Note on
$\mathscr{D}$-modules
(on complex manifolds)

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Abstract

This is my reading and thought notes on $\mathscr{D}$-modules in the context of complex geometry. It contains standard materials of definitions and conclusions in this field at beginner level. In addition, it also contains funny, cumbersome and maybe highly non-necessary materials (in small fonts) basically around my confusions and brainstorms.

Contents

Conventions

Throughout this note, unless specify otherwise, all objects are over the complex field $\mathbb{C}$. For example, by a vector space, we mean a vector space over $\mathbb{C}$; by a sheaf, we mean a sheaf of vector spaces over $\mathbb{C}$.

To invalid potential issue of infinity, we require charts to be connected.

We will widely use the notation $\in$. So, $x \in X$ could means x is an element of X, or x is an object of X, or x is a section of X, depending on the context.

There are many sheaves canonically defined on every complex manifolds M, and usually have notations of the form $\mathscr{F}_M$. When the manifold M is unambiguous, we simplify the notation to $\mathscr{F}$.

For $(X, \mathscr{O}_X)$ a locally ringed space, we will use $\mathcal{M}(\mathscr{O}_X)$ or simply $\mathcal{M}(X)$ to denote the $\mathbb{C}$-linear category of $\mathscr{O}_X$-modules.

We also use the following conventions from analysis: whenever we have $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{N}^m$, then

$$|\lambda| = \sum_{i=1}^{m} \lambda_i, \qquad \lambda! = \prod_{i=1}^{m} (\lambda_i!);$$

if we have another $\mu = (\mu_1, \cdots, \mu_m) \in \mathbb{N}^m$, then

$$\binom{\lambda}{\mu} = \begin{cases} \dfrac{\lambda!}{\mu!(\lambda-\mu)!} & \text{if } \lambda \geqslant \mu, \\ 0 & \text{if not,} \end{cases}$$

where $\lambda \geqslant \mu$ means $\lambda_i \geqslant \mu_i$ for all $i = 1, \cdots, m$. We use ϵ_i to denote the multi-index whose i-th term is 1 and all the other terms are 0. Let $X = (X_1, \cdots, X_m)$ be a m-tuple of *pairwise commutative* elements in a ring, then by X^λ, we mean the unambiguous product

$$X_1^{\lambda_1} \cdots X_m^{\lambda_m}.$$

Let $(X^p)_{p \in I}$ (resp. $(X_p)_{p \in I}$) be a family of elements in a ring *parametrized* by a subset I of $\mathbb{N}$ (for example, $I = [1, m] := \{1, \cdots, m\}$), then by X^λ (resp. X_λ), we mean the ordered product

$$X^{\lambda_1} \cdots X^{\lambda_m} \qquad (\text{resp. } X_{\lambda_1} \cdots X_{\lambda_m}).$$

Note that these two conventions would not cause ambiguity as long as we distinguish the cases that X^p is a power of the element X and that X^p is a member in the family X.

§ 1 Basic constructions

One can skip hard section (marked by ¶) and remarks in small font during first read.

1.1 The sheaf of holomorphic differential operators

Let M be a complex manifold, $\mathscr{O}_M$ its its structural sheaf, that is, the sheaf of holomorphic functions on M. Suppose M is of complex dimension m, then locally, one can always find a local coordinate system $(z^i)_{1\leqslant i\leqslant m}$. Let's keep such convention.

Let $\mathbb{C}_M$ be the constant sheaf with values $\mathbb{C}$ on M. It is where everything lives on in this notes, so the tensor product and the internal Hom-sheaf over it is denoted by $-\otimes-$ and $\mathscr{H}om(-,-)$. For $\mathscr{F}$ a sheaf on M, we use $\mathscr{E}nd(\mathscr{F})$ to denote $\mathscr{H}om(\mathscr{F},\mathscr{F})$.

Let Θ_M be the sheaf of holomorphic vector fields on M and note that it is a locally free $\mathscr{O}_M$-modules with local basis $(\partial_{z^i})_{1\leqslant i\leqslant m}$ (or simply denoted by $(\partial_i)_{1\leqslant i\leqslant m}$) under the local coordinate system $(z^i)_{1\leqslant i\leqslant m}$. Note that it is a sheaf of Lie algebras and that each ∂_i acting on a function f gives $\frac{\partial f}{\partial z^i}$.

Note that the pair $(\mathscr{O},\Theta)$ satisfies the following properties:

(a) Θ is an $\mathscr{O}$-modules,

(b) $\mathscr{O}$ is a Θ-module and

(c) those two actions give rise to an $\mathscr{O}$-linear monomorphism of sheaves of ($\mathbb{C}$-linear) Lie algebras from Θ to $\mathscr{D}er(\mathscr{O})$, the sheaf of ($\mathbb{C}$-linear) derivations of $\mathscr{O}$.

This means they form a sheaf of faithful *Lie–Rinehart algebras*.

Remark Indeed, a *Lie–Rinehart algebra* is a pair $(A,\mathfrak{g})$ of a commutative ring A and a Lie algebra $\mathfrak{g}$ subject to the following axioms:

(LR1) $\mathfrak{g}$ is an A-modules;

(LR2) A is a $\mathfrak{g}$-module;

(LR3) $\mathfrak{g}$ acts as derivations of A;

(LR4) A acts on $\mathfrak{g}$ by the following *Leibniz rule*:

$$a[v,w]=[av,w]+w(a)v, \qquad \forall a\in A, v,w\in\mathfrak{g}.$$

If $\mathfrak{g}$ acts faithfully on A, that is, the Lie algebra homomorphism $\mathfrak{g}\to\mathrm{Der}(A)$, where $\mathrm{Der}(A)$ denotes the set of derivations of A which is both a Lie algebra and an A-module, is injective, then (LR4) is equivalent to say that the homomorphism $\mathfrak{g}\to\mathrm{Der}(A)$ is A-linear.

Since Θ acts faithfully on $\mathscr{O}$, we have the following embedding:

$$\Theta \hookrightarrow \mathscr{D}er(\mathscr{O}) \hookrightarrow \mathscr{E}nd(\mathscr{O}).$$

On the other hand, since $\mathscr{O}$ a sheaf of commutative rings, it can be canonically embedded into $\mathscr{E}nd(\mathscr{O})$ as its center. Note that, in each case, we have a monomorphism of $\mathscr{O}$-modules. Then, we meet the following definition:

1.1.1 Definition The $\mathscr{O}$-subalgebra of $\mathscr{E}nd(\mathscr{O})$ generated by the images of above two embeddings is called the *sheaf of differential operators on M*, denoted by $\mathscr{D}_M$.

Note that this makes $\mathscr{D}$ into the universal algebra of $(\mathscr{O}, \Theta)$.

Remark Indeed, first note that if R is a ring and B is a commutative subring of R, then (B, R), where R is equipped with the standard Lie bracket and acts on B by adjoint actions, is a Lie–Rinehart algebra.

A *homomorphism of Lie–Rinehart algebras* $(A, \mathfrak{g}) \to (B, \mathfrak{h})$ is a pair (φ, ψ) of a ring homomorphism $\varphi\colon A \to B$ and a Lie algebra homomorphism $\psi\colon \mathfrak{g} \to \mathfrak{h}$ such that

(a) φ makes ψ into a homomorphism of A-modules;

(b) ψ makes φ into a homomorphism of $\mathfrak{g}$-modules.

Then, a *homomorphism* from a Lie–Rinehart algebra $(A, \mathfrak{g})$ to a ring R is a pair (φ, ψ) of a ring homomorphism $\varphi\colon A \to R$ and a Lie algebra homomorphism $\psi\colon \mathfrak{g} \to R$ such that (φ, ψ) is a homomorphism of Lie–Rinehart algebras from $(A, \mathfrak{g})$ to $(\mathrm{Im}(\varphi), R)$.

Finally, the *universal algebra* of a Lie–Rinehart algebra $(A, \mathfrak{g})$ is the ring $\mathcal{U}(A, \mathfrak{g})$ (equipped with a homomorphism (ι, ρ) from $(A, \mathfrak{g})$ to it) satisfying the following universal property:

> Whenever there is a homomorphism (φ, ψ) from $(A, \mathfrak{g})$ to a ring R, there exists a unique ring homomorphism ϕ from $\mathcal{U}(A, \mathfrak{g})$ to R such that $\varphi = \phi \circ \iota$ and $\psi = \phi \circ \rho$.

Note that in particular, there exists a unique representation $\vartheta\colon \mathcal{U}(A, \mathfrak{g}) \to \mathrm{End}(A)$ such that $\varphi \circ \iota$ is the canonical representation of A and $\varphi \circ \rho$ is the action of $\mathfrak{g}$ on A. In this way, we can always identify A and its image in $\mathcal{U}(A, \mathfrak{g})$.

Note that, using a local coordinate system $(z^i)_{1 \leqslant i \leqslant m}$, any differential operator can be locally uniquely written as

$$\sum_{\lambda \in \mathbb{N}^m} f_\lambda \partial^\lambda,$$

where $f_\lambda \in \mathscr{O}$ and all but finitely many of them are zero. This can be shown using the following lemma, which relates the left and right $\mathscr{O}$-module structures on $\mathscr{D}$ (the left (resp. right) $\mathscr{O}$-module structure is given by left (resp. right) multiplication).

1.1.2 Proposition *Let U be a chart of M with coordinate system $(z^i)_{1\leqslant i\leqslant m}$ and $(\partial_i)_{1\leqslant i\leqslant m}$ the corresponding basis of $\Gamma(U,\Theta)$. Then for any $f\in\Gamma(U,\mathscr{O})$ and $i,j\in[1,m]$, we have*

$$[\partial_i,f]=\partial_i(f),\quad [\partial_i,\partial_j]=\delta_{ij}.$$

Moreover, for any $\lambda\in\mathbb{N}^m$, we have

$$\partial^\lambda f=\sum_{\mu\leqslant\lambda}\binom{\lambda}{\mu}\partial^{\lambda-\mu}(f)\partial^\mu,$$
$$f\partial^\lambda=\sum_{\mu\leqslant\lambda}\binom{\lambda}{\mu}(-1)^{|\lambda-\mu|}\partial^\mu\partial^{\lambda-\mu}(f).$$

Then, for such an open set U, we get an exhaustive filtration on $\Gamma(U,\mathscr{D})$:

$$F_0\Gamma(U,\mathscr{D})\subset F_1\Gamma(U,\mathscr{D})\subset F_2\Gamma(U,\mathscr{D})\subset\cdots,$$

where for each $p\in\mathbb{N}$,

$$F_p\Gamma(U,\mathscr{D})=\left\{\sum_{|\lambda|\leqslant p}f_\lambda\partial^\lambda;f_\lambda\in\Gamma(U,\mathscr{O})\right\}.$$

A differential operator $P\in F_p\Gamma(U,\mathscr{D})\setminus F_{p-1}\Gamma(U,\mathscr{D})$ is said to be of *order* p, denoted by $\operatorname{ord}(P)$. We always keep the convention that $\operatorname{ord}(0)=-\infty$. Note that if we write P as $\sum_{\lambda\in\mathbb{N}^m}f_\lambda\partial^\lambda$, then $\operatorname{ord}(P)$ is precisely the integer $\max\{|\lambda|;f_\lambda\neq 0\}$.

These filtrations can be glued into an exhaustive filtration on $\mathscr{D}$:

$$F_0\mathscr{D}\subset F_1\mathscr{D}\subset F_2\mathscr{D}\subset\cdots.$$

To see this, we need the following lemma:

1.1.3 Lemma *The order of a differential operator on a chart U does not depend on the choice of coordinate system. Consequently, the order of a differential operator on M is locally constant.*

Proof: The first assertion follows by apply the differential operator on polynomials. The second assertion follows from the *Identity Principle*: if two holomorphic functions on a connected open set coincide on a nonempty open subset, then they are the same. Indeed, this implies that the order of the restriction of a differential operator on a connected chart to another smaller chart remains the same. □

This lemma justify the notation $\operatorname{ord}(P)$. Moreover, since $\operatorname{ord}(P)$ is a locally constant, the presheaves

$$F_p\mathscr{D}\colon U\longmapsto\{P\in\Gamma(U,\mathscr{D});\operatorname{ord}(P)\leqslant p\}$$

are in fact sheaves and furthermore $\mathscr{O}$-submodules of $\mathscr{D}$. Moreover, by same reason, the filtration of stalks $(F_p\mathscr{D})_x$ at each point $x \in M$ is exhaustive, hence so is the filtration of sheaves $F_p\mathscr{D}$.

Remark Let $(A, \mathfrak{g})$ be a Lie–Rinehart algebra. Since $\mathcal{U}(A, \mathfrak{g})$ is a ring extension of A generated by $\rho(\mathfrak{g})$, we get a natural filtration on $\mathcal{U}(A, \mathfrak{g})$:

$$\mathcal{U}_0(A, \mathfrak{g}) \subset \mathcal{U}_1(A, \mathfrak{g}) \subset \mathcal{U}_2(A, \mathfrak{g}) \subset \cdots ,$$

where $\mathcal{U}_p(A, \mathfrak{g})$ is the A-submodules of $\mathcal{U}(A, \mathfrak{g})$ generated by $\bigoplus_{q \leqslant p} \rho(\mathfrak{g})^q$ (with the convention that $\rho(\mathfrak{g})^0 = A$). The filtration $F_p\mathscr{D}$ can be understood in this way.

Example (Differential operator of infinite order) Let M be the disjoint union of countable copies of $\mathbb{C}$. On the p-th copy, consider the differential operator ∂^p, the p-th power of the standard vector field on $\mathbb{C}$. Since those copies are disjoint with each other, one can glue these differential operators together to get a differential operator on M. However, this global differential operator wouldn't be of any finite order.

By computation using local coordinate system, we have:

1.1.4 Proposition *If we identify $\mathscr{O}$, Θ with their images in $\mathscr{D}$, then we have*

(i) $F_0\mathscr{D} = \mathscr{O}$,

(ii) $F_1\mathscr{D} = \mathscr{O} \oplus \Theta$,

(iii) $F_p\mathscr{D} \circ F_q\mathscr{D} \subset F_{p+q}\mathscr{D}$,

(iv) $[F_p\mathscr{D}, F_q\mathscr{D}] \subset F_{p+q-1}\mathscr{D}$.

Note that this implies that $(\mathscr{D}, F_\bullet)$ is a sheaf of *almost commutative rings*, hence its *associated graded algebra*

$$\mathrm{gr}_\bullet(\mathscr{D}) = \mathrm{gr}_\bullet^F(\mathscr{D}) := \bigoplus_{p=0}^{\infty} F_p\mathscr{D}/F_{p-1}\mathscr{D}$$

is commutative (here and from now on, we keep the convention that if $F_\bullet$ is an increasing filtration start from 0, then $F_p = 0$ for negative p).

Remark A *filtered ring* $(R, F_\bullet)$ is a ring R equipped with an exhaustive increasing filtration of subspaces $F_\bullet$ on it satisfying the following axioms:

(a) $1 \in F_0R$;

(b) $F_pR \cdot F_qR \subset F_{p+q}R$.

Any filtered ring $(R, F_\bullet)$ admits an *associated graded algebra*

$$\mathrm{gr}_\bullet(R) = \mathrm{gr}^F_\bullet(R) := \bigoplus_{p=0}^{\infty} F_pR/F_{p-1}R,$$

whose multiplication is induced from that of R in an obvious way. If a filtered ring $(R, F_\bullet)$ furthermore satisfies

(c) $[F_pR, F_qR] \subset F_{p+q-1}R$,

then its associated graded algebra is commutative. Such a filtered ring is called a *almost commutative ring*.

Note that we have $\mathrm{gr}_0(\mathscr{D}) = F_0\mathscr{D} = \mathscr{O}$, hence $\mathrm{gr}_\bullet(\mathscr{D})$ is a commutative graded $\mathscr{O}$-algebra. Since $\mathscr{D}$ is generated by $F_1\mathscr{D}$, the $\mathscr{O}$-algebra $\mathrm{gr}_\bullet(\mathscr{D})$ is generated by $\mathrm{gr}_1(\mathscr{D})$, which is isomorphic to Θ by Proposition 1.1.4. Then the commutative multiplication on $\mathrm{gr}_\bullet(\mathscr{D})$ induces the following surjective homomorphisms of $\mathscr{O}$-modules

$$\mathbb{S}^p_{\mathscr{O}}(\Theta) \longrightarrow \mathrm{gr}_p(\mathscr{D}),$$

which give rise to a surjective homomorphism of graded $\mathscr{O}$-algebras:

$$\mathbb{S}^\bullet_{\mathscr{O}}(\Theta) \longrightarrow \mathrm{gr}_\bullet(\mathscr{D}). \tag{1.1.5}$$

Using a local coordinate system and notice that $\{\partial^\lambda; \lambda \in \mathbb{N}^m\}$ form an $\mathscr{O}$-basis, it is straightforward to see that the above homomorphism is an isomorphism.

Note that $\mathscr{O}$ is noetherian by *Rückert Basis Theorem*, which can be shown by *Weierstrass Preparation Theorem*. From this and that Θ is a locally free $\mathscr{O}$-module of finite rank, we conclude that $\mathbb{S}_{\mathscr{O}}(\Theta)$, as well as $\mathrm{gr}(\mathscr{D})$, is noetherian. Then, we have

1.1.6 Theorem *$\mathscr{D}$ is left and right noetherian.*

To see this, we need the following lemma:

1.1.7 Lemma *Let $(R, F_\bullet)$ be an almost commutative ring. Suppose furthermore:*

(i) $\mathrm{gr}^F_1(R)$ generates $\mathrm{gr}^F_\bullet(R)$ as a F_0R-algebra,

(ii) $\mathrm{gr}^F_\bullet(R)$ is noetherian.

Then R is left and right noetherian.

We leave the proof later.

1.2 The sheaf of differential operators

In this subsection, $(X, \mathscr{O}_X)$ is a (commutative) locally ringed space. For any pair of $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, we define a sequence of $\mathscr{O}_X$-submodules of $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ (not $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$) recursively as follows:

- $\mathscr{D}iff_X^p(\mathscr{F}, \mathscr{G}) = 0$ for negative p;
- for $p \geqslant 0$, $\mathscr{D}iff_X^p(\mathscr{F}, \mathscr{G})$ maps each open set U to

  $$\left\{P\colon\ \mathscr{F}|_U \to \mathscr{G}|_U\,;\,[P|_V\,, a] \in \Gamma(V, \mathscr{D}iff_X^{p-1}(\mathscr{F}, \mathscr{G})), \forall a \in \Gamma(V, \mathscr{O})\right\},$$

  where the morphism $[P|_V\,, a]$ maps each section t of $\mathscr{F}|_V$ to the section $P(a.t) - a.P(t)$ of $\mathscr{G}|_V$.

Then the *sheaf of differential operators from $\mathscr{F}$ to $\mathscr{G}$* is the sheaf union

$$\mathscr{D}iff_X(\mathscr{F}, \mathscr{G}) = \bigcup \mathscr{D}iff_X^p(\mathscr{F}, \mathscr{G}).$$

A section of $\mathscr{D}iff_X^p(\mathscr{F}, \mathscr{G})$ is called a *differential operator of order $\leqslant p$*. The order of a differential operator can also be characterized by the following construction: for P an endomorphism of $\mathscr{O}$ and p a natural number, let $\sigma_p(P)\colon \mathscr{O}_X^{\otimes p} \to \mathscr{H}om(\mathscr{F}, \mathscr{G})$ be the homomorphism

$$a_1 \otimes a_2 \otimes \cdots \otimes a_n \mapsto [\cdots[[P, a_1], a_2], \cdots, a_n],$$

where $a_1, a_2, \cdots, a_n$ are sections of $\mathscr{O}_X$ (note that when $p = 0$, $\sigma_0 = \mathrm{id}$). For a differential operator P, $\sigma_p(P)$ is called its *p-th symbol*. In particular, the *principal symbol* of P is its symbol of order $\mathrm{ord}(P)$, denoted by $\sigma(P)$. Then, we have

1.2.1 Lemma *(i) Every $\sigma_p(P)$ is symmetric, hence from $\mathbb{S}^p(\mathscr{O}_X)$.*

(ii) $\mathrm{Ker}(\sigma_p) = \mathscr{D}iff_X^{p-1}(\mathscr{F}, \mathscr{G})$*, hence if $Q \in \mathscr{D}iff_X^q(\mathscr{F}, \mathscr{G})$, then $\sigma_p(Q)$ lands in $\mathscr{D}iff_X^{q-p}(\mathscr{F}, \mathscr{G})$.*

Proof: (i) follows from the Jacobi identity and the that $\mathscr{O}_X$ is commutative. (ii) follows from expansion of the recursive definition of $\mathscr{D}iff_X$. □

From this lemma and the observation that $\mathscr{D}iff_X^0(\mathscr{F}, \mathscr{G}) = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$, each σ_p induces a monomorphism of $\mathscr{O}_X$-modules

$$\mathrm{gr}_p(\mathscr{D}iff_X(\mathscr{F}, \mathscr{G})) \longrightarrow \mathscr{H}om(\mathbb{S}^p(\mathscr{O}_X), \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})).$$

From now on, we identify each $\mathrm{gr}_p(\mathscr{D}iff_X(\mathscr{F}, \mathscr{G}))$ with its image under above monomorphism and view σ_p as the projection from $\mathscr{D}iff_X^p(\mathscr{F}, \mathscr{G})$ to $\mathrm{gr}_p(\mathscr{D}iff_X(\mathscr{F}, \mathscr{G}))$.

See [EGA4, Chapter 16] and [SGA3, Exposé VII] for more details. From now on, we only forces on the special one $\mathscr{D}iff_X = \mathscr{D}iff_X(\mathscr{O}_X, \mathscr{O}_X)$.

1.2.2 Proposition *If we identify $\mathscr{O}_X$ with its image in $\mathscr{E}nd(\mathscr{O}_X)$, then we have*

(i) $\mathscr{D}iff^0_X = \mathscr{O}_X$,

(ii) $\mathscr{D}iff^1_X = \mathscr{O}_X \oplus \mathscr{D}er(\mathscr{O}_X)$,

(iii) $\mathscr{D}iff^p_X \circ \mathscr{D}iff^q_X \subset \mathscr{D}iff^{p+q}_X$,

(iv) $[\mathscr{D}iff^p_X, \mathscr{D}iff^q_X] \subset \mathscr{D}iff^{p+q-1}_X$.

PROOF: Note that we only need to verify these at stalks. So, the assertions follow from those in commutative algebra. $\square$

Then, we have an almost commutative $\mathscr{O}_X$-algebra $\mathscr{D}iff^\bullet_X$. Hence, there is a canonical homomorphism of graded $\mathscr{O}_X$-algebras

$$\mathbb{S}^\bullet_{\mathscr{O}_X}(\mathscr{D}er(\mathscr{O}_X)) \longrightarrow \mathrm{gr}_\bullet(\mathscr{D}iff_X). \tag{1.2.3}$$

However, this is not surjective, *a fortiori* an isomorphism in general.

Remark Let A be a commutative ring and M, N two A-modules, then the filtered A-module $\mathrm{Diff}^\bullet_A(M,N)$ can be defined recursively as follows:

- $\mathrm{Diff}^p_A(M,N) = 0$ for negative p;
- for $p \geqslant 0$, $\mathrm{Diff}^p_A(M,N)$ is the following submodule of $\mathrm{Hom}(M,N)$

$$\{P \in \mathrm{Hom}(M,N); [P,a] \in \mathrm{Diff}^{p-1}_A(M,N), \forall a \in A\},$$

  where the homomorphism $[P,a]$ maps each $t \in M$ to $P(a.t) - a.P(t)$.

In particular, $\mathrm{Diff}^\bullet_A(A,A)$ is simply denoted by $\mathrm{Diff}^\bullet_A$. It is not difficult to show that $\mathrm{Diff}^\bullet_A$ is an almost commutative ring and $\mathrm{Diff}^1_A = A \oplus \mathrm{Der}(A)$. However, it is not true in general that the subalgebra of $\mathrm{End}(A)$ generated by $A \oplus \mathrm{Der}(A)$ is the entire Diff_A.

Now, we go back to the case on a complex manifold M. We have

1.2.4 Lemma *The image of Θ_M in $\mathscr{E}nd(\mathscr{O}_M)$ is $\mathscr{D}er(\mathscr{O}_M)$.*

PROOF: Since the problem is local, we may assume we are working on an open set of $\mathbb{C}^m$ with coordinate system $(z^i)_{1\leqslant i\leqslant m}$. First, it is clear that every holomorphic vector field defines a derivation. Conversely, let D be a derivation, then it comes from the vector field $\theta = \sum_{i=1}^m D(z^i)\partial_i$. Indeed, the *Hadamard lemma* shows that, if f is a holomorphic function nearby a point x, then there exist holomorphic functions $(f_i)_{1\leqslant i\leqslant m}$ nearby x such that

$$f = f(x) + \sum_{i=1}^m (z^i - z^i(x)) f_i$$

and that $f_i(x) = \frac{\partial f}{\partial z^i}(x)$ for all i. Then we have

$$D(f) = D(\sum_{i=1}^{m}(z^i - z^i(x))f_i) = \sum_{i=1}^{m}(D(z^i)f_i + (z^i - z^i(x))D(f_i)).$$

Hence $D(f)(x) = \theta(f)(x)$. Since x is arbitrary, we conclude that D comes from the holomorphic vector field θ. □

More general, we have

1.2.5 Theorem *For each p, $F_p\mathscr{D}_M = \mathscr{Diff}^p_M$.*

PROOF: It reduces to show $(F_p\mathscr{D}_M)_x = (\mathscr{Diff}^p_M)_x$ at every point $x \in M$. We keep the same assumption as previous, then we may assume $x = 0 \in \mathbb{C}^m$. Let A be the ring of germs of holomorphic functions at 0. Then $(\mathscr{D}_M)_0$ equals the subalgebra of $\mathrm{End}(A)$ generated by A and $\mathrm{Der}(A)$, which is also the universal algebra $\mathcal{U}(A, \mathrm{Der}(A))$ by the discussion in previous subsection. In addition, we have $(\mathscr{Diff}^p_M)_0 = \mathrm{Diff}_A$ from the definition. Then it remains to show that $\mathcal{U}_p = \mathcal{U}_p(A, \mathrm{Der}(A))$ equals Diff^p_A.

It is clear that $\mathcal{U}_p \subset \mathrm{Diff}^p_A$. To prove the converse, we do induction on p. The $p = 1$ case is Lemma 1.2.4. To do the inductive step, we need a lemma:

1.2.6 Lemma *For positive p, if there are $P_1, \cdots, P_m \in \mathcal{U}_{p-1}$ satisfying*

$$[P_i, z^j] = [P_j, z^i], \qquad \forall i, j \in [1, m],$$

then there exists a $Q \in \mathcal{U}_p$ such that

$$[Q, z^i] = P_i, \qquad i \in [1, m].$$

PROOF: First, it is not difficult to find a $Q_m \in \mathcal{U}_p$ such that $[Q_m, z^m] = P_m$. Indeed, if we write P_m as

$$P_m = \sum_{|\lambda| \leqslant p-1} f_\lambda \partial^\lambda,$$

then

$$Q_m = \sum_{|\lambda| \leqslant p-1} \frac{f_\lambda}{\lambda_m} \partial^{\lambda + \epsilon_m}$$

works:

$$[Q_m, z^m] = \sum_{|\lambda| \leqslant p-1} \frac{f_\lambda}{\lambda_m}[\partial^{\lambda+\epsilon_m}, z^m] = \sum_{|\lambda| \leqslant p-1} \frac{f_\lambda}{\lambda_m} \lambda_m \partial^\lambda = P_m.$$

Suppose we already find $Q_{k+1} \in \mathcal{U}_p$ ($k \in [1, m-1]$) such that

$$[Q_{k+1}, z^i] = P_i, \qquad i \in [k+1, m].$$

Then we want to find a $Q_k \in \mathcal{U}_p$ such that

$$[Q_k, z^i] = P_i, \qquad i \in [k, m].$$

To do this, we can first let $P_{k+\frac{1}{2}}$ be $[Q_{k+1}, z^k] - P_k$. Then, for each $i \in [k+1, m]$, we have

$$\begin{aligned}
[P_{k+\frac{1}{2}}, z^i] &= [[Q_{k+1}, z^k] - P_k, z^i] \\
&= [[Q_{k+1}, z^i], z^k] - [P_k, z^i] \\
&= [P_i, z^k] - [P_k, z^i] = 0.
\end{aligned}$$

On the other hand, if we write $P_{k+\frac{1}{2}}$ as

$$P_{k+\frac{1}{2}} = \sum_{|\lambda| \leqslant p-1} g_\lambda \partial^\lambda,$$

then

$$[P_{k+\frac{1}{2}}, z^i] = \sum_{|\lambda| \leqslant p-1} g_\lambda [\partial^\lambda, z^i] = \sum_{|\lambda| \leqslant p-1} g_\lambda \lambda_i \partial^{\lambda - \epsilon_i}.$$

Hence $g_\lambda = 0$ if $\lambda_i \neq 0$ for $i \in [k+1, m]$. Then $P_{k+\frac{1}{2}}$ is a operator built up only from $\partial_1, \cdots, \partial_k$. Hence if we put

$$Q_{k+\frac{1}{2}} = \sum_{|\lambda| \leqslant p-1} \frac{g_\lambda}{\lambda_k} \partial^{\lambda + \epsilon_k}$$

Then, we have

$$[Q_{k+\frac{1}{2}}, z^k] = P_{k+\frac{1}{2}}$$

and, for each $i \in [k+1, m]$,

$$[Q_k, z_i] = 0.$$

Then, $Q_k = Q_{k+1} - Q_{k+\frac{1}{2}}$ works.

Therefore, by induction, we can find the required $Q \in \mathcal{U}_p$. $\square$

Now, we go back to the proof of Theorem 1.2.5. Suppose we already have $\mathcal{U}_{p-1} = \mathrm{Diff}_A^{p-1}$. Let P be a differential operator of order $\leqslant p$. Then for each $i \in [1, m]$, $P_i = [P, z^i]$ is of order $p - 1$, hence $P_i \in \mathcal{U}_{p-1}$. Note that for any $i, j \in [1, m]$, we have

$$[P_i, z^j] = [[P, z^i], z^j] = [[P, z^j], z^i] = [P_j, z^i].$$

Hence the lemma applies and there exists a $Q \in \mathcal{U}_p$ such that

$$[Q, z^i] = P_i = [P, z^i], \qquad i \in [1, m].$$

Now, we need another lemma:

1.2.7 Lemma *If D is a differential operator such that*

$$[D, z^i] = 0, \qquad i \in [1, m].$$

Then D is of order 0*.*

PROOF: Let f be an arbitrary holomorphic function at 0 and x a point nearby 0 such that f is also holomorphic at x. Then, by *Hadamard lemma*, there exist holomorphic functions $(f_i)_{1 \leqslant i \leqslant m}$ nearby x such that

$$f = f(x) + \sum_{i=1}^{m} (z^i - z^i(x)) f_i.$$

Then we have (notice that D commutes with any number)

$$\begin{aligned}
[D, f] &= [D, \sum_{i=1}^{m} (z^i - z^i(x)) f_i] \\
&= \sum_{i=1}^{m} ([D, z^i] f_i + (z^i - z^i(x))[D, f_i]) \\
&= \sum_{i=1}^{m} (z^i - z^i(x))[D, f_i].
\end{aligned}$$

Apply both sides to arbitrary $g \in A$ and evaluate at x, we see that the function $D(fg) - fD(g)$ vanishes at x. By arbitrarily choosing x, we see that $D(fg) = fD(g)$. Hence

$$[D, f] = 0.$$

Since f is arbitrary, this means D is of order 0. □

Now $Q \in \mathfrak{U}_p$, $P - Q \in \mathrm{Diff}_A^0 \subset \mathfrak{U}_p$, hence $P = Q + (P - Q) \in \mathfrak{U}_p$. Therefore $\mathfrak{U}_p = \mathrm{Diff}_A^p$ as desired. □

Remark One may expect another proof of Theorem 1.2.5 using *sheaf of principle parts*: if the sheaf $\Omega^1_{M/\mathbb{C}}$ of *Kahler differentials* on $\mathscr{O}_M$ is locally free of finite rank, then M is *differentially smooth* (*Jacobi Criterion*, see [EGA4, Theorem 16.12.2]) and consequently, $\mathscr{D}\!\mathit{iff}_M$ is generated by $\mathscr{D}\!\mathit{iff}^1_M$ by [EGA4, Theorem 16.11.2]. However, although $\mathscr{D}\!\mathit{er}(\mathscr{O}_M)$ is locally free of rank m, it is not true that so is $\Omega^1_{M/\mathbb{C}}$. Be careful that $\Omega^1_{M/\mathbb{C}}$ is *NOT* equal to Ω^1_M, the sheaf of holomorphic differentials: the later is just a quotient of the first. However, one may still use a similar strategy since we still have

$$\Omega^1_M = (\Omega^1_{M/\mathbb{C}})^{**},$$

where $()^*$ means the dual $\mathscr{O}_M$-module operation, and that the canonical morphism $\Omega^1_{M/\mathbb{C}} \to (\Omega^1_{M/\mathbb{C}})^{**}$ is surjective. See this $n$lab term and this MO post for more details about the relation between $\Omega^1_{M/\mathbb{C}}$ and Ω^1_M.

1.2.8 Corollary *The homomorphism* (1.2.3) *is an isomorphism.*

PROOF: Follows from Lemma 1.2.4, Theorem 1.2.5 and the fact that (1.1.5) is an isomorphism. □

1.3 $\mathscr{D}$-modules and flat connections

Let Ω^1_M be the sheaf of holomorphic 1-forms, which is the dual $\mathscr{O}_M$-module of Θ_M. Then $\Omega^p_M = \bigwedge^p \Omega^1_M$ is the sheaf of holomorphic p-forms. Note that $\Omega^0_M = \mathscr{O}_M$. Let $d\colon \Omega^p_M \to \Omega^{p+1}_M$ be the *exterior derivative*, which can be locally defined as

$$d(\sum_\lambda f_\lambda dz^\lambda) = \sum_\lambda df_\lambda \wedge dz^\lambda, \quad \forall f_\lambda \in \mathscr{O}_M, \lambda \in [1,m]^p$$

and (globally) characterized by the following property:

$$d(\omega \wedge \upsilon) = d\omega \wedge \upsilon + (-1)^q \omega \wedge d\upsilon,$$

where $\omega \in \Omega^q_M$ (in other words, d is a *graded derivation of degree* 1 on $\Omega^\bullet_M$). Note that then we have a complex (the *de Rham complex*)

$$(1.3.1) \qquad 0 \longrightarrow \mathbb{C}_M \longrightarrow \mathscr{O}_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m_M \longrightarrow 0.$$

By *Holomorphic Poincaré Lemma*, the above complex is *acyclic*.

The sheaf Θ_M can acts on $\Omega^\bullet_M$ by the *interior derivatives*: for θ a vector field on M, the interior derivative of a p-form ω along θ is defined as

$$\iota_\theta \omega(\theta_1, \cdots, \theta_{p-1}) := \omega(\theta, \theta_1, \cdots, \theta_{p-1}).$$

It can also be characterized by the following property

$$\iota_\theta(\omega \wedge \upsilon) = (\iota_\theta \omega) \wedge \upsilon + (-1)^q \omega \wedge (\iota_\theta \upsilon),$$

where $\omega \in \Omega^q_M$, together with the condition

$$\iota_\theta \omega = \langle v, w \rangle, \quad \forall \omega \in \Omega^1_M.$$

The sheaf Θ_M can also acts on $\Omega^\bullet_M$ by the *Lie derivatives*: for θ a vector field on M, the Lie derivative $\mathcal{L}_\theta\colon \Omega^\bullet_M \to \Omega^\bullet_M$ is the unique chain map (of degree 0) characterized by the following property

$$\mathcal{L}_\theta(\omega \wedge \upsilon) = (\mathcal{L}_\theta \omega) \wedge \upsilon + \omega \wedge (\mathcal{L}_\theta \upsilon)$$

and the condition

$$\mathcal{L}_\theta(f) = \theta(f), \quad \forall f \in \mathscr{O}_M.$$

The two actions of Θ_M on $\Omega^\bullet_M$ are related by the *Cartan's magic formula*:

$$\mathcal{L}_\theta = d \circ \iota_\theta + \iota_\theta \circ d.$$

In particular, we have

$$\mathcal{L}_\theta(d\omega) = d\mathcal{L}_\theta \omega, \qquad \mathcal{L}_{f\theta}\omega = f\mathcal{L}_\theta \omega + df \wedge \iota_\theta \omega.$$

Let $\boldsymbol{\omega}_M$ denote the sheaf Ω^m_M of *volume forms*. Locally, it has a basis $dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m$ and therefore is an invertible $\mathscr{O}_M$-module. The Lie derivatives gives a right action of the Lie algebra Θ_M on $\boldsymbol{\omega}_M$ by

$$\theta.\omega := -\mathcal{L}_\theta \omega.$$

1.3.2 Lemma *Let $\mathscr{M}$ be an $\mathscr{O}$-module. Then an left (resp. right) $\mathscr{D}$-module structure on $\mathscr{M}$ is equivalent to (in the sense that the $\mathscr{D}$-module structure extends the action of Θ) an action $\alpha\colon \Theta \to \mathscr{E}nd(\mathscr{M})$ satisfying the followings:*

(a) $\alpha(f\theta) = f \circ \alpha(\theta)$ *(resp.* $\alpha(f\theta) = \alpha(\theta) \circ f$*);*

(b) $[\alpha(\theta), f] = \theta(f)$ *(resp.* $[\alpha(\theta), f] = -\theta(f)$*);*

(c) $[\alpha(\theta), \alpha(\theta')] = \alpha([\theta, \theta'])$ *(resp.* $[\alpha(\theta), \alpha(\theta')] = -\alpha([\theta, \theta'])$*).*

PROOF: Follows from the universal property of the universal algebra of a Lie-Rinehart algebra. □

We use the notation $\mathcal{M}^l(\mathscr{D})$ to denote the category of left $\mathscr{D}$-modules and $\mathcal{M}^r(\mathscr{D})$ the category of right $\mathscr{D}$-modules.

1.3.3 Corollary *The Lie derivatives induces a left $\mathscr{D}$-module structure on $\mathscr{O}$ and a right $\mathscr{D}$-module structure on ω. Moreover, they are simple modules.*

PROOF: To see $\mathscr{O}$ is simple, let $\mathscr{I}$ be any $\mathscr{D}$-submodule of $\mathscr{O}$. Suppose $\mathscr{I} \neq 0$ and consider the inclusion $\mathscr{I} \hookrightarrow \mathscr{O}$. At each point, by choosing a local coordinate, it becomes the inclusion $I \to A$ of D-modules, where A is the ring of germs of holomorphic functions at origin, D is the ring of differential operators of A. Then since for any nonzero holomorphic function f, there exists a multi-index λ such that $(\partial^\lambda f)$ doesn't vanish at origin hence is invertible in A, $D.I = A$ for any nonzero I. This shows $\mathscr{I} = \mathscr{O}$ as desired. Similar reasoning shows that ω is simple. □

Let $\mathscr{M}$ be an $\mathscr{O}$-module, a *connection* on $\mathscr{M}$ is a linear map $\nabla\colon \mathscr{M} \to \Omega^1 \otimes_{\mathscr{O}} \mathscr{M}$ satisfying the *Leibniz rule*:

$$\nabla(fm) = df \otimes m + f\nabla m, \qquad \forall f \in \mathscr{O}, m \in \mathscr{M}.$$

Note that once we have a connection ∇ on $\mathscr{M}$, we also have the following connections on $\Omega^\bullet \otimes_{\mathscr{O}} \mathscr{M}$:

$$\nabla\colon \Omega^p \otimes_{\mathscr{O}} \mathscr{M} \longrightarrow \Omega^{p+1} \otimes_{\mathscr{O}} \mathscr{M}$$

characterized by the *Leibniz rule*:

$$\nabla(\omega \otimes m) = d\omega \otimes m + (-1)^p \omega \wedge \nabla m, \qquad \forall \omega \in \Omega^p, m \in \mathscr{M}.$$

Note that, from the Leibniz rule, we immediately have

1.3.4 Proposition (Affine space of connections) *Let $\mathscr{M}$ be an $\mathscr{O}$-module, then the set of connections on $\mathscr{M}$ form an affine space with transition group* $\mathrm{Hom}_{\mathscr{O}}(\mathscr{M}, \Omega^1 \otimes_{\mathscr{O}} \mathscr{M})$*. In other words, any two connections are differed by an $\mathscr{O}$-linear map $\mathscr{M} \to \Omega^1 \otimes_{\mathscr{O}} \mathscr{M}$ and conversely any connection added with an $\mathscr{O}$-linear map $\mathscr{M} \to \Omega^1 \otimes_{\mathscr{O}} \mathscr{M}$ gives another connection. Consequently, the sheaf of connections on $\mathscr{M}$ is a $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \Omega^1 \otimes_{\mathscr{O}} \mathscr{M})$-torsor.*

Proof: Let ∇, ∇' be two connections on $\mathscr{M}$. Then for any $f \in \mathscr{O}$ and $m \in \mathscr{M}$, we have

$$\begin{aligned}(\nabla - \nabla')(fm) &= \nabla(fm) - \nabla'(fm)\\ &= (df \otimes m + f\nabla m) - (df \otimes m + f\nabla' m)\\ &= f\nabla m - f\nabla' m\\ &= f(\nabla - \nabla')(m).\end{aligned}$$

On the other hand, let ∇ be a connection on $\mathscr{M}$ and $\varphi\colon \mathscr{M} \to \Omega^1 \otimes_{\mathscr{O}} \mathscr{M}$ be an $\mathscr{O}$-linear map. Then for any $f \in \mathscr{O}$ and $m \in \mathscr{M}$, we have

$$\begin{aligned}(\nabla + \varphi)(fm) &= \nabla(fm) + \varphi(fm)\\ &= df \otimes m + f\nabla m + f\varphi(m)\\ &= df \otimes m + f(\nabla + \varphi)(m).\end{aligned}$$

Then the statements follows. $\square$

1.3.5 Proposition *Let $(\mathscr{M}, \nabla^{\mathscr{M}})$ and $(\mathscr{N}, \nabla^{\mathscr{N}})$ be two $\mathscr{O}$-modules with connections. Then*

(i) $\nabla^{\mathscr{M}} \oplus \nabla^{\mathscr{N}}$ *defines a connection on* $\mathscr{M} \oplus \mathscr{N}$*;*

(ii) $\nabla^{\mathscr{M}} \otimes 1 + 1 \otimes \nabla^{\mathscr{N}}$ *defines a connection on* $\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}$*;*

(iii) $\varphi \mapsto \nabla^{\mathscr{N}} \circ \varphi - \varphi \circ \nabla^{\mathscr{M}}$ *defines a connection on* $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$*.*

Proof: Let $\nabla^{\mathscr{M}\oplus\mathscr{N}}$ denote $\nabla^{\mathscr{M}} \oplus \nabla^{\mathscr{N}}$. For any $f \in \mathscr{O}$ and $m \in \mathscr{M}, n \in \mathscr{N}$, we have

$$\begin{aligned}\nabla^{\mathscr{M}\oplus\mathscr{N}}(f(m \oplus n)) &= \nabla^{\mathscr{M}}(fm) \oplus \nabla^{\mathscr{N}}(fn)\\ &= (df \otimes m + f\nabla^{\mathscr{M}} m) \oplus (df \otimes n + f\nabla^{\mathscr{N}} n)\\ &= df \otimes (m \oplus n) + f(\nabla^{\mathscr{M}} m \oplus \nabla^{\mathscr{N}} n)\\ &= df \otimes (m \oplus n) + f\nabla^{\mathscr{M}\oplus\mathscr{N}}(m \oplus n).\end{aligned}$$

Hence $\nabla^{\mathscr{M}\oplus\mathscr{N}}$ is a connection on $\mathscr{M} \oplus \mathscr{N}$.

Let $\nabla^{\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}}$ denote $\nabla^{\mathscr{M}} \otimes 1 + 1 \otimes \nabla^{\mathscr{N}}$. Note that this notation looks somehow ambiguous and meaningless. First, since $\nabla^{\mathscr{M}}$ and $\nabla^{\mathscr{N}}$ are not $\mathscr{O}$-linear, $\nabla^{\mathscr{M}} \otimes 1$ and $1 \otimes \nabla^{\mathscr{N}}$ are not well-defined on $\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}$. Instead, they are well-defined on $\mathscr{M} \otimes \mathscr{N}$. Second, $\nabla^{\mathscr{M}} \otimes 1$ lands on $\Omega^1 \otimes_{\mathscr{O}} \mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}$ while $1 \otimes \nabla^{\mathscr{N}}$ lands on $\mathscr{M} \otimes_{\mathscr{O}} \Omega^1 \otimes_{\mathscr{O}} \mathscr{N}$, so one need to apply the transport operation $\rho_{12}\colon \mathscr{M} \otimes_{\mathscr{O}} \Omega^1 \otimes_{\mathscr{O}} \mathscr{N} \to \Omega^1 \otimes_{\mathscr{O}} \mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}$ after $1 \otimes \nabla^{\mathscr{N}}$ and before we sum up it with $\nabla^{\mathscr{M}} \otimes 1$. So $\nabla^{\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}}$ is *a priori* a $\mathbb{C}$-linear map

$$\begin{aligned}\mathscr{M} \otimes \mathscr{N} &\longrightarrow \Omega^1 \otimes_{\mathscr{O}} \mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}\\ m \otimes n &\longmapsto \nabla^{\mathscr{M}} m \otimes n + \rho_{12}(m \otimes \nabla^{\mathscr{N}} n).\end{aligned}$$

Then to show it induces a well-defined map on $\mathscr{M} \otimes \mathscr{N}$, we need to show

$$\nabla^{\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}}(fm \otimes n) = \nabla^{\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}}(m \otimes fn)$$

for any $f \in \mathscr{O}$ and $m \in \mathscr{M}, n \in \mathscr{N}$. In fact, we have

$$\begin{aligned}
\nabla^{\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}}(fm \otimes n) &= \nabla^{\mathscr{M}}(fm) \otimes n + \rho_{12}(fm \otimes \nabla^{\mathscr{N}} n) \\
&= df \otimes m \otimes n + f\nabla^{\mathscr{M}} m \otimes n + f\rho_{12}(m \otimes \nabla^{\mathscr{N}} n) \\
&= df \otimes (m \otimes n) + f(\nabla^{\mathscr{M}} m \otimes n + \rho_{12}(m \otimes \nabla^{\mathscr{N}} n)) \\
&= df \otimes (m \otimes n) + f\nabla^{\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}}(m \otimes n), \\
\nabla^{\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}}(m \otimes fn) &= \nabla^{\mathscr{M}} m \otimes fn + \rho_{12}(m \otimes \nabla^{\mathscr{N}}(fn)) \\
&= f\nabla^{\mathscr{M}} m \otimes n + \rho_{12}(m \otimes (df \otimes n + f\nabla^{\mathscr{N}} n)) \\
&= f\nabla^{\mathscr{M}} m \otimes n + df \otimes m \otimes n + \rho_{12}(m \otimes f\nabla^{\mathscr{N}} n) \\
&= df \otimes (m \otimes n) + f\nabla^{\mathscr{M}} m \otimes n + f\rho_{12}(m \otimes \nabla^{\mathscr{N}} n) \\
&= df \otimes (m \otimes n) + f(\nabla^{\mathscr{M}} m \otimes n + \rho_{12}(m \otimes \nabla^{\mathscr{N}} n)) \\
&= df \otimes (m \otimes n) + f\nabla^{\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}}(m \otimes n).
\end{aligned}$$

Therefore $\nabla^{\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}}$ is a well-defined connection on $\mathscr{M} \otimes \mathscr{N}$.

Let $\nabla^{\mathscr{H}\!om_{\mathscr{O}}(\mathscr{M},\mathscr{N})}$ denote the map defined in (iii). Again, this notation looks somehow ambiguous and meaningless. Let φ be a section of $\mathscr{H}\!om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$. To simply notations, we omit restrictions on local sections and treat them as global sections. First, $\nabla^{\mathscr{M}}$ lands on $\Omega^1 \otimes_{\mathscr{O}} \mathscr{M}$. Therefore $\varphi \circ \nabla^{\mathscr{M}}$ is actually the composition

$$\mathscr{M} \xrightarrow{\nabla^{\mathscr{M}}} \Omega^1 \otimes_{\mathscr{O}} \mathscr{M} \xrightarrow{1\otimes\varphi} \Omega^1 \otimes_{\mathscr{O}} \mathscr{N}.$$

Then both $\varphi \circ \nabla^{\mathscr{M}}$ and $\nabla^{\mathscr{N}} \circ \varphi$ are from $\mathscr{M}$ to $\Omega^1 \otimes_{\mathscr{O}} \mathscr{N}$, hence are sections of $\mathscr{H}\!om(\mathscr{M}, \Omega^1 \otimes_{\mathscr{O}} \mathscr{N})$. Moreover, since $\nabla^{\mathscr{M}}$ and $\nabla^{\mathscr{N}}$ are not $\mathscr{O}$-linear, $\varphi \circ \nabla^{\mathscr{M}}$ and $\nabla^{\mathscr{N}} \circ \varphi$ are not $\mathscr{O}$-linear, hence not sections of $\mathscr{H}\!om_{\mathscr{O}}(\mathscr{M}, \Omega^1 \otimes_{\mathscr{O}} \mathscr{N})$. So $\nabla^{\mathscr{H}\!om_{\mathscr{O}}(\mathscr{M},\mathscr{N})}$ is *a priori* a $\mathbb{C}$-linear map

$$\begin{aligned}
\mathscr{H}\!om_{\mathscr{O}}(\mathscr{M}, \mathscr{N}) &\longrightarrow \mathscr{H}\!om(\mathscr{M}, \Omega^1 \otimes_{\mathscr{O}} \mathscr{N}) \\
\varphi &\longmapsto \nabla^{\mathscr{N}} \circ \varphi - \varphi \circ \nabla^{\mathscr{M}},
\end{aligned}$$

where the composition $\varphi \circ \nabla^{\mathscr{M}}$ should be viewed as $(1 \otimes \varphi) \circ \nabla^{\mathscr{M}}$. Then, we need to show

(a) $\nabla^{\mathscr{H}\!om_{\mathscr{O}}(\mathscr{M},\mathscr{N})}$ lands on $\mathscr{H}\!om_{\mathscr{O}}(\mathscr{M}, \Omega^1 \otimes_{\mathscr{O}} \mathscr{N})$;

(b) $\Omega^1 \otimes_{\mathscr{O}} \mathscr{H}\!om_{\mathscr{O}}(\mathscr{M}, \mathscr{N}) \cong \mathscr{H}\!om_{\mathscr{O}}(\mathscr{M}, \Omega^1 \otimes_{\mathscr{O}} \mathscr{N})$;

(c) $\nabla^{\mathscr{H}\!om_{\mathscr{O}}(\mathscr{M},\mathscr{N})}\colon \mathscr{H}\!om_{\mathscr{O}}(\mathscr{M}, \mathscr{N}) \to \Omega^1 \otimes_{\mathscr{O}} \mathscr{H}\!om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$ is a connection.

Indeed, for any $f \in \mathscr{O}$, $\varphi \in \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$ and $m \in \mathscr{M}$, we have

$$\begin{aligned}
(\nabla^{\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N})}\varphi)(fm) &= \nabla^{\mathscr{N}}(\varphi(fm)) - (1 \otimes \varphi)(\nabla^{\mathscr{M}}(fm)) \\
&= \nabla^{\mathscr{N}}(f\varphi(m)) - (1 \otimes \varphi)(df \otimes m + f\nabla^{\mathscr{M}} m) \\
&= df \otimes \varphi(m) + f\nabla^{\mathscr{N}}\varphi(m) \\
&\quad - df \otimes \varphi(m) - (1 \otimes \varphi)(f\nabla^{\mathscr{M}} m) \\
&= f\nabla^{\mathscr{N}}\varphi(m) - f(1 \otimes \varphi)(\nabla^{\mathscr{M}} m) \\
&= f(\nabla^{\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N})}\varphi)(m).
\end{aligned}$$

This shows (a). Note that we always have a morphism

$$\Omega^1 \otimes_{\mathscr{O}} \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N}) \longrightarrow \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \Omega^1 \otimes_{\mathscr{O}} \mathscr{N})$$

which maps each $\omega \otimes \varphi$ to the morphism $m \mapsto \omega \otimes \varphi(m)$. To see it is an isomorphism, we only need to verify at stalks. Under a local coordinate system (z^i), this morphism can be written as

$$\sum_i dz^i \otimes \varphi_i \longmapsto (m \mapsto \sum_i dz^i \otimes \varphi_i(m)),$$

which is clearly an isomorphism. This shows (b). Finally, for any $f \in \mathscr{O}$, $\varphi \in \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$ and $m \in \mathscr{M}$, we have

$$\begin{aligned}
(\nabla^{\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N})}f\varphi)(m) &= \nabla^{\mathscr{N}}(f\varphi(m)) - (1 \otimes f\varphi)(\nabla^{\mathscr{M}} m) \\
&= df \otimes \varphi(m) + f\nabla^{\mathscr{N}}\varphi(m) - f(1 \otimes \varphi)(\nabla^{\mathscr{M}} m) \\
&= df \otimes \varphi(m) + f(\nabla^{\mathscr{N}}\varphi(m) - (1 \otimes \varphi)(\nabla^{\mathscr{M}} m)) \\
&= df \otimes \varphi(m) + f\nabla^{\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N})} m.
\end{aligned}$$

This shows (c). Therefore $\nabla^{\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N})}$ is a connection on $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$. $\square$

The *curvature* $\mathcal{R}_\nabla$ of a connection ∇ is the composition

$$\mathcal{R}_\nabla : \mathscr{M} \xrightarrow{\nabla} \Omega^1 \otimes_{\mathscr{O}} \mathscr{M} \xrightarrow{\nabla} \Omega^2 \otimes_{\mathscr{O}} \mathscr{M}.$$

A connection ∇ is *flat/integrable* if $\mathcal{R}_\nabla = 0$. Note that, if this is the case, then we have a complex

$$0 \longrightarrow \mathscr{M} \xrightarrow{\nabla} \Omega^1 \otimes_{\mathscr{O}} \mathscr{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \omega \otimes_{\mathscr{O}} \mathscr{M} \longrightarrow 0, \tag{1.3.6}$$

called the *de Rham complex* of $\mathscr{M}$, denoted by $dR^\bullet(\mathscr{M})$.

Note that if $\mathscr{M}$ has a connection ∇, then it defines a left action of Θ on $\mathscr{M}$ satisfying (a) and (b) in Lemma 1.3.2 as follows: for θ a vector field on M, its action is given by the composition

$$\nabla_\theta : \mathscr{M} \xrightarrow{\nabla} \Omega^1 \otimes_{\mathscr{O}} \mathscr{M} \xrightarrow{\iota_\theta \otimes 1} \mathscr{M}. \tag{1.3.7}$$

Then, the condition (c) is equivalent to that ∇ is flat.

Proof: It is clear that condition (c) is equivalent to

$$\nabla_\theta \nabla_{\theta'} - \nabla_{\theta'} \nabla_\theta - \nabla_{[\theta,\theta']} = 0, \quad \forall \theta, \theta' \in \Theta.$$

The left hand side is precisely $\mathcal{R}_\nabla(\theta, \theta') := \iota_{\theta'}\iota_\theta \mathcal{R}_\nabla$. To see this, one can choose a local coordinate system (z^i) and we have

$$\begin{aligned} \nabla m &= \sum_i dz^i \otimes \nabla_{\partial_i} m, \\ \mathcal{R}_\nabla m &= \sum_{i,j} dz^i \wedge dz^j \otimes \nabla_{\partial_i} \nabla_{\partial_j} m. \end{aligned}$$

Therefore, for any vector fields $\theta = \sum_i f_i \partial_i$ and $\theta' = \sum_i g_i \partial_i$, we have

$$\begin{aligned} \nabla_\theta m &= \sum_i f_i \nabla_{\partial_i} m; \\ \mathcal{R}_\nabla m &= \sum_{i,j} (f_i g_j - g_i f_j) \otimes \nabla_{\partial_i} \nabla_{\partial_j} m. \end{aligned}$$

Since

$$\begin{aligned} [\theta, \theta'] &= \sum_{i,j} [f_i \partial_i, g_j \partial_j] \\ &= \sum_{i,j} f_i \partial_i g_j \partial_j - g_j \partial_j f_i \partial_i \\ &= \sum_{i,j} (f_i \partial_i (g_j) \partial_j + f_i g_j \partial_i \partial_j - g_j \partial_j (f_i) \partial_i + f_i g_j \partial_j \partial_i) \\ &= \sum_i (\sum_j f_j \partial_j (g_i) - g_j \partial_j (f_i)) \partial_i, \end{aligned}$$

we have

$$\nabla_{[\theta,\theta']} m = \sum_i (\sum_j f_j \partial_j (g_i) - g_j \partial_j (f_i)) \nabla_{\partial_i} m$$

Therefore,

$$\begin{aligned} \nabla_\theta \nabla_{\theta'} m &= \sum_i f_i \nabla_{\partial_i} \nabla_{\theta'} m \\ &= \sum_i f_i \nabla_{\partial_i} (\sum_j g_j \nabla_{\partial_j} m) \\ &= \sum_{i,j} f_i \nabla_{\partial_i} (g_j \nabla_{\partial_j} m) \\ &= \sum_{i,j} (f_i \partial_i (g_j) \nabla_{\partial_j} m + f_i g_j \nabla_{\partial_i} \nabla_{\partial_j} m). \end{aligned}$$

Similarly,
$$\nabla_{\theta'}\nabla_\theta m = \sum_{i,j}(g_i\partial_i(f_j)\nabla_{\partial_j}m + g_i f_j \nabla_{\partial_i}\nabla_{\partial_j}m)$$
Then it is clear now
$$\nabla_\theta\nabla_{\theta'} - \nabla_{\theta'}\nabla_\theta - \nabla_{[\theta,\theta']} = \mathcal{R}_\nabla(\theta,\theta').$$
Then the claim follows. □

If this is the case, we obtain a left $\mathscr{D}$-module structure on $\mathscr{M}$. Conversely, any left $\mathscr{D}$-module admits a flat connection. To see this, first consider $\mathscr{M} = \mathscr{D}$. Note that

1.3.8 Lemma *Let $\mathscr{E}$ be a locally free $\mathscr{O}$-module and $\mathscr{E}^\vee$ its dual $\mathscr{O}$-module. Let $(e_i)_{1\leqslant i\leqslant m}$ be a local basis and $(e_i^\vee)_{1\leqslant i\leqslant m}$ its dual basis. Then the section $\sum_{i=1}^m e_i \otimes e_i^\vee$ is independent on the choice of local basis, hence extends to a global section on $\mathscr{E}\otimes_{\mathscr{O}}\mathscr{E}^\vee$.*

In particular, $\sum_{i=1}^m dz^i \otimes \partial_i$ is such a global section on $\Omega^1 \otimes_{\mathscr{O}} \Theta$. Let $\nabla(1)$ be this section, then by the action (left multiplication) of Θ on $\mathscr{D}$, we have the following connection $\nabla\colon \mathscr{D} \to \Omega^1 \otimes \mathscr{D}$:

$$\nabla(P) = \nabla(1)P. \tag{1.3.9}$$

it is then straightforward to verify that it is a flat connection. This connection is called the *universal connection*. Let $\mathscr{M}$ be a left $\mathscr{D}$-module whose action denoted by $\alpha\colon \mathscr{D}\otimes_{\mathscr{O}}\mathscr{M} \to \mathscr{M}$. Then the corresponding flat connection is given by the composition

$$\mathscr{M} \longrightarrow \mathscr{D}\otimes_{\mathscr{O}}\mathscr{M} \xrightarrow{\nabla\otimes 1} \Omega^1 \otimes_{\mathscr{O}} \mathscr{D}\otimes_{\mathscr{O}}\mathscr{M} \xrightarrow{1\otimes\alpha} \Omega^1 \otimes_{\mathscr{O}} \mathscr{M}, \tag{1.3.10}$$

where the first map is $m \mapsto 1\otimes m$ and ∇ is the universal connection.

Conclusively, we have

1.3.11 Theorem *The category of left $\mathscr{D}$-modules is isomorphic to the category of $\mathscr{O}$-modules with flat connections.*

Proof: The equivalence is clear if we explain what a morphism between $\mathscr{O}$-modules with flat connections means. For $(\mathscr{M},\nabla^{\mathscr{M}})$ and $(\mathscr{N},\nabla^{\mathscr{N}})$ two $\mathscr{O}$-modules with connections, a morphism between them is an $\mathscr{O}$-module homomorphism $\varphi\colon \mathscr{M} \to \mathscr{N}$ making the following diagram commutative

$$\begin{tikzcd}
\mathscr{M} \arrow[r, "\varphi"] \arrow[d, "\nabla^{\mathscr{M}}"'] & \mathscr{N} \arrow[d, "\nabla^{\mathscr{N}}"] \\
\Omega^1\otimes_{\mathscr{O}}\mathscr{M} \arrow[r, "1\otimes\varphi"] & \Omega^1\otimes_{\mathscr{O}}\mathscr{N}
\end{tikzcd}$$

Then one can verify the equivalence. □

1.4 Basic operations of $\mathscr{D}$-modules

Let $\mathscr{M}$ and $\mathscr{N}$ be two $\mathscr{O}$-modules, we have $\mathscr{O}$-modules $\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}$ and $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$. The following propositions discuss what happens when $\mathscr{M}$ or $\mathscr{N}$ or both of them are $\mathscr{D}$-modules.

1.4.1 Proposition *Let $\mathscr{M}^l$ be a left $\mathscr{D}$-module, $\mathscr{M}^r$ a right $\mathscr{D}$-module and $\mathscr{N}$ a $\mathscr{O}$-module. Then*

(i) $\mathscr{M}^l \otimes_{\mathscr{O}} \mathscr{N}$ has a left $\mathscr{D}$-module structure given by

$$P.(m \otimes n) = (P.m) \otimes n, \qquad \forall P \in \mathscr{D}, m \in \mathscr{M}^l, n \in \mathscr{N};$$

(ii) $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}^l, \mathscr{N})$ has a left $\mathscr{D}$-module structure given by

$$(P.\varphi)(m) = \varphi(P.m), \qquad \forall P \in \mathscr{D}, \varphi \in \mathscr{H}om_{\mathscr{O}}(\mathscr{M}^l, \mathscr{N}), m \in \mathscr{M}^l;$$

(iii) $\mathscr{N} \otimes_{\mathscr{O}} \mathscr{M}^r$ has a right $\mathscr{D}$-module structure given by

$$(n \otimes m).P = n \otimes (m.P), \qquad \forall P \in \mathscr{D}, m \in \mathscr{M}^r, n \in \mathscr{N};$$

(iv) $\mathscr{H}om_{\mathscr{O}}(\mathscr{N}, \mathscr{M}^r)$ has a right $\mathscr{D}$-module structure given by

$$(\varphi.P)(n) = \varphi(n).P, \qquad \forall P \in \mathscr{D}, \varphi \in \mathscr{H}om_{\mathscr{O}}(\mathscr{N}, \mathscr{M}^r), n \in \mathscr{N};$$

Those $\mathscr{D}$-module structures are called *trivial* $\mathscr{D}$-module structures in the sense that it essentially has nothing to do with (possible existed) $\mathscr{D}$-module structures on the $\mathscr{N}$.

However, if both of $\mathscr{M}$ and $\mathscr{N}$ are $\mathscr{D}$-modules, there do exist nontrivial $\mathscr{D}$-module structures on $\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}$ and $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$.

1.4.2 Proposition *Let $\mathscr{M}^l$, $\mathscr{N}^l$ be two left $\mathscr{D}$-modules and $\mathscr{M}^r, \mathscr{N}^r$ two right $\mathscr{D}$-modules. Then*

(i) $\mathscr{M}^l \otimes_{\mathscr{O}} \mathscr{N}^l$ has a left $\mathscr{D}$-module structure induced by

$$\theta.(m \otimes n) = (\theta.m) \otimes n + m \otimes (\theta.n), \qquad \forall \theta \in \Theta, m \in \mathscr{M}^l, n \in \mathscr{N}^l;$$

(ii) $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}^l, \mathscr{N}^l)$ has a left $\mathscr{D}$-module structure induced by

$$(\theta.\varphi)(m) = \theta.(\varphi(m)) - \varphi(\theta.m), \qquad \forall \theta \in \Theta, \varphi \in \mathscr{H}om_{\mathscr{O}}(\mathscr{M}^l, \mathscr{N}^l), m \in \mathscr{M}^l;$$

(iii) $\mathscr{M}^l \otimes_{\mathscr{O}} \mathscr{N}^r$ has a right $\mathscr{D}$-module structure induced by

$$(m \otimes n).\theta = m \otimes (n.\theta) - (\theta.m) \otimes n, \qquad \forall \theta \in \Theta, m \in \mathscr{M}^l, n \in \mathscr{N}^r;$$

(iv) $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}^l, \mathscr{N}^r)$ has a right $\mathscr{D}$-module structure induced by

$$(\varphi.\theta)(m) = \varphi(m).\theta + \varphi(\theta.m), \qquad \forall \theta \in \Theta, \varphi \in \mathscr{H}om_{\mathscr{O}}(\mathscr{M}^l, \mathscr{N}^r), m \in \mathscr{M}^l;$$

(v) $\mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{N}^l$ *has a right* $\mathscr{D}$*-module structure induced by*

$$(m \otimes n).\theta = (m.\theta) \otimes n - m \otimes (\theta.n), \qquad \forall \theta \in \Theta, m \in \mathscr{M}^r, n \in \mathscr{N}^l;$$

(vi) $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}^r, \mathscr{N}^r)$ *has a left* $\mathscr{D}$*-module structure induced by*

$$(\theta.\varphi)(m) = \varphi(m.\theta) - \varphi(m).\theta, \qquad \forall \theta \in \Theta, \varphi \in \mathscr{H}om_{\mathscr{O}}(\mathscr{M}^r, \mathscr{N}^r), m \in \mathscr{M}^r;$$

Remark The above can be summarized into the following *Oda's rule* [Oda83]: left $= 0$, right $= 1$, $a \otimes_{\mathscr{O}} b = a + b$ and $\mathscr{H}om_{\mathscr{O}}(a, b) = -a + b$. Of cause one can always define trivial $\mathscr{D}$-module structures in any situation, but we keep using above setting whenever Oda's rule apply.

Proof: One can use Lemma 1.3.2 to verify above constructions. On the other hand, one can also use Proposition 1.3.5 to get (i) and (ii), and then apply the *left-right transformation* to get the others. □

Since $\mathscr{D}$ is non-commutative and $\mathbb{C}$ is its center, we have the following bifunctors:

$$\begin{aligned} \mathscr{H}om_{\mathscr{D}} &: \mathcal{M}^l(\mathscr{D})^{\mathrm{op}} \times \mathcal{M}^l(\mathscr{D}) \longrightarrow \mathcal{M}(\mathbb{C}); \\ \mathscr{H}om_{\mathscr{D}^{\mathrm{op}}} &: \mathcal{M}^r(\mathscr{D})^{\mathrm{op}} \times \mathcal{M}^r(\mathscr{D}) \longrightarrow \mathcal{M}(\mathbb{C}); \\ \otimes_{\mathscr{D}} &: \mathcal{M}^r(\mathscr{D}) \times \mathcal{M}^l(\mathscr{D}) \longrightarrow \mathcal{M}(\mathbb{C}). \end{aligned}$$

Note that the first two bifunctors give $\mathcal{M}^l(\mathscr{D})$ and $\mathcal{M}^r(\mathscr{D})$ structures of $\mathbb{C}_M$-*linear category* and the third gives a $\mathbb{C}_M$-*linear pairing* of them. Therefore, we have a $\mathbb{C}_M$-*linear category* together with a forgetful functor

$$\mathcal{M}^r(\mathscr{D}) \otimes_{\mathbb{C}_M} \mathcal{M}^l(\mathscr{D}) \xrightarrow{\otimes_{\mathscr{D}}} \mathcal{M}(\mathbb{C})$$

Note that $\mathcal{M}^l(\mathscr{D})$ (resp. $\mathcal{M}^r(\mathscr{D})$) can be embedded into it by sending each left $\mathscr{D}$-module $\mathscr{M}$ to $(\mathscr{D}, \mathscr{M})$ (resp. each right $\mathscr{D}$-module $\mathscr{M}$ to $(\mathscr{M}, \mathscr{D})$).

Then, the following proposition describe the adjunctive relation of $\otimes_{\mathscr{O}}$ and $\mathscr{H}om_{\mathscr{O}}$ in those categories.

1.4.3 Proposition *Let* $\mathscr{M}^l, \mathscr{N}^l, \mathscr{P}^l$ *be three left* $\mathscr{D}$*-modules and* $\mathscr{M}^r, \mathscr{N}^r, \mathscr{P}^r$ *three right* $\mathscr{D}$*-modules. We have canonical isomorphisms*

$$\begin{aligned} (\mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{N}^l) \otimes_{\mathscr{D}} \mathscr{P}^l &\cong \mathscr{M}^r \otimes_{\mathscr{D}} (\mathscr{N}^l \otimes_{\mathscr{O}} \mathscr{P}^l), \\ \mathscr{H}om_{\mathscr{D}}(\mathscr{M}^l \otimes_{\mathscr{O}} \mathscr{N}^l, \mathscr{P}^l) &\cong \mathscr{H}om_{\mathscr{D}}(\mathscr{M}^l, \mathscr{H}om_{\mathscr{O}}(\mathscr{N}^l, \mathscr{P}^l)), \\ \mathscr{H}om_{\mathscr{D}}(\mathscr{M}^l \otimes_{\mathscr{O}} \mathscr{N}^r, \mathscr{P}^r) &\cong \mathscr{H}om_{\mathscr{D}}(\mathscr{M}^l, \mathscr{H}om_{\mathscr{O}}(\mathscr{N}^r, \mathscr{P}^r)), \\ \mathscr{H}om_{\mathscr{D}}(\mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{N}^l, \mathscr{P}^r) &\cong \mathscr{H}om_{\mathscr{D}}(\mathscr{M}^r, \mathscr{H}om_{\mathscr{O}}(\mathscr{N}^l, \mathscr{P}^r)). \end{aligned}$$

Proof: It is clear that, by viewing $\mathscr{M}^r, \mathscr{N}^l, \mathscr{P}^l$ as $\mathscr{O}$-modules, we have

$$(\mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{N}^l) \otimes_{\mathscr{O}} \mathscr{P}^l \cong \mathscr{M}^r \otimes_{\mathscr{O}} (\mathscr{N}^l \otimes_{\mathscr{O}} \mathscr{P}^l),$$

To show it induces an isomorphism

$$(\mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{N}^l) \otimes_{\mathscr{D}} \mathscr{P}^l \cong \mathscr{M}^r \otimes_{\mathscr{D}} (\mathscr{N}^l \otimes_{\mathscr{O}} \mathscr{P}^l),$$

it suffices to verify the following diagram is commutative

$$\begin{tikzcd}
(\mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{N}^l) \otimes_{\mathscr{O}} \mathscr{D} \otimes_{\mathscr{O}} \mathscr{P}^l \arrow[r, "\cong"] \arrow[d] & \mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{D} \otimes_{\mathscr{O}} (\mathscr{N}^l \otimes_{\mathscr{O}} \mathscr{P}^l) \arrow[d] \\
(\mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{N}^l) \otimes_{\mathscr{O}} \mathscr{P}^l \arrow[r, "\cong"] & \mathscr{M}^r \otimes_{\mathscr{O}} (\mathscr{N}^l \otimes_{\mathscr{O}} \mathscr{P}^l)
\end{tikzcd}$$

where the vertical arrows denote the difference of actions of $\mathscr{D}$ on modules on its left hand side and right hand side. Moreover, by Lemma 1.3.2, it suffices to consider only the actions of Θ. For any $m \in \mathscr{M}, n \in \mathscr{N}, p \in \mathscr{P}$ and $\theta \in \Theta$, through the left vertical morphism, we have

$$\begin{aligned}
(m \otimes n) \otimes \theta \otimes p \mapsto & ((m \otimes n).\theta) \otimes p - (m \otimes n) \otimes (\theta.p) \\
&= ((m.\theta) \otimes n - m \otimes (\theta.n)) \otimes p - (m \otimes n) \otimes (\theta.p) \\
&= (m.\theta) \otimes n \otimes p - m \otimes (\theta.n) \otimes p - m \otimes n \otimes (\theta.p);
\end{aligned}$$

through the right vertical morphism, we have

$$\begin{aligned}
m \otimes \theta \otimes (n \otimes p) \mapsto & (m.\theta) \otimes (n \otimes p) - m \otimes (\theta.(n \otimes p)) \\
&= (m.\theta) \otimes (n \otimes p) - m \otimes ((\theta.n) \otimes p + n \otimes (\theta.p)) \\
&= (m.\theta) \otimes (n \otimes p) - m \otimes (\theta.n) \otimes p - m \otimes n \otimes (\theta.p).
\end{aligned}$$

This shows the commutativity of the diagram and the desired isomorphism then follows.

Recall that for any three $\mathscr{O}$-modules $\mathscr{M}, \mathscr{N}, \mathscr{P}$, we have the canonical isomorphism

$$\begin{aligned}
\mathscr{H}om_{\mathscr{O}}(\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}, \mathscr{P}) & \xrightarrow{\sim} \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{H}om_{\mathscr{O}}(\mathscr{N}, \mathscr{P})) \\
\varphi & \longmapsto \tilde{\varphi} = (m \mapsto \varphi(m \otimes))
\end{aligned}$$

where $\varphi(m\otimes) \colon \mathscr{N} \to \mathscr{P}$ is the morphism maps each $n \in \mathscr{N}$ to $\varphi(m \otimes n)$. Then, to show it induces the last three isomorphisms, it suffices to verify it maps $\mathscr{D}$-linear morphisms to $\mathscr{D}$-linear morphisms in each case. Note that, by Lemma 1.3.2, it suffices to verify it commutes with the action of Θ.

If $\mathscr{M}, \mathscr{N}, \mathscr{P}$ are all left $\mathscr{D}$-modules, then for any $\theta \in \Theta$, $m \in \mathscr{M}$, $n \in \mathscr{N}$ and $\varphi \in \mathscr{H}om_{\mathscr{D}}(\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}, \mathscr{P})$, we have

$$\begin{aligned}
(\theta.\tilde{\varphi}(m))(n) &= \theta.(\tilde{\varphi}(m)(n)) - \tilde{\varphi}(m)(\theta.n) \\
&= \theta.(\varphi(m \otimes n)) - \varphi(m \otimes (\theta.n)) \\
&= \varphi(\theta.(m \otimes n)) - \varphi(m \otimes (\theta.n)) \\
&= \varphi((\theta.m) \otimes n) = \tilde{\varphi}(\theta.m)(n).
\end{aligned}$$

Hence $\tilde{\varphi} \in \mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{H}om_{\mathscr{O}}(\mathscr{N}, \mathscr{P}))$.

If $\mathscr{M}$ is a left $\mathscr{D}$-module and $\mathscr{N}, \mathscr{P}$ are right $\mathscr{D}$-modules, then for any $\theta \in \Theta$, $m \in \mathscr{M}$, $n \in \mathscr{N}$ and $\varphi \in \mathscr{H}om_{\mathscr{D}}(\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}, \mathscr{P})$, we have

$$\begin{aligned}(\theta.\tilde{\varphi}(m))(n) &= \tilde{\varphi}(m)(n.\theta) - (\tilde{\varphi}(m)(n)).\theta \\ &= \varphi(m \otimes (n.\theta)) - (\varphi(m \otimes n)).\theta \\ &= \varphi(m \otimes (n.\theta)) - \varphi((m \otimes n).\theta) \\ &= \varphi((\theta.m) \otimes n) = \tilde{\varphi}(\theta.m)(n).\end{aligned}$$

Hence $\tilde{\varphi} \in \mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{H}om_{\mathscr{O}}(\mathscr{N}, \mathscr{P}))$.

If $\mathscr{M}, \mathscr{P}$ are right $\mathscr{D}$-modules and $\mathscr{N}$ is a left $\mathscr{D}$-module, then for any $\theta \in \Theta$, $m \in \mathscr{M}$, $n \in \mathscr{N}$ and $\varphi \in \mathscr{H}om_{\mathscr{D}}(\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}, \mathscr{P})$, we have

$$\begin{aligned}(\tilde{\varphi}(m).\theta)(n) &= \tilde{\varphi}(m)(\theta.n) + (\tilde{\varphi}(m)(n)).\theta \\ &= \varphi(m \otimes (\theta.n)) + (\varphi(m \otimes n)).\theta \\ &= \varphi(m \otimes (\theta.n)) + \varphi((m \otimes n).\theta) \\ &= \varphi((m.\theta) \otimes n) = \tilde{\varphi}(m.\theta)(n).\end{aligned}$$

Hence $\tilde{\varphi} \in \mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{H}om_{\mathscr{O}}(\mathscr{N}, \mathscr{P}))$. □

Note that $\mathscr{D}$ itself is both a left and right $\mathscr{D}$-module. The following two propositions describe the $\mathscr{D}$-bimodule structures on tensor product with $\mathscr{D}$.

1.4.4 Proposition *Let $\mathscr{M}$ be a left $\mathscr{D}$-module.*

(i) $\mathscr{M} \otimes \mathscr{D}$ has a $\mathscr{D}$-bimodule structure:

- *the left $\mathscr{D}$-module structure is given by viewing $\mathscr{D}$ as a left $\mathscr{D}$-module and apply Proposition 1.4.2.(i);*
- *the right $\mathscr{D}$-module structure is the trivial one given by viewing $\mathscr{D}$ as a right $\mathscr{D}$-module.*

(ii) $\mathscr{D} \otimes \mathscr{M}$ has a $\mathscr{D}$-bimodule structure:

- *the left $\mathscr{D}$-module structure is the trivial one given by viewing $\mathscr{D}$ as a left $\mathscr{D}$-module;*
- *the right $\mathscr{D}$-module structure is given by viewing $\mathscr{D}$ as a right $\mathscr{D}$-module and apply Proposition 1.4.2.(v).*

(iii) The following two natural morphisms are $\mathscr{D}$-bilinear and inverse to each other:

$$\begin{aligned}\mathscr{M} \otimes \mathscr{D} &\longrightarrow \mathscr{D} \otimes \mathscr{M} \\ m \otimes P &\longmapsto (1 \otimes m).P\end{aligned} \qquad \begin{aligned}\mathscr{D} \otimes \mathscr{M} &\longrightarrow \mathscr{M} \otimes \mathscr{D} \\ P \otimes m &\longmapsto P.(m \otimes 1)\end{aligned}$$

(iv) Let $\mathscr{N}$ be an $\mathscr{O}$-module, then we have canonical isomorphisms (keep eyes on the $\mathscr{D}$-module structures on them) of left $\mathscr{D}$-modules

$$\mathscr{M}\otimes_{\mathscr{O}}(\mathscr{D}\otimes_{\mathscr{O}}\mathscr{N}) \cong (\mathscr{M}\otimes_{\mathscr{O}}\mathscr{D})\otimes_{\mathscr{O}}\mathscr{N} \cong (\mathscr{D}\otimes_{\mathscr{O}}\mathscr{M})\otimes_{\mathscr{O}}\mathscr{N} \cong \mathscr{D}\otimes_{\mathscr{O}}(\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}).$$

Moreover, if $\mathscr{M}$ and $\mathscr{N}$ are locally free $\mathscr{O}$-modules, then the above is a locally free $\mathscr{D}$-module.

Proof: Note that, by Lemma 1.3.2, to verify a $\mathscr{D}$-bimodule structure, it suffices to verify the left and right action of Θ are compatible.

For any $m \in \mathscr{M}$, $P \in \mathscr{D}$ and $\theta, \theta' \in \Theta$, we have

$$\begin{aligned}(\theta.(m \otimes P)).\theta' &= ((\theta.m) \otimes P + m \otimes (\theta P)).\theta' \\ &= (\theta.m) \otimes (P\theta') + m \otimes (\theta P \theta') \\ &= \theta.(m \otimes (P\theta')) \\ &= \theta.((m \otimes P).\theta').\end{aligned}$$

This shows (i). Similarly, we have

$$\begin{aligned}(\theta.(P \otimes m)).\theta' &= ((\theta P) \otimes m).\theta' \\ &= (\theta P \theta') \otimes m - (\theta P) \otimes (\theta'.m) \\ &= \theta.((P\theta') \otimes m - P \otimes (\theta'.m)) \\ &= \theta.((P \otimes m).\theta').\end{aligned}$$

This shows (ii).

Let $\mathcal{L}$ and $\mathcal{R}$ denote the two morphisms in (iii), we have

$$\begin{aligned}\mathcal{L}(\theta.(m \otimes P)) &= \mathcal{L}((\theta.m) \otimes P + m \otimes (\theta P)) \\ &= (1 \otimes (\theta.m)).P + (1 \otimes m).(\theta P) \\ &= (1 \otimes (\theta.m) + (1 \otimes m).\theta).P \\ &= (\theta \otimes m).P \\ &= \theta.(1 \otimes m).P \\ &= \theta.\mathcal{L}(m \otimes P); \\ \mathcal{L}((m \otimes P).\theta) &= \mathcal{L}(m \otimes (P\theta)) \\ &= (1 \otimes m).P\theta \\ &= \mathcal{L}(m \otimes P).\theta.\end{aligned}$$

Therefore $\mathcal{L}$ is $\mathscr{D}$-bilinear. Similarly, so is $\mathcal{R}$. We also have

$$\begin{aligned}\mathcal{R}\mathcal{L}(m \otimes P) = \mathcal{R}((1 \otimes m).P) = \mathcal{R}(1 \otimes m).P = (m \otimes 1).P = m \otimes P, \\ \mathcal{L}\mathcal{R}(P \otimes m) = \mathcal{L}(P.(m \otimes 1)) = P.\mathcal{L}(m \otimes 1) = P.(1 \otimes m) = P \otimes m.\end{aligned}$$

This shows (iii).

Finally, (iv) can be shown by verify those isomorphisms of $\mathscr{O}$-modules commute with the action of Θ. □

Since Oda's rule doesn't give a canonical nontrivial $\mathscr{D}$-module structure on the tensor product of two right $\mathscr{D}$-modules, the tensor product of a right $\mathscr{D}$-module with $\mathscr{D}$ is not expected to be a $\mathscr{D}$-bimodule. However, we still have another story about it.

1.4.5 Proposition *Let $\mathscr{M}$ be a right $\mathscr{D}$-module.*

(i) $\mathscr{M} \otimes \mathscr{D}$ has two right $\mathscr{D}$-module structures:

- *one is the trivial one by viewing $\mathscr{D}$ as a right $\mathscr{D}$-module;*
- *another one is given by viewing $\mathscr{D}$ as a left $\mathscr{D}$-module and apply Proposition 1.4.2.(v).*

(To distinguish them, we always use $(m \otimes P).Q$ to denote the right action of Q on $m \otimes P$ by the nontrivial module structure, and $(m \otimes P)Q$ for the trivial one).

(ii) The morphism

$$\begin{aligned} \iota \colon \mathscr{M} \otimes \mathscr{D} &\longrightarrow \mathscr{M} \otimes \mathscr{D} \\ m \otimes P &\longmapsto (m \otimes 1).P \end{aligned}$$

is an involution, exchanges the two right $\mathscr{D}$-module structures and is the identity on the submodule $\mathscr{M} \otimes 1$. Moreover it is the unique one satisfying those properties.

(iii) Recall that there are two $\mathscr{O}$-module structures (called left *and* right*) on $\mathscr{D}$, hence on each $F_p\mathscr{D}$, and on the tensor products $\mathscr{M} \otimes_{\mathscr{O}} F_p\mathscr{D}$. Let $(\mathscr{M} \otimes_{\mathscr{O}} F_p\mathscr{D})_l$ (resp. $(\mathscr{M} \otimes_{\mathscr{O}} F_p\mathscr{D})_r$) denote the $\mathscr{O}$-module whose module structure comes from the left (resp. right) $\mathscr{O}$-module structure on $\mathscr{D}$. On the other hand, the two right $\mathscr{D}$-module structures induce two $\mathscr{O}$-module structures on each $\mathscr{M} \otimes_{\mathscr{O}} F_p\mathscr{D}$: the one from the trivial right $\mathscr{D}$-module structure coincide with $(\mathscr{M} \otimes_{\mathscr{O}} F_p\mathscr{D})_r$, while the one from the nontrivial right $\mathscr{D}$-module structure coincide with $(\mathscr{M} \otimes_{\mathscr{O}} F_p\mathscr{D})_l$. Moreover, ι induces isomorphisms of $\mathscr{O}$-modules between $(\mathscr{M} \otimes_{\mathscr{O}} F_p\mathscr{D})_l$ and $(\mathscr{M} \otimes_{\mathscr{O}} F_p\mathscr{D})_r$.*

Proof: For any $m \in \mathscr{M}$, $P, Q \in \mathscr{D}$, we have

$$\iota((m \otimes P)Q) = \iota(m \otimes PQ) = (m \otimes 1).PQ = \iota(m \otimes P).Q.$$

On the other hand, for any $\theta \in \Theta$, we have

$$\begin{aligned} (m \otimes P).\theta &= (m.\theta) \otimes P - m \otimes (\theta P) \\ &= ((m.\theta) \otimes 1)P - (m \otimes \theta)P \\ &= ((m \otimes 1).\theta)P = \iota(m \otimes \theta)P \end{aligned}$$

Since $\mathscr{D}$ is generated by $\mathscr{O}$ and Θ, the above implies

$$(m \otimes P).Q = \iota(m \otimes Q)P.$$

Then we have

$$\begin{aligned}\iota((m \otimes P).\theta) &= \iota((m.\theta) \otimes P - m \otimes (\theta P)) \\ &= ((m.\theta) \otimes 1).P - (m \otimes 1).\theta P \\ &= (m \otimes \theta).P = \iota(m \otimes P)\theta.\end{aligned}$$

This shows ι exchanges the two right $\mathscr{D}$-module structures. It is clear that ι is the identity on the submodule $\mathscr{M} \otimes 1$. Then, we have

$$\iota\iota(m \otimes P) = \iota\iota((m \otimes 1)P) = \iota(\iota(m \otimes 1).P) = \iota\iota(m \otimes 1)P = m \otimes P.$$

This shows ι is an involution. The uniqueness is clear.

(iii) is clear since by Proposition 1.1.2, for any $f \in \mathscr{O}$ and $P \in \mathscr{D}$, fP and Pf are in the same filtrations. $\square$

1.5 Left-right transformation

Let $\mathscr{M}$ be a left $\mathscr{D}$-module, we have seen (Proposition 1.4.2.(v)) that $\omega \otimes_{\mathscr{O}} \mathscr{M}$ is a right $\mathscr{D}$-module. Conversely, let $\mathscr{N}$ be a right $\mathscr{D}$-module, $\mathscr{H}om_{\mathscr{O}}(\omega, \mathscr{N})$ is a left $\mathscr{D}$-module. In this way, we get two functors, one from $\mathcal{M}^l(\mathscr{D})$ to $\mathcal{M}^r(\mathscr{D})$ and another goes conversely. It turns out that they are inverse to each other.

1.5.1 Theorem *The functors $\omega \otimes_{\mathscr{O}} -$ and $\mathscr{H}om_{\mathscr{O}}(\omega, -)$ form a pair of adjoint equivalences between $\mathcal{M}^l(\mathscr{D})$ and $\mathcal{M}^r(\mathscr{D})$.*

To prove the theorem, first notice that ω is an invertible $\mathscr{O}$-module. Indeed, the canonical evaluation morphism

$$\mathscr{H}om_{\mathscr{O}}(\omega, \mathscr{O}) \otimes_{\mathscr{O}} \omega \longrightarrow \mathscr{O}$$

is an isomorphism. To see this, one only needs to choose a local coordinate system (z^i), and then the morphism is bijective since it admits an inverse

$$f \longmapsto \varphi \otimes (f dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m)$$

where φ maps $f dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m$ to f. For this reason, we use ω^{-1} to denote its inverse sheaf $\mathscr{H}om_{\mathscr{O}}(\omega, \mathscr{O})$.

In general, we have

1.5.2 Lemma *Let $\mathscr{L}$ be an invertible $\mathscr{O}$-module (with inverse sheaf $\mathscr{L}^{-1}$) and $\mathscr{R}$ an $\mathscr{O}$-algebra. Then $\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}$ is also an $\mathscr{O}$-algebra. Moreover, $\mathscr{L} \otimes_{\mathscr{O}} -$ defines an equivalence of categories from the category $\mathcal{M}(\mathscr{R})$ of left $\mathscr{R}$-modules to the category $\mathcal{M}(\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1})$ of left $\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}$-modules.*

Proof: The product is given by the composition

$$(\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}) \otimes_{\mathscr{O}} (\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}) \xrightarrow{\langle,\rangle} \mathscr{L} \otimes_{\mathscr{O}} (\mathscr{R} \otimes_{\mathscr{O}} \mathscr{R}) \otimes_{\mathscr{O}} \mathscr{L}^{-1} \longrightarrow \mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}$$

where the first comes from the pairing $\mathscr{L}^{-1} \otimes_{\mathscr{O}} \mathscr{L} \to \mathscr{O}$ and the second comes from the product of $\mathscr{R}$.

Let $\mathscr{M}$ be a left $\mathscr{R}$-module, then the left $\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}$-module structure on $\mathscr{L} \otimes_{\mathscr{O}} \mathscr{M}$ is given by

$$(\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}) \otimes_{\mathscr{O}} (\mathscr{L} \otimes_{\mathscr{O}} \mathscr{M}) \xrightarrow{\langle,\rangle} \mathscr{L} \otimes_{\mathscr{O}} (\mathscr{R} \otimes_{\mathscr{O}} \mathscr{M}) \longrightarrow \mathscr{L} \otimes_{\mathscr{O}} \mathscr{M}$$

where the first comes from the pairing and the second comes from the left $\mathscr{R}$-module structure on $\mathscr{M}$. Then it is clear $\mathscr{L} \otimes_{\mathscr{O}} -$ defines a functor from $\mathcal{M}(\mathscr{R})$ to $\mathcal{M}(\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1})$.

Conversely, let $\mathscr{N}$ be a left $\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}$-module, then we define the left $\mathscr{R}$-module structure on $\mathscr{L}^{-1} \otimes_{\mathscr{O}} \mathscr{N}$ is given by

$$\mathscr{R} \otimes_{\mathscr{O}} (\mathscr{L}^{-1} \otimes_{\mathscr{O}} \mathscr{N}) \longrightarrow (\mathscr{L}^{-1} \otimes_{\mathscr{O}} \mathscr{L}) \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1} \otimes_{\mathscr{O}} \mathscr{N} \longrightarrow \mathscr{L}^{-1} \otimes_{\mathscr{O}} \mathscr{N}$$

where the first comes from the inverse of the pairing and the second comes from the left $\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}$-module structure on $\mathscr{N}$. Therefore $\mathscr{L}^{-1} \otimes_{\mathscr{O}} -$ defines a functor from $\mathcal{M}(\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1})$ to $\mathcal{M}(\mathscr{R})$.

Then it is easy to verify those functors are inverse to each other hence equivalence of categories. □

1.5.3 Lemma *Let $\mathscr{L}$ be an invertible $\mathscr{O}$-module (with inverse sheaf $\mathscr{L}^{-1}$). Then $\mathscr{L} \otimes_{\mathscr{O}} -$ and $\mathscr{L}^{-1} \otimes_{\mathscr{O}} -$ are adjoint to each other.*

Proof: In fact, they are inverse to each other, *a fortiori* adjoint. □

1.5.4 Lemma *Let $\mathscr{L}$ be an invertible $\mathscr{O}$-module (with inverse sheaf $\mathscr{L}^{-1}$). Then we have natural isomorphisms*

$$\begin{aligned} \mathscr{L}^{-1} \otimes_{\mathscr{O}} - &\cong \mathscr{H}om_{\mathscr{O}}(\mathscr{L}, -), \\ \mathscr{L} \otimes_{\mathscr{O}} - &\cong \mathscr{H}om_{\mathscr{O}}(\mathscr{L}^{-1}, -). \end{aligned}$$

Proof: Because the adjoint functor of a given functor is unique up to a unique natural isomorphism. □

In our case, we additionally have

1.5.5 Lemma *There is an isomorphism of $\mathscr{O}$-algebras*

$$\mathscr{D}^{\mathrm{op}} \longrightarrow \omega \otimes_{\mathscr{O}} \mathscr{D} \otimes_{\mathscr{O}} \omega^{-1}.$$

Proof: First, the right $\mathscr{D}$-module structure on ω defines a homomorphism

$$\mathscr{D}^{\mathrm{op}} \longrightarrow \mathscr{E}nd(\omega).$$

On the other hand, the homomorphism

$$\omega \otimes_{\mathscr{O}} \mathscr{D} \otimes_{\mathscr{O}} \omega^{-1} \longrightarrow \omega \otimes_{\mathscr{O}} \mathscr{E}nd(\mathscr{O}) \otimes_{\mathscr{O}} \omega^{-1} \longrightarrow \mathscr{E}nd(\omega).$$

where the first comes from the embedding $\mathscr{D} \to \mathscr{E}nd(\mathscr{O})$ and the second comes from the composition

$$\mathscr{H}om(\mathscr{O}, \omega) \otimes \mathscr{H}om(\mathscr{O}, \mathscr{O}) \otimes \mathscr{H}om(\omega, \mathscr{O}) \longrightarrow \mathscr{H}om(\omega, \omega).$$

Then it is not difficult to verify the above two homomorphisms are into $\mathscr{E}nd(\omega)$. Then, to show there is a desired isomorphism, it suffices to show their image in $\mathscr{E}nd(\omega)$ coincide. But this is clear since both of $\mathscr{D}^{\mathrm{op}}$ and $\omega \otimes_{\mathscr{O}} \mathscr{D} \otimes_{\mathscr{O}} \omega^{-1}$ are generated by the images of $\mathscr{O}$ and Θ in them. □

Now, combining the previous lemmas, we proves Theorem 1.5.1.

From now on, for a left $\mathscr{D}$-module $\mathscr{M}$, we will use $\mathscr{M}^r$ to denote the right $\mathscr{D}$-module $\omega \otimes_{\mathscr{O}} \mathscr{M}$; for a right $\mathscr{D}$-module $\mathscr{N}$, we will use $\mathscr{N}^l$ to denote the right $\mathscr{D}$-module $\mathscr{H}om_{\mathscr{O}}(\omega, \mathscr{N})$. We call such corresponding *left-right transformation*.

On a local coordinate chart, one can choose a volume form dV. Then we locally have an isomorphism

$$\omega \xrightarrow{\cong} \mathscr{O}: \quad f dV \longmapsto f. \tag{1.5.6}$$

Then the *left-right transformation* has a local expression as follows.

1.5.7 Proposition *Let (z^i) be a local coordinate system. Fix a volume form*

$$dV = A dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m,$$

then after the identification $\omega \otimes_{\mathscr{O}} \mathscr{D} \otimes_{\mathscr{O}} \omega^{-1} \cong \mathscr{D}$, the isomorphism in Lemma 1.5.5 can be expressed as

$$(P = \sum_{\lambda} f_\lambda \partial^\lambda) \longmapsto P^\sim = \sum_{\lambda} \frac{1}{A}(-\partial)^\lambda \circ (f_\lambda A).$$

In particularly, for $\mathscr{M}$ a left $\mathscr{D}$-module (resp. right $\mathscr{D}$-module), the left action (resp. right action) of P on $\mathscr{M}$ is the same as the right action (resp. left action) of $P^\sim$ on $\omega \otimes_{\mathscr{O}} \mathscr{M}$ (resp. $\mathscr{H}om_{\mathscr{O}}(\omega, \mathscr{M})$) being identified with $\mathscr{M}$ through (1.5.6).

PROOF: First note that, the right $\mathscr{D}$-module structure on ω corresponds each differential operator

$$P = \sum_\lambda f_\lambda \partial^\lambda$$

to the endomorphism

$$gdz^1 \wedge dz^2 \wedge \cdots \wedge dz^m \longmapsto \sum_\lambda (-\mathcal{L}_\partial)^\lambda (f_\lambda g dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m).$$

Then through the identification (1.5.6), the above endomorphism becomes the composition

$$\begin{aligned} g \longmapsto & gdV = gAdz^1 \wedge dz^2 \wedge \cdots \wedge dz^m \\ \longmapsto & \sum_\lambda (-\mathcal{L}_\partial)^\lambda (f_\lambda gAdz^1 \wedge dz^2 \wedge \cdots \wedge dz^m) \\ & = \sum_\lambda (-\partial)^\lambda (f_\lambda gA) dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m \\ & = \left(\sum_\lambda \frac{1}{A} (-\partial)^\lambda (f_\lambda gA) \right) dV \\ \longmapsto & \sum_\lambda \frac{1}{A} (-\partial)^\lambda (f_\lambda gA). \end{aligned}$$

This shows the desired expression. $\square$

1.5.8 Proposition *Let $\mathscr{M}^l, \mathscr{N}^l$ be two left $\mathscr{D}$-modules and $\mathscr{M}^r, \mathscr{N}^r$ their corresponding right $\mathscr{D}$-modules. Then there is a canonical isomorphism of right $\mathscr{D}$-modules.*

$$\mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{N}^l \cong \mathscr{N}^r \otimes_{\mathscr{O}} \mathscr{M}^l.$$

PROOF: This isomorphism is given by

$$(\omega \otimes m) \otimes n \longmapsto (\omega \otimes n) \otimes m.$$

To show it is a homomorphism of right $\mathscr{D}$-modules, it suffices to show it commutes with the right action of Θ. Indeed, for any $\theta \in \Theta$, we have

$$\begin{aligned} ((\omega \otimes m) \otimes n).\theta &= (\omega \otimes m).\theta \otimes n - (\omega \otimes m) \otimes (\theta.n) \\ &= (\omega.\theta \otimes m) \otimes n - (\omega \otimes (\theta.m)) \otimes n - (\omega \otimes m) \otimes (\theta.n) \\ &\mapsto (\omega.\theta \otimes n) \otimes m - (\omega \otimes n) \otimes (\theta.m) - (\omega \otimes (\theta.n)) \otimes m \\ &= (\omega \otimes n).\theta \otimes m - (\omega \otimes n) \otimes (\theta.m) \\ &= ((\omega \otimes n) \otimes m).\theta. \end{aligned}$$

Hence the claim follows. $\square$

1.6 ¶The sheaf of principal parts

In this subsection, we deal with the notion of sheaves of principal parts. This notion has been well-studied in [TCGA] and [Kan77].

Let M be a complex manifold. Then we have the following canonical morphisms:

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\mathrm{pr}_i} M$$

where M is the diagonal morphism and pr_i is the projection to i-th factor. Let $\mathscr{I}_\Delta$ be the kernel of the canonical homomorphism $\Delta^\sharp \colon \Delta^{-1}(\mathscr{O}_{M\times M}) \to \mathscr{O}_M$. Let $M_\Delta^{(p)}$ be the locally ringed space whose underlying topological space is M and whose structure sheaf is $\Delta^{-1}(\mathscr{O}_{M\times M})/\mathscr{I}_\Delta^{p+1}$. Then we have an inductive system of locally ringed spaces

$$M = M_\Delta^{(0)} \longrightarrow M_\Delta^{(1)} \longrightarrow M_\Delta^{(2)} \longrightarrow \cdots$$

over $M \times M$ such that $\Delta \colon M \to M \times M$ factors through each structure morphism $\Delta^{(p)} \colon M_\Delta^{(p)} \to M \times M$. Therefore, $M_\Delta^{(p)}$ is called the *p-th infinitesimal neighborhood* of M with respect to Δ.

Note that we have morphisms

$$M_\Delta^{(p)} \xrightarrow{\Delta^{(p)}} M \times M \xrightarrow{\mathrm{pr}_i} M$$

where $(\Delta^{(p)})^\sharp$ is the quotient homomorphism. Let pr_i^* be the homomorphism $(\mathrm{pr}_i \circ \Delta^{(p)})^\sharp \colon \mathscr{O}_M \to \mathscr{O}_{M_\Delta^{(p)}}$. Then each pr_i^* gives $\mathscr{O}_{M_\Delta^{(p)}}$ an *augmented $\mathscr{O}_M$-algebra* structure. Then, the *sheaf of principal parts* of M is sheaf $\mathscr{O}_{M_\Delta^{(p)}}$ equipped with the $\mathscr{O}_M$-algebra structure from pr_1^*, denoted by $\mathscr{P}_M^{(p)}$. From now on, we will identify $\mathscr{O}_M$ with its image in $\mathscr{P}_M^{(p)}$. On the other hand, $d^{(p)} := \mathrm{pr}_2^* \colon \mathscr{O}_M \to \mathscr{P}_M^{(p)}$ is called the *universal differential operator of order p*. For any section f of $\mathscr{O}_M$, the section $df = d^{(1)}(f) - f$ of $\mathscr{P}_M^{(1)}$ is called the *(holomorphic) differential* of f.

1.6.1 Lemma *As $\mathscr{O}_M$-algebras, $\mathscr{P}_M^{(p)} \cong \mathscr{O}_M \oplus \mathscr{I}_\Delta/\mathscr{I}_\Delta^{p+1}$.*

Proof: This follows from the fact that the augmented $\mathscr{O}_M$-algebra structure on $\mathscr{P}_M^{(p)}$ makes the following exact sequence split:

$$0 \longrightarrow \mathscr{I}_\Delta/\mathscr{I}_\Delta^{p+1} \longrightarrow \mathscr{P}_M^{(p)} \longrightarrow \mathscr{O}_M \longrightarrow 0 \qquad \square$$

In particular, we can see that for any section f of $\mathscr{O}_M$, the section df belongs to $\mathscr{I}_\Delta/\mathscr{I}_\Delta^2$. In this sense, we call $\mathscr{I}_\Delta/\mathscr{I}_\Delta^2$ the *sheaf of (holomorphic) differentials*. Later, we sill show that it is isomorphic to the sheaf of holomorphic 1-forms.

Note that the ideal $\mathscr{J}_\Delta$ gives $\Delta^{-1}(\mathscr{O}_{M\times M})$ a $\mathscr{J}_\Delta$-adic filtration, hence we have the associated graded algebra

$$\mathscr{G}r_\bullet(\mathscr{P}_M) := \bigoplus_{p\geqslant 0} \mathscr{J}_\Delta^p/\mathscr{J}_\Delta^{p+1}. \tag{1.6.2}$$

Since $\mathscr{G}r_0(\mathscr{P}_M) = \Delta^{-1}(\mathscr{O}_{M\times M})/\mathscr{J}_\Delta \cong \mathscr{O}_M$, we see that $\mathscr{G}r_\bullet(\mathscr{P}_M)$ is a graded $\mathscr{O}_M$-algebra. Moreover, this $\mathscr{O}_M$-algebra structure coincides with those from pr_1^* and pr_2^*. Then the $\mathscr{O}_M$-linear multiplication of $\mathscr{G}r_\bullet(\mathscr{P}_M)$ induces a surjective homomorphism of graded $\mathscr{O}_M$-algebras

$$\mathbb{S}^\bullet_{\mathscr{O}_M}(\mathscr{G}r_1(\mathscr{P}_M)) \longrightarrow \mathscr{G}r_\bullet(\mathscr{P}_M).$$

We will show

1.6.3 Theorem *Each $\mathscr{P}_M^{(p)}$ is a locally free $\mathscr{O}_M$-module of finite rank.*

PROOF: It suffices to show that $\mathscr{P}_{M,x}^{(p)}$ is a free $\mathscr{O}_{M,x}$-module of finite rank for any point $x \in M$. By choosing a local coordinate (z,w) of $M\times M$ we reduce to the case where x is the origin $(0,0) \in \mathbb{C}^m \times \mathbb{C}^m$.

Let $f(z,w)$ be a germ of holomorphic functions at $(0,0) \in \mathbb{C}^m \times \mathbb{C}^m$. Then since we have invertible holomorphic linear transformation $(z,w) \mapsto (z, w-z)$, it can be uniquely written as

$$f(z,w) = \sum_{\lambda\in\mathbb{N}^m} f_\lambda(z)(w-z)^\lambda,$$

where $f_\lambda(z)$ are germs of holomorphic functions at $0 \in \mathbb{C}^m$. In this way, we obtain an injective homomorphism of $\mathscr{O}_{M,x}$-algebras

$$\mathscr{O}_{M\times M,(x,x)} \longrightarrow \mathscr{O}_{M,x}[[w-z]],$$

where $\mathscr{O}_{M,x}[[w-z]]$ denotes the formal power series ring over $\mathscr{O}_{M,x}$. We identify $\mathscr{O}_{M\times M,(x,x)}$ with its image. Note that $\Delta_x^\sharp$ maps $f(z,w)$ to $f(z,z)$. Then we have

$$\mathscr{J}_{\Delta,x} = \{f(z,w); f_0(z) = 0\} \subset \left\{ \sum_{|\lambda|\geqslant 1} f_\lambda(z)(w-z)^\lambda \right\}.$$

Consequently, we have

$$\mathscr{J}_{\Delta,x}^p = \{f(z,w); f_\lambda(z) = 0, \forall|\lambda| < p\} \subset \left\{ \sum_{|\lambda|\geqslant p} f_\lambda(z)(w-z)^\lambda \right\}.$$

Therefore

$$\mathscr{P}_{M,x}^{(p)} \cong \{f(z,w); f_\lambda(z) = 0, \forall|\lambda| > p\} = \left\{ \sum_{|\lambda|\leqslant p} f_\lambda(z)(w-z)^\lambda \right\},$$

which is a free $\mathscr{O}_{M,x}$-module with the finite basis

$$\{(w-z)^\lambda; |\lambda| \leqslant p\}. \qquad \square$$

From the above proof, it is clear that

1.6.4 Corollary *The homomorphism* (1.6.2) *is an isomorphism.*

1.6.5 Corollary *$\mathscr{G}r_1(\mathscr{P}_M)$ is isomorphic to the sheaf of holomorphic* 1*-forms.*

PROOF: From previous reasoning, we see that being restricted to a coordinate chart, $\mathscr{G}r_1(\mathscr{P}_M)$ has a basis $\{w^i - z^i; 1 \leqslant i \leqslant m\}$. Then, the map $w^i - z^i \mapsto dz^i$ gives an isomorphism to Ω^1_M which is compatible with the transition maps. Hence the conclusion. $\square$

Note that, after identify $\mathscr{G}r_1(\mathscr{P}_M)$ with Ω^1_M, the morphism $d\colon \mathscr{O}_M \to \Omega^1_M$ has the following explicit expression in local coordinate:

$$f \longmapsto \sum_{i=1}^m \frac{\partial f}{\partial z^i} dz^i.$$

One can see it coincides with the usual differential.

1.6.6 Theorem *For each p, we have*

$$\mathscr{H}om_{\mathscr{O}_M}(\mathscr{P}^{(p)}_M, \mathscr{O}_M) \cong \mathscr{D}iff^p_M.$$

PROOF: We may assume we are working on a coordinate chart. Then $d^{(p)}\colon \mathscr{O}_M \to \mathscr{P}^{(p)}_M$ maps each f to

$$d^{(p)}(f) = \sum_{|\lambda| \leqslant p} \frac{\partial^\lambda (f)(z)}{\lambda!}(w-z)^\lambda.$$

Then it is straightforward to verify that $d^{(p)}$ is a differential operator of order p from $\mathscr{O}_M$ to $\mathscr{P}^{(p)}_M$. Consequently, for each homomorphism of $\mathscr{O}_M$-modules $P\colon \mathscr{P}^{(p)}_M \to \mathscr{O}_M$, the composition $P \circ d^{(p)}$ is a differential operator of order p on $\mathscr{O}_M$. This gives the desired homomorphism.

It remains to show it is bijective. To show this, we construct its inverse as follows: for any P a differential operator of order p on $\mathscr{O}_M$, let $\widetilde{P}\colon \mathscr{P}^{(p)}_M \to \mathscr{O}_M$ be defined by

$$\widetilde{P}(w^\lambda) = P(z^\lambda)$$

(note that $\{w^\lambda; |\lambda| \leqslant p\}$ is also a basis of $\mathscr{P}_M^{(p)}$). Then, we have

$$\begin{aligned}
\widetilde{P}(d^{(p)}(f)) &= \widetilde{P}(\sum_{|\lambda|\leqslant p} \frac{\partial^\lambda(f)(z)}{\lambda!}(w-z)^\lambda) \\
&= \sum_{|\lambda|\leqslant p} \frac{\partial^\lambda(f)(z)}{\lambda!} \sum_{\mu\leqslant\lambda} \binom{\lambda}{\mu}(-z)^{\lambda-\mu}\widetilde{P}(w^\mu) \\
&= \sum_{|\lambda|\leqslant p} \frac{\partial^\lambda(f)(z)}{\lambda!} \sum_{\mu\leqslant\lambda} \binom{\lambda}{\mu}(-z)^{\lambda-\mu}P(z^\mu) \\
&= \sum_{|\lambda|\leqslant p} \frac{\partial^\lambda(f)(z)}{\lambda!}[P, z^\lambda](1).
\end{aligned}$$

If we know that $\mathscr{D}\!iff_M$ is generated by $\mathscr{D}\!iff^1_M$, then we only need to verify the last line equals $P(f)$ for $P = \partial^\mu$ where $|\mu| = p$. Hence the conclusion.□

The previous theorem can also be obtained from the following one.

1.6.7 Theorem *For each p, we have*

$$\mathscr{H}\!om_{\mathscr{O}_M}(\mathscr{D}\!iff^p_M, \mathscr{O}_M) \cong \mathscr{P}_M^{(p)}.$$

PROOF: We may assume we are working on a coordinate chart. For any $\varphi\colon \mathscr{D}\!iff^p_M \to \mathscr{O}_M$, we define the corresponding section in $\mathscr{P}_M^{(p)}$ as

$$\sum_{|\lambda|\leqslant p} \varphi(\partial^\lambda)(w-z)^\lambda.$$

Then, the bijectivity follows from the fact that $\{\partial^\lambda; |\lambda| \leqslant p\}$ form a basis of the free $\mathscr{O}_M$-module $\mathscr{D}\!iff^p_M$. □

Remark Note that the product $M \times M$ is took in the category of complex manifolds. If one takes products in the category of locally ringed spaces, then one will have

$$\Delta^{-1}(\mathscr{O}_{M\times M}) = \mathscr{O}_M \otimes_{\mathbb{C}} \mathscr{O}_M$$

and the result sheaves, denoted by $\mathscr{P}^{(p)}_{M/\mathbb{C}}$ instead of $\mathscr{P}^{(p)}_M$, is not locally free. However, one can see that

$$\mathscr{H}\!om_{\mathscr{O}_M}(\mathscr{P}^{(p)}_{M/\mathbb{C}}, \mathscr{O}_M) \cong \mathscr{D}\!iff^p_M$$

by the recursive definition. Therefore $\mathscr{P}^{(p)}_{M/\mathbb{C}}$ and $\mathscr{P}^{(p)}_M$ have the same dual $\mathscr{O}_M$-modules, hence the later is the double dual of the first.

§ 2 Coherent $\mathscr{D}$-modules

2.1 Coherence

Let $\mathscr{R}$ be a sheaf of rings on a space X and $\mathscr{M}$ a (left) $\mathscr{R}$-module. Then, we say

- $\mathscr{M}$ is *locally free of finite rank* if for any point x, there exists an open neighborhood U on which there is an isomorphism

$$\mathscr{M}_{|U} \cong \mathscr{R}^s_{|U},$$

  where s is a finite number;

- $\mathscr{M}$ is *(locally) of finite type* if for any point x, there exists an open neighborhood U on which there is an exact sequence

$$\mathscr{R}^s_{|U} \longrightarrow \mathscr{M}_{|U} \longrightarrow 0,$$

  where s is a finite number;

- $\mathscr{M}$ is *(locally) of finite presentation* if for any point x, there exists an open neighborhood U on which there is an exact sequence

$$\mathscr{R}^r_{|U} \longrightarrow \mathscr{R}^s_{|U} \longrightarrow \mathscr{M}_{|U} \longrightarrow 0,$$

  where r, s are finite numbers;

- $\mathscr{M}$ is *pseudo-coherent* if for any open set U, any $\mathscr{R}_{|U}$-submodule of $\mathscr{M}_{|U}$ which is of finite type is of finite presentation;

- $\mathscr{M}$ is *coherent* if it is of finite type and pseudo-coherent;

- $\mathscr{R}$ is *coherent* if $\mathscr{R}$ is coherent as an $\mathscr{R}$-module;

- $\mathscr{M}$ is *noetherian* if it is coherent, each stalk is noetherian and for any open set U, any ascending chain of coherent $\mathscr{R}_{|U}$-submodules of $\mathscr{M}$ is locally stationary;

- $\mathscr{R}$ is *noetherian* if $\mathscr{R}$ is noetherian as an $\mathscr{R}$-module.

Remark All of those notions are *local*: to show a module satisfies certain property, one only need to verify it on small opens.

Remark This notion of noetherian is from [KS02] and [KS03]. One can see it is weaker than what one expects. In particular, it doesn't imply that any ideal of a noetherian ring is of finite type. Indeed, every stalk is of finite type doesn't imply the module itself is of finite type. See [Tag 065J] for an example of a module not of finite type while its stalks are of finite type.

2.1.1 Proposition *Let $\mathscr{M}$ be an $\mathscr{R}$-module. The following are equivalent:*

(i) *$\mathscr{M}$ is locally of finite presentation;*

(ii) *$\mathscr{M}$ is locally of finite type and for any open set U and any epimorphism $\varphi\colon \mathscr{N} \to \mathscr{M}_{|U}$ on U, if $\mathscr{N}$ is locally free of finite rank, then $\ker(\varphi)$ is locally of finite type;*

(iii) *$\mathscr{M}$ is locally of finite type and for any open set U and any epimorphism $\varphi\colon \mathscr{N} \to \mathscr{M}_{|U}$ on U, if $\mathscr{N}$ is locally of finite type, then $\ker(\varphi)$ is locally of finite type.*

Proof: Suppose (i). Since the problem is local we may assume $U = X$ and by shrinking U, we may assume we have an exact sequence

$$\mathscr{R}^r \longrightarrow \mathscr{R}^s \xrightarrow{\psi} \mathscr{M} \longrightarrow 0,$$

and an isomorphism $\mathscr{N} \cong \mathscr{R}^t$ where r, s, t are finite numbers. Note that a morphism $\varphi\colon \mathscr{N} = \mathscr{R}^t \to \mathscr{M}$ corresponds to t sections $e_1, \cdots, e_t$ of $\mathscr{M}$ on U. Since we have epimorphism $\psi\colon \mathscr{R}^s \to \mathscr{M}$, By shrinking U finitely many times, we can find t sections of $\mathscr{R}^s$ which are mapped to $e_1, \cdots, e_t$. In this way, we have a morphism $\alpha\colon \mathscr{R}^t \to \mathscr{R}^s$ making the following diagram commute.

$$\begin{tikzcd} & \mathscr{R}^t \arrow[dl, dashed, "\alpha"'] \arrow[d, "\varphi"] & \\ \mathscr{R}^s \arrow[r, "\psi"] & \mathscr{M} \arrow[r] & 0 \end{tikzcd}$$

Similarly, we have another morphism $\beta\colon \mathscr{R}^s \to \mathscr{R}^t$ making the following diagram commute.

$$\begin{tikzcd} & \mathscr{R}^s \arrow[dl, dashed, "\beta"'] \arrow[d, "\psi"] & \\ \mathscr{R}^t \arrow[r, "\varphi"] & \mathscr{M} \arrow[r] & 0 \end{tikzcd}$$

Note that the morphism $\alpha\colon \mathscr{R}^t \to \mathscr{R}^s$ and the morphism $\mathscr{R}^r \to \mathscr{R}^s$ gives rise to a unique morphism $\phi\colon \mathscr{R}^{r+t} \to \mathscr{R}^s$ (by the universal property of direct sum) and the following morphism of exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{R}^r \arrow[r] \arrow[d] & \mathscr{R}^{r+t} \arrow[r, "\rho"] \arrow[d, dashed, "\phi"'] & \mathscr{R}^t \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & \ker(\psi) \arrow[r] & \mathscr{R}^s \arrow[r, "\psi"] & \mathscr{M} \arrow[r] & 0 \end{tikzcd}$$

Then we have the following morphism of exact sequences, which turns out to be surjective by *snake lemma*.

$$\begin{tikzcd} 0 \arrow[r] & \ker(\phi) \arrow[r] \arrow[d] & \mathscr{R}^{r+t} \arrow[r, "\phi"] \arrow[d, "\rho"'] & \mathscr{R}^s \arrow[r] \arrow[d, "\psi"] & 0 \\ 0 \arrow[r] & \ker(\varphi) \arrow[r] & \mathscr{R}^t \arrow[r, "\varphi"] & \mathscr{M} \arrow[r] & 0 \end{tikzcd}$$

Note that
$$\varphi \circ \rho = \psi \circ \phi = \varphi \circ \beta \circ \phi,$$
therefore the morphism $\gamma := \rho - \beta \circ \phi$ lands on $\ker(\varphi)$. Then it is cleat that (γ, β) form a homotopy between this epimorphism and the zero morphism:

$$\begin{tikzcd}
0 \arrow[r] & \ker(\phi) \arrow[r] \arrow[d] & \mathscr{R}^{r+t} \arrow[r, "\phi"] \arrow[d, "\rho"'] \arrow[dl, dashed, "\gamma"'] & \mathscr{R}^s \arrow[r] \arrow[d, "\psi"] \arrow[dl, "\beta"'] & 0 \\
0 \arrow[r] & \ker(\varphi) \arrow[r] & \mathscr{R}^t \arrow[r, "\varphi"] & \mathscr{M} \arrow[r] & 0
\end{tikzcd}$$

Therefore γ is an epimorphism and $\ker(\varphi)$ is of finite type. This proves (ii).

Suppose (ii). Since the problem is local we may assume $U = X$ and by shrinking U, we may assume we have an epimorphism $\psi\colon \mathscr{R}^s \to \mathscr{N}$ with s a finite number. Then we have the following morphism of exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \ker(\varphi \circ \psi) \arrow[r] \arrow[d] & \mathscr{R}^s \arrow[r] \arrow[d, "\psi"'] & \mathscr{M} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker(\varphi) \arrow[r] & \mathscr{N} \arrow[r, "\varphi"] & \mathscr{M} \arrow[r] & 0
\end{tikzcd}$$

By *snake lemma*, we see that the morphism $\ker(\varphi \circ \psi) \to \ker(\varphi)$ is surjective. By our assumption, $\ker(\varphi \circ \psi)$ is locally of finite type, hence so is $\ker(\varphi)$. This proves (iii).

Finally, (iii) implies (i) is clear. □

2.1.2 Proposition *Let $\mathscr{M}$ be an $\mathscr{R}$-module. The following are equivalent:*

(i) $\mathscr{M}$ is pseudo-coherent;

(ii) for any open set U and any morphism $\varphi\colon \mathscr{N} \to \mathscr{M}_{|U}$ on U, if $\mathscr{N}$ is locally free of finite rank, then $\ker(\varphi)$ *is locally of finite type;*

(iii) for any open set U and any morphism $\varphi\colon \mathscr{N} \to \mathscr{M}_{|U}$ on U, if $\mathscr{N}$ is locally of finite type, then $\ker(\varphi)$ *is locally of finite type.*

Proof: Suppose (i). Since the problem is local we may assume $U = X$ and by shrinking U, we may assume we have $\mathscr{N} = \mathscr{R}^s$ with s a finite number. Then the submodule $\mathrm{im}(\varphi)$ of $\mathscr{M}$ is of finite type. By our assumption, it is locally of finite presentation, hence $\ker(\varphi) = \ker(\mathscr{N} \to \mathrm{im}(\varphi))$ is locally of finite type by Proposition 2.1.1.(ii). This proves (i).

Suppose (ii). Since the problem is local we may assume $U = X$ and by shrinking U, we may assume we have an epimorphism $\psi\colon \mathscr{R}^s \to \mathscr{N}$ with s a finite number. Then we have the following morphism of exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \ker(\varphi \circ \psi) \arrow[r] \arrow[d] & \mathscr{R}^s \arrow[r] \arrow[d, "\psi"'] & \mathscr{M} \arrow[d, equal] \\
0 \arrow[r] & \ker(\varphi) \arrow[r] & \mathscr{N} \arrow[r, "\varphi"] & \mathscr{M}
\end{tikzcd}$$

Since both $\mathrm{im}(\varphi) = \mathrm{im}(\varphi \circ \psi)$, by *snake lemma*, we see that the morphism $\ker(\varphi \circ \psi) \to \ker(\varphi)$ is surjective. By our assumption, $\ker(\varphi \circ \psi)$ is locally of finite type, hence so is $\ker(\varphi)$. This proves (iii).

Finally, (iii) implies (i) is clear. $\square$

2.1.3 Proposition *Let $\mathscr{M}, \mathscr{N}$ be two $\mathscr{R}$-modules. For any point x, consider the canonical morphism*

$$\Phi_x \colon (\mathscr{H}om_{\mathscr{R}}(\mathscr{M}, \mathscr{N}))_x \longrightarrow \mathrm{Hom}_{\mathscr{R}_x}(\mathscr{M}_x, \mathscr{N}_x).$$

If $\mathscr{M}$ is locally of finite type (resp. locally of finite presentation), then Φ_x is injective (resp. bijective).

Proof: Since the problem is local we may assume we have an epimorphism $\psi \colon \mathscr{R}^s \to \mathscr{M}$ with s a finite number. Then there are sections $e_1, \cdots, e_s$ of $\mathscr{M}$ corresponding to ψ. Let $\varphi \colon \mathscr{M} \to \mathscr{N}$ represents a germ in $(\mathscr{H}om_{\mathscr{R}}(\mathscr{M}, \mathscr{N}))_x$ which is mapped to 0 under Φ_x. Then, by shrinking U finitely many times, we may assume $\varphi(e_i) = 0$ for $i = 1, \cdots, s$. It follows that $\varphi = 0$ hence so is its germ in $(\mathscr{H}om_{\mathscr{R}}(\mathscr{M}, \mathscr{N}))_x$. This proves the injectivity of Φ_x.

Suppose furthermore $\ker(\psi)$ is locally of finite type. By shrinking U, we may assume it is generated by sections $f_1, \cdots, f_r$. Then $\mathscr{M}$ is generated by $e_1, \cdots, e_s$ subject to relations $f_1(e_1, \cdots, e_s) = \cdots = f_r(e_1, \cdots, e_s) = 0$. Let $\varphi_x \colon \mathscr{M}_x \to \mathscr{N}_x$ be any homomorphism of $\mathscr{R}_x$-modules. Let n_i be a section of $\mathscr{N}$ such that its germ at x is $\varphi_x(e_{i,x})$, the image of germ of e_i at x under the homomorphism φ_x. Then, since we have

$$f_1(e_{1,x}, \cdots, e_{s,x}) = \cdots = f_r(e_{1,x}, \cdots, e_{s,x}) = 0$$

and hence

$$f_1(\varphi_x(e_{1,x}), \cdots, \varphi_x(e_{s,x})) = \cdots = f_r(\varphi_x(e_{1,x}), \cdots, \varphi_x(e_{s,x})) = 0,$$

by shrinking U finitely many times, we may assume $n_1, \cdots, n_s$ satisfy

$$f_1(n_1, \cdots, n_s) = \cdots = f_r(n_1, \cdots, n_s) = 0,$$

and hence $e_i \mapsto n_i$ defines a morphism $\varphi \colon \mathscr{M} \to \mathscr{N}$ whose germ at x is φ_x. This proves the bijectivity of Φ_x. $\square$

2.1.4 Corollary *Let $\mathscr{M}$ be an $\mathscr{R}$-module which is locally of finite type. If $\mathscr{M}_x = 0$, then there exists an open neighborhood U of x such that $\mathscr{M}_{|U} = 0$.*

Proof: Consider the germ of morphism $\mathrm{id} \colon \mathscr{M} \to \mathscr{M}$ at x, noting that $\mathscr{M}_x = 0$ and apply Proposition 2.1.3. $\square$

2.1.5 Corollary *Let $\mathscr{M}$ be an $\mathscr{R}$-module which is locally of finite presentation. If $\mathscr{M}_x \cong \mathscr{R}_x^s$ for some finite number s, then there exists an open neighborhood U of x such that $\mathscr{M}_{|U} \cong \mathscr{R}_{|U}^s$.*

PROOF: Apply Proposition 2.1.3 to the isomorphism $\mathscr{M}_x \cong \mathscr{R}_x^s$. $\square$

2.1.6 Corollary *Let $\mathscr{M}$ be a coherent $\mathscr{R}$-module, $\mathscr{N}$ a finite type $\mathscr{R}$-module and $\varphi\colon \mathscr{N} \to \mathscr{M}$ a morphism. If $\varphi_x\colon \mathscr{N}_x \to \mathscr{M}_x$ is injective, then there exists an open neighborhood U of x such that $\varphi_{|U}\colon \mathscr{N}_{|U} \to \mathscr{M}_{|U}$ is injective.*

PROOF: It follows that $\ker(\varphi)$ is of finite type. Then the result follows from Corollary 2.1.4. $\square$

We use $\mathcal{M}_{coh}(\mathscr{R})$ to denote the full subcategory of $\mathcal{M}(\mathscr{R})$ consisting of all coherent $\mathscr{R}$-modules. It is then natural to ask if it is an abelian subcategory. It turns out that it is a *weak Serre subcategory.*

2.1.7 Proposition *Let $\mathscr{R}$ be a sheaf of rings.*

(i) Any finite type submodule of a coherent $\mathscr{R}$-module is coherent.
(ii) Let $\varphi\colon \mathscr{M} \to \mathscr{N}$ be a morphism of coherent $\mathscr{R}$-modules. Then $\ker(\varphi)$ *and* $\operatorname{coker}(\varphi)$ *are coherent.*
(iii) Given a short exact sequence of $\mathscr{R}$-modules $0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$ if two out of three are coherent so is the third.
(iv) The category $\mathcal{M}_{coh}(\mathscr{R})$ is a weak Serre subcategory of $\mathcal{M}(\mathscr{R})$. In particular, $\mathcal{M}_{coh}(\mathscr{R})$ is abelian and the inclusion functor $\mathcal{M}_{coh}(\mathscr{R}) \to \mathcal{M}(\mathscr{R})$ is exact.

PROOF: (i) is clear. Let $\varphi\colon \mathscr{M} \to \mathscr{N}$ be a morphism of coherent $\mathscr{R}$-modules. Then, by Proposition 2.1.2.(iii), $\ker(\varphi)$ is of finite type. It is a submodule of the coherent $\mathscr{R}$-module $\mathscr{M}$, hence is coherent by (i).

Let's prove $\operatorname{coker}(\varphi)$ is coherent. Since it is a quotient of the finite type $\mathscr{R}$-module $\mathscr{N}$, it is of finite type. Let U be an open set and $\psi\colon \mathscr{R}^s_{|U} \to \operatorname{coker}(\varphi)_{|U}$ a morphism with s a finite number. By Proposition 2.1.2.(ii), it suffices to show $\ker(\psi)$ is of finite type. Since the problem is local, we may assume $U = X$. Since $\mathscr{N} \to \operatorname{coker}(\varphi)$ is surjective, after shrinking U finitely many times, we have a morphism $\alpha\colon \mathscr{R}^s \to \mathscr{N}$ making the following diagram commute.

$$\begin{tikzcd} & \mathscr{R}^s \arrow[dl, dashed, "\alpha"'] \arrow[d, "\psi"] & \\ \mathscr{N} \arrow[r] & \operatorname{coker}(\varphi) \arrow[r] & 0 \end{tikzcd}$$

The kernel of $\mathscr{N} \to \operatorname{coker}(\varphi)$, that is $\operatorname{im}(\varphi)$, is of finite type, since it is a quotient of the finite type $\mathscr{R}$-module $\mathscr{M}$. So by shrinking U, we may assume there is an epimorphism $\mathscr{R}^r \to \operatorname{im}(\varphi)$. Then we have a morphism of exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{R}^r \arrow[r] \arrow[d] & \mathscr{R}^{r+s} \arrow[r] \arrow[d, "\bar{\psi}"'] & \mathscr{R}^s \arrow[r] \arrow[d, "\psi"] & 0 \\ 0 \arrow[r] & \operatorname{im}(\varphi) \arrow[r] & \mathscr{N} \arrow[r] & \operatorname{coker}(\varphi) \arrow[r] & 0 \end{tikzcd}$$

Then, by *snake lemma*, we get an exact sequence

$$\ker(\bar{\psi}) \longrightarrow \ker(\psi) \longrightarrow 0.$$

Since $\mathscr{N}$ is coherent, $\ker(\bar{\psi})$ is of finite type. Then it follows that $\ker(\psi)$ is also of finite type.

Being proved (ii), to show (iii), it suffices to show that for any short exact sequence of $\mathscr{R}$-modules $0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$ with $\mathscr{M}_1$ and $\mathscr{M}_3$ coherent, so is $\mathscr{M}_2$. First, $\mathscr{M}_2$ is of finite type since both $\mathscr{M}_1$ and $\mathscr{M}_3$ are of finite type. Then, let U be any open set and $\psi\colon \mathscr{R}^s_{|U} \to \mathscr{M}_{2|U}$ a morphism with s a finite number. By Proposition 2.1.2.(ii), it suffices to show $\ker(\psi)$ is of finite type. Consider the following morphism of exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \ker(\varphi \circ \psi) \arrow[r] \arrow[d, "\bar{\psi}"'] & \mathscr{R}^s \arrow[r] \arrow[d, "\psi"] & \mathscr{M}_3 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{M}_1 \arrow[r] & \mathscr{M}_2 \arrow[r, "\varphi"] & \mathscr{M}_3 \arrow[r] & 0
\end{tikzcd}$$

Then, by *snake lemma*, we get an exact sequence

$$\ker(\bar{\psi}) \longrightarrow \ker(\psi) \longrightarrow 0.$$

Since $\mathscr{M}_3$ is coherent, $\ker(\varphi \circ \psi)$ is of finite type. Then, since $\mathscr{M}_1$ is coherent, $\ker(\bar{\psi})$ is of finite type. Then it follows that $\ker(\psi)$ is also of finite type.

Finally, (iv) follows from (ii) and (iii). $\square$

Remark Let $\mathcal{A}$ be an abelian category. A *weak Serre subcategory* is a nonempty full subcategory $\mathcal{C}$ such that given an exact sequence

$$\mathscr{M}_1 \longrightarrow \mathscr{M}_2 \longrightarrow \mathscr{M}_3 \longrightarrow \mathscr{M}_4 \longrightarrow \mathscr{M}_5,$$

in $\mathcal{A}$, with all but $\mathscr{M}_3$ belong to $\mathcal{C}$, then so does $\mathscr{M}_3$. This condition is equivalent to the follows:

(a) $0 \in \mathcal{C}$;

(b) $\mathcal{C}$ is a *strictly full* subcategory of $\mathcal{A}$, that means $\mathcal{C}$ is a subcategory and is closed under isomorphisms;

(c) if φ is a morphism between objects of $\mathcal{C}$, then its kernel and cokernel in $\mathcal{A}$ are in $\mathcal{C}$;

(d) extensions of objects of $\mathcal{C}$ are also in $\mathcal{C}$, that means if there is an exact sequence $0 \to A \to B \to C \to 0$ with A and C are in $\mathcal{C}$, then so is B.

Since all the properties, *locally free*, *of finite type*, *of finite presentation*, *coherent* and *noetherian* are local, when dealing with relation between them, we may always restrict to a small locus and replace the entire space by it.

2.1.8 Corollary *Let $\mathscr{R}$ be coherent. Then an $\mathscr{R}$-module is coherent if and only if it is locally of finite presentation.*

PROOF: It is clear that a coherent $\mathscr{R}$-module is of finite presentation.

Conversely, suppose $\mathscr{M}$ is of finite presentation. Since being coherent is a local problem, we may assume there is an exact sequence

$$\mathscr{R}^r \longrightarrow \mathscr{R}^s \longrightarrow \mathscr{M} \longrightarrow 0$$

with r, s finite numbers. By Proposition 2.1.7.(iii), since $\mathscr{R}$ is coherent, both $\mathscr{R}^r$ and $\mathscr{R}^s$ are also coherent. Then, by Proposition 2.1.7.(ii), $\mathscr{M}$ is coherent. $\square$

The following proposition shows that noetherian modules also form a weak Serre subcategory.

2.1.9 Proposition *Let $\mathscr{R}$ be a sheaf of rings.*

(i) Any finite type submodule of a noetherian $\mathscr{R}$-module is noetherian.
(ii) Let $\varphi\colon \mathscr{M} \to \mathscr{N}$ be a morphism of notherian $\mathscr{R}$-modules. Then $\ker(\varphi)$ *and* $\operatorname{coker}(\varphi)$ *are noetherian.*
(iii) Given a short exact sequence of $\mathscr{R}$-modules $0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$ if two out of three are noetherian so is the third.
(iv) The category of noetherian $\mathscr{R}$-modules is a weak Serre subcategory of $\mathcal{M}(\mathscr{R})$.

PROOF: Let $\mathscr{M}$ be a noetherian module and $\mathscr{N}$ a submodule of $\mathscr{M}$. Since $\mathscr{M}$ is coherent, by Proposition 2.1.7.(i), $\mathscr{N}$ is coherent. At any point x, the stalk $\mathscr{N}_x$ is a submodule of the noetherian module $\mathscr{M}_x$, hence is also noetherian. For any open set U, any ascending chain of coherent submodules of $\mathscr{N}$ is also an ascending chain of coherent submodules of $\mathscr{M}$, hence is locally stationary. This proves (i).

Let $\varphi\colon \mathscr{M} \to \mathscr{N}$ be a morphism of notherian $\mathscr{R}$-modules. Since $\mathscr{M}$ and $\mathscr{N}$ are coherent, by Proposition 2.1.7.(ii), $\ker(\varphi)$ and $\operatorname{coker}(\varphi)$ are coherent. Then $\ker(\varphi)$ is a finite type submodule of the noetherian module $\mathscr{M}$, hence it is noetherian.

Let's prove $\operatorname{coker}(\varphi)$ is noetherian. At any point x, the stalk $\operatorname{coker}(\varphi)$ is a quotient module of the noetherian module $\mathscr{N}_x$, hence is also noetherian. Note that Proposition 2.1.7.(iv) implies that fired products of coherent modules are coherent. Therefore, any ascending chain of coherent submodules of $\operatorname{coker}(\varphi)$ can be pullbacked along $\mathscr{N} \to \operatorname{coker}(\varphi)$ and becomes an ascending chain of coherent submodules of $\mathscr{N}$. But $\mathscr{N}$ is noetherian, hence this chain has to be locally stationary. This proves (ii).

Being proved (ii), to show (iii), it suffices to show that for any short exact sequence of $\mathscr{R}$-modules $0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$ with $\mathscr{M}_1$ and $\mathscr{M}_3$ noetherian, so is $\mathscr{M}_2$. First, $\mathscr{M}_2$ is coherent since both $\mathscr{M}_1$ and $\mathscr{M}_3$ are coherent (by Proposition 2.1.7.(iii)). At any point x, we have a short exact sequence $0 \to \mathscr{M}_{1,x} \to \mathscr{M}_{2,x} \to \mathscr{M}_{3,x} \to 0$ with $\mathscr{M}_{1,x}$ and $\mathscr{M}_{3,x}$ noetherian, hence so is $\mathscr{M}_{2,x}$. Let $\mathscr{N}_2^{(i)}$ be an ascending chain of coherent submodules of $\mathscr{M}_2$. Then, by pullback along $\mathscr{M}_1 \to \mathscr{M}_2$, we have an ascending chain of coherent submodules $\mathscr{N}_1^{(i)}$ of $\mathscr{M}_1$. Then by taking cokernel of each $\mathscr{N}_1^{(i)} \to \mathscr{N}_2^{(i)}$, we get an ascending chain of coherent submodules $\mathscr{N}_3^{(i)}$ of $\mathscr{M}_3$. Since $\mathscr{M}_1$ and $\mathscr{M}_3$ are noetherian, the chains $\mathscr{N}_1^{(\bullet)}$ and $\mathscr{N}_3^{(\bullet)}$ are locally stationary, hence so is the chain $\mathscr{N}_2^{(\bullet)}$. This proves (iii).

Finally, (iv) follows from (ii) and (iii). □

2.1.10 Corollary *Let $\mathscr{R}$ be noetherian. Then an $\mathscr{R}$-module is noetherian if and only if it is coherent if and only if it is of finite presentation.*

Proof: It is clear that a noetherian $\mathscr{R}$-module is coherent and hence of finite presentation.

Conversely, suppose $\mathscr{M}$ is of finite presentation. Since being coherent is a local problem, we may assume there is an exact sequence

$$\mathscr{R}^r \longrightarrow \mathscr{R}^s \longrightarrow \mathscr{M} \longrightarrow 0$$

with r, s finite numbers. By Proposition 2.1.9.(iii), since $\mathscr{R}$ is noetherian, both $\mathscr{R}^r$ and $\mathscr{R}^s$ are also noetherian. Then, by Proposition 2.1.9.(ii), $\mathscr{M}$ is noetherian. □

The following proposition is useful to determine coherence of modules on a complicated ring from a base ring.

2.1.11 Proposition *Let $\mathscr{O}$ be a noetherian ring and $\mathscr{R}$ be an $\mathscr{O}$-algebra which is coherent as an $\mathscr{O}$-module. Then $\mathscr{R}$ is also noetherian. Moreover, any $\mathscr{R}$-module is coherent if and only if it is coherent as an $\mathscr{O}$-module.*

Proof: First, $\mathscr{R}$ is coherent. Indeed, let $\varphi\colon \mathscr{R}^r \to \mathscr{R}$ be a morphism. Then consider the following morphism of exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{I}^r \arrow[r] \arrow[d] & \mathscr{O}^r \arrow[r] \arrow[d, "\phi"'] & \mathscr{R}^r \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathscr{R} \arrow[r, equal] & \mathscr{R} \arrow[r] & 0
\end{tikzcd}$$

where $\mathscr{I}$ is the kernel of the structure morphism $\mathscr{O} \to \mathscr{R}$ and ϕ is the restriction of φ to $\mathscr{O}$. By *snake lemma*, we have the following exact sequence

$$0 \longrightarrow \mathscr{I}^r \longrightarrow \ker(\phi) \longrightarrow \ker(\varphi) \longrightarrow 0.$$

Since $\mathscr{R}$ is a coherent $\mathscr{O}$-modules, $\ker(\phi)$ is of finite type. Then so is $\ker(\varphi)$.

At any point x, since $\mathscr{O}$ is noetherian, $\mathscr{O}_x$ is a noetherian ring. Then since $\mathscr{R}$ is a coherent $\mathscr{O}$-module, it is a noetherian $\mathscr{O}$-module. Then $\mathscr{R}_x$ is a noetherian $\mathscr{O}_x$-module, hence a noetherian $\mathscr{R}_x$-module.

Any coherent $\mathscr{R}$-submodule of $\mathscr{R}$ is a coherent $\mathscr{O}$-submodule. Indeed, let $\mathscr{N}$ be such a module, then it is of finite type. Then it is a coherent $\mathscr{O}$-module since it is a finite type submodule of a coherent $\mathscr{O}$-module.

Then any ascending chain of coherent $\mathscr{R}$-submodules of $\mathscr{R}$ is also an ascending chain of coherent $\mathscr{O}$-submodules, hence is stationary. This finishes proving $\mathscr{R}$ is noetherian.

Let $\mathscr{M}$ be an $\mathscr{R}$-module. If it is coherent as an $\mathscr{R}$-module. Then, we have an exact sequence

$$\mathscr{R}^r \longrightarrow \mathscr{R}^s \longrightarrow \mathscr{M} \longrightarrow 0.$$

But $\mathscr{R}^r$ and $\mathscr{R}^s$ are coherent $\mathscr{O}$-modules. Hence so is $\mathscr{M}$. Conversely, if $\mathscr{M}$ is a coherent $\mathscr{O}$-module. Then for any morphism $\varphi\colon \mathscr{R}^t \to \mathscr{M}$, consider the following morphism of exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{I}^r \arrow[r] \arrow[d] & \mathscr{O}^r \arrow[r] \arrow[d, "\phi"'] & \mathscr{R}^r \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathscr{M} \arrow[r, equal] & \mathscr{M} \arrow[r] & 0
\end{tikzcd}$$

where ϕ is the restriction of φ to $\mathscr{O}$. By *snake lemma*, we have the following exact sequence

$$0 \longrightarrow \mathscr{I}^r \longrightarrow \ker(\phi) \longrightarrow \ker(\varphi) \longrightarrow 0.$$

Since $\mathscr{M}$ is a coherent $\mathscr{O}$-modules, $\ker(\phi)$ is of finite type. Then so is $\ker(\varphi)$. This shows $\mathscr{M}$ is a coherent $\mathscr{R}$-module. □

2.1.12 Corollary *Let $\mathscr{R}$ be a noetherian ring and $\mathscr{I}$ a finite type ideal of $\mathscr{R}$. Then, $\mathscr{R}/\mathscr{I}$ is a noetherian ring.*

We also have the following lemma to deduce noetherian property from the quotient ring.

2.1.13 Proposition *Let $\mathscr{R}$ be a sheaf of rings and $\mathscr{I}$ a two-sided ideal of $\mathscr{R}$. If*

(a) $\mathscr{I}^2 = 0$;

(b) $\mathscr{I}$ is a coherent $\mathscr{R}/\mathscr{I}$-module;

(c) $\mathscr{R}/\mathscr{I}$ is a coherent (resp. noetherian) ring;

Then $\mathscr{R}$ is a coherent (resp. noetherian) ring and $\mathscr{I}$ is a coherent ideal.

Proof: Note that, by the adjunction of extending ans restricting scales, an $\mathscr{R}/\mathscr{I}$-module is of finite type if and only if it is of finite type as an $\mathscr{R}$-module. Therefore in this proof, we will not distinguish them.

First, $\mathscr{R}/\mathscr{I}$ is a coherent $\mathscr{R}$-module. Indeed, any morphism $\varphi\colon \mathscr{R}^r \to \mathscr{R}/\mathscr{I}$ can be factorized into the projection $\mathscr{R}^r \to (\mathscr{R}/\mathscr{I})^r$ and a morphism of $\mathscr{R}/\mathscr{I}$-modules $\bar{\varphi}\colon (\mathscr{R}/\mathscr{I})^r \to \mathscr{R}/\mathscr{I}$. Since $\mathscr{R}/\mathscr{I}$ is coherent, $\ker(\bar{\varphi})$ is of finite type. Now, consider the following morphism of exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{I}^r \arrow[r] \arrow[d] & \mathscr{R}^r \arrow[r] \arrow[d, "\varphi"'] & (\mathscr{R}/\mathscr{I})^r \arrow[r] \arrow[d, "\bar{\varphi}"] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \mathscr{R}/\mathscr{I} \arrow[r, equal] & \mathscr{R}/\mathscr{I} \arrow[r] & 0 \end{tikzcd}$$

By *snake lemma*, we have the following exact sequence

$$0 \longrightarrow \mathscr{I}^r \longrightarrow \ker(\varphi) \longrightarrow \ker(\bar{\varphi}) \longrightarrow 0.$$

where $\mathscr{I}^r$ is a coherent $\mathscr{R}/\mathscr{I}$-module, hence of finite type, $\ker(\bar{\varphi})$ is of finite type as shown. Therefore $\ker(\varphi)$ is of finite type.

Then, $\mathscr{R}$ is a coherent ring. Indeed, for any morphism $\varphi\colon \mathscr{R}^r \to \mathscr{R}$, consider the following morphism of exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{I}^r \arrow[r] \arrow[d, "\phi"] & \mathscr{R}^r \arrow[r] \arrow[d, "\varphi"'] & (\mathscr{R}/\mathscr{I})^r \arrow[r] \arrow[d, "\bar{\varphi}"] & 0 \\ 0 \arrow[r] & \mathscr{I} \arrow[r] & \mathscr{R} \arrow[r] & \mathscr{R}/\mathscr{I} \arrow[r] & 0 \end{tikzcd}$$

where $\bar{\varphi}$ is the induced morphism of $\mathscr{R}/\mathscr{I}$-modules and ϕ is the restriction of φ to $\mathscr{I}$. By *snake lemma*, we have the following exact sequence

$$0 \longrightarrow \ker(\phi) \longrightarrow \ker(\varphi) \longrightarrow \ker(\bar{\varphi}) \longrightarrow \operatorname{coker}(\phi).$$

Since $\mathscr{I}$ is a coherent $\mathscr{R}/\mathscr{I}$-module, hence so is $\mathscr{I}^r$. Then $\ker(\phi)$ and $\operatorname{coker}(\phi)$ are coherent $\mathscr{R}/\mathscr{I}$-modules. Similarly, $\ker(\bar{\varphi})$ is a coherent $\mathscr{R}/\mathscr{I}$-module. Then $\ker(\ker(\bar{\varphi}) \to \operatorname{coker}(\phi))$ is a coherent $\mathscr{R}/\mathscr{I}$-module. In the short exact sequence

$$0 \longrightarrow \ker(\phi) \longrightarrow \ker(\varphi) \longrightarrow \ker(\ker(\bar{\varphi}) \to \operatorname{coker}(\phi)) \longrightarrow 0,$$

both $\ker(\phi)$ and $\ker(\ker(\bar{\varphi}) \to \operatorname{coker}(\phi))$ are of finite type, hence $\ker(\varphi)$ is of finite type.

Now, since both $\mathscr{R}$ and $\mathscr{R}/\mathscr{I}$ are coherent $\mathscr{R}$-modules, hence so is $\mathscr{I}$.

Now, let's suppose $\mathscr{R}/\mathscr{I}$ is a noetherian ring. Then $\mathscr{I}$ is a noetherian $\mathscr{R}/\mathscr{I}$-module since it is coherent. Therefore, at any point x, $\mathscr{R}_x/\mathscr{I}_x$ and $\mathscr{I}_x$ are noetherian $\mathscr{R}_x/\mathscr{I}_x$-modules. But then they are noetherian $\mathscr{R}_x$-modules.

Notice that any coherent $\mathscr{R}$-submodule of $\mathscr{R}/\mathscr{I}$ (resp. $\mathscr{I}$) is also a coherent $\mathscr{R}/\mathscr{I}$-submodule. Indeed, let $\mathscr{N}$ be such a module, then it is of finite type. Then it is a coherent $\mathscr{R}/\mathscr{I}$-submodule since it is a finite type submodule of a coherent $\mathscr{R}/\mathscr{I}$-module.

Therefore any ascending chain of coherent $\mathscr{R}$-submodules of $\mathscr{R}/\mathscr{I}$ (resp. $\mathscr{I}$) is an ascending chain of coherent $\mathscr{R}/\mathscr{I}$-submodules, hence is stationary. This shows both $\mathscr{R}/\mathscr{I}$ and $\mathscr{I}$ are noetherian $\mathscr{R}$-module, hence so is $\mathscr{R}$. $\square$

2.1.14 Proposition *Let $X = Y \times M$ be a product of topological spaces and let $f\colon X \to Y$ be the projection. Let $\mathscr{R}$ be a sheaf of rings on Y.*

(i) If $\mathscr{R}$ is coherent, so is $f^{-1}\mathscr{R}$.
(ii) If $\mathscr{R}$ is noetherian and moreover M is a topological manifold, then $f^{-1}\mathscr{R}$ is noetherian.

2.1.15 Theorem (Oka-Cartan Theorem) *Let M be a complex manifold. Then $\mathscr{O}_M$ is noetherian.*

2.2 Filtrations

We begin by reviewing some notions on graded sheaves.

A *graded sheaf* is a function from $\mathbb{Z}$ to the category of sheaves. In other word, a graded sheaf $\mathscr{M}_\bullet$ is just a family of sheaves $(\mathscr{M}_p)$ indexed by integers $p \in \mathbb{Z}$. A *shifted graded sheaf* $\mathscr{M}_\bullet^{[n]}$ of a graded sheaf $\mathscr{M}_\bullet$ is the one obtained by shift the index: $\mathscr{M}_p^{[n]} = \mathscr{M}_{n+p}$. A *morphism* between graded sheaves $\varphi_\bullet \colon \mathscr{M}_\bullet \to \mathscr{N}_\bullet$ is then a family of morphisms of sheaves between each components $\varphi_p \colon \mathscr{M}_p \to \mathscr{N}_p$. A *morphism* from a graded sheaf $\mathscr{M}_\bullet$ to a sheaf $\mathscr{N}$ is a family of morphisms of sheaves from each component $\mathscr{M}_p$ to the target sheaf $\mathscr{N}$. By the universal property of direct sum, such a morphism can be represented by a morphism of sheaves $\bigoplus_{p\in\mathbb{Z}} \mathscr{M}_p \to \mathscr{N}$. The sheaf $\bigoplus_{p\in\mathbb{Z}} \mathscr{M}_p$ is called the *total sheaf* of the graded sheaf $\mathscr{M}_\bullet$. Then we get a faithful and exact functor from the category of graded sheaves to the category sheaves. Note that shifted graded sheaf has the same total sheaf with the original graded sheaf. Therefore it's better to involve *twisted morphisms*: a *twisted morphism of degree n* between graded sheaves $\mathscr{M}_\bullet \to \mathscr{N}_\bullet$ is merely a morphism from $\mathscr{M}_\bullet$ to $\mathscr{N}_\bullet^{[n]}$. Under the total sheaf functor, the total group of the graded abelian group of twisted morphisms between graded sheaves become a subgroup of the abelian group of all morphisms between corresponding total sheaves. In general, it is a proper subgroup. A morphism in this subgroup is called a *homogeneous morphism*.

A *sheaf of graded rings*, or simply *graded ring* is a graded sheaf $\mathscr{R}_\bullet$ equipped with the following structures:

- a identity $1 \in \mathscr{R}_0$;
- for each $p, q \in \mathbb{Z}$, a multiplication $\mathscr{R}_p \times \mathscr{R}_q \to \mathscr{R}_{p+q}$;

and those structures satisfy certain commutative diagrams similar to those in the definition of a sheaf of rings. Note that by this definition,

1. $\mathscr{R}_0$ is a ring;
2. each $\mathscr{R}_p$ is an $\mathscr{R}_0$-module;
3. the graded ring structure of $\mathscr{R}_\bullet$ induces a ring structure on the total sheaf $\mathscr{R} = \bigoplus_{p\in\mathbb{Z}} \mathscr{R}_p$;
4. under such a ring structure, $\mathscr{R}_0$, $\mathscr{R}_{\geqslant 0} := \bigoplus_{p\geqslant 0} \mathscr{R}_p$ and $\mathscr{R}_{\leqslant 0} := \bigoplus_{p\leqslant 0} \mathscr{R}_p$ are subrings of $\mathscr{R}$.

Given a graded ring $\mathscr{R}_\bullet$, a *graded module* over $\mathscr{R}_\bullet$, or simply an *$\mathscr{R}_\bullet$-module* is a graded sheaf $\mathscr{M}_\bullet$ equipped with a family of actions $\mathscr{R}_p \times \mathscr{M}_q \to \mathscr{M}_{p+q}$ satisfying certain commutative digram similar to those in the definition of a sheaf of modules. Note that by this definition,

1. each $\mathscr{M}_p$ is an $\mathscr{R}_0$-module;
2. the graded module structure of $\mathscr{M}_\bullet$ induces a $\mathscr{R}$-module structure on the total sheaf $\mathscr{M} = \bigoplus_{p\in\mathbb{Z}} \mathscr{M}_p$.

Note that not every $\mathscr{R}$-submodule of $\mathscr{M}$ is a total sheaf of an $\mathscr{R}_\bullet$-submodule of $\mathscr{M}_\bullet$. Such a submodule of $\mathscr{M}$ is called a *homogeneous submodule*.

Let $\mathscr{M}_\bullet$ be an $\mathscr{R}_\bullet$-module. We say

- $\mathscr{M}_\bullet$ is *locally free of finite rank* if for any point x, there exists an open neighborhood U on which there is an isomorphism
$$(\mathscr{M}_{|U})_\bullet \cong \bigoplus_{i=1}^{s} (\mathscr{R}_{|U}^{[n_i]})_\bullet,$$
where s is a finite number and $n_i \in \mathbb{Z}$;
- $\mathscr{M}_\bullet$ is *(locally) of finite type* if for any point x, there exists an open neighborhood U on which there is an exact sequence
$$\bigoplus_{i=1}^{s} (\mathscr{R}_{|U}^{[n_i]})_\bullet \longrightarrow (\mathscr{M}_{|U})_\bullet \longrightarrow 0,$$
where s is a finite number and $n_i \in \mathbb{Z}$;
- $\mathscr{M}_\bullet$ is *(locally) of finite presentation* if for any point x, there exists an open neighborhood U on which there is an exact sequence
$$\bigoplus_{i=1}^{r} (\mathscr{R}_{|U}^{[m_i]})_\bullet \longrightarrow \bigoplus_{j=1}^{s} (\mathscr{R}_{|U}^{[n_j]})_\bullet \longrightarrow (\mathscr{M}_{|U})_\bullet \longrightarrow 0,$$
where r, s are finite numbers and $m_i, n_j \in \mathbb{Z}$;
- $\mathscr{M}_\bullet$ is *pseudo-coherent* if for any open set U, any $(\mathscr{R}_{|U})_\bullet$-submodule of $(\mathscr{M}_{|U})_\bullet$ which is of finite type is of finite presentation;
- $\mathscr{M}_\bullet$ is *coherent* if it is of finite type and pseudo-coherent;
- $\mathscr{R}_\bullet$ is *coherent* if $\mathscr{R}_\bullet$ is coherent as an $\mathscr{R}_\bullet$-module;
- $\mathscr{M}_\bullet$ is *noetherian* if it is coherent, each stalk is noetherian and for any open set U, any ascending chain of coherent $(\mathscr{R}_{|U})_\bullet$-submodules of $\mathscr{M}_\bullet$ is locally stationary;
- $\mathscr{R}_\bullet$ is *noetherian* if $\mathscr{R}_\bullet$ is noetherian as an $\mathscr{R}_\bullet$-module.

Note that those properties are *a priori* not equivalent to the corresponding properties for the $\mathscr{R}$-module $\mathscr{M}$. For example, this post shows that $\mathscr{M}_\bullet$ *is free* is stronger than $\mathscr{M}$ *is free*. However, we have

2.2.1 Proposition *Let $\mathscr{M}_\bullet$ be an $\mathscr{R}_\bullet$-module. Then $\mathscr{M}_\bullet$ is of finite type (resp. of finite presentation) over $\mathscr{R}_\bullet$, if and only if $\mathscr{M}$ is of finite type (resp. of finite presentation) over $\mathscr{R}$.*

PROOF: If $e_1, \cdots, e_r$ is a finite family of generators of $\mathscr{M}$. Then each of them can be written as sum of homogeneous sections of $\mathscr{M}$. Then it is clear those homogeneous sections form a finite family of generators of the graded module $\mathscr{M}_\bullet$.

Suppose $\mathscr{M}$ is of finite presentation. We may assume there is an epimorphism $\varphi\colon \mathscr{R}^s \to \mathscr{M}$ which is homogeneous, hence is the total morphism of a twisted morphism of graded sheaves $\varphi_\bullet\colon (\mathscr{R}^s)_\bullet \to \mathscr{M}_\bullet$. Then $\ker(\varphi)$ is the total sheaf of $\ker(\varphi_\bullet)$. Since $\mathscr{M}$ is of finite presentation, $\ker(\varphi)$ is of finite type. Hence we have homogeneous epimorphism $\mathscr{R}^r \to \ker(\varphi)$ and thus a homogeneous right exact sequence $\mathscr{R}^r \to \mathscr{R}^s \to \mathscr{M} \to 0$.

It is clear that if $\mathscr{M}_\bullet$ is of finite type (resp. of finite presentation), the $\mathscr{M}$ is of finite type (resp. of finite presentation). □

Therefore we don't need to distinguish *of finite type* (resp. *of finite presentation*) over $\mathscr{R}_\bullet$ and *of finite type* (resp. *of finite presentation*) over $\mathscr{R}$ for graded $\mathscr{R}$-modules.

2.2.2 Proposition *Let $\mathscr{M}_\bullet$ be an $\mathscr{R}_\bullet$-module. If $\mathscr{M}$ is pseudo-coherent over $\mathscr{R}$, then $\mathscr{M}_\bullet$ is pseudo-coherent over $\mathscr{R}_\bullet$.*

PROOF: Let $\mathscr{N}_\bullet$ is a graded submodule of $\mathscr{M}_\bullet$. If $\mathscr{N}_\bullet$ is of finite type, then $\mathscr{N}$ is of finite type. Then $\mathscr{N}$ is of finite presentation since $\mathscr{M}$ is pseudo-coherent. Then $\mathscr{N}_\bullet$ is of finite presentation. □

By reviewing proofs of Propositions 2.1.1, 2.1.2 and 2.1.7, we see that the graded versions of them and hence of Corollary 2.1.8 also hold.

2.2.3 Proposition *Let $\mathscr{M}_\bullet$ be an $\mathscr{R}_\bullet$-module and suppose the total sheaf $\mathscr{R}$ is coherent. Then $\mathscr{M}_\bullet$ is coherent (resp. noetherian) over $\mathscr{R}_\bullet$ if and only if $\mathscr{M}$ is coherent (resp. noetherian) over $\mathscr{R}$.*

PROOF: Since $\mathscr{R}$ is coherent, $\mathscr{R}_\bullet$ is coherent.

If $\mathscr{M}$ is coherent over $\mathscr{R}$, then $\mathscr{M}_\bullet$ is coherent over $\mathscr{R}_\bullet$. Conversely, if $\mathscr{M}_\bullet$ is coherent over $\mathscr{R}_\bullet$, then it is of finite presentation. Hence $\mathscr{M}$ is of finite presentation over $\mathscr{R}$. Then $\mathscr{M}$ is coherent over $\mathscr{R}$.

The equivalence of noetherian and graded noetherian at stalks is a result in ring theory, see Corollary 2.2.16 for instance.

If $\mathscr{M}$ is noetherian over $\mathscr{R}$. To show $\mathscr{M}_\bullet$ is noetherian over $\mathscr{R}_\bullet$, it remains to show any ascending chain of coherent $\mathscr{R}_\bullet$-submodules is locally stationary. But we know that the chain of their total sheaves is a chain of coherent $\mathscr{R}$-submodule of $\mathscr{M}$, hence is locally stationary. But this implies that the original chain of coherent $\mathscr{R}_\bullet$-submodules is locally stationary. The proof for converse is similar. □

Before going further, let us review the some notions on filtered sheaves.

Let $\mathscr{M}$ be a sheaf. A *(increasing) filtration* F on $\mathscr{M}$ is a family of subsheaves $(F_p\mathscr{M})_{p\in\mathbb{Z}}$ such that

$$F_p\mathscr{M} \subset F_{p+1}\mathscr{M}, \qquad \forall p \in \mathbb{Z}.$$

A sheaf with a filtration is called a *filtered sheaf*. Given a filtered sheaf $(\mathscr{M}, F)$, the *shifted filtration* $F^{[n]}$ is the filtration on $\mathscr{M}$ given by

$$F_p^{[n]}\mathscr{M} = F_{n+p}\mathscr{M}, \qquad \forall p \in \mathbb{Z}.$$

A *subfiltration* F' of a filtration F on $\mathscr{M}$ is a filtration on $\mathscr{M}$ such that

$$F'_p\mathscr{M} \subset F_p\mathscr{M}, \qquad \forall p \in \mathbb{Z}.$$

Two filtrations are *equivalent* if they are subfiltrations of a shifted filtration of each other. Note that equivalent filtrations share the same limit and colimit.

Let $(\mathscr{M}, F)$ and $(\mathscr{N}, F)$ be two filtered sheaves. A *(filtered) morphism* between them is a morphism of sheaves $\varphi\colon \mathscr{M} \to \mathscr{N}$ compatible with all the inclusions in the filtrations, that is

$$\varphi(F_p\mathscr{M}) \subset F_p\mathscr{N}, \qquad \forall p \in \mathbb{Z}.$$

It is a *strict morphism* if furthermore

$$\varphi(F_p\mathscr{M}) = F_p\mathscr{N} \cap \operatorname{im}(\varphi), \qquad \forall p \in \mathbb{Z},$$

or equivalently,

$$\varphi^{-1}(F_p\mathscr{N}) = F_p\mathscr{M} + \ker(\varphi), \qquad \forall p \in \mathbb{Z}.$$

Given a filtered sheaf $(\mathscr{M}, F)$ and a subsheaf $\mathscr{N}$ and a quotient sheaf $\mathscr{P}$ of it. The *induced filtration* on $\mathscr{N}$ is the unique filtration such that the inclusion $\mathscr{N} \to \mathscr{M}$ is a strict morphism of filtered sheaves. The *quotient filtration* on $\mathscr{P}$ is the unique filtration such that the projection $\mathscr{M} \to \mathscr{P}$ is a strict morphism of filtered sheaves.

Given a morphism of filtered sheaves $\varphi\colon \mathscr{M} \to \mathscr{N}$, its *kernel* is the kernel sheaf $\ker(\varphi)$ together with the induced filtration; its *cokernel* is the cokernel sheaf $\operatorname{coker}(\varphi)$ together with the quotient filtration. Similarly we have the notions of *image* $\operatorname{im}(\varphi)$ and *coimage* $\operatorname{coim}(\varphi)$. Then we see that

2.2.4 Proposition *The filtered sheaves form a pre-abelian category. Moreover, for a morphism of filtered sheaves $\varphi\colon \mathscr{M} \to \mathscr{N}$, the canonical morphism* $\operatorname{coim}(\varphi) \to \operatorname{im}(\varphi)$ *is an isomorphism if and only if φ is strict.*

Remark But the composition of strict morphisms is *not* strict in general. Therefore one can not expect an abelian category by restricting to strict morphisms only.

Remark Since the category of filtered sheaves is not an abelian category, the notion of exact sequences is not well-behaved. So, when a sequence of filtered sheaves is called *exact*, it actually means exact in the category of sheaves.

Given a filtered sheaf $(\mathscr{M}, F)$, the *Rees sheaf* of it is the graded sheaf $F_\bullet \mathscr{M}$, i.e. viewing each $F_p\mathscr{M}$ as a component of the Rees sheaf. The total sheaf of the Rees sheaf is denoted by $F\mathscr{M}$. A sequence of filtered sheaves $\mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3$ is *filtered exact* if the corresponding sequence of Rees sheaves

$$F_\bullet \mathscr{M}_1 \longrightarrow F_\bullet \mathscr{M}_2 \longrightarrow F_\bullet \mathscr{M}_3$$

is exact. A morphism $\varphi\colon \mathscr{M} \to \mathscr{N}$ of filtered sheaves is *filtered surjective* (resp. *filtered injective*, *filtered bijective*) if the corresponding morphism of Rees sheaves $F_\bullet\varphi\colon F_\bullet\mathscr{M} \to F_\bullet\mathscr{N}$ is surjective (resp. injective, bijective).

Remark *Filtered injective* morphisms are precisely the injective morphisms and are precisely the monomorphisms in the category of filtered sheaves. *Filtered bijective* morphisms are precisely the bijective morphisms and are precisely the isomorphisms in the category of filtered sheaves. Therefore those notions are useless. However, although *filtered surjective* morphisms are surjective morphisms and hence are epimorphisms in the category of filtered sheaves. This notion is more than that: a morphism is filtered surjective if and only if it is surjective and strict.

Remark It is clear that an exact sequence consisting of only strict morphisms is filtered exact.

Given a filtered sheaf $(\mathscr{M}, F)$, the *associated graded sheaf* $\operatorname{gr}^F_\bullet \mathscr{M}$ (or simply $\operatorname{gr}_\bullet \mathscr{M}$) is defined by

$$\operatorname{gr}_p \mathscr{M} = F_p\mathscr{M}/F_{p-1}\mathscr{M}, \qquad \forall p \in \mathbb{Z}.$$

Its total sheaf is denoted by $\operatorname{gr}\mathscr{M}$. The projection $\sigma_p\colon F_p\mathscr{M} \to \operatorname{gr}_p\mathscr{M}$ is called the *symbol of order p*, The morphism $\sigma\colon \mathscr{M} \to \operatorname{gr}\mathscr{M}$ deduced from σ_p is called the *principal symbol*.

Taking associated graded sheaf is kind like an exact functor:

2.2.5 Proposition *Let $\mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to \mathscr{M}_4$ be a sequence of filtered sheaves. If either*

(i) $\mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3$ is an exact sequence and both morphisms are strict;
(ii) $\mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to \mathscr{M}_4$ is filtered exact.

Then the associated sequence of graded sheaves

$$\operatorname{gr}_\bullet \mathscr{M}_1 \longrightarrow \operatorname{gr}_\bullet \mathscr{M}_2 \longrightarrow \operatorname{gr}_\bullet \mathscr{M}_3$$

is exact.

For the converse, we need some conditions.

2.2.6 Definition A filtration F on a sheaf $\mathscr{M}$ is said to be

- *positive* if $F_p\mathscr{M} = 0$ for sufficiently negative p;
- *separated* if $\varprojlim F_p\mathscr{M} = 0$;
- *exhaustive* if $\varinjlim F_p\mathscr{M} = \mathscr{M}$;
- *regular* if it is both separated and exhaustive.

2.2.7 Proposition *Let $\mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3$ be a complex of filtered sheaves. Suppose the associated sequence of graded sheaves*

$$\operatorname{gr}_\bullet \mathscr{M}_1 \longrightarrow \operatorname{gr}_\bullet \mathscr{M}_2 \longrightarrow \operatorname{gr}_\bullet \mathscr{M}_3$$

is exact.

(i) If the filtration on $\mathscr{M}_2$ is positive, then the sequence $\mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3$ is filtered exact.
(ii) If furthermore the three filtrations are exhaustive, then the sequence $\mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3$ is exact and both of the two morphisms are strict.

From now on, all filtrations are assume to be positive and exhaustive without specification otherwise.

Remark Under this assumption, taking Rees sheaf is a faithful additive functor. However, it is not full: a morphisms $\varphi_\bullet \colon F_\bullet\mathscr{M} \to F'_\bullet\mathscr{N}$ is given by a filtered morphism $(\mathscr{M}, F) \to (\mathscr{N}, F')$ if and only if the components φ_p commutate with the inclusions in filtrations F and F'. Taking associated graded sheaf is merely an additive functor and is not full or faithful in general.

2.2.8 Definition Let $\mathscr{R}$ be a sheaf of rings. A *ring filtration* F on $\mathscr{R}$ is a filtration on $\mathscr{R}$ such that

Fl.1 $1 \in F_0\mathscr{R}$;

Fl.2 $F_p\mathscr{R} \cdot F_q\mathscr{R} \subset F_{p+q}\mathscr{R}$ for all $p, q \in \mathbb{Z}$.

A sheaf of rings with a ring filtration is called a *filtered sheaf of rings*, *sheaf of filtered rings*, or simply *filtered ring*. To simplicity, we always assume $F_p\mathscr{R} = 0$ for $p < 0$.

Given a filtered ring $(\mathscr{R}, F)$, the Rees sheaf $F_\bullet\mathscr{R}$ has a structure of graded rings. This graded ring is called the *Rees ring* of $\mathscr{R}$. Likewise, the associated graded sheaf $\operatorname{gr}_\bullet \mathscr{R}$ has a structure of graded rings. This graded ring is called the *associated graded ring* of $\mathscr{R}$.

Remark If $(\mathscr{R}, F)$ is a filtered ring, then $F_0\mathscr{R}$ is a subring of $\mathscr{R}$. Moreover, each $F_p\mathscr{R}$ is an $F_0\mathscr{R}$-submodule of $\mathscr{R}$.

Given a filtered ring $(\mathscr{R}, F)$. Let $\mathscr{M}$ be an $\mathscr{R}$-module. A *module filtration* (with respect to F) on $\mathscr{M}$ is a filtration Fl on $\mathscr{M}$ such that

$$F_p\mathscr{R}.Fl_q\mathscr{M} \subset Fl_{p+q}\mathscr{M}, \qquad \forall p, q \in \mathbb{Z}.$$

An $\mathscr{R}$-module with a module filtration is called a *filtered $(\mathscr{R}, F)$-module*, or simply *filtered $\mathscr{R}$-module*.

Given a filtered $\mathscr{R}$-module $(\mathscr{M}, Fl)$, the Rees sheaf $Fl_\bullet\mathscr{M}$ has a structure of graded modules over the Rees ring $F_\bullet\mathscr{R}$. This graded module is called the *Rees module* of $\mathscr{M}$. Likewise, the associated graded sheaf $\mathrm{gr}_\bullet\mathscr{M}$ has a structure of graded modules over the associated graded ring $\mathrm{gr}_\bullet\mathscr{R}$. This graded module is called the *associated graded module* of $\mathscr{M}$.

Let $(\mathscr{M}, Fl)$ and $(\mathscr{N}, Fl')$ be two filtered $\mathscr{R}$-modules. A *(filtered) $\mathscr{R}$-morphism* between them is an $\mathscr{R}$-morphism $\varphi \colon \mathscr{M} \to \mathscr{N}$ which is also a filtered morphism. Note that the induced morphism between Rees modules of φ is then an $F_\bullet\mathscr{R}$-morphism and the induced morphism between associated graded modules of φ is an $\mathrm{gr}_\bullet\mathscr{R}$-morphism.

Let $(\mathscr{M}, Fl)$ and $(\mathscr{N}, Fl')$ be two filtered $\mathscr{R}$-modules such that $\mathscr{N}$ is a $\mathscr{R}$-submodule of $\mathscr{M}$. $(\mathscr{N}, Fl')$ is called a *filtered submodule* of $(\mathscr{M}, Fl)$ if Fl' is a subfiltration of the induced filtration (from F) on $\mathscr{N}$. If this is the case, then $Fl'_\bullet\mathscr{N}$ is a $F_\bullet\mathscr{R}$-submodule of $Fl_\bullet\mathscr{M}$. However, this doesn't implies that $\mathrm{gr}^{Fl'}_\bullet\mathscr{N}$ is a $\mathrm{gr}^F_\bullet\mathscr{R}$-submodule of $\mathrm{gr}^{Fl}_\bullet\mathscr{M}$ unless Fl' is precisely the induced filtration.

Remark If $(\mathscr{M}, Fl)$ is a filtered $\mathscr{R}$-module, then each $Fl_p\mathscr{M}$ is a $F_0\mathscr{R}$-submodule of $\mathscr{M}$. Moreover, the graded module structure on $Fl_\bullet\mathscr{M}$ and the ring filtration on $\mathscr{R}$ recover the inclusions in the filtration Fl: indeed, the inclusion $Fl_p\mathscr{M} \subset Fl_{p+1}\mathscr{M}$ is coincide with the composition $Fl_p\mathscr{M} = F_0\mathscr{R}.Fl_p\mathscr{M} \subset F_1\mathscr{R}.Fl_p\mathscr{M} \subset Fl_{p+1}\mathscr{M}$. Therefore, *taking Rees modules is a fully faithful additive functor and surjective on the class of submodules.*

Remark One need to distinguish *module filtrations* and *ascending chain of submodules* of a module: the later notion has nothing to do with the ring filtration and is not assumed to be exhaustive.

2.2.9 Definition Let $(\mathscr{R}, F)$ be a filtered ring and $(\mathscr{M}, Fl)$ a filtered $\mathscr{R}$-module. Then, we say

- $(\mathscr{M}, Fl)$, or Fl, is *locally free of finite rank* if $Fl_\bullet\mathscr{M}$ is locally free of finite rank over $F_\bullet\mathscr{R}$. That is, for any point x, there exists an open neighborhood U on which there is an isomorphism

$$Fl_\bullet\mathscr{M}_{|U} \cong \bigoplus_{i=1}^{s} F^{[n_i]}_\bullet\mathscr{R}_{|U},$$

where s is a finite number and $n_i \in \mathbb{Z}$;

- $(\mathscr{M}, Fl)$, or Fl, is *(locally) of finite type* if $Fl_\bullet\mathscr{M}$ is of finite type over $F_\bullet\mathscr{R}$. That is, for any point x, there exists an open neighborhood U on which there is an exact sequence

$$\bigoplus_{i=1}^{s} F_\bullet^{[n_i]}\mathscr{R}_{|U} \longrightarrow Fl_\bullet\mathscr{M}_{|U} \longrightarrow 0,$$

  where s is a finite number and $n_i \in \mathbb{Z}$;

- $(\mathscr{M}, Fl)$, or Fl, is *(locally) of finite presentation* if $Fl_\bullet\mathscr{M}$ is of finite presentation over $F_\bullet\mathscr{R}$. That is, for any point x, there exists an open neighborhood U on which there is an exact sequence

$$\bigoplus_{i=1}^{r} F_\bullet^{[m_i]}\mathscr{R}_{|U} \longrightarrow \bigoplus_{j=1}^{s} F_\bullet^{[n_j]}\mathscr{R}_{|U} \longrightarrow Fl_\bullet\mathscr{M}_{|U} \longrightarrow 0,$$

  where r, s are finite numbers and $m_i, n_j \in \mathbb{Z}$.

Remark Note that in the second and third definitions, the Rees sheaves can be replaced by their total sheaves in virtue of Proposition 2.2.1.

By Proposition 2.2.5, we immediately have:

2.2.10 Proposition *Let $(\mathscr{R}, F)$ be a filtered ring and $(\mathscr{M}, Fl)$ a filtered $\mathscr{R}$-module. If Fl is locally free of finite rank (resp. of finite type, of finite presentation), then $\mathrm{gr}_\bullet\mathscr{M}$, as well as $\mathscr{M}$, is locally free of finite rank (resp. of finite type, of finite presentation).*

Conversely, we have

2.2.11 Proposition *Let $(\mathscr{R}, F)$ be a filtered ring and $\mathscr{M}$ an $\mathscr{R}$-module. If $\mathscr{M}$ is of finite type, then $\mathscr{M}$ locally has a finite type filtration.*

Proof: We may assume there is an epimorphism $\mathscr{R}^r \to \mathscr{M}$. Then the quotient filtration on $\mathscr{M}$ is a finite type filtration. □

Remark This doesn't means that if a filtered module is of finite type as modules, its filtration is of finite type.

2.2.12 Proposition *Let $(\mathscr{R}, F)$ be a filtered ring and $(\mathscr{M}, Fl)$ a filtered $\mathscr{R}$-module. If $\mathrm{gr}_\bullet\mathscr{M}$ is of finite type, then Fl is a finite type filtration.*

Proof: Let $e_1, \cdots, e_s$ be sections of $\mathscr{M}$ such that $e_i \in Fl_{p_i}\mathscr{M} \setminus Fl_{p_i-1}\mathscr{M}$ and $\sigma(e_i)$ form a family of generators of $\mathrm{gr}_\bullet\mathscr{M}$ over $\mathrm{gr}_\bullet\mathscr{R}$. Then we have a morphism of $F_\bullet\mathscr{M}$-modules $\varphi_\bullet\colon \bigoplus_{i=1}^{s} F_\bullet^{[-p_i]}\mathscr{R} \to Fl_\bullet\mathscr{M}$ mapping each $1 \in F_{p_i}^{[-p_i]}\mathscr{R}$ to $e_i \in Fl_{p_i}\mathscr{M}$. Then it is clear that the induced morphism $\bar{\varphi}_\bullet\colon \bigoplus_{i=1}^{s} \mathrm{gr}_\bullet^{[-p_i]}\mathscr{R} \to \mathrm{gr}_\bullet\mathscr{M}$ is precisely the epimorphism mapping each $1 \in \mathrm{gr}_{p_i}^{[-p_i]}\mathscr{R}$ to $\sigma(e_i) \in Fl_{p_i}\mathscr{M}$. Then by induction, we see that each φ_p is surjective. Then $\varphi_\bullet$ is an epimorphism and hence Fl is a finite type filtration. □

The sections $e_1, \cdots, e_s$ in the proof is called a *system of local generators* of $\mathscr{M}$ of *type* $(p_1, \cdots, p_s)$. The following fact is very useful:

2.2.13 Lemma *Let $(\mathscr{M}, Fl)$ be a filtered $\mathscr{R}$-module of finite type. Let $e_1, \cdots, e_s$ be a system of local generators of $\mathscr{M}$ of type $(p_1, \cdots, p_s)$. Then locally we have $Fl_p\mathscr{M} = \sum_{i=1}^s F_{p-p_i}\mathscr{R}.e_i$ for all p.*

PROOF: It is clear that $\mathrm{gr}_p \mathscr{M} = \sum_{i=1}^s \mathrm{gr}_{p-p_i} \mathscr{R}.\sigma(e_i)$. Then the statement follows by induction on p. □

2.2.14 Proposition *The finite type filtration is the finest filtration on a finite type module. In particular, any two finite type filtrations on a module is equivalent.*

PROOF: Let Fl and Fl' be two filtrations on an $\mathscr{R}$-module $\mathscr{M}$ and assume Fl is of finite type. Let $e_1, \cdots, e_s$ be a system of local generators of $\mathscr{M}$ of type $(p_1, \cdots, p_s)$ with respect to the filtration Fl. Suppose $e_i \in Fl'_{q_i}\mathscr{M}$ and let $n = \max_i(q_i - p_i)$. Then for any p, we have

$$\begin{aligned} Fl_p\mathscr{M} &= \sum_{i=1}^s F_{p-p_i}\mathscr{R}.e_i \subset \sum_{i=1}^s F_{p-p_i}\mathscr{R}.Fl'_{q_i}\mathscr{M} \\ &\subset \sum_{i=1}^s Fl'_{p-p_i+q_i}\mathscr{M} \subset Fl'_{p+m}\mathscr{M}. \end{aligned}$$

This proves the first assertion. □

Remark We see that finite type filtrations behave very well. In many materials, they are called *good filtrations*. In the ring theory, good filtrations are used to prove the following useful fact.

2.2.15 Theorem *Let (R, F) be a filtered ring. Then it is* filtered noetherian *(that means any filtered submodule of a finite type filtered module is of finite type) if and only if the Rees ring $F_\bullet R$ is a noetherian graded ring if and only if the associated graded ring $\mathrm{gr}_\bullet R$ is a noetherian graded ring. If this is the case, then R is noetherian.*

PROOF (SKETCH): The critical thing one need to show is that the good filtration, if it exists, is unique up to equivalence of filtrations. Then the statement is clear. □

2.2.16 Corollary *A graded ring $R_\bullet$ is noetherian if and only if its total sheaf R is noetherian.*

PROOF: The total sheaf R admits a standard filtration $F_pR = \bigoplus_{q \leqslant p} R_q$ whose associated graded ring is again $R_\bullet$. □

However, in the context of sheaf theory, the noetherian property is not merely an analogy of that in ring theory.

2.3 Good filtrations

2.3.1 Definition Let $(\mathscr{R}, F)$ be a filtered ring and $(\mathscr{M}, Fl)$ a filtered $\mathscr{R}$-module. Then, we say

- Fl is *good* if the total sheaf of the Rees module $Fl_\bullet \mathscr{M}$ is of finite type over the total sheaf of the Rees ring $F_\bullet \mathscr{R}$;
- Fl is *of finite presentation* (resp. *coherent*, *noetherian*) if the total sheaf of the Rees module $Fl_\bullet \mathscr{M}$ is of finite presentation (resp. coherent, noetherian) over the total sheaf of the Rees ring $F_\bullet \mathscr{R}$.

The behavior of finite presentation (resp. coherent, noetherian) filtrations is not good in general. To go further, we need some assumptions on the filtered ring $(\mathscr{R}, F)$.

2.3.2 Theorem *Let $(\mathscr{R}, F)$ be a filtered ring. Suppose*

(i) $F_0\mathscr{R}$ is a noetherian ring,
(ii) each $\mathrm{gr}_p \mathscr{R}$ is coherent over $F_0\mathscr{R}$ and
(iii) for any open U, if an ideal $\mathscr{I}$ of $\mathscr{R}_{|U}$ has the property that each $F_p\mathscr{I}$ is coherent over $F_0\mathscr{R}_{|U}$ then $\mathscr{I}$ is of finite type over $\mathscr{R}_{|U}$.

Then $\mathscr{R}$ is a noetherian ring.

Proof: First note that the condition (ii) implies that each $F_p\mathscr{R}$ is coherent over $F_0\mathscr{R}$. Then $\mathscr{R}$, as well as $\mathscr{R}^r$ for any finite r, is *pseudo-coherent*.

2.3.3 Lemma *For any open U and any submodule $\mathscr{M}$ of $\mathscr{R}^r_{|U}$, $\mathscr{M}$ is of finite type over $\mathscr{R}_{|U}$ if and only if each $\mathscr{M} \cap (F_p\mathscr{R}_{|U})^r$ is coherent over $F_0\mathscr{R}_{|U}$.*

Proof: The problem is local, so we may assume $U = X$. Suppose $\mathscr{M}$ is of finite type over $\mathscr{R}$. Then it is generated by a finite type $F_0\mathscr{R}$-submodule $\mathscr{M}_0$ of $\mathscr{R}^r$. Since $\mathscr{R}^r$ is pseudo-coherent and $F_0\mathscr{R}$ is a coherent ring, $\mathscr{M}_0$ is coherent. Then each $\mathscr{M}_q := (F_q\mathscr{R})\mathscr{M}_0$ is coherent and hence so is each $\mathscr{M}_q \cap (F_p\mathscr{R})^r$. Note that $\mathscr{M}_\bullet$ is an exhaustive $\mathscr{R}$-filtration and in particular an exhaustive ascending chain of submodules of $\mathscr{M}$. Then $\mathscr{M}_\bullet \cap (F_p\mathscr{R})^r$ is an ascending chain of coherent submodules of $(F_p\mathscr{R})^r$ whose colimit is $\mathscr{M} \cap (F_p\mathscr{R})^r$. Since $(F_q\mathscr{R})^r$ is coherent over a noetherian ring $F_0\mathscr{R}$, it is noetherian. Then this chain must be locally stationary. By shrinking U, we may assume it is stationary. Then $\mathscr{M} \cap (F_p\mathscr{R})^r = \mathscr{M}_q \cap (F_p\mathscr{R})^r$ for some large q and hence is coherent.

We prove the converse by induction on r. Then $r = 1$ is precisely the condition (iii). Suppose this property holds for any number less than r. Let $\mathscr{M}$ be a submodule of $\mathscr{R}^r$ such that each $\mathscr{M} \cap (F_p\mathscr{R})^r$ is coherent over $F_0\mathscr{R}$. Let i be inclusion $\mathscr{R} \to \mathscr{R}^r$ to the first factor. Let $\mathscr{I}$ be the kernel of the composition $\mathscr{R} \xrightarrow{i} \mathscr{R}^r \longrightarrow \mathscr{R}^r/\mathscr{M}$. Then we have an injective morphism

$i_0 \colon \mathscr{I} \to \mathscr{M}$, which is precisely the pullback of i along $\mathscr{M} \subset \mathscr{R}^r$. Let $\mathscr{N}$ be the cokernel of it. Then we have the following morphism of exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{I} \arrow[r, "i_0"] \arrow[d] & \mathscr{M} \arrow[r] \arrow[d] & \mathscr{N} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{R} \arrow[r, "i"] & \mathscr{R}^r \arrow[r] & \mathscr{R}^{r-1} \arrow[r] & 0 \end{tikzcd}$$

Since left square is a Cartesian square, the morphism $\mathscr{N} \to \mathscr{R}^{r-1}$ is injective. Then, we can identify $\mathscr{N}$ with a submodule of $\mathscr{R}^{r-1}$. Note that the above constructions are compatible with taking intersection of $F_p\mathscr{R}$. That is, for each p, we have a morphism of exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{I} \cap (F_p\mathscr{R}) \arrow[r, "i_0"] \arrow[d] & \mathscr{M} \cap (F_p\mathscr{R})^r \arrow[r] \arrow[d] & \mathscr{N} \cap (F_p\mathscr{R})^{r-1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F_p\mathscr{R} \arrow[r, "i"] & (F_p\mathscr{R})^r \arrow[r] & (F_p\mathscr{R})^{r-1} \arrow[r] & 0 \end{tikzcd}$$

where the left square is a Cartesian square. In this square, $F_p\mathscr{R}$, $(F_p\mathscr{R})^r$ and $\mathscr{M} \cap (F_p\mathscr{R})^r$ are coherent, hence so is $\mathscr{I} \cap (F_p\mathscr{R})$. In the upper exact sequence, both $\mathscr{M} \cap (F_p\mathscr{R})^r$ and $\mathscr{I} \cap (F_p\mathscr{R})$ are coherent, hence so is $\mathscr{N} \cap (F_p\mathscr{R})^{r-1}$. Then, by inductive hypothesis, $\mathscr{I}$ and $\mathscr{N}$ are of finite type over $\mathscr{R}$, hence so is $\mathscr{M}$. □

2.3.4 Lemma *Let $\mathscr{M}$ be a finite type $\mathscr{R}$-module. If $\mathscr{M}$ is pseudo-coherent over $F_0\mathscr{R}$, then it is of finite presentation over $\mathscr{R}$.*

PROOF: We may assume there is an epimorphism $\varphi \colon \mathscr{R}^r \to \mathscr{M}$. Then we have $(F_p\mathscr{R})^r \cap \ker(\varphi) = \ker((F_p\mathscr{R})^r \xrightarrow{\varphi} \mathscr{M})$. Since $\mathscr{M}$ is pseudo-coherent and $(F_p\mathscr{R})^r$ is coherent, $(F_p\mathscr{R})^r \cap \ker(\varphi)$ is coherent over $F_0\mathscr{R}$. Therefore $\ker(\varphi)$ is of finite type by Lemma 2.3.3. □

2.3.5 Lemma *The ring $\mathscr{R}$ is coherent.*

PROOF: Let $\mathscr{I}$ be a finite type ideal of $\mathscr{R}$. By Lemma 2.3.3, each $\mathscr{I} \cap (F_p\mathscr{R})^r$ is coherent over $F_0\mathscr{R}$. Hence $\mathscr{I}$ is pseudo-coherent over $F_0\mathscr{R}$. Then it is of finite presentation over $\mathscr{R}$ by Lemma 2.3.4. □

Now, we go back to prove Theorem 2.3.2. To show $\mathscr{R}$ is noetherian at each stalk, we recall the following lemma from algebra:

2.3.6 Lemma *Let (R, F) be a filtered ring. Suppose*

(i) F_0R is a noetherian ring,
(ii) each $\operatorname{gr}_p R$ is finite type over F_0R and

(iii) if an ideal I of R has the property that if each F_pI is of finite type over F_0R then I is of finite type over R.

Then R is a noetherian ring.

PROOF: Let $I_\bullet$ be an ascending chain of finite type ideals of R and let I be their colimit. Then each $I_q \cap F_pR$ is of finite type over F_0R. Since F_pR is noetherian over F_0R, the chain $I_\bullet \cap F_pR$ is stationary and thus $I \cap F_pR$ is of finite type over F_0R. Then I is of finite type over R. Therefore the chain $I_\bullet$ must be stationary. □

It is straightforward to verify that each stalk of $(\mathscr{R}, F)$ satisfies the conditions in the above lemma.

Let $\mathscr{I}_\bullet$ be an ascending chain of coherent ideals of $\mathscr{R}$. It remains to show it is locally stationary. Let $\mathscr{I}$ be their colimit, which is pseudo-coherent. Since each $\mathscr{I}_q$ is coherent over $\mathscr{R}$, by Lemma 2.3.3, $\mathscr{I}_q \cap F_p\mathscr{R}$ is coherent over $F_0\mathscr{R}$. Then $\mathscr{I}_\bullet \cap F_p\mathscr{R}$ is an ascending chain of coherent $F_0\mathscr{R}$-submodule of $F_p\mathscr{R}$, hence is locally stationary. Then $\mathscr{I} \cap F_p\mathscr{R}$ is coherent over $F_0\mathscr{R}$. Then, by Lemma 2.3.3, $\mathscr{I}$ is of finite type over $\mathscr{R}$, and thus coherent over $\mathscr{R}$. Then the chain $\mathscr{I}_\bullet$ has to be locally stationary. This proves $\mathscr{R}$ is a noetherian ring. □

Let's use Theorem 2.3.2 to prove the following generalization of famous *Hilbert's basis theorem.*

2.3.7 Theorem *If $\mathscr{R}$ is noetherian, then so is $\mathscr{R}[T]$.*

PROOF: We define the filtration by

$$F_p\mathscr{R}[T] = \bigoplus_{i=0}^{p} \mathscr{R}T^i.$$

That is, we consider the *degree filtration*. Then each $\operatorname{gr}_p \mathscr{R}[T]$ is isomorphic to $\mathscr{R}$ as an $\mathscr{R}$-module, hence is coherent. It remains to show condition (iii) in Theorem 2.3.2.

Let $\mathscr{I}$ be an ideal of $\mathscr{R}[T]$ such that each $F_p\mathscr{I}$ is coherent over $\mathscr{R}$. Then each $\operatorname{gr}_p \mathscr{I}$ is coherent over $\mathscr{R}$. Since $\operatorname{gr}_p \mathscr{I} \subset \operatorname{gr}_p \mathscr{R}[T] = \mathscr{R}T^p$. We obtain an ascending chain of coherent ideals of $\mathscr{R}$:

$$\cdots \subset T^{1-p} \operatorname{gr}_{p-1} \mathscr{I} \subset T^{-p} \operatorname{gr}_p \mathscr{I} \subset \cdots .$$

Then locally there exists an integer P such that

$$T^{1-p} \operatorname{gr}_{p-1} \mathscr{I} = T^{-p} \operatorname{gr}_p \mathscr{I}$$

for any $p > P$. Then $F_p\mathscr{I} \subset TF_{p-1}\mathscr{I} + F_{p-1}\mathscr{I}$ for any $p > P$. Therefore $F_p\mathscr{I} \subset \mathscr{R}[T]F_P\mathscr{I}$ for any $p > P$ and thus $\mathscr{I} = \mathscr{R}[T]F_P\mathscr{I}$. Then it is clear that $\mathscr{I}$ is of finite type over $\mathscr{R}[T]$ □

2.3.8 Corollary *If $\mathscr{R}$ is noetherian, then so is $\mathscr{R}[T_1, \cdots, T_r]$.*

This doesn't imply that if $\mathscr{R}$ is noetherian, then any $\mathscr{R}$-algebra of finite type is noetherian. However, we have

2.3.9 Corollary *If $\mathscr{R}$ is noetherian, then any $\mathscr{R}$-algebra of finite presentation is noetherian.*

PROOF: Recall that an $\mathscr{R}$-algebra of finite presentation is a quotient of the polynomial ring $\mathscr{R}[T_1, \cdots, T_r]$ for some finite r by a finite type ideal. Then, apply Corollary 2.1.12. $\square$

The following theorem is also useful in many situations and is important to study filtrations.

2.3.10 Theorem *Let $(\mathscr{R}, F)$ be a filtered ring. Suppose*

(i) $F_0\mathscr{R}$ is a noetherian ring,
(ii) each $\operatorname{gr}_p \mathscr{R}$ is of finite type over $F_0\mathscr{R}$ and
(iii) the total sheaf of $\operatorname{gr}_\bullet \mathscr{R}$ is a noetherian ring.

Then $(\mathscr{R}, F)$ satisfies the conditions of Theorem 2.3.2. Consequently, $\mathscr{R}$ is noetherian. Moreover, the total sheaf of the Rees ring $F_\bullet\mathscr{R}$ is noetherian.

PROOF: The condition (i) is precisely the same as in Theorem 2.3.2.

Let's prove $(\mathscr{R}, F)$ satisfies condition (ii) of Theorem 2.3.2. We may assume there exists an epimorphism of $F_0\mathscr{R}$-modules

$$\varphi\colon (F_0\mathscr{R})^r \longrightarrow \operatorname{gr}_p \mathscr{R}$$

with r a finite number. Note that this extends to a morphism of $\operatorname{gr}_\bullet \mathscr{R}$-modules

$$\widetilde{\varphi}\colon (\operatorname{gr}_\bullet \mathscr{R})^r \longrightarrow \operatorname{gr}_\bullet \mathscr{R}.$$

Then, since $\operatorname{gr}_\bullet \mathscr{R}$ is a coherent ring, $\ker(\widetilde{\varphi})_\bullet$ is of finite type. Suppose there is an epimorphism of $\operatorname{gr}_\bullet \mathscr{R}$-modules

$$(\operatorname{gr}_\bullet \mathscr{R})^s \longrightarrow \ker(\widetilde{\varphi})_\bullet$$

with s a finite number. Then there is an epimorphism of $F_0\mathscr{R}$-modules

$$\bigoplus_{i=1}^{s} \operatorname{gr}_{q_i} \mathscr{R} \longrightarrow \ker(\widetilde{\varphi})_0.$$

Since each $\operatorname{gr}_{q_i} \mathscr{R}$ is of finite type over $F_0\mathscr{R}$, the above epimorphism induces an epimorphism of $F_0\mathscr{R}$-modules

$$(F_0\mathscr{R})^t \longrightarrow \ker(\varphi)$$

with t a finite number. This shows $\operatorname{gr}_p \mathscr{R}$ is of finite presentation over $F_0\mathscr{R}$. Then it is coherent since $F_0\mathscr{R}$ is a coherent ring.

Let's prove $(\mathscr{R}, F)$ satisfies condition (iii) of Theorem 2.3.2. Let $\mathscr{I}$ be an ideal of $\mathscr{R}$ such that each $F_p\mathscr{I}$ is coherent over $F_0\mathscr{R}$. Then each $\operatorname{gr}_p \mathscr{I}$ is coherent over $F_0\mathscr{R}$. Then each $(\operatorname{gr}_\bullet \mathscr{R})(\operatorname{gr}_p \mathscr{I})$ is a coherent ideal of $\operatorname{gr}_\bullet \mathscr{R}$. Since $\operatorname{gr}_\bullet \mathscr{R}$ is a noetherian ring, $\operatorname{gr}_\bullet \mathscr{I} = \sum_p (\operatorname{gr}_\bullet \mathscr{R})(\operatorname{gr}_p \mathscr{I})$ is also a coherent ideal. In particular, it is of finite type. Then, $\mathscr{I}$ is of finite type over $\mathscr{R}$ by Propositions 2.2.10 and 2.2.12.

Let $\mathscr{F}$ be the total sheaf of the Rees ring $F_\bullet\mathscr{R}$. We identify it with a subring of $\mathscr{R}[T]$ by putting

$$\mathscr{F}_p = (F_p\mathscr{R})T^p$$

Then we give a ring filtration on $\mathscr{F}$ by putting

$$Fl_p\mathscr{F} = \bigoplus_k \Big(\sum_{q\leqslant k,p} F_q\mathscr{R}\Big)T^k.$$

Then we have

$$\operatorname{gr}_p \mathscr{F} = \bigoplus_{k\geqslant p} (\operatorname{gr}_p \mathscr{R})T^k.$$

In particular,

$$Fl_0\mathscr{F} = \bigoplus_{k\geqslant 0} (F_0\mathscr{R})T^k = (F_0\mathscr{R})[T],$$

which is a noetherian ring by Theorem 2.3.7.

On the other hand, we have

$$\begin{aligned}
\operatorname{gr} \mathscr{F} &= \bigoplus_p \operatorname{gr}_p \mathscr{F} \\
&= \bigoplus_p \bigoplus_{k\geqslant p} (\operatorname{gr}_p \mathscr{R})T^k \\
&= \bigoplus_k \Big(\bigoplus_p (\operatorname{gr}_p \mathscr{R})T^p\Big)T^k \\
&\cong \bigoplus_k (\operatorname{gr} \mathscr{R})T^k = (\operatorname{gr} \mathscr{R})[T],
\end{aligned}$$

which is a noetherian ring by Theorem 2.3.7.

Under the above identification, we have

$$\operatorname{gr}_p \mathscr{F} = \bigoplus_k ((\operatorname{gr}_p \mathscr{R})T^p)T^k \cong \bigoplus_k (\operatorname{gr}_p \mathscr{R})T^k = (Fl_0\mathscr{F})(\operatorname{gr}_p \mathscr{R}).$$

Thence $\operatorname{gr}_p \mathscr{F}$ is of finite type over $Fl_0\mathscr{F}$. $\square$

Under the assumption of Theorem 2.3.10, the notion *coherent filtration* behaves very well. In the rest of this subsection, the filtered ring $(\mathscr{R}, F)$ is assumed to satisfying the conditions of Theorem 2.3.10.

2.3.11 Proposition *A filtration is noetherian if and only if it is coherent if and only if it is of finite presentation.*

Proof: Because $F_\bullet$ is noetherian. □

So we don't distinguish those notions and simply call it *coherent filtration*.

2.3.12 Proposition *Let $(\mathscr{M}, Fl)$ be a coherent filtered $\mathscr{R}$-module and $(\mathscr{N}, Fl')$ a filtered submodule of $(\mathscr{M}, Fl)$. If each $Fl'_p\mathscr{N}$ is of finite type over $F_0\mathscr{R}$, then Fl' is a coherent filtration.*

Proof: For each p, define:

$$Fl_q^{(p)}\mathscr{N}^{(p)} := \begin{cases} Fl'_q\mathscr{N} & (q \leqslant p), \\ F_{q-p}Fl'_p\mathscr{N} & (q > p). \end{cases}$$

Then each $Fl_\bullet^{(p)}\mathscr{N}^{(p)}$ is a finite type submodule of $Fl_\bullet\mathscr{M}$, hence is coherent over $F_\bullet\mathscr{R}$. Then $\{Fl_\bullet^{(p)}\mathscr{N}^{(p)}\}_p$ form an ascending chain of coherent submodules of $Fl_\bullet\mathscr{M}$. Since $F_\bullet\mathscr{R}$ is noetherian, $Fl_\bullet\mathscr{M}$ is noetherian. Then the chain is locally stationary. Therefore its colimit $Fl'_\bullet\mathscr{N}$ is coherent. □

By Propositions 2.2.10 and 2.3.11, we have

2.3.13 Corollary *If an $\mathscr{R}$-module $\mathscr{M}$ has a coherent filtration, then it is coherent.*

Conversely, we have

2.3.14 Proposition *Let $\mathscr{M}$ be a coherent $\mathscr{R}$-module. Then*

(i) locally, $\mathscr{M}$ has a coherent filtration;
(ii) $\mathscr{M}$ is pseudo-coherent over $F_0\mathscr{R}$.

Proof: Locally, we have a short exact sequence

$$0 \longrightarrow \mathscr{N} \longrightarrow \mathcal{R}^r \longrightarrow \mathscr{M} \longrightarrow 0.$$

Then $\mathscr{N} \cap (F_p\mathscr{R})^r$ defines a coherent filtration on $\mathscr{N}$. Indeed, since $\mathscr{N}$ is of finite type, by Lemma 2.3.3, each $\mathscr{N} \cap (F_p\mathscr{R})^r$ is coherent hence of finite type over $F_0\mathscr{R}$. This defines a coherent filtration by Proposition 2.3.12. Then $\mathscr{M}$ has a coherent filtration Fl.

By the short exact sequence

$$0 \longrightarrow \mathscr{N} \cap (F_\bullet\mathscr{R})^r \longrightarrow (F_\bullet\mathcal{R})^r \longrightarrow Fl_\bullet\mathscr{M} \longrightarrow 0,$$

each $Fl_p\mathscr{M}$ is coherent over $F_0\mathscr{R}$. Hence $\mathscr{M}$, as the colimit of them, is pseudo-coherent over $F_0\mathscr{R}$. □

2.3.15 Proposition *Let $\mathscr{M}$ be a coherent $\mathscr{R}$-module and $\mathscr{N}$ a submodule of $\mathscr{M}$. If Fl is a good filtration on $\mathscr{N}$. Then Fl is a coherent filtration on $\mathscr{N}$.*

PROOF: Let Fl' be a coherent filtration on $\mathscr{M}$. Then by Proposition 2.2.14, we have

$$Fl_p\mathscr{N} \subset Fl'_p\mathscr{M}$$

for any p after certain shifting of Fl'. Then $Fl_\bullet\mathscr{N}$ is a finite type submodule of the coherent module $Fl'_\bullet\mathscr{M}$, hence is coherent. $\square$

2.3.16 Proposition *Let $0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$ be a short exact sequence of coherent $\mathscr{R}$-modules and let Fl be a coherent filtration on $\mathscr{M}_2$. Then the induced filtration on $\mathscr{M}_1$ and quotient filtration on $\mathscr{M}_3$ are coherent.*

PROOF: The quotient filtration on $\mathscr{M}_3$ is a good filtration. Hence it is a coherent filtration by Proposition 2.3.15. Then the induced filtration on $\mathscr{M}_1$ is coherent. $\square$

A filtration Fl on an $\mathscr{R}$-module $\mathscr{M}$ is said to be *locally stable*, if locally there exists a q_0 such that

$$F_p\mathscr{R}.Fl_q\mathscr{M} = Fl_{p+q}\mathscr{M}$$

for all $p \geqslant 0$ and $q \geqslant q_0$. The following characterization is useful in the case of $\mathscr{D}$-modules.

2.3.17 Proposition *Suppose $(\mathscr{R}, F)$ satisfies that the canonical morphism*

$$\mathbb{S}^\bullet_{F_0\mathscr{R}}(\mathrm{gr}_1\,\mathscr{R}) \longrightarrow \mathrm{gr}_\bullet\,\mathscr{R}$$

is surjective. Let $\mathscr{M}$ be an $\mathscr{R}$-module and Fl a filtration on $\mathscr{M}$. Then the followings are equivalent:

(i) Fl is a good filtration.
(ii) Fl satisfies the following:

1. *each $Fl_p\mathscr{M}$ is of finite type over $F_0\mathscr{R}$;*
2. *Fl is locally stable.*

PROOF: Suppose (i), we may assume there is an epimorphism

$$\bigoplus_{i=1}^{s} F_\bullet^{[n_i]}\mathscr{R} \longrightarrow Fl_\bullet\mathscr{M}.$$

Since each $F_{p+n_i}\mathscr{R}$ is of finite type over $F_0\mathscr{R}$, $Fl_p\mathscr{M}$ is of finite type over $F_0\mathscr{R}$. Since Fl is a good filtration, $\mathscr{M}$ has a system of local generators of type $(p_1, \cdots, p_s)$, then we have

$$Fl_p\mathscr{M} = \sum_{i=1}^{s} F_{p-p_i}\mathscr{R}.Fl_{p_i}\mathscr{M},$$

and hence

$$\operatorname{gr}_p \mathscr{M} = \sum_{i=1}^{s} \operatorname{gr}_{p-p_i} \mathscr{R}. \operatorname{gr}_{p_i} \mathscr{M}.$$

Let $p_0 = \max(p_1, \cdots, p_s)$. Then the surjectivity of the canonical morphism implies that $\operatorname{gr}_{p-p_i} \mathscr{R} = (\operatorname{gr}_1 \mathscr{R})^{p-p_i}$ for $p \geqslant p_0$. Then, we have

$$\operatorname{gr}_p \mathscr{M} = \operatorname{gr}_{p-p_0} \mathscr{R}. \operatorname{gr}_{p_0} \mathscr{M}, \qquad \forall p \geqslant p_0.$$

Hence

$$Fl_p\mathscr{M} = F_{p-p_0}\mathscr{R}.Fl_{p_0}\mathscr{M}, \qquad \forall p \geqslant p_0,$$

by induction on p.

Conversely, suppose (ii). Then there is a p_0 such that

$$Fl_p\mathscr{M} = F_{p-p_0}\mathscr{R}.Fl_{p_0}\mathscr{M}$$

for $p \geqslant p_0$. Since $Fl_{p_0}\mathscr{M}$ is of finite type over $F_0\mathscr{R}$, we may assume there is a system of generators $e_1, \cdots, e_s$ of $Fl_{p_0}\mathscr{M}$ over $F_0\mathscr{R}$. Then it is clear that $e_1, \cdots, e_s$ form a system of generators of $\mathscr{M}$ of type $(p, p, \cdots, p)$. □

2.3.18 Proposition *Suppose $(\mathscr{R}, F)$ satisfies that the canonical morphism*

$$\mathbb{S}^{\bullet}_{F_0\mathscr{R}}(\operatorname{gr}_1 \mathscr{R}) \longrightarrow \operatorname{gr}_{\bullet} \mathscr{R}$$

is surjective. Let $\mathscr{M}$ be an $\mathscr{R}$-module and Fl a filtration on $\mathscr{M}$. Then the followings are equivalent:

(i) Fl is a coherent filtration.

(ii) Fl satisfies the following:

1. *each $Fl_p\mathscr{M}$ is coherent over $F_0\mathscr{R}$;*
2. *Fl is locally stable.*

Proof: Suppose (i), each $Fl_p\mathscr{M}$ is of finite type over $F_0\mathscr{R}$, hence coherent by Proposition 2.3.14.(ii). The proof of local stability is the same as before.

Suppose (ii), Fl is a good filtration and hence $\mathscr{M}$ is of finite type. Since each $Fl_p\mathscr{M}$ is coherent over $F_0\mathscr{R}$, $\mathscr{M}$ is pseudo-coherent over $F_0\mathscr{R}$. Then $\mathscr{M}$ is of finite presentation over $\mathscr{R}$ by Lemma 2.3.4. Hence $\mathscr{M}$ is coherent. Then Fl is a coherent filtration by Proposition 2.3.15. □

2.4 The symplectic structure of the cotangent bundle

Note that since $[F_p\mathscr{D}, F_q\mathscr{D}] \subset F_{p+q-1}\mathscr{D}$, for any differential operators P and Q, we have

$$\operatorname{ord}([P,Q]) \leqslant \operatorname{ord}(P) + \operatorname{ord}(Q) - 1.$$

Therefore $(P,Q) \mapsto \sigma([P,Q])$ defines homomorphisms

$$F_p\mathscr{D} \otimes F_q\mathscr{D} \longrightarrow \mathscr{H}om(\mathbb{S}^{p+q-1}(\mathscr{O}), \mathscr{O})$$

factorizing through $\operatorname{gr}_{p+q-1}(\mathscr{D})$ and annihilating $F_{p-1}\mathscr{D} \otimes F_q\mathscr{D}$ as well as $F_p\mathscr{D} \otimes F_{q-1}\mathscr{D}$. Therefore they induces homomorphisms

$$\operatorname{gr}_p(\mathscr{D}) \otimes \operatorname{gr}_q(\mathscr{D}) \longrightarrow \operatorname{gr}_{p+q-1}(\mathscr{D}).$$

In this way, we obtain a graded binary operation of degree -1

$$\{-,-\}\colon \operatorname{gr}_\bullet(\mathscr{D}) \otimes \operatorname{gr}_\bullet(\mathscr{D}) \longrightarrow \operatorname{gr}_\bullet(\mathscr{D}) \tag{2.4.1}$$

satisfying

$$\{\sigma(P), \sigma(Q)\} = \sigma([P,Q])$$

for arbitrary differential operators P and Q. Note that it has the following properties

(a) $\{-,-\}$ makes $\operatorname{gr}(\mathscr{D})$ a Lie algebra.

(b) For any sections f, g, h of $\operatorname{gr}(\mathscr{D})$, we have the *Leibniz rule*:

$$\{f, gh\} = \{f,g\}h + g\{f,h\}.$$

In this way, we get a *Coisson algebra* (i.e. *commutative Poisson algebra*) structure on $\operatorname{gr}(\mathscr{D})$.

Remark A *Poisson algebra* is an associative algebra A equipped with a binary bracket $\{-,-\}$ such that

(P1) $\{-,-\}$ makes A a Lie algebra.

(P2) For any $f, g, h \in A$, we have the *Leibniz rule*:

$$\{f, gh\} = \{f,g\}h + g\{f,h\}.$$

A *Coisson algebra* is a Poisson algebra whose underlying associative algebra is commutative.

The Coisson structure can be obtained in another way: the *symplectic structure* of the *cotangent bundle* T^*M.

Like the *tangent bundle* is the vector bundle associated to the locally free $\mathscr{O}_M$-module Θ_M, the cotangent bundle is the vector bundle $\pi\colon T^*M \to M$ associated to the locally free $\mathscr{O}_M$-module Ω^1_M. Let $(z^i)_{1\leqslant i\leqslant m}$ be a local coordinate system of M on a chart U. Since $(dz^i)_{1\leqslant i\leqslant m}$ is a basis of Ω^1_M on U, we see that T^*M has a local coordinate system $(z^i;\xi^j)_{1\leqslant i,j\leqslant m}$ on chart $\pi^{-1}(U)$ such that: any local section $\omega = \sum_{i=1}^m f_i dz^i$ of the sheaf Ω^1_M on U is corresponding to the following section of the projection π on U:

$$z = (z^1, z^2, \cdots, z^m) \longmapsto (z,\omega) := (z^1, z^2, \cdots, z^m; f_1, f_2, \cdots, f_m)$$

written under this coordinate system.

Note that, for any morphism $\varphi\colon N \to M$, we have a canonical homomorphism $\varphi^*\Omega^1_M \to \Omega_N$, hence a morphism $N \times_M T^*M \to T^*N$. Taking φ to be $\pi\colon T^*M \to M$, we obtained a morphism

$$\pi_\pi\colon T^*M \times_M T^*M \to T^*T^*M.$$

Then we get a section of the projection $\pi\colon T^*T^*M \to T^*M$ by compositing above with the diagonal morphism $T^*M \to T^*M \times_M T^*M$. In other words, we obtain a canonical 1-form α_M on T^*M, called the *tautological form*. Let $\omega_M = -d\alpha_M$. Note that, under a local coordinate system $(z^i;\xi^j)_{1\leqslant i,j\leqslant m}$, these forms can be written as

$$\alpha_M = \sum_{i=1}^m \xi^i dz^i, \qquad \omega_M = \sum_{i=1}^m dz^i \wedge d\xi^i. \tag{2.4.2}$$

Then it is clear that ω_M is a *symplectic form* on T^*M, called the *canonical symplectic form*.

Remark (A little bit sympletic geometry) On a manifold of even dimension $2m$, a *symplectic form* is a closed 2-form σ such that $\omega^m = \omega \wedge \omega \wedge \cdots \wedge \omega$ vanishes nowhere (hence is a volume form).

On a vector space, a *symplectic structure* is a skew-symmetric non-degenerate bilinear form. A closed 2-form is symplectic if and only if it induces a symplectic strucutre on the tangent space at every point.

Given a *symplectic manifold*, that is a manifold M with a symplectic form ω, we can obtain a Poisson algebra structure on $\mathscr{O}_M$ as follows. First note that, at each point x, the skew-symmetric non-degenerate bilinear form ω_x gives a linear isomorphism from T_xM to T^*_xM. Consequently, we have a linear isomorphism

$$H\colon \Theta_M \to \Omega^1_M.$$

Let f be a section of $\mathscr{O}_M$, then its *Hamiltonian vector field* is the unique vector field H_f such that

$$H_f = H(df).$$

For any two sections of $\mathscr{O}_M$, their *Poisson bracket* is

$$\{f, g\} := H_f(g).$$

Then, one can verify that this does define a Coisson algebra structure on $\mathscr{O}_M$ and that

$$[H_f, H_g] = H_{\{f,g\}}.$$

Or, one can deduce aboves by verifying that

$$\{f, g\} = \omega_M(H_g, H_f).$$

Now, we have a sympletic manifold (T^*M, ω_M). Under a local coordinate system $(z^i; \xi^j)_{1 \leqslant i,j \leqslant m}$, we have

$$H_{z^i} = -\partial_{\xi^i}, \qquad H_{\xi^i} = \partial_{z^i}.$$

Therefore the Poisson bracket on T^*M can be written as

$$\{f, g\} = \sum_{i=1}^{m} \left(\frac{\partial f}{\partial \xi^i} \frac{\partial g}{\partial z^i} - \frac{\partial f}{\partial z^i} \frac{\partial g}{\partial \xi^i}\right). \tag{2.4.3}$$

To see how the above constructions relate to (2.4.1), consider follows. First, Θ_M *can be identified with functions on T^*M which are linear in the fibers.* Indeed, given a section s of $\pi \colon T^*M \to M$ with ω_s the corresponding 1-form on M, then a vector field θ acts on a point $y \in s(M)$ as evaluating the function $\langle \omega_s, \theta \rangle$ at $\pi(y)$. Conversely, if f is a function on T^*M which is linear in the fibers, then for each 1-form ω_s with corresponding section s of π, we define the action of f on ω_s as

$$(f.\omega_s)(x) = f(s(x)).$$

The linear in fibers property implies that such action is $\mathscr{O}_M$-linear, hence defines a derivation on M. Furthermore, we obtain a monomorphism

$$\mathbb{S}_{\mathscr{O}_M}(\Theta_M) \longrightarrow \pi_* \mathscr{O}_{T^*M} \tag{2.4.4}$$

and hence can identify the first with its image in the later, which precisely consists of the sections which are *polynomials in the fibers*, i.e. sections of the polynomial algebra $\mathscr{O}_M[\xi^1, \cdots, \xi^m]$. Then, since (1.1.5) is a graded isomorphism, we obtain a homomorphism of $\mathscr{O}_M$-modules

$$\mathscr{D}_M \longrightarrow \mathrm{gr}_\bullet(\mathscr{D}_M) \cong \mathbb{S}_{\mathscr{O}_M}(\Theta_M) \longrightarrow \pi_* \mathscr{O}_{T^*M}.$$

Now, the Coisson algebra structure (2.4.1) on $\mathrm{gr}_\bullet(\mathscr{D}_M)$ is precisely that pullback from one on the symplectic manofold T^*M.

Remark One can also obtain above constructions as follows. Let φ be a section of $\mathscr{H}om(\mathbb{S}^p(\mathscr{O}_M), \mathscr{O}_M)$, define its action on a point $y \in s(M)$ as the function

$$y \longmapsto \frac{1}{p!}\varphi(\ell_\xi(y), \ell_\xi(y), \cdots, \ell_\xi(y))(\pi(y)),$$

where $\ell_\xi \colon T^*M \to \mathscr{O}_M$ acts on y as

$$\ell_\xi(y) := \sum_{i=1}^m \xi^i(y) z^i.$$

Above construction applies to those from differential operators of order p, hence induces a homomorphism of $\mathscr{O}_M$-modules

$$F_p\mathscr{D}_M \longrightarrow \mathscr{H}om(\mathbb{S}^p(\mathscr{O}_M), \mathscr{O}_M) \longrightarrow \pi_*\mathscr{O}_{T^*M}.$$

The image of a differential operator P is also called the *p-th symbol* and denoted by $\sigma_p(P)$. Then we also have the notion of *principal symbol* and the notation $\sigma(P)$. (We have already defined variants of the notion of *symbols of differential operators*, they are closely related to each other and share the same terminology and notations. However, ambiguity can be avoided if where those symbols take values are clear.)

It is straightforward to show that if $\varphi = \sigma(\theta_1\theta_2\cdots\theta_p)$, where $\theta_1, \theta_2, \cdots, \theta_p$ are vector fields on M, then

$$\varphi.y = \theta_1(\ell_\xi(y))\theta_2(\ell_\xi(y))\cdots\theta_p(\ell_\xi(y)), \quad \forall y \in T^*M.$$

It is also clear that for any vector field θ on M,

$$\theta(\ell_\xi(y)) = \theta.y, \quad \forall y \in T^*M,$$

where the action of θ on y is defined as before. In this way, we obtain a sequence of homomorphisms of $\mathscr{O}_M$-modules

$$\mathbb{S}^p_{\mathscr{O}_M}(\Theta_M) \longrightarrow \mathrm{gr}_p(\mathscr{D}_M) \longrightarrow \mathscr{H}om_{\mathscr{O}_M}(\mathbb{S}^p(\mathscr{O}_M), \mathscr{O}_M) \longrightarrow \pi_*\mathscr{O}_{T^*M}$$

whose composition agree with p-th component of (2.4.4). Note that this gives another way to show (1.1.5) is injective even without knowing $\mathscr{D}_M$ is generated by $\mathscr{O}_M \oplus \Theta_M$ and applies to singular case (i.e. complex analytic spaces which are not manifolds).

Any way, we can conclude that

2.4.5 Theorem *We have isomorphisms of graded $\mathscr{O}_M$-algebras*

$$\mathrm{gr}_\bullet(\mathscr{D}_M) \cong \mathbb{S}^\bullet_{\mathscr{O}_M}(\Theta_M) \cong \mathscr{O}_M[\xi^1, \cdots, \xi^m].$$

2.4.6 Corollary $\mathrm{gr}_\bullet(\mathscr{D}_M)$ *is a noetherian ring. Moreover, $\mathscr{D}$ is noetherian.*

Proof: $\mathscr{O}_M$ is noetherian by *Rückert Basis Theorem*. Then the statements follow from Theorems 2.4.5, 2.3.7 and 2.3.10. ☐

2.5 Characteristic varieties

2.6 De Rham and Spencer

§ 3 Homological algebra on $\mathscr{D}$-modules

§ 4 Holonomic $\mathscr{D}$-modules

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