SuppNote on $\mathscr{D}$-modules

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Conventions

- Unless specify otherwise, all rings are commutative and unitary.

§ 1 Sheaves of differential operators

1.1 Differential operators for commutative algebras

Let K be a ring and A be a K-algebra. Recall that, a K*-derivation* on A is a K-endomorphism D of the K-module A such that for any $a, b \in A$, we have the Leibniz rule:

$$D(ab) = D(a)b + aD(b).$$

The collection of all K-derivations on A from an A-submodule $\mathrm{Der}_K(A)$ of the module $\mathrm{End}_K(A)$ consisting of all K-endomorphisms on A. Moreover, $\mathrm{End}_K(A)$ is an associative algebra and hence a Lie algebra over K. Then

- $\mathrm{Der}_K(A)$ is then a Lie subalgebra of $\mathrm{End}_K(A)$;
- and A can be identified with the subalgebra $\mathrm{End}_A(A)$ of $\mathrm{End}_K(A)$ via the left (or right) multiplication operators.

Those gives a prototype of the notion *Lie-Rinehart algebras*, the algebraic version of *Lie algebroids*.

Using the Lie bracket of $\mathrm{End}_K(A)$, one can see that a K-endomorphism D of A is a K-derivation if and only if for any $a \in A$, we have

$$[D, a] = D(a),$$

where we identify A with $\mathrm{End}_A(A)$ via left multiplication operators. Motivated by this, we have the following inductive definition.

1.1.1 Definition A K-endomorphism D of A is called a ***K-linear differential operator of order*** $\leqslant n$, where $n \in \mathbb{N}$, if for any $a \in A$, $[D, a]$ is a K-linear differential operator of order $\leqslant n-1$. We use the convention that the zero endomorphism is of order $\leqslant -1$. The set of all K-linear differential operator of order $\leqslant n$ on A is denoted by $\mathrm{Diff}_K^n(A)$ and their union is denoted by $\mathrm{Diff}_K(A)$.

1.1.2 Proposition *We have the followings.*

1. *Each* $\mathrm{Diff}_K^n(A)$ *is an* A*-submodule of* $\mathrm{End}_K(A)$*.*
2. $\mathrm{Diff}_K^0(A) = A$*.*
3. $\mathrm{Diff}_K^1(A) = A \oplus \mathrm{Der}_K(A)$*.*
4. *For any* $m, n \in \mathbb{N}$*,* $\mathrm{Diff}_K^m(A) \circ \mathrm{Diff}_K^n(A) \subset \mathrm{Diff}_K^{m+n}(A)$*.*
5. *For any* $m, n \in \mathbb{N}$*,* $[\mathrm{Diff}_K^m(A), \mathrm{Diff}_K^n(A)] \subset \mathrm{Diff}_K^{m+n-1}(A)$

In this proposition, *1* shows that $\mathrm{Diff}_K^{\bullet}(A)$ form a positive increasing exhaustive filtration of A-submodules of $\mathrm{Diff}_K(A)$. *2, 3, 4* show that such a filtration makes $\mathrm{Diff}_K(A)$ a filtered non-commutative ring, which extends the Lie-Rinehart algebra $(A, \mathrm{Der}_K(A))$. Finally, *5* shows that $\mathrm{Diff}_K(A)$ is moreover an *almost commutative ring*, i.e. its associated graded ring

$$\mathrm{Gr}_{\bullet}(\mathrm{Diff}_K(A)) = \bigoplus_{n=0}^{\infty} \mathrm{Diff}_K^n(A)/\,\mathrm{Diff}_K^{n-1}(A)$$

is a graded commutative ring. Note that $\mathrm{Gr}_0(\mathrm{Diff}_K(A)) = \mathrm{Diff}_K^0(A) = A$. Hence $\mathrm{Gr}_{\bullet}(\mathrm{Diff}_K(A))$ is moreover a graded A-algebra.

1.2 First definition of the sheaf $\mathscr{D}$

How to define?

1.3 Kähler differentials

1.4 Second definition of the sheaf $\mathscr{D}$

How to use the higher conormal sheaves?