Note on

Cohomology

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Conventions

Capital letters like F, G, H and small letters like u, v, w will denote functors, the later ones often used for functors between sites. Small letters like f, g, h and ϕ, ψ will denote morphisms in suitable context, the later ones often used for morphisms serving as a part of some structure. The script letters like $\mathscr{F}, \mathscr{G}, \mathscr{H}$ will denote presheaves and sheaves. The fraktur letters like $\mathfrak{U}, \mathfrak{V}, \mathfrak{W}$ will denote coverings.

$*$ denotes the terminal object in a category, usually the singleton.

The coproducts are also called *direct sums* and denoted by the direct sum symbol $\oplus$.

I

Philosophy of higher category theory

§ 1 Philosophy of higher category theory

1 What should be an ∞-category?

The notion of category can be extended to involve higher morphisms, such as 2-*morphisms* between the original 1-morphisms and 3-*morphisms* between 2-morphisms. Continuing this process, the result notion should be called ***∞-category***. There should also be notions of various ***compositions*** between those higher morphisms and all those data should satisfy suitable ***coherence laws*** such as associative and unitary laws.

To give an intuition, here is the definition of *strict* 2-*categories*.

1.1 Example (Strict 2-categories) A ***(strict)* 2-*category*** $\mathcal{C}$ consists of

- 0-morphisms, i.e. objects,
- 1-morphisms between objects, and their compositions

$$\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \qquad \rightsquigarrow \qquad \cdot \xrightarrow{g\circ f} \cdot$$

- 2-morphisms between 1-morphisms, and the ***vertical*** and ***horizontal compositions***

$$\begin{tikzcd}
\cdot \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M, above}, ""{name=M2, below}] \arrow[r, bend right=60, ""{name=D, above}] & \cdot
\arrow[Rightarrow, from=U, to=M, "\alpha"] \arrow[Rightarrow, from=M2, to=D, "\beta"]
\end{tikzcd}
\qquad \rightsquigarrow \qquad
\begin{tikzcd}
\cdot \arrow[r, bend left=50, ""{name=U, below}] \arrow[r, bend right=50, ""{name=D, above}] & \cdot
\arrow[Rightarrow, from=U, to=D, "\beta\circ\alpha"]
\end{tikzcd}$$

$$\begin{tikzcd}
\cdot \arrow[r, bend left=50, ""{name=U1, below}] \arrow[r, bend right=50, ""{name=D1, above}] & \cdot \arrow[r, bend left=50, ""{name=U2, below}] \arrow[r, bend right=50, ""{name=D2, above}] & \cdot
\arrow[Rightarrow, from=U1, to=D1, "\beta"] \arrow[Rightarrow, from=U2, to=D2, "\alpha"]
\end{tikzcd}
\qquad \rightsquigarrow \qquad
\begin{tikzcd}
\cdot \arrow[r, bend left=50, ""{name=U, below}] \arrow[r, bend right=50, ""{name=D, above}] & \cdot
\arrow[Rightarrow, from=U, to=D, "\beta * \alpha"]
\end{tikzcd}$$

and satisfies the following *coherence laws*:

1. the compositions satisfy the associative law,
2. every object X admits an 1-morphism id_X called the identity of X, which is the identity under the composition operation of 1-morphisms,
3. every 1-morphism f admits a 2-morphism id_f called the identity of f, which is the identity under the vertical composition operation of 2-morphisms,
4. for every object X, the 2-morphism $\mathrm{id}_{\mathrm{id}_X}$ is the identity under the horizontal composition operation of 2-morphisms,

5. the vertical and horizontal compositions satisfy the ***interchange law***: for all quadruples $(\alpha, \alpha', \beta, \beta')$ of 2-morphisms of the form

$$\begin{tikzcd} \cdot \arrow[r, bend left=80] \arrow[r] \arrow[r, bend right=80] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta"] & \cdot \arrow[r, bend left=80] \arrow[r] \arrow[r, bend right=80] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha'"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta'"] & \cdot \end{tikzcd}$$

the following equality holds.

$$(\beta \circ \alpha) * (\beta' \circ \alpha') = (\beta * \beta') \circ (\alpha * \alpha').$$

Now, we can similarly define *strict n-categories* for larger n. However, the strict version of higher categories is too special to encode interesting mathematical phenomenons such as homotopy theory of topological spaces.

1.2 Example (∞-category of topological spaces) Consider the category of topological spaces. We have continuous maps between them as morphisms. But their are hided higher morphisms: the homotopies between continuous maps are 2-morphisms, furthermore, since every homotopy can be represented by a continuous map, there are also higher morphisms. However, the compositions of those higher morphisms are just up to homotopy *a fortiori* the coherence laws.

So the general notion should be much weaker. Indeed, we only require the coherence laws hold as *weak equivalences* instead of equalities.

The typical examples of weak equivalences are the isomorphisms. Recall that two objects are said to be isomorphic when there exists an isomorphism between them and a morphism is called an isomorphism if it admits an inverse. The general notion of weak equivalences can be viewed as a weak version of isomorphisms in the sense the equality can be replaced by a weak equivalence.

The following is a sketch of what an ∞-category should be and one can get some idea from it. However, this is *NOT* a serious definition.

2 (∞-categories) An ***∞-category*** should consists of the following data:

- 0-*morphisms* i.e. *objects*,
- 1-*morphisms* between objects,
- 2-*morphisms* between 1-morphisms,
- et cetera;
- for each k-morphism x, there exists a $k+1$-morphism id_x, called the ***identity*** on x;

- for any diagram which can be composed (for instance, a path), there are ***compositions***;

  Remark For $f\colon x \to y$ and $g\colon y \to z$ two k-morphisms, one may agree there should be a composition $g \circ f\colon x \to z$. However, there are other kinds of diagrams can be composed, for instance, the *horizontal compositions* and the mixed composition mentioned in the *interchange law.*

- a special class of morphisms called ***weak equivalences***;

- two k-morphisms x and y are said to be ***weakly equivalent*** when there exists a weak equivalence $f\colon x \to y$.

The above data should satisfy the following ***coherence laws***:

1. The compositions of the same diagram should be weakly equivalent to each other. In this sense, we can still use the notation $f \circ g$ while it refers to a weak equivalent class of higher morphisms instead of a specific one.

2. The identities serve as ***weak identities*** under compositions in the sense that the composition of a morphism with a identity is weak equivalent to the morphism itself.

3. A k-morphism $f\colon x \to y$ is a weak equivalence when it admits a ***weak inverse***, which is a k-morphism $g\colon y \to x$ such that $g \circ f$ is weakly equivalent to id_x and $f \circ g$ is weakly equivalent to id_y.

To who may have been overwhelmed by those staff, here is a slogan:

> *Whenever you think about something in an ∞-category, think it up to weak equivalence.*

Remark However, the strict notions may still make sense. It may be the case that there is only one possible way to composite two k-morphisms. In this case, their composition is a k-morphism rather than a weak equivalent class. For instance, if all k-morphisms with $k > n$ are identities, then the compositions of n-morphisms are unique.

In the case the compositions $f \circ g, g \circ f$ of $f\colon x \to y$ and $g\colon y \to x$ are unique, it makes sense to ask if $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$. If this is the case, we say f and g are ***isomorphisms*** which are ***inverse*** to each other and that x and y are ***isomorphic***.

To distinguish, we use the symbol $\simeq$ for weak equivalences and $\cong$ for isomorphisms.

2.1 Example ((n, r)-categories) An (∞, r)***-category*** is an ∞-category in which all k-morphisms with $k > r$ are weak equivalences. A (n, r)***-category*** is an (∞, r)-category in which all k-morphisms with $k > n$ are identities.

2.2 Example (Hom spaces) In an ∞-category, all morphisms from an object X to another Y together with higher morphisms between them form an ∞-category, in which

- objects are morphisms from X to Y;
- 1-morphisms are 2-morphisms between the above morphisms;
- et cetera.

This ∞-category $\mathcal{H}\mathrm{om}(X, Y)$ is called the ***Hom space***.

Note that a Hom space in an (∞, r)-category is an $(\infty, r-1)$-category. In this sense, study $(\infty, 1)$-categories and $(\infty, 0)$-categories are enough for understanding (∞, r)-categories.

Remark Note that an (∞, r)-category can NOT be viewed as a category enriched in the category of $(\infty, r-1)$-categories. Instead, it should be viewed as enriched up to weak equivalences.

2.3 Example (Higher groupoids) The $(\infty, 0)$-categories are usually called ∞***-groupoids***. In particular, $(n, 0)$-categories are called n***-groupoids***. One can see they are the generalizations of sets and groupoids.

Each topological space admits an ∞-groupoid (the ***fundamental*** ∞***-groupoid***) which records its homotpy-theoretic information.

3 Homotopies

From now on, we will force on $(\infty, 1)$-categories. Without specification, an ∞-category is always an $(\infty, 1)$-category. Under this setting, we will also use some topology-style terminology:

- A ***homotopy*** is a k-morphism with $k > 1$. Note that in our case, they are all weak equivalences. Two k-morphisms x, y with $k > 0$ are said to be ***homotopic*** if there exists a $k+1$-morphism $x \to y$.
- Two k-morphisms are said to be ***connected*** if there exists a *zigzag* of $k+1$-morphisms connecting them.
- For $\mathcal{G}$ an ∞-groupoid, its ***set of components*** $\pi_0\mathcal{G}$ is obtained by identifying all connected objects in $\mathcal{G}$. If $\mathcal{G}$ serves as a Hom space in some ∞-category, we also call $\pi_0\mathcal{G}$ as the ***set of homotopy classes***.
- An ∞-groupoid $\mathcal{G}$ is said to be ***connected*** if $\pi_0\mathcal{G} = *$.
- For $\mathcal{C}$ an ∞-category, its ***homotopy category*** $\mathrm{h}\mathcal{C}$ is obtained by identifying homotopic morphisms. Therefore we also denote the set of homotopy classes $\pi_0 \mathcal{H}\mathrm{om}_{\mathcal{C}}(X, Y)$ as $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$, or even more simply $H(X, Y)$.

4 (∞-functors) There should be a notion of ***∞-functors*** and furthermore higher morphisms between them such that for each pair of ∞-categories $\mathcal{C}, \mathcal{D}$, there is an ∞-category $\mathrm{Fun}[\mathcal{C}, \mathcal{D}]$ of ∞-functors between them. In this way, there should be an $(\infty, 2)$-*category* $\infty\mathcal{C}\mathrm{at}$ consisting of

- ∞-categories as objects;
- the ∞-category $\mathrm{Fun}[\mathcal{C}, \mathcal{D}]$ as the Hom space $\mathcal{H}\mathrm{om}(\mathcal{C}, \mathcal{D})$.

An ∞-functor $F\colon \mathcal{C} \to \mathcal{D}$ should be something like this: it endows each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$, each pair of objects X, Y in $\mathcal{C}$ a morphism of ∞-groupoids:

$$\mathcal{H}\mathrm{om}_{\mathcal{C}}(X, Y) \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{D}}(F(X), F(Y)),$$

each object X in $\mathcal{C}$ a homotopy between $F(\mathrm{id}_X)$ to $\mathrm{id}_{F(X)}$ and so forth. One can see this notion is too complicate to be defined in this naive way. Anyhow, we obtain a *strict* 2-*category* $\mathrm{h}\infty\mathcal{C}\mathrm{at}$ consisting of

- ∞-categories as objects;
- the homotopy category of $\mathrm{Fun}[\mathcal{C}, \mathcal{D}]$ as the Hom space $\mathcal{H}\mathrm{om}(\mathcal{C}, \mathcal{D})$.

We say an ∞-functor is an ***equivalence*** if it is an weak equivalence in the $(\infty, 2)$-category $\infty\mathcal{C}\mathrm{at}$, equivalently, an equivalence in the 2-category $\mathrm{h}\infty\mathcal{C}\mathrm{at}$.

5 ($\infty\mathcal{G}$pd) The ∞-category $\infty\mathcal{G}\mathrm{pd}$ of ∞-groupoids plays an important role in ∞-category theory. Indeed, one can see its role is just like $\mathcal{S}\mathrm{et}$ in category theory. Therefore having an intuition on this ∞-category is necessary even for who do not care general ∞-category theory.

It is helpful to treat an ∞-groupoid $\mathcal{G}$ as a nice topological space, at least as its data of *homotopy type*: the objects in $\mathcal{G}$ are points, morphisms are paths and homotopies are homotopies. Indeed, one consistency condition for a good ∞-category theory is the *homotopy hypothesis*: the ∞-category $\infty\mathcal{G}\mathrm{pd}$ is equivalent to the ∞-category of nice topological spaces.

The homotopy category of this fundamental ∞-category is called the ***homotopy category of spaces*** and denoted by $\mathcal{H}$.

6 (Homotopy categories) Any ∞-functor induces a functor between the homotopy categories. Moreover, equivalent ∞-functors induce isomorphic functors. Thus there is always an essentially surjective functor

$$\mathrm{h}\,\mathrm{Fun}[\mathcal{C}, \mathcal{D}] \longrightarrow \mathrm{Fun}[\mathrm{h}\mathcal{C}, \mathrm{h}\mathcal{D}].$$

However, the naive definition of homotopy categories may not make the above functor an equivalence: although the homotopy categories remembers what are homotopic in the ∞-categories, they still forget how those things homotopic. To patch up this, we need to bring up some richer structure on the functor categories between homotopy categories.

One way to do this is to view the homotopy category $h\mathcal{C}$ as a $\mathcal{H}$-enriched category. In other words, $h\mathcal{C}$ is almost $\mathcal{C}$ expect the composition laws of morphisms are strict. Now the $\mathcal{H}$-enriched functor category $\mathrm{Fun}[h\mathcal{C}, h\mathcal{D}]$ should remember how equivalent things are equivalent.

But as I said, the original category $h\mathcal{C}$ is enough to understand what are homotopic in the ∞-categories $\mathcal{C}$. So we use the following mixed definition:

- the ***homotopy functor*** between two homotopy categories is the $\mathcal{H}$-enriched functor between them;
- while the homotopy categories are viewed as original categories.

7 Homotopy limits

We now generalize the notion of limits to ∞-categories.

First, we consider the terminal and initial objects.

8 (Homotopy terminal and initial objects) Recall that a *terminal object* in a category $\mathcal{C}$ is an object $*$ such that for any object X in $\mathcal{C}$, there exists a unique morphism $X \to *$. Note that the terminal object is unique up to isomorphisms.

Analogously, we define a ***homotopy terminal object*** in an ∞-category $\mathcal{C}$ is an object $*$ such that for any object X in $\mathcal{C}$, the Hom space $\mathcal{H}\mathrm{om}(X, *)$ is ***(weakly) contractible***, meaning it is homotopic to the trivial ∞-groupoid. Note that the homotopy terminal object is unique up to homotopies.

Note that from the above definition, we see that an object is a homotopy terminal object if and only if it is a terminal object in the homotopy category.

One can similarly define the notion of ***homotopy initial objects***.

A object 0 which is both a homotopy terminal object and a homotopy initial object is called a ***homotopy zero object***.

Remark It is often the case that the ∞-category we are studying is obtained by enlarging a category, revealing its hiding ∞-natural. Such a construction may be called a ***category with hiding homotopies***.

If this is the case, then the notion of equalities make sense for 1-morphisms. Then we have two notions of terminal (resp. initial) objects, the *strict* one and the *homotopy* one. However, it is easy to see that the strict terminal (resp. initial) object is homotopic to the homotopy terminal (resp. initial) object. In this sense, we will simply say ***terminal objects***, ***initial objects*** and ***zero objects***.

9 (Pointed objects) Any category $\mathcal{C}$ admits a terminal object $*$ admits a ***category of pointed objects*** $\mathcal{C}^{*/}$ whose objects are objects X in $\mathcal{C}$ together with a morphism (the ***point***) from the terminal object to X and whose morphisms are morphisms $f\colon X \to Y$ in $\mathcal{C}$ preserving the points in the sense that the following diagram commutes.

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

Note that the category $\mathcal{C}^{*/}$ of pointed objects admits a zero object $\mathrm{id}\colon * \to *$.

The similar things also happen in the ∞-world except every notion should be weaken by its homotopy version. But what is the homotopy version of *commutative diagram*?

10 (Commutative diagram) Recall that a diagram consists of several paths, each of them is a composible diagram. A diagram is said to *commute* if all such paths with same start and end give rise to equal compositions.

In an ∞-category, the equality of compositions rarely make sense. So to weak this notion, the requirement "equal" should be replaced by "weakly equivalent". To avoid ambiguities, we also say the diagram ***commute up to homotopies*** if needed.

Remark Note that the data of how those compositions are homotopic is part of a ***homotopy commutative diagram.***

11 (Homotopy limits) The notion of *homotopy limits* should be like this:

> Let $D\colon \mathcal{I} \to \mathcal{C}$ be an ∞-functor. Then a ***homotopy limit*** $\underleftarrow{\mathrm{hlim}}\, D$ of D is a terminal object in the ∞-category of *homotopy cones*.

A *cone* over a diagram $D\colon \mathcal{I} \to \mathcal{C}$ in category theory means a functor F from the $* \star \mathcal{I}$, the category obtained from $\mathcal{I}$ by adding an initial object, to $\mathcal{C}$ such that F factors through D. Thus, a ***homotopy cone*** should be an ∞-analogy of this.

The notions of ***homotopy cocones*** and ***homotopy colimits*** are defined similarly.

Remark Note that, for any diagram D and any object t, we have

$$\mathcal{H}\mathrm{om}(t, \underleftarrow{\mathrm{hlim}}\, D) \simeq \underleftarrow{\mathrm{hlim}}\, \mathcal{H}\mathrm{om}(t, D),$$
$$\mathcal{H}\mathrm{om}(\underrightarrow{\mathrm{hlim}}\, D, t) \simeq \underleftarrow{\mathrm{hlim}}\, \mathcal{H}\mathrm{om}(D, t).$$

where the homotopy limits on the right are taking in the ∞-category $\infty\mathcal{G}\mathrm{pd}$.

12 (Homotopy cones over a 1-dimensional diagram) To give intuitions, let's consider the homotopy cones over a 1-*dimensional diagram* $D\colon \mathcal{I} \to \mathcal{C}$, meaning $\mathcal{I}$ is a 1-category. Equivalently, we consider homotopy cones over a graph consisting of *vertexes* and *arrows*.

Recall that a *cone* over a diagram D is merely a family of morphisms $\{f_i\}$, each from a fixed object X to a vertex i in D, such that for any arrow $e\colon i \to j$ in D, the following diagram commutes.

$$\begin{tikzcd}
& X \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\
D_i \arrow[rr, "D_e"] & & D_j
\end{tikzcd}$$

A morphism between two cones $\{f_i\colon X \to D_i\}$ and $\{g_i\colon Y \to D_i\}$ is a morphism $\phi\colon X \to Y$ such that the following diagrams commute.

$$\begin{tikzcd}
X \arrow[rr, "\phi"] \arrow[dr, "f_i"'] & & Y \arrow[dl, "g_i"] \\
& D_i &
\end{tikzcd}$$

In this way, cones over a diagram form a category and the limit of the diagram is the terminal object in this category.

A ***homotopy cone*** over a diagram D consists of the following data:

- an object X;
- a family of (homotopy classes of) morphisms $\{f_i\}$, each from X to a vertex D_i in D;
- a family of (homotopy classes of) homotopies $f_e\colon f_j \Rightarrow D_e \circ f_i$.

$$\begin{tikzcd}
& X \arrow[dl, "f_i"'] \arrow[dr, "f_j"] \arrow[d, phantom, "\overset{f_e}{\Leftarrow}"] & \\
D_i \arrow[rr, "D_e"'] & {} & D_j
\end{tikzcd}$$

A *morphism* $\phi\colon (X, f_i, f_e) \to (Y, g_i, g_e)$ between homotopy cones consists of the following data:

- a morphism $\phi\colon X \to Y$;
- a family of (homotopy classes of) homotopies $\{\phi_i\}$;

$$\begin{tikzcd}
X \arrow[rr, "\phi"] \arrow[dr, "f_i"'] & {} \arrow[d, phantom, "\overset{\phi_i}{\Rightarrow}"] & Y \arrow[dl, "g_i"] \\
& D_i &
\end{tikzcd}$$

- a family of (homotopy classes of) 2-homotopies $\{\phi_e\}$.

$$
\begin{tikzcd}
X \arrow[rr, "f_i"] \arrow[dr, "\phi"'] \arrow[rrr, bend left=50, "f_j"] \arrow[rrr, phantom, bend left=25, "\Downarrow f_e"] & & D_i \arrow[r, "D_e"] & D_j \arrow[r, phantom, "\overset{\phi_e}{\rightsquigarrow}"] & X \arrow[r, "\phi"] \arrow[rrr, bend left=50, "f_j"] \arrow[rrr, phantom, bend left=25, "\Downarrow \phi_j"] & Y \arrow[rr, "g_j"] \arrow[dr, "g_i"'] & & D_j \\
 & Y \arrow[ur, "g_i"'] \arrow[u, phantom, "\Downarrow \phi_i"] & & & & & D_i \arrow[ur, "D_e"'] \arrow[u, phantom, "\Downarrow g_e"] &
\end{tikzcd}
$$

Next, a homotopy $\Psi\colon \phi \Rightarrow \phi'$ consists of the following data:

- a homotopy $\Psi\colon \phi \Rightarrow \phi'$;
- a family of (homotopy classes of) 2-homotopies $\{\Psi_i\}$;

$$
\begin{tikzcd}
X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\phi'"'] \arrow[r, phantom, "\Downarrow\Psi"] \arrow[rr, bend left=60, "f_i"] & Y \arrow[r, "g_i"] \arrow[r, phantom, bend left=30, "\Downarrow\phi_i"] & D_i \arrow[r, phantom, "\overset{\Psi_i}{\rightsquigarrow}"] & X \arrow[r, "\phi'"] \arrow[rr, bend left=60, "f_i"] & Y \arrow[r, "g_i"] \arrow[phantom, u, "\Downarrow\phi'_i"] & D_i
\end{tikzcd}
$$

- a family of (homotopy classes of) 3-homotopies $\{\Psi_e\}$,

$$
\begin{tikzcd}
X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\phi'"'] \arrow[r, phantom, "\Downarrow\Psi"] \arrow[rr, bend left=60, "f_i"] \arrow[rrr, bend left=70, "f_j"] \arrow[rrr, phantom, bend left=55, "\Downarrow f_e"] & Y \arrow[r, "g_i"'] \arrow[r, phantom, bend left=30, "\Downarrow\phi_i"] & D_i \arrow[r, "D_e"] & D_j \arrow[r, phantom, "\overset{\widehat{\phi_e}}{\rightsquigarrow}"] & X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\phi'"'] \arrow[r, phantom, "\Downarrow\Psi"] \arrow[rrr, bend left=50, "f_j"] \arrow[rrr, phantom, bend left=25, "\Downarrow \phi_j"] & Y \arrow[rr, "g_j"] \arrow[dr, "g_i"'] & & D_j \\
 & & & & & & D_i \arrow[ur, "D_e"'] \arrow[u, phantom, "\Downarrow g_e"] &
\end{tikzcd}
$$

$$
\Big\downarrow \widehat{\Psi_i} \qquad \Swarrow \Psi_e \qquad \Big\downarrow \widehat{\Psi_j}
$$

$$
\begin{tikzcd}
X \arrow[rr, "f_i"] \arrow[dr, "\phi'"'] \arrow[rrr, bend left=50, "f_j"] \arrow[rrr, phantom, bend left=25, "\Downarrow f_e"] & & D_i \arrow[r, "D_e"] & D_j \arrow[r, phantom, "\overset{\phi'_e}{\rightsquigarrow}"] & X \arrow[r, "\phi'"] \arrow[rrr, bend left=50, "f_j"] \arrow[rrr, phantom, bend left=25, "\Downarrow \phi'_j"] & Y \arrow[rr, "g_j"] \arrow[dr, "g_i"'] & & D_j \\
 & Y \arrow[ur, "g_i"'] \arrow[u, phantom, "\Downarrow \phi'_i"] & & & & & D_i \arrow[ur, "D_e"'] \arrow[u, phantom, "\Downarrow g_e"] &
\end{tikzcd}
$$

where $\widehat{\phi_e}, \widehat{\Psi_i}, \widehat{\Psi_j}$ in the above square are constructed from ϕ_e, Ψ_i, Ψ_j respectively.

Keep going the process, the homotopy cones over a diagram form an ∞-category. The ***homotopy limit*** $\varprojlim^{\mathrm{h}} D$ of the diagram D is then the homotopy terminal object in this ∞-category.

13 Intrinsic cohomology theory

We will introduce the notion of *intrinsic cohomologies.* To do this, we need some special kinds of homotopy limits.

14 (Homotopy fibred products) We now consider diagrams like this

$$A \xrightarrow{f} B \xleftarrow{g} C.$$

The homotopy limit of this diagram is called the ***homotopy fibred product*** of A and C over B, denoted by $A \times^{\mathrm{h}}_B C$, or the ***homotopy pullback*** of f along g, denoted by $g^{\mathrm{h}} f$, or the ***homotopy pullback*** of g along f, denoted by $f^{\mathrm{h}} g$.

Remark In a category with hiding homotopies, the strict fibred products are rarely homotopic to the homotopy fibred products. However, there is always a canonical morphism from the homotopy one to the strict one.

Like the strict pullbacks, we have the pasting lemma.

14.1 Lemma (Pasting lemma) *In an ∞-category having homotopy pullbacks, consider the following diagram of morphisms.*

$$\begin{tikzcd} \cdot \arrow[r, "g'"] \arrow[d, "h''"'] & \cdot \arrow[r, "f'"] \arrow[d, "h'"'] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "g"] & \cdot \arrow[r, "f"] & \cdot \end{tikzcd}$$

1. *If h' is the homotopy pullback of h along f and h'' is the homotopy pullback of h' along g, then h'' is the homotopy pullback of h along $f \circ g$.*

2. *If h'' is the homotopy pullback of h along $f \circ g$ and the homotopy pullback of h' along g, then h' is the homotopy pullback of h along f.*

Proof: The proof is almost the same as the strict version except a careful treat on homotopies. $\square$

15 (Homotopy fibers) Let $f\colon X \to Y$ be a morphism in an ∞-category of pointed objects. Then the ***homotopy fiber*** is the homotopy pullback of the point $* \to Y$ along f.

A sequence of pointed objects

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called a ***fiber sequence*** if f is a homotopy fiber of g.

Remark The homotopy fibers can be viewed as the homotopy version of kernels. in this sense, fiber sequences can be viewed as the homotopy version of left exact sequences. However, unlike the strict case, the homotopy fiber of a homotopy fiber need NOT be trivial.

16 (Looping and delooping) Let X be a pointed object. Its ***looping*** ΩX is the homotopy fiber of the point $* \to X$. If Y is the looping of a pointed object X, then we say X is the ***delooping*** of Y and denote it by $\mathbf{B}X$.

Remark Note that this is no corresponding notion in category theory since the kernel of a kernel must be trivial. Therefore the information determining a looping must come from the higher structures on the point $* \to X$.

16.1 Example (Loop spaces) In the ∞-category Top of topological spaces, the loop spaces are precisely the loopings. in fact, the notion of loopings is motivated by loop spaces.

17 (Connected objects) Let X be an object in an ∞-category $\mathcal{C}$ admitting terminal object $*$, its ***space of points*** is the Hom space $\mathcal{H}\mathrm{om}(*, X)$. A ***connected object*** is an object X whose space of points is connected. In other word, the information of homotopies is encoded in higher morphisms in $\mathcal{H}\mathrm{om}(*, X)$. Thus, the sub ∞-category of connected objects is denoted by $\mathcal{C}_{\geqslant 1}$.

Let $\mathcal{C}^{*/}_{\geqslant 1}$ denote the ∞-category of pointed connected objects. Note that, this ∞-category is distinct from $\mathcal{C}_{\geqslant 1}$. Although each connected object admits a unique point up to homotopies, the Hom space between two connected objects is different from the Hom space between the corresponding pointed objects. Consider, for instance, a group, viewed as an ∞-groupoid, in $\infty\mathcal{G}\mathrm{pd}$.

18 (Group objects) Recall that a ***group object*** in a category $\mathcal{C}$ is a product-preserving functor from the theory of groups to $\mathcal{C}$. Equivalently, a *group object* is an object such that the contravariant functor $\mathrm{Hom}(-, X)$ is from $\mathcal{C}$ to the category of groups.

Similarly, a ***group object*** in an ∞-category $\mathcal{C}$ is an object such that the contravariant ∞-functor $\mathcal{H}\mathrm{om}(-, X)$ is from $\mathcal{C}$ to the ∞-category of ∞-***groups***. But what are ∞-*groups*?

Recall that a group can be viewed as a groupoid having only one object. Viewing the 2-category of categories as an ∞-category, this is to say groups are loopings of pointed connected groupoids. Motivated by this, the ∞-***groups*** must be homotopic to loopings of pointed connected ∞-groupoids.

Let $\mathcal{G}\mathrm{rp}(\mathcal{C})$ denote the ∞-category of group objects in an ∞-category $\mathcal{C}$. Then the looping construction extends to an ∞-functor

$$\Omega \colon \mathcal{C}^{*/} \longrightarrow \mathcal{G}\mathrm{rp}(\mathcal{C}),$$

which induces an equivalence

$$\Omega \colon \mathcal{C}^{*/}_{\geqslant 1} \xrightarrow{\simeq} \mathcal{G}\mathrm{rp}(\mathcal{C})$$

whose weak inverse is given by deloopings.

18.1 Lemma *In an ∞-category having homotopy pullbacks, whenever there is a fiber sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

there is a long fiber sequence

$$\cdots \longrightarrow \Omega A \xrightarrow{\Omega f} \Omega B \xrightarrow{\Omega g} \Omega C \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

18.2 Lemma *For any fiber sequence of ∞-groupoids*

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C},$$

the corresponding sequence of sets of components

$$\pi_0\mathcal{A} \xrightarrow{\pi_0 f} \pi_0\mathcal{B} \xrightarrow{\pi_0 g} \pi_0\mathcal{C}$$

is exact at $\pi_0\mathcal{B}$.

19 (Intrinsic cohomology) Let X be an object in an ∞-category $\mathcal{C}$ and G a group object in $\mathcal{C}$ admits n-th delooping. Then the ***n-th cohomology*** of X with coefficients in G is the pointed set

$$H^n(X, G) := \pi_0 \,\mathcal{H}\mathrm{om}(X, \mathbf{B}^n G).$$

Since $\mathbf{B}^{n-1}G = \Omega\mathbf{B}^n G$, it is a group object in $\mathcal{C}$. Hence $\mathcal{H}\mathrm{om}(X, \mathbf{B}^{n-1}G)$ is an ∞-group and therefore $H^{n-1}(X, G)$ is a group rather than a pointed set. Moreover, $\mathbf{B}^{n-2}G$ admits two group object structures, one comes from the functor Ω, one inherits from $\mathbf{B}^{n-1}G$. The two structures are compatible (meaning the two multiplications satisfy the *interchange law*), at least up to homotopy. Hence, the pointed set $H^{n-2}(X, G)$ admits two compatible multiplications. By the Eckmann-Hilton argument, they must be the same and commutative. Consequently, $H^k(X, G)$ is a group whenever $H^{k+1}(X, G)$ exists, and is an abelian group whenever $H^{k+2}(X, G)$ exists.

Let A, B, C be three group objects in $\mathcal{C}$ admits n-th deloopings. By Lemma 18.1, we have a long fiber sequence of ∞-groupoids

$$\cdots \longrightarrow \mathcal{H}\mathrm{om}(X, \mathbf{B}^n A) \longrightarrow \mathcal{H}\mathrm{om}(X, \mathbf{B}^n B) \longrightarrow \mathcal{H}\mathrm{om}(X, \mathbf{B}^n C) \longrightarrow \cdots.$$

Thus, by Lemma 18.2, we have a long exact sequence of pointed sets (and groups):

$$\cdots \longrightarrow H^{n-1}(X, C) \longrightarrow H^n(X, A) \longrightarrow H^n(X, B) \longrightarrow H^n(X, C) \longrightarrow \cdots.$$

20 Stable ∞-categories

Recall that an object 0 is called a ***zero object*** if it is both a terminal object and an initial object. An ∞-category is said to be ***pointed*** if it has a zero object. In such an ∞-category, any pair of objects A, B admits a morphism obtained by compositing the unique morphisms $A \to 0$ and $0 \to B$, called the ***zero morphism***.

A ***triangle*** is a pair of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

together with a homotopy from 0 to $g \circ f$.

An ∞-category $\mathcal{C}$ is said to be ***stable*** if it satisfies the following conditions.

1. $\mathcal{C}$ is pointed.
2. Every morphism in $\mathcal{C}$ admits a homotopy fiber and a homotopy cofiber.
3. A triangle in $\mathcal{C}$ is a fiber sequence if and only if it a cofiber sequence.

One can see this notion is similar to the notion of abelian categories.

The homotopy category of a stable ∞-category is a triangulated category.

§ 2 The category of topological spaces

(In this section, we should discuss the ∞-category Top of nice topology)

1 The category of nice topology

2 (CW complexes) content

3 (*k*-spaces) content

4 (compactly generated spaces) content

5 Homotopy limits in Top

Although Top has better categorical property than the whole category of topological spaces, the limits (and colimits) in Top may not coincide with the usual ones in the whole category. However, they are weakly homotopic, meaning they share isomorphic homotopy groups.

In this subsection, we will discuss homotopy limits and colimits in Top.

5.1 Example (Homotopy pullbacks) To give intuition, we describe the universal property of *homotopy pullbacks*, especially in the ∞-category Top of nice topology spaces. Note that Top is equivalent to $\infty\mathcal{G}\mathrm{pd}$.

First a *homotopy square* from an object X to $A \xrightarrow{f} B \xleftarrow{g} C$ is a square with a homotopy.

$$\begin{tikzcd} X \arrow[r, "X_0"] \arrow[d, "X_1"'] \arrow[dr, phantom, "\alpha\swarrow"] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

A morphism $\phi\colon (X,\alpha) \to (Y,\beta)$ between homotopy squares consists of the following data:

- a morphism $\phi\colon X \to Y$;
- two (homotopy classes of) homotopies ϕ_0, ϕ_1;

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr] & \arrow[d, phantom, "\phi_0 \Rrightarrow"] & Y \arrow[dl] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr] & \arrow[d, phantom, "\phi_1 \Rrightarrow"] & Y \arrow[dl] \\ & C & \end{tikzcd}$$

- a (homotopy class of) 2-homotopies ϕ_2.

$$
\begin{tikzcd}
& & A \arrow[dr, "f"] & & & A \arrow[dr, "f"] & \\
X \arrow[r, "\phi"] \arrow[urr, bend left] \arrow[drr, bend right] \arrow[ur, phantom, "\Downarrow\phi_0"] \arrow[dr, phantom, "\Downarrow\phi_1^{-1}"] & Y \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow\beta"] & & B \arrow[r, squiggly, "\phi_2"] & X \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow\alpha"] & & B \\
& & C \arrow[ur, "g"'] & & & C \arrow[ur, "g"'] &
\end{tikzcd}
$$

Next, a homotopy $\Psi\colon \phi \Rightarrow \phi'$ consists of the following data:

- a homotopy $\Psi\colon \phi \Rightarrow \phi'$;
- two (homotopy classes of) 2-homotopies Ψ_0, Ψ_1;

$$
\begin{tikzcd}
X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\phi'"'] \arrow[r, phantom, "\Downarrow\Psi"] \arrow[rr, bend left=60] & Y \arrow[r] \arrow[u, phantom, "\Downarrow\phi_0"] & A & \rightsquigarrow^{\Psi_0} & X \arrow[r, "\phi'"] \arrow[rr, bend left=60] & Y \arrow[r] \arrow[u, phantom, "\Downarrow\phi_0'"] & A
\end{tikzcd}
$$

- a (homotopy class of) 3-homotopies Ψ_2,

$$
\begin{tikzcd}
& & A \arrow[dr, "f"] & & & & A \arrow[dr, bend left, "f"] & \\
X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\phi'"'] \arrow[r, phantom, "\Downarrow\Psi"] \arrow[urr, bend left] \arrow[drr, bend right] \arrow[ur, phantom, "\Downarrow\phi_0"] \arrow[dr, phantom, "\Downarrow{\phi'_1}^{-1}"] & Y \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow\beta"] & & B \arrow[r, squiggly, "\widehat{\phi_2}"] & X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\phi'"'] \arrow[r, phantom, "\Downarrow\Psi"] \arrow[urr, bend left] \arrow[rr, bend left=60] \arrow[rr, bend right=60] & Y \arrow[r] \arrow[ur, phantom, "\Downarrow\phi_1"] \arrow[dr, phantom, "\Downarrow{\phi'_1}^{-1}"] & C \arrow[r, "g"] \arrow[u, phantom, "\Downarrow\alpha"] \arrow[ddd, squiggly, "\widehat{\Psi_1}"] & B \\
& & C \arrow[ur, "g"'] \arrow[dd, squiggly, "\widehat{\Psi_0}"] & & & & & \\
& & & & \Swarrow\Psi_2 & & & \\
& & A \arrow[dr, "f"] & & & & A \arrow[dr, "f"] & \\
X \arrow[r, "\phi'"] \arrow[urr, bend left] \arrow[drr, bend right] \arrow[ur, phantom, "\Downarrow\phi_0'"] \arrow[dr, phantom, "\Downarrow{\phi'_1}^{-1}"] & Y \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow\beta"] & & B \arrow[rr, squiggly, "\phi_2'"] & & X \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow\alpha"] & & B \\
& & C \arrow[ur, "g"'] & & & & C \arrow[ur, "g"'] &
\end{tikzcd}
$$

where $\widehat{\phi_2}, \widehat{\Psi_0}, \widehat{\Psi_1}$ in the above square are constructed from ϕ_2, Ψ_0, Ψ_1 respectively.

Now, the condition of two morphisms $\phi, \phi' \colon (X, \alpha) \to (Y, \beta)$ being homotopic in Top reads

1. there exists a continuous map $\Psi \colon I \times X \to Y$ such that $\Psi(0, x) = \phi(x)$ and $\Psi(1, x) = \phi'(x)$;

2. there exist continuous maps $\Psi_0 \colon I \times I \times X \to A, \Psi_1 \colon I \times I \times X \to C$ such that

$$
\begin{aligned}
\Psi_0(0, t, x) &= \begin{cases} \phi_0(2t, x) & 0 \leqslant t < \frac{1}{2}, \\ Y_0 \Psi(2t - 1, x) & \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \\
\Psi_0(1, t, x) &= \phi_0'(t, x), \\
\Psi_1(0, t, x) &= \begin{cases} \phi_1(2t, x) & 0 \leqslant t < \frac{1}{2}, \\ Y_1 \Psi(2t - 1, x) & \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \\
\Psi_1(1, t, x) &= \phi_1'(t, x);
\end{aligned}
$$

3. there exists a continuous mapa $\Psi_2 \colon I \times I \times I \times X \to B$ such that

$$
\begin{aligned}
\Psi_2(0, t, s, x) &= \begin{cases} \widehat{\phi_2}(t, s, x) & 0 \leqslant t < \frac{1}{2}, \\ \widehat{\Psi_1}(t, s, x) & \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \\
\Psi_2(1, t, s, x) &= \begin{cases} \widehat{\Psi_0}(t, s, x) & 0 \leqslant t < \frac{1}{2}, \\ \phi_2'(t, s, x) & \frac{1}{2} \leqslant t \leqslant 1, \end{cases}
\end{aligned}
$$

   where

$$
\begin{aligned}
\widehat{\phi_2}(t, s, x) &= \begin{cases} \widetilde{\phi_2}(2t, s, x) & 0 \leqslant t < \frac{1}{2}, \\ \widetilde{\eta}(2t - 1, s, x) & \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \\
\widehat{\Psi_0}(t, s, x) &= \begin{cases} \Psi_0(t, 4s, x) & 0 \leqslant s < \frac{1}{4}, \\ \beta(4s - 1, \phi'(x)) & \frac{1}{4} \leqslant s < \frac{1}{2}, \\ \phi_1'(2 - 2s, x) & \frac{1}{2} \leqslant s \leqslant 1, \end{cases} \\
\widehat{\Psi_1}(t, s, x) &= \begin{cases} \widetilde{\Psi_1}(2t, s, x) & 0 \leqslant t < \frac{1}{2}, \\ \widetilde{\eta}'(2 - 2t, s, x) & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}
\end{aligned}
$$

   Here, $\widetilde{\phi_2}, \widetilde{\eta}, \widetilde{\Psi_1}, \widetilde{\eta}'$ are the obtained by horizontal composting $\phi_2, \eta, \Psi_1, \eta'$ with something respectively. Note that the homotopy η (resp. η') is just a homotopy from the trivial homotopy of ϕ (resp. ϕ') to the composition $\phi_1 \circ \phi_1^{-1}$ (resp. $\phi_1' \circ {\phi_1'}^{-1}$), i.e.

$$\eta(0, s, x) = \phi(x)$$

$$\eta(1,s,x) = \begin{cases} \phi_1(1-2s,x) & 0 \leqslant s < \frac{1}{2}, \\ \phi_1(2s-1,x) & \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

The follows are explicit formulas of $\widetilde{\phi_2}, \widetilde{\eta}, \widetilde{\Psi_1}, \widetilde{\eta'}$.

$$\widetilde{\phi_2}(t,s,x) = \begin{cases} \phi_2(t,8s,x) & 0 \leqslant s < \frac{1}{8}, \\ g\phi_1(8s-1,x) & \frac{1}{8} \leqslant s < \frac{1}{4}, \\ gY_1\Psi(4s-1,x) & \frac{1}{4} \leqslant s < \frac{1}{2}, \\ g\phi_1'(2-2s,x) & \frac{1}{2} \leqslant s \leqslant 1, \end{cases}$$

$$\widetilde{\eta}(t,s,x) = \begin{cases} f\phi_0(16s,x) & 0 \leqslant s < \frac{1}{16}, \\ \beta(16s-1,\phi(x)) & \frac{1}{16} \leqslant s < \frac{1}{8}, \\ g\eta(t,8s-1,x) & \frac{1}{8} \leqslant s < \frac{1}{4}, \\ gY_1\Psi(4s-1,x) & \frac{1}{4} \leqslant s < \frac{1}{2}, \\ g\phi_1'(2-2s,x) & \frac{1}{2} \leqslant s \leqslant 1, \end{cases}$$

$$\widetilde{\Psi_1}(t,s,x) = \begin{cases} \alpha(4s,x) & 0 \leqslant s < \frac{1}{4}, \\ g\Psi_1(t,4s-1,x) & \frac{1}{4} \leqslant s < \frac{1}{2}, \\ g\phi_1'(2-2s,x) & \frac{1}{2} \leqslant s \leqslant 1, \end{cases}$$

$$\widetilde{\eta'}(t,s,x) = \begin{cases} \alpha(2s,x) & 0 \leqslant s < \frac{1}{2}, \\ g\eta'(t,2s-1,x) & \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

6 Problem *If (L,λ) is a homotopy square from L to $A \xrightarrow{f} B \xleftarrow{g} C$ such that for any homotopy square (X,α) to $A \xrightarrow{f} B \xleftarrow{g} C$, there exists a unique homotopy class of morphisms $[\phi]$ of objects such that the horizontal composition of λ with ϕ is homotopic to α. Must (L,λ) be a homotopy fibred product over $A \xrightarrow{f} B \xleftarrow{g} C$?*

6.1 Example (Homotopy products) content

6.2 Example (Homotopy fibers in Top) To give intuition, we describe the universal property of *homotopy fibers*, especially in the ∞-category Top of nice topology spaces. Note that Top is equivalent to $\infty\mathcal{G}$pd.

For convenience, we work in the category of pointed objects and omit the pointing. Let 0 denote the zero object id_*. Note that it is often the case the terminal object can be chosen to be strict and hence the Hom spaces $\mathcal{H}\mathrm{om}(X,*)$ are just the singleton. However, even in this case, the homotopies on zero morphisms need not to be trivial.

First a *homotopy annihilation* of a morphism $f\colon A \to B$ consists of the following data:

- a morphism $a\colon X \to A$;

- a (homotopy class of) homotopies α.

$$
\begin{tikzcd}
X \arrow[rr, "a"] \arrow[dr, "0"'] \arrow[rr, phantom, shift right=4ex, "\alpha \Rrightarrow"] & & A \arrow[dl, "f"] \\
& B &
\end{tikzcd}
$$

A morphism $\phi\colon (X, a, \alpha) \to (Y, b, \beta)$ between homotopy annihilations consists of the following data:

- a morphism $\phi\colon X \to Y$;
- two (homotopy classes of) homotopies ϕ_0, ϕ_1;

$$
\begin{tikzcd}
X \arrow[rr, "\phi"] \arrow[dr, "a"'] \arrow[rr, phantom, shift right=4ex, "\phi_0 \Rrightarrow"] & & Y \arrow[dl, "b"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "\phi"] \arrow[dr, "0"'] \arrow[rr, phantom, shift right=4ex, "\phi_1 \Rrightarrow"] & & Y \arrow[dl, "0"] \\
& B &
\end{tikzcd}
$$

- a (homotopy class of) 2-homotopies ϕ_2.

$$
\begin{tikzcd}
X \arrow[rr, "a"] \arrow[rrr, bend left=60, "0"] \arrow[rr, phantom, shift left=6ex, "\Downarrow\alpha"] \arrow[dr, "\phi"'] \arrow[rr, phantom, shift right=5ex, "\Downarrow\phi_0"] & & A \arrow[r, "f"] & B \\
& Y \arrow[ur, "b"'] & &
\end{tikzcd}
\overset{\phi_2}{\rightsquigarrow}
\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[rrr, bend left=60, "0"] \arrow[rrr, phantom, shift left=6ex, "\Downarrow\phi_1"] & Y \arrow[rr, "0"] \arrow[dr, "b"'] \arrow[rr, phantom, shift right=5ex, "\Downarrow\beta"] & & B \\
& & A \arrow[ur, "f"'] &
\end{tikzcd}
$$

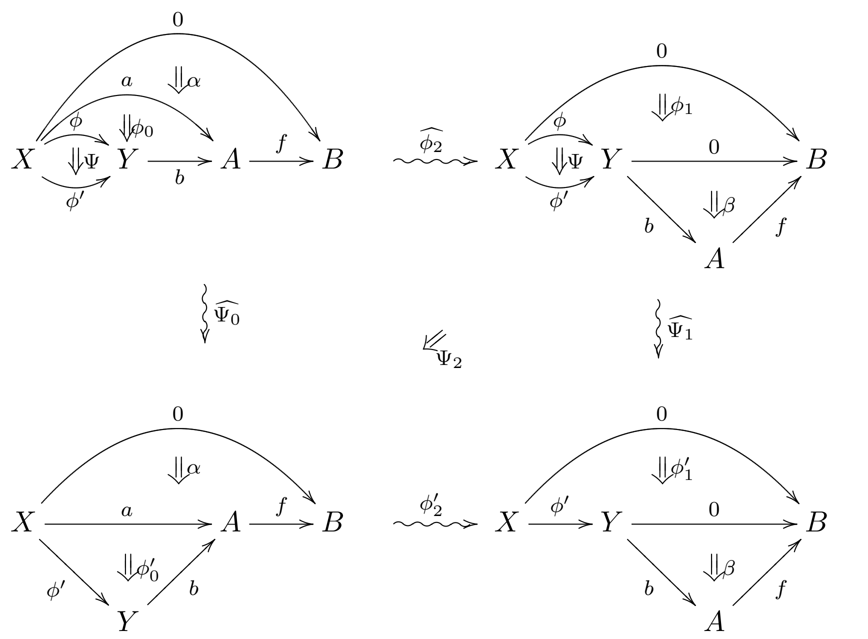
Next, a homotopy $\Psi\colon \phi \Rightarrow \phi'$ consists of the following data:

- a homotopy $\Psi\colon \phi \Rightarrow \phi'$;
- two (homotopy classes of) 2-homotopies Ψ_0, Ψ_1;

$$
\begin{tikzcd}
X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\phi'"'] \arrow[r, phantom, "\Downarrow\Psi"] \arrow[rr, bend left=70, "a"] \arrow[rr, phantom, shift left=5ex, "\Downarrow\phi_0"] & Y \arrow[r, "b"] & A
\end{tikzcd}
\overset{\Psi_0}{\rightsquigarrow}
\begin{tikzcd}
X \arrow[r, "\phi'"] \arrow[rr, bend left=70, "a"] \arrow[rr, phantom, shift left=5ex, "\Downarrow\phi'_0"] & Y \arrow[r, "b"] & A
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\phi'"'] \arrow[r, phantom, "\Downarrow\Psi"] \arrow[rr, bend left=70, "0"] \arrow[rr, phantom, shift left=5ex, "\Downarrow\phi_1"] & Y \arrow[r, "0"] & B
\end{tikzcd}
\overset{\Psi_1}{\rightsquigarrow}
\begin{tikzcd}
X \arrow[r, "\phi'"] \arrow[rr, bend left=70, "0"] \arrow[rr, phantom, shift left=5ex, "\Downarrow\phi'_1"] & Y \arrow[r, "0"] & B
\end{tikzcd}
$$

- a (homotopy classes of) 3-homotopies Ψ_2,



  where $\widehat{\phi_2}, \widehat{\Psi_0}, \widehat{\Psi_1}$ in the above square are constructed from ϕ_2, Ψ_0, Ψ_1 respectively.

Now, the condition of two morphisms $\phi, \phi' \colon (X, a, \alpha) \to (Y, b, \beta)$ being homotopic in Top reads

1. there exists a continuous map $\Psi \colon I \times X \to Y$ such that $\Psi(0, x) = \phi(x)$ and $\Psi(1, x) = \phi'(x)$;

2. there exist continuous maps $\Psi_0 \colon I \times I \times X \to A, \Psi_1 \colon I \times I \times X \to C$ such that

$$\begin{aligned}
\Psi_0(0,t,x) &= \begin{cases} \phi_0(2t,x) & 0 \leqslant t < \frac{1}{2}, \\ b\Psi(2t-1,x) & \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \\
\Psi_0(1,t,x) &= \phi_0'(t,x), \\
\Psi_1(0,t,x) &= \begin{cases} \phi_1(2t,x) & 0 \leqslant t < \frac{1}{2}, \\ 0 & \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \\
\Psi_1(1,t,x) &= \phi_1'(t,x);
\end{aligned}$$

3. there exists a continuous mapa $\Psi_2 \colon I \times I \times I \times X \to B$ such that

$$\Psi_2(0,t,s,x) = \begin{cases} \widehat{\phi_2}(t,s,x) & 0 \leqslant t < \frac{1}{2}, \\ \widehat{\Psi_1}(t,s,x) & \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

$$\Psi_2(1,t,s,x) = \begin{cases} \widehat{\Psi_0}(t,s,x) & 0 \leqslant t < \frac{1}{2}, \\ \phi_2'(t,s,x) & \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

where

$$\widehat{\phi_2}(t,s,x) = \begin{cases} \phi_2(t,2s,x) & 0 \leqslant s < \frac{1}{2}, \\ b\Psi(2s-1,x) & \frac{1}{2} \leqslant s \leqslant 1, \end{cases}$$

$$\widehat{\Psi_0}(t,s,x) = \begin{cases} \alpha(2s,x) & 0 \leqslant s < \frac{1}{2}, \\ f\Psi_0(t,2s-1,0) & \frac{1}{2} \leqslant s \leqslant 1, \end{cases}$$

$$\widehat{\Psi_1}(t,s,x) = \begin{cases} \Psi_1(t,2s,x) & 0 \leqslant s < \frac{1}{2}, \\ \beta(2s-1,\phi'(x)) & \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

7 Problem *If (L,l,λ) is a homotopy annihilation of $f\colon A \to B$ such that for any homotopy annihilation (X,a,α), there exists a unique homotopy class of morphisms $[\phi]$ of objects such that the horizontal composition of λ with ϕ is homotopic to α. Must (L,l,λ) be a homotopy fiber of $f\colon A \to B$?*

7.1 Example (Loop spaces) To give intuition, we describe the universal property of *loopings*, especially in the ∞-category Top of nice topology spaces. Note that Top is equivalent to $\infty\mathcal{G}$pd.

For convenience, we work in the category of pointed objects and omit the pointing. Let 0 denote the zero object id_*. Note that it is often the case the terminal object can be chosen to be strict and hence the Hom spaces $\mathcal{H}\mathrm{om}(X,*)$ are just the singleton. However, even in this case, the homotopies on zero morphisms need not to be trivial.

First a *looped zero morphism* is a zero morphism $a\colon X \to A$ with a (homotopy class of) homotopies $\alpha\colon a \Rightarrow a$.

A morphism $\phi\colon (a,\alpha) \to (b,\beta)$ between looped zero morphisms consists of the following data:

- a morphism $\phi\colon X \to Y$;
- a (homotopy class of) 2-homotopies ϕ_2.

$$\begin{tikzcd} X \arrow[r, "\phi"] & Y \arrow[r, bend left, "b"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\beta"] & A \arrow[r, rightsquigarrow, "\phi_2"] & X \arrow[r, bend left, "a"] \arrow[r, bend right, "a"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$$

Next, a homotopy $\Psi\colon \phi \Rightarrow \phi'$ consists of the following data:

- a homotopy $\Psi\colon \phi \Rightarrow \phi'$;

- a (homotopy class of) 2-homotopies Ψ_1;

$$
\begin{tikzcd}
X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\phi'"'] \arrow[r, phantom, "\Downarrow\Psi"] & Y \arrow[r, "b"] & A \arrow[r, squiggly, "\Psi_1"] & X \arrow[r, "\phi'"] & Y \arrow[r, "b"] & A
\end{tikzcd}
$$

- a (homotopy classes of) 3-homotopies Ψ_2,

$$
\begin{tikzcd}
X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\phi'"'] \arrow[r, phantom, "\Downarrow\Psi"] & Y \arrow[r, bend left, "b"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\beta"] \arrow[dd, squiggly, "\widehat{\Psi}_0"] & A \arrow[rr, squiggly, "\widehat{\phi_2}"] & & X \arrow[r, "\phi"] \arrow[r, bend right=60, "\phi'"'] \arrow[r, phantom, shift right=3, "\Downarrow\Psi"] \arrow[rr, bend left=50, "a"] \arrow[rr, phantom, shift left=3, "\Downarrow\alpha"] & Y \arrow[r, "b"] \arrow[dl, squiggly, "\widehat{\Psi}_1"] & A \\
& & & \Swarrow \Psi_2 & & & \\
X \arrow[r, "\phi'"] & Y \arrow[r, bend left, "b"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\beta"] & A \arrow[rr, squiggly, "\phi'_2"] & & X \arrow[r, bend left, "a"] \arrow[r, bend right, "a"'] \arrow[r, phantom, "\Downarrow\alpha"] & A &
\end{tikzcd}
$$

  where $\widehat{\phi_2}, \widehat{\Psi}_0, \widehat{\Psi}_1$ in the above square are constructed from ϕ_2 and Ψ_1 respectively.

Now, the condition of two morphisms $\phi, \phi' \colon (X, a, \alpha) \to (Y, b, \beta)$ being homotopic in Top reads

1. there exists a continuous map $\Psi \colon I \times X \to Y$ such that $\Psi(0, x) = \phi(x)$ and $\Psi(1, x) = \phi'(x)$;

2. there exist continuous maps $\Psi_1 \colon I \times I \times X \to A$ such that

$$
\begin{aligned}
\Psi_1(0, t, x) &= b\Psi(t, x), \\
\Psi_1(1, t, x) &= a(x);
\end{aligned}
$$

3. there exists a continuous mapa $\Psi_2 \colon I \times I \times I \times X \to A$ such that

$$
\begin{aligned}
\Psi_2(0, t, s, x) &= \begin{cases} \widehat{\phi_2}(t, s, x) & 0 \leqslant t < \frac{1}{2}, \\ \widehat{\Psi_1}(t, s, x) & \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \\
\Psi_2(1, t, s, x) &= \begin{cases} \widehat{\Psi_0}(t, s, x) & 0 \leqslant t < \frac{1}{2}, \\ \phi_2'(t, s, x) & \frac{1}{2} \leqslant t \leqslant 1, \end{cases}
\end{aligned}
$$

   where

$$
\widehat{\phi_2}(t, s, x) = \begin{cases} \phi_2(t, 2s, x) & 0 \leqslant s < \frac{1}{2}, \\ b\Psi(2s - 1, x) & \frac{1}{2} \leqslant s \leqslant 1, \end{cases}
$$

$$\widehat{\Psi}_0(t,s,x) = \begin{cases} \beta(2s,\phi(x)) & 0 \leqslant s < \frac{1}{2}, \\ \Psi_1(t,2s-1,x) & \frac{1}{2} \leqslant s \leqslant 1, \end{cases}$$
$$\widehat{\Psi}_1(t,s,x) = \begin{cases} \alpha(2s,x) & 0 \leqslant s < \frac{1}{2}, \\ \Psi_1(t,2s-1,x) & \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

8 Problem *If (L,λ) is a looped zero morphism to A such that for any looped zero morphism (X,α) to A, there exists a unique homotopy class of morphisms $[\phi]$ of objects such that the horizontal composition of λ with ϕ is homotopic to α. Must (L,λ) be a looping of A?*

9 Fibrations

10 (Quasi-fibrations) Although the *homotopy fibers* are in general not the fibers, it may be the case they are weakly equivalent. We call a morphism f between pointed objects in an ∞-category a ***quasi-fibration*** when the fiber of f is weakly equivalent to the homotopy fiber.

Whenever there is a quasi-fibration $f\colon A \to B$, we obtain a long exact sequence of pointed sets (and groups):

$$\cdots \longrightarrow H^n(X, \ker f) \longrightarrow H^n(X,A) \longrightarrow H^n(X,B) \longrightarrow \cdots .$$

The notion of quasi-fibrations can help us understand homotopy limits or even more general homotopy phenomena via strict limits or strict constructions.

(Example?)

However, it may be too difficult to determine which morphisms are quasi-fibrations. In practice, people often choose a class of morphisms which can be proved to be quasi-fibrations and the class has good properties. Such a class is called a ***class of fibrations***. After choose a class of fibrations, an object X is said to be ***fibrant*** if the morphism $X \to *$ is a fibration.

THe notion of ***cofibrations*** and ***cofibrant objects*** are similarly defined.

II

Homological algebra

In this chapter, we will review basic homological algebra through an ∞-categorical viewpoint.

In §1, we will review some basic notions about chain complexes. Then, in §2, we will show that there are natural higher structures on the category of complexes and the homotopy data of those higher structures can be encoded into chain complexes of abelian groups. This fact leads to the notion of *dg-categories*, which can be viewed as a convenience context to do basic ∞-category theory. However, the correct definition of equivalences between dg-categories suggests that the correct higher structure in the category of complexes should be much finer than the obvious one such that the quasi-isomorphisms are homotopy equivalences. This leads to the notion of derived categories.

In order to construct the derived categories, we need some preliminaries. In §3, we define the notion of fibers and cofibers. In §4, we discuss the long exact sequences. In §5, we introduce the notion of projective and injective objects to preserve our fibrant objects. After those preparations, we construct various of derived categories in §6. Then, we can define the correct notion of functor categories in the context of dg-categories in §7. Finally, we will give a framework of dg-category theory with more discussion on derived categories in §8.

§ 1 Chain complexes

1 Graded and twisted objects

Let G be an additive monoid and $\mathcal{C}$ a category. A ***G-graded object*** in $\mathcal{C}$ is a family of objects $\{X_g\}_{g\in G}$ indexed by G. We denote it by $X_\bullet$ or simply X if there is no ambiguity. A G-graded object should be viewed as a functor from the discrete category of the underlying set of G to $\mathcal{C}$. Because of this, the category of G-graded objects is denoted by $\mathcal{C}^G$. In the case $\mathcal{C}$ has direct sums, a G-graded object $X_\bullet$ can be represented by the direct sum

$$X_\bullet = \bigoplus_{g\in G} X_g.$$

Convention By a ***graded object***, we mean a $\mathbb{Z}$-graded object.

A morphism between G-graded objects is merely a family of morphisms in $\mathcal{C}$ indexed by G. Thus we have

$$\mathrm{Hom}_{\mathcal{C}^G}(X,Y) = \prod_{g\in G} \mathrm{Hom}_{\mathcal{C}}(X_g, Y_g).$$

Convention By a ***homogeneous morphism***, we mean a morphism in $\mathcal{C}^G$, or the morphism in $\mathcal{C}$ representing it.

Let $X_\bullet$ be a G-graded object and g an element of G. The ***twisted object of degree*** g of $X_\bullet$ is the G-graded object $X(g)_\bullet$ defined as

$$X(g)_u := X_{g+u}.$$

Let $X_\bullet, Y_\bullet$ be two G-graded objects. A homogeneous morphism from $X_\bullet$ to $Y(g)_\bullet$ is called a ***twisted morphism of degree*** g from $X_\bullet$ to $Y_\bullet$. The G-graded abelian group defined by

$$\mathrm{Hom}(X,Y)_g := \mathrm{Hom}_{\mathcal{C}^G}(X, Y(g))$$

is called the ***G-graded Hom-group***.

From now on, let $\mathcal{C}$ be an *abelian tensor category*. For $X_\bullet, Y_\bullet$ two G-graded objects, their ***tensor product*** is defined by

$$(X\otimes Y)_g := \bigoplus_{u+v=g} (X_u \otimes Y_v).$$

In this way, $\mathcal{C}^G$ is again an abelian tensor category. Note that the limits and colimits in $\mathcal{C}^G$ are computed degree-wisely.

Now, further assume $\mathcal{C}$ is *closed*. Then $\mathcal{C}^G$ can be viewed as a $\mathcal{C}$-enriched category by setting the *Hom-object* as

$$\underline{\mathrm{Hom}}_{\mathcal{C}^G}(X, Y) := \prod_{g \in G} \underline{\mathrm{Hom}}(X_g, Y_g),$$

where $\underline{\mathrm{Hom}}(-,-)$ denotes the internal Hom-object in $\mathcal{C}$.

Similarly, we define the ***internal G-graded Hom-object*** by

$$\underline{\mathrm{Hom}}(X, Y)_g := \underline{\mathrm{Hom}}_{\mathcal{C}^G}(X, Y(g)).$$

Note that the internal G-graded Hom-objects are the *internal Hom-objects* in $\mathcal{C}^G$ since the functor $\underline{\mathrm{Hom}}(-, X)_\bullet$ is right adjoint to the functor $(- \otimes X)_\bullet$.

2 Chain complexes

Let $\mathcal{A}$ be a pointed category, meaning a category having *zero object*.

- A ***chain complex*** in $\mathcal{A}$ is a graded object $C_\bullet$ endowed with a twisted morphism $\partial\colon C_\bullet \to C_\bullet$ of degree -1 (called the ***differential*** or ***boundary operator***), such that $\partial \circ \partial = 0$.
- Dually, a ***cochain complex*** in $\mathcal{A}$ is a graded object $C_\bullet$ endowed with a twisted morphism $\partial\colon C_\bullet \to C_\bullet$ of degree 1 (called the ***differential*** or ***coboundary operator***), such that $\partial \circ \partial = 0$.

A ***chain map*** $f\colon C_\bullet \to D_\bullet$ is a homogeneous morphism such that the following diagrams commute.

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_n \arrow[r, "\partial_n"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Dually, a ***cochain map*** $f\colon C^\bullet \to D^\bullet$ is a homogeneous morphism such that the following diagrams commute.

$$\begin{tikzcd}
\cdots \arrow[r] & C^n \arrow[r, "\partial^n"] \arrow[d, "f_n"] & C^{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & \cdots \\
\cdots \arrow[r] & D^n \arrow[r, "\partial^n"] & D^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

The category of chain complexes (resp. cochain complexes) in $\mathcal{A}$ with chain maps (resp. cochain maps) between them is denoted by $\mathcal{C}\mathrm{h}_*(\mathcal{A})$ (resp. $\mathcal{C}\mathrm{h}^*(\mathcal{A})$).

Remark The categories $\mathcal{C}\mathrm{h}_*(\mathcal{A})$ and $\mathcal{C}\mathrm{h}^*(\mathcal{A})$ are also pointed. Furthermore, if $\mathcal{A}$ is additive, so are $\mathcal{C}\mathrm{h}_*(\mathcal{A})$ and $\mathcal{C}\mathrm{h}^*(\mathcal{A})$.

Convention When $\mathcal{A} = \mathcal{A}\mathrm{b}$, the category of abelian groups, we simply denote $\mathcal{C}\mathrm{h}_*(\mathcal{A}\mathrm{b})$ (resp. $\mathcal{C}\mathrm{h}^*(\mathcal{A}\mathrm{b})$) by $\mathcal{C}\mathrm{h}_*$ (resp. $\mathcal{C}\mathrm{h}^*$). When $\mathcal{A} = k\mathcal{M}\mathrm{od}$, the category of k-modules, we simply denote $\mathcal{C}\mathrm{h}_*(k\mathcal{M}\mathrm{od})$ (resp. $\mathcal{C}\mathrm{h}^*(k\mathcal{M}\mathrm{od})$) by $\mathcal{C}\mathrm{h}_*(k)$ (resp. $\mathcal{C}\mathrm{h}^*(k)$).

A chain complex $C_\bullet$ is said to be

- ***connective***, or ***nonnegatively graded***, if $C_n = 0$ for all $n < 0$;
- ***coconnective***, or ***nonpositively graded***, if $C_n = 0$ for all $n > 0$;
- ***bounded above*** if $C_n = 0$ for sufficiently large n;
- ***bounded below*** if $C_n = 0$ for sufficiently small n;
- ***bounded*** if it is both bounded above and bounded below.

Convention The full subcategory of $\mathcal{C}\mathrm{h}_*(\mathcal{A})$ spanned by connective (resp. coconnective, bounded above, bounded below, bounded) chain complexes is denoted by $\mathcal{C}\mathrm{h}_{\geqslant 0}(\mathcal{A})$ (resp. $\mathcal{C}\mathrm{h}_{\leqslant 0}(\mathcal{A})$, $\mathcal{C}\mathrm{h}_-(\mathcal{A})$, $\mathcal{C}\mathrm{h}_+(\mathcal{A})$, $\mathcal{C}\mathrm{h}_b(\mathcal{A})$).

Dually, a cochain complex $C^\bullet$ is said to be

- ***connective***, or ***nonpositively graded***, if $C^n = 0$ for all $n > 0$;
- ***coconnective***, or ***nonnegatively graded***, if $C^n = 0$ for all $n < 0$;
- ***bounded above*** if $C^n = 0$ for sufficiently large n;
- ***bounded below*** if $C^n = 0$ for sufficiently small n;
- ***bounded*** if it is both bounded above and bounded below.

Convention The full subcategory of $\mathcal{C}\mathrm{h}_*(\mathcal{A})$ spanned by connective (resp. coconnective, bounded above, bounded below, bounded) chain complexes is denoted by $\mathcal{C}\mathrm{h}^{\leqslant 0}(\mathcal{A})$ (resp. $\mathcal{C}\mathrm{h}^{\geqslant 0}(\mathcal{A})$, $\mathcal{C}\mathrm{h}^-(\mathcal{A})$, $\mathcal{C}\mathrm{h}^+(\mathcal{A})$, $\mathcal{C}\mathrm{h}^b(\mathcal{A})$).

Remark Any chain complex $C_\bullet$ becomes a cochain complex after reindex

$$C^n := C_{-n} \quad \partial^n := \partial_{-n+1}$$

and vice versa. Thus we have

$$\mathcal{C}\mathrm{h}_*(\mathcal{A}) \cong \mathcal{C}\mathrm{h}^*(\mathcal{A}).$$

In this sense, we may use the term ***complex*** to refer to both chain complexes and cochain complexes, term ***morphism of complexes*** to refer to both chain maps and cochain maps and use the notation $\mathcal{C}\mathrm{h}(\mathcal{A})$ to refer to both $\mathcal{C}\mathrm{h}_*(\mathcal{A})$ and $\mathcal{C}\mathrm{h}^*(\mathcal{A})$.

On the other hand, one can see that chain complexes in $\mathcal{A}$ are the same as cochain complexes in $\mathcal{A}^{\text{opp}}$, hence

$$\mathcal{C}\mathrm{h}_*(\mathcal{A})^{\text{opp}} = \mathcal{C}\mathrm{h}^*(\mathcal{A}^{\text{opp}}).$$

So, $\mathcal{C}\mathrm{h}(\mathcal{A}^{\text{opp}}) \cong \mathcal{C}\mathrm{h}(\mathcal{A})^{\text{opp}}$.

Restricting the reindexing to the full subcategories mentioned before, we have natural isomorphisms

$$\begin{aligned}
\mathcal{C}\mathrm{h}_{\geqslant 0}(\mathcal{A})^{\text{opp}} &= \mathcal{C}\mathrm{h}^{\geqslant 0}(\mathcal{A}^{\text{opp}}) \cong \mathcal{C}\mathrm{h}_{\leqslant 0}(\mathcal{A}^{\text{opp}}),\\
\mathcal{C}\mathrm{h}_{\leqslant 0}(\mathcal{A})^{\text{opp}} &= \mathcal{C}\mathrm{h}^{\leqslant 0}(\mathcal{A}^{\text{opp}}) \cong \mathcal{C}\mathrm{h}^{\geqslant 0}(\mathcal{A}^{\text{opp}}),\\
\mathcal{C}\mathrm{h}_{-}(\mathcal{A})^{\text{opp}} &= \mathcal{C}\mathrm{h}^{-}(\mathcal{A}^{\text{opp}}) \cong \mathcal{C}\mathrm{h}_{+}(\mathcal{A}^{\text{opp}}),\\
\mathcal{C}\mathrm{h}_{+}(\mathcal{A})^{\text{opp}} &= \mathcal{C}\mathrm{h}^{+}(\mathcal{A}^{\text{opp}}) \cong \mathcal{C}\mathrm{h}_{-}(\mathcal{A}^{\text{opp}}),\\
\mathcal{C}\mathrm{h}_{b}(\mathcal{A})^{\text{opp}} &= \mathcal{C}\mathrm{h}^{b}(\mathcal{A}^{\text{opp}}) \cong \mathcal{C}\mathrm{h}_{b}(\mathcal{A}^{\text{opp}}).
\end{aligned}$$

We can identify connective (resp. coconnective) chain complexes with connective (resp. coconnective) cochain complexes and call them *connective (resp. coconnective) complexes.* In practice, the term ***connective complex*** often refers to connective chain complexes and ***coconnective complex*** to coconnective cochain complexes.

Next, we can identify bounded above (resp. bounded below,) chain complexes with bounded below (resp. bounded above) cochain complexes and call them *bounded back (resp. bounded front) complexes.* In practice, the term ***bounded back complex*** often refers to bounded below cochain complexes and ***bounded front complex*** to bounded below chain complexes.

Finally, we can identify bounded chain complexes with bounded cochian complexes and call them simply ***bounded complexes***.

3 Homology objects

Let $C_\bullet$ be a chain complex (and dually let $C^\bullet$ be a cochain complex) in an abelian category $\mathcal{A}$.

- The n-th ***cycle object*** of $C_\bullet$ is $Z_n(C) := \operatorname{Ker} \partial_n$, whose elements are called ***n-cycles***.
- The n-th ***boundary object*** of $C_\bullet$ is $B_n(C) := \operatorname{Im} \partial_{n+1}$, whose elements are called ***n-boundaries***.
- The n-th ***cocycle object*** of $C^\bullet$ is $Z^n(C) := \operatorname{Ker} \partial^n$, whose elements are called ***n-cocycles***.
- The n-th ***coboundary object*** of $C^\bullet$ is $B^n(C) := \operatorname{Im} \partial^{n-1}$, whose elements are called ***n-coboundaries***.

Since $\partial \circ \partial = 0$, the inclusion $B_n(C) \hookrightarrow C_n$ factors through $Z_n(C)$. Likewise, the inclusion $B^n(C) \hookrightarrow C^n$ factors through $Z^n(C)$.

- The cokernel of the resulted inclusion $B_n(C) \hookrightarrow Z_n(C)$ is called the n-th ***homology object*** of $C_\bullet$ and denoted by $H_n(C)$. The elements of $H_n(C)$ are called ***homology classes***.
- The cokernel of the resulted inclusion $B^n(C) \hookrightarrow Z^n(C)$ is called the n-th ***cohomology object*** of $C^\bullet$ and denoted by $H^n(C)$. The elements of $H^n(C)$ are called ***cohomology classes***.

Any chain map $f\colon A_\bullet \to B_\bullet$ induces a homogeneous morphism

$$H(f)\colon H_\bullet(A) \to H_\bullet(B).$$

Obviously, if f is an isomorphism, then so is $H(f)$. But the converse may not be true. A chain map f is called a ***quasi-isomorphism*** if $H(f)$ is an isomorphism. A chain complex $C_\bullet$ is said to be ***acyclic*** if it is *quasi-isomorphic* to 0, equivalently, $H_\bullet(C) = 0$.

4 Theorem *Let $\mathcal{A}$ be an abelian tensor category. Then*

1. *the finite limits and colimits in* $\mathrm{Ch}(\mathcal{A})$ *are computed degree-wisely;*
2. $\mathrm{Ch}(\mathcal{A})$ *is an abelian tensor category;*
3. *for each integer n, we have additive functors $B_n, Z_n, H_n\colon \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$.*

Moreover, if $\mathcal{A}$ is a cocomplete abelian tensor category, then

4. *the colimits in* $\mathrm{Ch}(\mathcal{A})$ *are computed degree-wisely;*
5. $\mathrm{Ch}(\mathcal{A})$ *is a cocomplete abelian tensor category.*

Furthermore, if $\mathcal{A}$ is closed, then

6. $\mathrm{Ch}(\mathcal{A})$ *is also closed.*

Finally, if $\mathcal{A}$ satisfies AB5, that means any filtered colimit of exact sequences is again exact, then

7. $\mathrm{Ch}(\mathcal{A})$ *satisfies AB5;*
8. *the functors B_n, Z_n, H_n preserve filtered colimits.*

Proof: We only show how to construct the tensor products and the internal Hom objects and then prove 8. The rests are straightforward.

For $C_\bullet, D_\bullet$ two chain complexes, we should give a natural boundary operator ∂ on $(C \otimes D)_\bullet$ (the resulted chain complex is called the ***Koszul product*** of $C_\bullet$ and $D_\bullet$). By its construction, we only need to define the following morphisms

$$C_p \otimes D_q \xrightarrow{\partial^{(1)}_{p,q}} C_{p-1} \otimes D_q, \qquad C_p \otimes D_q \xrightarrow{\partial^{(2)}_{p,q}} C_p \otimes D_{q-1}.$$

Note that the condition $\partial \circ \partial = 0$ requires that the following two morphisms must be negative to each other.

$$C_p \otimes D_q \xrightarrow{\partial^{(2)}_{p,q}} C_p \otimes D_{q-1} \xrightarrow{\partial^{(1)}_{p,q-1}} C_{p-1} \otimes D_{q-1},$$
$$C_p \otimes D_q \xrightarrow{\partial^{(1)}_{p,q}} C_{p-1} \otimes D_q \xrightarrow{\partial^{(2)}_{p-1,q}} C_{p-1} \otimes D_{q-1}.$$

Hence we define

$$\partial^{(1)}_{p,q} := \partial_p \otimes \mathrm{id}_{D_q}, \qquad \partial^{(2)}_{p,q} := (-1)^p \,\mathrm{id}_{C_p} \otimes \partial_q.$$

Next, we should give a natural boundary operator ∂ on $\underline{\mathrm{Hom}}(C, D)_\bullet$ (the resulted chain complex is called the ***Koszul Hom-complex***). By the construction, we only need to define the following morphisms

$$\underline{\mathrm{Hom}}(C_{p-1}, D_{q-1}) \xrightarrow{\partial^{(1)}_{p,q-1}} \underline{\mathrm{Hom}}(C_p, D_{q-1}),$$
$$\underline{\mathrm{Hom}}(C_p, D_q) \xrightarrow{\partial^{(2)}_{p,q-1}} \underline{\mathrm{Hom}}(C_p, D_{q-1}).$$

Note that the condition $\partial \circ \partial = 0$ requires that the following two morphisms must be negative to each other.

$$\underline{\mathrm{Hom}}(C_{p-1}, D_q) \xrightarrow{\partial^{(2)}_{p-1,q-1}} \underline{\mathrm{Hom}}(C_{p-1}, D_{q-1}) \xrightarrow{\partial^{(1)}_{p,q-1}} \underline{\mathrm{Hom}}(C_p, D_{q-1}),$$
$$\underline{\mathrm{Hom}}(C_{p-1}, D_q) \xrightarrow{\partial^{(1)}_{p,q}} \underline{\mathrm{Hom}}(C_p, D_q) \xrightarrow{\partial^{(2)}_{p,q-1}} \underline{\mathrm{Hom}}(C_p, D_{q-1}).$$

Hence we define

$$\partial^{(1)}_{p,q} := (-1)^{q-p}\, \underline{\mathrm{Hom}}(\partial_p, D_q), \qquad \partial^{(2)}_{p,q-1} := \underline{\mathrm{Hom}}(C_p, \partial_q).$$

We now show that

$$\mathrm{Hom}_{\mathcal{Ch}(\mathcal{A})}(A \otimes B, C) \cong \mathrm{Hom}_{\mathcal{Ch}(\mathcal{A})}(A, \underline{\mathrm{Hom}}(B, C)).$$

For any $p, q \in \mathbb{Z}$, let $f_{p,q} \colon A_p \otimes B_q \to C_{p+q}$ and $g_{p,q} \colon A_p \to \underline{\mathrm{Hom}}(B_q, C_{p+q})$ be any pair of *adjoint transposes*, i.e. a pair corresponded under the natural isomorphism

$$\mathrm{Hom}(A_p \otimes B_q, C_{p+q}) \cong \mathrm{Hom}(A_p, \underline{\mathrm{Hom}}(B_q, C_{p+q})).$$

Then $f_{p,q}$ are compatible with the boundary operators of $(A \otimes B)_\bullet$ and $C_\bullet$ if and only if $g_{p,q}$ are compatible with the boundary operators of $A_\bullet$ and $[B, C]_\bullet$. Let $f_\bullet \colon (A \otimes B)_\bullet \to C_\bullet$ and $g_\bullet \colon A_\bullet \to [B, C]_\bullet$ be the morphisms defined by $f_{p,q}$ and $g_{p,q}$ respectively. Then f is a chain map if and only if so is g. Hence the conclusion follows.

Finally, since filtered colimits commute with finite limits and all colimits, it follows that filtered colimits commute with the constructions of the functors B_m, Z_n and H_n. $\square$

Subcategories of special complexes

Let $\mathcal{A}$ be an abelian tensor category. We have seen that so is $\mathcal{C}\mathrm{h}(\mathcal{A})$. Moreover, since the full subcategories $\mathcal{C}\mathrm{h}_-(\mathcal{A})$, $\mathcal{C}\mathrm{h}_+(\mathcal{A})$, $\mathcal{C}\mathrm{h}_b(\mathcal{A})$ are closed under finite limits and colimits and Koszul product, they are also abelian tensor categories. As for the full subcategories $\mathcal{C}\mathrm{h}_{\geqslant 0}(\mathcal{A})$ and $\mathcal{C}\mathrm{h}_{\leqslant 0}(\mathcal{A})$, we use the following proposition.

5 Proposition *Let $\mathcal{A}$ be an abelian category. Then*

1. *the inclusion $\mathcal{C}\mathrm{h}_{\geqslant 0}(\mathcal{A}) \hookrightarrow \mathcal{C}\mathrm{h}(\mathcal{A})$ admits a left adjoint $\mathrm{sk}_{\geqslant 0}$ and a right adjoint $\tau_{\geqslant 0}$ and hence is exact;*

2. *the inclusion $\mathcal{C}\mathrm{h}_{\leqslant 0}(\mathcal{A}) \hookrightarrow \mathcal{C}\mathrm{h}(\mathcal{A})$ admits a left adjoint $\mathrm{sk}_{\leqslant 0}$ and a right adjoint $\tau_{\leqslant 0}$ and hence is exact.*

In particular, $\mathcal{C}\mathrm{h}_{\geqslant 0}(\mathcal{A})$ and $\mathcal{C}\mathrm{h}_{\leqslant 0}(\mathcal{A})$ are abelian categories.

Proof: The functors $\mathrm{sk}_{\geqslant 0}$ and $\tau_{\geqslant 0}$ are defined as follows.

$$\mathrm{sk}_{\geqslant 0}(C)_n = \begin{cases} C_n & n \geqslant 0, \\ 0 & n < 0; \end{cases}$$

$$\tau_{\geqslant 0}(C)_n = \begin{cases} C_n & n > 0, \\ Z_0(C) & n = 0, \\ 0 & n < 0. \end{cases}$$

The functors $\mathrm{sk}_{\leqslant 0}$ and $\tau_{\leqslant 0}$ are defined similarly. □

Remark Note that this lemma shows that $\tau_{\geqslant 0}$ is a lax functor. It is usually called the ***0-th truncation functor***. The general ***n-th truncation functor*** is defined similarly.

§ 2 Higher structures on $\mathcal{C}\mathrm{h}$

1 Chain homotopies

A ***chain homotopy*** $\psi\colon f \Rightarrow g$ between chain maps $f, g\colon C_\bullet \to D_\bullet$ is a twisted morphism $\psi\colon C_\bullet \to D_\bullet$ of degree 1 such that

$$\psi \circ \partial + \partial \circ \psi = f - g.$$

Two chain maps $f, g\colon C_\bullet \rightrightarrows D_\bullet$ are said to be ***chain homotopic***, denoted by $f \simeq g$, if there exists a chain homotopy $\psi\colon f \Rightarrow g$ if and only if $f - g \simeq 0$.

A chain map $f\colon C_\bullet \to D_\bullet$ is called a ***chain homotopy equivalence*** if there exists a chain map $g\colon D_\bullet \to C_\bullet$ such that $g \circ f \simeq \mathrm{id}_C$ and $f \circ g \simeq \mathrm{id}_D$.

Two chain complexes $C_\bullet, D_\bullet$ are said to be ***chain homotopy equivalent*** if there exists a chain homotopy equivalence $f\colon C_\bullet \to D_\bullet$.

The ***vertical composition*** of two chain homotopies is then the sum of them. The ***horizontal composition*** of two chain homotopies $\psi\colon f \Rightarrow g$ and $\psi'\colon f' \Rightarrow g'$ is defined as

$$\psi' * \psi := f' \circ \psi + \psi' \circ g.$$

One should be careful since this definition releases on the sources and targets of the chain homotopies.

Remark Note that for chain homotopies $\psi\colon f \Rightarrow g, \psi'\colon f' \Rightarrow g', \phi\colon g \Rightarrow h$ and $\phi'\colon g' \Rightarrow h'$, we have

$$\begin{aligned}(\psi + \phi) * (\psi' + \phi') &= f' \circ \psi + f' \circ \phi + \psi' \circ h + \phi' \circ h,\\ (\psi * \psi') + (\phi * \phi') &= f' \circ \psi + \psi' \circ g + g' \circ \phi + \phi' \circ h.\end{aligned}$$

Hence the interchange law does not hold on the nose. However, the difference of them is

$$(f' - g') \circ \phi - \psi' \circ (g - h),$$

which is $\psi' * \phi$ if we treat ψ' as from $f' - g'$ to 0, ϕ as from 0 to $g - h$ and $\psi' * \phi$ as from 0 to 0.

In this way, $\mathcal{C}\mathrm{h}(\mathcal{A})$ is already a 2-category although not a strict one. But we can do more.

Let $I_\bullet$ be the following chain complex (called the ***standard interval***)

$$\cdots \longrightarrow \mathbf{1} \xrightarrow{(\mathrm{id},-\mathrm{id})} \mathbf{1} \oplus \mathbf{1} \longrightarrow \cdots$$

where $\mathbf{1} \oplus \mathbf{1}$ is of degree 0.

Note that the Koszul product $(I \otimes C)_\bullet$ is the following chain complex

$$\cdots \longrightarrow C_{n-1} \oplus C_n \oplus C_n \xrightarrow{\partial_n} C_{n-2} \oplus C_{n-1} \oplus C_{n-1} \longrightarrow \cdots$$

where the boundary operator is induced by the follows.

$$\begin{tikzcd} C_{n-1} \arrow[r, "-\partial_{n-1}"] \arrow[dr, "(\mathrm{id},-\mathrm{id})"] & C_{n-2} \\ C_n \oplus C_n \arrow[r, "(\partial_n,\partial_n)"] & C_{n-1} \oplus C_{n-1} \end{tikzcd}$$

Therefore, there are canonical inclusions

$$C_\bullet \rightrightarrows (I \otimes C)_\bullet \quad \text{and} \quad C(-1)_\bullet \to (I \otimes C)_\bullet$$

induced by the inclusions

$$C_n \rightrightarrows C_{n-1} \oplus C_n \oplus C_n \quad \text{and} \quad C_{n-1} \to C_{n-1} \oplus C_n \oplus C_n.$$

Convention For convenience, we use the matrix notation such as

$$\begin{pmatrix} f & g & h \\ f' & g' & h' \end{pmatrix}$$

to denote the morphism $A \oplus B \oplus C \to X \oplus Y$ induced by

$$f \colon A \to X, \quad g \colon B \to X, \quad h \colon C \to X,$$

and

$$f' \colon A \to Y, \quad g' \colon B \to Y, \quad h' \colon C \to Y.$$

We also use *element notations.* In this setting, the above operator reads

$$(a, b, c) \longmapsto (f(a) + g(b) + h(c), f'(a) + g'(b) + h'(c)).$$

1.1 Lemma *A chain homotopy $\psi \colon f \Rightarrow g$ is equivalently a chain map*

$$(\psi, f, g) \colon (I \otimes C)_\bullet \longrightarrow D_\bullet$$

fitting the commuting diagram

$$\begin{tikzcd} (I \otimes C)_\bullet \arrow[dr, "{(\psi,f,g)}"] & C_\bullet \arrow[l] \arrow[d, "g"] \\ C_\bullet \arrow[u] \arrow[r, "f"] & D_\bullet \end{tikzcd}$$

where the unlabeled morphisms are the canonical inclusions.

Proof: For $\psi \colon f \Rightarrow g$ a chain homotopy, the chain map (ψ, f, g) is given by

$$C_{n-1} \oplus C_n \oplus C_n \xrightarrow{(\psi_{n-1}, f_n, g_n)} D_n.$$

To show (ψ, f, g) is a chain map, it suffices to show

$$\partial_{n+1} \circ (\psi, f, g)_{n+1} = (\psi, f, g)_n \circ \partial_{n+1},$$

which is equivalent to show the following equalities:

$$\begin{aligned} \partial_{n+1} \circ \psi_n &= -\psi_{n-1} \circ \partial_n + f_n - g_n, \\ \partial_{n+1} \circ f_{n+1} &= \partial_n \circ f_n, \\ \partial_{n+1} \circ g_{n+1} &= \partial_n \circ g_n, \end{aligned}$$

which follow from that f, g are chain maps and ψ is a chain homotopy between them.

Conversely, let Ψ be a chain map $(I \otimes C)_\bullet \to D_\bullet$ fitting the commutative diagram. Then the restriction of Ψ along the inclusion $C(-1)_\bullet \hookrightarrow (I \otimes C)_\bullet$ defines a chain homotopy between f and g. $\square$

Remark A chain homotopy ϕ from a chain map $f\colon C_\bullet \to D_\bullet$ to itself is merely a twisted morphism of degree 1 such that

$$\partial \circ \phi + \phi \circ \partial = 0,$$

which is equivalent to say that ϕ is a chain map from the chain complex $(C(-1)_\bullet, -\partial)$ to $D_\bullet$.

Note that the chain map $(\phi, f, f)\colon (I \otimes C)_\bullet \to D_\bullet$ is uniquely determined by the chain maps f and the above chain map. So, for chain endo-homotopies, we may use the above chain map to replace (ϕ, f, f).

Using the above lemma, one can furthermore define *chain homotopies between chain homotopies*. The existence of higher homotopies indicates that there should be an ∞-category structure on $\mathcal{C}\mathrm{h}(\mathcal{A})$.

Remark The chain complex $(C(-1)_\bullet, -\partial)$ is called the ***suspension*** of $C_\bullet$ and denoted by $(\Sigma C)_\bullet$. One should note that the ***suspension*** of a cochain complex $C^\bullet$ is $(C(1)^\bullet, -\partial)$.

2 Encoding chain homotopies into a complex

Let $C_\bullet, D_\bullet$ be two chain complexes. The Hom-space $\mathcal{H}\mathrm{om}_{\mathcal{C}\mathrm{h}(\mathcal{A})}(C, D)$ is an ∞-groupoid, which by the philosophy should be uniquely determined by its homotopy groups. We now construct a chain complex, which encodes the data of chain homotopies and provides the homotopy groups.

A natural candidate is *graded Hom-group* $\mathrm{Hom}(C, D)_\bullet$. First, we have

$$Z_0(\mathrm{Hom}(C, D)) = \mathrm{Hom}_{\mathcal{C}\mathrm{h}(\mathcal{A})}(C, D)$$

since the condition of a graded morphism $f\colon C_\bullet \to D_\bullet$ to be a 0-cycle in $\mathrm{Hom}(C, D)_\bullet$ is

$$\partial_n \circ f_n - f_{n-1} \circ \partial_n = 0,$$

which is equivalent to say that f is a chain map. Moreover, we have

$$H_0(\mathrm{Hom}(C, D)) = \pi_0\, \mathcal{H}\mathrm{om}_{\mathcal{C}\mathrm{h}(\mathcal{A})}(C, D)$$

since the condition of a homogeneous morphism $f\colon C_\bullet \to D_\bullet$ being a 0-boundary in $\mathrm{Hom}(C, D)_\bullet$ is that there exists a twisted morphism $\psi\colon C_\bullet \to D_\bullet$ of degree 1 such that

$$\partial_{n+1} \circ \psi_n + \psi_{n-1} \circ \partial_n = f_n,$$

which is equivalent to say that $f \simeq 0$.

In general, a n-homotopy from a n-morphism to itself in $\mathcal{C}\mathrm{h}(\mathcal{A})$ can be represented as a chain map from $C_\bullet$ to the chain complex $\Sigma^n D$. Thus all n-endohomotopies in $\mathcal{H}\mathrm{om}_{\mathcal{C}\mathrm{h}(\mathcal{A})}(C, D)$ can be encoded into the group $Z_n(\mathrm{Hom}(C, D))$. Moreover, we have

$$H_n(\mathrm{Hom}(C, D)) = \pi_n(\mathcal{H}\mathrm{om}_{\mathcal{C}\mathrm{h}(\mathcal{A})}(C, D))$$

since the condition of a twisted morphism $f\colon C_\bullet \to D_\bullet$ of degree n to be a n-boundary in $\mathrm{Hom}(C, D)_\bullet$ is that there exists a twisted morphism $\psi\colon C_\bullet \to D_\bullet$ of degree $n+1$ such that

$$(-1)^n \partial_{d+n+1} \circ \psi_d + \psi_{d-1} \circ \partial_d = f_d,$$

which is equivalent to say that $f \simeq 0$.

From the above, we see that the homotopy data of $\mathcal{H}\mathrm{om}_{\mathcal{C}\mathrm{h}(\mathcal{A})}(C, D)$ are encoded in the chain complex

$$\tau_{\geqslant 0}(\mathrm{Hom}(C, D))_n = \begin{cases} \mathrm{Hom}(C, D)_n & n > 0, \\ Z_0(\mathrm{Hom}(C, D)) & n = 0, \\ 0 & n < 0. \end{cases}$$

We will call it the ***(∞-categorical) truncated Hom complex***. As this complex is obtained by applying the 0-truncated functor $\tau_{\geqslant 0}$, we also use the term ***(∞-categorical) Hom complex*** to refer to the complex $\mathrm{Hom}(C, D)_\bullet$. The notation $\mathcal{H}\mathrm{om}_{\mathcal{C}\mathrm{h}(\mathcal{A})}(C, D)$ or $\mathcal{H}\mathrm{om}(C, D)$ can refer to either of them, depending on the context.

We also denote $H_n(\mathcal{H}\mathrm{om}_{\mathcal{C}\mathrm{h}(\mathcal{A})}(C, D))$ by $H_n(C, D)$, called the ***intrinsic homology*** of C with coefficients in D.

Remark The above discussion also work for cochain complexes. Therefore, $\mathcal{C}\mathrm{h}(\mathcal{A})$ can be viewed as enriched over $\mathcal{C}\mathrm{h}_{\geqslant 0}$ when view complexes as chain complexes, enriched over $\mathcal{C}\mathrm{h}^{\geqslant 0}$ when view complexes as cochain complexes, and enriched over $\mathcal{C}\mathrm{h}$ in any case.

Remark We also have the notion of ***intrinsic cohomology*** $H^n(C, D)$, which is $H^n(\mathcal{H}\mathrm{om}_{\mathcal{C}\mathrm{h}(\mathcal{A})}(C, D))$ where $\mathcal{H}\mathrm{om}_{\mathcal{C}\mathrm{h}(\mathcal{A})}(C, D)$ is viewed as a cochain complex. Therefore, we have

$$H^n(C, D) = H_{-n}(C, D).$$

3 Dg-categories

The fact that the higher homotopy data can be encoded into the Hom complex leads to a model of ∞-category theory, namely the *dg-category theory*. Here the prefix "dg" is short for "differential graded".

A ***dg-category*** is a category enriched over $\mathcal{C}\mathrm{h}$. (Of course, one can slightly generalize this notion by replacing $\mathcal{C}\mathrm{h}$ with $\mathcal{C}\mathrm{h}(k)$). A *morphism*

$f: A \to B$ between two objects in a dg-category is a 0-cycle in $\mathcal{H}\mathrm{om}(A, B)$. Two morphisms are said to be *homotopic* if their difference is a 0-boundary.

Any dg-category $\mathcal{C}$ admits a category $\mathcal{C}_0$ (its ***underlying category***) obtained by applying the *change of base categories* $Z_0: \mathcal{C}\mathrm{h} \to \mathcal{A}\mathrm{b}$ and another category $\mathrm{h}\mathcal{C}$ (its ***homotopy category***) obtained by applying the *change of base categories* $H_0: \mathcal{C}\mathrm{h} \to \mathcal{A}\mathrm{b}$.

3.1 Example For any additive category $\mathcal{A}$, $\mathcal{C}\mathrm{h}(\mathcal{A})$ is a dg-category, whose Hom spaces are given by the Hom complexes. The underlying category of $\mathcal{C}\mathrm{h}(\mathcal{A})$ is the classical category of complexes. The homotopy category of $\mathcal{C}\mathrm{h}(\mathcal{A})$ is usually denoted by $\mathcal{K}(\mathcal{A})$. The similar conventions apply to the subcategories $\mathcal{C}\mathrm{h}^{\geqslant 0}(\mathcal{A})$, $\mathcal{C}\mathrm{h}_{\leqslant 0}(\mathcal{A})$, $\mathcal{C}\mathrm{h}^{\leqslant 0}(\mathcal{A})$, $\mathcal{C}\mathrm{h}_{\geqslant 0}(\mathcal{A})$, $\mathcal{C}\mathrm{h}^{-}(\mathcal{A})$, $\mathcal{C}\mathrm{h}_{+}(\mathcal{A})$, $\mathcal{C}\mathrm{h}^{+}(\mathcal{A})$, $\mathcal{C}\mathrm{h}_{-}(\mathcal{A})$, $\mathcal{C}\mathrm{h}^{b}(\mathcal{A})$, $\mathcal{C}\mathrm{h}_{b}(\mathcal{A})$.

Remark Note that the Hom spaces which can be represented by a complex is very special: the homotopy groups of such a Hom space is independent of the choice of base point. Therefore, one should not expect the dg-category theory is a full version of ∞-category theory. Instead, it is an ∞-version of additive categories.

A ***dg-functor*** between dg-categories $F: \mathcal{C} \to \mathcal{D}$ is an enriched functor. Equivalently, a dg-functor F consists of the following data

- a mapping between objects $\mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$,
- a family of chain maps $F_{A,B}: \mathcal{H}\mathrm{om}_{\mathcal{C}}(A, B) \to \mathcal{H}\mathrm{om}_{\mathcal{D}}(F(A), F(B))$, functoral in $A, B \in \mathrm{ob}\,\mathcal{C}$,

satisfying the obvious associative and unitary laws.

In this way, we have a category $\mathrm{dg}\mathcal{C}\mathrm{at}$. Using the canonical monoidal constructions on enriched categories over a symmetric monoidal category, we see $\mathrm{dg}\mathcal{C}\mathrm{at}$ is moreover a closed symmetric monoidal category. In this way, we obtain the notion of *functor dg-categories* between dg-categories and in particular, the equivalences in the functor dg-category induce the equivalences between dg-categories. However, this is not a correct notion of equivalences of ∞-categories.

Note that the ∞-groupoids should be determined by and only by its homotopy data. We give the following definitions. A dg-functor $F: \mathcal{C} \to \mathcal{D}$ is said to be

- ***quasi-fully faithful***, if for each pair of objects $A, B \in \mathcal{C}$, the canonical morphism

$$\mathcal{H}\mathrm{om}_{\mathcal{C}}(A, B) \to \mathcal{H}\mathrm{om}_{\mathcal{D}}(F(A), F(B))$$

is a quasi-isomorphism;

- ***essentially surjective***, if its *homotopy functor* $\mathrm{h}F\colon \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is essentially surjective;
- a ***quasi-equivalence***, if it is both quasi-fully faithful and essentially surjective.

Motivated by this, we may redefine dg-categories as enriched categories over such a base category, in which the weak equivalences are generated by the quasi-isomorphisms, not chain homotopy equivalences. Such a category is the *derived category* $\mathcal{D}(\mathcal{A}\mathrm{b})$ of $\mathcal{A}\mathrm{b}$.

We will continue this discussion later. Before that, let's introduce some results on homotopy fibers.

§ 3 Homotopy fibers

1 Suspension and looping

Recall that for a chain complex $C_\bullet$, its ***suspension*** $(\Sigma C)_\bullet$ is the chain complex

$$(\Sigma C)_n := C_{n-1}, \qquad \partial_n := -\partial_{n-1}^C.$$

Dually, the ***looping*** $(\Omega C)_\bullet$ of $C_\bullet$ is the chain complex

$$(\Omega C)_n := C_{n+1}, \qquad \partial_n := -\partial_{n+1}^C.$$

Moreover, taking suspension extends to be a functor which maps each chain map f to the chain map

$$(\Sigma f)_n := f_{n-1}.$$

Likewise, taking looping extends to be a functor which maps each chain map f to the chain map

$$(\Omega f)_n := f_{n+1}.$$

It is easy to see that

$$H_n(\Omega C) = H_{n+1}(C), \qquad H_n(\Sigma C) = H_{n-1}(C).$$

Moreover, we can see (through direct computation or Proposition 5) that for any chain complex $X_\bullet$,

$$\begin{aligned} \mathcal{H}\mathrm{om}(X, \Omega C) &\cong \Omega\, \mathcal{H}\mathrm{om}(X, C), & \mathcal{H}\mathrm{om}(X, \Sigma C) &\cong \Sigma\, \mathcal{H}\mathrm{om}(X, C); \\ \mathcal{H}\mathrm{om}(\Omega C, X) &\cong \Sigma\, \mathcal{H}\mathrm{om}(C, X), & \mathcal{H}\mathrm{om}(\Sigma C, X) &\cong \Omega\, \mathcal{H}\mathrm{om}(C, X). \end{aligned}$$

Moreover, we have

$$\begin{aligned} H_n(X, \Omega C) &\cong H_{n+1}(X, C), & H_n(X, \Sigma C) &\cong H_{n-1}(X, C); \\ H_n(\Omega C, X) &\cong H_{n-1}(C, X), & H_n(\Sigma C, X) &\cong H_{n+1}(C, X). \end{aligned}$$

In particular, $H^n(X, C) = H_{-n}(X, C) = H_0(X, \Sigma^n C)$, which satisfies our general philosophy.

Remark Note that the ***suspension*** of a cochain complex $C^\bullet$ is the cochain complex
$$(\Sigma C)^n := C^{n+1}, \qquad \partial^n := -\partial_C^{n+1}.$$
Dually, the ***looping*** $(\Omega C)^\bullet$ is the cochain complex
$$(\Omega C)^n := C^{n-1}, \qquad \partial^n := -\partial_C^{n-1}.$$
We also have
$$\begin{aligned} H^n(\Omega C) &= H^{n-1}(C), & H^n(\Sigma C) &= H^{n+1}(C);\\ H^n(X, \Omega C) &\cong H^{n-1}(X, C), & H^n(X, \Sigma C) &\cong H^{n+1}(X, C);\\ H^n(\Omega C, X) &\cong H^{n+1}(C, X), & H^n(\Sigma C, X) &\cong H^{n-1}(C, X). \end{aligned}$$

By the general philosophy of ∞-category theory, the suspension should be the cofiber of zero morphism $C \to 0$, while the looping should be the fiber of zero morphism $0 \to C$. So, to see the suspension (resp. looping) defined above satisfies the general philosophy, it suffices to define the correct notion of homotopy fibers and cofibers. We will show this in Proposition 3.

2 Homotopy fibers and cofibers

Let $f\colon A_\bullet \to B_\bullet$ be a chain map. We now compute its *homotopy fibers* in $\mathcal{C}\mathrm{h}(\mathcal{A})$.

First, a *homotopy annihilation* of f consists of the following data

- a chain map $a\colon X_\bullet \to A_\bullet$;
- a chain homotopy $\alpha\colon 0 \Rightarrow f \circ a$.

Note that the α can be interpreted as a twisted morphism $\alpha\colon X_\bullet \to B(1)_\bullet$ such that
$$\partial_{n+1} \circ \alpha_n + \alpha_{n-1} \circ \partial_n = -f_n \circ a_n.$$
Therefore any homotopy annihilation of f defines a twisted morphism
$$(a, \alpha)\colon X_\bullet \to (A \oplus B(1))_\bullet$$
such that
$$(a_{n-1}, \alpha_{n-1}) \circ \partial_n = (\partial_n \circ a_n, -f_n \circ a_n - \partial_{n+1} \circ \alpha_n).$$
Thus, we may consider the chain complex
$$\mathrm{Fib}(f)_n := A_n \oplus B_{n+1}$$
with the boundary operator
$$\partial_n := \begin{pmatrix} \partial_n & 0 \\ -f_n & -\partial_{n+1} \end{pmatrix}.$$

In element notations, the boundary operator reads

$$\partial(a,b) = (\partial(a), -f(a) - \partial(b)).$$

This is really a boundary operator since

$$\begin{aligned}\partial^2(a,b) &= \partial(\partial(a), -f(a) - \partial(b)) \\ &= (\partial^2(a), -f\partial(a) + \partial f(a) + \partial^2(b)).\end{aligned}$$

We write $\mathrm{fib}(f)\colon \mathrm{Fib}(f)_\bullet \to A_\bullet$ for the evident projection. Then the other evident projection $\mathrm{pr}\colon \mathrm{Fib}(f)_\bullet \to B(1)_\bullet$ is a chain homotopy $0 \Rightarrow f \circ \mathrm{fib}(f)$:

$$(\partial \circ \mathrm{pr} + \mathrm{pr} \circ \partial)(a,b) = \partial(b) - f(a) - \partial(b) = -f(a).$$

We claim that $(\mathrm{Fib}(f)_\bullet, \mathrm{fib}(f))$ is the ***homotopy fiber*** of f. As we have seen that for any homotopy annihilation (X, a, α) of f, there is already a chain map $(a,\alpha)\colon X_\bullet \to \mathrm{Fib}(f)_\bullet$ making the following diagram commutes

$$\begin{tikzcd} X_\bullet \arrow[r, "{(a,\alpha)}"] \arrow[dr, "a"'] & \mathrm{Fib}(f)_\bullet \arrow[d, "\mathrm{fib}(f)"] \\ & A \end{tikzcd}$$

and the following homotopies coincide.

$$\begin{tikzcd} X_\bullet \arrow[r, "{(a,\alpha)}"] & \mathrm{Fib}(f)_\bullet \arrow[r, bend left, "0"] \arrow[r, phantom, "\Downarrow\mathrm{pr}"] \arrow[r] & B \end{tikzcd} \quad = \quad \begin{tikzcd} X_\bullet \arrow[r, bend left, "0"] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[r, "f\circ a"'] & B \end{tikzcd}$$

For $\phi\colon X_\bullet \to \mathrm{Fib}(f)_\bullet$ a chain map making the following diagram commutes

$$\begin{tikzcd} X_\bullet \arrow[r, "\phi"] \arrow[dr, "a"'] & \mathrm{Fib}(f)_\bullet \arrow[d, "\mathrm{fib}(f)"] \\ & A \end{tikzcd}$$

up to a homotopy $\Phi_1\colon a \Rightarrow \mathrm{fib}(f) \circ \phi$, and the following homotopies coincide,

$$\begin{tikzcd} X_\bullet \arrow[r, "\phi"] \arrow[rr, bend right, "a"'] & \mathrm{Fib}(f)_\bullet \arrow[d, phantom, "\Downarrow\Phi_1"] \arrow[r, "\mathrm{fib}(f)"] \arrow[rr, bend left, "0"] & A \arrow[l, phantom, "\Downarrow\mathrm{pr}"] \arrow[r, "f"] & B \end{tikzcd} \quad \simeq \quad \begin{tikzcd} X_\bullet \arrow[r, bend left, "0"] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[r, "f\circ a"'] & B \end{tikzcd}$$

up to a homotopy Φ_2, we have

$$\partial_A \circ \Phi_1 + \Phi_1 \circ \partial_X = a - \mathrm{fib}(f) \circ \phi,$$
$$-\partial_B \circ \Phi_2 + \Phi_2 \circ \partial_X = \alpha - \mathrm{pr} \circ \phi + f \circ \Phi_1,$$

which imply that (Φ_1, Φ_2) is a chain homotopy from (a, α) to ϕ.

Similar discussion shows that the ***homotopy cofiber*** of f is the chain complex

$$\mathrm{Cofib}(f)_n := A_{n-1} \oplus B_n$$

with the boundary operator

$$\partial_n := \begin{pmatrix} -\partial_{n-1} & 0 \\ -f_{n-1} & \partial_n \end{pmatrix}$$

In element notations, the boundary operator reads

$$\partial(a, b) = (-\partial(a), -f(a) + \partial(b)).$$

We write $\mathrm{cofib}(f)\colon B_\bullet \to \mathrm{Cofib}(f)_\bullet$ for the evident inclusion.

Remark Let $f\colon A^\bullet \to B^\bullet$ be a cochain map. One can see from above that

1. the ***homotopy fiber*** of f is the cochain complex

   $$\mathrm{Fib}(f)^n := A^n \oplus B^{n-1}$$

   with the coboundary operator

   $$\partial_n := \begin{pmatrix} \partial_n & 0 \\ -f_n & -\partial_{n+1} \end{pmatrix}.$$

   In element notations, the boundary operator reads

   $$\partial(a, b) = (\partial(a), -f(a) - \partial(b)).$$

   We write $\mathrm{fib}(f)\colon \mathrm{Fib}(f)^\bullet \to A^\bullet$ for the evident projection.

2. the ***homotopy cofiber*** of f is the cochain complex

   $$\mathrm{Cofib}(f)^n := A^{n+1} \oplus B^n$$

   with the boundary operator

   $$\partial^n := \begin{pmatrix} -\partial^{n+1} & 0 \\ -f^{n+1} & \partial^n \end{pmatrix}$$

   In element notations, the boundary operator reads

   $$\partial(a, b) = (-\partial(a), -f(a) + \partial(b)).$$

   We write $\mathrm{cofib}(f)\colon B^\bullet \to \mathrm{Cofib}(f)^\bullet$ for the evident inclusion.

It is now clear that

3 Proposition *Let C be a complex. Then*

$$\Omega C \cong \mathrm{Fib}(0 \to C), \qquad \Sigma C \cong \mathrm{Cofib}(C \to 0).$$

Fiber and cofiber sequences

A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is called a ***fiber sequence*** (resp. ***cofiber sequence***) if A is homotopy equivalent to the homotopy fiber of g (resp. C is homotopy equivalent to the homotopy cofiber of f). A ***long fiber sequence*** (resp. ***long cofiber sequence***) is a sequence in which any two adjoining morphisms form a fiber sequence (resp. cofiber sequence).

4 Proposition (Stability of complexes) *Let $\mathcal{A}$ be an abelian category. Then for any sequence of complexes*

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

it is a fiber sequence if and only if it is a cofiber sequence.

To prove this, we need some lemmas.

4.1 Lemma *Let $\mathcal{A}$ be an abelian category and let $f\colon A \to B$ be a morphism in $\mathcal{C}\mathrm{h}$. Then*

1. *There is a natural isomorphism $\mathrm{Fib}(f) \cong \Omega\,\mathrm{Cofib}(f)$.*
2. *There is a natural isomorphism $\Sigma\,\mathrm{Fib}(f) \cong \mathrm{Cofib}(f)$*

Proof: There is a natural homogeneous isomorphism $\zeta\colon\ \mathrm{Fib}(f) \to \Omega\,\mathrm{Cofib}(f)$ given by

$$\zeta_n = \begin{pmatrix} -\mathrm{id}_n & 0 \\ 0 & \mathrm{id}_{n+1} \end{pmatrix}.$$

It reminds to show it is a chain map.

Since

$$\zeta_{n-1} \circ \partial_n = \begin{pmatrix} -\mathrm{id}_{n-1} & 0 \\ 0 & \mathrm{id}_n \end{pmatrix} \circ \begin{pmatrix} \partial_n & 0 \\ -f_n & -\partial_{n+1} \end{pmatrix} = \begin{pmatrix} -\partial_n & 0 \\ -f_n & -\partial_{n+1} \end{pmatrix},$$
$$\partial_n \circ \zeta_n = \begin{pmatrix} \partial_n & 0 \\ f_n & -\partial_{n+1} \end{pmatrix} \circ \begin{pmatrix} -\mathrm{id}_n & 0 \\ 0 & \mathrm{id}_{n+1} \end{pmatrix} = \begin{pmatrix} -\partial_n & 0 \\ -f_n & -\partial_{n+1} \end{pmatrix},$$

ζ is a chain map.

The natural isomorphism $\Sigma\,\mathrm{Fib}(f) \cong \mathrm{Cofib}(f)$ is obtained from ζ by the adjunction $\Sigma \dashv \Omega$. □

4.2 Lemma *Any fiber sequence*

$$\mathrm{Fib}(f) \xrightarrow{\mathrm{fib}(f)} A \xrightarrow{f} B$$

extends to a long fiber sequence

$$\cdots \longrightarrow \Omega B \xrightarrow{\delta} \mathrm{Fib}(f) \xrightarrow{\mathrm{fib}(f)} A \xrightarrow{f} B \longrightarrow \Sigma\,\mathrm{Fib}(f) \longrightarrow \cdots.$$

Likewise, Any cofiber sequence

$$A \xrightarrow{f} B \xrightarrow{\operatorname{cofib}(f)} \operatorname{Cofib}(f)$$

extends to a long cofiber sequence

$$\cdots \longrightarrow \Omega \operatorname{Cofib}(f) \longrightarrow A \xrightarrow{f} B \xrightarrow{\operatorname{cofib}(f)} \operatorname{Cofib}(f) \longrightarrow \Sigma A \longrightarrow \cdots .$$

Proof: This follows from the pasting lemmas. For the fiber sequence, it suffices to show the pasting lemma of the form

> For $f \colon A \to B$ a morphism of complexes, the homotopy fiber of $\operatorname{fib}(f)$ is homotopy equivalent to ΩB.

First, the homotopy fiber of $\operatorname{fib}(f)$ should be the complex

$$C_n := A_n \oplus B_{n+1} \oplus A_{n+1},$$

with the boundary operator

$$\partial_n := \begin{pmatrix} \partial_n & 0 & 0 \\ -f_n & -\partial_{n+1} & 0 \\ -\operatorname{id}_n & 0 & -\partial_{n+1} \end{pmatrix}.$$

In element notations, the boundary operator reads

$$\partial(a, b, a') = (\partial(a), -f(a) - \partial(b), -a - \partial(a')).$$

Then, one can see that the inclusions $B_{n+1} \hookrightarrow C_n$ defines a chain map $i \colon \Omega B \to C$. On the other hand,

$$(a, b, a') \longmapsto b - f(a')$$

defines a chain map $\epsilon \colon C \to \Omega B$.

It is easy to see $\epsilon \circ i = \operatorname{id}$. As for $i \circ \epsilon$, consider the twisted morphism

$$\begin{aligned} \alpha \colon C &\longrightarrow C(1) \\ (a, b, a') &\longmapsto (a', 0, 0). \end{aligned}$$

One can see that

$$\partial \circ \alpha + \alpha \circ \partial = i \circ \epsilon - \operatorname{id},$$

and hence α is a chain homotopy from $i \circ \epsilon$ to id. Therefore ΩB and C are chain homotopy equivalent.

The story for the cofiber sequence is similar. $\square$

Remark The canonical morphism $\delta \colon \Omega B \to \operatorname{Fib}(f)$ now can be written as $\operatorname{fib}(\operatorname{fib}(f)) \circ i$.

Now Proposition 4 is clear.

Proof (Proposition 4): We show that any fiber sequence is a cofiber sequence. The converse is similar.

By Lemma 4.2, the fiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

extends to a long fiber sequence

$$\cdots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \Sigma A \longrightarrow \cdots .$$

By Lemma 4.1, ΣA is homotopy equivalent to the homotopy cofiber of g. Hence the above long sequence is also the one extending the cofiber sequence

$$B \xrightarrow{g} C \longrightarrow \Sigma A.$$

In particular,

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

is a cofiber sequence. □

Fiber and cofiber sequences are preserved by Hom

By the general philosophy, the covariant functor $\mathcal{H}\mathrm{om}(X,-)$ should preserve fiber sequences, the contravariant functor $\mathcal{H}\mathrm{om}(-,X)$ should translate cofiber sequences to fiber sequences.

5 **Proposition** *Let $f\colon A \to B$ be a morphism of complexes and X be a complex. Then the Hom complex $\mathcal{H}\mathrm{om}(X, \mathrm{Fib}(f))$ is isomorphic to the homotopy fiber of the morphism $\mathcal{H}\mathrm{om}(X,f)\colon \mathcal{H}\mathrm{om}(X,A) \to \mathcal{H}\mathrm{om}(X,B)$. Dually, the Hom complex $\mathcal{H}\mathrm{om}(\mathrm{Cofib}(f), X)$ is isomorphic to the homotopy fiber of the morphism $\mathcal{H}\mathrm{om}(f,X)\colon \mathcal{H}\mathrm{om}(B,X) \to \mathcal{H}\mathrm{om}(A,X)$.*

Proof: We prove the first statement only. The second follows by duality.

First, they are obviously isomorphic as graded objects via the homogeneous isomorphism

$$\theta = (\theta_1, \theta_2)\colon \mathcal{H}\mathrm{om}(X, \mathrm{Fib}(f)) \longrightarrow \mathcal{H}\mathrm{om}(X,A) \oplus \mathcal{H}\mathrm{om}(X,B)(1)$$

induced from

$$(\mathrm{fib}(f), \mathrm{pr})\colon \mathrm{Fib}(f) \longrightarrow A \oplus B(1).$$

It reminds to show it is a chain map.

Given any $x \in \mathrm{Hom}(X_p, \mathrm{Fib}(f)_q)$, we have

$$\begin{aligned}\theta_1\partial(x) &= \theta_1((-1)^{q-p-1} x \circ \partial, \partial \circ x) \\ &= ((-1)^{q-p-1}\,\mathrm{fib}(f) \circ x \circ \partial, \mathrm{fib}(f) \circ \partial \circ x)\end{aligned}$$

$$
\begin{aligned}
&\in \operatorname{Hom}(X_{p+1}, A_q) \times \operatorname{Hom}(X_p, A_{q-1}), \\
\theta_2 \partial(x) &= ((-1)^{q-p-1} \operatorname{pr} \circ x \circ \partial, \operatorname{pr} \circ \partial \circ x) \\
&\in \operatorname{Hom}(X_{p+1}, B_{q+1}) \times \operatorname{Hom}(X_p, B_q), \\
\partial\theta(x) &= \partial(\operatorname{fib}(f) \circ x, \operatorname{pr} \circ x) \\
&= (\partial(\operatorname{fib}(f) \circ x), -f \circ \operatorname{fib}(f) \circ x - \partial(\operatorname{pr} \circ x)), \\
\partial(\operatorname{fib}(f) \circ x) &= ((-1)^{q-p-1} \operatorname{fib}(f) \circ x, \partial \circ \operatorname{fib}(f) \circ x) \\
&\in \operatorname{Hom}(X_{p+1}, A_q) \times \operatorname{Hom}(X_p, A_{q-1}), \\
f \circ \operatorname{fib}(f) \circ x &\in \operatorname{Hom}(X_p, B_q), \\
\partial(\operatorname{pr} \circ x) &= ((-1)^{q-p-1} \operatorname{pr} \circ x \circ \partial, \partial \circ \operatorname{pr} \circ x) \\
&\in \operatorname{Hom}(X_{p+1}, B_{q+1}) \times \operatorname{Hom}(X_p, B_q).
\end{aligned}
$$

We see that $\theta\partial(x)$ and $\partial\theta(x)$ agree on $\operatorname{Hom}(X_{p+1}, A_q) \times \operatorname{Hom}(X_{p+1}, B_{q+1})$. Since $\operatorname{fib}(f)$ is a chain map and pr is a chain homotopy from 0 to $f \circ \operatorname{fib}(f)$, $\theta\partial(x)$ and $\partial\theta(x)$ also agree on $\operatorname{Hom}(X_p, A_{q-1}) \times \operatorname{Hom}(X_p, B_q)$. Thus $\theta \circ \partial = \partial \circ \theta$ as desired. □

Remark Note that, since each cofiber sequence is a fiber sequence, we conclude that the two Hom functors preserve fiber sequences.

§ 4 Exact sequences

In this section, we recall some basic knowledge on exact sequences.

Long exact sequences of fiber sequences

We now discuss the long exact sequences associated to a short sequence, which can be a fiber sequence or a short exact sequence.

1 Proposition *Any fiber sequence*

$$\operatorname{Fib}(f)_\bullet \xrightarrow{\operatorname{fib}(f)} A_\bullet \xrightarrow{f} B_\bullet$$

induces a long exact sequence of homology objects

$$\cdots \longrightarrow H_{n+1}(B) \longrightarrow H_n(\operatorname{Fib}(f)) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow \cdots .$$

Proof: We have seen that any fiber sequence

$$\operatorname{Fib}(f) \xrightarrow{\operatorname{fib}(f)} A \xrightarrow{f} B$$

extends to a long fiber sequence

$$\cdots \longrightarrow \Omega B \xrightarrow{\delta} \operatorname{Fib}(f) \xrightarrow{\operatorname{fib}(f)} A \xrightarrow{f} B \longrightarrow \Sigma \operatorname{Fib}(f) \longrightarrow \cdots .$$

Applying the homology functor H_n, we obtain a long sequence

$$\cdots \longrightarrow H_{n+1}(B) \longrightarrow H_n(\mathrm{Fib}(f)) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow \cdots .$$

Now, it reminds to prove the following lemma. □

1.1 Lemma *For any fiber sequence*

$$\mathrm{Fib}(f)_\bullet \overset{\mathrm{fib}(f)}{\longrightarrow} A_\bullet \overset{f}{\longrightarrow} B_\bullet,$$

the sequence of homology

$$H_n(\mathrm{Fib}(f)) \overset{H_n(\mathrm{fib}(f))}{\longrightarrow} H_n(A) \overset{H_n(f)}{\longrightarrow} H_n(B)$$

is exact at $H_n(A)$.

Proof: We may assume $\mathcal{A} = \mathcal{A}\mathrm{b}$.

For any $(a, b) \in Z_n(\mathrm{Fib}(f))$, we have

$$(\partial(a), -f(a) - \partial(b)) = (0, 0).$$

Thus $f\,\mathrm{fib}(f)(a) = f(a) = -\partial(b) \in B_n(B)$. Therefore

$$\mathrm{Im}(H_n(\mathrm{fib}(f)) \subset \mathrm{Ker}(H_n(f)).$$

Conversely, for any $a \in Z_n(A)$ such that $f(a) \in B_n(B)$, we have $f(a) = \partial(-b)$ for some $b \in B_{n+1}$. Then

$$\partial(a, b) = (\partial(a), -f(a) - \partial(b)) = (0, 0),$$

and hence $(a, b) \in Z_n(\mathrm{Fib}(f))$ and $\mathrm{fib}(f)(a, b) = a$. Therefore

$$\mathrm{Ker}(H_n(f)) \subset \mathrm{Im}(H_n(\mathrm{fib}(f)).$$

Now, we have $\mathrm{Ker}(H_n(f)) = \mathrm{Im}(H_n(\mathrm{fib}(f))$ as desired. □

1.2 Corollary *The homotopy fiber of a quasi-isomorphism* $f\colon A \to B$ *is acyclic.*

Proof: There are exact sequences

$$H_{n+1}(A) \overset{H_{n+1}(f)}{\longrightarrow} H_{n+1}(B) \longrightarrow H_n(\mathrm{Fib}(f)) \longrightarrow H_n(A) \overset{H_n(f)}{\longrightarrow} H_n(B)$$

where both $H_{n+1}(f)$ and $H_n(f)$ are isomorphisms. Thus $H_n(\mathrm{Fib}(f)) = 0$ as desired. □

Remark The similar results hold for cofiber sequences as they coincide with fiber sequences (Proposition 3.4).

1.3 Remark Note that, since the Hom functors preserve fiber sequences, each fiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

induces two fiber sequences of hom complexes

$$\mathcal{H}\mathrm{om}(X, A) \xrightarrow{f\circ} \mathcal{H}\mathrm{om}(X, B) \xrightarrow{g\circ} \mathcal{H}\mathrm{om}(X, C)$$

and

$$\mathcal{H}\mathrm{om}(C, X) \xrightarrow{\circ g} \mathcal{H}\mathrm{om}(B, X) \xrightarrow{\circ f} \mathcal{H}\mathrm{om}(A, X).$$

Apply Proposition 1 to these fiber sequences, we obtain two long exact sequences

$$\cdots \longrightarrow H_{n+1}(X, C) \longrightarrow H_n(X, A) \longrightarrow H_n(X, B) \longrightarrow H_n(X, C) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow H_{n+1}(A, X) \longrightarrow H_n(C, X) \longrightarrow H_n(B, X) \longrightarrow H_n(A, X) \longrightarrow \cdots .$$

Long exact sequences from short exact sequences

2 Proposition *Any short exact sequence of complexes*

$$0 \longrightarrow A_\bullet \longrightarrow B_\bullet \longrightarrow C_\bullet \longrightarrow 0$$

induces a long exact sequence of homology objects.

$$\cdots \longrightarrow H_{n+1}(C) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow \cdots .$$

Proof: Denote $f\colon A \to B$ and $g\colon B \to C$. Let h be the twisted morphism from $\mathrm{Fib}(f)$ to C of degree 1 induced by the chain maps $0\colon A \to C$ and $-g\colon B \to C$. Note that

$$h_n \circ \partial_n = \begin{pmatrix} 0 & -g_n \end{pmatrix} \circ \begin{pmatrix} \partial_n^A & 0 \\ -f_n & -\partial_{n+1}^B \end{pmatrix} = \begin{pmatrix} 0 & g_n \circ \partial_{n+1}^B \end{pmatrix} = -\partial_{n+1}^C \circ h_{n+1}.$$

Thus $h\colon \mathrm{Fib}(f) \to \Omega C$ is also a chain map.

Now, we have a commutative diagram

$$\begin{tikzcd}
H_n(\Omega B) \arrow[r, "H_n(\delta)"] \arrow[d, equal] & H_n(\mathrm{Fib}(f)) \arrow[r] \arrow[d, "H_n(h)"] & H_n(A) \arrow[r, "H_n(f)"] & H_n(B) \arrow[r, "H_n(g)"] & H_n(C) \\
H_n(\Omega B) \arrow[r, "H_n(\Omega g)"] \arrow[d, equal] & H_n(\Omega C) \arrow[d, equal] & & & \\
H_{n+1}(B) \arrow[r, "H_{n+1}(g)"] & H_{n+1}(C) & & &
\end{tikzcd}$$

where the first row is exact except at $H_n(B)$. Then, it reminds to show the following two lemmas. $\square$

2.1 Lemma *h is a quasi-isomorphism.*

Proof: We may assume $\mathcal{A} = \mathcal{A}\mathrm{b}$.

For any $(a, b) \in Z_n(\mathrm{Fib}(f))$, if $h(a, b) \in B_{n+1}(C)$, there exists a $c \in C_{n+2}$ such that

$$h(a, b) = -g(b) = -\partial_C(c).$$

Since g is surjective, there exists a $b' \in B_{n+2}$ such that $-g(b') = c$. Then, $-\partial_B(b') - b \in \mathrm{Ker}\,(g) = \mathrm{Im}\,(f)$. Thus there exists an $a' \in A_{n+1}$ such that $f(a') = -\partial_B(b') - b$. Note that

$$f(\partial_A(a')) = \partial_B(f(a')) = -\partial_B(b) = f(a).$$

Thus $\partial(a') = a$ since f is injective. Now, we have

$$\partial(a', b') = (\partial_A(a'), -f(a') - \partial_B(b')) = (a, b).$$

Hence $(a, b) \in B_n(\mathrm{Fib}(f))$ and $H(h)$ is injective.

Conversely, consider any $c \in Z_n(\Omega C) = Z_{n+1}(C)$. Since g is surjective, there exists a $b \in B_{n+1}$ such that $g(b) = -c$. Note that

$$g(-\partial_B(b)) = -\partial_C(g(b)) = \partial_C(c) = 0.$$

Thus $-\partial_B(b) \in \mathrm{Ker}(g) = \mathrm{Im}(f)$. Then there exists an $a \in A_n$ such that $f(a) = -\partial_B(b)$. Note that

$$f(\partial_A(a)) = \partial_B(f(a)) = 0.$$

Thus $\partial_A(a) = 0$ since f is injective. Now, we have

$$\partial(a, b) = (\partial(a), -f(a) - \partial(b)) = (0, 0).$$

Thus $(a, b) \in Z_n(\mathrm{Fib}(f))$. Since $h(a, b) = -g(b) = c$, we see that $H(h)$ is surjective. $\square$

Remark The proof can be encoded into the following two diagram chases.

$$\begin{tikzcd}
n+2 & & b' \arrow[rr, two heads, "-g"] \arrow[d, "-\partial_B"'] & & c \arrow[d, "-\partial_C"] \\
n+1 & a' \arrow[r, "f"] \arrow[d, "\partial_A"] & -\partial_B(b') \; - \arrow[rr, "-g"] \arrow[d, "\partial_B"] & b \arrow[r, "-g"'] & -\partial_C(c) \\
n & a \arrow[r, hook, "f"] & -\partial_B(b) & &
\end{tikzcd}$$

$$
\begin{tikzcd}
n+1 & & b \arrow[r, two heads, "-g"] \arrow[d, "-\partial_B"'] & c \arrow[d, "-\partial_C"] \\
n & a \arrow[r, "f"] \arrow[d, "\partial_A"] & -\partial_B(b) \arrow[r, "-g"] \arrow[d, "\partial_B"] & 0 \\
n-1 & 0 \arrow[r, hook, "f"] & 0 &
\end{tikzcd}
$$

2.2 Lemma *For any short exact sequence of complexes*

$$0 \longrightarrow A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \longrightarrow 0,$$

the sequence of homology

$$H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C)$$

is exact at $H_n(B)$.

Proof: We may assume $\mathcal{A} = \mathcal{A}\mathrm{b}$.

Since $g \circ f = 0$, $\mathrm{Im}(H(f)) \subset \mathrm{Ker}(H(g))$. Conversely, for any $b \in Z_n(B)$ such that $g(b) \in B_n(C)$, we have $g(b) = \partial(c)$ for some $c \in C_{n+1}$. Since g is surjective, there exists a $b' \in B_{n+1}$ such that $g(b') = c$. Note that

$$g(\partial(b')) = \partial(g(b')) = \partial(c) = g(b).$$

Then $\partial(b') - b \in \mathrm{Ker}(g) = \mathrm{Im}(f)$. Thus there exists an $a \in A_n$ such that $f(a) = \partial(b') - b$. Note that

$$f(\partial(a)) = \partial(f(a)) = -\partial(b) = 0.$$

Thus $\partial(a) = 0$ since f is injective. This shows $a \in Z_n(A)$ and it defines a homology class $[a]$. Since $f(a) + b = \partial(b') \in B_n(B)$, $H(f)(-[a]) = [b]$. This shows $\mathrm{Im}(H(f)) \supset \mathrm{Ker}(H(g))$ as desired. $\square$

Remark The proof can be encoded into the following diagram chase.

$$
\begin{tikzcd}
n+1 & & b' \arrow[rr, two heads, "g"] \arrow[d, "\partial"'] & & c \arrow[d, "\partial"] \\
n & a \arrow[r, "f"] \arrow[d, "\partial"'] & \partial(b') \arrow[rr, bend left=10, "g"] \arrow[d, phantom, "-"] & & \partial(c) \\
 & & {} & b \arrow[ur, "g"'] & \\
n-1 & 0 \arrow[r, hook, "f"] & 0 & &
\end{tikzcd}
$$

3 Split exact sequences

Recall that

- A monomorphism $f\colon A \to B$ is said to be ***split*** if it admits a ***retraction***, meaning a morphism $r\colon B \to A$ such that $r \circ f = \mathrm{id}_A$.
- An epimorphism $f\colon A \to B$ is said to be ***split*** if it admits a ***section***, meaning a morphism $s\colon B \to A$ such that $f \circ s = \mathrm{id}_B$.

We have the following useful proposition.

4 Proposition (Splitting Lemma) *Consider a short exact sequence in an abelian category $\mathcal{A}$:*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

The following are equivalent.

1. *f is a split monomorphism.*
2. *g is a split epimorphism.*
3. *f admits a retraction r anf g admits a section s such that $r \circ s = 0$ and that*
$$f \circ r + s \circ g = \mathrm{id}_B\,.$$

In particular, B is the direct sum of A and C in such a case.

Proof: *1 ⇒ 2.* Assume $r \circ f = \mathrm{id}_A$. Then

$$(\mathrm{id}_B - f \circ r) \circ f = f - f = 0.$$

Thus, there exists a unique morphism $s\colon C \to B$ such that $s \circ g = \mathrm{id}_B - f \circ r$. Then

$$g \circ s \circ g = g \circ (\mathrm{id}_B - f \circ r) = g.$$

Since g is an epimorphism, $g \circ s = \mathrm{id}_C$.

2 ⇒ 1. Assume $g \circ s = \mathrm{id}_C$. Then

$$g \circ (\mathrm{id}_B - s \circ g) = g - g = 0.$$

Thus, there exists a unique morphism $r\colon B \to A$ such that $f \circ r = \mathrm{id}_B - s \circ g$. Then

$$f \circ r \circ f = (\mathrm{id}_B - s \circ g) \circ f = f.$$

Since f a monomorphism, $r \circ f = \mathrm{id}_A$.

1,2 ⇒ 3. Assume we have r, s as above. Then

$$f \circ r + s \circ g = \mathrm{id}_B$$

by our construction. Furthermore,

$$r \circ s = r \circ (f \circ r + s \circ g) \circ s = r \circ f \circ r \circ s + r \circ s \circ g \circ s = r \circ s + r \circ s.$$

Thus $r \circ s = 0$ as desired.

3 $\Rightarrow$ *1,2* is obvious. $\square$

A short exact sequence satisfying the conditions of the above lemma is called a ***split sequence***. A ***degreewise split sequence*** is a short exact sequence of complexes such that at each degree, it is a split sequence.

5 Proposition *Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a degreewise split sequence of complexes. Then there exists a morphism of complexes

$$\delta \colon C \longrightarrow \Sigma A$$

such that the morphisms $H_n(\delta)$ give the long exact sequence in Proposition 2.

Proof: Let's consider chain complexes. Let r_n and s_n be the retraction and section of f_n and g_n respectively. Then,

$$\delta_n := r_{n-1} \circ \partial_B \circ s_n$$

defines a twisted morphism of degree -1. Note that

$$f_{n-1} \circ r_{n-1} \circ \partial_B \circ s_n = \partial_B \circ s_n - s_{n-1} \circ \partial_C.$$

Therefore,

$$\begin{aligned}
\delta_n \circ \partial_C &= r_{n-1} \circ \partial_B \circ s_n \circ \partial_C \\
&= r_{n-1} \circ \partial_B \circ (\partial_B \circ s_{n+1} - f_n \circ r_n \circ \partial_B \circ s_{n+1}) \\
&= -r_{n-1} \circ \partial_B \circ f_n \circ r_n \circ \partial_B \circ s_{n+1} \\
&= -r_{n-1} \circ f_{n-1} \circ \partial_A \circ r_n \circ \partial_B \circ s_{n+1} \\
&= -\partial_A \circ \delta_{n+1}.
\end{aligned}$$

This shows δ is a chain map from C to ΣA.

Consider the following diagram.

$$\begin{tikzcd}
H_n(\mathrm{Fib}(f)) \arrow[r] \arrow[d, "H_n(h)"] & H_n(A) \\
H_n(\Omega C) \arrow[ur, "H_n(\Omega\delta)"'] &
\end{tikzcd}$$

We need to show it is commutative. Then, the proof of Proposition 2 shows that δ induces such a long exact sequence. As we are dealing with homology, we may assume $\mathcal{A} = \mathcal{A}\mathrm{b}$. Let $(a,b) \in Z_n(\mathrm{Fib}(f))$, then $f(a) = -\partial_B(b)$. The morphisms in the above diagram maps it as follows.

$$\begin{tikzcd}
(a,b) \arrow[r, mapsto] \arrow[d, mapsto] & a & -\delta_{n+1}g_{n+1}(b) \\
-g_{n+1}(b) \arrow[urr, mapsto] & &
\end{tikzcd}$$

Note that

$$f_n \circ \delta_{n+1} \circ g_{n+1} = \partial_B - f_n \circ \partial_A \circ r_{n+1} - s_n \circ \partial_C \circ g_{n+1}.$$

Therefore

$$\begin{aligned}
f_n(a + \delta_{n+1}g_{n+1}(b)) &= f_n(a) + f_n\delta_{n+1}g_{n+1}(b) \\
&= -\partial_B(b) + \partial_B(b) - f_n\partial_A r_{n+1}(b) - s_n\partial_C g_{n+1}(b) \\
&= f_n(\partial_A(-r_{n+1}(b))) - s_n g_n \partial_B(b) \\
&= f_n(\partial_A(-r_{n+1}(b))).
\end{aligned}$$

Since f_n is injective, $a + \delta_{n+1}g_{n+1}(b) = \partial_A(-r_{n+1}(b)) \in B_n(A)$ and thus $[a] = [-\delta_{n+1}g_{n+1}(b)]$ as desired. $\square$

Remark This δ depends on the choice of splitting. However, it is unique up to unique homotopy.

Up to quasi-isomorphisms

Homotopy fibers and cofibers are determined up to quasi-isomorphisms.

6 Proposition *Consider the following commutative diagram of complexes.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
A' \arrow[r, "f'"] & B'
\end{tikzcd}$$

If the vertical arrows are quasi-isomorphisms, then:

1. *the induced morphism* $\mathrm{Fib}(f) \to \mathrm{Fib}(f')$ *is a quasi-isomorphism;*
2. *the induced morphism* $\mathrm{Cofib}(f) \to \mathrm{Cofib}(f')$ *is a quasi-isomorphism.*

Proof: Let α, β be the vertical morphisms and θ the induced morphism $\mathrm{Fib}(f) \to \mathrm{Fib}(f')$. Now, the commutative diagram induces the following commutative diagram of long exact sequences.

$$\begin{tikzcd}
H_{n+1}(A) \arrow[r] \arrow[d, "H_{n+1}(\alpha)"] & H_{n+1}(B) \arrow[r] \arrow[d, "H_{n+1}(\beta)"] & H_n(\mathrm{Fib}(f)) \arrow[r] \arrow[d, "H_n(\theta)"] & H_n(A) \arrow[r] \arrow[d, "H_n(\alpha)"] & H_n(B) \arrow[d, "H_n(\beta)"] \\
H_n(A') \arrow[r] & H_{n+1}(B') \arrow[r] & H_n(\mathrm{Fib}(f')) \arrow[r] & H_n(A') \arrow[r] & H_n(B')
\end{tikzcd}$$

Since $H_{n+1}(\alpha), H_{n+1}(\beta), H_n(\alpha), H_n(\beta)$ are all isomorphisms, by 5-lemma, so is $H_n(\theta)$. $\square$

§ 5 Triangular categories and localizations

§ 6 Projective and injective objects

Let $\mathcal{A}$ be an abelian category.

An object P is said to be ***projective*** if it satisfies the following *left lift property*:

> For any $f\colon A \to B$ an epimorphism and $b\colon P \to B$ a morphism, there exists a morphism $a\colon P \to A$ making the following diagram commutes.

$$\begin{tikzcd} P \arrow[d, dashed, "a"'] \arrow[dr, "b"] & \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Dually, an object I is said to be ***injective*** if it satisfies the following *right extension property*:

> For any $f\colon A \to B$ a monomorphism and $a\colon A \to I$ a morphism, there exists a morphism $b\colon B \to I$ making the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[dl, dashed, "b"] \\ I & \end{tikzcd}$$

1 Proposition *Let $\mathcal{A}$ be an abelian category and P an object in $\mathcal{A}$. The following are equivalent.*

1. *P is a projective object.*
2. *$\mathrm{Hom}_{\mathcal{A}}(P, -)$ is an exact functor.*
3. *Ant short exact sequence of the form*

$$0 \to A \to B \to P \to 0$$

is split.

Dually, for I an object in $\mathcal{A}$, the following are equivalent.

1'. *I is an injective object.*

2'. *$\mathrm{Hom}_{\mathcal{A}}(-, I)$ is an exact functor.*

3'. Ant short exact sequence of the form

$$0 \to I \to A \to B \to 0$$

is split.

Proof: Omit. $\square$

Let $\mathcal{A}$ be an abelian category.

- We say $\mathcal{A}$ ***has enough projectives*** if for any object A, there is an epimorphism $P \to A$ from a projective object P to it.
- Dually, we say $\mathcal{A}$ ***has enough injectives*** if for any object A, there is a monomorphism $A \to I$ from it to an injective object I.
- We say $\mathcal{A}$ ***has functorial projective surjections*** if there is a functor

$$\begin{aligned} P\colon \mathcal{A} &\longrightarrow \mathcal{A}^{\to} \\ A &\longmapsto (P(A) \to A) \end{aligned}$$

  sending every object A to an epimorphism from a projective object $P(A)$ to A.
- We say $\mathcal{A}$ ***has functorial injective embeddings*** if there is a functor

$$\begin{aligned} I\colon \mathcal{A} &\longrightarrow \mathcal{A}^{\to} \\ A &\longmapsto (A \to I(A)) \end{aligned}$$

  sending every object A to a monomorphism from A to an injective object $I(A)$.

2 Fibrations

Let $\mathcal{A}$ be an abelian category.

- By a ***projective fibration***, we mean an epimorphism in $\mathcal{C}\mathrm{h}(\mathcal{A})$.
- By an ***injective cofibration***, we mean a monomorphism in $\mathcal{C}\mathrm{h}(\mathcal{A})$.
- By a ***projective trivial fibration***, we mean a projective fibration which is also a quasi-isomorphism.
- By an ***injective trivial cofibration***, we mean an injective cofibration which is also a quasi-isomorphism.
- By a ***split-projective fibration***, we mean a degreewise split epimorphism.

- By a ***split-inejctive cofibration***, we mean a degreewise split monomorphism.
- By a ***split-projective trivial fibration***, we mean a split-projective fibration which is also a quasi-isomorphism.
- By a ***split-injective trivial cofibration***, we mean a split-injective cofibration which is also a quasi-isomorphism.

The following propositions explain the above terminology.

3 Proposition *Let $f\colon A \to B$ be a morphism of complexes. Then*

1. *If f is a projective fibration, then* $\mathrm{Ker}(f)$ *is quasi-isomorphic to* $\mathrm{Fib}(f)$*. If f is furthermore trivial, then both* $\mathrm{Ker}(f)$ *and* $\mathrm{Fib}(f)$ *are acyclic.*
2. *If f is an injective cofibration, then* $\mathrm{Coker}(f)$ *is quasi-isomorphic to* $\mathrm{Cofib}(f)$*. If f is furthermore trivial, then both* $\mathrm{Coker}(f)$ *and* $\mathrm{Cofib}(f)$ *are acyclic.*

Proof: If f is a projective fibration, then we have a short exact sequence

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$

and a fiber sequence

$$\mathrm{Fib}(f) \longrightarrow A \xrightarrow{f} B.$$

Therefore, by Propositions 4.1 and 4.2, we have a diagram of long exact sequences

$$\begin{tikzcd}
H_{n+1}(A) \arrow[r] \arrow[d, equal] & H_{n+1}(B) \arrow[r] \arrow[d, equal] & H_n(\mathrm{Ker}(f)) \arrow[r] \arrow[d, "H_n(\theta)"] & H_n(A) \arrow[r] \arrow[d, equal] & H_n(B) \arrow[d, equal] \\
H_n(A) \arrow[r] & H_{n+1}(B) \arrow[r] & H_n(\mathrm{Fib}(f)) \arrow[r] & H_n(A) \arrow[r] & H_n(B)
\end{tikzcd}$$

where θ is the canonical morphism $\mathrm{Ker}(f) \to \mathrm{Fib}(f)$, which can be obtained from the evident inclusion $k\colon \mathrm{Ker}(f) \hookrightarrow A$ and the zero morphism. If the above diagram is commutative, then θ is a quasi-isomorphism by the 5-lemma. If f is furthermore a quasi-isomorphism, then one can see that $H_n(\mathrm{Ker}(f)) = H_n(\mathrm{Fib}(f)) = 0$.

By the proof of Proposition 4.2, to show the above diagram is commutative, it suffices to check that the following diagram commutes.

$$\begin{tikzcd}
H_n(\mathrm{Fib}(k)) \arrow[r] \arrow[d, "H_n(h)"] & H_n(\mathrm{Ker}(f)) \arrow[d, "H_n(\theta)"] \\
H_{n+1}(B) \arrow[r] & H_n(\mathrm{Fib}(f))
\end{tikzcd}$$

Since we are dealing with homology, we may assume $\mathcal{A} = \mathcal{A}\mathrm{b}$. Indeed, for any $(x, a) \in Z_n(\mathrm{Fib}(k))$, the above morphisms send it as follows.

$$\begin{tikzcd}
(x,a) \arrow[rr, mapsto] \arrow[d, mapsto] & & x \arrow[d, mapsto] \\
-f(a) \arrow[r, mapsto] & (0,-f(a)) & (k(x),0)
\end{tikzcd}$$

Since $k(x) = -\partial(a)$, we have

$$(0, -f(a)) - (k(x), 0) = (-k(x), -f(a)) = (\partial(a), -f(a)) = \partial(a, 0).$$

Thus $[0, -f(a)] = [k(x), 0]$ as desired. □

4 Proposition *Let $f\colon A \to B$ be a morphism of complexes. Then*

1. *If f is a split-projective fibration, then* $\mathrm{Ker}(f)$ *is homotopy equivalent to* $\mathrm{Fib}(f)$*. If f is furthermore trivial, then both* $\mathrm{Ker}(f)$ *and* $\mathrm{Fib}(f)$ *are acyclic.*

2. *If f is a split-injective cofibration, then* $\mathrm{Coker}(f)$ *is homotopy equivalent to* $\mathrm{Cofib}(f)$*. If f is furthermore trivial, then both* $\mathrm{Coker}(f)$ *and* $\mathrm{Cofib}(f)$ *are acyclic.*

Proof: Let f be a split-projective fibration. Let $k\colon \mathrm{Ker}(f) \to A$ be its kernel. Let θ denote the canonical morphism from $\mathrm{Ker}(f)$ to $\mathrm{Fib}(f)$. Let r_n and s_n be the retraction and section of k_n and f_n respectively. Then,

$$\delta_n := r_{n-1} \circ \partial_A \circ s_n$$

defines a chain map from B to $\Sigma\,\mathrm{Ker}(f)$. Let ϕ be the homogeneous morphism from $\mathrm{Fib}(f)$ to $\mathrm{Ker}(f)$ defined as

$$\phi_n = \begin{pmatrix} r_n & \delta_{n+1} \end{pmatrix}.$$

Note that

$$r_{n-1} \circ \partial_n - \partial_n \circ r_n = \delta_n \circ f_n$$

and that

$$\begin{aligned}
\partial_n \circ \phi_n &= \begin{pmatrix} \partial_n \circ r_n & \partial_n \circ \delta_{n+1} \end{pmatrix}, \\
\phi_{n-1} \circ \partial_n &= \begin{pmatrix} r_{n-1} & \delta_n \end{pmatrix} \circ \begin{pmatrix} \partial_n & 0 \\ -f_n & -\partial_{n+1} \end{pmatrix} \\
&= \begin{pmatrix} r_{n-1} \circ \partial_n - \delta_n \circ f_n & -\delta_n \circ \partial_{n+1} \end{pmatrix}.
\end{aligned}$$

Therefore ϕ is a chain map.

Now, the composition $\phi \circ \theta$ is given by

$$\phi_n \circ \theta_n = \begin{pmatrix} r_n & \delta_{n+1} \end{pmatrix} \circ \begin{pmatrix} k_n \\ 0 \end{pmatrix} = r_n \circ k_n = \mathrm{id}_n \,.$$

On the other hand, the composition $\theta \circ \phi$ is given by

$$\theta_n \circ \phi_n = \begin{pmatrix} k_n \\ 0 \end{pmatrix} \circ \begin{pmatrix} r_n & \delta_{n+1} \end{pmatrix} = \begin{pmatrix} k_n \circ r_n & k_n \circ \delta_{n+1} \\ 0 & 0 \end{pmatrix}.$$

To see it is homotopic to id, consider the twisted morphism of degree 1 given as follows.

$$\alpha_n := \begin{pmatrix} 0 & -s_{n+1} \\ 0 & 0 \end{pmatrix} : \mathrm{Fib}(f)_n \longrightarrow \mathrm{Fib}(f)_{n+1}.$$

Then

$$\begin{aligned}
\alpha_{n-1} \circ \partial_n + \partial_{n+1} \circ \alpha_n &= \begin{pmatrix} 0 & -s_n \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} \partial_n & 0 \\ -f_n & -\partial_{n+1} \end{pmatrix} \\
&\quad + \begin{pmatrix} \partial_{n+1} & 0 \\ -f_{n+1} & -\partial_{n+2} \end{pmatrix} \circ \begin{pmatrix} 0 & -s_{n+1} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} s_n \circ f_n & s_n \circ \partial_{n+1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\partial_{n+1} \circ s_{n+1} \\ 0 & f_{n+1} \circ s_{n+1} \end{pmatrix} \\
&= \begin{pmatrix} s_n \circ f_n & s_n \circ \partial_{n+1} - \partial_{n+1} \circ s_{n+1} \\ 0 & f_{n+1} \circ s_{n+1} \end{pmatrix} \\
&= \mathrm{id} - \begin{pmatrix} k_n \circ r_n & k_n \circ \delta_{n+1} \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

This shows $\theta \circ \phi \simeq \mathrm{id}$. Therefore ϕ is the weak inverse of θ and $\mathrm{Ker}(f)$ is homotopy equivalent to $\mathrm{Fib}(f)$. $\square$

5 Complexes of projectives and injectives

Now, we consider complexes of projectives (resp. injectives).

The following lemma will be used in the proofs of propositions later.

5.1 Lemma *Let K be an acyclic complex and P a bounded front complex of projectives.*

1. *Then any morphism $P \to K$ is homotopic to zero.*
2. *Moreover, the intrinsic homology $H_\bullet(P, K)$ is zero.*

Dually, let I be a bounded back complex of injectives.

1'. *Then any morphism $K \to I$ is homotopic to zero.*

2'. *Moreover, the intrinsic cohomology $H^\bullet(K, I)$ is zero.*

Proof: We may assume P is a connective chain complex. Let $f\colon P_\bullet \to K_\bullet$ be any chain map. Since $K_\bullet$ is acyclic, $\mathrm{Im}(\partial_n) = \mathrm{Ker}(\partial_{n-1})$ for every n.

We now construct a homotopy α from f to 0 by induction. Then f is homotopic to 0 and *1* is proved. Note that $H_n(P, K) = H_0(P, \Sigma^n K)$, so applying *1* to every $\Sigma^n K$ shows *2*.

First, since $\partial_0 \circ f_0 = 0$, we have the following commutative diagram by the universal property of kernel.

$$\begin{tikzcd} & P_0 \arrow[dl, dashed, "g_0"'] \arrow[d, "f_0"] & \\ \mathrm{Ker}(\partial_0) \arrow[r, hook] & K_0 \arrow[r, "\partial_0"] & K_{-1} \end{tikzcd}$$

Now, consider the following diagram

$$\begin{tikzcd} & P_0 \arrow[dl, dashed, "\alpha_0"'] \arrow[d, "g_0"] \\ K_1 \arrow[r, two heads] & \mathrm{Im}(\partial_1) \end{tikzcd}$$

where the bottom is an epimorphism. Thus, by the definition of projective objects, there exists a morphism α_0 making the diagram commute.

Now, assume we have constructed morphisms $\alpha_i\colon P_i \to K_{i+1}$ for each $i = 0, 1, \cdots, n-1$ such that

$$\alpha_{i-1} \circ \partial_i + \partial_{i+1} \circ \alpha_i = f_i, \quad i = 1, \cdots, n-1.$$

Then, consider the diagram.

$$\begin{tikzcd} & P_n \arrow[r, "\partial_n"] \arrow[dl, dashed, "g_n"'] \arrow[d, "f_n"] & P_{n-1} \arrow[dl, "\alpha_{n-1}"] \arrow[d, "f_{n-1}"] \\ \mathrm{Ker}(\partial_n) \arrow[r, hook] & K_n \arrow[r, "\partial_n"] & K_{n-1} \end{tikzcd}$$

Since we have

$$\begin{aligned} \partial_n \circ (f_n - \alpha_{n-1} \circ \partial_n) &= \partial_n \circ f_n - (\alpha_{n-2} \circ \partial_{n-1} + \partial_n \circ \alpha_{n-1}) \circ \partial_n \\ &= \partial_n \circ f_n - f_{n-1} \circ \partial_n = 0, \end{aligned}$$

there exists a g_n such that

$$\ker(\partial_n) \circ g_n = f_n - \alpha_{n-1} \circ \partial_n.$$

Now, consider the diagram

$$\begin{tikzcd} & P_n \arrow[dl, dashed, "\alpha_n"'] \arrow[d, "g_n"] \\ K_{n+1} \arrow[r, two heads] & \mathrm{Im}(\partial_{n+1}) \end{tikzcd}$$

where the bottom is an epimorphism. Thus, by the definition of projective objects, there exists a morphism α_n making the diagram commute. Then

$$\alpha_{n-1} \circ \partial_n + \partial_{n+1} \circ \alpha_n = f_n$$

as desired. $\square$

6 Proposition *Consider the diagram*

$$\begin{tikzcd} & P \arrow[dl, dashed, "g"'] \arrow[d] \\ K \arrow[r, "f"] & L \end{tikzcd}$$

where P is a bounded front complex of projectives, K, L are two complexes and f is a quasi-isomorphism.

1. *There exists a morphism of complexes g making the diagram commute up to homotopy. Moreover, this morphism is unique up to homotopy.*
2. *If f is further a projective trivial fibration, then g can be chosen to make the diagram commute.*

Proof: Consider the fiber sequence

$$\mathrm{Fib}(f) \longrightarrow K \xrightarrow{f} L.$$

Since f is a quasi-isomorphism, $\mathrm{Fib}(f)$ is acyclic. Then the exact sequences

$$H_n(P, \mathrm{Fib}(f)) \longrightarrow H_n(P, K) \longrightarrow H_n(P, L) \longrightarrow H_{n-1}(P, \mathrm{Fib}(f))$$

imply that $H_n(P, K) \cong H_n(P, L)$ via compositing with f. This shows *1.*

We now prove *2.* We may assume P is a connective chain complex. Let p denote the chain map $p\colon P_\bullet \to L_\bullet$. We construct the required chain map $g\colon P_\bullet \to K_\bullet$ by induction.

First, let $g_n = 0$ for $n < 0$. Now, assume we have constructed g_i for $i < n$ such that

$$\partial_i \circ g_i = g_{i-1} \circ \partial_i, \qquad f_i \circ g_i = p_i, \qquad \forall i < n.$$

Since both $f_n\colon K_n \to L_n$ and the canonical morphism $K_n \to \partial(K_n)$ are epimorphisms, there exists morphisms $g_n^{(1)}, g_n^{(2)}\colon P_n \rightrightarrows K_n$ making the following diagrams commute.

$$\begin{tikzcd} P_n \arrow[r] \arrow[d, dashed, "g_n^{(1)}"'] & \partial(P_n) \arrow[d, "g_{n-1}"] \\ K_n \arrow[r, two heads] & \partial(K_n) \end{tikzcd} \qquad \begin{tikzcd} P_n \arrow[d, dashed, "g_n^{(2)}"'] \arrow[dr, "p_n"] & \\ K_n \arrow[r, two heads, "f_n"] & L_n \end{tikzcd}$$

Note that both $\partial(K_n)$ and L_n are quotient objects of K_n. Let U be the fibred coproduct of them and V the fibred product of $\partial(K_n) \twoheadrightarrow U$ and $L_n \twoheadrightarrow U$. Then the canonical morphism $K_n \to V$ is also a epimorphism.

$$
\begin{tikzcd}
K_n \arrow[dr, dashed] \arrow[drr, bend left, ->>] \arrow[ddr, bend right, ->>] & & \\
& V \arrow[r] \arrow[d] & L_n \arrow[d, ->>] \\
& \partial(K_n) \arrow[r, ->>] & U
\end{tikzcd}
$$

Thus, if the morphisms $P_n \to L_n \to U$ and $P_n \to \partial(K_n) \to U$ agree, then there exists a unique morphism $P_n \to V$ making the following diagram commute.

$$
\begin{tikzcd}
P_n \arrow[dr, dashed] \arrow[drr, bend left, ->>] \arrow[ddr, bend right, ->>] & & \\
& V \arrow[r] \arrow[d] & L_n \arrow[d, ->>] \\
& \partial(K_n) \arrow[r, ->>] & U
\end{tikzcd}
$$

As $K_n \to V$ is an epimorphism, there exists a morphism g_n making the following diagram commute.

$$
\begin{tikzcd}
& P_n \arrow[dl, dashed, "g_n"'] \arrow[d] \\
K_n \arrow[r] & V
\end{tikzcd}
$$

Combining the above three commutative diagrams, we see that this g_n satisfies the requirements.

Now, it reminds to show that the two morphisms $P_n \to L_n \to U$ and $P_n \to \partial(K_n) \to U$ agree. Note that we have the following commutative diagram.

$$
\begin{tikzcd}
P_n \arrow[dr, shift left] \arrow[dr, shift right] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& K_n \arrow[r] \arrow[d] & L_n \arrow[d] \\
& \partial(K_n) \arrow[r] & \partial(L_n)
\end{tikzcd}
$$

Therefore, it suffices to show $U \cong \partial(L_n)$, which is Lemma 6.1. $\square$

6.1 Lemma *Let $f\colon K \to L$ be a projective trivial fibration. Then the followings are fibred coproduct diagrams.*

$$\begin{tikzcd}
K_n \arrow[r, "f_n"] \arrow[d] & L_n \arrow[d] \\
\partial(K_n) \arrow[r] & \partial(L_n)
\end{tikzcd}$$

Proof: Let T be a test object and $x\colon L_n \to T, y\colon \partial(K_n) \to T$ two test morphisms making the following diagram commute.

$$\begin{tikzcd}
K_n \arrow[r] \arrow[d] & L_n \arrow[d] \\
\partial(K_n) \arrow[r] & T
\end{tikzcd}$$

Then, consider the following commutative diagram.

$$\begin{tikzcd}
B_n(K) \arrow[r, two heads] \arrow[d] & B_n(L) \arrow[d] \\
Z_n(K) \arrow[r] \arrow[d] & Z_n(L) \arrow[d] \\
K_n \arrow[r] \arrow[d] & L_n \arrow[d, "x"] \\
\partial(K_n) \arrow[r, "y"] & T
\end{tikzcd}$$

The composition of left vertical morphisms is 0 and so is the composition of right vertical morphisms since the top morphism is an epimorphism. Thus, since $H_n = \mathrm{Coker}(B_n \hookrightarrow Z_n)$, we get the following commutative diagram.

$$\begin{tikzcd}
Z_n(K) \arrow[rr] \arrow[dr, two heads] \arrow[dd] & & Z_n(L) \arrow[dr, two heads] \arrow[dd] & \\
& H_n(K) \arrow[rr, "H_n(f)"] \arrow[dd] & & H_n(L) \arrow[dd] \\
K_n \arrow[rr, "f_n"] \arrow[dr] & & L_n \arrow[dr, "x"] & \\
& \partial(K_n) \arrow[rr, "y"] & & T
\end{tikzcd}$$

Since the composition $Z_n(K) \to K_n \to \partial(K_n)$ is a zero morphism, so is the composition $H_n(K) \to T$. Since $H_n(f)$ is an isomorphism, we see that the morphism $H_n(L) \to T$ is a zero morphism. Thus so is the composition $Z_n(L) \to L_n \to T$. Since $\partial(L_n)$ is the cokernel of $Z_n(L) \to L_n$, we see that x factors through $\partial(L_n)$. Now consider the following diagram

$$\begin{tikzcd}
K_n \arrow[r] \arrow[d] & L_n \arrow[d] \arrow[ddr, bend left, "x"] & \\
\partial(K_n) \arrow[r] \arrow[drr, bend right, "y"'] & \partial(L_n) \arrow[dr] & \\
 & & T
\end{tikzcd}$$

where all but the lower triangle commute. Since the left vertical morphism is an epimorphism, we see that the lower triangle also commutes. This proves the lemma. $\square$

§ 7 Derived categories

1 Derived categories and localizations

We need the derived category $\mathcal{D}(\mathcal{A}\mathrm{b})$. By our expectation, it should be almost $\mathcal{C}\mathrm{h}$, except the weak equivalences are generated by the quasi-isomorphisms, not chain homotopy equivalences. We can also consider the general notion of ***derived category*** $\mathcal{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$. We also consider the notions of derived categories $\mathcal{D}^{\geqslant 0}(\mathcal{A})$, $\mathcal{D}_{\leqslant 0}(\mathcal{A})$, $\mathcal{D}^{\leqslant 0}(\mathcal{A})$, $\mathcal{D}_{\geqslant 0}(\mathcal{A})$, $\mathcal{D}^{-}(\mathcal{A})$, $\mathcal{D}_{+}(\mathcal{A})$, $\mathcal{D}^{+}(\mathcal{A})$, $\mathcal{D}_{-}(\mathcal{A})$, $\mathcal{D}^{b}(\mathcal{A})$, $\mathcal{D}_{b}(\mathcal{A})$, which are full subcategories of $\mathcal{D}(\mathcal{A})$ spanned by $\mathcal{C}\mathrm{h}^{\geqslant 0}(\mathcal{A})$, $\mathcal{C}\mathrm{h}_{\leqslant 0}(\mathcal{A})$, $\mathcal{C}\mathrm{h}^{\leqslant 0}(\mathcal{A})$, $\mathcal{C}\mathrm{h}_{\geqslant 0}(\mathcal{A})$, $\mathcal{C}\mathrm{h}^{-}(\mathcal{A})$, $\mathcal{C}\mathrm{h}_{+}(\mathcal{A})$, $\mathcal{C}\mathrm{h}^{+}(\mathcal{A})$, $\mathcal{C}\mathrm{h}_{-}(\mathcal{A})$, $\mathcal{C}\mathrm{h}^{b}(\mathcal{A})$, $\mathcal{C}\mathrm{h}_{b}(\mathcal{A})$ respectively.

Since all chain homotopy equivalences are already quasi-isomorphisms, the derived category $\mathcal{D}(\mathcal{A}\mathrm{b})$ should be obtained from $\mathcal{C}\mathrm{h}$ by formally inverting each quasi-isomorphism which is not a chain homotopy equivalence. This process sounds like the notion of *localizations* in commutative algebra theory.

Motivated by this, we give the following philosophic definition.

> The *localization* of an ∞-category $\mathcal{C}$ at a collection of morphisms W is the ∞-category $\mathcal{D}$ universally satisfies the property the functor $\mathcal{C} \to \mathcal{D}$ maps each morphism in W to a weak equivalence in $\mathcal{D}$.

In the dg-setting, the above notion can be realized as follows. Let $\mathcal{C}$ be a dg-category and W a collection of morphisms in $\mathcal{C}$. A ***localization*** of $\mathcal{C}$ at W is a dg-category $\mathcal{D}$ with a dg-functor $F\colon \mathcal{C} \to \mathcal{D}$ such that for each dg-category $\mathcal{E}$, composition with F gives a quasi-fully faithful embedding $\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ such that the essential image contains the functors

$\mathcal{C} \to \mathcal{E}$ mapping each morphism in W to a weak equivalence in $\mathcal{E}$. Such a dg-category $\mathcal{D}$ is unique up to quasi-equivalences and is denoted by $\mathcal{C}[W^{-1}]$.

Remark Here the notation $\mathrm{Fun}(-,-)$ should refer to the correct notion of *functor ∞-categories*. However, by our discussion before, the naive functor dg-categories are not the correct ones.

If the notion of localization has been correctly defined, then the *derived category* of $\mathcal{A}\mathrm{b}$ is just the localization $\mathcal{C}\mathrm{h}[W^{-1}]$ where W is the collection of quasi-isomorphisms in $\mathcal{C}\mathrm{h}$. Conversely, if the derived category $\mathcal{D}(\mathcal{A}\mathrm{b})$ has been correctly defined, then the correct notion of functor dg-categories and furthermore the notion of localizations can be defined.

Now, we will give a dg-category, which is equivalent to the should-be-existed derived category $\mathcal{D}(\mathcal{A}\mathrm{b})$. To achieve this, we need some results on projective or injective objects.

2 Bounded derived categories

Let $\mathcal{A}$ be an abelian category having enough projective objects. Let $\mathcal{P}$ denote the full subcategory of $\mathcal{A}$ spanned by projective objects.

2.1 Lemma *In $\mathcal{C}\mathrm{h}_+(\mathcal{P})$, the homotopy equivalences and quasi-isomorphisms coincide.*

Proof: It suffices to show that any quasi-isomorphism $f\colon P \to Q$ between bounded below complexes of projectives admits a weak inverse $g\colon Q \to P$.

Apply Proposition 6.6 to the quasi-isomorphism f, we see that there is a commutative square

$$\begin{tikzcd} H_0(Q,P) \arrow[r, "f\circ"] \arrow[d, "\circ f"'] & H_0(Q,Q) \arrow[d, "\circ f"] \\ H_0(P,P) \arrow[r, "f\circ"] & H_0(P,Q) \end{tikzcd}$$

in which both horizontal morphisms are isomorphisms. Then, there exists a morphism $g\colon Q \to P$ such that $f \circ g \simeq \mathrm{id}_Q$. It remains to show $g \circ f \simeq \mathrm{id}_P$. Now, consider $[g \circ f] \in H_0(P,P)$. Obviously, it image in $H_0(P,Q)$ is $[f]$. On the other hand, the image of $[\mathrm{id}_P] \in H_0(P,P)$ in $H_0(P,Q)$ is also $[f]$. Since the bottom morphism is an isomorphism, we conclude that $g \circ f \simeq \mathrm{id}_P$. $\square$

From this theorem, we see that the dg-category $\mathcal{C}\mathrm{h}_+(\mathcal{P})$ is a full subcategory of the should-be-existed derived category $\mathcal{D}_+(\mathcal{A})$. Conversely, it is natural to ask wither this dg-category is equivalent to the entire derived category $\mathcal{D}_+(\mathcal{A})$. To do this, we need to know what are the objects in this category.

2.2 Lemma *A complex X is quasi-isomorphic to a bounded front complex if and only if $H_n(K) = 0$ for sufficient small n.*

Proof: Just apply the truncation functor τ_n. $\square$

Therefore, to show $\mathcal{C}\mathrm{h}_+(\mathcal{P})$ is essentially the derived category $\mathcal{D}_+(\mathcal{A})$, it suffices to show the following lemma.

2.3 Lemma *Any bounded front complex X admits a projective resolution, which means a projective trivial fibration $P \to X$.*

Proof: We construct this chain map by induction. To simply notations, we assume X is connective.

Since $\mathcal{A}$ has enough projective objects, there is an epimorphism $f_0\colon P_0 \to X_0$ from a projective object P_0 in $\mathcal{A}$. Assume we already have epimorphisms $f_k\colon P_k \to X_k$ from projective objects P_k and morphisms $d_k\colon P_k \to P_{k-1}$ in $\mathcal{A}$ for all $0 < k < n$ such that

1. $d_{k-1} \circ d_k = 0$ for all $1 < k < n$;
2. $\partial_k \circ f_k = f_{k-1} \circ d_k$ for all $0 < k < n$;
3. those f_k induces isomorphisms $\mathrm{Coker}(\mathrm{Im}(d_{k+1}) \hookrightarrow \mathrm{Ker}(d_k)) \cong H_k(X)$ for all $0 < k < n-1$.

Now, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(d_{n-1}) \arrow[r] \arrow[d, dashed, "g"] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & P_{n-2} \arrow[d, "f_{n-2}"] \\
0 \arrow[r] & \mathrm{Ker}(\partial_{n-1}) \arrow[r] & K_{n-1} \arrow[r] & K_{n-2}
\end{tikzcd}$$

where the rows are exact sequences and the morphism g is obtained from the universal property of kernel. Let the following be a Cartesian diagram.

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "g'"] & \mathrm{Ker}(d_{n-1}) \arrow[d, "g"] \\
K_n \arrow[r] & \mathrm{Ker}(\partial_{n-1})
\end{tikzcd}$$

Then we have a commutative diagram

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "g'"] & \mathrm{Ker}(d_{n-1}) \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, dashed, "h"] & 0 \\
K_n \arrow[r] & \mathrm{Ker}(\partial_{n-1}) \arrow[r] & H_{n-1}(K) \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact sequences and the morphism h is obtained from the universal property of cokernel. If g is an epimorphism, then so is g' and the previous Cartesian diagram is also co-Cartesian. Then h is an isomorphism. Now, let $P_n \to B$ the an arbitrary epimorphism from a projective object P_n, let f_n be the composition $P_n \to B \to K_n$ and let d_n be the composition $P_n \to B \to \mathrm{Ker}(d_{n-1}) \to P_{n-1}$, then we have

1. $d_{n-1} \circ d_n = 0$;
2. $\partial_n \circ f_n = f_{n-1} \circ d_n$;
3. $\mathrm{Coker}(\mathrm{Im}(d_n) \hookrightarrow \mathrm{Ker}(d_{n-1})) = C \cong H_{n-1}(K)$.

Continuing this process, we obtain the desired complex P.

It remains to show that at each step, the induced morphism $\mathrm{Ker}(d_n) \to \mathrm{Ker}(\partial_n)$ is an epimorphism. Indeed, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(d_n) \arrow[r] \arrow[d, dashed] & P_n \arrow[r] \arrow[d] & \mathrm{Im}\, d_n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\partial_n) \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, "g'"] & \mathrm{Im}\, d_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\partial_n) \arrow[r] & K_n \arrow[r] & \mathrm{Im}\, \partial_n \arrow[r] & 0
\end{tikzcd}$$

where each row is an exact sequence. Using 5-lemma, we conclude that the induced morphism $\mathrm{Ker}(d_n) \to \mathrm{Ker}(\partial_n)$ is an epimorphism. □

Summarizing the above results, we give the following definition.

> Let $\mathcal{A}$ be an abelian category having enough projective objects. Let $\mathcal{P}$ denote the full subcategory of $\mathcal{A}$ spanned by projective objects. Then the ***bounded front derived category*** $\mathcal{D}_+(\mathcal{A})$ is the dg-category $\mathcal{C}\mathrm{h}_+(\mathcal{P})$.

Remark It is widely called the ***bounded above derived category*** and denoted by $\mathcal{D}^-(\mathcal{A})$.

The same story happens for injective objects. To be explicit, let $\mathcal{A}$ be an abelian category having enough injective objects and let $\mathcal{I}$ denote the full subcategory of $\mathcal{A}$ spanned by injective objects. Then we also have the follows.

2.4 Lemma *In $\mathcal{C}\mathrm{h}^+(\mathcal{I})$, the homotopy equivalences and quasi-isomorphisms coincide.*

2.5 Lemma *A complex X is quasi-isomorphic to a bounded back complex if and only if $H^n(K) = 0$ for sufficient small n.*

2.6 Lemma *Any bounded back complex X admits a injective resolution, which means a injective trivial fibration $X \to I$.*

Therefore, we also meet the following definition.

> Let $\mathcal{A}$ be an abelian category having enough injective objects. Let $\mathcal{I}$ denote the full subcategory of $\mathcal{A}$ spanned by injective objects. Then the ***bounded back derived category*** $\mathcal{D}^+(\mathcal{A})$ is the dg-category $\mathcal{C}h^+(\mathcal{I})$.

Remark It is widely called the ***bounded below derived category***.

Now, assume that $\mathcal{A}$ has both enough projective objects and injective objects. Then both $\mathcal{D}^-(\mathcal{A})$ and $\mathcal{D}^+(\mathcal{A})$ exist. The intersection of them is the ***bounded derived category*** $\mathcal{D}^b(\mathcal{A})$.

3 Unbounded derived category

Now, we consider the entire derived category $\mathcal{D}(\mathcal{A})$. To allow some constructions, we need some assumption on the abelian category $\mathcal{A}$.

First, motivated by Lemma 6.5.1, we define a ***K-injective complex*** to be a complex I such that for any acyclic complex K, the intrinsic cohomology $H^\bullet(K, I)$ is zero. Note that this equivalent to say that any morphism $K \to I$ is homotopic to zero.

3.1 Lemma *Let $\mathcal{A}$ be an abelian category and I be a complex. The following are equivalent.*

1. *I is K-injective.*
2. *The intrinsic cohomology functor $H^\bullet(-, I)$ maps quasi-isomorphisms to isomorphisms.*

Proof: Assume *1.* Let $f \colon X \to Y$ be a quasi-isomorphism. By Proposition 4.1, $\mathrm{Fib}(f)$ is acyclic and by Remark 4.1.3, we have a long exact sequence

$$\longrightarrow H^{n-1}(\mathrm{Fib}(f), I) \longrightarrow H^n(Y, I) \longrightarrow H^n(X, I) \longrightarrow H^n(\mathrm{Fib}(f), I) \longrightarrow$$

which implies $H^\bullet(X, I) \cong H^\bullet(Y, I)$.

Assume *2.* Let K be acyclic, then there is a quasi-isomorphism $K \to 0$. Then, apply the intrinsic cohomology functor $H^\bullet(-, I)$, we see that $H^\bullet(K, I) \cong H^\bullet(0, I) = 0$. $\square$

3.2 Lemma *In the subcategory of K-injective complexes, quasi-isomorphisms are homotopy equivalences.*

Proof: Let $f\colon K \to I$ be a quasi-isomorphism between K-injective complexes, we need to show it admits a weak inverse $g\colon I \to K$. Consider the following commutative diagram.

$$\begin{tikzcd}
H^0(I,K) \arrow[r, "f\circ"] \arrow[d, "\circ f"'] & H^0(I,I) \arrow[d, "\circ f"] \\
H^0(K,K) \arrow[r, "f\circ"] & H^0(K,I)
\end{tikzcd}$$

Then the result is clear. $\square$

AB5 $\mathcal{A}$ has colimits and filtered colimits of exact sequences are exact.

An abelian category is called a ***Grothendeick category*** if it satisfies AB5 and has a *generator* S in the sense that the functor $\mathrm{Hom}_{\mathcal{A}}(S,-)$ is faithful.

Plan:

AB5 -> left resolution. K-inj -> right resolution.

§ 8 Homotopy limits

§ 9 Dg-category theory

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