningful since the dg-functor D and the functor D are just a pair of adjuncts. As a comparison, the functor

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}_0, \mathcal{D}_0)$$

sending a dg-functor to its underlying functor is neither fully faithful not essentially surjective.

Remark One may wonder why we expect natural quasi-isomorphisms rather than natural equivalences, the first is certainly weaker then the later. If we use the equivalence-version universal property, we can guarantee the the universal object is unique up to equivalence. How about the quasi-isomorphism-version?

Suppose we have quasi-isomorphisms

$$\mathcal{H}om_{\mathcal{C}}(-, L) \xrightarrow{qis} \operatorname{holim} \mathcal{H}om_{\mathcal{C}}(-, D),$$
$$\mathcal{H}om_{\mathcal{C}}(-, L') \xrightarrow{qis} \operatorname{holim} \mathcal{H}om_{\mathcal{C}}(-, D).$$

Then we $\Upsilon(L)$ and $\Upsilon(L')$ are quasi-isomorphic. Then by the Yoneda lemma (Proposition 3.21), L and L' are equivalent and a natural quasi-isomorphism $\Upsilon(L) \to \Upsilon(L')$ is induced by an equivalence $L \to L'$.

4.2 Let $\mathcal{C}$ be a dg-category. Let $D \colon \mathcal{I} \to \mathcal{C}$ be a diagram. Since we have the dg-category $\mathcal{F}\mathrm{un}(\mathcal{I}, \mathcal{C})$, it seems reasonable to define a *naïve homotopy cone* from an object X to D as a (general) morphism from the constant functor with value X, hence also denoted by X, to D. So the *space of naïve homotopy cones* from X to D is merely the complex $\mathcal{H}om_{\mathcal{F}\mathrm{un}(\mathcal{I},\mathcal{C})}(X, D)$. Let ϕ be a homotopy cone from object L. Then we have a canonical transformation

$$\mathcal{H}om_{\mathcal{C}}(-, L) \xrightarrow{\phi_*} \mathcal{H}om_{\mathcal{F}\mathrm{un}(\mathcal{I},\mathcal{C})}(-, D).$$

Then since we require

$$\mathcal{H}om_{\mathcal{C}}(-, \operatorname{holim} D) \xrightarrow{qis} \operatorname{holim} \mathcal{H}om_{\mathcal{C}}(-, D),$$

the universal property for ϕ being a *naïve homotopy limit* (resp. *stupid homotopy limit*) of D is that the dg-transformation ϕ_* is a natural quasi-isomorphism (resp. natural isomorphism).

Let ϕ be a naïve homotopy cone exhibits an object L as a stupid homotopy limit of D, then the following composition is an isomorphism.

$$\mathrm{Hom}_{\mathcal{C}_0}(-, L) \xrightarrow{Z^0(\phi_*)} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(-, D) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C}_0)}(-, D)$$

(notice that the isomorphism comes from the adjunction of building free dg-category and taking underlying category). Hence, a stupid limit is just an ordinary limit!

Remark One should be not surprise since by carefully looking at the above definition, a naïve homotopy cone is still a strict commutative diagram, although not necessary of 1-morphisms, not as wild as merely commutative up to homotopy. This drawback suggests that $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})$ is still not the full higher category of dg-functors.

Remark Another way is to define a homotopy cone from an object X to D as cone on the category $\mathrm{h}\mathcal{C}$ instead of $\mathcal{C}$. This makes more sense as now a cone is a diagram commutative up to homotopy. However, since the category $\mathrm{h}\mathcal{C}$ forgets higher homotopies, this definition is still not the one we expect! In practice is even worse, there are many reasonable homotopy categories just don't have enough limit!

4.3 Before giving the correct definition of homotopy limits, let's try to give a reasonable definition for special diagrams inspired by algebraic topology.

(i) D is the empty diagram. Then a homotopy cone is an object. So the homotopy limit of D, no matter what it is, must equivalent to the terminal object.

(ii) D is merely an object. Then a cone is a morphism to D. So the homotopy limit of D, no matter what it is, must equivalent to the object D.

(iii) D is a family of objects $\{D_i\}_{i \in I}$. Then a cone is a family of morphisms to each D_i with no requirement of commutativity. So the homotopy limit of D, no matter what it is, must equivalent to the product $\prod_{i \in I} D_i$.

(iv) D is a morphism $f : C \to D$. Then a homotopy cone is a triangle

$$\begin{tikzcd} X \arrow[d] \arrow[dr] & \\ C \arrow[r, "f"'] & D \arrow[from=1-1, to=2-1, phantom, shift left=4ex, "\swarrow"] \end{tikzcd}$$

commuting up to a homotopy. As in algebraic topology, we call the homotopy limit of this diagram, no matter what it is, the **mapping path object** of f.

(v) D is a pair of morphisms $f : C \to D$ and $g : E \to D$. Then a homotopy cone is a square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & E \arrow[d, "g"] \arrow[dl, Rightarrow, dashed] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

commuting up to a homotopy. As in algebraic topology, we call the homotopy limit of this diagram, no matter what it is, the **homotopy fiber product** or the **homotopy pullback** of g along f.

4.4 Let $f\colon C^\bullet \to D^\bullet$ be a cochain map between complexes of abelian groups. Then the data of a homotopy cone of this diagram (a *homotopy triangle*) consists of

- a complex $X^\bullet$;
- two cochain maps $x_0\colon X^\bullet \to C^\bullet$ and $x_1\colon X^\bullet \to D^\bullet$; and
- a homotopy $\Phi\colon x_1 \Rightarrow f \circ x_0$.

The above data can be organized into the following commutative diagram of complexes

$$\begin{tikzcd}
X^\bullet \arrow[r, "\Phi"] \arrow[d, "x_0"'] & \langle I, D\rangle^\bullet \arrow[d, "\mathrm{ev}_1"] \\
C^\bullet \arrow[r, "f"] & D^\bullet
\end{tikzcd}$$

where the cochain map x_1 is hidden in the diagram by $x_1 = \mathrm{ev}_0 \circ \Phi$. Therefore, a homotopy triangle is equivalent to a commutative square as above, hence equivalent to a cochain map

$$x\colon X^\bullet \longrightarrow \mathrm{Path}(f)^\bullet$$

where $\mathrm{Path}(f)^\bullet$ is the fiber product of $C^\bullet$ and $\langle I, D\rangle^\bullet$ over $D^\bullet$. More elementarily, $\mathrm{Path}(f)^\bullet$ is the complex

$$\mathrm{Path}(f)^n = D^{n-1}e^* \oplus D^n v_0^* \oplus C^n v_1^*,$$
$$\mathrm{d}^n = \begin{pmatrix} -\mathrm{d}_D^{n-1} & -1 & f \\ & \mathrm{d}_D^n & \\ & & \mathrm{d}_C^n \end{pmatrix}.$$

Under this description, the cochain map x has components

$$x^n = (\phi^n, x_1^n, x_0^n)^{\mathrm{t}}$$

where ϕ is the cochain homotopy presenting Φ. One can think this as a kind of *universal property*: whenever one has a homotopy triangle (X, x_0, x_1, ϕ), one gets a unique cochain map $x\colon X^\bullet \to \mathrm{Path}(f)^\bullet$ such that the compositions of x with the three projections from $\mathrm{Path}(f)^\bullet$ give ϕ, x_1 and x_0 respectively. Then the complex of homotopy triangles with vertex X is essentially the complex

$$[X, \mathrm{Path}(f)]^\bullet.$$

Let $[X, f]$ denote the cochain map

$$[X, C]^\bullet \longrightarrow [X, D]^\bullet$$

induced by f. Then it is easy to show that

$$[X, \mathrm{Path}(f)]^\bullet \cong \mathrm{Path}([X, f])^\bullet.$$

Now, let $f\colon C \to D$ be a 1-morphism in any dg-category $\mathcal{C}$. For any object X, let $\mathcal{H}om_{\mathcal{C}}(X, f)$ denote the cochain map

$$\mathcal{H}om_{\mathcal{C}}(X, C) \longrightarrow \mathcal{H}om_{\mathcal{C}}(X, D)$$

induced by f. Then a **mapping path object** of f is an object $\mathrm{Path}(f)$ of $\mathcal{C}$ together with a natural quasi-isomorphism

$$\mathcal{H}om_{\mathcal{C}}\left(-, \mathrm{Path}(f)\right) \xrightarrow{qis} \mathrm{Path}\big(\mathcal{H}om_{\mathcal{C}}(-, f)\big).$$

By Yoneda lemma (Theorem 3.17), this is equivalent to a 0-cocycle π of $\mathrm{Path}\big(\mathcal{H}om_{\mathcal{C}}(\mathrm{Path}(f), f)\big)$ (called the **universal homotopy triangle** exhibiting $\mathrm{Path}(f)$ as a mapping path object of f), such that π induces above natural quasi-isomorphism. If the above natural quasi-isomorphism is further an natural isomorphism, then we say $\mathrm{Path}(f)$ is a **strong mapping path object** of f.

Dually, for any object X, let $\mathcal{H}om_{\mathcal{C}}(f, X)$ denote the cochain map

$$\mathcal{H}om_{\mathcal{C}}(D, X) \longrightarrow \mathcal{H}om_{\mathcal{C}}(C, X)$$

induced by f. Then a **mapping cylinder** of f is an object $\mathrm{Cly}(f)$ of $\mathcal{C}$ together with a natural quasi-isomorphism

$$\mathcal{H}om_{\mathcal{C}}\left(\mathrm{Cly}(f), -\right) \xrightarrow{qis} \mathrm{Path}\big(\mathcal{H}om_{\mathcal{C}}(f, -)\big).$$

By Yoneda lemma (Theorem 3.17), this is equivalent to a 0-cocycle π of $\mathrm{Cly}\big(\mathcal{H}om_{\mathcal{C}}(f, \mathrm{Cly}(f))\big)$ (called the **universal homotopy cotriangle** exhibiting $\mathrm{Cly}(f)$ as a mapping cylinder of f), such that π induces above natural quasi-isomorphism. If the above natural quasi-isomorphism is further an natural isomorphism, then we say $\mathrm{Cly}(f)$ is a **strong mapping cylinder** of f.

Let $f\colon C^\bullet \to D^\bullet$ be a cochain morphism of complexes in arbitrary abelian category $\mathcal{A}$. Then, the *strong mapping path object* of f is the complex

$$\mathrm{Path}(f)^n = D^{n-1}e^* \oplus D^n v_0^* \oplus C^n v_1^*,$$
$$\mathrm{d}^n = \begin{pmatrix} -\mathrm{d}_D^{n-1} & -1 & f \\ & \mathrm{d}_D^n & \\ & & \mathrm{d}_C^n \end{pmatrix}.$$

The *strong mapping cylinder* of f is the complex

$$\mathrm{Cly}(f)^n = C^{n+1}e \oplus C^n v_0 \oplus D^n v_1,$$
$$\mathrm{d}^n = \begin{pmatrix} -\mathrm{d}_C^{n+1} & & \\ -1 & \mathrm{d}_C^n & \\ f & & \mathrm{d}_D^n \end{pmatrix}.$$

Dually, if $f\colon C_\bullet \to D_\bullet$ is a chain morphism of complexes in $\mathcal{A}$, then the *strong mapping path object* of f is the complex

$$\begin{aligned}\mathrm{Path}(f)_n &= D_{n+1}e^* \oplus D_n v_0^* \oplus C_n v_1^*,\\ \partial_n &= \begin{pmatrix} -\partial^D_{n+1} & -1 & f \\ & \partial^D_n & \\ & & \partial^C_n \end{pmatrix},\end{aligned}$$

and the *strong mapping cylinder* of f is the complex

$$\begin{aligned}\mathrm{Cly}(f)_n &= C_{n-1}e \oplus C_n v_0 \oplus D_n v_1,\\ \partial_n &= \begin{pmatrix} -\partial^C_{n-1} & & \\ -1 & \partial^C_n & \\ f & & \partial^D_n \end{pmatrix}.\end{aligned}$$

4.5 Let $f\colon C^\bullet \to D^\bullet$ and $g\colon E^\bullet \to D^\bullet$ be two cochain maps between complexes of abelian groups. Then the data of a homotopy cone (a *homotopy square*) consists of

- a complex $X^\bullet$;
- two cochain maps $x_0\colon X^\bullet \to C^\bullet$, and $x_1\colon X^\bullet \to E^\bullet$; and
- a homotopy $\Phi\colon g \circ x_1 \Rightarrow f \circ x_0$.

The above data can be organized into the following commutative diagram of complexes

$$\begin{tikzcd} E^\bullet \arrow[r, "g"] & D^\bullet \\ X^\bullet \arrow[u, "x_1"] \arrow[d, "x_0"'] \arrow[r, dashed, "\Phi"] & \langle I, D\rangle^\bullet \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ C^\bullet \arrow[r, "f"] & D^\bullet \end{tikzcd}$$

hence is equivalent to a cochain map

$$x\colon X^\bullet \longrightarrow (C \times^h_D E)^\bullet$$

where $(C \times^h_D E)^\bullet$ is the limit of the diagram

$$C^\bullet \xrightarrow{f} D^\bullet \xleftarrow{\mathrm{ev}_1} \langle I, D\rangle^\bullet \xrightarrow{\mathrm{ev}_0} D^\bullet \xleftarrow{g} E^\bullet$$

More elementarily, $(C \times^h_D E)^\bullet$ is the complex

$$\begin{aligned}(C \times^h_D E)^n &= D^{n-1}e^* \oplus E^n v_0^* \oplus C^n v_1^*,\\ \mathrm{d}^n &= \begin{pmatrix} -\mathrm{d}^{n-1}_D & -g & f \\ & \mathrm{d}^n_E & \\ & & \mathrm{d}^n_C \end{pmatrix}.\end{aligned}$$

Under this description, the cochain map x has components

$$x^n = (\phi^n, x_1^n, x_0^n)^{\mathrm{t}},$$

where ϕ is the cochain homotopy presenting Φ. Again, one can think this as a *universal property*: whenever one has a homotopy square (X, x_0, x_1, ϕ), one gets a unique cochain map $x\colon X^\bullet \to (C \times_D^h E)^\bullet$ such that the compositions of x with the three projections from $(C \times_D^h E)^\bullet$ give ϕ, x_1 and x_0 respectively. Then the complex of homotopy square with vertex X is essentially the complex

$$[X, C \times_D^h E]^\bullet.$$

Then it is easy to show that

$$[X, C \times_D^h E]^\bullet \cong \Big([X, C] \times_{[X,D]}^h [X, E]\Big)^\bullet.$$

Let g' and f' be the projections from $(C \times_D^h E)^\bullet$ to $C^\bullet$ and $E^\bullet$ respectively. Then we have

$$[X, f'] = [X, f]', \qquad \text{and} \qquad [X, g'] = [X, g]'.$$

Now, let $f\colon C \to D$ and $g\colon E \to D$ be two 1-morphisms in any dg-category $\mathcal{C}$. Then a **homotopy fiber product** of C and E over D is an object $C \times_D^h E$ of $\mathcal{C}$ together with a natural quasi-isomorphism

$$\mathcal{H}om_{\mathcal{C}}\left(-, C \times_D^h E\right) \xrightarrow{qis} \mathcal{H}om_{\mathcal{C}}(-, C) \times_{\mathcal{H}om_{\mathcal{C}}(-,D)}^h \mathcal{H}om_{\mathcal{C}}(-, E).$$

By Yoneda lemma (Theorem 3.17), this is equivalent to a 0-cocycle π of the complex $\mathcal{H}om_{\mathcal{C}}(-, C) \times_{\mathcal{H}om_{\mathcal{C}}(-,D)}^h \mathcal{H}om_{\mathcal{C}}(-, E)$ (called the **homotopy Cartesian diagram** exhibiting $C \times_D^h E$ as a homotopy fiber product of C and D over E), such that π induces above natural quasi-isomorphism. If the above natural quasi-isomorphism is further an natural isomorphism, then we say $C \times_D^h E$ is a **strong homotopy fiber product**. The 1-morphism $g'\colon C \times_D^h E \to C$ given by the projection

$$\mathcal{H}om_{\mathcal{C}}(C \times_D^h E, C) \times_{\mathcal{H}om_{\mathcal{C}}(C \times_D^h E, D)}^h \mathcal{H}om_{\mathcal{C}}(C \times_D^h E, E) \longrightarrow \mathcal{H}om_{\mathcal{C}}(C \times_D^h E, C)$$

is called the **homotopy pullback of** g along f.

Dually, let $f\colon D \to C$ and $g\colon D \to E$ be two 1-morphisms in $\mathcal{C}$. Then a **homotopy fiber coproduct** of C and E rel D is an object $C \amalg_D^h E$ of $\mathcal{C}$ together with a natural quasi-isomorphism

$$\mathcal{H}om_{\mathcal{C}}\left(C \amalg_D^h E, -\right) \xrightarrow{qis} \mathcal{H}om_{\mathcal{C}}(C, -) \times_{\mathcal{H}om_{\mathcal{C}}(D,-)}^h \mathcal{H}om_{\mathcal{C}}(E, -).$$

By Yoneda lemma (Theorem 3.17), this is equivalent to a 0-cocycle π of the complex $\mathcal{H}om_{\mathcal{C}}(C, -) \times_{\mathcal{H}om_{\mathcal{C}}(D,-)}^h \mathcal{H}om_{\mathcal{C}}(E, -)$ (called the **homotopy co-Cartesian diagram** exhibiting $C \amalg_D^h E$ as a homotopy fiber coproduct of

C and D rel E), such that π induces above natural quasi-isomorphism. If the above natural quasi-isomorphism is further an natural isomorphism, then we say $C \amalg_D^h E$ is a **strong homotopy fiber coproduct**. The 1-morphism $g' \colon C \to C \amalg_D^h E$ given by the projection

$$\mathcal{H}om_{\mathcal{C}}(C, C \amalg_D^h E) \times^h_{\mathcal{H}om_{\mathcal{C}}(D, C \amalg_D^h E)} \mathcal{H}om_{\mathcal{C}}(E, C \amalg_D^h E) \longrightarrow \mathcal{H}om_{\mathcal{C}}(C, C \amalg_D^h E)$$

is called the **homotopy pushout of** g along f.

Let $f \colon C^\bullet \to D^\bullet$ and $g \colon E^\bullet \to D^\bullet$ be two cochain morphisms between complexes in arbitrary abelian category $\mathcal{A}$. Then, the *strong homotopy fiber product* $C \times_D^h E$ is the complex

$$\begin{aligned} (C \times_D^h E)^n &= D^{n-1} e^* \oplus E^n v_0^* \oplus C^n v_1^*, \\ \mathrm{d}^n &= \begin{pmatrix} -\mathrm{d}_D^{n-1} & -g & f \\ & \mathrm{d}_E^n & \\ & & \mathrm{d}_C^n \end{pmatrix}. \end{aligned}$$

Dually, let $f \colon D^\bullet \to C^\bullet$ and $g \colon D^\bullet \to E^\bullet$ be two cochain morphisms between complexes in $\mathcal{A}$. Then, the *strong homotopy fiber coproduct* $C \amalg_D^h E$ is the complex

$$\begin{aligned} (C \amalg_D^h E)^n &= D^{n+1} e \oplus E^n v_0 \oplus C^n v_1, \\ \mathrm{d}^n &= \begin{pmatrix} -\mathrm{d}_D^{n+1} & & \\ -g & \mathrm{d}_E^n & \\ f & & \mathrm{d}_C^n \end{pmatrix}. \end{aligned}$$

Dually, let $f \colon C_\bullet \to D_\bullet$ and $g \colon E_\bullet \to D_\bullet$ be two chain morphisms between complexes in $\mathcal{A}$. Then, the *strong homotopy fiber product* $C \times_D^h E$ is the complex

$$\begin{aligned} (C \times_D^h E)_n &= D_{n+1} e^* \oplus E_n v_0^* \oplus C_n v_1^*, \\ \partial_n &= \begin{pmatrix} -\partial_{n+1}^D & -g & f \\ & \partial_n^E & \\ & & \partial_n^C \end{pmatrix}. \end{aligned}$$

Dually, let $f \colon D^\bullet \to C^\bullet$ and $g \colon D^\bullet \to E^\bullet$ be two cochain maps between complexes in $\mathcal{A}$. Then, the *strong homotopy fiber coproduct* $C \amalg_D^h E$ is the complex

$$\begin{aligned} (C \amalg_D^h E)_n &= D_{n-1} e \oplus E_n v_0 \oplus C_n v_1, \\ \partial_n &= \begin{pmatrix} -\partial_{n-1}^D & & \\ -g & \partial_n^E & \\ f & & \partial_n^C \end{pmatrix}. \end{aligned}$$

4.6 Proposition (Pasting lemma) *Suppose we have the following diagram in a dg-category $\mathcal{C}$:*

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] \arrow[dl, Rightarrow, dashed] & \bullet \arrow[d] \arrow[dl, Rightarrow, dashed] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

where the dashed arrows denote homotopies.

(i) If both square are homotopy Cartesian diagrams (resp. homotopy co-Cartesian diagrams), then so is the rectangle.

(ii) If both the right square and the rectangle are homotopy Cartesian diagrams, then so is the left square.

(iii) If both the left square and the rectangle are homotopy co-Cartesian diagrams, then so is the right square.

Proof: The key point is: the two squares and the rectangle are homotopy squares. Hence the statements follows from the universal property. If one is satisfied with this argument, one can just skip the following proof.

To spell out an explicit proof, first note that it is sufficient to prove the statements of homotopy Cartesian diagrams in **Ch**.

Suppose we have the diagram

$$\begin{tikzcd}
C \times^h_D (D \times^h_E F) \arrow[r, "f'"] \arrow[d, "h''"'] & D \times^h_E F \arrow[r, "g'"] \arrow[d, "h'"'] \arrow[dl, Rightarrow, dashed, "\alpha"] & F \arrow[d, "h"] \arrow[dl, Rightarrow, dashed, "\beta"'] \\
C \arrow[r, "f"'] & D \arrow[r, "g"'] & E
\end{tikzcd}$$

where both squares are homotopy Cartesian diagrams. Then, by composing g' with f, g with f and α with β, we get a homotopy square

$$\begin{tikzcd}
C \times^h_D (D \times^h_E F) \arrow[r, "g' \circ f'"] \arrow[d, "h''"'] & F \arrow[d, "h"] \arrow[dl, Rightarrow, dashed, "\Phi"] \\
C \arrow[r, "g \circ f"'] & E
\end{tikzcd}$$

where the homotopy Φ is

$$(g * \alpha) \dotplus (\beta * f').$$

Therefore, the corresponding cochain map

$$p \colon C \times^h_D (D \times^h_E F) \longrightarrow C \times^h_E F$$

is $(\phi, g' \circ f', h'')^{\mathrm{t}}$ which can be spelled out as

$$p^n = \begin{pmatrix} g & 1 & & & \\ & 0 & 1 & 0 & \\ & & & & 1 \end{pmatrix}$$

under the decomposition

$$
\begin{aligned}
(C \times^h_D (D \times^h_E F))^n &= D^{n-1}e^* \oplus E^{n-1}e^* v_0^* \oplus F^n v_0^* v_0^* \oplus D^n v_1^* v_0^* \oplus C^n v_1^*,\\
(C \times^h_E F)^n &= E^{n-1}e^* \oplus F^n v_0^* \oplus C^n v_1^*.
\end{aligned}
$$

Conversely, suppose we have homotopy Cartesian diagrams.

$$
\begin{tikzcd}
C \times^h_E F \arrow[r, "(g\circ f)'"] \arrow[d, "h'''"'] & F \arrow[d, "h"] \arrow[dl, Rightarrow, dashed, "\Psi"'] \\
C \arrow[r, "g\circ f"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
D \times^h_E F \arrow[r, "g'"] \arrow[d, "h'"'] & F \arrow[d, "h"] \arrow[dl, Rightarrow, dashed, "\beta"'] \\
D \arrow[r, "g"'] & E
\end{tikzcd}
$$

Then, the left one gives a homotopy square

$$
\begin{tikzcd}
C \times^h_E F \arrow[r, "(g\circ f)'"] \arrow[d, "f\circ h'''"'] & F \arrow[d, "h"] \arrow[dl, Rightarrow, dashed, "\Psi"'] \\
D \arrow[r, "g"'] & E
\end{tikzcd}
$$

The corresponding cochain map

$$
f'' \colon C \times^h_E F \longrightarrow D \times^h_E F
$$

is $(\psi, (g\circ f)', f\circ h''')^{\mathrm{t}}$ which can be spelled out as

$$
f''^n = \begin{pmatrix} 1 & & \\ & 1 & \\ & & f \end{pmatrix}
$$

under the decomposition

$$
\begin{aligned}
(C \times^h_E F)^n &= E^{n-1}e^* \oplus F^n v_0^* \oplus C^n v_1^*,\\
(D \times^h_E F)^n &= E^{n-1}e^* \oplus F^n v_0^* \oplus D^n v_1^*.
\end{aligned}
$$

Now, we have a commutative square

$$
\begin{tikzcd}
C \times^h_E F \arrow[r, "f''"] \arrow[d, "h'''"'] & D \times^h_E F \arrow[d, "h'"] \\
C \arrow[r, "f"'] & D
\end{tikzcd}
$$

Therefore, the corresponding cochain map

$$
q \colon C \times^h_E F \longrightarrow C \times^h_D (D \times^h_E F)
$$

is $(0, f'', h''')^{\mathrm{t}}$ which can be spelled out as

$$q^n = \begin{pmatrix} 0 & & \\ 1 & & \\ & 1 & \\ & & f \\ & & 1 \end{pmatrix}.$$

Now, it is clear that

$$p \circ q = \mathrm{id}_{C \times^h_E F}$$

On the other hand, α gives a homotopy from $\mathrm{id}_{C \times^h_D (D \times^h_E F)}$ to $q \circ p$ by composing with a section of $h' \circ f'$ □

4.7 Let $f\colon C^\bullet \to D^\bullet$ be a cochain map between complexes of abelian groups. Consider the diagram $C^\bullet \xrightarrow{f} D^\bullet \longleftarrow 0$. Note that this is a special case of 4.5. However, since the special properties of 0, one expects a more concentrate expression of the strong homotopy limit of this diagram. A homotopy cone of this diagram (a *homotopy annihilation*) consists of the following data

- a complex $X^\bullet$;
- a cochain map $x_0\colon X^\bullet \to C^\bullet$; and
- a homotopy $\Phi\colon 0 \Rightarrow f \circ x_0$.

The above data can be organized into the following commutative diagram of complexes

$$\begin{tikzcd} & D^\bullet \\ X^\bullet \arrow[ur, "0"] \arrow[r, dashed, "\Phi"] \arrow[d, "x_0"'] & \langle I, D\rangle^\bullet \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ C^\bullet \arrow[r, "f"] & D^\bullet \end{tikzcd}$$

hence is equivalent to a cochain map

$$x\colon X^\bullet \longrightarrow \mathrm{Fib}(f)^\bullet$$

where $\mathrm{Fib}(f)^\bullet$ is the limit of the diagram

$$C^\bullet \xrightarrow{f} D^\bullet \xleftarrow{\mathrm{ev}_1} \langle I, D\rangle^\bullet \xrightarrow{\mathrm{ev}_0} D^\bullet \longleftarrow 0$$

More elementarily, $(C \times^h_D E)^\bullet$ is the complex

$$\begin{aligned} \mathrm{Fib}(f)^n &= D^{n-1}e^* \oplus C^n v_1^*, \\ \mathrm{d}^n &= \begin{pmatrix} -\mathrm{d}_D^{n-1} & f \\ & \mathrm{d}_C^n \end{pmatrix}. \end{aligned}$$

Under this description, the cochain map x has components

$$x^n = (\phi^n, x_0^n)^{\mathrm{t}},$$

where ϕ is the cochain homotopy presenting Φ. Again, one can think this as a *universal property*: whenever one has a homotopy annihilation (X, x_1, ϕ), one gets a unique cochain map $x\colon X^\bullet \to (C \times_D^h E)^\bullet$ such that the compositions of x with the two projections from $\mathrm{Fib}(f)^\bullet$ give ϕ and x_0 respectively. Then the complex of homotopy annihilation with vertex X is the complex

$$[X, \mathrm{Fib}(f)]^\bullet.$$

Then it is easy to show that

$$[X, \mathrm{Fib}(f)]^\bullet \cong \mathrm{Fib}([X, f])^\bullet.$$

Now, let $f\colon C \to D$ be a 1-morphisms in any dg-category $\mathcal{C}$. Then a **homotopy fiber** of f is an object $\mathrm{Fib}(f)$ of $\mathcal{C}$ together with a natural quasi-isomorphism

$$\mathcal{H}om_{\mathcal{C}}\left(-, \mathrm{Fib}(f)\right) \xrightarrow{qis} \mathrm{Fib}\big(\mathcal{H}om_{\mathcal{C}}(-, f)\big).$$

By Yoneda lemma (Theorem 3.17), this is equivalent to a 0-cocycle π of the complex $\mathrm{Fib}\big(\mathcal{H}om_{\mathcal{C}}(-, f)\big)$ such that π induces above natural quasi-isomorphism. If the above natural quasi-isomorphism is further an natural isomorphism, then we say $\mathrm{Fib}(f)$ is a **strong homotopy fiber**.

Dually, a **homotopy cofiber**, or **mapping cone** of f is an object $\mathrm{Cofib}(f)$ of $\mathcal{C}$ together with a natural quasi-isomorphism

$$\mathcal{H}om_{\mathcal{C}}\left(\mathrm{Cofib}(f), -\right) \xrightarrow{qis} \mathrm{Fib}\big(\mathcal{H}om_{\mathcal{C}}(f, -)\big).$$

By Yoneda lemma (Theorem 3.17), this is equivalent to a 0-cocycle π of the complex $\mathrm{Fib}\big(\mathcal{H}om_{\mathcal{C}}(f, -)\big)$ such that π induces above natural quasi-isomorphism. If the above natural quasi-isomorphism is further an natural isomorphism, then we say $\mathrm{Cofib}(f)$ is a **strong homotopy cofiber**.

Let $f\colon C^\bullet \to D^\bullet$ be a cochain morphism between complexes in arbitrary abelian category $\mathcal{A}$. Then, the *strong homotopy fiber* $\mathrm{Fib}(f)$ is the complex

$$\begin{aligned} \mathrm{Fib}(f)^n &= D^{n-1}e^* \oplus C^n v_1^*, \\ \mathrm{d}^n &= \begin{pmatrix} -\mathrm{d}_D^{n-1} & f \\ & \mathrm{d}_C^n \end{pmatrix}, \end{aligned}$$

and the *strong homotopy cofiber* $\mathrm{Cofib}(f)$ is the complex

$$\begin{aligned} \mathrm{Cofib}(f)^n &= D^{n+1}e \oplus C^n v_1, \\ \mathrm{d}^n &= \begin{pmatrix} -\mathrm{d}_D^{n+1} & \\ f & \mathrm{d}_C^n \end{pmatrix}. \end{aligned}$$

Dually, let $f\colon C_\bullet \to D_\bullet$ be a chain morphism between complexes in $\mathcal{A}$. Then, the *strong homotopy fiber* $\operatorname{Fib}(f)$ is the complex

$$\begin{aligned} \operatorname{Fib}(f)_n &= D_{n+1}e^* \oplus C_n v_1^*, \\ \partial_n &= \begin{pmatrix} -\partial^D_{n+1} & f \\ & \partial^C_n \end{pmatrix}, \end{aligned}$$

and the *strong homotopy cofiber* $\operatorname{Cofib}(f)$ is the complex

$$\begin{aligned} \operatorname{Cofib}(f)_n &= D_{n-1}e \oplus C_n v_1, \\ \partial_n &= \begin{pmatrix} -\partial^D_{n-1} & \\ f & \partial^C_n \end{pmatrix}. \end{aligned}$$

4.8 Let $C^\bullet$ be a cochain complex of abelian groups. Consider the diagram $0 \to C^\bullet \leftarrow 0$. Note that this is a special case of 4.7. However, since the special properties of 0, one expects a more concentrated expression of the strong homotopy limit of this diagram. A homotopy cone of this diagram (a *homotopy loop*) consists of the following data

- a complex $X^\bullet$; and
- a homotopy $\Phi\colon 0 \Rightarrow 0\colon X^\bullet \to C^\bullet$.

The above data can be organized into the following commutative diagram of complexes

$$\begin{tikzcd} & C^\bullet \\ X^\bullet \arrow[ur, "0"] \arrow[r, dashed, "\Phi"] \arrow[dr, "0"'] & \langle I, C\rangle^\bullet \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & C^\bullet \end{tikzcd}$$

hence is equivalent to a cochain map

$$x\colon X^\bullet \longrightarrow \Omega C^\bullet$$

where $\Omega C^\bullet$ is the limit of the diagram

$$\begin{tikzcd} \langle I, C\rangle^\bullet \arrow[r, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_1"'] & C^\bullet \\ C^\bullet & 0 \arrow[l] \arrow[u] \end{tikzcd}$$

More elementarily, $\Omega C^\bullet$ is the complex

$$\Omega C^n = C^{n-1}e^*, \qquad \mathrm{d}^n = -\mathrm{d}_C^{n-1}.$$

In other words, $\Omega C^\bullet = C[-1]^\bullet$. Under this description, the cochain map x is precisely the cochain homotopy presenting Φ. Then the complex of homotopy loops with vertex X is the complex

$$[X, \Omega C]^\bullet.$$

Then it is easy to show that

$$[X, \Omega C]^\bullet \cong \Omega[X, C]^\bullet.$$

Now, let C be an object in a dg-category $\mathcal{C}$. Then a **loop space object**, or **looping**, of C is an object ΩC of $\mathcal{C}$ together with a natural quasi-isomorphism

$$\mathcal{H}om_{\mathcal{C}}(-, \Omega C) \xrightarrow{qis} \Omega\, \mathcal{H}om_{\mathcal{C}}(-, C).$$

By Yoneda lemma (Theorem 3.17), this is equivalent to a 0-cocycle π of the complex $\Omega\, \mathcal{H}om_{\mathcal{C}}(-, C)$ such that π induces above natural quasi-isomorphism. If the above natural quasi-isomorphism is further an natural isomorphism, then we say ΩC is a **strong loop space object**.

Dually, a **suspension** of C is an object ΣC of $\mathcal{C}$ together with a natural quasi-isomorphism

$$\mathcal{H}om_{\mathcal{C}}(\Sigma C, -) \xrightarrow{qis} \Omega\, \mathcal{H}om_{\mathcal{C}}(C, -).$$

By Yoneda lemma (Theorem 3.17), this is equivalent to a 0-cocycle π of the complex $\Omega\, \mathcal{H}om_{\mathcal{C}}(C, -)$ such that π induces above natural quasi-isomorphism. If the above natural quasi-isomorphism is further an natural isomorphism, then we say ΣC is a **strong suspension**.

Let $C^\bullet$ be a cochain complex in arbitrary abelian category $\mathcal{A}$. Then, the *strong loop space object* ΩC is the complex

$$\Omega C^n = C^{n-1} e^*, \qquad \mathrm{d}^n = -\mathrm{d}_C^{n-1},$$

i.e. $\Omega C^\bullet = C[-1]^\bullet$. The *strong suspension* ΣC is the complex

$$\Sigma C^n = C^{n+1} e, \qquad \mathrm{d}^n = -\mathrm{d}_C^{n+1},$$

i.e. $\Sigma C^\bullet = C[1]^\bullet$.

Dually, let $C_\bullet$ be a chain complex in $\mathcal{A}$. Then, the *strong loop space object* ΩC is the complex

$$\Omega C_n = C_{n+1} e^*, \qquad \partial_n = -\partial_{n+1}^C,$$

i.e. $\Omega C_\bullet = C[1]_\bullet$. The *strong suspension* ΣC is the complex

$$\Sigma C_n = C_{n-1} e, \qquad \partial_n = -\partial_{n-1}^C,$$

i.e. $\Sigma C_\bullet = C[-1]_\bullet$.

§ 5 Fiber sequences and exact sequences

References

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Abandoned drafts

.1 Proposition *Let $f\colon C \to D$ be a 1-morphism in a dg-category. Let f_* be the induced dg-transformation*

$$f_*\colon \mathcal{H}om_{\mathcal{C}}(-,C) \longrightarrow \mathcal{H}om_{\mathcal{C}}(-,D).$$

Then the followings are equivalent.

(i) f_* is a natural equivalence.
(ii) f_* is a natural quasi-isomorphism.
(iii) $H^0(f_*)$ is a natural isomorphism.
(iv) f is a homotopy equivalence.

The similar statement also holds for f^*.

Proof: Let's first prove (iii) implies (iv). Indeed, if $H^0(f_*)$ is a natural isomorphism, then in particular we have isomorphisms

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(D,C) \xrightarrow{f_*} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(D,D),$$
$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C,C) \xrightarrow{f_*} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C,D).$$

The first isomorphism gives a 1-morphism $g\colon D \to C$ such that $f \circ g \simeq \mathrm{id}_D$. The second isomorphism deduces $g \circ f \simeq \mathrm{id}_C$ from $f \circ g \circ f \simeq f$.

It remains to prove (iv) implies (i). So, let $f\colon C \to D$ be a homotopy equivalence with weak inverse $g\colon D \to C$ and a pair of homotopies $\phi\colon \mathrm{id} \Rightarrow g \circ f$ and $\psi\colon f \circ g \Rightarrow \mathrm{id}$. Then let's prove that ϕ_* and ψ_* are cochain homotopies.

Indeed, for any object X and any general morphism $h\colon X \to C$ of degree n, we have (note that the composition rule in $\mathcal{C}$ is a cochain map and that ϕ is of degree -1)

$$\begin{aligned}
&(\mathrm{d}^{n-1}_{\mathcal{H}om_{\mathcal{C}}(X,D)} \circ \phi^n_* + \phi^{n+1}_* \circ \mathrm{d}^n_{\mathcal{H}om_{\mathcal{C}}(X,C)})(h) \\
&= \mathrm{d}^{n-1}_{\mathcal{H}om_{\mathcal{C}}(X,D)}(\phi \circ h) + \phi \circ \mathrm{d}^n_{\mathcal{H}om_{\mathcal{C}}(X,C)}(h) \\
&= \mathrm{d}^{-1}_{\mathcal{H}om_{\mathcal{C}}(C,D)}(\phi) \circ h - \phi \circ \mathrm{d}^n_{\mathcal{H}om_{\mathcal{C}}(X,C)}(h) + \phi \circ \mathrm{d}^n_{\mathcal{H}om_{\mathcal{C}}(X,C)}(h) \\
&= \mathrm{d}^{-1}_{\mathcal{H}om_{\mathcal{C}}(C,D)}(\phi) \circ h \\
&= (g \circ f - \mathrm{id}) \circ h \\
&= (g_* \circ f_* - \mathrm{id}_{\mathcal{H}om_{\mathcal{C}}(X,X)})(h).
\end{aligned}$$

Hence ϕ_* is a cochain homotopy from id to $g_* \circ f_*$. The proof for ψ_* being cochain homotopy is similar. $\square$