Let \mathcal{C} be a category and \mathcal{I} be a small category. A functor from \mathcal{I} to \mathcal{C} is called a **diagram** in \mathcal{C} of shape \mathcal{I} .

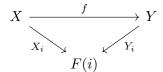
Let F be a diagram in \mathcal{C} of shape \mathcal{I} . The category over F, denoted by $\mathcal{C}_{/F}$, consists of the following data:

• a object in $\mathcal{C}_{/F}$ is a object (the *vertex*) X in \mathcal{C} together with a family of morphisms $x_i \colon X \to F(i)$ for each object $i \in \mathcal{I}$ satisfying the commutative diagram

$$F(i) \xrightarrow{X_i} F(\phi) \xrightarrow{F(\phi)} F(j)$$

for each morphism ϕ in \mathfrak{I} ;

• a morphism in $\mathcal{C}_{/F}$ is a morphism $f\colon X\to Y$ in \mathcal{C} satisfying the commutative diagram



for each object $i \in \mathcal{I}$.

The terminal object in $\mathcal{C}_{/F}$ is called the **limit** of the diagram F, denoted by $\lim F$.

Dually, the category under F, denoted by $\mathfrak{C}_{F/}$, consists of the following data:

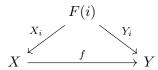
• a object in $\mathcal{C}_{F/}$ is a object (the *vertex*) X in \mathcal{C} together with a family of morphisms $x_i \colon F(i) \to X$ for each object $i \in \mathcal{I}$ satisfying the commutative diagram

$$F(i) \xrightarrow{F(\phi)} F(j)$$

$$X_i \xrightarrow{X_j} X$$

for each morphism ϕ in \mathfrak{I} ;

• a morphism in $\mathcal{C}_{F/}$ is a morphism $f\colon X\to Y$ in \mathcal{C} satisfying the commutative diagram



for each object $i \in \mathcal{I}$.

The initial object in $\mathcal{C}_{F/}$ is called the **colimit** of the diagram F, denoted by colim F.

- **1.1** Show that colimit of $F: \mathcal{I} \to \mathcal{C}$ is the same as limit of $F^{\text{op}}: \mathcal{I}^{\text{op}} \to \mathcal{C}^{\text{op}}$.
- **1.2** For any object X in \mathcal{C} , the composition of $F: \mathcal{I} \to \mathcal{C}$ with $\mathrm{Hom}_{\mathcal{C}}(X,-)$ defines a diagram in **Set**. Let $\mathrm{Hom}_{\mathcal{C}}(X,F)$ denote this diagram. Show that there is a natural isomorphism:

$$\operatorname{Hom}_{\mathfrak{C}}(-, \lim F) \cong \lim \operatorname{Hom}_{\mathfrak{C}}(-, F).$$

Dually, the composition of $F^{\text{op}}: \mathcal{I}^{\text{op}} \to \mathcal{C}^{\text{op}}$ with $\text{Hom}_{\mathcal{C}}(-, X)$ defines another diagram in **Set**. Let $\text{Hom}_{\mathcal{C}}(F, X)$ denote this diagram. Show that there is a natural isomorphism:

$$\operatorname{Hom}_{\mathfrak{C}}(\operatorname{colim} F, -) \cong \lim \operatorname{Hom}_{\mathfrak{C}}(F, -).$$

Now, let \mathcal{C} be a **dg-category**.

Let $f: X \to Y$ be a morphism in \mathcal{C} . A **homotopy triangle above** f is a object (the *vertex*) T in \mathcal{C} together with a triangle



where the bold arrow denoted a homotopy. If $S \to T$ is a morphism in \mathcal{C} , then by composing it with a homotopy triangle above f with vertex T, we obtain a homotopy triangle above f with vertex S. A morphism between homotopy triangles is such a morphism in \mathcal{C} .

Dually, a **homotopy triangle below** f is a object (the *vertex*) T in $\mathcal C$ together with a triangle

$$X \xrightarrow{f} Y$$

$$\downarrow$$

$$T$$

where the bold arrow denoted a homotopy.

- **2.1** Find the terminal object in the category of homotopy triangles above f.
- **2.2** Find the initial object in the category of homotopy triangles below f.

Let $f: X \to Y$ and $g: Z \to Y$ be two morphisms in \mathcal{C} . A **homotopy** square over f, g is a object (the *vertex*) T in \mathcal{C} together with a square

$$T \xrightarrow{T} Z$$

$$\downarrow \qquad \downarrow \qquad \downarrow g$$

$$X \xrightarrow{f} Y$$

where the dashed arrow denoted a homotopy. If $S \to T$ is a morphism in \mathbb{C} , then by composing it with a homotopy square over f, g with vertex T, we obtain a homotopy square over f, g with vertex S. A morphism between homotopy triangles is such a morphism in \mathbb{C} .

- **2.3** Find the terminal object in the category of homotopy square over f, g with Z the zero object.
- **2.4** Find the terminal object in the category of homotopy square over f, g.

Let $f: X \to Y$ and $g: X \to Z$ be two morphisms in \mathcal{C} . A **homotopy** square under f, g is a object (the *vertex*) T in \mathcal{C} together with a square

$$X \xrightarrow{f} Y$$

$$\downarrow \qquad \qquad \downarrow$$

$$Z \xrightarrow{f \mid f \mid f \mid f} T$$

where the dashed arrow denoted a homotopy. If $T \to S$ is a morphism in \mathbb{C} , then by composing it with a homotopy square under f, g with vertex T, we obtain a homotopy square under f, g with vertex S. A morphism between homotopy triangles is such a morphism in \mathbb{C} .

- **2.5** Find the initial object in the category of homotopy square under f, g with Z the zero object.
- **2.6** Find the initial object in the category of homotopy square under f, g.

The aboves are examples of **homotopy limits/colimits**. Let's ignore the general definition. In the following problems, only consider the above special types of homotopy limits/colimits.

- **2.7** Show that homotopy colimit of $F: \mathcal{I} \to \mathcal{C}$ is equivalent to homotopy limit of $F^{\text{op}}: \mathcal{I}^{\text{op}} \to \mathcal{C}^{\text{op}}$.
- **2.8** For any object T in \mathcal{C} , the composition of $F: \mathcal{I} \to \mathcal{C}$ with $\mathcal{H}om_{\mathcal{C}}(T,-)$ defines a diagram in \mathbf{Ch} . Let $\mathcal{H}om_{\mathcal{C}}(T,F)$ denote this diagram. Show that there is a natural isomorphism:

$$\mathcal{H}om_{\mathbb{C}}(-,\operatorname{HoLim} F)\cong\operatorname{HoLim}\mathcal{H}om_{\mathbb{C}}(-,F).$$

Dually, the composition of $F^{\text{op}}: \mathcal{I}^{\text{op}} \to \mathcal{C}^{\text{op}}$ with $\mathcal{H}om_{\mathcal{C}}(-,T)$ defines another diagram in **Ch**. Let $\mathcal{H}om_{\mathcal{C}}(F,T)$ denote this diagram. Show that there is a natural isomorphism:

$$\mathcal{H}om_{\mathcal{C}}(\operatorname{HoColim} F, -) \cong \operatorname{HoLim} \mathcal{H}om_{\mathcal{C}}(F, -).$$