Note on

Cohomology

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Notations

Capital letters like F, G, H and small letters like u, v, w will denote functors, the later ones often used for functors between sites. Small letters like f, g, h and ϕ, ψ will denote morphisms in suitable context, the later ones often used for morphisms serving as a part of some structure. The script letters like $\mathscr{F}, \mathscr{G}, \mathscr{H}$ will denote presheaves and sheaves. The fraktur letters like $\mathfrak{U}, \mathfrak{V}, \mathfrak{W}$ will denote coverings.

$*$ denotes the terminal object in a category, usually the singleton.

§ 1 Review on cohomology of topological spaces

We start from the following famous observation:

> For X a nice (for instance, homotopically equivalent to a CW complex) topological space, A an abelian group and n a natural number, we have
>
> $$H^n(X, A) \cong [X, K(A, n)],$$
>
> where the right is the set of homotopy classes from X to the ***Eilenberg-Mac Lane space*** $K(A, n)$, which is a nice space (for instance a CW complex) determined up to homotopy by the conditions
>
> $$\pi_k K(A, n) = \begin{cases} A & \text{if } k = n, \\ 0 & \text{if not.} \end{cases}$$
>
> Furthermore, we have
>
> $$H^{n-k}(X, A) \cong \pi_k \operatorname{Hom}(X, K(A, n)).$$

This story is not a coincidence. If one knows the notion of ***loop spaces***, one can see that

$$\Omega K(A, n) = K(A, n - 1),$$

where ΩX denotes the loop space on some point of X. If one further notice that loop spaces are given as kind of limits in the nice category of topology spaces, then it is obvious that

$$\Omega[X, K(A, n)] = [X, \Omega K(A, n)] = [X, K(A, n - 1)].$$

Recall that loop space construction shifts the degree of homotopy of a space, thus

$$\pi_0 \Omega^k [X, K(A, n)] = \pi_k [X, K(A, n)].$$

By Eilenberg-Mac Lane axioms, to show

$$H^n(X, A) \cong [X, K(A, n)],$$

it suffices to show $h^n(X) := \langle X, K(A, n) \rangle$ (here $\langle X, K(A, n) \rangle$ denotes the *reduced* 0*-th homotopy set*) defines a reduced cohomology theory such that $h^n(S^0) = 0$ for $n \neq 0$ and $h^0(S^0) = A$.

This can be done as one notice that there exists an adjoint on the nice category of pointed topological spaces

$$[\Sigma X, Y] \cong [X, \Omega Y],$$

and that Σ gives rise to a sequence for each CW pair (X, U):

$$U \to X \to X/U \to \Sigma U \to \Sigma X \to \Sigma X/U \to \cdots$$

which induces a sequence of groups

$$\langle U, K(A,n)\rangle \leftarrow \langle X, K(A,n)\rangle \leftarrow \langle X/U, K(A,n)\rangle \leftarrow \langle \Sigma U, K(A,n)\rangle \leftarrow \cdots$$

and it is not difficult to show this sequence is exact.

Details or proofs can be found in textbook on algebraic topology like

- Hatcher, Allen, *Algebraic topology.* Cambridge University Press, 2002.

§ 2 Philosophy of higher category theory

The notion of category can be extended to involve higher morphisms, such as 2-*morphisms* between the original 1-morphisms and 3-*morphisms* between 2-morphisms. Continuing this process, the result notion should be called ∞***-category***. There should also be notions of various ***compositions*** between those higher morphisms and all those data should satisfy suitable ***coherence laws*** such as associative and unitary laws.

To give an intuition, here is the definition of *strict* 2-*categories*.

2.1 Example (Strict 2-categories) A ***(strict)*** 2-***category*** $\mathcal{C}$ consists of

- 0-morphisms, i.e. objects,
- 1-morphisms between objects, and their compositions

$$\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \qquad \rightsquigarrow \qquad \cdot \xrightarrow{g \circ f} \cdot$$

- 2-morphisms between 1-morphisms, and the ***vertical*** and ***horizontal compositions***

$$\begin{tikzcd} \cdot \arrow[r, bend left=70] \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta"] & \cdot \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & \cdot \end{tikzcd}$$

$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\beta"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha"] & \cdot \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\beta * \alpha"] & \cdot \end{tikzcd}$$

and satisfies the following *coherence laws*:

1. the compositions satisfy the associative law,
2. every object X admits an 1-morphism id_X called the identity of X, which is the identity under the composition operation of 1-morphisms,
3. every 1-morphism f admits a 2-morphism id_f called the identity of f, which is the identity under the vertical composition operation of 2-morphisms,
4. for every object X, the 2-morphism $\mathrm{id}_{\mathrm{id}_X}$ is the identity under the horizontal composition operation of 2-morphisms,

5. the vertical and horizontal compositions satisfy the ***interchange law***: for all quadruples $(\alpha, \alpha', \beta, \beta')$ of 2-morphisms of the form

$$\begin{tikzcd}
\cdot \arrow[r, bend left=80] \arrow[r] \arrow[r, bend right=80] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \cdot \arrow[r, bend left=80] \arrow[r] \arrow[r, bend right=80] \arrow[r, phantom, shift left=3, "\Downarrow\alpha'"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & \cdot
\end{tikzcd}$$

   the following equality holds.

$$(\beta \circ \alpha) * (\beta' \circ \alpha') = (\beta * \beta') \circ (\alpha * \alpha').$$

Now, we can similarly define *strict n-categories* for larger n. However, the strict version of higher categories is too special to encode interesting mathematical phenomenons such as homotopy theory of topological spaces.

2.2 Example (∞-category of topological spaces) Consider the category of topological spaces. We have continuous maps between them as morphisms. But their are hided higher morphisms: the homotopies between continuous maps are 2-morphisms, furthermore, since every homotopy can be represented by a continuous map, there are also higher morphisms. However, the compositions of those higher morphisms are just up to homotopy *a fortiori* the coherence laws.

So the general notion should be much weaker. Indeed, we only require the coherence laws hold as *weak equivalences* instead of equalities.

The typical examples of weak equivalences are the isomorphisms. Recall that two objects are said to be isomorphic when there exists an isomorphism between them and a morphism is called an isomorphism if it admits an inverse. The general notion of weak equivalences can be viewed as a weak version of isomorphisms in the sense the equality can be replaced by a weak equivalence.

The following is a sketch of what an ∞-category should be and one can get some idea from it. However, this is *NOT* a serious definition.

2.3 (∞-categories) An ***∞-category*** should consists of the following data:

- 0-*morphisms* i.e. *objects*,
- 1-*morphisms* between objects,
- 2-*morphisms* between 1-morphisms,
- et cetera;
- for each k-morphism x, there exists a $k+1$-morphism id_x, called the ***identity*** on x;

- for any diagram which can be composed (for instance, a path), there are ***compositions***;

  Remark For $f\colon x \to y$ and $g\colon y \to z$ two k-morphisms, one may agree there should be a composition $g \circ f\colon x \to z$. However, there are other kinds of diagrams can be composed, for instance, the *horizontal compositions* and the mixed composition mentioned in the *interchange law*.

- a special class of morphisms called ***weak equivalences***;
- two k-morphisms x and y are said to be ***weakly equivalent*** when there exists a weak equivalence $f\colon x \to y$.

The above data should satisfy the following ***coherence laws***:

1. The compositions of the same diagram should be weakly equivalent to each other. In this sense, we can still use the notation $f \circ g$ while it refers to a weak equivalent class of higher morphisms instead of a specific one.
2. The identities serve as ***weak identities*** under compositions in the sense that the composition of a morphism with a identity is weak equivalent to the morphism itself.
3. A k-morphism $f\colon x \to y$ is a weak equivalence when it admits a ***weak inverse***, which is a k-morphism $g\colon y \to x$ such that $g \circ f$ is weakly equivalent to id_x and $f \circ g$ is weakly equivalent to id_y.

To who may have been overwhelmed by those staff, here is a slogan:

> *Whenever you think about something in an ∞-category, think it up to weak equivalence.*

Remark However, the strict notions may still make sense. It may be the case that there is only one possible way to composite two k-morphisms. In this case, their composition is a k-morphism rather than a weak equivalent class. For instance, if all k-morphisms with $k > n$ are identities, then the compositions of n-morphisms are unique.

In the case the compositions $f \circ g, g \circ f$ of $f\colon x \to y$ and $g\colon y \to x$ are unique, it makes sense to ask if $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$. If this is the case, we say f and g are ***isomorphisms*** which are ***inverse*** to each other and that x and y are ***isomorphic***.

To distinguish, we use the symbol $\simeq$ for weak equivalences and $\cong$ for isomorphisms.

2.4 Example ((n, r)-categories) An (∞, r)***-category*** is an ∞-category in which all k-morphisms with $k > r$ are weak equivalences. A (n, r)***-category*** is an (∞, r)-category in which all k-morphisms with $k > n$ are identities.

2.5 Example (Hom spaces) In an ∞-category, all morphisms from an object X to another Y together with higher morphisms between them form an ∞-category, in which

- objects are morphisms from X to Y;
- 1-morphisms are 2-morphisms between the above morphisms;
- et cetera.

This ∞-category $\mathcal{H}\mathrm{om}(X, Y)$ is called the ***Hom space***.

Note that a Hom space in an (∞, r)-category is an $(\infty, r-1)$-category. In this sense, study $(\infty, 1)$-categories and $(\infty, 0)$-categories are enough for understanding (∞, r)-categories.

Remark Note that an (∞, r)-category can NOT be viewed as a category enriched in the category of $(\infty, r-1)$-categories. Instead, it should be viewed as enriched up to weak equivalences.

2.6 Example (Higher groupoids) The $(\infty, 0)$-categories are usually called ∞***-groupoids***. In particular, $(n, 0)$-categories are called n***-groupoids***. One can see they are the generalizations of sets and groupoids.

Each topological space admits an ∞-groupoid (the ***fundamental*** ∞***-groupoid***) which records its homotpy-theoretic information.

Homotopies

From now on, we will force on $(\infty, 1)$-categories. Without specification, an ∞-category is always an $(\infty, 1)$-category. Under this setting, we will also use some topology-style terminology:

- A ***homotopy*** is a k-morphism with $k > 1$. Note that in our case, they are all weak equivalences. Two k-morphisms x, y with $k > 0$ are said to be ***homotopic*** if there exists a $k+1$-morphism $x \to y$.
- Two k-morphisms are said to be ***connected*** if there exists a *zigzag* of $k+1$-morphisms connecting them.
- For $\mathcal{G}$ an ∞-groupoid, its ***set of components*** $\pi_0\mathcal{G}$ is obtained by identifying all connected objects in $\mathcal{G}$. If $\mathcal{G}$ serves as a Hom space in some ∞-category, we also call $\pi_0\mathcal{G}$ as the ***set of homotopy classes***.
- An ∞-groupoid $\mathcal{G}$ is said to be ***weakly contractible*** if $\pi_0\mathcal{G} = *$.
- For $\mathcal{C}$ an ∞-category, its ***homotopy category*** $\mathrm{h}\mathcal{C}$ is obtained by identifying homotopic morphisms. Therefore we also denote the set of homotopy classes $\pi_0 \,\mathcal{H}\mathrm{om}_{\mathcal{C}}(X, Y)$ as $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$, or even more simply $H(X, Y)$.

2.7 (∞-functors) There should be a notion of ***∞-functors*** and furthermore higher morphisms between them such that for each pair of ∞-categories $\mathcal{C}, \mathcal{D}$, there is an ∞-category $\mathrm{Fun}[\mathcal{C}, \mathcal{D}]$ of ∞-functors between them. In this way, there should be an $(\infty, 2)$-*category* $\infty\mathcal{C}\mathrm{at}$ consisting of

- ∞-categories as objects;
- the ∞-category $\mathrm{Fun}[\mathcal{C}, \mathcal{D}]$ as the Hom space $\mathcal{H}\mathrm{om}(\mathcal{C}, \mathcal{D})$.

An ∞-functor $F\colon \mathcal{C} \to \mathcal{D}$ should be something like this: it endows each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$, each pair of objects X, Y in $\mathcal{C}$ a morphism of ∞-groupoids:

$$\mathcal{H}\mathrm{om}_{\mathcal{C}}(X, Y) \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{D}}(F(X), F(Y)),$$

each object X in $\mathcal{C}$ a homotopy between $F(\mathrm{id}_X)$ to $\mathrm{id}_{F(X)}$ and so forth. One can see this notion is too complicate to be defined in this naive way. Anyhow, we obtain a *strict* 2-*category* $\mathrm{h}\infty\mathcal{C}\mathrm{at}$ consisting of

- ∞-categories as objects;
- the homotopy category of $\mathrm{Fun}[\mathcal{C}, \mathcal{D}]$ as the Hom space $\mathcal{H}\mathrm{om}(\mathcal{C}, \mathcal{D})$.

We say an ∞-functor is an ***equivalence*** if it is an weak equivalence in the $(\infty, 2)$-category $\infty\mathcal{C}\mathrm{at}$, equivalently, an equivalence in the 2-category $\mathrm{h}\infty\mathcal{C}\mathrm{at}$.

2.8 ($\infty\mathcal{G}\mathrm{pd}$) The ∞-category $\infty\mathcal{G}\mathrm{pd}$ of ∞-groupoids plays an important role in ∞-category theory. Indeed, one can see its role is just like $\mathcal{S}\mathrm{et}$ in category theory. Therefore having an intuition on this ∞-category is necessary even for who do not care general ∞-category theory.

It is helpful to treat an ∞-groupoid $\mathcal{G}$ as a nice topological space, at least as its data of *homotopy type*: the objects in $\mathcal{G}$ are points, morphisms are paths and homotopies are homotopies. Indeed, one consistency condition for a good ∞-category theory is the *homotopy hypothesis*: the ∞-category $\infty\mathcal{G}\mathrm{pd}$ is equivalent to the ∞-category of nice topological spaces.

The homotopy category of this fundamental ∞-category is called the ***homotopy category of spaces*** and denoted by $\mathcal{H}$.

2.9 (Homotopy categories) Any ∞-functor induces a functor between the homotopy categories. Moreover, equivalent ∞-functors induce isomorphic functors. Thus there is always an essentially surjective functor

$$\mathrm{h}\,\mathrm{Fun}[\mathcal{C}, \mathcal{D}] \longrightarrow \mathrm{Fun}[\mathrm{h}\mathcal{C}, \mathrm{h}\mathcal{D}].$$

However, the naive definition of homotopy categories may not make the above functor an equivalence: although the homotopy categories remembers what are homotopic in the ∞-categories, they still forget how those things homotopic. To patch up this, we need to bring up some richer structure on the functor categories between homotopy categories.

One way to do this is to view the homotopy category $h\mathcal{C}$ as a $\mathcal{H}$-enriched category. In other words, $h\mathcal{C}$ is almost $\mathcal{C}$ expect the composition laws of morphisms are strict. Now the $\mathcal{H}$-enriched functor category $\mathrm{Fun}[h\mathcal{C}, h\mathcal{D}]$ should remember how equivalent things are equivalent.

But as I said, the original category $h\mathcal{C}$ is enough to understand what are homotopic in the ∞-categories $\mathcal{C}$. So we use the following mixed definition:

- the ***homotopy functor*** between two homotopy categories is the $\mathcal{H}$-enriched functor between them;

- while the homotopy categories are viewed as original categories.

Homotopy limits

We now generalize the notion of limits to ∞-categories.

First, we consider the terminal and initial objects.

2.10 (Homotopy terminal and initial objects) Recall that a *terminal object* in a category $\mathcal{C}$ is an object $*$ such that for any object X in $\mathcal{C}$, there exists a unique morphism $X \to *$. Note that the terminal object is unique up to isomorphisms.

Analogously, we define a ***homotopy terminal object*** in an ∞-category $\mathcal{C}$ is an object $*$ such that for any object X in $\mathcal{C}$, the Hom space $\mathcal{H}\mathrm{om}(X, *)$ is weakly contractible. Note that the homotopy terminal object is unique up to homotopies.

Note that from the above definition, we see that an object is a homotopy terminal object if and only if it is a terminal object in the homotopy category.

One can similarly define the notion of ***homotopy initial objects***.

Remark It is often the case that the ∞-category we are studying is obtained by enlarging a category, revealing its hiding ∞-natural. Such a construction may be called ***a category with hiding homotopies***.

If this is the case, then the notion of equalities make sense for 1-morphisms. Then we have two notions of terminal (resp. initial) objects, the *strict* one and the *homotopy* one. However, it is easy to see that the strict terminal (resp. initial) object is homotopic to the homotopy terminal (resp. initial) object. In this sense, we will simply say ***terminal objects*** and ***initial objects***.

2.11 (Pointed objects) Any category $\mathcal{C}$ admits a terminal object $*$ admits a ***category of pointed objects*** $\mathcal{C}^{*/}$ whose objects are objects X in $\mathcal{C}$ together with a morphism (the ***point***) from the terminal object to X and whose morphisms are morphisms $f\colon X \to Y$ in $\mathcal{C}$ preserving the points in the sense that the following diagram commutes.

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

The similar things also happen in the ∞-world except every notion should be weaken by its homotopy version. But what is the homotopy version of *commutative diagram*?

2.12 (Commutative diagram) Recall that a diagram consists of several paths, each of them is a composible diagram. A diagram is said to *commute* if all such paths with same start and end give rise to equal compositions.

In an ∞-category, the equality of compositions rarely make sense. So to weak this notion, the requirement "equal" should be replaced by "weakly equivalent". To avoid ambiguities, we also say the diagram ***commute up to homotopies*** if needed.

Recall that a *cone* over a diagram D is merely a family of morphisms $\{f_i\}$, each from a fixed object X to a vertex i in D, such that for any arrow $e\colon i \to j$ in D, the following diagram commutes.

$$\begin{tikzcd} & X \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\ D_i \arrow[rr, "D_e"] & & D_j \end{tikzcd}$$

A morphism between two cones $\{f_i\colon X \to D_i\}$ and $\{g_i\colon Y \to D_i\}$ is a morphism $\phi\colon X \to Y$ such that the following diagrams commute.

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "f_i"'] & & Y \arrow[dl, "g_i"] \\ & D_i & \end{tikzcd}$$

In this way, cones over a diagram form a category and the limit of the diagram is the terminal object in this category.

Now, we can define the homotopy version of limits.

2.13 (Homotopy limits) A ***homotopy cone*** over a diagram D now can be defined as a family of morphisms $\{f_i\}$, each from a fixed object X to a vertex D_i in D, such that for any arrow $e\colon i \to j$ in D, the following diagram commutes up to homotopy.

$$\begin{tikzcd} & X \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\ D_i \arrow[rr, "D_e"] & & D_j \end{tikzcd}$$

The morphisms between homotopy cones is defined similar to the morphisms between cones except the requirements of commutativity should be weaken to up to homotopy.

In this way, the homotopy cones over a diagram form an ∞-category. The ***homotopy limit*** $\underleftarrow{\mathrm{hlim}}\, D$ of the diagram D is merely the homotopy terminal object in this ∞-category.

The notions of ***homotopy cocones*** and ***homotopy colimits*** are defined similarly.

Remark From the above definitions, one can see that a family of morphisms $\{f_i\colon X \to D_i\}$ is a homotopy cone if and only if it is a cone in the homotopy category, that morphism between apexes of two homotopy cones is a morphism between homotopy cones if and only if it is a morphism between cones in the homotopy category and that a homotopy cone is a homotopy limit of a diagram if and only if it is a limit in the homotopy category.

Remark Note that, for any diagram D and any object t, we have

$$\mathcal{H}\mathrm{om}(t, \underleftarrow{\mathrm{hlim}}\, D) \simeq \underleftarrow{\mathrm{hlim}}\, \mathcal{H}\mathrm{om}(t, D),$$
$$\mathcal{H}\mathrm{om}(\underrightarrow{\mathrm{hlim}}\, D, t) \simeq \underleftarrow{\mathrm{hlim}}\, \mathcal{H}\mathrm{om}(D, t).$$

where the homotopy limits on the right are taking in the ∞-category $\infty\mathcal{G}\mathrm{pd}$.

2.14 (Homotopy fibred products) We now consider diagrams like this

$$A \xrightarrow{f} B \xleftarrow{g} C.$$

The homotopy limit of this diagram is called the ***homotopy fibred product*** of A and C over B, denoted by $A \times^{\mathrm{h}}_B C$, or the ***homotopy pullback*** of f along g, denoted by $g^{\mathrm{h}} f$, or the ***homotopy pullback*** of g along f, denoted by $f^{\mathrm{h}} g$.

Remark In a category with hiding homotopies, the strict fibred products are rarely homotopic to the homotopy fibred products. However, there is always a canonical morphism from the homotopy one to the strict one.

2.15 (Homotopy fibers) Let $f\colon X \to Y$ be a morphism in an ∞-category of pointed objects. Then the ***homotopy fiber*** is the homotopy pullback of the point $* \to Y$ along f.

A sequence of pointed objects

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called a ***fibration sequence*** if f is a homotopy fiber of g.

Remark The homotopy fibers can be viewed as the homotopy version of kernels. in this sense, fibration sequences can be viewed as the homotopy version of left exact sequences. However, unlike the strict case, the homotopy fiber of a homotopy fiber need not be trivial.

2.16 **(Looping and delooping)** Let X be a pointed object. Its ***looping*** ΩX is the homotopy fiber of the point $* \to X$. If Y is the looping of a pointed object X, then we say X is the ***delooping*** of Y denoted by $\mathbf{B}X$.