Let $\mathcal{C}$ be a category and $\mathcal{J}$ be a small category. A functor from $\mathcal{J}$ to $\mathcal{C}$ is called a **diagram** in $\mathcal{C}$ of shape $\mathcal{J}$.

Let F be a diagram in $\mathcal{C}$ of shape $\mathcal{J}$. The category over F, denoted by $\mathcal{C}_{/F}$, consists of the following data:

- a object in $\mathcal{C}_{/F}$ is a object (the *vertex*) X in $\mathcal{C}$ together with a family of morphisms $x_i \colon X \to F(i)$ for each object $i \in \mathcal{J}$ satisfying the commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "X_i"'] \arrow[dr, "X_j"] & \\ F(i) \arrow[rr, "F(\phi)"] & & F(j) \end{tikzcd}$$

  for each morphism ϕ in $\mathcal{J}$;

- a morphism in $\mathcal{C}_{/F}$ is a morphism $f \colon X \to Y$ in $\mathcal{C}$ satisfying the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "X_i"'] & & Y \arrow[dl, "Y_i"] \\ & F(i) & \end{tikzcd}$$

  for each object $i \in \mathcal{J}$.

The terminal object in $\mathcal{C}_{/F}$ is called the **limit** of the diagram F, denoted by $\lim F$.

Dually, the category under F, denoted by $\mathcal{C}_{F/}$, consists of the following data:

- a object in $\mathcal{C}_{F/}$ is a object (the *vertex*) X in $\mathcal{C}$ together with a family of morphisms $x_i \colon F(i) \to X$ for each object $i \in \mathcal{J}$ satisfying the commutative diagram

$$\begin{tikzcd} F(i) \arrow[rr, "F(\phi)"] \arrow[dr, "X_i"'] & & F(j) \arrow[dl, "X_j"] \\ & X & \end{tikzcd}$$

  for each morphism ϕ in $\mathcal{J}$;

- a morphism in $\mathcal{C}_{F/}$ is a morphism $f \colon X \to Y$ in $\mathcal{C}$ satisfying the commutative diagram

$$\begin{tikzcd} & F(i) \arrow[dl, "X_i"'] \arrow[dr, "Y_i"] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

  for each object $i \in \mathcal{J}$.

The initial object in $\mathcal{C}_{F/}$ is called the **colimit** of the diagram F, denoted by $\operatorname{colim} F$.

1.1 Show that colimit of $F\colon \mathcal{I} \to \mathcal{C}$ is the same as limit of $F^{\mathrm{op}}\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$.

1.2 For any object X in $\mathcal{C}$, the composition of $F\colon \mathcal{I} \to \mathcal{C}$ with $\operatorname{Hom}_{\mathcal{C}}(X, -)$ defines a diagram in **Set**. Let $\operatorname{Hom}_{\mathcal{C}}(X, F)$ denote this diagram. Show that there is a natural isomorphism:

$$\operatorname{Hom}_{\mathcal{C}}(-, \lim F) \cong \lim \operatorname{Hom}_{\mathcal{C}}(-, F).$$

Dually, the composition of $F^{\mathrm{op}}\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ with $\operatorname{Hom}_{\mathcal{C}}(-, X)$ defines another diagram in **Set**. Let $\operatorname{Hom}_{\mathcal{C}}(F, X)$ denote this diagram. Show that there is a natural isomorphism:

$$\operatorname{Hom}_{\mathcal{C}}(\operatorname{colim} F, -) \cong \lim \operatorname{Hom}_{\mathcal{C}}(F.-).$$

Now, let $\mathcal{C}$ be a **dg-category**.

Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. A **homotopy triangle above** f is a object (the *vertex*) T in $\mathcal{C}$ together with a triangle

$$\begin{tikzcd} T \arrow[d] \arrow[dr] \arrow[dr, phantom, "\Swarrow"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where the bold arrow denoted a homotopy. If $S \to T$ is a morphism in $\mathcal{C}$, then by composing it with a homotopy triangle above f with vertex T, we obtain a homotopy triangle above f with vertex S. A *morphism* between homotopy triangles is such a morphism in $\mathcal{C}$.

Dually, a **homotopy triangle below** f is a object (the *vertex*) T in $\mathcal{C}$ together with a triangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] \arrow[dr, phantom, "\Nearrow"] & Y \arrow[d] \\ & T \end{tikzcd}$$

where the bold arrow denoted a homotopy.

2.1 Find the terminal object in the category of homotopy triangles above f.

2.2 Find the initial object in the category of homotopy triangles below f.

Let $f\colon X \to Y$ and $g\colon Z \to Y$ be two morphisms in $\mathcal{C}$. A **homotopy square** over f, g is a object (the *vertex*) T in $\mathcal{C}$ together with a square

$$\begin{tikzcd} T \arrow[r] \arrow[d] & Z \arrow[d, "g"] \arrow[dl, dashed, Rightarrow] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where the dashed arrow denoted a homotopy. If $S \to T$ is a morphism in $\mathcal{C}$, then by composing it with a homotopy square over f, g with vertex T, we obtain a homotopy square over f, g with vertex S. A *morphism* between homotopy triangles is such a morphism in $\mathcal{C}$.

2.3 Find the terminal object in the category of homotopy square over f, g with Z the zero object.

2.4 Find the terminal object in the category of homotopy square over f, g.

Let $f\colon X \to Y$ and $g\colon X \to Z$ be two morphisms in $\mathcal{C}$. A **homotopy square** under f, g is a object (the *vertex*) T in $\mathcal{C}$ together with a square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d] \\ Z \arrow[r] \arrow[ur, Rightarrow, dashed] & T \end{tikzcd}$$

where the dashed arrow denoted a homotopy. If $T \to S$ is a morphism in $\mathcal{C}$, then by composing it with a homotopy square under f, g with vertex T, we obtain a homotopy square under f, g with vertex S. A *morphism* between homotopy triangles is such a morphism in $\mathcal{C}$.

2.5 Find the initial object in the category of homotopy square under f, g with Z the zero object.

2.6 Find the initial object in the category of homotopy square under f, g.

The aboves are examples of **homotopy limits/colimits**. Let's ignore the general definition. In the following problems, only consider the above special types of homotopy limits/colimits.

2.7 Show that homotopy colimit of $F\colon \mathcal{I} \to \mathcal{C}$ is equivalent to homotopy limit of $F^{\mathrm{op}}\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$.

2.8 For any object T in $\mathcal{C}$, the composition of $F\colon \mathcal{I} \to \mathcal{C}$ with $\mathcal{H}om_{\mathcal{C}}(T, -)$ defines a diagram in **Ch**. Let $\mathcal{H}om_{\mathcal{C}}(T, F)$ denote this diagram. Show that there is a natural isomorphism:

$$\mathcal{H}om_{\mathcal{C}}(-, \operatorname{HoLim} F) \cong \operatorname{HoLim} \mathcal{H}om_{\mathcal{C}}(-, F).$$

Dually, the composition of $F^{\mathrm{op}}\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ with $\mathcal{H}om_{\mathcal{C}}(-, T)$ defines another diagram in **Ch**. Let $\mathcal{H}om_{\mathcal{C}}(F, T)$ denote this diagram. Show that there is a natural isomorphism:

$$\mathcal{H}om_{\mathcal{C}}(\operatorname{HoColim} F, -) \cong \operatorname{HoLim} \mathcal{H}om_{\mathcal{C}}(F.-).$$