Note on

Spectral Sequences

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§ 1 Spectral sequences

Let $\mathcal{A}$ be an abelian category. Note that any abelian category an be exactly embedded into a category of modules over a commutative ring.

1.1 A **spectral sequence** consists of the following data:

- a sequence of **pages** $\{E_r\}_r$, which are objects in $\mathcal{A}$;
- a sequence of **differentials** $\{d_r \colon E_r \to E_r\}_r$, which are morphisms in $\mathcal{A}$ satisfying $d_r^2 = 0$;

- for each n, an automorphism $H(E_r) \to E_{r+1}$, where $H(E_r)$ means $\operatorname{Ker} d_r / \operatorname{Im} d_r$.

Let (E_r, d_r) and (E'_r, d'_r) be two spectral sequences, a **morphism** between them is a sequence of morphism $u_r \colon E_r \to E'_r$ commuting with the differentials and isomorphisms, i.e. the following diagrams commute.

$$\begin{tikzcd} E_r \arrow[r, "u_r"] \arrow[d, "d_r"'] & E'_r \arrow[d, "d'_r"] \\ E_r \arrow[r, "u_r"] & E'_r \end{tikzcd} \qquad \begin{tikzcd} H(E_r) \arrow[r, "H(u_r)"] \arrow[d, "\cong"] & H(E'_r) \arrow[d, "\cong"] \\ E_{r+1} \arrow[r, "u_{r+1}"] & E'_{r+1} \end{tikzcd}$$

Remark A **differential object** is an object E equipped with a morphism $d \colon E \to E$ such that $d^2 = 0$. For a differential object, its *cohomology* is the quotient $\operatorname{Ker} d / \operatorname{Im} d$. One can verify that taking cohomology H is an additive functor. In the above diagrams, the left one shows u_r is an endomorphism of the differential object (E_r, d_r), thus induces the morphism $H(u_r)$ in the right diagram.

In a spectral sequence, each page is the cohomology of the previous page. One can expect there is the *final page*, by infinitely tracking the pages. However, this is ill-defined for infinite times of operations is ill-behaved in logic. But if for enough large r, the differentials $d_r, d_{r+1}, \cdots$ vanish, then all pages after one finite number stay the same. In this case, we can safely define the **limit object** to be any page after that number. We say the spectral sequence **collapses** if this is the case.

Remark The spectral sequences in $\mathcal{A}$ from an additive category. It is almost always NOT an abelian category. But we have:

1.2 Lemma *If F is an exact functor and $(E_r, d_r)_r$ is a spectral sequence, then so is $(F(E_r), F(d_r))_r$.*

1.3 Given a spectral sequence $(E_r, d_r)_{r \geqslant 0}$, there is an **associated filtration**

$$0 = B_0 \subset B_1 \subset \cdots \subset B_r \subset \cdots \subset B_\infty \subset Z_\infty \subset \cdots \subset Z_r \subset \cdots \subset Z_1 \subset Z_0 = E_0$$

where

1. $B_1 = \operatorname{Im} d_0, Z_1 = \operatorname{Ker} d_0$ and thus $Z_1/B_1 \cong E_1$, so we can identify d_1 as a morphism from Z_1/B_1 to Z_1/B_1;

2. for each r, assuming we have constructed B_r, Z_r and identify d_r as a morphism from Z_r/B_r to Z_r/B_r, then B_{r+1} is the preimage of $\operatorname{Im} d_r$ in E_0 and Z_{r+1} is the preimage of $\operatorname{Ker} d_r$ in E_0:

$$\begin{tikzcd} B_{r+1} \arrow[d, two heads] & \subset & Z_{r+1} \arrow[d, two heads] & \subset & E_0 \arrow[d, two heads] \\ \operatorname{Im} d_r & \subset & \operatorname{Ker} d_r & \subset & E_0/B_r \end{tikzcd}$$

more precisely, we have following Cartesian diagrams:

$$\begin{tikzcd} B_r \arrow[r, phantom, "\subset"] \arrow[d, two heads] & B_{r+1} \arrow[r, phantom, "\subset"] \arrow[d, two heads] & Z_{r+1} \arrow[r, phantom, "\subset"] \arrow[d, two heads] & Z_r \arrow[d, two heads] \\ 0 \arrow[r, phantom, "\subset"] & \operatorname{Im} d_r \arrow[r, phantom, "\subset"] & \operatorname{Ker} d_r \arrow[r, phantom, "\subset"] & Z_r/B_r \end{tikzcd}$$

3. $B_\infty = \bigcup_r B_r, Z_\infty = \bigcap_r Z_r$.

Remark The objects B_∞, Z_∞ may NOT exist. If they do exist,we say the spectral sequence **weakly converges** and call the quotient $E_\infty := Z_\infty / B_\infty$ the **limit object** of it. One can see that if the spectral sequence collapses, then the limit object exists and coincide with the one we defined before.

Note that if we start from n-th page, we also have a spectral sequence and its associated filtration

$$0 = B'_n \subset \cdots \subset B'_r \subset \cdots \subset Z'_r \subset \cdots \subset Z'_n = E_n.$$

What is the relation between it and the previous one? In fact the answer lies in following Cartesian diagrams

$$\begin{tikzcd} B_n \arrow[r, phantom, "\subset"] \arrow[d, two heads] & B_r \arrow[r, phantom, "\subset"] \arrow[d, two heads] & Z_r \arrow[r, phantom, "\subset"] \arrow[d, two heads] & Z_n \arrow[d, two heads] \\ 0 \arrow[r, phantom, "\subset"] & B'_r \arrow[r, phantom, "\subset"] & Z'_r \arrow[r, phantom, "\subset"] & E_n \end{tikzcd}$$

Exact couples

A common modern tool to construct a spectral sequence is the follow one.

1.4 An **exact couple** consists of the following data:

- two objects A, E;
- three morphisms $f\colon A \to E, g\colon E \to A$ and $\alpha\colon A \to A$;

such that

1. $\operatorname{Ker} f = \operatorname{Im} \alpha$;
2. $\operatorname{Ker} g = \operatorname{Im} f$;
3. $\operatorname{Ker} \alpha = \operatorname{Im} g$.

In other words, an exact couple is a special type of exact sequence:

$$\cdots \xrightarrow{f} E \xrightarrow{g} A \xrightarrow{\alpha} A \xrightarrow{f} E \xrightarrow{g} \cdots$$

Remark Therefore, together with the obvious notion of morphisms, the category of exact couples form a full subcategory of the abelian category of complexes. What are the kernels and cokernels?

1.5 An exact couple encodes a spectral sequence.

1. First, let $d = f \circ g$, then it is clear that $d^2 = 0$. Let the differential object (E, d) be the first page.

2. Let E' be the quotient $\operatorname{Ker} d / \operatorname{Im} d$, it is supposed to be the next page.

3. Since $\operatorname{Im} d \subset \operatorname{Im} f = \operatorname{Ker} g$, the morphism g induces a morphism from E' to A. Note that $f(g(\operatorname{Ker} d)) = 0$, hence the above morphism lands in $\operatorname{Ker} f = \operatorname{Im} \alpha$. Denote this morphism as g'.

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r] & A \arrow[r, "f"] & E \\
 & E' \arrow[u, dashed] \arrow[ul, dotted, "g'"] & \operatorname{Ker} d \arrow[l, two heads] \arrow[ul, "g"']
\end{tikzcd}$$

4. Let $A' = \operatorname{Im} \alpha$. We denote the restriction of α on it by α'.

5. Obviously $\operatorname{Ker} g \subset \operatorname{Ker} d$, hence we can treat f as a morphism from A to $\operatorname{Ker} d$. Since $f(\operatorname{Ker} \alpha) = f(\operatorname{Im} g) = \operatorname{Im} d$, the morphism f induces a morphism $f' \colon A' \to E'$.

$$\begin{tikzcd}
\operatorname{Ker} \alpha \arrow[r, hook] & A \arrow[r, two heads, "\alpha"] \arrow[d, "f"'] & A' \arrow[d, dashed, "f'"] \\
 & \operatorname{Ker} d \arrow[r, two heads] & E'
\end{tikzcd}$$

6. Now we get another couple $(A', E', \alpha', f', g')$.

7. The preimage of $\operatorname{Ker} f'$ under α is the kernel of the composition

$$A \xrightarrow{f} \operatorname{Ker} d$$

which is A', thus $\operatorname{Ker} f' = \operatorname{Im} \alpha'$.

8. The preimage of $\operatorname{Ker} g'$ under the projection $\operatorname{Ker} d \to E'$ is $\operatorname{Ker} g$ which equals $\operatorname{Im} f$, thus by the definition of f', $\operatorname{Ker} g' = \operatorname{Im} f'$.

9. $\operatorname{Im} g'$ equals the image of $\operatorname{Ker} d$ under g, which by the definition of d is the same as $\operatorname{Ker} f \cap \operatorname{Im} g$, which equals $\operatorname{Im} \alpha \cap \operatorname{Ker} \alpha = \operatorname{Ker} \alpha'$.

10. Therefore, $(A', E', \alpha', f', g')$ is an exact couple. Let $d' = f' \circ g'$, we get the second page (E', d'). Moreover, we can repeat the above process to get further pages.

This spectral sequence is called the **associated spectral sequence** of the exact couple.

Remark The above construction extends to an additive functor from the category of exact couples to the category of spectral sequences.

1.6 Now, we consider the *associated filtration* of it.

1. $B_1 = \operatorname{Im} d$ which equals $f(\operatorname{Ker}\alpha)$, seeing 5 in the above.

2. $Z_1 = \operatorname{Ker} d$ which equals $g^{-1}(\operatorname{Im}\alpha)$, seeing 9 in the above.

3. For any element $x\colon T \to Z_1$, the image of its class in E_1 under d_1 is given as follow: choose $y\colon T \to A$ such that $g \circ x = \alpha \circ y$, then $d_1 \circ [x] = [f \circ y]$, seeing the following digram.

$$\begin{tikzcd}
Z_1 \arrow[d] \arrow[dr, "g"] & A \arrow[r, "f"] \arrow[d, "\alpha"] & Z_1 \arrow[d] \\
E_1 \arrow[r, "g'"] & A' \arrow[r, "f'"] & E_1
\end{tikzcd}$$

   Remark Note that an *element* in an abelian category is not just a generalized element $T \to X$ but an equivalent class, meaning all its refinements.

4. Using induction, in general we have $B_r = f(\operatorname{Ker}\alpha^r), Z_r = g^{-1}(\operatorname{Im}\alpha^r)$ and the morphism d_r works as follows: for any $x\colon T \to Z_r, y\colon T \to A$ such that $g \circ x = \alpha^r \circ y$, we have $d_r \circ [x] = [f \circ y]$.

$$\begin{tikzcd}
Z_r \arrow[d] \arrow[dr, "g"] & A \arrow[r, "f"] \arrow[d, "\alpha^r"] & Z_r \arrow[d] \\
E_r \arrow[r] & \operatorname{Im}\alpha^r \arrow[r] & E_r
\end{tikzcd}$$

5. It is clear now that $B_\infty = f(\bigcup_r \operatorname{Ker}\alpha^r)$, $Z_\infty = g^{-1}(\bigcap_r \operatorname{Im}\alpha^r)$.

Spectral sequences with shifts

1.7 A **spectral sequence with shifts** consists of the following data:

- a sequence of **pages** $\{E_r\}_r$, which are objects in $\mathcal{A}$;
- a sequence of **shifts** τ_r, which are isomorphisms of $\mathcal{A}$;
- a sequence of **differentials** $\{d_r\colon E_r \to \tau_r E_r\}_r$, which are morphisms in $\mathcal{A}$ satisfying $d_r \circ \tau_r^{-1} d_r = 0$; (hence, in each page, we have a cochain complex

$$\cdots \longrightarrow \tau_r^{-1}E_r \xrightarrow{\tau_r^{-1}d_r} E_r \xrightarrow{d_r} \tau_r E_r \longrightarrow \cdots$$

  then it makes sense to call the quotient $\operatorname{Ker}\tau_r^k d_r / \operatorname{Im}\tau_r^{k-1}d_r$ as the **k-th cohomology** of E_r, denoted by $H^k(E_r)$)

- for each n, an automorphism $H^0(E_r) \to E_{r+1}$.

A **morphism** of spectral sequences with shifts (E_r, τ_r, d_r) and (E'_r, τ'_r, d'_r) is then a sequence of morphisms $u_r \colon E_r \to E'_r$ commute with differentials and the isomorphisms.

1.8 Like the no-shifts case, a spectral sequence with shifts $(E_r, \tau_r, d_r)_{r\geqslant 0}$ admits an **associated filtration**

$$0 = B_0 \subset B_1 \subset \cdots \subset B_r \subset \cdots \subset B_\infty \subset Z_\infty \subset \cdots \subset Z_r \subset \cdots \subset Z_1 \subset Z_0 = E_0$$

where

1. $B_1 = \operatorname{Im} \tau_0^{-1} d_0, Z_1 = \operatorname{Ker} d_0$ and thus $Z_1/B_1 \cong E_1$, so we can identify d_1 as a morphism from Z_1/B_1 to Z_1/B_1;
2. for each r, assuming we have constructed B_r, Z_r and identify d_r as a morphism from Z_r/B_r to Z_r/B_r, then B_{r+1} is the preimage of $\operatorname{Im} \tau_r^{-1} d_r$ in E_0 and Z_{r+1} is the preimage of $\operatorname{Ker} d_r$ in E_0:
$$\begin{tikzcd} B_{r+1} \arrow[d, two heads] \arrow[r, phantom, "\subset"] & Z_{r+1} \arrow[d, two heads] \arrow[r, phantom, "\subset"] & E_0 \arrow[d, two heads] \\ \operatorname{Im} \tau_r^{-1} d_r \arrow[r, phantom, "\subset"] & \operatorname{Ker} d_r \arrow[r, phantom, "\subset"] & E_0/B_r \end{tikzcd}$$
more precisely, we have following Cartesian diagrams:
$$\begin{tikzcd} B_r \arrow[d, two heads] \arrow[r, phantom, "\subset"] & B_{r+1} \arrow[d, two heads] \arrow[r, phantom, "\subset"] & Z_{r+1} \arrow[d, two heads] \arrow[r, phantom, "\subset"] & Z_r \arrow[d, two heads] \\ 0 \arrow[r, phantom, "\subset"] & \operatorname{Im} \tau_r^{-1} d_r \arrow[r, phantom, "\subset"] & \operatorname{Ker} d_r \arrow[r, phantom, "\subset"] & Z_r/B_r \end{tikzcd}$$
3. $B_\infty = \bigcup_r B_r, Z_\infty = \bigcap_r Z_r$.

Exact couples with shifts

Now, we consider the shifting version of exact couples and their related spectral sequences.

1.9 An **(twisted) exact couple** consists of the following data:

- two objects A, E;
- two shifts σ, τ, i.e. isomorphisms of $\mathcal{A}$;
- three morphisms $f \colon A \to \sigma E, g \colon E \to A$ and $\alpha \colon \tau A \to A$;

such that the following sequence is exact.

$$\tau E \xrightarrow{\tau g} \tau A \xrightarrow{\alpha} A \xrightarrow{f} \sigma E \xrightarrow{\sigma g} \sigma A$$

We visualize this as follows (i.e. the shift σ is "*go downstairs*" while τ is "*go backward*", in particular, we call the red data as the primitive ones)

$$\begin{tikzcd}
\ddots & & \ddots & \\
\tau E \arrow[rr, "\tau g"] \arrow[d, dotted, "\tau d"'] & & \tau A \arrow[dll, "\tau f"'] \arrow[ddr, bend left, red, "\alpha"] & \\
\tau\sigma E \arrow[rr, "\tau\sigma g"'] & & \tau\sigma A & \\
& \color{red}{E} \arrow[rr, red, "g"] \arrow[d, dotted, "d"'] & & \color{red}{A} \arrow[dll, red, "f"] \\
& \sigma E \arrow[rr, "\sigma g"'] & & \sigma A
\end{tikzcd}$$

1.10 Then, we draw the **associated spectral sequence** from a given exact couple $(A, E, \sigma, \tau, f, g, \alpha)$.

1. First, let $d = f \circ g$, then it is clear that $\sigma d \circ d = 0$. Let the triple (E, σ, d) be the first page.

2. Let E' be the quotient $\operatorname{Ker} d / \operatorname{Im} \sigma^{-1} d$, it is supposed to be the next page.

3. Since $\operatorname{Im} d \subset \operatorname{Im} f = \operatorname{Ker} \sigma g$, we have $\operatorname{Im} \sigma^{-1} d \subset \operatorname{Im} \sigma^{-1} f = \operatorname{Ker} g$. Then the morphism g induces a morphism from E' to A. Note that $f(g(\operatorname{Ker} d)) = 0$, hence the above morphism lands in $\operatorname{Ker} f = \operatorname{Im} \alpha$. Denote this morphism as g'.

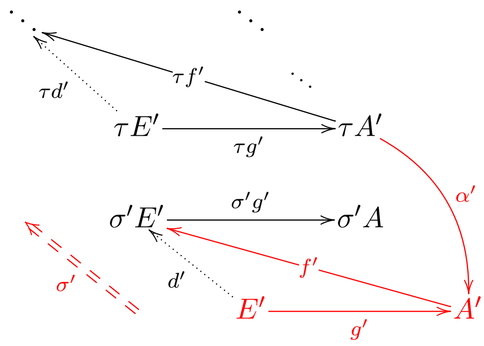
$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r] & A \arrow[r, "f"] & \sigma E \\
& E' \arrow[u, dashed] \arrow[ul, dotted, "g'"] & \operatorname{Ker} d \arrow[l, two heads] \arrow[ul, "g"']
\end{tikzcd}$$

4. Let $A' = \operatorname{Im} \alpha$. We denote the restriction of α on $\tau A'$ by α'.

5. Obviously $\operatorname{Ker} g \subset \operatorname{Ker} d$. Thus, $\operatorname{Im} f = \operatorname{Ker} \sigma g \subset \operatorname{Ker} \sigma d$. Hence we can treat f as a morphism from A to $\operatorname{Ker} \sigma d$. Consider the composition of $\tau f \colon \tau A \to \tau \operatorname{Ker} \sigma d$ with the inclusion $\operatorname{Ker} \alpha \subset \tau A$, since $\operatorname{Ker} \alpha = \operatorname{Im} \tau g$, the compostion lands in $\operatorname{Im} \tau d$. Thus the composition of the left and below morphisms in the following diagram equals 0. This gives rise to a unique $f' \colon A' \to \sigma E'$ making the diagram commute.

$$\begin{tikzcd}
\operatorname{Ker} \alpha \arrow[r, hook] & \tau A \arrow[r, two heads, "\alpha"] \arrow[d, "\tau f"'] & A' \arrow[d, dashed, "f'"] \\
& \tau \operatorname{Ker} \sigma d \arrow[r, two heads] & \tau\sigma E'
\end{tikzcd}$$

6. Now we get another couple $(A', E', \tau\sigma, \tau, \alpha', f', g')$. We visualize it as follows



7. The preimage of $\operatorname{Ker} f'$ under α is the kernel of the composition

$$\tau A \xrightarrow{\tau f} \tau \operatorname{Ker} \sigma d$$

which is $\tau A'$, thus $\operatorname{Ker} f' = \operatorname{Im} \alpha'$.

8. The preimage of $\operatorname{Ker} g'$ under the projection $\operatorname{Ker} d \to E'$ is $\operatorname{Ker} g$. So, the preimage of $\operatorname{Ker} \tau\sigma g'$ under the projection $\tau \operatorname{Ker} \sigma d \to \tau\sigma E'$ is $\operatorname{Ker} \tau\sigma g$ which equals $\operatorname{Im} \tau f$. Therefore, by the definition of f', $\operatorname{Ker} \tau\sigma g' = \operatorname{Im} f'$.

9. $\operatorname{Im} g'$ equals the image of $\operatorname{Ker} d$ under g, which by the definition of d is the same as $\operatorname{Ker} f \cap \operatorname{Im} g$, which equals $\operatorname{Im} \alpha \cap \operatorname{Ker} \tau^{-1}\alpha = \operatorname{Ker} \tau^{-1}\alpha'$.

10. Therefore, $(A', E', \tau\sigma, \tau, \alpha', f', g')$ is an exact couple. Let $d' = f' \circ g'$ and $\sigma' = \tau\sigma$, we get the second page (E', σ', d'). Moreover, we can repeat the above process to get further pages.

1.11 Now, we consider the *associated filtration* of it.

1. $B_1 = \operatorname{Im} \sigma^{-1} d$ and $\operatorname{Im} d$ equals $f(\operatorname{Ker} \tau^{-1}\alpha)$, seeing 5 in the above.

2. $Z_1 = \operatorname{Ker} d$ which equals $g^{-1}(\operatorname{Im} \alpha)$, seeing 9 in the above.

3. For any element $x \colon T \to Z_1$, the image of its class in E_1 under d_1 is given as follow: choose $y \colon T \to \tau A$ such that $g \circ x = \alpha \circ y$, then $d_1 \circ [x] = [\tau f \circ y]$, seeing the following digram.

$$\begin{tikzcd}
Z_1 \arrow[d] \arrow[dr, "g"] & \tau A \arrow[r, "\tau f"] \arrow[d, "\alpha"] & \tau\sigma Z_1 \arrow[d] \\
E_1 \arrow[r, "g'"] & A' \arrow[r, "f'"] & \tau\sigma E_1
\end{tikzcd}$$

4. Using induction, in general we have

$$\sigma B_r = f(\mathrm{Ker}(\tau^{-r}\alpha \circ \cdots \circ \tau^{-1}\alpha))$$
$$Z_r = g^{-1}(\mathrm{Im}(\alpha \circ \cdots \circ \tau^{r-1}\alpha))$$

and the morphism d_r works as follows: (to simplify notations, denote $\alpha \circ \cdots \circ \tau^{r-1}\alpha$ by $\alpha^{[r]}$) for any $x\colon T \to Z_r, y\colon T \to \tau^r A$ such that $g \circ x = \alpha^{[r]} \circ y$, we have $d_r \circ [x] = [\tau^r f \circ y]$.

$$\begin{tikzcd}
Z_r \arrow[d, two heads] \arrow[dr, "g"] & \tau^r A \arrow[r, "\tau^r f"] \arrow[d, two heads, "\alpha^{[r]}"] & \tau^r \sigma Z_r \arrow[d, two heads] \\
E_r \arrow[r, two heads] & \mathrm{Im}\,\alpha^{[r]} \arrow[r, tail] & \tau^r \sigma E_r
\end{tikzcd}$$

5. It is clear now that $B_\infty = f(\bigcup_r \mathrm{Ker}\,\tau^{-r}\alpha^{[r]})$, $Z_\infty = g^{-1}(\bigcap_r \mathrm{Im}\,\alpha^{[r]})$.

1.12 Given two exact couples $(A, E, \sigma, \tau, f, g, \alpha)$ and $(B, F, \sigma, \tau, h, k, \beta)$ twisted by same shifts σ, τ, a **morphism** between them is a pair (u, u^0) of morphisms making the following diagram commute.

$$\begin{tikzcd}
\tau E \arrow[r, "\tau g"] \arrow[d, "\tau u^0"'] & \tau A \arrow[r, "\alpha"] \arrow[d, "\tau u"'] & A \arrow[r, "f"] \arrow[d, "u"'] & \sigma E \arrow[r, "\sigma g"] \arrow[d, "\sigma u^0"'] & \sigma A \arrow[d, "\sigma u"'] \\
\tau F \arrow[r, "\tau k"] & \tau B \arrow[r, "\beta"] & B \arrow[r, "h"] & \sigma F \arrow[r, "\sigma k"] & \sigma B
\end{tikzcd}$$

Now, apply the process in 1.10 to those exact couples, we get another two exact couples $(A', E', \tau\sigma, \tau, f', g', \alpha')$ and $(B', F', \tau\sigma, \tau, h', k', \beta')$ twisted by same shifts. Furthermore, one can verify that

1. being restricted to A', u lands in B';
2. being restricted to $\mathrm{Ker}\,(f \circ g)$, u^0 lands in $\mathrm{Ker}\,(h \circ k)$, moreover, it induces a morphism $u^1\colon E' \to F'$;
3. the pair (u, u^1) gives a morphism between exact couples (A', E') and (B', F').

In this way, we see that a morphism of exact couples induces a morphism of associated spectral sequences. Moreover, such a construction together with our process of drawing a spectral sequence from an exact couple give rise to an additive functor from the category of exact couples to spectral sequences.

§ 2 Spectral sequences for differential objects

2.1 A **differential object** is an object A equipped with an endomorphism d such that $d^2 = 0$. For a differential object, it **cohomology** $H(A, d)$ is the quotient $\operatorname{Ker} d/\operatorname{Im} d$. A **morphism** between differential objects is a morphism commutes with the differentials.

One can verify that the differential objects form an abelian category and taking homology is an additive functor.

2.2 Lemma *Given a short exact sequence of differential objects*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

there is an associated long exact sequence

$$\cdots \longrightarrow H(A) \longrightarrow H(B) \longrightarrow H(C) \longrightarrow H(A) \longrightarrow \cdots.$$

Proof: This is a special case of the general long exact sequence associated to a short exact sequence of complexes. See any homology algebra reference.$\square$

2.3 (Associated spectral sequence) Let (A, d) be a differential object and let α be an injective endomorphism of it. Then we have a short exact sequence

$$0 \longrightarrow (A, d) \xrightarrow{\alpha} (A, d) \longrightarrow (A/\alpha, \bar{d}) \longrightarrow 0.$$

Apply the lemma to it, we get an exact couple

$$\begin{tikzcd}
H(A,d) \arrow[rr, "H(\alpha)"] & & H(A,d) \arrow[dl, "f"] \\
& H(A/\alpha, \bar{d}) \arrow[ul, "g"] &
\end{tikzcd}$$

where f is induced by the quotient map and g is the boundary map. From this exact couple, we immediately (·‿·)get a spectral sequence:

1. $E_0 = H(A/\alpha, \bar{d})$, $d_0 = f \circ g$.
2. Backward a page, $E_{-1} = A/\alpha$ and $d_{-1} = \bar{d}$.

Since start from -1-th page is a little weird, we shift the page numbers as follows. The spectral sequence start from $E_0 = \operatorname{Coker} \alpha, d_0 = \bar{d}$ and there is an associated filtration

$$0 = B_0 \subset B_1 \subset \cdots B_r \subset \cdots \subset Z_r \subset \cdots \subset Z_1 \subset Z_0 = E_0$$

such that $E_r = Z_r/B_r$. For the positive terms, we can get the desired filtration from the one obtained from the discussion under "exact couples":

1. $B_1 = \operatorname{Im} \bar{d}, Z_1 = \operatorname{Ker} \bar{d}$ and $d_1 = f \circ g$ works as follows: for any $z \in Z_1$, let z' be its lifting in A, choose $y \in A$ such that $d(z') = \alpha(y)$, then $d_1([z]) = [y]$.

2. B_r is the preimage of $f(\operatorname{Ker} H(\alpha)^{r-1})$ under the projection $Z_1 \to E_1$. To see what is it, first consider the preimage of $\operatorname{Ker} H(\alpha)^{r-1}$ under the projection $\operatorname{Ker} d \to H(A,d)$, which is not difficult to see equals $(\alpha^{r-1})^{-1}(\operatorname{Im} d)$.
$$\begin{tikzcd} (\alpha^{r-1})^{-1}(\operatorname{Im} d) \arrow[d, two heads] \arrow[r, phantom, "\subset"] & \operatorname{Ker} d \arrow[d, two heads] \\ \operatorname{Ker} H(\alpha)^{r-1} \arrow[r, phantom, "\subset"] & H(A,d) \end{tikzcd}$$
Composite it with the Cartesian diagram
$$\begin{tikzcd} \operatorname{Ker} d \arrow[r, two heads] \arrow[d, two heads] & Z_1 \arrow[d, two heads] \\ H(A,d) \arrow[r, two heads, "f"] & E_1 \end{tikzcd}$$
We see that the preimage of $f(\operatorname{Ker} H(\alpha)^{r-1})$ under the projection $Z_1 \to E_1$ is exactly the image of $(\alpha^{r-1})^{-1}(\operatorname{Im} d)$ under the projection $A \to A/\alpha$.

3. Z_r is the preimage of $g^{-1}(\operatorname{Im} H(\alpha)^{r-1})$ under the projection $Z_1 \to E_1$, which is preimage of $\operatorname{Im} H(\alpha)^{r-1}$ under the composition
$$Z_1 \longrightarrow E_1 \xrightarrow{g} H(A,d).$$
By looking at the definition of g carefully, one can show this is nothing but the image of $d^{-1}(\operatorname{Im} \alpha^r)$ under the projection $A \to A/\alpha$. (· ‿ ·)

4. Now E_r can be written as
$$E_r = \frac{d^{-1}(\operatorname{Im} \alpha^r) + \operatorname{Im} \alpha}{(\alpha^{r-1})^{-1}(\operatorname{Im} d) + \operatorname{Im} \alpha}.$$
Note that although $(\alpha^{r-1})^{-1}(\operatorname{Im} d) \subset d^{-1}(\operatorname{Im} \alpha^r)$, it is NOT always true that $\operatorname{Im} \alpha \subset d^{-1}(\operatorname{Im} \alpha^r)$.

5. The morphism d_r works as follows. For any $x + \operatorname{Im} \alpha \in Z_r$, consider its class $[x + \operatorname{Im} \alpha] \in g^{-1}(\operatorname{Im} H(\alpha)^{r-1})$. We know how d_r works as differential of
$$E_r' = \frac{g^{-1}(\operatorname{Im} H(\alpha)^{r-1})}{f(\operatorname{Ker} H(\alpha)^{r-1})},$$
which is: choose $y \in A$ such that $g[x + \operatorname{Im} \alpha] = [\alpha^{r-1}(y)]$, then d_r maps the class of $x + \operatorname{Im} \alpha$ in E_r' to $y + \operatorname{Im} \alpha$ in E_r'. By carefully review the definition of g, we may choose y such that
$$d(x) = \alpha^r(y).$$
Then, we have $d_r(x + \operatorname{Im} \alpha + B_r) = y + \operatorname{Im} \alpha + B_r$.

In twisted context

Now, we play the same game in twisted context. Let σ, τ be two isomorphisms of $\mathcal{A}$ and assume that $\tau\sigma = \sigma\tau$.

2.4 A **differential object (twisted by σ)** is an object A equipped with an morphism $d\colon A \to \sigma A$ such that $d \circ \sigma^{-1} d = 0$. Hence, given a differential object (A, d), we have a cochain complex

$$\cdots \longrightarrow \sigma^{-1} A \xrightarrow{\sigma^{-1} d} A \xrightarrow{d} \sigma A \longrightarrow \cdots$$

then for any k, it makes sense to call the quotient $\operatorname{Ker} \sigma^k d / \operatorname{Im} \sigma^{k-1} d$ as the **k-th cohomology** of (A, d), denoted by $H^k(A)$. For intuition, we also denote $\operatorname{Ker} \sigma^k d$ and $\operatorname{Im} \sigma^{k-1} d$ by $Z^k(A)$ and $B^k(A)$ respectively.

A **morphism** between differential objects is a morphism commutes with the differentials. One can verify that the differential objects form an abelian category and taking homologies are additive functors.

Since $\tau\sigma = \sigma\tau$, the functor τ defines a shift on the category of differential objects (twisted by σ) by setting $\tau(A, d) := (\tau A, \tau d)$.

We also have

2.5 Lemma *Given a short exact sequence of differential objects*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

there is an associated long exact sequence.

$$\cdots \longrightarrow H^{k-1}(C) \longrightarrow H^k(A) \longrightarrow H^k(B) \longrightarrow H^k(C) \longrightarrow H^{k+1}(A) \longrightarrow \cdots$$

2.6 (**Associated spectral sequence**) Now, given a differential object (A, d) and a monomorphism $\alpha\colon \tau(A, d) \to (A, d)$ with cokernel $(C, \bar{d})$. Then we can apply Lemma 2.5 to the short exact sequence

$$0 \longrightarrow \tau(A, d) \xrightarrow{\alpha} (A, d) \longrightarrow (C, \bar{d}) \longrightarrow 0.$$

Then, we get an exact couple twisted by shifts $(\tau\sigma, \tau)$ (the primitive data are emphasized by red color)

$$\begin{tikzcd}
\ddots & \ddots \\
H^{-1}(C) \arrow[r, "\sigma^{-1}g"] \arrow[d, dotted, "\sigma^{-1}d_0"'] & H^0(\tau A) \arrow[dl, "\sigma^{-1}f"'] \arrow[ddd, dashed, bend left, "H^0(\alpha)"] \\
\tau H^0(C) \arrow[r, "\tau g"] \arrow[d, dotted, "\tau d_0"'] & \tau H^1(\tau A) = \tau^2 H^1(A) \arrow[dl, "\tau f"'] \arrow[ddd, dashed, "\widehat{\alpha}"] \\
\tau^2 H^1(C) \arrow[r, "\tau^2\sigma g"'] & \tau^2 H^2(\tau A) \\
\tau^{-1}H^{-1}(C) \arrow[r, "(\tau\sigma)^{-1}g"] \arrow[d, dotted, "(\tau\sigma)^{-1}d_0"'] & H^0(A) \arrow[dl, "(\tau\sigma)^{-1}f"'] \\
H^0(C) \arrow[r, "g"] \arrow[d, dotted, "d_0"'] & H^1(\tau A) = \tau H^1(A) \arrow[dl, "f"'] \\
\tau H^1(C) \arrow[r, "\tau\sigma g"'] & \tau H^2(\tau A)
\end{tikzcd}$$

where f is obtained from the quotient map $A \to C$ by apply τH^1 and *g is the boundary map*. From this exact couple, we immediately ($\cdot$ ‿ $\cdot$) get a spectral sequence:

1. $E_0 = H^0(C)$ with $\tau_0 = \tau\sigma$, $d_0 = f \circ g$.

2. Backward a page, $E_{-1} = C$ and $\tau_{-1} = \sigma, d_{-1} = \bar{d}$.

Since start from -1-th page is a little weird, we shift the page numbers as follows. The spectral sequence start from $E_0 = C, d_0 = \bar{d}$ and there is an associated filtration

$$0 = B_0 \subset B_1 \subset \cdots B_r \subset \cdots \subset Z_r \subset \cdots \subset Z_1 \subset Z_0 = E_0$$

such that $E_r = Z_r/B_r$. For the positive terms, we can get the desired filtration from the one obtained from the discussion under "exact couples":

1. $B_1 = \operatorname{Im} \sigma^{-1}\bar{d}, Z_1 = \operatorname{Ker} \bar{d}$, $\tau_1 = \tau\sigma$ and $d_1 = f \circ g$ works as follows: for any $z \in Z_1$, let z' be its lifting in A, choose $y \in \tau\sigma A$ such that $d(z') = (\tau\alpha)(y)$, then $d_1([z]) = [y]$.

2. B_r is the preimage of $(\tau\sigma)^{-1}(f(\operatorname{Ker}(\tau^{1-r}\widehat{\alpha}^{[r-1]})))$ (here, we use $\widehat{\alpha}^{[k]}$ to denote the composition $\widehat{\alpha} \circ \cdots \circ \tau^{k-1}\widehat{\alpha}$) under the projection $Z_1 \to E_1$.

   To see what is it, as the shift $(\tau\sigma)^{-1}$ suggests, we go upstairs, focusing on the blue morphisms in the big diagram. Let's first consider the preimage of $(\tau\sigma)^{-1}\operatorname{Ker}(\tau^{1-r}\widehat{\alpha}^{[r-1]})$, which equals $\operatorname{Ker}(H^0(\tau^{1-r}\alpha^{[r-1]}))$, under the projection $Z^0(A) \to H^0(A)$. It is not difficult to see that it equals $(\tau^{1-r}\alpha^{[r-1]})^{-1}(B^0(\tau^{1-r}A))$.

$$\tau^{1-r}\alpha^{[r-1]}: \quad A \xrightarrow{\tau^{-1}\alpha} \tau^{-1}A \xrightarrow{\tau^{-2}\alpha} \cdots \xrightarrow{\tau^{r-1}\alpha} \tau^{r-1}A.$$

Now, we get the Cartesian diagram

$$\begin{tikzcd}
(\tau^{1-r}\alpha^{[r-1]})^{-1}(B^0(\tau^{1-r}A)) \arrow[r, phantom, "\subset"] \arrow[d, two heads] & Z^0(A) \arrow[d, two heads] \\
\operatorname{Ker}(H^0(\tau^{1-r}\alpha^{[r-1]})) \arrow[r, phantom, "\subset"] & H^0(A)
\end{tikzcd}$$

Composite it with the Cartesian diagram

$$\begin{tikzcd}
Z^0(A) \arrow[r] \arrow[d, two heads] & Z_1 \arrow[d, two heads] \\
H^0(A) \arrow[r, "f"] & E_1
\end{tikzcd}$$

We see that the preimage of $f(\operatorname{Ker}(H^0(\tau^{1-r}\alpha^{[r-1]})))$ under the projection $Z_1 \to E_1$ is exactly the image of $(\tau^{1-r}\alpha^{[r-1]})^{-1}(B^0(\tau^{1-r}A))$ under the projection $A \to C$.

3. Z_r is the preimage of $g^{-1}(\operatorname{Im}\widehat{\alpha}^{[r-1]})$ under the projection $Z_1 \to E_1$, which is preimage of $\operatorname{Im}\widehat{\alpha}^{[r-1]}$ under the composition

$$Z_1 \longrightarrow E_1 \xrightarrow{g} H^1(\tau A).$$

Note that preimage of $\operatorname{Im}\widehat{\alpha}^{[r-1]} \subset H^1(\tau A)$ in $\tau\sigma A$ is $\operatorname{Im}\tau\sigma\alpha^{[r-1]}$. By looking at the definition of g carefully, one can show this is nothing but the image of $d^{-1}(\operatorname{Im}\sigma\alpha^{[r]})$ under the projection $A \to C$. (· ‿ ·)

4. Now E_r can be written as

$$E_r = \frac{d^{-1}(\operatorname{Im}\sigma\alpha^{[r]}) + \operatorname{Im}\alpha}{(\tau^{1-r}\alpha^{[r-1]})^{-1}(B^0(\tau^{1-r}A)) + \operatorname{Im}\alpha}.$$

Again, note that it is NOT necessary that $\operatorname{Im}\alpha \subset d^{-1}(\operatorname{Im}\sigma\alpha^{[r]})$.

5. The morphism d_r works as follows. For $x + \operatorname{Im}\alpha \in Z_r$, we know how d_r works on its class $[x + \operatorname{Im}\alpha] \in g^{-1}(\operatorname{Im}\widehat{\alpha}^{[r-1]})$ as a differential of

$$E'_r = \frac{g^{-1}(\operatorname{Im}\widehat{\alpha}^{[r-1]})}{(\tau\sigma)^{-1}(f(\operatorname{Ker}(\tau^{1-r}\widehat{\alpha}^{[r-1]})))}$$

respect to the shift $\tau_r = \tau^r\sigma$. That is: choose $y \in \tau^r\sigma A$ such that $g[x + \operatorname{Im}\alpha] = [\tau\sigma\alpha^{[r-1]}(y)]$ in $H^1(\tau A)$, then d_r maps $[x + \operatorname{Im}\alpha] \in E'_r$

to $[y + \operatorname{Im}\alpha] \in \tau^r\sigma E'_r$. By carefully review the definition of g, we may choose y such that

$$d(x) = \sigma\alpha^{[r]}(y).$$

Then, we have $d_r(x + \operatorname{Im}\alpha + B_r) = y + \tau^r\sigma(\operatorname{Im}\alpha + B_r)$.

2.7 Given the following commutative diagram of differential objects

$$\begin{tikzcd}
0 \arrow[r] & \tau A \arrow[r, "\alpha"] \arrow[d, "\tau v"'] & A \arrow[r] \arrow[d, "v"'] & C \arrow[r] \arrow[d, "u"'] & 0 \\
0 \arrow[r] & \tau B \arrow[r, "\beta"] & B \arrow[r] & D \arrow[r] & 0
\end{tikzcd}$$

Suppose that the rows are exact. Then, by apply Lemma 2.5, we get a morphism $(\tau H^1(v), H^0(u))$ between exact couples. Then, we immediately (· ‿ ·) get a morphism between the associated spectral sequences.

$$(u, H^0(u), \cdots).$$

§ 3 Spectral sequences for filtered differential objects

Filtered objects

3.1 In an abelian category $\mathcal{A}$, we have the followings.

1. A **decreasing filtration** F on an object A is a family $(F^pA)_{p\in\mathbb{Z}}$ of subobjects of A such that

$$A \supset \cdots F^pA \supset F^{p+1}A \supset \cdots \supset 0.$$

2. A **filtered object** (A, F) is an object A equipped with a decreasing filtration F on it. A morphism between filtered objects is a **morphism** $f\colon A \to B$ compatible with the filtrations, meaning $f(F^pA) \subset F^pB$ for all $p \in \mathbb{Z}$. The category of filtered objects is denoted as $\operatorname{Fil}(\mathcal{A})$.

3. Given a filtered object (A, F) and a subobject $X \subset A$, the **induced filtration** on X is the filtration $F^pX := X \cap F^pA$.

4. Given a filtered object (A, F) and an epimorphism $\pi\colon A \to Y$, the **quotient filtration** on Y is the filtration $F^pY := \pi(F^pA)$.

5. A filtration F on an object A is said to be
   - **discrete** if it stays the same when p is large enough, i.e there exist m such that for all $p \geqslant m$,

$$F^pA = F^mA;$$

- **codiscrete** if it stays the same when p is small enough, i.e there exist m such that for all $p \leqslant m$,

$$F^p A = F^m A;$$

- **finite** if it is both discrete and codiscrete;

  Remark Usually, a *finite* filtration is also required to be *regular*.

- **separated** (or **Hausdorff**) if $\varprojlim F^p A = 0$;
- **exhaustive** if $\varinjlim F^p A = A$;

  Remark Note that the arrows in the directed system is given by inclusions of subobjects NOT the order of indexes p.

- **regular** if it is both separated and exhaustive;
- **complete** if $\varprojlim^1 F^p A = 0$.

  Remark Here $\varprojlim^1$ denotes the *first derived limit*, that is the first right derived functor of the limit functor over sequential diagrams.

6. A morphism between filtered objects is said to be **injective** (resp. **surjective**) if it is injective (resp. surjective) in $\mathcal{A}$. One can see this is equivalent to being a *monomorphism* (resp. *epimorphism*) in $\mathrm{Fil}(\mathcal{A})$.

7. It is clear that $\mathrm{Fil}(\mathcal{A})$ is an *additive category* with the direct sum $(A, F) \oplus (B, F)$ defined as $(A \oplus B, F)$ where the filtration F is given by $F^p(A \oplus B) = F^p A \oplus F^p B$.

8. $\mathrm{Fil}(\mathcal{A})$ has zero object 0 whose filtration is the trivial filtration. $\mathrm{Fil}(\mathcal{A})$ also has kernels and cokernels, whose filtrations are the induced filtrations and quotient filtrations.

9. However, $\mathrm{Fil}(\mathcal{A})$ is in general NOT abelian.

   Example Let $\mathcal{A}$ be the category of vector spaces over a field k. Let V, W both be k but the filtrations on them are given as follows:

   $$F^{-1}V = V \supset F^0 V = 0, \qquad F^0 W = W \supset F^1 W = 0.$$

   Let f be the morphism corresponding to id_k. Then this morphism is both injective and surjective but not an isomorphism.

3.2 Let $\mathcal{A}$ be an abelian category and $\mathrm{Gr}(\mathcal{A})$ the abelian category of graded objects in $\mathcal{A}$.

1. For any filtered object (A, F), let $\mathrm{gr}^p A$ denote quotient $F^p A / F^{p+1} A$. This gives rise to an additive functor $\mathrm{gr}^p \colon \mathrm{Fil}(\mathcal{A}) \to \mathcal{A}$.

2. Then, we have an additive functor $\mathrm{gr}\colon \mathrm{Fil}(\mathcal{A}) \to \mathrm{Gr}(\mathcal{A})$ which maps a filtered object (A, F) to a graded object $\mathrm{gr}\, A$ whose p-th component is $\mathrm{gr}^p A$.

Remark Note that for an abelian category $\mathcal{A}$ with countable direct sums, the functor $\mathrm{Gr}(\mathcal{A}) \to \mathcal{A} : (A_n) \mapsto \bigoplus_n A_n$ is NOT exact.

The following is a useful fact.

3.3 Lemma *Let A be a filtered object and X a subobject of A. Given X and A/X the induced and quotient filtrations respectively, we have the following exact sequences for each p:*

$$0 \longrightarrow \mathrm{gr}^p X \longrightarrow \mathrm{gr}^p A \longrightarrow \mathrm{gr}^p A/X \longrightarrow 0$$

Proof: First, since $F^{p+1}A \cap F^p X = F^{p+1}X$, the morphism $\mathrm{gr}^p X \to \mathrm{gr}^p A$ is injective. Dually, $\mathrm{gr}^p A \to \mathrm{gr}^p A/X$ is surjective. Finally, the kernel of $\mathrm{gr}^p A \to \mathrm{gr}^p A/X$ is $((F^{p+1}A + X) \cap F^p A)/F^{p+1}A$, which equals $F^p X/F^{p+1}X$. Hence the middle is exact. □

3.4 Lemma *Let $f\colon A \to B$ be a morphism of discrete and regular filtered objects. If $\mathrm{gr}\, f\colon \mathrm{gr}\, A \to \mathrm{gr}\, B$ is an isomorphism, then f itself is an isomorphism of filtered objects.*

Proof: Since F is exhaustive, it suffices to verify that $f(F^p A) = F^p B$ for all p. Since F is discrete and separated, we can prove it by descending induction on p: just apply 5-lemma to the commutative diagrams:

$$\begin{tikzcd}
0 \arrow[r] & F^{p+1}A \arrow[r] \arrow[d] & F^p A \arrow[r] \arrow[d] & \mathrm{gr}^p A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^{p+1}B \arrow[r] & F^p B \arrow[r] & \mathrm{gr}^p B \arrow[r] & 0
\end{tikzcd}$$

□

3.5 A morphism $f\colon (A, F) \to (B, F)$ of filtered objects is said to be **strict** if $f(F^p A) = \mathrm{Im}\, f \cap F^p B$ for all $p \in \mathbb{Z}$. Note that this is equivalent to say $f^{-1}(F^p B) = F^p A + \mathrm{Ker}\, f$ for all $p \in \mathbb{Z}$.

Moreover, Consider the two filtrations on $\mathrm{Im}\, f$ (note that, as objects in $\mathcal{A}$, $\mathrm{Coim}\, f$ is canonically isomorphic to $\mathrm{Im}\, f$), one is the quotient filtration on $\mathrm{Coim}\, f$, as a quotient of (A, F), and another is the induced filtration on $\mathrm{Im}\, F$, as subobject of (B, F). It is not difficult to see that, on $\mathrm{Im}\, f$, the previous filtration is given by

$$F_1^p \,\mathrm{Im}\, f = f(F^p A),$$

and the later filtration is given by

$$F_2^p \,\mathrm{Im}\, f = F^p B \cap \mathrm{Im}\, f.$$

In this way, we see that the following are equivalent:

1. $f(F^pA) = \operatorname{Im} f \cap F^pB$;

2. $f^{-1}(F^pB) = F^pA + \operatorname{Ker} f$;

3. The isomorphism $\operatorname{Coim} f \to \operatorname{Im} f$ preserves the filtrations.

Let $f\colon A \to B$ be a morphism of filtered objects. Then, by Lemma 3.3, there naturally exist two short exact sequences

$$0 \longrightarrow \operatorname{gr}\operatorname{Ker} f \longrightarrow \operatorname{gr} A \longrightarrow \operatorname{gr}\operatorname{Coim} f \longrightarrow 0,$$
$$0 \longrightarrow \operatorname{gr}\operatorname{Im} f \longrightarrow \operatorname{gr} B \longrightarrow \operatorname{gr}\operatorname{Coker} f \longrightarrow 0.$$

It is clear that the canonical morphism $\operatorname{Coim} f \to \operatorname{Im} f$ becomes an isomorphism after applying the functor gr if and only if these two short exact sequences can be pieced together to get the following exact sequence.

$$0 \longrightarrow \operatorname{gr}\operatorname{Ker} f \longrightarrow \operatorname{gr} A \xrightarrow{\operatorname{gr} f} \operatorname{gr} B \longrightarrow \operatorname{gr}\operatorname{Coker} f \longrightarrow 0.$$

Hence we have the following criteria of strictness.

3.6 Proposition *Let $f\colon A \to B$ be a morphism of discrete and regular filtered objects. Then the following are equivalent:*

1. *f is strict;*

2. *the canonical morphism $\operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism of filtered objects;*

3. *the morphism $\operatorname{gr}\operatorname{Coim} f \to \operatorname{gr}\operatorname{Im} f$ is an isomorphism;*

4. *the sequence*

$$0 \longrightarrow \operatorname{gr}\operatorname{Ker} f \longrightarrow \operatorname{gr} A \xrightarrow{\operatorname{gr} f} \operatorname{gr} B \longrightarrow \operatorname{gr}\operatorname{Coker} f \longrightarrow 0.$$

is exact.

Proof: It remains to show *3.⇒2.* which follows from Lemma 3.4. □

Remark Note that, from this proposition, if $f\colon A \to B$ is a strict morphism of filtered objects, then

$$\operatorname{Ker}\operatorname{gr} f = \operatorname{gr}\operatorname{Ker} f, \qquad \operatorname{Coker}\operatorname{gr} f = \operatorname{gr}\operatorname{Coker} f.$$

It is natural to ask if we restrict to strict morphisms only, is the resulted category abelian. At least, we have

3.7 Lemma *Let $f\colon A \to B$ be a morphism of filtered objects.*

1. *If f is injective, then it is strict if and only if the filtration on A is the induced filtration.*

2. *If f is surjective, then it is strict if and only if the filtration on B is the quotient filtration.*

Proof: This is immediate from definition. ☐

3.8 Lemma *Let $f\colon A \to B$ and $g\colon B \to C$ be strict morphisms of filtered objects.*

1. *If g is injective, then $g \circ f$ is strict.*

2. *If f is surjective, then $g \circ f$ is strict.*

Proof: Suppose g is injective, then we have

$$\begin{aligned} g(f(F^pA)) &= g(F^pB \cap f(A)) && \text{since } g \text{ is strict} \\ &= g(F^pB) \cap g(f(A)) && \text{since } g \text{ is injective} \\ &= F^pC \cap g(B) \cap g(f(A)) && \text{since } f \text{ is strict} \\ &= F^pC \cap g(f(A)). \end{aligned}$$

This shows $g \circ f$ is strict.

Suppose f is surjective, then we have

$$\begin{aligned} (g \circ f)^{-1}(F^pC) &= f^{-1}(F^pB + \operatorname{Ker} g) && \text{since } g \text{ is strict} \\ &= f^{-1}(F^pB) + f^{-1}(\operatorname{Ker} g) && \text{since } f \text{ is surjective} \\ &= F^pA + \operatorname{Ker} f + \operatorname{Ker} g \circ f && \text{since } f \text{ is strict} \\ &= F^pA + \operatorname{Ker} g \circ f. \end{aligned}$$

This shows $g \circ f$ is strict. ☐

However, the composition of strict morphisms may NOT be strict.

Example Let B be a vector spaces k^2 with basis e_1, e_2. Define the filtration F on B as

$$F^{-1}B = B \supset F^0B = ke_1 \supset F^1B = 0.$$

Let A be the subspace $k(e_1 + e_2)$ and C be the quotient B/ke_2 with the induced and quotient filtration respectively. By Lemma 3.7, this gives two strict morphisms

$$A \hookrightarrow B \twoheadrightarrow C.$$

However, the image of $F^0A = 0$ in C is 0, while the image of A in $F^0C = C$ is C. Therefore, the composition is not strict.

The following is a positive example, where the composition of two strict morphisms is again strict.

3.9 Lemma *Let (A, F) be a filtered object. Let $X \subset Y$ be subobjects of A. Then*

1. *On the object $Y/X \cong \operatorname{Ker}(A/X \to A/Y)$, the quotient filtration (as quotient of Y) and the induced filtration (as subobject of A/X) agree.*

2. *The composition $Y \to Y/X \to A/X$ is strict.*

Proof: The quotient filtration on Y/X is given by

$$F_1^p(Y/X) = F^pY/(X \cap F^pY) = F^pY/F^pX.$$

The induced filtration on Y/X is given by

$$\begin{aligned} F_2^p(Y/X) &= F^p(A/X) \cap (Y/X) \\ &= (F^pA + X)/X \cap (Y/X) \\ &= (F^pA \cap Y)/(F^pA \cap X) && \text{since } Y \supset X \\ &= F^pY/F^pX. \end{aligned}$$

Under the composition $Y \to Y/X \to A/X$, it is clear that the image of F^pY is F^pY/F^pX, which equals $F^p(A/X) \cap (Y/X)$ as shown above, $\square$

3.10 Lemma *The pullback of strict morphism is again strict. The pushforward of strict morphism is again strict.*

Proof: Since the two statements are dual to each other, it suffices to show the first one. Let $f\colon A \to B$ and $g\colon C \to B$ be two morphisms. Let D denote the fibred product of them. Then the pullback of f can be written as the composition

$$f'\colon D \longrightarrow A \oplus C \xrightarrow{\text{pr}} C.$$

Note that F^pD is the kernel of $\langle f, -g\rangle\colon F^pA \oplus F^pC \to F^pB$, thus we have

$$f'(F^pD) = \operatorname{pr}((F^pA \oplus F^pC) \cap \operatorname{Ker}\langle f, -g\rangle) = F^pC \cap g^{-1}(f(F^pA)).$$

Since f is strict, $f(F^pA) = F^pB \cap f(A)$. Then

$$f'(F^pD) = F^pC \cap g^{-1}(F^pB \cap f(A)) = F^pC \cap g^{-1}(f(A)) = F^pC \cap F'(D).$$

This shows f' is also strict. $\square$

Let $A \to B \to C$ be a complex of filtered objects with both $f\colon A \to B$ and $g\colon B \to C$ strict. Then, applying the functor gr, we have a complex of graded objects $\operatorname{gr} A \to \operatorname{gr} B \to \operatorname{gr} C$. Furthermore, by Proposition 3.6, we have

$$\operatorname{gr}(\operatorname{Ker} g/\operatorname{Im} f) = \operatorname{Ker}(\operatorname{gr} g)/\operatorname{Im}(\operatorname{gr} f).$$

In particular, if $A \to B \to C$ is exact, then so is $\operatorname{gr} A \to \operatorname{gr} B \to \operatorname{gr} C$.

Conversely, we have

3.11 Proposition *Let $A \to B \to C$ be a complex of filtered objects. Assume that the filtrations on A, B, C are finite, regular (or, discrete, exhaustive and $\mathcal{A}$ is an AB5 category[1]) and that the sequence $\operatorname{gr} A \longrightarrow \operatorname{gr} B \longrightarrow \operatorname{gr} C$ is exact. Then*

1. *the sequences $F^pA \to F^pB \to F^pC$ are exact;*
2. *the sequences $A/F^pA \to B/F^pB \to C/F^pC$ are exact;*
3. *the sequence $A \to B \to C$ is exact with both $A \to B$ and $B \to C$ strict.*

Proof: First, since A, B, C are discrete, we can prove *1* by descending induction on p: just apply 9-lemma to the following short exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] & F^{p+1}A \arrow[r] \arrow[d] & F^pA \arrow[r] \arrow[d] & \operatorname{gr}^p A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^{p+1}B \arrow[r] \arrow[d] & F^pB \arrow[r] \arrow[d] & \operatorname{gr}^p B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^{p+1}C \arrow[r] & F^pC \arrow[r] & \operatorname{gr}^p C \arrow[r] & 0
\end{tikzcd}$$

where, the right complex is exact by assumption, left complex is exact by induction hypothesis.

Now, if the filtrations on A, B, C are finite, we already get the exactness of $A \to B \to C$. If the filtrations are exhaustive and $\mathcal{A}$ satisfies AB5, then, by taking filtered colimit of the direct system given by filtrations, we have exact sequence

$$\varinjlim F^pA \longrightarrow \varinjlim F^pB \longrightarrow \varinjlim F^pC,$$

which equals the exact sequence $A \to B \to C$.

As for the exactness of $A/F^pA \to B/F^pB \to C/F^pC$, just apply 9-lemma to the following short exact sequence of complexes.

$$\begin{tikzcd}
0 \arrow[r] & F^pA \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A/F^pA \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^pB \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B/F^pB \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^pC \arrow[r] & C \arrow[r] & C/F^pC \arrow[r] & 0
\end{tikzcd}$$

Finally, it remains to show that $f\colon A \to B$ and $g\colon B \to C$ are strict. Note that $f(A) = \operatorname{Ker} g$. So, $f(F^pA) = \operatorname{Ker}(F^pB \to F^pC) = F^pB \cap \operatorname{Ker} g = F^pB \cap f(A)$. The proof for strictness of g is dual to this. $\square$

[1] which means: all (small) colimit exist and commute with finite limits.

Spectral sequences for filtered differential objects

Now, it's time to apply the machinery of spectral sequence to filtered differential objects, which are very close to our interest: filtered complexes.

3.12 A **filtered differential object** in $\mathcal{A}$ is a differential object in $\mathrm{Fil}(\mathcal{A})$. A **morphism** between filtered differential objects is then a morphism compatible with both the differentials and filtrations.

3.13 Before going on, let's recall a convention on graded objects: for any graded object $A = (A_n)$, $A[k]$ denote the **k-th shifted** of A, which is a graded object defined by $A[k]_n = A_{n+k}$; a morphism from A to $A[k]$ is called a **twisted endomorphism** of **degree** k, or *degree k morphism* for short.

Given a filtered differential object (X, F, d), let $A_n = F^n X$, we get a graded object A. Moreover, d induces a (degree 0) differential on A, hence making it a differential graded object. The inclusions $F^n X \subset F^{n-1} X$ then forms a degree -1 injective endomorphism of (A, d). Note that such a twisted endomorphism can also be viewed as a monomorphism $\alpha\colon A[1] \to A$. Then, we have a short exact sequence

$$0 \longrightarrow A[1] \xrightarrow{\alpha} A \longrightarrow \operatorname{gr} X \longrightarrow 0$$

where $\operatorname{gr} X$ equipped with the induced differential from (X, F, d).

Now, we can apply the process in previous section to get a spectral sequence start from $\operatorname{gr} X$, called the **associated spectral sequence** of (X, F, d).

3.14 Before expand the details of the associated spectral sequence, let first do this in lazy way. Temporarily, assume $\mathcal{A}$ satisfies AB5. In this case, the functor

$$A = (A_n) \longmapsto \bigoplus_n A_n$$

is exact. In this case, we can always identify a graded object A as the its image under this functor equipped with the given decomposition

$$A = \bigoplus_n A_n.$$

Note that, under such identification, a morphism of graded objects is a morphism preserves decompositions on the nod while a twisted morphism preserves decomposition up to shifting of degrees.

Use this convention, we can identify the exact sequence

$$0 \longrightarrow A[1] \xrightarrow{\alpha} A \longrightarrow \operatorname{gr} X \longrightarrow 0$$

as an exact sequence of differential objects

$$0 \longrightarrow A \xrightarrow{\alpha} A \longrightarrow \operatorname{gr} X \longrightarrow 0$$

with extra condition that α shifts the degrees by -1 and the quotient preserves the degrees.

Now, apply the process for differential objects, we immediately (· ‿ ·) get the associated spectral sequence:

1. $E_0 = \operatorname{gr} X, d_0 = \operatorname{gr} d$.

2. $Z_1 = \operatorname{Ker}(\operatorname{gr} d), B_1 = \operatorname{Im}(\operatorname{gr} d), E_1 = H(\operatorname{gr} X, \operatorname{gr} d)$ and d_1 works as follows: for any $z \in Z_1$, let z' be its lifting in A, choose $y \in A$ such that $d(z') = \alpha(y)$, then $d_1([z]) = [y]$.

3. B_r is the image of $(\alpha^{r-1})^{-1}(\operatorname{Im} d)$ under the projection $A \to \operatorname{gr} X$.

4. Z_r is the image of $d^{-1}(\operatorname{Im} \alpha^r)$ under the projection $A \to \operatorname{gr} X$.

5. E_r can be written as

$$E_r = \frac{d^{-1}(\operatorname{Im} \alpha^r) + \operatorname{Im} \alpha}{(\alpha^{r-1})^{-1}(\operatorname{Im} d) + \operatorname{Im} \alpha}.$$

6. The morphism d_r works as follows. For any $x + \operatorname{Im} \alpha \in Z_r$, choose $y \in A$ such that $d(x) = \alpha^r(y)$, then $d_r(x + \operatorname{Im} \alpha + B_r) = y + \operatorname{Im} \alpha + B_r$.

Ok, I just copy it to here (· ‿ ·). To make this more precise, we should consider the whole process finer, degree by degree.

1. First, $E_0 = \operatorname{gr} X$ is graded, so let E_0^p be its p-th degree, i.e. $\operatorname{gr}^p X$.

2. Since $\operatorname{gr} d$ preserves the grading, it makes sense to put $Z_1^p = \operatorname{Ker}(\operatorname{gr}^p d)$, $B_1^p = \operatorname{Im}(\operatorname{gr}^p d)$ and $E_1 = H(\operatorname{gr}^p X, \operatorname{gr}^p d)$. Straight computation shows

$$\begin{aligned} Z_1^p &= \frac{F^p X \cap d^{-1}(F^{p+1} X) + F^{p+1} X}{F^{p+1} X}, \\ B_1^p &= \frac{F^p X \cap d(F^p X) + F^{p+1} X}{F^{p+1} X}, \\ E_1^p &= \frac{F^p X \cap d^{-1}(F^{p+1} X) + F^{p+1} X}{F^p X \cap d(F^p X) + F^{p+1} X}. \end{aligned}$$

let d_1^p denote the restriction of d_1 to E_1^p. Let's see how it works. Taking any $\bar{x} \in Z_1^p$, let x be its lifting in A, more precisely, in $F^p X$. In other words, $\bar{x} = x + F^{p+1} X$. Choose $y \in A$ such that $d(x) = \alpha(y)$. By the definition of α, this means $y = d(x)$ and $y \in F^{p+1}$, hence its image in E_1 lies in E_1^{p+1}, NOT E_1^p. In other words, d_1 shifts the degree by 1. Then we have

$$\begin{aligned} d_1^p \colon E_1^p &\longrightarrow E_1^{p+1} \\ [x + F^{p+1} X] &\longmapsto [d(x) + F^{p+2} X]. \end{aligned}$$

3. As d_1 shifts the degree by 1, it makes more sense to put $Z_2^p = \operatorname{Ker} d_1^p$ and $B_2^p = \operatorname{Im} d_1^{p-1}$. Then $E_2^p = Z_2^p / B_2^p$. One may expect that d_2 will shift degree by 2 without further computation. In general, d_r should shift degree by r and hence we should put $Z_{r+1}^p = \operatorname{Ker} d_r^p$ and $B_{r+1}^p = \operatorname{Im} d_r^{p-1}$.

4. Anyway, Z_r should inherit the grading of $\operatorname{gr} X$. Therefore
$$Z_r^p := \operatorname{gr}^p X \cap Z_r = \operatorname{gr}^p X \cap \frac{d^{-1}(\operatorname{Im} \alpha^r) + \operatorname{Im} \alpha}{\operatorname{Im} \alpha}.$$
To see what it is, we go back to $A_p = F^p X$ instead of $\operatorname{gr}^p X$, where the preimage is $F^p X \cap d^{-1}(\operatorname{Im} \alpha^r)$. Note that α shifts degree by -1, so this term must comes from $F^{p+r} X$ only. Hence, we have
$$Z_r^p = \frac{F^p X \cap d^{-1}(F^{p+r}) + F^{p+1} X}{F^{p+1} X}.$$

5. Likewise, B_r should inherit the grading of $\operatorname{gr} X$. Therefore
$$B_r^p := \operatorname{gr}^p X \cap B_r = \operatorname{gr}^p X \cap \frac{(\alpha^{r-1})^{-1}(\operatorname{Im} d) + \operatorname{Im} \alpha}{\operatorname{Im} \alpha}.$$
To see what it is, we go back to $A_p = F^p X$ instead of $\operatorname{gr}^p X$, where the preimage is $F^p X \cap (\alpha^{r-1})^{-1}(\operatorname{Im} d)$. Note that α shifts degree by -1, so this term must comes from $F^{p-r+1} X$ only. Hence, we have
$$B_r^p = \frac{F^p X \cap d(F^{p-r+1} X) + F^{p+1} X}{F^{p+1} X}.$$

6. Now, $E_r^p := Z_r^p / B_r^p$ can be written as
$$E_r^p = \frac{F^p X \cap d^{-1}(F^{p+r}) + F^{p+1} X}{F^p X \cap d(F^{p-r+1} X) + F^{p+1} X}.$$

7. The morphism d_r^p works as follows. For any $x + F^{p+1} X \in Z_r^p$, choose $y \in A$ such that $d(x) = \alpha^r(y)$. Note that this means $y = d(x)$ and $y \in F^{p+r} X$, hence d_r shifts degree by r. Then we have
$$\begin{aligned} d_r^p \colon E_r^p &\longrightarrow E_r^{p+r} \\ x + F^{p+1} X + B_r^p &\longmapsto d(x) + F^{p+r+1} X + B_r^{p+r}. \end{aligned}$$

8. Obviously each B_r and Z_r become graded objects
$$B_r = \bigoplus_p B_r^p, \qquad Z_r = \bigoplus_p Z_r^p.$$
Furthermore, E_r also becomes graded via the canonical isomorphism
$$E_r = \bigoplus_p E_r^p.$$

3.15 However, not any abelian category satisfies AB5. So, for general case, we should use the twisted version of the machinery of associated spectral sequence of differential object. Note that in our case, $\sigma = \mathrm{id}$ and τ is given by $\tau A = A[1]$.

By the general process, we explain the details as follows.

1. $E_0 = \operatorname{gr} X$, $\tau_0 = \mathrm{id}$ and $d_0 = \operatorname{gr} d$. Note that d_0 is twisted of degree 0.

2. $Z_1 = \operatorname{Ker}(\operatorname{gr} d)$, $B_1 = \operatorname{Im}(\operatorname{gr} d)$, $\tau_1 = \tau$ and d_1 is twisted of degree 1. To see how d_1 works, we focus on each degree. First, we have

$$\begin{aligned} Z_1^p &= \operatorname{Ker}(\operatorname{gr}^p d) = \frac{F^pX \cap d^{-1}(F^{p+1}X) + F^{p+1}X}{F^{p+1}X}, \\ B_1^p &= \operatorname{Im}(\operatorname{gr}^p d) = \frac{F^pX \cap d(F^pX) + F^{p+1}X}{F^{p+1}X}, \\ E_1^p &= Z_1^p / B_1^p = \frac{F^pX \cap d^{-1}(F^{p+1}X) + F^{p+1}X}{F^pX \cap d(F^pX) + F^{p+1}X}. \end{aligned}$$

Then, d_1^p, the p-th component of d_1, works as follows: for any $x + F^{p+1}X \in \operatorname{gr}^p X$, choose $y \in A[1]_p = F^{p+1}X$ such that $d(x) = \alpha(y)$, which means $y = d(x)$ and $d(x) \in F^{p+1}X$, then we have

$$\begin{aligned} d_1^p \colon E_1^p &\longrightarrow E_1^{p+1} \\ [x + F^{p+1}X] &\longmapsto [d(x) + F^{p+2}X]. \end{aligned}$$

3. Z_r is the image of $(\operatorname{gr} d)^{-1}(\operatorname{Im} \alpha^{[r]})$ under the projection $A \to \operatorname{gr} X$. In detail, Z_r^p is the image of $(\operatorname{gr}^p d)^{-1}(\alpha^{[r]}(A[r]_p))$ under the projection $A \to \operatorname{gr} X$, that is

$$Z_r^p = \frac{F^pX \cap d^{-1}(F^{p+r}X) + F^{p+1}X}{F^{p+1}X}.$$

4. B_r is the image of $(\tau^{1-r}\alpha^{[r-1]})^{-1} d(A[1-r])$ under the projection $A \to \operatorname{gr} X$. In detail, B_r^p is the image of $(\tau^{1-r}\alpha^{[r-1]})^{-1} d(A[1-r]_p)$ under the projection $A \to \operatorname{gr} X$, that is

$$B_r^p = \frac{F^pX \cap d(F^{p-r+1}X) + F^{p+1}X}{F^{p+1}X}.$$

5. $E_r = Z_r / B_r$. In particular,

$$E_r^p = Z_r^p / B_r^p = \frac{F^pX \cap d^{-1}(F^{p+r}X) + F^{p+1}X}{F^pX \cap d(F^{p-r+1}X) + F^{p+1}X}.$$

6. d_r is of degree r. On p-th components, d_r^p works as follows: for any $x + F^{p+1}X + B_r^p \in E_r^p$, choose $y \in A[r]_p = F^{p+r}X$ such that $d(x) = \alpha^r(y)$, which means $y = d(x)$ and $d(x) \in F^{p+r}X$, then we have

$$\begin{aligned} d_r^p \colon E_r^p &\longrightarrow E_r^{p+r} \\ x + F^{p+1}X + B_r^p &\longmapsto d(x) + F^{p+r+1}X + B_r^{p+r}. \end{aligned}$$

3.16 Let's take a look at the limit object. First, we have

$$\begin{aligned} Z_\infty^p &= \bigcap_r Z_r^p = \frac{\bigcap_r F^pX \cap d^{-1}(F^{p+r}X) + F^{p+1}X}{F^{p+1}X}, \\ B_\infty^p &= \bigcup_r B_r^p = \frac{\bigcup_r F^pX \cap d(F^{p-r+1}X) + F^{p+1}X}{F^{p+1}X}, \\ E_\infty^p &= Z_\infty^p / B_\infty^p = \frac{\bigcap_r F^pX \cap d^{-1}(F^{p+r}X) + F^{p+1}X}{\bigcup_r F^pX \cap d(F^{p-r+1}X) + F^{p+1}X}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} F^pX \cap \operatorname{Ker} d &\subset \bigcap_r F^pX \cap d^{-1}(F^{p+r}X), \\ F^pX \cap \operatorname{Im} d &\supset \bigcup_r F^pX \cap d(F^{p-r+1}X). \end{aligned}$$

Therefore, we have

$$\frac{F^pX \cap \operatorname{Ker} d + F^{p+1}X}{F^pX \cap \operatorname{Im} d + F^{p+1}X} \subset E_\infty^p.$$

Recall that given a filtered differential object (X, F, d), by Lemma 3.9, its cohomology $H(X)$ inherits an *induced filtration*:

$$F^pH(X) = \frac{F^pX \cap \operatorname{Ker} d}{F^pX \cap \operatorname{Im} d} = \frac{F^pX \cap \operatorname{Ker} d + \operatorname{Im} d}{\operatorname{Im} d}.$$

Then, its corresponding graded object is given by

$$\operatorname{gr}^p H(X) = \frac{F^pX \cap \operatorname{Ker} d + \operatorname{Im} d}{F^{p+1}X \cap \operatorname{Ker} d + \operatorname{Im} d}.$$

Recall the following formula:

3.17 Lemma (Zassenhaus' Butterfly lemma) *If A, B, C, D are subobjects of something in an abelian category and $A \supset B, C \supset D$, then*

$$\frac{A \cap C + B}{A \cap D + B} = \frac{A \cap C}{A \cap D + B \cap C} = \frac{A \cap C + D}{B \cap C + D}.$$

Proof: This is clear by chasing the following (Hasse) diagram.

$$
\begin{tikzcd}[column sep=small]
B \cap C \arrow[drrr] \arrow[ddr, bend right] & & & & & & A \cap D \arrow[dlll] \arrow[ddl, bend left] \\
& & & B \cap C + A \cap D \arrow[ddl, bend right] \arrow[ddr, bend left] \arrow[ddd, blue] & & & \\
& B \arrow[dr] & & & & D \arrow[dl] & \\
& & B + A \cap D \arrow[ddl, blue, bend right] & & B \cap C + D \arrow[ddr, blue, bend left] & & \\
& & & A \cap C \arrow[dll, bend left] \arrow[drr, bend right] & & & \\
& B + A \cap C \arrow[d] & & & & A \cap C + D \arrow[d] & \\
& A & & & & C &
\end{tikzcd}
$$

Indeed, it follows the third isomorphism theorem that each half and each front wing is a (co-)Cartesian digram. Then so are the back wings and hence the three blue vertical morphisms share isomorphic cokernels.

What a beautiful butterfly! $\square$

3.18 By this butterfly lemma, $\operatorname{gr} H(X)$ can be viewed as a subobject of E_∞. It is then natural to ask if they coincide. We say the associated spectral sequence of (X, F, d)

- **weakly converges** to $H(X)$ if $\operatorname{gr} H(X) = E_\infty$ through the above identification.
- **converges** to $H(X)$ if it weakly converges to $H(X)$ and the filtration on $H(X)$ is *regular*, i.e. both *separated* and *exhaustive*.

It is not difficult to see that

1. The associated spectral sequence weakly converges to $H(X)$ if and only if for each p,

$$F^p X \cap \operatorname{Ker} d = \bigcap_r F^p X \cap d^{-1}(F^{p+r} X),$$
$$F^p X \cap \operatorname{Im} d = \bigcup_r F^p X \cap d(F^{p-r+1} X).$$

2. The associated spectral sequence converges to $H(X)$ if and only if it weakly converges to $H(X)$ and

$$\bigcap_p (F^p X \cap \operatorname{Ker} d + \operatorname{Im} d) = \operatorname{Im} d, \quad \bigcup_p (F^p X \cap \operatorname{Ker} d + \operatorname{Im} d) = \operatorname{Ker} d.$$

3.19 Given a morphism of filtered differential objects $u\colon (X, F, d) \to (Y, F, d)$, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A[1] \arrow[r, "\alpha"] \arrow[d, "u[1]"'] & A \arrow[r] \arrow[d, "u"'] & \operatorname{gr} X \arrow[r] \arrow[d, "\operatorname{gr} u"'] & 0 \\
0 \arrow[r] & B[1] \arrow[r, "\beta"] & B \arrow[r] & \operatorname{gr} Y \arrow[r] & 0
\end{tikzcd}$$

where B is defined by $B_p = F^p Y$, β is given by the inclusion $F^{p+1}Y \subset F^p Y$ and each row is exact. Then, we have a morphism between associated spectral sequences

$$v = (v_0, v_1, \cdots),$$

where $v_0 = \operatorname{gr} u$, and for each r, v_r works as follows:

$$\begin{aligned} v_r^p \colon E(X)_r^p &\longrightarrow E(Y)_r^p \\ x + F^{p+1}X + B(X)_r^p &\longmapsto u(x) + F^{p+1}Y + B(Y)_r^p. \end{aligned}$$

§ 4 Bigraded spectral sequences

4.1 In the category of bigraded objects, there are shifts $[s, t]$ which maps a bigraded object $A = (A^{p,q})$ to a bigraded object $A[s, t]$ whose (p, q)-term is $A^{p+s,q+t}$. So, a morphism from A to $B[s, t]$ can usually be called a **twisted morphism of degree** (s, t).

By a **bigraded spectral sequence**, we mean a spectral sequence of bigraded objects $(E_r, d_r)_{r \geqslant r_0}$ the differential d_r is of degree $(r, 1 - r)$. In other word, it is a spectral sequence with shifts $\tau_r = [r, 1 - r]$.

For such a spectral sequence, we say it is

- **regular** if for each p, q, there exists an R such that for all $r \geqslant R$, $E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}$ are zero;
- **coregular** if for each p, q, there exists an R such that for all $r \geqslant R$, $E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q}$ are zero;
- **biregular** if it is both regular and coregular;
- **bounded** if for all n, there are only finitely many nonzero $E_{r_0}^{p,n-p}$;
- **bounded below** if for all n there exists an B such that for all $p \geqslant B$, $E_{r_0}^{p,n-p} = 0$;
- **bounded above** if for all n there exists an B such that for all $p \leqslant B$, $E_{r_0}^{p,n-p} = 0$.

It is not difficult to see that

1. The spectral sequence is *regular* if and only if for all p, q, the sequence
$$Z_{r_0}^{p,q} \supset Z_{r_0+1}^{p,q} \supset \cdots \supset Z_r^{p,q} \supset \cdots$$
is *stationary*, i.e. $Z_r^{p,q} = Z_{r+1}^{p,q}$ for large r.

2. The spectral sequence is *coregular* if and only if for all p, q, the sequence
$$B_{r_0}^{p,q} \subset B_{r_0+1}^{p,q} \subset \cdots \subset B_r^{p,q} \subset \cdots$$
is *stationary*, i.e. $B_r^{p,q} = B_{r+1}^{p,q}$ for large r.

3. If the spectral sequence is bounded below, then it is regular.

4. If the spectral sequence is bounded above, then it is coregular.

5. The spectral sequence is *bounded* if and only if it is both bounded below and bounded above.

4.2 Given a bigraded spectral sequence $(E_r, d_r)_{r \geqslant r_0}$, we say

- it **weakly converges** if the limit object E_∞ exists.

Let (H, F) be a filtered graded object. We say the spectral sequence $(E_r, d_r)_{r \geqslant r_0}$

- **weakly converges** to H if for any p, q, there are isomorphisms
$$E_\infty^{p,q} \cong \operatorname{gr}^p H^{p+q};$$
- **converges** to H if it weakly converges to H and the filtration F on H is *regular*, i.e. both *separated* and *exhaustive*;
- **strongly converges** to H if it converges to H and the filtration F on H is *complete*.

Remark The following notation is widely used to denote that the spectral sequence $(E_r, d_r)_{r \geqslant r_0}$ weakly converges to a filtered graded object H:
$$E_r^{p,q} \Rightarrow H^{p+q}.$$
In many case, one only writes the interesting page on the left, for instance:
$$E_1^{p,q} \Rightarrow H^{p+q} \quad \text{or} \quad E_2^{p,q} \Rightarrow H^{p+q}.$$

The following lemma may answers why one would like to consider the notion of strong convergence.

4.3 Lemma *Let (X, F) be a filtered object in an AB5 category. Suppose that the filtration is regular and complete, then we have*
$$X \cong \varprojlim_p X/F^pX \cong \varprojlim_p (\varinjlim_q (F^{p+q}X/F^pX)).$$

Proof: Consider the following exact sequences:

$$0 \longrightarrow F^pX \longrightarrow X \longrightarrow X/F^pX \longrightarrow 0.$$

By passing to the inverse limit, we obtain the exact sequence:

$$0 \longrightarrow \varprojlim_p F^pX \longrightarrow X \longrightarrow \varprojlim_p X/F^pX \longrightarrow \varprojlim_p^1 F^pX.$$

Since F is separated and complete, $\varprojlim_p F^pX = 0$, $\varprojlim^1{}_p F^pX = 0$. Then

$$X \cong \varprojlim_p X/F^pX.$$

Fix p, consider the following exact sequences:

$$0 \longrightarrow F^{p+q}X \longrightarrow X \longrightarrow X/F^{p+q}X \longrightarrow 0.$$

By passing to the inverse limit, we obtain the exact sequence:

$$0 \longrightarrow \varinjlim_q F^{p+q}X \longrightarrow X \longrightarrow \varinjlim_q X/F^{p+q}X \longrightarrow 0.$$

Since F is exhaustive, $\varinjlim_q F^{p+q}X = X$. Then

$$\varinjlim_q X/F^{p+q}X = 0.$$

Now, consider the following exact sequences:

$$0 \longrightarrow F^{p+q}X/F^pX \longrightarrow X/F^pX \longrightarrow X/F^{p+q}X \longrightarrow 0.$$

By passing to the direct limit respecting to q, we get the exact sequence:

$$0 \longrightarrow \varinjlim_q F^{p+q}X/F^pX \longrightarrow X/F^pX \longrightarrow \varinjlim_q X/F^{p+q}X \longrightarrow 0.$$

Since $\varinjlim_q X/F^{p+q}X = 0$, we have

$$\varinjlim_q F^{p+q}X/F^pX \cong X/F^pX.$$

□

Spectral sequences for filtered complexes

4.4 A **filtered complex** in $\mathcal{A}$ is a complex in $\mathrm{Fil}(\mathcal{A})$, or equivalently, a filtered object in the category of complexes in $\mathcal{A}$. Hence for a filtered complex $(K^\bullet, F)$, $\mathrm{gr}\, K^\bullet$ is a **graded complex**, meaning graded object in the category of complexes in $\mathcal{A}$.

Similarly to what we do for filtered differential objects, we associate a filtered complex $(K^\bullet, F)$ a graded complex

$$A_p := F^p K^\bullet.$$

Such a graded complex is naturally bigraded, but we regrade it as

$$A^{p,q} := F^p K^{p+q},$$

where p is called the **filtration degree**, q the **complementary degree** and $p+q$ the **total degree**. Recall that for a graded object $A = (A_n)$, the notation $A[k]$ denotes the graded object defined by $A[k]_n = A_{n+k}$. Similarly, for a bigraded object $A = (A^{p,q})$, the notation $A[s,t]$ denotes the bigraded object defined by

$$A[s,t]^{p,q} = A^{p+s,q+t}.$$

So, for our $A = (F^p K^{p+q})$, $[s,t]$ can also be viewed as shift the filtration degree and complementary degree by s and t respectively.

In the language of twisted differential objects, a complex is a differential objects twisted by the shift σ

$$\sigma(K^\bullet) = K[1]^\bullet = K^{\bullet+1}.$$

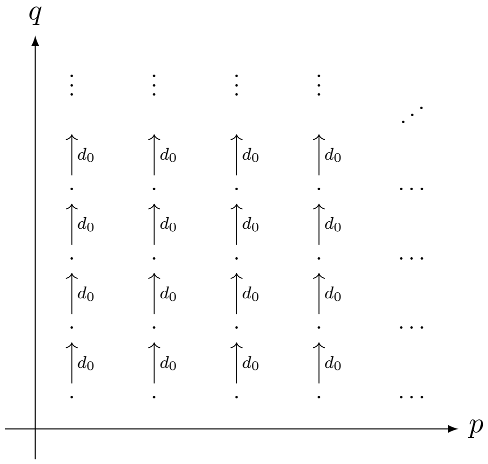
So, we extend it to a shift for bigraded object which maps each $A = (A^{p,q})$ to $A[0,1]$. Then, a graded complex is a differential bigraded object twisted by σ, in other word, differentials are of degree $(0,1)$. Besides the differentials, the inclusions $F^{p+1}K^\bullet \subset F^p K^\bullet$ is a morphism of degree $(-1,1)$. So, we let τ denote the shift $[1,-1]$.

Now, we can apply the machinery of associated spectral sequence of twisted differential objects. First, we have a short exact sequence of graded complexes:

$$0 \longrightarrow A[1,-1] \xrightarrow{\alpha} A \longrightarrow \mathrm{gr}\, K^\bullet \longrightarrow 0,$$

where α is given by the inclusions $F^{p+1}K^\bullet \subset F^p K^\bullet$ and $\mathrm{gr}\, K^\bullet$ equipped with the filtration-complementary bigrading. Then, through the general process, we immediately (· ‿ ·) get the following spectral sequence:

1. $E_0 = \mathrm{gr}\, K^\bullet$ and $d_0 = \mathrm{gr}\, d$ which is of degree $(0,1)$.



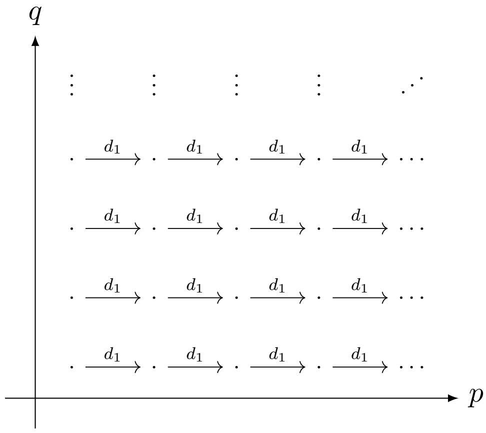
Graph of E_0

2. $Z_1 = \mathrm{Ker}\,(\mathrm{gr}\, d)$, $B_1 = \mathrm{Im}\,(\mathrm{gr}\, d)$ and d_1 is of degree $(1, 0)$. To see how it works, we focus on each degree. First, we have

$$\begin{aligned}
Z_1^{p,q} &= \mathrm{Ker}\left(\mathrm{gr}^p K^{p+q} \xrightarrow{\mathrm{gr}^p d} \mathrm{gr}^p K^{p+q+1}\right)\\
&= \frac{F^p K^{p+q} \cap d^{-1}(F^{p+1}K^{p+q+1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}},\\
B_1^{p,q} &= \mathrm{Im}\left(\mathrm{gr}^p K^{p+q-1} \xrightarrow{\mathrm{gr}^p d} \mathrm{gr}^p K^{p+q}\right)\\
&= \frac{F^p K^{p+q} \cap d(F^p K^{p+q-1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}},\\
E_1^{p,q} &= Z_1^{p,q}/B_1^{p,q}\\
&= \frac{F^p K^{p+q} \cap d^{-1}(F^{p+1}K^{p+q+1}) + F^{p+1}K^{p+q}}{F^p K^{p+q} \cap d(F^p K^{p+q-1}) + F^{p+1}K^{p+q}}.
\end{aligned}$$

Then, $d_1^{p,q}$, the (p, q)-th component of d_1, works as follows: for any $x + F^{p+1}K^{p+q} \in \mathrm{gr}^p K^{p+q}$, choose $y \in (\tau\sigma A)^{p,q} = A[1, 0]^{p,q} = F^{p+1}K^{p+q+1}$ such that $d(x) = \alpha(y)$, which means $y = d(x)$ and $d(x) \in F^{p+1}K^{p+q+1}$, then we have

$$\begin{aligned}
d_1^{p,q} \colon E_1^{p,q} &\longrightarrow E_1^{p+1,q}\\
[x + F^{p+1}K^{p+q}] &\longmapsto [d(x) + F^{p+2}K^{p+q+1}].
\end{aligned}$$



Graph of E_1

3. In general, Z_r is the image of $(\operatorname{gr} d)^{-1}(\operatorname{Im} \sigma\alpha^{[r]})$ under the projection $A \to \operatorname{gr} K^\bullet$. In detail, $Z_r^{p,q}$ is the image of $(\operatorname{gr}^p d)^{-1}(\sigma\alpha^{[r]}(A[r, 1-r])^{p,q})$ under the projection $A \to \operatorname{gr} K^\bullet$, that is

$$Z_r^{p,q} = \frac{F^p K^{p+q} \cap d^{-1}(F^{p+r} K^{p+q+1}) + F^{p+1} K^{p+q}}{F^{p+1} K^{p+q}}.$$

4. B_r is the image of $(\tau^{1-r}\alpha^{[r-1]})^{-1} d(A[1-r, r-2])$ under the projection $A \to \operatorname{gr} K^\bullet$. In detail, $B_r^{p,q}$ is $(\tau^{1-r}\alpha^{[r-1]})^{-1} d(A[1-r, r-2]^{p,q})$ under the projection $A \to \operatorname{gr} K^\bullet$, that is

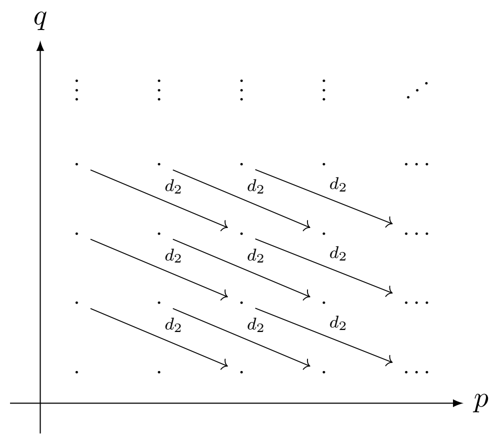
$$B_r^{p,q} = \frac{F^p K^{p+q} \cap d(F^{p-r+1} K^{p+q-1}) + F^{p+1} K^{p+q}}{F^{p+1} K^{p+q}}.$$

5. $E_r = Z_r / B_r$. In particular,

$$E_r^{p,q} = \frac{Z_r^{p,q}}{B_r^{p,q}} = \frac{F^p K^{p+q} \cap d^{-1}(F^{p+r} K^{p+q+1}) + F^{p+1} K^{p+q}}{F^p K^{p+q} \cap d(F^{p-r+1} K^{p+q-1}) + F^{p+1} K^{p+q}}.$$

6. d_r is of degree $(r, 1-r)$. In details, on (p,q)-th components, $d_r^{p,q}$ works as follows: for any $x + F^{p+1}K^{p+q} + B_r^{p,q} \in E_r^{p,q}$, choose $y \in A[r, 1-r]^{p,q} = F^{p+r}K^{p+q+1}$ such that $d(x) = \alpha^r(y)$, which means $y = d(x)$ and $d(x) \in F^{p+r}K^{p+q+1}$, then we have

$$\begin{aligned} d_r^{p,q} \colon E_r^{p,q} &\longrightarrow E_r^{p+r, q-r+1} \\ x + F^{p+1}K^{p+q} + B_r^{p,q} &\longmapsto d(x) + F^{p+r+1}K^{p+q+1} + B_r^{p+r, q-r+1}. \end{aligned}$$



Graph of E_2

7. As for the limit object, we have:

$$
\begin{aligned}
Z_\infty^{p,q} &= \bigcap_r Z_r^{p,q} = \frac{\bigcap_r F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}},\\
B_\infty^{p,q} &= \bigcup_r B_r^{p,q} = \frac{\bigcup_r F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}},\\
E_\infty^{p,q} &= \frac{Z_\infty^{p,q}}{B_\infty^{p,q}} = \frac{\bigcap_r F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q}}{\bigcup_r F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q}}.
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
F^pK^{p+q} \cap \operatorname{Ker} d &\subset \bigcap_r F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}),\\
F^pK^{p+q} \cap \operatorname{Im} d &\supset \bigcup_r F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}).
\end{aligned}
$$

4.5 For $(K^\bullet, F)$ a filtered complex, we also define the induced filtration on $H^n(K^\bullet)$ by Lemma 3.9, i.e.

$$F^pH^n(K^\bullet) := \frac{F^pK^n \cap \operatorname{Ker} d}{F^pK^n \cap \operatorname{Im} d} = \frac{F^pK^n \cap \operatorname{Ker} d + K^n \cap \operatorname{Im} d}{K^n \cap \operatorname{Im} d}.$$

Then, its corresponding graded object is given by

$$\operatorname{gr}^p H^n(K^\bullet) = \frac{F^pK^n \cap \operatorname{Ker} d + K^n \cap \operatorname{Im} d}{F^{p+1}K^n \cap \operatorname{Ker} d + K^n \cap \operatorname{Im} d}.$$

By Zassenhaus' Butterfly lemma, we can always identify $\operatorname{gr}^p H^n(K^\bullet)$ as a subobject of $E_\infty^{p,n-p}$. Then, it is then natural to ask if they coincide.

We say the associated spectral sequence of $(K^\bullet, F)$

- **weakly converges** to $H^*(K^\bullet)$ if $\operatorname{gr}^p H^n(K^\bullet) = E_\infty^{p,n-p}$ through the above identification for all p, n.
- **converges** to $H^*(K^\bullet)$ if it weakly converges to $H^*(K^\bullet)$ and the filtration on $H^*(K^\bullet)$ is *regular*, i.e. both *separated* and *exhaustive*.

If the associated spectral sequence weakly converges to $H^*(K^\bullet)$, we write $E_r^{p,q} \Rightarrow H^{p+q}(K^\bullet)$.

It is not difficult to see that

1. **(Condition 1)** The associated spectral sequence weakly converges to $H^*(K^\bullet)$ if and only if for each p, q,
$$F^pK^{p+q} \cap \operatorname{Ker} d = \bigcap_r F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}),$$
$$F^pK^{p+q} \cap \operatorname{Im} d = \bigcup_r F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}).$$

2. **(Condition 2)** The associated spectral sequence converges to $H^*(K^\bullet)$ if and only if it weakly converges to $H^*(K^\bullet)$ and for each n,
$$\bigcap_p (F^pK^n \cap \operatorname{Ker} d + K^n \cap \operatorname{Im} d) = K^n \cap \operatorname{Im} d,$$
$$\bigcup_p (F^pK^n \cap \operatorname{Ker} d + K^n \cap \operatorname{Im} d) = K^n \cap \operatorname{Ker} d.$$

The first example is for finite filtration case.

4.6 Proposition *Let $(K^\bullet, F)$ be a filtered complex. Assume that the filtration F on each K^n is finite. Then*

1. *the associated spectral sequence of $(K^\bullet, F)$ is bounded;*
2. *the associated spectral sequence of $(K^\bullet, F)$ is biregular;*
3. *the filtration on each $H^n(K^\bullet)$ is finite;*
4. *the associated spectral sequence of $(K^\bullet, F)$ strongly converges to $H^*(K^\bullet)$.*

Proof: *1.* Since $E_0^{p,n-p} = \operatorname{gr}^p K^n$ and the filtration on K^n is finite, E is bounded. Then *2* follows. *3* is also clear by the definition of $F^pH^n(K^\bullet)$. As for *4*, it is not difficult to verify the condition 1 and 2 from the finiteness of filtration on each K^n. Finally, the filtration on $H^n(K^\bullet)$ is complete since it is finite. □

The next case is that, instead of the filtration is finite, it becomes finite after applying the cohomology functors.

4.7 Proposition *Let $(K^\bullet, F)$ be a filtered complex. Assume that for any n, the sequence*

$$\cdots \longrightarrow H^n(F^{p+1}K^\bullet) \longrightarrow H^n(F^pK^\bullet) \longrightarrow H^n(F^{p-1}K^\bullet) \longrightarrow \cdots$$

obtained by applying $H^n(-)$ to the filtration F, is finite *in the sense that $H^n(F^pK^\bullet) = 0$ for large p and $H^n(F^pK^\bullet) \cong H^n(K^\bullet)$ for small p. Then*

1. *the associated spectral sequence of $(K^\bullet, F)$ is bounded;*
2. *the associated spectral sequence of $(K^\bullet, F)$ is biregular;*
3. *the filtration on each $H^n(K^\bullet)$ is finite;*
4. *the associated spectral sequence of $(K^\bullet, F)$ strongly converges to $H^*(K^\bullet)$.*

Proof: Consider the long exact sequence associated to the short exact sequence of complexes

$$0 \longrightarrow F^{p+1}K^\bullet \longrightarrow F^pK^\bullet \longrightarrow \operatorname{gr}^p K^\bullet \longrightarrow 0$$

we find that $E_1^{p,n-p} = 0$ for large and small p. This shows *1*, hence *2*. Note that our condition implies *3*. As for *4*, one only needs to verify condition 1 and 2. To do this, one can look at $d_r^{p,n-p}$ for sufficiently large r such that the complex $F^{p+r}K^\bullet$ is exact at degree $n+1$ and that $H^{n-1}(F^{p-r+1}K^\bullet) \cong H^{n-1}(K^\bullet)$. Then the equalities are clear. $\square$

4.8 Given a morphism of filtered differential complexes $u\colon (K^\bullet, F) \to (L^\bullet, F)$, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A[1,-1] \arrow[r, "\alpha"] \arrow[d, "{u[1,-1]}"'] & A \arrow[r] \arrow[d, "u"'] & \operatorname{gr} K^\bullet \arrow[r] \arrow[d, "\operatorname{gr} u"'] & 0 \\
0 \arrow[r] & B[1,-1] \arrow[r, "\beta"] & B \arrow[r] & \operatorname{gr} L^\bullet \arrow[r] & 0
\end{tikzcd}$$

where B is defined by $B^{p,q} = F^pL^{p+q}$, β is given by the inclusions $F^{p+1}L^\bullet \subset F^pL^\bullet$ and each row is exact. Then, we have a morphism between associated spectral sequences

$$v = (v_0, v_1, \cdots),$$

where $v_0 = \operatorname{gr} u$, and for each r, v_r works as follows:

$$\begin{aligned} v_r^{p,q}\colon E(K^\bullet)_r^{p,q} &\longrightarrow E(L^\bullet)_r^{p,q} \\ x + F^{p+1}K^{p+q} + B(K^\bullet)_r^{p,q} &\longmapsto u(x) + F^{p+1}L^{p+q} + B(L^\bullet)_r^{p,q}. \end{aligned}$$

Spectral sequences for bicomplexes

4.9 A **bicomplex** in $\mathcal{A}$ is a complex in the category of complexes in $\mathcal{A}$. So, concretely, a bicomplex $K^{\bullet,\bullet}$ consists of the following data

- a bigraded object $(K^{p,q})_{p,q}$,
- the **horizontal differential** d_{I}, which is a differential of degree $(1,0)$,
- the **vertical differential** d_{II}, which is a differential of degree $(0,1)$,

such that the two differentials commute. We visualize a bicomplex as follows.

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \iddots \\
\cdot \arrow[u, "d_{\mathrm{II}}"'] \arrow[r, "d_{\mathrm{I}}"] & \cdot \arrow[u, "d_{\mathrm{II}}"'] \arrow[r, "d_{\mathrm{I}}"] & \cdot \arrow[u, "d_{\mathrm{II}}"'] \arrow[r, "d_{\mathrm{I}}"] & \cdot \arrow[u, "d_{\mathrm{II}}"'] \arrow[r, "d_{\mathrm{I}}"] & \cdots \\
\cdot \arrow[u, "d_{\mathrm{II}}"'] \arrow[r, "d_{\mathrm{I}}"] & \cdot \arrow[u, "d_{\mathrm{II}}"'] \arrow[r, "d_{\mathrm{I}}"] & \cdot \arrow[u, "d_{\mathrm{II}}"'] \arrow[r, "d_{\mathrm{I}}"] & \cdot \arrow[u, "d_{\mathrm{II}}"'] \arrow[r, "d_{\mathrm{I}}"] & \cdots \\
\cdot \arrow[u, "d_{\mathrm{II}}"'] \arrow[r, "d_{\mathrm{I}}"] & \cdot \arrow[u, "d_{\mathrm{II}}"'] \arrow[r, "d_{\mathrm{I}}"] & \cdot \arrow[u, "d_{\mathrm{II}}"'] \arrow[r, "d_{\mathrm{I}}"] & \cdot \arrow[u, "d_{\mathrm{II}}"'] \arrow[r, "d_{\mathrm{I}}"] & \cdots \\
\cdot \arrow[u, "d_{\mathrm{II}}"'] \arrow[r, "d_{\mathrm{I}}"] & \cdot \arrow[u, "d_{\mathrm{II}}"'] \arrow[r, "d_{\mathrm{I}}"] & \cdot \arrow[u, "d_{\mathrm{II}}"'] \arrow[r, "d_{\mathrm{I}}"] & \cdot \arrow[u, "d_{\mathrm{II}}"'] \arrow[r, "d_{\mathrm{I}}"] & \cdots
\end{tikzcd}
$$

q p

Given a bicomplex $K^{\bullet,\bullet}$, there are two natural ways to view it as complex of complex:

- We view $K^{\bullet,\bullet}$ as a complex whose p-th term is a complex $(K^{p,\bullet}, d_{\mathrm{II}})$. In this viewpoint, it is customary to use $H^q_{\mathrm{II}}(K^{\bullet,\bullet})$ denote the complex whose p-th term is $H^q(K^{p,\bullet}, d_{\mathrm{II}})$ and whose differentials are induced by d_{I}. Then, we use $H^p_{\mathrm{I}}(H^q_{\mathrm{II}}(K^{\bullet,\bullet}))$ to denote its p-th cohomology.
- We view $K^{\bullet,\bullet}$ as a complex whose q-th term is a complex $(K^{\bullet,q}, d_{\mathrm{I}})$. In this viewpoint, it is customary to use $H^p_{\mathrm{I}}(K^{\bullet,\bullet})$ denote the complex whose q-th term is $H^p(K^{\bullet,q}, d_{\mathrm{I}})$ and whose differentials are induced by d_{II}. Then, we use $H^q_{\mathrm{II}}(H^p_{\mathrm{I}}(K^{\bullet,\bullet}))$ to denote its q-th cohomology.

4.10 In the case the involved countable direct sum exists, the **associated total complex** $sK^{\bullet}$ of a bicomplex $K^{\bullet,\bullet}$ is defined as

$$sK^n := \bigoplus_{p+q=n} K^{p,q},$$

with the differential

$$d_{sK}^n := \sum_{p+q=n} (d_{\mathrm{I}}^{p,q} + (-1)^p d_{\mathrm{II}}^{p,q}).$$

There are two natural filtrations on the total complex:

$$F_{\mathrm{I}}^p(sK^n) = \bigoplus_{i+j=n, i\geqslant p} K^{i,j}, \qquad F_{\mathrm{II}}^q(sK^n) = \bigoplus_{i+j=n, j\geqslant q} K^{i,j},$$

making $sK^\bullet$ filtered complex in two ways. Then, by previous section, each of them associates a spectral sequence. We use $('E_r, 'd_r)$ to denote the one associated to F_{I}, and $(''E_r, ''d_r)$ to F_{II}.

In details, we have

1. $'E_0^{p,q} = K^{p,q}$ with $'d_0^{p,q} = (-1)^p d_{\mathrm{II}}^{p,q} \colon K^{p,q} \to K^{p,q+1}$;
2. $''E_0^{q,p} = K^{p,q}$ with $'d_0^{q,p} = d_{\mathrm{I}}^{p,q} \colon K^{p,q} \to K^{p+1,q}$;
3. $'E_1^{p,q} = H^q(K^{p,\bullet})$ with $'d_1^{p,q} = H^q(d_{\mathrm{I}}^{p,\bullet})$, in other words, $('E_1^{\bullet,q}, 'd_1^{\bullet,q})$ is precisely the complex $H_{\mathrm{II}}^q(K^{\bullet,\bullet})$;
4. $''E_1^{q,p} = H^p(K^{\bullet,q})$ with $''d_1^{q,p} = (-1)^p H^p(d_{\mathrm{II}}^{\bullet,q})$, hence $(''E_1^{\bullet,p}, ''d_1^{\bullet,p})$ is precisely the complex $\mathbf{1} \otimes H_{\mathrm{I}}^p(K^{\bullet,\bullet})$, which equals $H_{\mathrm{I}}^p(K^{\bullet,\bullet})$ up to signs;
5. hence, we have $'E_2^{p,q} = H_{\mathrm{I}}^p H_{\mathrm{II}}^q(K^{\bullet,\bullet})$ and $''E_2^{q,p} = H_{\mathrm{II}}^q H_{\mathrm{I}}^p(K^{\bullet,\bullet})$;
6. for the rest pages, one can follow the general process, but in many practical cases, they are irrelevant.

4.11 It is cleat that the limit object of $('E_r, 'd_r)$ (resp. $(''E_r, ''d_r)$), if it exists, is $H^*(sK^\bullet)$. Then it make sense to say if $('E_r, 'd_r)$ (resp. $(''E_r, ''d_r)$) **weakly converges/converges/strongly converges** to $H^*(sK^\bullet)$. The following notations are widely used to denote that $('E_r, 'd_r)$ and $(''E_r, ''d_r)$ weakly converges to $H^*(sK^\bullet)$ respectively:

$$H_{\mathrm{I}}^p H_{\mathrm{II}}^q(K^{\bullet,\bullet}) \Rightarrow H^{p+q}(sK^\bullet),$$
$$H_{\mathrm{II}}^q H_{\mathrm{I}}^p(K^{\bullet,\bullet}) \Rightarrow H^{p+q}(sK^\bullet).$$

4.12 Proposition *Let $K^{\bullet,\bullet}$ be a bicomplex. Assume that for each n, there are only finitely many (p,q) such that $p+q=n$ and $K^{p,q} \neq 0$. Then*

1. *the total complex $sK^\bullet$ exists;*
2. *the associated spectral sequences $('E_r, 'd_r)$ and $(''E_r, ''d_r)$ are bounded;*
3. *the filtrations $F_{\mathrm{I}}, F_{\mathrm{II}}$ on each $H^n(sK^\bullet)$ are finite;*
4. *the spectral sequences $('E_r, 'd_r)$ and $(''E_r, ''d_r)$ strongly converges to $H^*(sK^\bullet)$.*

Proof: Follows from Proposition 4.6. □

§ 5 Read out the limit

The notions in previous sections are somehow nonstandard, in particular, they are different from what in EGA. They can be thought as a modern simplification of the original theorem. From this section, we will go to the classical material. First, let's introduce the notions coinciding with those from EGA.

5.1 A **spectral sequence with target** consists of the following data:

- a bigraded spectral sequence $(E_r, d_r)_{r \geqslant r_0}$;
- a filtered graded object (H, F);
- a family of isomorphisms:

$$E_\infty^{p,q} \cong \operatorname{gr}_F^p H^{p+q}.$$

In other words, a spectral sequence with target is a bigraded spectral sequence $(E_r, d_r)_{r \geqslant r_0}$ weakly converges to a filtered graded object (H, F) equipped the specific isomorphisms. In practical case, the bigraded spectral sequence and the filtered graded object usually arise from the same object and the specific isomorphisms emerge naturally.

Such a spectral sequence with target is usually denoted as

$$E_r^{p,q} \Rightarrow H^{p+q}.$$

In many case, one only writes the interesting page on the left, for instance:

$$E_1^{p,q} \Rightarrow H^{p+q} \quad \text{or} \quad E_2^{p,q} \Rightarrow H^{p+q}.$$

A **morphism** between spectral sequences with target

$$E_r^{p,q} \Rightarrow H^{p+q} \quad \text{and} \quad E_r'^{p,q} \Rightarrow H'^{p+q}$$

consists of a morphism between spectral sequences $(u_r \colon E_r \to E_r')_{r \geqslant r_0}$ and a morphism between filtered graded objects $u \colon H \to H'$, such that the diagram commutes:

$$\begin{tikzcd}
E_\infty^{p,q} \arrow[r, "\cong"] \arrow[d, "u_\infty^{p,q}"'] & \operatorname{gr}^p H^{p+q} \arrow[d, "\operatorname{gr}^p u^{p+q}"] \\
E_\infty'^{p,q} \arrow[r, "\cong"] & \operatorname{gr}^p H'^{p+q}
\end{tikzcd}$$

where $u_\infty^{p,q}$ is the limit of the morphisms $u_r^{p,q}$.

5.2 A spectral sequence with target $E_r^{p,q} \Rightarrow H^{p+q}$ is said to be

- **regular** if the spectral sequence E_r is regular and the filtration on H is discrete and regular;
- **coregular** if the spectral sequence E_r is coregular and the filtration on H is codiscrete and reuglar;
- **biregular** if the spectral sequence E_r is biregular and the filtration on H is finite (and regular).

In many case, a spectral sequence collapses at some page r and hence is biregular. Finding the collapsed page is a good way to work out the useful information in such a spectral sequence.

5.3 Proposition *Let $u\colon (E,H) \to (E',H')$ be a morphism between spectral sequences with target. Suppose that the filtration on H is discrete and regular. If u_{r_0} is an isomorphism of differential bigraded objects, then u is an isomorphism.*

Proof: The statement for the spectral sequence part is somehow trivial and holds for arbitrary kind of spectral sequences. Indeed, whenever u_r is an isomorphism, it follows that the upper three morphisms in the following diagram are isomorphisms

$$\begin{tikzcd} H(E_r) \arrow[r, "H(u_r)"] \arrow[d, "\cong"] & H(E'_r) \arrow[d, "\cong"] \\ E_{r+1} \arrow[r, "u_{r+1}"] & E'_{r+1} \end{tikzcd}$$

then so is the bottom one. Therefore, using induction, one can see that whenever there is a k such that u_k is an isomorphism, then so are all u_r with $r \geqslant k$.

As for the statement for the limit objects, note that u_r are isomorphisms for large r implies that so is u_∞. Then so are $\operatorname{gr} u^n\colon \operatorname{gr} H^n \to \operatorname{gr} H'^n$. Then, since the filtrations on H^n and H'^n are discrete and regular, by Lemma 3.4, $u^n\colon H^n \to H'^n$ are isomorphisms of filtered objects. $\square$

5.4 Proposition *Let $\mathcal{A}$ be an AB5 category. Then the abelian category of spectral sequences with target in $\mathcal{A}$ is also AB5. In particular, if $(E_{r,\lambda}, H_\lambda)_\lambda$ be a* filtered system *of spectral sequences with target, then the spectral sequence with target*

$$E_r^{p,q} := \varinjlim E_{r,\lambda}^{p,q} \Rightarrow H^{p+q} := \varinjlim H_\lambda^{p+q}$$

is the direct limit of the previous system.

Proof: Note that taking filtered colimit is an exact functor in AB5 category, hence the conclusion follow. $\square$

Special type of spectral sequences

5.5 Proposition *Let $E_r^{p,q} \Rightarrow H^{p+q}$ be a biregular spectral sequence with target. Suppose that for some r, $E_r^{p,q} = 0$ unless $p = p_0$ (resp. $q = q_0$), then $E_r^{p_0,n-p_0} \cong H^n$ (resp. $E_r^{n-q_0,q_0} \cong H^n$).*

Remark Such kind of spectral sequence is called **degenerate**.

Proof: The condition implies that the spectral sequence $E_r^{p,q}$ collapses at the page r. Hence we have $E_\infty^{p,q} = 0$ unless $p = p_0$, then $\operatorname{gr}^p H^n = 0$ unless $p = p_0$. But the filtration on H^n is finite (and regular), so $F^{p_0} H^n = H^n$ and $F^{p_0+1} H^n = 0$ and hence $H^n = \operatorname{gr}^{p_0} H^n \cong E_\infty^{p_0,n-p_0} = E_r^{p_0,n-p_0}$. $\square$

5.6 Proposition *Let $E_r^{p,q} \Rightarrow H^{p+q}$ be a biregular spectral sequence with target. Suppose $E_2^{p,q} = 0$ when $p < 0$ or $q < 0$. Then $E_r^{0,0} \cong H^0$ and we have an exact sequence*

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1 \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow H^2.$$

Remark Such kind of spectral sequence is called **first-quadrant**.

Proof: First, note that if $p < 0$ or $q < 0$, $E_r^{p,q} = 0$ for all $r \geqslant 2$. Then $\operatorname{gr}^p H^0 = 0$ when $p \neq 0$. This implies that $H^0 = \operatorname{gr}^0 H^0 \cong E_\infty^{0,0}$. Also, we have $\operatorname{gr}^p H^1 = 0$ when $p \neq 0, 1$. Hence the filtration of H^1 is of the form

$$H^1 = F^0 H^1 \supset F^1 H^1 \supset F^2 H^1 = 0.$$

So, $F^1 H^1 = \operatorname{gr}^1 H^1 \cong E_\infty^{1,0}$ and we have a short exact sequence

$$0 \longrightarrow E_\infty^{1,0} \longrightarrow H^1 \longrightarrow E_\infty^{0,1} \longrightarrow 0. \tag{5.1}$$

Also, we have $\operatorname{gr}^p H^2 = 0$ when $p \neq 0, 1, 2$. Hence the filtration of H^1 is of the form

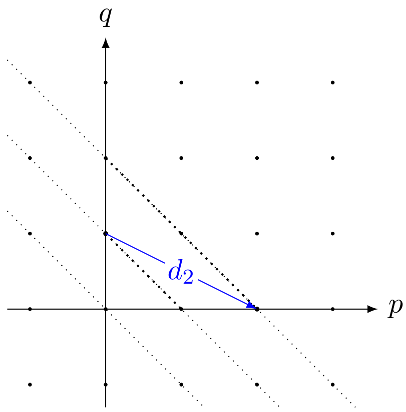
$$H^2 = F^0 H^2 \supset F^1 H^2 \supset F^2 H^2 \supset F^3 H^2 = 0.$$

So, $F^2 H^2 = \operatorname{gr}^2 H^2 \cong E_\infty^{2,0}$ and we have short exact sequences

$$0 \longrightarrow E_\infty^{2,0} \longrightarrow F^1 H^2 \longrightarrow E_\infty^{1,1} \longrightarrow 0,$$
$$0 \longrightarrow F^1 H^2 \longrightarrow H^2 \longrightarrow E_\infty^{0,2} \longrightarrow 0.$$

Hence, we have the left short exact sequence

$$0 \longrightarrow E_\infty^{2,0} \longrightarrow H^2. \tag{5.2}$$



Graph of E_2, dotted lines suggests the filtrations.

For all $r \geqslant 2$, since both $E_r^{r,1-r}$ and $E_r^{-r,r-1}$ are 0, $E_r^{0,0} = E_2^{0,0}$. Therefore

$$H^0 \cong E_\infty^{0,0} = E_2^{0,0}.$$

For all $r \geqslant 2$, since both $E_r^{1+r,1-r}$ and $E_r^{1-r,r-1}$ are 0, $E_r^{1,0} = E_2^{1,0}$. In particular, we have

$$E_\infty^{1,0} = E_2^{1,0}. \tag{5.3}$$

Since $E_2^{4,-1} = 0$ and $E_2^{-2,2} = 0$, we have $B_3(E_2^{0,1}) = 0, Z_3(E_2^{2,0}) = E_2^{2,0}$, and hence an exact sequence

$$0 \longrightarrow E_3^{0,1} \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow E_3^{2,0} \longrightarrow 0.$$

For all $r \geqslant 3$, since both $E_r^{r,2-r}$ and $E_r^{-r,r}$ are 0, $E_r^{0,1} = E_3^{r,0}$. Similarly, $E_r^{2,0} = E_3^{2,0}$ for all $r \geqslant 3$. Thus we have an exact sequence

$$0 \longrightarrow E_\infty^{0,1} \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow E_\infty^{2,0} \longrightarrow 0. \tag{5.4}$$

Combine (5.1)–(5.4), we get the following exact sequence:

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1 \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow H^2. \qquad \square$$

The following is a generalization of above with similar proof.

5.7 Proposition *Let $E_r^{p,q} \Rightarrow H^{p+q}$ be a biregular spectral sequence with target. Suppose $E_2^{p,q} = 0$ when $p < 0$, or $q < 0$, or $0 < q < n$. Then $E_r^{i,0} \cong H^i$ for any $i < n$ and we have an exact sequence*

$$0 \longrightarrow E_2^{n,0} \longrightarrow H^n \longrightarrow E_2^{0,n} \longrightarrow E_2^{n+1,0} \longrightarrow H^{n+1}.$$

Proof: First, for $p < 0$, or $q < 0$, or $0 < q < n$, $E_r^{p,q} = 0$ for all $r \geqslant 2$. Therefore for any $i < n$, $\mathrm{gr}^p H^i \cong E_\infty^{p,i-p} = 0$ when $p \neq i$. This implies that $H^i = \mathrm{gr}^i H^i \cong E_\infty^{i,0}$. Also $\mathrm{gr}^p H^n = 0$ when $p \neq 0, n$. Hence the filtration on H^n is of the form

$$H^n = F^0 H^n \supset F^1 H^n = F^2 H^n = \cdots = F^n H^n \supset F^{n+1} H^n = 0.$$

So, $F^1 H^n = F^2 H^n = \cdots = F^n H^n = \mathrm{gr}^n H^n \cong E_\infty^{n,0}$ and we have a short exact sequence

$$0 \longrightarrow E_\infty^{n,0} \longrightarrow H^n \longrightarrow E_\infty^{0,n} \longrightarrow 0. \tag{5.5}$$

Also $\mathrm{gr}^p H^{n+1} = 0$ when $p \neq 0, 1, n+1$. Hence the filtration on H^{n+1} is of the form

$$H^{n+1} = F^0 H^{n+1} \supset F^1 H^{n+1} \supset F^2 H^{n+1} = \cdots = F^{n+1} H^{n+1} \supset F^{n+1} H^{n+1} = 0.$$

So, $F^2 H^{n+1} = \cdots = F^{n+1} H^{n+1} = \mathrm{gr}^{n+1} H^{n+1} \cong E_\infty^{n+1,0}$ and we have short exact sequences

$$0 \longrightarrow E_\infty^{n+1,0} \longrightarrow F^1 H^{n+1} \longrightarrow E_\infty^{1,n} \longrightarrow 0,$$
$$0 \longrightarrow F^1 H^{n+1} \longrightarrow H^{n+1} \longrightarrow E_\infty^{0,n+1} \longrightarrow 0.$$

Hence, we have the left short exact sequence

$$0 \longrightarrow E_\infty^{n+1,0} \longrightarrow H^{n+1}. \tag{5.6}$$

For every $i < n$ and for all $r \geqslant 2$, since both $E_r^{i+r,1-r}$ and $E_r^{i-r,r-1}$ are 0, $E_r^{i,0} = E_2^{i,0}$. Therefore

$$H^i \cong E_\infty^{i,0} = E_2^{i,0}.$$

For all $r \geqslant 2$, since both $E_r^{n+r,1-r}$ and $E_r^{n-r,r-1}$ are 0, $E_r^{n,0} = E_2^{n,0}$. In particular, we have

$$E_\infty^{n,0} = E_2^{n,0}. \tag{5.7}$$

For all $2 \leqslant r < n+1$, since $E_r^{r,n+1-r}$, $E_r^{-r,n+r-1}$, $E_r^{n+1+r,1-r}$ and $E_r^{n+1-r,r-1}$ are all 0, $E_{r+1}^{0,n} = E_2^{0,n}$ and $E_{r+1}^{n+1,0} = E_2^{n+1,0}$. Similarly, for all $r > n+1$, $E_r^{0,n} = E_{n+2}^{0,n}$ and $E_r^{n+1,0} = E_{n+2}^{n=1,0}$. As for $r = n+1$, since $E^{-r,n+r-1} = 0$, we have $B_{n+2}(E_{n+1}^{0,n}) = 0$ and hence $E_{n+2}^{0,n} = Z_{n+2}(E_{n+1}^{0,n})$. Similarly, $E_{n+2}^{n+1,0} = E_{n+1}^{n+1,0}/B_{n+2}(E_{n+1}^{n+1,0})$. In this way, we get a left exact sequence

$$0 \longrightarrow E_\infty^{0,n} \longrightarrow E_2^{0,n} \xrightarrow{d_{n+1}^{0,n}} E_2^{n+1,0} \longrightarrow E_\infty^{n+1,0} \longrightarrow 0. \tag{5.8}$$

Combine (5.5)–(5.8), we get the following exact sequence:

$$0 \longrightarrow E_2^{n,0} \longrightarrow H^n \longrightarrow E_2^{0,n} \longrightarrow E_2^{n+1,0} \longrightarrow H^{n+1}. \qquad \square$$

5.8 Proposition *Let $E_r^{p,q} \Rightarrow H^{p+q}$ be a biregular spectral sequence with target. Suppose $E_2^{p,q} = 0$ when $p \neq p_1, p_2$, where $p_1 < p_2$.*

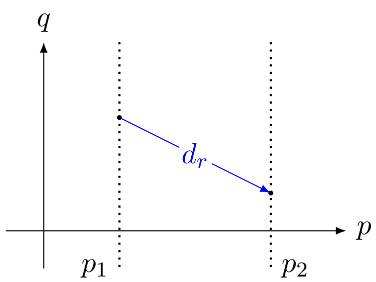
1. *If $p_2 - p_1 = 1$, then there is a short exact sequence*
$$0 \longrightarrow E_2^{p_2, n-p_2} \longrightarrow H^n \longrightarrow E_2^{p_1, n-p_1} \longrightarrow 0.$$

2. *If $p_2 - p_1 \geqslant 2$, then there is a long exact sequence*
$$\cdots \longrightarrow E_2^{p_2, n-p_2} \longrightarrow H^n \longrightarrow E_2^{p_1, n-p_1} \longrightarrow E_2^{p_2, n+1-p_2} \longrightarrow H^{n+1} \longrightarrow \cdots.$$

Remark The result sequence is sometimes called *Wang sequence.*

Proof: First, we have $E_\infty^{p,q} = 0$ when $p \neq p_1, p_2$. Hence the filtration on H^n is of the form
$$H^n = F^{p_1} H^n \supset F^{p_1+1} H^n = \cdots F^{p_2} H^n \supset F^{p_2+1} H^n = 0.$$
So, $F^{p_1+1} H^n = \cdots F^{p_2} H^n = \operatorname{gr}^{p_2} H^n \cong E_\infty^{p_2, n-p_2}$ and we have a short exact sequence
$$0 \longrightarrow E_\infty^{p_2, n-p_2} \longrightarrow H^n \longrightarrow E_\infty^{p_1, n-p_1} \longrightarrow 0. \tag{5.9}$$

If $p_2 - p_1 = 1$, then for any $r \geqslant 2$, the followings are all 0:
$$E_r^{p_2+r, n-p_2+1-r}, E_r^{p_2-r, n-p_2+r-1}, E_r^{p_1+r, n-p_1+1-r}, E_r^{p_1-r, n-p_1+r-1}.$$
Therefore, $E_r^{p_2, n-p_2} = E_2^{p_2, n-p_2}$ and $E_r^{p_1, n-p_1} = E_2^{p_1, n-p_1}$ for all $r \geqslant 2$. In particular, we have the following short exact sequence:
$$0 \longrightarrow E_2^{p_2, n-p_2} \longrightarrow H^n \longrightarrow E_2^{p_1, n-p_1} \longrightarrow 0.$$



Graph of E_r, all nonzero terms lie on $p = p_1$ and $p = p_2$.

If $p_2 - p_1 = k \geqslant 2$, then by similar argument as before, we have exact sequences
$$0 \longrightarrow E_\infty^{p_1, n-p_1} \longrightarrow E_2^{p_1, n-p_1} \xrightarrow{d_k^{p_1, n-p_1}} E_2^{p_2, n+1-p_2} \longrightarrow E_\infty^{p_2, n+1-p_2} \longrightarrow 0.$$
Combine these with (5.9), we get a long exact sequence
$$\cdots \longrightarrow E_2^{p_2, n-p_2} \longrightarrow H^n \longrightarrow E_2^{p_1, n-p_1} \longrightarrow E_2^{p_2, n+1-p_2} \longrightarrow H^{n+1} \longrightarrow \cdots$$
as desired. $\square$