Note on
$\mathscr{D}$-modules
(on complex manifolds)

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Abstract

This is my reading and thought notes on $\mathscr{D}$-modules in the context of complex geometry. It contains standard materials of definitions and conclusions in this field at beginner level. In addition, it also contains funny, cumbersome and maybe highly non-necessary materials (in small fonts) basically around my confusions and brainstorms.

Contents

Conventions

Throughout this note, unless specify otherwise, all objects are over the complex field $\mathbb{C}$. For example, by a vector space, we mean a vector space over $\mathbb{C}$; by a sheaf, we mean a sheaf of vector spaces over $\mathbb{C}$.

To invalid potential issue of infinity, we require charts to be connected.

We will widely use the notation $\in$. So, $x \in X$ could means x is an element of X, or x is an object of X, or x is a section of X, depending on the context.

There are many sheaves canonically defined on every complex manifolds M, and usually have notations of the form $\mathscr{F}_M$. When the manifold M is unambiguous, we simplify the notation to $\mathscr{F}$.

For $(X, \mathscr{O}_X)$ a locally ringed space, we will use $\mathcal{M}(\mathscr{O}_X)$ or simply $\mathcal{M}(X)$ to denote the $\mathbb{C}$-linear category of $\mathscr{O}_X$-modules.

We also use the following conventions from analysis: whenever we have $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{N}^m$, then

$$|\lambda| = \sum_{i=1}^{m} \lambda_i, \qquad \lambda! = \prod_{i=1}^{m} (\lambda_i!);$$

if we have another $\mu = (\mu_1, \cdots, \mu_m) \in \mathbb{N}^m$, then

$$\binom{\lambda}{\mu} = \begin{cases} \dfrac{\lambda!}{\mu!(\lambda-\mu)!} & \text{if } \lambda \geqslant \mu, \\ 0 & \text{if not,} \end{cases}$$

where $\lambda \geqslant \mu$ means $\lambda_i \geqslant \mu_i$ for all $i = 1, \cdots, m$. We use ϵ_i to denote the multi-index whose i-th term is 1 and all the other terms are 0. Let $X = (X_1, \cdots, X_m)$ be a m-tuple of *pairwise commutative* elements in a ring, then by X^λ, we mean the unambiguous product

$$X_1^{\lambda_1} \cdots X_m^{\lambda_m}.$$

Let $(X^p)_{p \in I}$ (resp. $(X_p)_{p \in I}$) be a family of elements in a ring *parametrized* by a subset I of $\mathbb{N}$ (for example, $I = [1, m] := \{1, \cdots, m\}$), then by X^λ (resp. X_λ), we mean the ordered product

$$X^{\lambda_1} \cdots X^{\lambda_m} \qquad (\text{resp. } X_{\lambda_1} \cdots X_{\lambda_m}).$$

Note that these two conventions would not cause ambiguity as long as we distinguish the cases that X^p is a power of the element X and that X^p is a member in the family X.

§ 1 Basic constructions

One can skip hard section (marked by ¶) and remarks in small font during first read.

1.1 The sheaf of holomorphic differential operators

Let M be a complex manifold, $\mathscr{O}_M$ its its structural sheaf, that is, the sheaf of holomorphic functions on M. Suppose M is of complex dimension m, then locally, one can always find a local coordinate system $(z^i)_{1\leqslant i\leqslant m}$. Let's keep such convention.

Let $\mathbb{C}_M$ be the constant sheaf with values $\mathbb{C}$ on M. It is where everything lives on in this notes, so the tensor product and the internal Hom-sheaf over it is denoted by $-\otimes-$ and $\mathscr{H}om(-,-)$. For $\mathscr{F}$ a sheaf on M, we use $\mathscr{E}nd(\mathscr{F})$ to denote $\mathscr{H}om(\mathscr{F},\mathscr{F})$.

Let Θ_M be the sheaf of holomorphic vector fields on M and note that it is a locally free $\mathscr{O}_M$-modules with local basis $(\partial_{z^i})_{1\leqslant i\leqslant m}$ (or simply denoted by $(\partial_i)_{1\leqslant i\leqslant m}$) under the local coordinate system $(z^i)_{1\leqslant i\leqslant m}$. Note that it is a sheaf of Lie algebras and that each ∂_i acting on a function f gives $\frac{\partial f}{\partial z^i}$.

Note that the pair $(\mathscr{O},\Theta)$ satisfies the following properties:

(a) Θ is an $\mathscr{O}$-modules,

(b) $\mathscr{O}$ is a Θ-module and

(c) those two actions give rise to an $\mathscr{O}$-linear monomorphism of sheaves of ($\mathbb{C}$-linear) Lie algebras from Θ to $\mathscr{D}er(\mathscr{O})$, the sheaf of ($\mathbb{C}$-linear) derivations of $\mathscr{O}$.

This means they form a sheaf of faithful *Lie–Rinehart algebras.*

Remark Indeed, a *Lie–Rinehart algebra* is a pair $(A,\mathfrak{g})$ of a commutative ring A and a Lie algebra $\mathfrak{g}$ subject to the following axioms:

(LR1) $\mathfrak{g}$ is an A-modules;

(LR2) A is a $\mathfrak{g}$-module;

(LR3) $\mathfrak{g}$ acts as derivations of A;

(LR4) A acts on $\mathfrak{g}$ by the following *Leibniz rule*:

$$a[v,w]=[av,w]+w(a)v,\qquad \forall a\in A, v,w\in\mathfrak{g}.$$

If $\mathfrak{g}$ acts faithfully on A, that is, the Lie algebra homomorphism $\mathfrak{g}\to\mathrm{Der}(A)$, where $\mathrm{Der}(A)$ denotes the set of derivations of A which is both a Lie algebra and an A-module, is injective, then (LR4) is equivalent to say that the homomorphism $\mathfrak{g}\to\mathrm{Der}(A)$ is A-linear.

Since Θ acts faithfully on $\mathscr{O}$, we have the following embedding:

$$\Theta \hookrightarrow \mathscr{D}er(\mathscr{O}) \hookrightarrow \mathscr{E}nd(\mathscr{O}).$$

On the other hand, since $\mathscr{O}$ a sheaf of commutative rings, it can be canonically embedded into $\mathscr{E}nd(\mathscr{O})$ as its center. Note that, in each case, we have a monomorphism of $\mathscr{O}$-modules. Then, we meet the following definition:

1.1.1 Definition The $\mathscr{O}$-subalgebra of $\mathscr{E}nd(\mathscr{O})$ generated by the images of above two embeddings is called the ***sheaf of differential operators on*** M, denoted by $\mathscr{D}_M$.

Note that this makes $\mathscr{D}$ into the universal algebra of $(\mathscr{O}, \Theta)$.

Remark Indeed, first note that if R is a ring and B is a commutative subring of R, then (B, R), where R is equipped with the standard Lie bracket and acts on B by adjoint actions, is a Lie–Rinehart algebra.

A *homomorphism of Lie–Rinehart algebras* $(A, \mathfrak{g}) \to (B, \mathfrak{h})$ is a pair (φ, ψ) of a ring homomorphism $\varphi\colon A \to B$ and a Lie algebra homomorphism $\psi\colon \mathfrak{g} \to \mathfrak{h}$ such that

(a) φ makes ψ into a homomorphism of A-modules;

(b) ψ makes φ into a homomorphism of $\mathfrak{g}$-modules.

Then, a *homomorphism* from a Lie–Rinehart algebra $(A, \mathfrak{g})$ to a ring R is a pair (φ, ψ) of a ring homomorphism $\varphi\colon A \to R$ and a Lie algebra homomorphism $\psi\colon \mathfrak{g} \to R$ such that (φ, ψ) is a homomorphism of Lie–Rinehart algebras from $(A, \mathfrak{g})$ to $(\mathrm{Im}(\varphi), R)$.

Finally, the *universal algebra* of a Lie–Rinehart algebra $(A, \mathfrak{g})$ is the ring $\mathcal{U}(A, \mathfrak{g})$ (equipped with a homomorphism (ι, ρ) from $(A, \mathfrak{g})$ to it) satisfying the following universal property:

> Whenever there is a homomorphism (φ, ψ) from $(A, \mathfrak{g})$ to a ring R, there exists a unique ring homomorphism ϕ from $\mathcal{U}(A, \mathfrak{g})$ to R such that $\varphi = \phi \circ \iota$ and $\psi = \phi \circ \rho$.

Note that in particular, there exists a unique representation $\vartheta\colon \mathcal{U}(A, \mathfrak{g}) \to \mathrm{End}(A)$ such that $\varphi \circ \iota$ is the canonical representation of A and $\varphi \circ \rho$ is the action of $\mathfrak{g}$ on A. In this way, we can always identify A and its image in $\mathcal{U}(A, \mathfrak{g})$.

Note that, using a local coordinate system $(z^i)_{1 \leqslant i \leqslant m}$, any differential operator can be locally uniquely written as

$$\sum_{\lambda \in \mathbb{N}^m} f_\lambda \partial^\lambda,$$

where $f_\lambda \in \mathscr{O}$ and all but finitely many of them are zero. This can be shown using the following lemma, which relates the left and right $\mathscr{O}$-module structures on $\mathscr{D}$ (the left (resp. right) $\mathscr{O}$-module structure is given by left (resp. right) multiplication).

1.1.2 Lemma *Let U be a chart of M with coordinate system $(z^i)_{1\leqslant i\leqslant m}$ and $(\partial_i)_{1\leqslant i\leqslant m}$ the corresponding basis of $\Gamma(U,\Theta)$. Then for any $f\in\Gamma(U,\mathscr{O})$ and $i,j\in[1,m]$, we have*

$$[\partial_i,f]=\partial_i(f),\quad [\partial_i,\partial_j]=\delta_{ij}.$$

Moreover, for any $\lambda\in\mathbb{N}^m$, we have

$$\partial^\lambda f=\sum_{\mu\leqslant\lambda}\binom{\lambda}{\mu}\partial^{\lambda-\mu}(f)\partial^\mu,$$
$$f\partial^\lambda=\sum_{\mu\leqslant\lambda}\binom{\lambda}{\mu}(-1)^{|\lambda-\mu|}\partial^\mu\partial^{\lambda-\mu}(f).$$

Then, for such an open set U, we get an exhaustive filtration on $\Gamma(U,\mathscr{D})$:

$$F_0\Gamma(U,\mathscr{D})\subset F_1\Gamma(U,\mathscr{D})\subset F_2\Gamma(U,\mathscr{D})\subset\cdots,$$

where for each $p\in\mathbb{N}$,

$$F_p\Gamma(U,\mathscr{D})=\left\{\sum_{|\lambda|\leqslant p}f_\lambda\partial^\lambda;\,f_\lambda\in\Gamma(U,\mathscr{O})\right\}.$$

A differential operator $P\in F_p\Gamma(U,\mathscr{D})\setminus F_{p-1}\Gamma(U,\mathscr{D})$ is said to be of ***order*** p, denoted by $\mathrm{ord}(P)$. We always keep the convention that $\mathrm{ord}(0)=-\infty$. Note that if we write P as $\sum_{\lambda\in\mathbb{N}^m}f_\lambda\partial^\lambda$, then $\mathrm{ord}(P)$ is precisely the integer $\max\{|\lambda|;\,f_\lambda\neq0\}$.

These filtrations can be glued into an exhaustive filtration on $\mathscr{D}$:

$$F_0\mathscr{D}\subset F_1\mathscr{D}\subset F_2\mathscr{D}\subset\cdots.$$

To see this, we need the following lemma:

1.1.3 Lemma *The order of a differential operator on a chart U does not depend on the choice of coordinate system. Consequently, the order of a differential operator on M is locally constant.*

Proof: The first assertion follows by apply the differential operator on polynomials. The second assertion follows from the *Identity Principle*: if two holomorphic functions on a connected open set coincide on a nonempty open subset, then they are the same. Indeed, this implies that the order of the restriction of a differential operator on a connected chart to another smaller chart remains the same. □

This lemma justify the notation $\mathrm{ord}(P)$. Moreover, since $\mathrm{ord}(P)$ is a locally constant, the presheaves

$$F_p\mathscr{D}\colon U\longmapsto\{P\in\Gamma(U,\mathscr{D});\mathrm{ord}(P)\leqslant p\}$$

are in fact sheaves and furthermore $\mathscr{O}$-submodules of $\mathscr{D}$. Moreover, by same reason, the filtration of stalks $(F_p\mathscr{D})_x$ at each point $x \in M$ is exhaustive, hence so is the filtration of sheaves $F_p\mathscr{D}$.

Remark Let $(A, \mathfrak{g})$ be a Lie–Rinehart algebra. Since $\mathcal{U}(A, \mathfrak{g})$ is a ring extension of A generated by $\rho(\mathfrak{g})$, we get a natural filtration on $\mathcal{U}(A, \mathfrak{g})$:

$$\mathcal{U}_0(A, \mathfrak{g}) \subset \mathcal{U}_1(A, \mathfrak{g}) \subset \mathcal{U}_2(A, \mathfrak{g}) \subset \cdots,$$

where $\mathcal{U}_p(A, \mathfrak{g})$ is the A-submodules of $\mathcal{U}(A, \mathfrak{g})$ generated by $\bigoplus_{q \leqslant p} \rho(\mathfrak{g})^q$ (with the convention that $\rho(\mathfrak{g})^0 = A$). The filtration $F_p\mathscr{D}$ can be understood in this way.

Example (Differential operator of infinite order) Let M be the disjoint union of countable copies of $\mathbb{C}$. On the p-th copy, consider the differential operator ∂^p, the p-th power of the standard vector field on $\mathbb{C}$. Since those copies are disjoint with each other, one can glue these differential operators together to get a differential operator on M. However, this global differential operator wouldn't be of any finite order.

By computation using local coordinate system, we have:

1.1.4 Lemma *If we identify $\mathscr{O}$, Θ with their images in $\mathscr{D}$, then we have*

(i) $F_0\mathscr{D} = \mathscr{O}$,

(ii) $F_1\mathscr{D} = \mathscr{O} \oplus \Theta$,

(iii) $F_p\mathscr{D} \circ F_q\mathscr{D} \subset F_{p+q}\mathscr{D}$,

(iv) $[F_p\mathscr{D}, F_q\mathscr{D}] \subset F_{p+q-1}\mathscr{D}$.

Note that this implies that $(\mathscr{D}, F_\bullet)$ is a sheaf of *almost commutative rings*, hence its *associated graded algebra*

$$\mathrm{gr}_\bullet(\mathscr{D}) = \mathrm{gr}^F_\bullet(\mathscr{D}) := \bigoplus_{p=0}^{\infty} F_p\mathscr{D}/F_{p-1}\mathscr{D}$$

is commutative (here and from now on, we keep the convention that if $F_\bullet$ is an increasing filtration start from 0, then $F_p = 0$ for negative p).

Remark A ***filtered ring*** $(R, F_\bullet)$ is a ring R equipped with an exhaustive increasing filtration of subspaces $F_\bullet$ on it satisfying the following axioms:

(a) $1 \in F_0R$;

(b) $F_pR \cdot F_qR \subset F_{p+q}R$.

Any filtered ring $(R, F_\bullet)$ admits an *associated graded algebra*

$$\mathrm{gr}_\bullet(R) = \mathrm{gr}^F_\bullet(R) := \bigoplus_{p=0}^{\infty} F_pR/F_{p-1}R,$$

whose multiplication is induced from that of R in an obvious way. If a filtered ring $(R, F_\bullet)$ furthermore satisfies

(c) $[F_pR, F_qR] \subset F_{p+q-1}R$,

then its associated graded algebra is commutative. Such a filtered ring is called a ***almost commutative ring***.

Note that we have $\mathrm{gr}_0(\mathscr{D}) = F_0\mathscr{D} = \mathscr{O}$, hence $\mathrm{gr}_\bullet(\mathscr{D})$ is a commutative graded $\mathscr{O}$-algebra. Since $\mathscr{D}$ is generated by $F_1\mathscr{D}$, the $\mathscr{O}$-algebra $\mathrm{gr}_\bullet(\mathscr{D})$ is generated by $\mathrm{gr}_1(\mathscr{D})$, which is isomorphic to Θ by Lemma 1.1.4. Then the commutative multiplication on $\mathrm{gr}_\bullet(\mathscr{D})$ induces the following surjective homomorphisms of $\mathscr{O}$-modules

$$\mathbb{S}^p_{\mathscr{O}}(\Theta) \longrightarrow \mathrm{gr}_p(\mathscr{D}),$$

which give rise to a surjective homomorphism of graded $\mathscr{O}$-algebras:

$$\mathbb{S}^\bullet_{\mathscr{O}}(\Theta) \longrightarrow \mathrm{gr}_\bullet(\mathscr{D}). \tag{1.1.5}$$

Using a local coordinate system and notice that $\{\partial^\lambda; \lambda \in \mathbb{N}^m\}$ form an $\mathscr{O}$-basis, it is straightforward to see that the above homomorphism is an isomorphism.

Note that $\mathscr{O}$ is noetherian by *Rückert Basis Theorem*, which can be shown by *Weierstrass Preparation Theorem*. From this and that Θ is a locally free $\mathscr{O}$-module of finite rank, we conclude that $\mathbb{S}_{\mathscr{O}}(\Theta)$, as well as $\mathrm{gr}(\mathscr{D})$, is noetherian. Then, we have

1.1.6 Theorem *$\mathscr{D}$ is left and right noetherian.*

To see this, we need the following lemma:

1.1.7 Lemma *Let $(R, F_\bullet)$ be an almost commutative ring. Suppose furthermore:*

(i) $\mathrm{gr}^F_1(R)$ *generates* $\mathrm{gr}^F_\bullet(R)$ *as a F_0R-algebra,*

(ii) $\mathrm{gr}^F_\bullet(R)$ *is noetherian.*

Then R is left and right noetherian.

We leave the proof later.

1.2 The sheaf of differential operators

In this subsection, $(X, \mathscr{O}_X)$ is a (commutative) locally ringed space. For any pair of $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, we define a sequence of $\mathscr{O}_X$-submodules of $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ (not $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$) recursively as follows:

- $\mathscr{D}iff^p_X(\mathscr{F}, \mathscr{G}) = 0$ for negative p;
- for $p \geqslant 0$, $\mathscr{D}iff^p_X(\mathscr{F}, \mathscr{G})$ maps each open set U to

$$\left\{ P\colon\ \mathscr{F}|_U \to \mathscr{G}|_U\,; [P|_V\,, a] \in \Gamma(V, \mathscr{D}iff^{p-1}_X(\mathscr{F}, \mathscr{G})), \forall a \in \Gamma(V, \mathscr{O}) \right\},$$

  where the morphism $[P|_V\,, a]$ maps each section t of $\mathscr{F}|_V$ to the section $P(a.t) - a.P(t)$ of $\mathscr{G}|_V$.

Then the ***sheaf of differential operators from $\mathscr{F}$ to $\mathscr{G}$*** is the sheaf union

$$\mathscr{D}iff_X(\mathscr{F}, \mathscr{G}) = \bigcup \mathscr{D}iff^p_X(\mathscr{F}, \mathscr{G}).$$

A section of $\mathscr{D}iff^p_X(\mathscr{F}, \mathscr{G})$ is called a *differential operator of order* $\leqslant p$. The order of a differential operator can also be characterized by the following construction: for P an endomorphism of $\mathscr{O}$ and p a natural number, let $\sigma_p(P)\colon \mathscr{O}_X^{\otimes p} \to \mathscr{H}om(\mathscr{F}, \mathscr{G})$ be the homomorphism

$$a_1 \otimes a_2 \otimes \cdots \otimes a_n \mapsto [\cdots[[P, a_1], a_2], \cdots, a_n],$$

where $a_1, a_2, \cdots, a_n$ are sections of $\mathscr{O}_X$ (note that when $p = 0$, $\sigma_0 = \mathrm{id}$). For a differential operator P, $\sigma_p(P)$ is called its ***p-th symbol***. In particular, the ***principal symbol*** of P is its symbol of order $\mathrm{ord}(P)$, denoted by $\sigma(P)$. Then, we have

1.2.1 Lemma *(i) Every $\sigma_p(P)$ is symmetric, hence from $\mathbb{S}^p(\mathscr{O}_X)$.*

(ii) $\mathrm{Ker}(\sigma_p) = \mathscr{D}iff^{p-1}_X(\mathscr{F}, \mathscr{G})$, hence if $Q \in \mathscr{D}iff^q_X(\mathscr{F}, \mathscr{G})$, then $\sigma_p(Q)$ lands in $\mathscr{D}iff^{q-p}_X(\mathscr{F}, \mathscr{G})$.

PROOF: (i) follows from the Jacobi identity and the that $\mathscr{O}_X$ is commutative. (ii) follows from expansion of the recursive definition of $\mathscr{D}iff_X$. □

From this lemma and the observation that $\mathscr{D}iff^0_X(\mathscr{F}, \mathscr{G}) = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$, each σ_p induces a monomorphism of $\mathscr{O}_X$-modules

$$\mathrm{gr}_p(\mathscr{D}iff_X(\mathscr{F}, \mathscr{G})) \longrightarrow \mathscr{H}om(\mathbb{S}^p(\mathscr{O}_X), \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})).$$

From now on, we identify each $\mathrm{gr}_p(\mathscr{D}iff_X(\mathscr{F}, \mathscr{G}))$ with its image under above monomorphism and view σ_p as the projection from $\mathscr{D}iff^p_X(\mathscr{F}, \mathscr{G})$ to $\mathrm{gr}_p(\mathscr{D}iff_X(\mathscr{F}, \mathscr{G}))$.

See [EGA4, Chapter 16] and [SGA3, Exposé VII] for more details. From now on, we only forces on the special one $\mathscr{D}iff_X = \mathscr{D}iff_X(\mathscr{O}_X, \mathscr{O}_X)$.

1.2.2 Proposition *If we identify $\mathscr{O}_X$ with its image in $\mathscr{E}nd(\mathscr{O}_X)$, then we have*

(i) $\mathscr{D}iff^0_X = \mathscr{O}_X$,

(ii) $\mathscr{D}iff^1_X = \mathscr{O}_X \oplus \mathscr{D}er(\mathscr{O}_X)$,

(iii) $\mathscr{D}iff^p_X \circ \mathscr{D}iff^q_X \subset \mathscr{D}iff^{p+q}_X$,

(iv) $[\mathscr{D}iff^p_X, \mathscr{D}iff^q_X] \subset \mathscr{D}iff^{p+q-1}_X$.

Proof: Note that we only need to verify these at stalks. So, the assertions follow from those in commutative algebra. □

Then, we have an almost commutative $\mathscr{O}_X$-algebra $\mathscr{D}iff^\bullet_X$. Hence, there is a canonical homomorphism of graded $\mathscr{O}_X$-algebras

$$\mathbb{S}^\bullet_{\mathscr{O}_X}(\mathscr{D}er(\mathscr{O}_X)) \longrightarrow \mathrm{gr}_\bullet(\mathscr{D}iff_X). \tag{1.2.3}$$

However, this is not surjective, *a fortiori* an isomorphism in general.

Remark Let A be a commutative ring and M, N two A-modules, then the filtered A-module $\mathrm{Diff}^\bullet_A(M,N)$ can be defined recursively as follows:

- $\mathrm{Diff}^p_A(M,N) = 0$ for negative p;
- for $p \geqslant 0$, $\mathrm{Diff}^p_A(M,N)$ is the following submodule of $\mathrm{Hom}(M,N)$

  $$\{P \in \mathrm{Hom}(M,N); [P,a] \in \mathrm{Diff}^{p-1}_A(M,N), \forall a \in A\},$$

  where the homomorphism $[P,a]$ maps each $t \in M$ to $P(a.t) - a.P(t)$.

In particular, $\mathrm{Diff}^\bullet_A(A,A)$ is simply denoted by $\mathrm{Diff}^\bullet_A$. It is not difficult to show that $\mathrm{Diff}^\bullet_A$ is an almost commutative ring and $\mathrm{Diff}^1_A = A \oplus \mathrm{Der}(A)$. However, it is not true in general that the subalgebra of $\mathrm{End}(A)$ generated by $A \oplus \mathrm{Der}(A)$ is the entire Diff_A.

Now, we go back to the case on a complex manifold M. We have

1.2.4 Lemma *The image of Θ_M in $\mathscr{E}nd(\mathscr{O}_M)$ is $\mathscr{D}er(\mathscr{O}_M)$.*

Proof: Since the problem is local, we may assume we are working on an open set of $\mathbb{C}^m$ with coordinate system $(z^i)_{1\leqslant i\leqslant m}$. First, it is clear that every holomorphic vector field defines a derivation. Conversely, let D be a derivation, then it comes from the vector field $\theta = \sum_{i=1}^m D(z^i)\partial_i$. Indeed, the *Hadamard lemma* shows that, if f is a holomorphic function nearby a point x, then there exist holomorphic functions $(f_i)_{1\leqslant i\leqslant m}$ nearby x such that

$$f = f(x) + \sum_{i=1}^m (z^i - z^i(x)) f_i$$

and that $f_i(x) = \frac{\partial f}{\partial z^i}(x)$ for all i. Then we have

$$D(f) = D(\sum_{i=1}^{m}(z^i - z^i(x))f_i) = \sum_{i=1}^{m}(D(z^i)f_i + (z^i - z^i(x))D(f_i)).$$

Hence $D(f)(x) = \theta(f)(x)$. Since x is arbitrary, we conclude that D comes from the holomorphic vector field θ. □

More general, we have

1.2.5 Theorem *For each p, $F_p\mathscr{D}_M = \mathscr{D}\!\mathit{iff}^p_M$.*

Proof: It reduces to show $(F_p\mathscr{D}_M)_x = (\mathscr{D}\!\mathit{iff}^p_M)_x$ at every point $x \in M$. We keep the same assumption as previous, then we may assume $x = 0 \in \mathbb{C}^m$. Let A be the ring of germs of holomorphic functions at 0. Then $(\mathscr{D}_M)_0$ equals the subalgebra of $\mathrm{End}(A)$ generated by A and $\mathrm{Der}(A)$, which is also the universal algebra $\mathcal{U}(A, \mathrm{Der}(A))$ by the discussion in previous subsection. In addition, we have $(\mathscr{D}\!\mathit{iff}^p_M)_0 = \mathrm{Diff}_A$ from the definition. Then it remains to show that $\mathcal{U}_p = \mathcal{U}_p(A, \mathrm{Der}(A))$ equals Diff^p_A.

It is clear that $\mathcal{U}_p \subset \mathrm{Diff}^p_A$. To prove the converse, we do induction on p. The $p = 1$ case is Lemma 1.2.4. To do the inductive step, we need a lemma:

1.2.6 Lemma *For positive p, if there are $P_1, \cdots, P_m \in \mathcal{U}_{p-1}$ satisfying*

$$[P_i, z^j] = [P_j, z^i], \qquad \forall i, j \in [1, m],$$

then there exists a $Q \in \mathcal{U}_p$ such that

$$[Q, z^i] = P_i, \qquad i \in [1, m].$$

Proof: First, it is not difficult to find a $Q_m \in \mathcal{U}_p$ such that $[Q_m, z^m] = P_m$. Indeed, if we write P_m as

$$P_m = \sum_{|\lambda| \leqslant p-1} f_\lambda \partial^\lambda,$$

then

$$Q_m = \sum_{|\lambda| \leqslant p-1} \frac{f_\lambda}{\lambda_m} \partial^{\lambda+\epsilon_m}$$

works:

$$[Q_m, z^m] = \sum_{|\lambda| \leqslant p-1} \frac{f_\lambda}{\lambda_m}[\partial^{\lambda+\epsilon_m}, z^m] = \sum_{|\lambda| \leqslant p-1} \frac{f_\lambda}{\lambda_m}\lambda_m \partial^\lambda = P_m.$$

Suppose we already find $Q_{k+1} \in \mathcal{U}_p$ ($k \in [1, m-1]$) such that

$$[Q_{k+1}, z^i] = P_i, \qquad i \in [k+1, m].$$

Then we want to find a $Q_k \in \mathcal{U}_p$ such that

$$[Q_k, z^i] = P_i, \qquad i \in [k, m].$$

To do this, we can first let $P_{k+\frac{1}{2}}$ be $[Q_{k+1}, z^k] - P_k$. Then, for each $i \in [k+1, m]$, we have

$$\begin{aligned}
[P_{k+\frac{1}{2}}, z^i] &= [[Q_{k+1}, z^k] - P_k, z^i] \\
&= [[Q_{k+1}, z^i], z^k] - [P_k, z^i] \\
&= [P_i, z^k] - [P_k, z^i] = 0.
\end{aligned}$$

On the other hand, if we write $P_{k+\frac{1}{2}}$ as

$$P_{k+\frac{1}{2}} = \sum_{|\lambda| \leqslant p-1} g_\lambda \partial^\lambda,$$

then

$$[P_{k+\frac{1}{2}}, z^i] = \sum_{|\lambda| \leqslant p-1} g_\lambda [\partial^\lambda, z^i] = \sum_{|\lambda| \leqslant p-1} g_\lambda \lambda_i \partial^{\lambda - \epsilon_i}.$$

Hence $g_\lambda = 0$ if $\lambda_i \neq 0$ for $i \in [k+1, m]$. Then $P_{k+\frac{1}{2}}$ is a operator built up only from $\partial_1, \cdots, \partial_k$. Hence if we put

$$Q_{k+\frac{1}{2}} = \sum_{|\lambda| \leqslant p-1} \frac{g_\lambda}{\lambda_k} \partial^{\lambda + \epsilon_k}$$

Then, we have

$$[Q_{k+\frac{1}{2}}, z^k] = P_{k+\frac{1}{2}}$$

and, for each $i \in [k+1, m]$,

$$[Q_k, z_i] = 0.$$

Then, $Q_k = Q_{k+1} - Q_{k+\frac{1}{2}}$ works.

Therefore, by induction, we can find the required $Q \in \mathcal{U}_p$. $\square$

Now, we go back to the proof of Theorem 1.2.5. Suppose we already have $\mathcal{U}_{p-1} = \mathrm{Diff}_A^{p-1}$. Let P be a differential operator of order $\leqslant p$. Then for each $i \in [1, m]$, $P_i = [P, z^i]$ is of order $p-1$, hence $P_i \in \mathcal{U}_{p-1}$. Note that for any $i, j \in [1, m]$, we have

$$[P_i, z^j] = [[P, z^i], z^j] = [[P, z^j], z^i] = [P_j, z^i].$$

Hence the lemma applies and there exists a $Q \in \mathcal{U}_p$ such that

$$[Q, z^i] = P_i = [P, z^i], \qquad i \in [1, m].$$

Now, we need another lemma:

1.2.7 Lemma *If D is a differential operator such that*

$$[D, z^i] = 0, \qquad i \in [1, m].$$

Then D is of order 0.

Proof: Let f be an arbitrary holomorphic function at 0 and x a point nearby 0 such that f is also holomorphic at x. Then, by *Hadamard lemma*, there exist holomorphic functions $(f_i)_{1\leqslant i\leqslant m}$ nearby x such that

$$f = f(x) + \sum_{i=1}^{m}(z^i - z^i(x))f_i.$$

Then we have (notice that D commutes with any number)

$$\begin{aligned}
[D, f] &= [D, \sum_{i=1}^{m}(z^i - z^i(x))f_i] \\
&= \sum_{i=1}^{m}([D, z^i]f_i + (z^i - z^i(x))[D, f_i]) \\
&= \sum_{i=1}^{m}(z^i - z^i(x))[D, f_i].
\end{aligned}$$

Apply both sides to arbitrary $g \in A$ and evaluate at x, we see that the function $D(fg) - fD(g)$ vanishes at x. By arbitrarily choosing x, we see that $D(fg) = fD(g)$. Hence

$$[D, f] = 0.$$

Since f is arbitrary, this means D is of order 0. □

Now $Q \in \mathfrak{U}_p$, $P - Q \in \mathrm{Diff}^0_A \subset \mathfrak{U}_p$, hence $P = Q + (P - Q) \in \mathfrak{U}_p$. Therefore $\mathfrak{U}_p = \mathrm{Diff}^p_A$ as desired. □

Remark One may expect another proof of Theorem 1.2.5 using *sheaf of principle parts*: if the sheaf $\Omega^1_{M/\mathbb{C}}$ of *Kahler differentials* on $\mathscr{O}_M$ is locally free of finite rank, then M is *differentially smooth* (*Jacobi Criterion*, see [EGA4, Thm.16.12.2]) and consequently, $\mathscr{D}iff_M$ is generated by $\mathscr{D}iff^1_M$ by [EGA4, Thm.16.11.2]. However, although $\mathscr{D}er(\mathscr{O}_M)$ is locally free of rank m, it is not true that so is $\Omega^1_{M/\mathbb{C}}$. Be careful that $\Omega^1_{M/\mathbb{C}}$ is *NOT* equal to Ω^1_M, the sheaf of holomorphic differentials: the later is just a quotient of the first. However, one may still use a similar strategy since we still have

$$\Omega^1_M = (\Omega^1_{M/\mathbb{C}})^{**},$$

where $()^*$ means the dual $\mathscr{O}_M$-module operation, and that the canonical morphism $\Omega^1_{M/\mathbb{C}} \to (\Omega^1_{M/\mathbb{C}})^{**}$ is surjective. See this $n$lab term and this MO post for more details about the relation between $\Omega^1_{M/\mathbb{C}}$ and Ω^1_M.

1.2.8 Corollary *The homomorphism* (1.2.3) *is an isomorphism.*

Proof: Follows from Lemma 1.2.4, Theorem 1.2.5 and the fact that (1.1.5) is an isomorphism. □

1.3 $\mathscr{D}$-modules and flat connections

Let Ω_M^1 be the sheaf of holomorphic 1-forms, which is the dual $\mathscr{O}_M$-module of Θ_M. Then $\Omega_M^p = \bigwedge^p \Omega_M^1$ is the sheaf of holomorphic p-forms. Note that $\Omega_M^0 = \mathscr{O}_M$. Let $d\colon \Omega_M^p \to \Omega_M^{p+1}$ be the *exterior derivative*, which can be locally defined as

$$d(\sum_\lambda f_\lambda dz^\lambda) = \sum_\lambda df_\lambda \wedge dz^\lambda, \quad \forall f_\lambda \in \mathscr{O}_M, \lambda \in [1,m]^p$$

and (globally) characterized by the following property:

$$d(\omega \wedge \upsilon) = d\omega \wedge \upsilon + (-1)^q \omega \wedge d\upsilon,$$

where $\omega \in \Omega_M^q$ (in other words, d is a *graded derivation of degree* 1 on $\Omega_M^\bullet$). Note that then we have a complex (the *de Rham complex*)

$$0 \longrightarrow \mathbb{C}_M \longrightarrow \mathscr{O}_M \xrightarrow{d} \Omega_M^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_M^m \longrightarrow 0. \tag{1.3.1}$$

By *Holomorphic Poincaré Lemma*, the above complex is *acyclic*.

The sheaf Θ_M can acts on $\Omega_M^\bullet$ by the *interior derivatives*: for θ a vector field on M, the interior derivative of a p-form ω along θ is defined as

$$\iota_\theta\omega(\theta_1, \cdots, \theta_{p-1}) := \omega(\theta, \theta_1, \cdots, \theta_{p-1}).$$

It can also be characterized by the following property

$$\iota_\theta(\omega \wedge \upsilon) = (\iota_\theta\omega) \wedge \upsilon + (-1)^q \omega \wedge (\iota_\theta\upsilon),$$

where $\omega \in \Omega_M^q$, together with the condition

$$\iota_\theta\omega = \langle v, w\rangle, \quad \forall \omega \in \Omega_M^1.$$

The sheaf Θ_M can also acts on $\Omega_M^\bullet$ by the *Lie derivatives*: for θ a vector field on M, the Lie derivative $\mathcal{L}_\theta\colon \Omega_M^\bullet \to \Omega_M^\bullet$ is the unique chain map (of degree 0) characterized by the following property

$$\mathcal{L}_\theta(\omega \wedge \upsilon) = (\mathcal{L}_\theta\omega) \wedge \upsilon + \omega \wedge (\mathcal{L}_\theta\upsilon)$$

and the condition

$$\mathcal{L}_\theta(f) = \theta(f), \quad \forall f \in \mathscr{O}_M.$$

The two actions of Θ_M on $\Omega_M^\bullet$ are related by the *Cartan's magic formula*:

$$\mathcal{L}_\theta = d \circ \iota_\theta + \iota_\theta \circ d.$$

In particular, we have

$$\mathcal{L}_\theta(d\omega) = d\mathcal{L}_\theta\omega, \qquad \mathcal{L}_{f\theta}\omega = f\mathcal{L}_\theta\omega + df \wedge \iota_\theta\omega.$$

Let $\boldsymbol{\omega}_M$ denote the sheaf Ω_M^m of *volume forms*. Locally, it has a basis $dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m$ and therefore is an invertible $\mathscr{O}_M$-module. The Lie derivatives gives a right action of the Lie algebra Θ_M on $\boldsymbol{\omega}_M$ by

$$\theta.\omega := -\mathcal{L}_\theta\omega.$$

1.3.2 Lemma *Let $\mathscr{M}$ be an $\mathscr{O}$-module. Then an left (resp. right) $\mathscr{D}$-module structure on $\mathscr{M}$ is equivalent to (in the sense that the $\mathscr{D}$-module structure extends the action of Θ) an action $\alpha\colon \Theta \to \mathscr{E}nd(\mathscr{M})$ satisfying the followings:*

(a) $\alpha(f\theta) = f \circ \alpha(\theta)$ *(resp.* $\alpha(f\theta) = \alpha(\theta) \circ f$*);*

(b) $[\alpha(\theta), f] = \theta(f)$ *(resp.* $[\alpha(\theta), f] = -\theta(f)$*);*

(c) $[\alpha(\theta), \alpha(\theta')] = \alpha([\theta, \theta'])$ *(resp.* $[\alpha(\theta), \alpha(\theta')] = -\alpha([\theta, \theta'])$*).*

Proof: Follows from the universal property of the universal algebra of a Lie-Rinehart algebra. $\square$

We use the notation $\mathcal{M}^l(\mathscr{D})$ to denote the category of left $\mathscr{D}$-modules and $\mathcal{M}^r(\mathscr{D})$ the category of right $\mathscr{D}$-modules.

1.3.3 Corollary *The Lie derivatives induces a left $\mathscr{D}$-module structure on $\mathscr{O}$ and a right $\mathscr{D}$-module structure on ω. Moreover, they are simple modules.*

Proof: To see $\mathscr{O}$ is simple, let $\mathscr{I}$ be any $\mathscr{D}$-submodule of $\mathscr{O}$. Suppose $\mathscr{I} \neq 0$ and consider the inclusion $\mathscr{I} \hookrightarrow \mathscr{O}$. At each point, by choosing a local coordinate, it becomes the inclusion $I \to A$ of D-modules, where A is the ring of germs of holomorphic functions at origin, D is the ring of differential operators of A. Then since for any nonzero holomorphic function f, there exists a multi-index λ such that $(\partial^\lambda f)$ doesn't vanish at origin hence is invertible in A, $D.I = A$ for any nonzero I. This shows $\mathscr{I} = \mathscr{O}$ as desired. Similar reasoning shows that ω is simple. $\square$

Let $\mathscr{M}$ be an $\mathscr{O}$-module, a ***connection*** on $\mathscr{M}$ is a linear map $\nabla\colon \mathscr{M} \to \Omega^1 \otimes_{\mathscr{O}} \mathscr{M}$ satisfying the *Leibniz rule*:

$$\nabla(fm) = df \otimes m + f\nabla m, \qquad \forall f \in \mathscr{O}, m \in \mathscr{M}.$$

Note that once we have a connection ∇ on $\mathscr{M}$, we also have the following connections on $\Omega^\bullet \otimes_{\mathscr{O}} \mathscr{M}$:

$$\nabla\colon \Omega^p \otimes_{\mathscr{O}} \mathscr{M} \longrightarrow \Omega^{p+1} \otimes_{\mathscr{O}} \mathscr{M}$$

characterized by the *Leibniz rule*:

$$\nabla(\omega \otimes m) = d\omega \otimes m + (-1)^p \omega \wedge \nabla m, \qquad \forall \omega \in \Omega^p, m \in \mathscr{M}.$$

Note that, from the Leibniz rule, we immediately have

1.3.4 Proposition (Affine space of connections) *Let $\mathscr{M}$ be an $\mathscr{O}$-module, then the set of connections on $\mathscr{M}$ form an affine space with transition group $\mathrm{Hom}_{\mathscr{O}}(\mathscr{M}, \Omega^1 \otimes_{\mathscr{O}} \mathscr{M})$. In other words, any two connections are differed by an $\mathscr{O}$-linear map $\mathscr{M} \to \Omega^1 \otimes_{\mathscr{O}} \mathscr{M}$ and conversely any connection added with an $\mathscr{O}$-linear map $\mathscr{M} \to \Omega^1 \otimes_{\mathscr{O}} \mathscr{M}$ gives another connection. Consequently, the sheaf of connections on $\mathscr{M}$ is a $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \Omega^1 \otimes_{\mathscr{O}} \mathscr{M})$-torsor.*

Proof: Let ∇, ∇' be two connections on $\mathscr{M}$. Then for any $f \in \mathscr{O}$ and $m \in \mathscr{M}$, we have

$$\begin{aligned}(\nabla - \nabla')(fm) &= \nabla(fm) - \nabla'(fm) \\ &= (df \otimes m + f\nabla m) - (df \otimes m + f\nabla' m) \\ &= f\nabla m - f\nabla' m \\ &= f(\nabla - \nabla')(m).\end{aligned}$$

On the other hand, let ∇ be a connection on $\mathscr{M}$ and $\varphi\colon \mathscr{M} \to \Omega^1 \otimes_{\mathscr{O}} \mathscr{M}$ be an $\mathscr{O}$-linear map. Then for any $f \in \mathscr{O}$ and $m \in \mathscr{M}$, we have

$$\begin{aligned}(\nabla + \varphi)(fm) &= \nabla(fm) + \varphi(fm) \\ &= df \otimes m + f\nabla m + f\varphi(m) \\ &= df \otimes m + f(\nabla + \varphi)(m).\end{aligned}$$

Then the statements follows. $\square$

1.3.5 Proposition *Let $(\mathscr{M}, \nabla^{\mathscr{M}})$ and $(\mathscr{N}, \nabla^{\mathscr{N}})$ be two $\mathscr{O}$-modules with connections. Then*

(i) $\nabla^{\mathscr{M}} \oplus \nabla^{\mathscr{N}}$ defines a connection on $\mathscr{M} \oplus \mathscr{N}$;

(ii) $\nabla^{\mathscr{M}} \otimes 1 + 1 \otimes \nabla^{\mathscr{N}}$ defines a connection on $\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}$;

(iii) $\varphi \mapsto \nabla^{\mathscr{N}} \circ \varphi - \varphi \circ \nabla^{\mathscr{M}}$ defines a connection on $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$.

Proof: Let $\nabla^{\mathscr{M} \oplus \mathscr{N}}$ denote $\nabla^{\mathscr{M}} \oplus \nabla^{\mathscr{N}}$. For any $f \in \mathscr{O}$ and $m \in \mathscr{M}, n \in \mathscr{N}$, we have

$$\begin{aligned}\nabla^{\mathscr{M} \oplus \mathscr{N}}(f(m \oplus n)) &= \nabla^{\mathscr{M}}(fm) \oplus \nabla^{\mathscr{N}}(fn) \\ &= (df \otimes m + f\nabla^{\mathscr{M}} m) \oplus (df \otimes n + f\nabla^{\mathscr{N}} n) \\ &= df \otimes (m \oplus n) + f(\nabla^{\mathscr{M}} m \oplus \nabla^{\mathscr{N}} n) \\ &= df \otimes (m \oplus n) + f\nabla^{\mathscr{M} \oplus \mathscr{N}}(m \oplus n).\end{aligned}$$

Hence $\nabla^{\mathscr{M} \oplus \mathscr{N}}$ is a connection on $\mathscr{M} \oplus \mathscr{N}$.

Let $\nabla^{\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}}$ denote $\nabla^{\mathscr{M}} \otimes 1 + 1 \otimes \nabla^{\mathscr{N}}$. Note that this notation looks somehow ambiguous and meaningless. First, since $\nabla^{\mathscr{M}}$ and $\nabla^{\mathscr{N}}$ are not $\mathscr{O}$-linear, $\nabla^{\mathscr{M}} \otimes 1$ and $1 \otimes \nabla^{\mathscr{N}}$ are not well-defined on $\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}$. Instead, they are well-defined on $\mathscr{M} \otimes \mathscr{N}$. Second, $\nabla^{\mathscr{M}} \otimes 1$ lands on $\Omega^1 \otimes_{\mathscr{O}} \mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}$ while $1 \otimes \nabla^{\mathscr{N}}$ lands on $\mathscr{M} \otimes_{\mathscr{O}} \Omega^1 \otimes_{\mathscr{O}} \mathscr{N}$, so one need to apply the transport operation $\rho_{12}\colon \mathscr{M} \otimes_{\mathscr{O}} \Omega^1 \otimes_{\mathscr{O}} \mathscr{N} \to \Omega^1 \otimes_{\mathscr{O}} \mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}$ after $1 \otimes \nabla^{\mathscr{N}}$ and before we sum up it with $\nabla^{\mathscr{M}} \otimes 1$. So $\nabla^{\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}}$ is *a priori* a $\mathbb{C}$-linear map

$$\begin{aligned}\mathscr{M} \otimes \mathscr{N} &\longrightarrow \Omega^1 \otimes_{\mathscr{O}} \mathscr{M} \otimes_{\mathscr{O}} \mathscr{N} \\ m \otimes n &\longmapsto \nabla^{\mathscr{M}} m \otimes n + \rho_{12}(m \otimes \nabla^{\mathscr{N}} n).\end{aligned}$$

Then to show it induces a well-defined map on $\mathscr{M} \otimes \mathscr{N}$, we need to show

$$\nabla^{\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}}(fm \otimes n) = \nabla^{\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}}(m \otimes fn)$$

for any $f \in \mathscr{O}$ and $m \in \mathscr{M}, n \in \mathscr{N}$. In fact, we have

$$\begin{aligned}
\nabla^{\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}}(fm \otimes n) &= \nabla^{\mathscr{M}}(fm) \otimes n + \rho_{12}(fm \otimes \nabla^{\mathscr{N}} n) \\
&= df \otimes m \otimes n + f\nabla^{\mathscr{M}} m \otimes n + f\rho_{12}(m \otimes \nabla^{\mathscr{N}} n) \\
&= df \otimes (m \otimes n) + f(\nabla^{\mathscr{M}} m \otimes n + \rho_{12}(m \otimes \nabla^{\mathscr{N}} n)) \\
&= df \otimes (m \otimes n) + f\nabla^{\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}}(m \otimes n), \\
\nabla^{\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}}(m \otimes fn) &= \nabla^{\mathscr{M}} m \otimes fn + \rho_{12}(m \otimes \nabla^{\mathscr{N}}(fn)) \\
&= f\nabla^{\mathscr{M}} m \otimes n + \rho_{12}(m \otimes (df \otimes n + f\nabla^{\mathscr{N}} n)) \\
&= f\nabla^{\mathscr{M}} m \otimes n + df \otimes m \otimes n + \rho_{12}(m \otimes f\nabla^{\mathscr{N}} n) \\
&= df \otimes (m \otimes n) + f\nabla^{\mathscr{M}} m \otimes n + f\rho_{12}(m \otimes \nabla^{\mathscr{N}} n) \\
&= df \otimes (m \otimes n) + f(\nabla^{\mathscr{M}} m \otimes n + \rho_{12}(m \otimes \nabla^{\mathscr{N}} n)) \\
&= df \otimes (m \otimes n) + f\nabla^{\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}}(m \otimes n).
\end{aligned}$$

Therefore $\nabla^{\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}}$ is a well-defined connection on $\mathscr{M} \otimes \mathscr{N}$.

Let $\nabla^{\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N})}$ denote the map defined in (iii). Again, this notation looks somehow ambiguous and meaningless. Let φ be a section of $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$. To simply notations, we omit restrictions on local sections and treat them as global sections. First, $\nabla^{\mathscr{M}}$ lands on $\Omega^1 \otimes_{\mathscr{O}} \mathscr{M}$. Therefore $\varphi \circ \nabla^{\mathscr{M}}$ is actually the composition

$$\mathscr{M} \xrightarrow{\nabla^{\mathscr{M}}} \Omega^1 \otimes_{\mathscr{O}} \mathscr{M} \xrightarrow{1\otimes\varphi} \Omega^1 \otimes_{\mathscr{O}} \mathscr{N}.$$

Then both $\varphi \circ \nabla^{\mathscr{M}}$ and $\nabla^{\mathscr{N}} \circ \varphi$ are from $\mathscr{M}$ to $\Omega^1 \otimes_{\mathscr{O}} \mathscr{N}$, hence are sections of $\mathscr{H}om(\mathscr{M}, \Omega^1 \otimes_{\mathscr{O}} \mathscr{N})$. Moreover, since $\nabla^{\mathscr{M}}$ and $\nabla^{\mathscr{N}}$ are not $\mathscr{O}$-linear, $\varphi \circ \nabla^{\mathscr{M}}$ and $\nabla^{\mathscr{N}} \circ \varphi$ are not $\mathscr{O}$-linear, hence not sections of $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \Omega^1 \otimes_{\mathscr{O}} \mathscr{N})$. So $\nabla^{\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N})}$ is *a priori* a $\mathbb{C}$-linear map

$$\begin{aligned}
\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N}) &\longrightarrow \mathscr{H}om(\mathscr{M}, \Omega^1 \otimes_{\mathscr{O}} \mathscr{N}) \\
\varphi &\longmapsto \nabla^{\mathscr{N}} \circ \varphi - \varphi \circ \nabla^{\mathscr{M}},
\end{aligned}$$

where the composition $\varphi \circ \nabla^{\mathscr{M}}$ should be viewed as $(1 \otimes \varphi) \circ \nabla^{\mathscr{M}}$. Then, we need to show

(a) $\nabla^{\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N})}$ lands on $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \Omega^1 \otimes_{\mathscr{O}} \mathscr{N})$;

(b) $\Omega^1 \otimes_{\mathscr{O}} \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N}) \cong \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \Omega^1 \otimes_{\mathscr{O}} \mathscr{N})$;

(c) $\nabla^{\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N})}\colon \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N}) \to \Omega^1 \otimes_{\mathscr{O}} \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$ is a connection.

Indeed, for any $f \in \mathscr{O}$, $\varphi \in \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$ and $m \in \mathscr{M}$, we have

$$\begin{aligned}
(\nabla^{\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N})}\varphi)(fm) &= \nabla^{\mathscr{N}}(\varphi(fm)) - (1 \otimes \varphi)(\nabla^{\mathscr{M}}(fm)) \\
&= \nabla^{\mathscr{N}}(f\varphi(m)) - (1 \otimes \varphi)(df \otimes m + f\nabla^{\mathscr{M}} m) \\
&= df \otimes \varphi(m) + f\nabla^{\mathscr{N}}\varphi(m) \\
&\quad - df \otimes \varphi(m) - (1 \otimes \varphi)(f\nabla^{\mathscr{M}} m) \\
&= f\nabla^{\mathscr{N}}\varphi(m) - f(1 \otimes \varphi)(\nabla^{\mathscr{M}} m) \\
&= f(\nabla^{\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N})}\varphi)(m).
\end{aligned}$$

This shows (a). Note that we always have a morphism

$$\Omega^1 \otimes_{\mathscr{O}} \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N}) \longrightarrow \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \Omega^1 \otimes_{\mathscr{O}} \mathscr{N})$$

which maps each $\omega \otimes \varphi$ to the morphism $m \mapsto \omega \otimes \varphi(m)$. To see it is an isomorphism, we only need to verify at stalks. Under a local coordinate system (z^i), this morphism can be written as

$$\sum_i dz^i \otimes \varphi_i \longmapsto (m \mapsto \sum_i dz^i \otimes \varphi_i(m)),$$

which is clearly an isomorphism. This shows (b). Finally, for any $f \in \mathscr{O}$, $\varphi \in \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$ and $m \in \mathscr{M}$, we have

$$\begin{aligned}
(\nabla^{\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N})} f\varphi)(m) &= \nabla^{\mathscr{N}}(f\varphi(m)) - (1 \otimes f\varphi)(\nabla^{\mathscr{M}} m) \\
&= df \otimes \varphi(m) + f\nabla^{\mathscr{N}}\varphi(m) - f(1 \otimes \varphi)(\nabla^{\mathscr{M}} m) \\
&= df \otimes \varphi(m) + f(\nabla^{\mathscr{N}}\varphi(m) - (1 \otimes \varphi)(\nabla^{\mathscr{M}} m)) \\
&= df \otimes \varphi(m) + f\nabla^{\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N})} m.
\end{aligned}$$

This shows (c). Therefore $\nabla^{\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N})}$ is a connection on $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$. $\square$

The *curvature* $\mathcal{R}_\nabla$ of a connection ∇ is the composition

$$\mathcal{R}_\nabla \colon \mathscr{M} \xrightarrow{\nabla} \Omega^1 \otimes_{\mathscr{O}} \mathscr{M} \xrightarrow{\nabla} \Omega^2 \otimes_{\mathscr{O}} \mathscr{M}.$$

A connection ∇ is ***flat/integrable*** if $\mathcal{R}_\nabla = 0$. Note that, if this is the case, then we have a complex

(1.3.6) $$0 \longrightarrow \mathscr{M} \xrightarrow{\nabla} \Omega^1 \otimes_{\mathscr{O}} \mathscr{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \omega \otimes_{\mathscr{O}} \mathscr{M} \longrightarrow 0,$$

called the ***de Rham complex*** of $\mathscr{M}$, denoted by $dR^\bullet(\mathscr{M})$.

Note that if $\mathscr{M}$ has a connection ∇, then it defines a left action of Θ on $\mathscr{M}$ satisfying (a) and (b) in Lemma 1.3.2 as follows: for θ a vector field on M, its action is given by the composition

(1.3.7) $$\nabla_\theta \colon \mathscr{M} \xrightarrow{\nabla} \Omega^1 \otimes_{\mathscr{O}} \mathscr{M} \xrightarrow{\iota_\theta \otimes 1} \mathscr{M}.$$

Then, the condition (c) is equivalent to that ∇ is flat.

Proof: It is clear that condition (c) is equivalent to

$$\nabla_\theta \nabla_{\theta'} - \nabla_{\theta'} \nabla_\theta - \nabla_{[\theta,\theta']} = 0, \quad \forall \theta, \theta' \in \Theta.$$

The left hand side is precisely $\mathcal{R}_\nabla(\theta, \theta') := \iota_{\theta'} \iota_\theta \mathcal{R}_\nabla$. To see this, one can choose a local coordinate system (z^i) and we have

$$\nabla m = \sum_i dz^i \otimes \nabla_{\partial_i} m,$$
$$\mathcal{R}_\nabla m = \sum_{i,j} dz^i \wedge dz^j \otimes \nabla_{\partial_i} \nabla_{\partial_j} m.$$

Therefore, for any vector fields $\theta = \sum_i f_i \partial_i$ and $\theta' = \sum_i g_i \partial_i$, we have

$$\nabla_\theta m = \sum_i f_i \nabla_{\partial_i} m;$$
$$\mathcal{R}_\nabla m = \sum_{i,j} (f_i g_j - g_i f_j) \otimes \nabla_{\partial_i} \nabla_{\partial_j} m.$$

Since

$$\begin{aligned}
[\theta, \theta'] &= \sum_{i,j} [f_i \partial_i, g_j \partial_j] \\
&= \sum_{i,j} f_i \partial_i g_j \partial_j - g_j \partial_j f_i \partial_i \\
&= \sum_{i,j} (f_i \partial_i (g_j) \partial_j + f_i g_j \partial_i \partial_j - g_j \partial_j (f_i) \partial_i + f_i g_j \partial_j \partial_i) \\
&= \sum_i (\sum_j f_j \partial_j (g_i) - g_j \partial_j (f_i)) \partial_i,
\end{aligned}$$

we have

$$\nabla_{[\theta,\theta']} m = \sum_i (\sum_j f_j \partial_j (g_i) - g_j \partial_j (f_i)) \nabla_{\partial_i} m$$

Therefore,

$$\begin{aligned}
\nabla_\theta \nabla_{\theta'} m &= \sum_i f_i \nabla_{\partial_i} \nabla_{\theta'} m \\
&= \sum_i f_i \nabla_{\partial_i} (\sum_j g_j \nabla_{\partial_j} m) \\
&= \sum_{i,j} f_i \nabla_{\partial_i} (g_j \nabla_{\partial_j} m) \\
&= \sum_{i,j} (f_i \partial_i (g_j) \nabla_{\partial_j} m + f_i g_j \nabla_{\partial_i} \nabla_{\partial_j} m).
\end{aligned}$$

Similarly,

$$\nabla_{\theta'}\nabla_\theta m = \sum_{i,j}(g_i\partial_i(f_j)\nabla_{\partial_j}m + g_if_j\nabla_{\partial_i}\nabla_{\partial_j}m)$$

Then it is clear now

$$\nabla_\theta\nabla_{\theta'} - \nabla_{\theta'}\nabla_\theta - \nabla_{[\theta,\theta']} = \mathcal{R}_\nabla(\theta,\theta').$$

Then the claim follows. $\square$

If this is the case, we obtain a left $\mathscr{D}$-module structure on $\mathscr{M}$. Conversely, any left $\mathscr{D}$-module admits a flat connection. To see this, first consider $\mathscr{M} = \mathscr{D}$. Note that

1.3.8 Lemma *Let $\mathscr{E}$ be a locally free $\mathscr{O}$-module and $\mathscr{E}^\vee$ its dual $\mathscr{O}$-module. Let $(e_i)_{1\leqslant i\leqslant m}$ be a local basis and $(e_i^\vee)_{1\leqslant i\leqslant m}$ its dual basis. Then the section $\sum_{i=1}^m e_i\otimes e_i^\vee$ is independent on the choice of local basis, hence extends to a global section on $\mathscr{E}\otimes_{\mathscr{O}}\mathscr{E}^\vee$.*

In particular, $\sum_{i=1}^m dz^i\otimes\partial_i$ is such a global section on $\Omega^1\otimes_{\mathscr{O}}\Theta$. Let $\nabla(1)$ be this section, then by the action (left multiplication) of Θ on $\mathscr{D}$, we have the following connection $\nabla\colon\mathscr{D}\to\Omega^1\otimes\mathscr{D}$:

$$\nabla(P) = \nabla(1)P. \tag{1.3.9}$$

it is then straightforward to verify that it is a flat connection. This connection is called the ***universal connection***. Let $\mathscr{M}$ be a left $\mathscr{D}$-module whose action denoted by $\alpha\colon\mathscr{D}\otimes_{\mathscr{O}}\mathscr{M}\to\mathscr{M}$. Then the corresponding flat connection is given by the composition

$$\mathscr{M}\longrightarrow\mathscr{D}\otimes_{\mathscr{O}}\mathscr{M}\xrightarrow{\nabla\otimes 1}\Omega^1\otimes_{\mathscr{O}}\mathscr{D}\otimes_{\mathscr{O}}\mathscr{M}\xrightarrow{1\otimes\alpha}\Omega^1\otimes_{\mathscr{O}}\mathscr{M}, \tag{1.3.10}$$

where the first map is $m\mapsto 1\otimes m$ and ∇ is the universal connection.

Conclusively, we have

1.3.11 Theorem *The category of left $\mathscr{D}$-modules is isomorphic to the category of $\mathscr{O}$-modules with flat connections.*

Proof: The equivalence is clear if we explain what a morphism between $\mathscr{O}$-modules with flat connections means. For $(\mathscr{M},\nabla^{\mathscr{M}})$ and $(\mathscr{N},\nabla^{\mathscr{N}})$ two $\mathscr{O}$-modules with connections, a morphism between them is an $\mathscr{O}$-module homomorphism $\varphi\colon\mathscr{M}\to\mathscr{N}$ making the following diagram commutative

$$\begin{tikzcd} \mathscr{M} \arrow[r, "\varphi"] \arrow[d, "\nabla^{\mathscr{M}}"'] & \mathscr{N} \arrow[d, "\nabla^{\mathscr{N}}"] \\ \Omega^1\otimes_{\mathscr{O}}\mathscr{M} \arrow[r, "1\otimes\varphi"] & \Omega^1\otimes_{\mathscr{O}}\mathscr{N} \end{tikzcd}$$

Then one can verify the equivalence. $\square$

1.4 Basic operations of $\mathscr{D}$-modules

Let $\mathscr{M}$ and $\mathscr{N}$ be two $\mathscr{O}$-modules, we have $\mathscr{O}$-modules $\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}$ and $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$. The following propositions discuss what happens when $\mathscr{M}$ or $\mathscr{N}$ or both of them are $\mathscr{D}$-modules.

1.4.1 Proposition *Let $\mathscr{M}^l$ be a left $\mathscr{D}$-module, $\mathscr{M}^r$ a right $\mathscr{D}$-module and $\mathscr{N}$ a $\mathscr{O}$-module. Then*

(i) $\mathscr{M}^l \otimes_{\mathscr{O}} \mathscr{N}$ has a left $\mathscr{D}$-module structure given by

$$P.(m \otimes n) = (P.m) \otimes n, \qquad \forall P \in \mathscr{D}, m \in \mathscr{M}^l, n \in \mathscr{N};$$

(ii) $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}^l, \mathscr{N})$ has a left $\mathscr{D}$-module structure given by

$$(P.\varphi)(m) = \varphi(P.m), \qquad \forall P \in \mathscr{D}, \varphi \in \mathscr{H}om_{\mathscr{O}}(\mathscr{M}^l, \mathscr{N}), m \in \mathscr{M}^l;$$

(iii) $\mathscr{N} \otimes_{\mathscr{O}} \mathscr{M}^r$ has a right $\mathscr{D}$-module structure given by

$$(n \otimes m).P = n \otimes (m.P), \qquad \forall P \in \mathscr{D}, m \in \mathscr{M}^r, n \in \mathscr{N};$$

(iv) $\mathscr{H}om_{\mathscr{O}}(\mathscr{N}, \mathscr{M}^r)$ has a right $\mathscr{D}$-module structure given by

$$(\varphi.P)(n) = \varphi(n).P, \qquad \forall P \in \mathscr{D}, \varphi \in \mathscr{H}om_{\mathscr{O}}(\mathscr{N}, \mathscr{M}^r), n \in \mathscr{N};$$

Those $\mathscr{D}$-module structures are called ***trivial*** $\mathscr{D}$-module structures in the sense that it essentially has nothing to do with (possible existed) $\mathscr{D}$-module structures on the $\mathscr{N}$.

However, if both of $\mathscr{M}$ and $\mathscr{N}$ are $\mathscr{D}$-modules, there do exist nontrivial $\mathscr{D}$-module structures on $\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}$ and $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$.

1.4.2 Proposition *Let $\mathscr{M}^l$, $\mathscr{N}^l$ be two left $\mathscr{D}$-modules and $\mathscr{M}^r, \mathscr{N}^r$ two right $\mathscr{D}$-modules. Then*

(i) $\mathscr{M}^l \otimes_{\mathscr{O}} \mathscr{N}^l$ has a left $\mathscr{D}$-module structure induced by

$$\theta.(m \otimes n) = (\theta.m) \otimes n + m \otimes (\theta.n), \qquad \forall \theta \in \Theta, m \in \mathscr{M}^l, n \in \mathscr{N}^l;$$

(ii) $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}^l, \mathscr{N}^l)$ has a left $\mathscr{D}$-module structure induced by

$$(\theta.\varphi)(m) = \theta.(\varphi(m)) - \varphi(\theta.m), \qquad \forall \theta \in \Theta, \varphi \in \mathscr{H}om_{\mathscr{O}}(\mathscr{M}^l, \mathscr{N}^l), m \in \mathscr{M}^l;$$

(iii) $\mathscr{M}^l \otimes_{\mathscr{O}} \mathscr{N}^r$ has a right $\mathscr{D}$-module structure induced by

$$(m \otimes n).\theta = m \otimes (n.\theta) - (\theta.m) \otimes n, \qquad \forall \theta \in \Theta, m \in \mathscr{M}^l, n \in \mathscr{N}^r;$$

(iv) $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}^l, \mathscr{N}^r)$ has a right $\mathscr{D}$-module structure induced by

$$(\varphi.\theta)(m) = \varphi(m).\theta + \varphi(\theta.m), \qquad \forall \theta \in \Theta, \varphi \in \mathscr{H}om_{\mathscr{O}}(\mathscr{M}^l, \mathscr{N}^r), m \in \mathscr{M}^l;$$

(v) $\mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{N}^l$ *has a right* $\mathscr{D}$*-module structure induced by*

$$(m \otimes n).\theta = (m.\theta) \otimes n - m \otimes (\theta.n), \qquad \forall \theta \in \Theta, m \in \mathscr{M}^r, n \in \mathscr{N}^l;$$

(vi) $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}^r, \mathscr{N}^r)$ *has a left* $\mathscr{D}$*-module structure induced by*

$$(\theta.\varphi)(m) = \varphi(m.\theta) - \varphi(m).\theta, \qquad \forall \theta \in \Theta, \varphi \in \mathscr{H}om_{\mathscr{O}}(\mathscr{M}^r, \mathscr{N}^r), m \in \mathscr{M}^r;$$

Proof: One can use Lemma 1.3.2 to verify above constructions. On the other hand, one can also use Proposition 1.3.5 to get (i) and (ii), and then apply the *left-right transformation* to get the others. □

Remark The above can be summarized into the following *Oda's rule* [Oda83]: left $= 0$, right $= 1$, $a \otimes_{\mathscr{O}} b = a + b$ and $\mathscr{H}om_{\mathscr{O}}(a, b) = -a + b$. Of cause one can always define trivial $\mathscr{D}$-module structures in any situation, but we keep using above setting whenever Oda's rule apply.

Since $\mathscr{D}$ is non-commutative and $\mathbb{C}$ is its center, we have the following bifunctors:

$$\mathscr{H}om_{\mathscr{D}} \colon \mathcal{M}^l(\mathscr{D})^{\mathrm{op}} \times \mathcal{M}^l(\mathscr{D}) \longrightarrow \mathcal{M}(\mathbb{C});$$
$$\mathscr{H}om_{\mathscr{D}^{\mathrm{op}}} \colon \mathcal{M}^r(\mathscr{D})^{\mathrm{op}} \times \mathcal{M}^r(\mathscr{D}) \longrightarrow \mathcal{M}(\mathbb{C});$$
$$\otimes_{\mathscr{D}} \colon \mathcal{M}^r(\mathscr{D}) \times \mathcal{M}^l(\mathscr{D}) \longrightarrow \mathcal{M}(\mathbb{C}).$$

Note that the first two bifunctors give $\mathcal{M}^l(\mathscr{D})$ and $\mathcal{M}^r(\mathscr{D})$ structures of $\mathbb{C}_M$-*enriched category* and the third gives a $\mathbb{C}_M$-*enriched pairing* of them. Therefore, we have a $\mathbb{C}_M$-*enriched category* together with a forgetful functor

$$\mathcal{M}^r(\mathscr{D}) \otimes_{\mathbb{C}_M} \mathcal{M}^l(\mathscr{D}) \xrightarrow{\otimes_{\mathscr{D}}} \mathcal{M}(\mathbb{C})$$

Note that $\mathcal{M}^l(\mathscr{D})$ (resp. $\mathcal{M}^r(\mathscr{D})$) can be embedded into it by sending each left $\mathscr{D}$-module $\mathscr{M}$ to $(\mathscr{D}, \mathscr{M})$ (resp. each right $\mathscr{D}$-module $\mathscr{M}$ to $(\mathscr{M}, \mathscr{D})$).

Then, the following proposition describe the adjunctive relation of $\otimes_{\mathscr{O}}$ and $\mathscr{H}om_{\mathscr{O}}$ in those categories.

1.4.3 Proposition *Let* $\mathscr{M}^l, \mathscr{N}^l, \mathscr{P}^l$ *be three left* $\mathscr{D}$*-modules and* $\mathscr{M}^r, \mathscr{N}^r, \mathscr{P}^r$ *three right* $\mathscr{D}$*-modules. We have canonical isomorphisms*

$$(\mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{N}^l) \otimes_{\mathscr{D}} \mathscr{P}^l \cong \mathscr{M}^r \otimes_{\mathscr{D}} (\mathscr{N}^l \otimes_{\mathscr{O}} \mathscr{P}^l),$$
$$\mathscr{H}om_{\mathscr{D}}(\mathscr{M}^l \otimes_{\mathscr{O}} \mathscr{N}^l, \mathscr{P}^l) \cong \mathscr{H}om_{\mathscr{D}}(\mathscr{M}^l, \mathscr{H}om_{\mathscr{O}}(\mathscr{N}^l, \mathscr{P}^l)),$$
$$\mathscr{H}om_{\mathscr{D}}(\mathscr{M}^l \otimes_{\mathscr{O}} \mathscr{N}^r, \mathscr{P}^r) \cong \mathscr{H}om_{\mathscr{D}}(\mathscr{M}^l, \mathscr{H}om_{\mathscr{O}}(\mathscr{N}^r, \mathscr{P}^r)),$$
$$\mathscr{H}om_{\mathscr{D}}(\mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{N}^l, \mathscr{P}^r) \cong \mathscr{H}om_{\mathscr{D}}(\mathscr{M}^r, \mathscr{H}om_{\mathscr{O}}(\mathscr{N}^l, \mathscr{P}^r)).$$

Proof: It is clear that, by viewing $\mathscr{M}^r, \mathscr{N}^l, \mathscr{P}^l$ as $\mathscr{O}$-modules, we have

$$(\mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{N}^l) \otimes_{\mathscr{O}} \mathscr{P}^l \cong \mathscr{M}^r \otimes_{\mathscr{O}} (\mathscr{N}^l \otimes_{\mathscr{O}} \mathscr{P}^l),$$

To show it induces an isomorphism

$$(\mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{N}^l) \otimes_{\mathscr{D}} \mathscr{P}^l \cong \mathscr{M}^r \otimes_{\mathscr{D}} (\mathscr{N}^l \otimes_{\mathscr{O}} \mathscr{P}^l),$$

it suffices to verify the following diagram is commutative

$$\begin{tikzcd}
(\mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{N}^l) \otimes_{\mathscr{O}} \mathscr{D} \otimes_{\mathscr{O}} \mathscr{P}^l \arrow[r, "\cong"] \arrow[d] & \mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{D} \otimes_{\mathscr{O}} (\mathscr{N}^l \otimes_{\mathscr{O}} \mathscr{P}^l) \arrow[d] \\
(\mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{N}^l) \otimes_{\mathscr{O}} \mathscr{P}^l \arrow[r, "\cong"] & \mathscr{M}^r \otimes_{\mathscr{O}} (\mathscr{N}^l \otimes_{\mathscr{O}} \mathscr{P}^l)
\end{tikzcd}$$

where the vertical arrows denote the difference of actions of $\mathscr{D}$ on modules on its left hand side and right hand side. Moreover, by Lemma 1.3.2, it suffices to consider only the actions of Θ. For any $m \in \mathscr{M}, n \in \mathscr{N}, p \in \mathscr{P}$ and $\theta \in \Theta$, through the left vertical morphism, we have

$$\begin{aligned}
(m \otimes n) \otimes \theta \otimes p \mapsto & ((m \otimes n).\theta) \otimes p - (m \otimes n) \otimes (\theta.p) \\
&= ((m.\theta) \otimes n - m \otimes (\theta.n)) \otimes p - (m \otimes n) \otimes (\theta.p) \\
&= (m.\theta) \otimes n \otimes p - m \otimes (\theta.n) \otimes p - m \otimes n \otimes (\theta.p);
\end{aligned}$$

through the right vertical morphism, we have

$$\begin{aligned}
m \otimes \theta \otimes (n \otimes p) \mapsto & (m.\theta) \otimes (n \otimes p) - m \otimes (\theta.(n \otimes p)) \\
&= (m.\theta) \otimes (n \otimes p) - m \otimes ((\theta.n) \otimes p + n \otimes (\theta.p)) \\
&= (m.\theta) \otimes (n \otimes p) - m \otimes (\theta.n) \otimes p - m \otimes n \otimes (\theta.p).
\end{aligned}$$

This shows the commutativity of the diagram and the desired isomorphism then follows.

Recall that for any three $\mathscr{O}$-modules $\mathscr{M}, \mathscr{N}, \mathscr{P}$, we have the canonical isomorphism

$$\begin{aligned}
\mathscr{H}om_{\mathscr{O}}(\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}, \mathscr{P}) &\xrightarrow{\sim} \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{H}om_{\mathscr{O}}(\mathscr{N}, \mathscr{P})) \\
\varphi &\longmapsto \tilde{\varphi} = (m \mapsto \varphi(m \otimes))
\end{aligned}$$

where $\varphi(m\otimes) \colon \mathscr{N} \to \mathscr{P}$ is the morphism maps each $n \in \mathscr{N}$ to $\varphi(m \otimes n)$. Then, to show it induces the last three isomorphisms, it suffices to verify it maps $\mathscr{D}$-linear morphisms to $\mathscr{D}$-linear morphisms in each case. Note that, by Lemma 1.3.2, it suffices to verify it commutes with the action of Θ.

If $\mathscr{M}, \mathscr{N}, \mathscr{P}$ are all left $\mathscr{D}$-modules, then for any $\theta \in \Theta$, $m \in \mathscr{M}$, $n \in \mathscr{N}$ and $\varphi \in \mathscr{H}om_{\mathscr{D}}(\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}, \mathscr{P})$, we have

$$\begin{aligned}
(\theta.\tilde{\varphi}(m))(n) &= \theta.(\tilde{\varphi}(m)(n)) - \tilde{\varphi}(m)(\theta.n) \\
&= \theta.(\varphi(m \otimes n)) - \varphi(m \otimes (\theta.n)) \\
&= \varphi(\theta.(m \otimes n)) - \varphi(m \otimes (\theta.n)) \\
&= \varphi((\theta.m) \otimes n) = \tilde{\varphi}(\theta.m)(n).
\end{aligned}$$

Hence $\tilde{\varphi} \in \mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{H}om_{\mathscr{O}}(\mathscr{N}, \mathscr{P}))$.

If $\mathscr{M}$ is a left $\mathscr{D}$-module and $\mathscr{N}, \mathscr{P}$ are right $\mathscr{D}$-modules, then for any $\theta \in \Theta$, $m \in \mathscr{M}$, $n \in \mathscr{N}$ and $\varphi \in \mathscr{H}om_{\mathscr{D}}(\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}, \mathscr{P})$, we have

$$\begin{aligned}(\theta.\tilde{\varphi}(m))(n) &= \tilde{\varphi}(m)(n.\theta) - (\tilde{\varphi}(m)(n)).\theta \\ &= \varphi(m \otimes (n.\theta)) - (\varphi(m \otimes n)).\theta \\ &= \varphi(m \otimes (n.\theta)) - \varphi((m \otimes n).\theta) \\ &= \varphi((\theta.m) \otimes n) = \tilde{\varphi}(\theta.m)(n).\end{aligned}$$

Hence $\tilde{\varphi} \in \mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{H}om_{\mathscr{O}}(\mathscr{N}, \mathscr{P}))$.

If $\mathscr{M}, \mathscr{P}$ are right $\mathscr{D}$-modules and $\mathscr{N}$ is a left $\mathscr{D}$-module, then for any $\theta \in \Theta$, $m \in \mathscr{M}$, $n \in \mathscr{N}$ and $\varphi \in \mathscr{H}om_{\mathscr{D}}(\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}, \mathscr{P})$, we have

$$\begin{aligned}(\tilde{\varphi}(m).\theta)(n) &= \tilde{\varphi}(m)(\theta.n) + (\tilde{\varphi}(m)(n)).\theta \\ &= \varphi(m \otimes (\theta.n)) + (\varphi(m \otimes n)).\theta \\ &= \varphi(m \otimes (\theta.n)) + \varphi((m \otimes n).\theta) \\ &= \varphi((m.\theta) \otimes n) = \tilde{\varphi}(m.\theta)(n).\end{aligned}$$

Hence $\tilde{\varphi} \in \mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{H}om_{\mathscr{O}}(\mathscr{N}, \mathscr{P}))$. □

Note that $\mathscr{D}$ itself is both a left and right $\mathscr{D}$-module. The following two propositions describe the $\mathscr{D}$-bimodule structures on tensor product with $\mathscr{D}$.

1.4.4 Proposition *Let $\mathscr{M}$ be a left $\mathscr{D}$-module.*

(i) $\mathscr{M} \otimes \mathscr{D}$ has a $\mathscr{D}$-bimodule structure:

- *the left $\mathscr{D}$-module structure is given by viewing $\mathscr{D}$ as a left $\mathscr{D}$-module and apply Proposition 1.4.2.(i);*
- *the right $\mathscr{D}$-module structure is the trivial one given by viewing $\mathscr{D}$ as a right $\mathscr{D}$-module.*

(ii) $\mathscr{D} \otimes \mathscr{M}$ has a $\mathscr{D}$-bimodule structure:

- *the left $\mathscr{D}$-module structure is the trivial one given by viewing $\mathscr{D}$ as a left $\mathscr{D}$-module;*
- *the right $\mathscr{D}$-module structure is given by viewing $\mathscr{D}$ as a right $\mathscr{D}$-module and apply Proposition 1.4.2.(v).*

(iii) The following two natural morphisms are $\mathscr{D}$-bilinear and inverse to each other:

$$\begin{aligned}\mathscr{M} \otimes \mathscr{D} &\longrightarrow \mathscr{D} \otimes \mathscr{M} \\ m \otimes P &\longmapsto (1 \otimes m).P\end{aligned} \qquad \begin{aligned}\mathscr{D} \otimes \mathscr{M} &\longrightarrow \mathscr{M} \otimes \mathscr{D} \\ P \otimes m &\longmapsto P.(m \otimes 1)\end{aligned}$$

(iv) Let $\mathscr{N}$ be an $\mathscr{O}$-module, then we have canonical isomorphisms (keep eyes on the $\mathscr{D}$-module structures on them) of left $\mathscr{D}$-modules

$$\mathscr{M}\otimes_{\mathscr{O}}(\mathscr{D}\otimes_{\mathscr{O}}\mathscr{N}) \cong (\mathscr{M}\otimes_{\mathscr{O}}\mathscr{D})\otimes_{\mathscr{O}}\mathscr{N} \cong (\mathscr{D}\otimes_{\mathscr{O}}\mathscr{M})\otimes_{\mathscr{O}}\mathscr{N} \cong \mathscr{D}\otimes_{\mathscr{O}}(\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}).$$

Moreover, if $\mathscr{M}$ and $\mathscr{N}$ are locally free $\mathscr{O}$-modules, then the above is a locally free $\mathscr{D}$-module.

PROOF: Note that, by Lemma 1.3.2, to verify a $\mathscr{D}$-bimodule structure, it suffices to verify the left and right action of Θ are compatible.

For any $m \in \mathscr{M}$, $P \in \mathscr{D}$ and $\theta, \theta' \in \Theta$, we have

$$\begin{aligned}(\theta.(m \otimes P)).\theta' &= ((\theta.m) \otimes P + m \otimes (\theta P)).\theta' \\ &= (\theta.m) \otimes (P\theta') + m \otimes (\theta P \theta') \\ &= \theta.(m \otimes (P\theta')) \\ &= \theta.((m \otimes P).\theta').\end{aligned}$$

This shows (i). Similarly, we have

$$\begin{aligned}(\theta.(P \otimes m)).\theta' &= ((\theta P) \otimes m).\theta' \\ &= (\theta P \theta') \otimes m - (\theta P) \otimes (\theta'.m) \\ &= \theta.((P\theta') \otimes m - P \otimes (\theta'.m)) \\ &= \theta.((P \otimes m).\theta').\end{aligned}$$

This shows (ii).

Let $\mathcal{L}$ and $\mathcal{R}$ denote the two morphisms in (iii), we have

$$\begin{aligned}\mathcal{L}(\theta.(m \otimes P)) &= \mathcal{L}((\theta.m) \otimes P + m \otimes (\theta P)) \\ &= (1 \otimes (\theta.m)).P + (1 \otimes m).(\theta P) \\ &= (1 \otimes (\theta.m) + (1 \otimes m).\theta).P \\ &= (\theta \otimes m).P \\ &= \theta.(1 \otimes m).P \\ &= \theta.\mathcal{L}(m \otimes P); \\ \mathcal{L}((m \otimes P).\theta) &= \mathcal{L}(m \otimes (P\theta)) \\ &= (1 \otimes m).P\theta \\ &= \mathcal{L}(m \otimes P).\theta.\end{aligned}$$

Therefore $\mathcal{L}$ is $\mathscr{D}$-bilinear. Similarly, so is $\mathcal{R}$. We also have

$$\begin{aligned}\mathcal{R}\mathcal{L}(m \otimes P) = \mathcal{R}((1 \otimes m).P) = \mathcal{R}(1 \otimes m).P = (m \otimes 1).P = m \otimes P, \\ \mathcal{L}\mathcal{R}(P \otimes m) = \mathcal{L}(P.(m \otimes 1)) = P.\mathcal{L}(m \otimes 1) = P.(1 \otimes m) = P \otimes m.\end{aligned}$$

This shows (iii).

Finally, (iv) can be shown by verify those isomorphisms of $\mathscr{O}$-modules commute with the action of Θ. $\square$

Since Oda's rule doesn't give a canonical nontrivial $\mathscr{D}$-module structure on the tensor product of two right $\mathscr{D}$-modules, the tensor product of a right $\mathscr{D}$-module with $\mathscr{D}$ is not expected to be a $\mathscr{D}$-bimodule. However, we still have another story about it.

1.4.5 Proposition *Let $\mathscr{M}$ be a right $\mathscr{D}$-module.*

(i) $\mathscr{M} \otimes \mathscr{D}$ has two right $\mathscr{D}$-module structures:

- *one is the trivial one by viewing $\mathscr{D}$ as a right $\mathscr{D}$-module;*
- *another one is given by viewing $\mathscr{D}$ as a left $\mathscr{D}$-module and apply Proposition 1.4.2.(v).*

(To distinguish them, we always use $(m \otimes P).Q$ to denote the right action of Q on $m \otimes P$ by the nontrivial module structure, and $(m \otimes P)Q$ for the trivial one).

(ii) The morphism

$$\begin{aligned} \iota\colon \mathscr{M} \otimes \mathscr{D} &\longrightarrow \mathscr{M} \otimes \mathscr{D} \\ m \otimes P &\longmapsto (m \otimes 1).P \end{aligned}$$

is an involution, exchanges the two right $\mathscr{D}$-module structures and is the identity on the submodule $\mathscr{M} \otimes 1$. Moreover it is the unique one satisfying those properties.

(iii) Recall that there are two $\mathscr{O}$-module structures (called left *and* right*) on $\mathscr{D}$, hence on each $F_p\mathscr{D}$, and on the tensor products $\mathscr{M} \otimes_{\mathscr{O}} F_p\mathscr{D}$. Let $(\mathscr{M} \otimes_{\mathscr{O}} F_p\mathscr{D})_l$ (resp. $(\mathscr{M} \otimes_{\mathscr{O}} F_p\mathscr{D})_r$) denote the $\mathscr{O}$-module whose module structure comes from the left (resp. right) $\mathscr{O}$-module structure on $\mathscr{D}$. On the other hand, the two right $\mathscr{D}$-module structures induce two $\mathscr{O}$-module structures on each $\mathscr{M} \otimes_{\mathscr{O}} F_p\mathscr{D}$: the one from the trivial right $\mathscr{D}$-module structure coincide with $(\mathscr{M} \otimes_{\mathscr{O}} F_p\mathscr{D})_r$, while the one from the nontrivial right $\mathscr{D}$-module structure coincide with $(\mathscr{M} \otimes_{\mathscr{O}} F_p\mathscr{D})_l$. Moreover, ι induces isomorphisms of $\mathscr{O}$-modules between $(\mathscr{M} \otimes_{\mathscr{O}} F_p\mathscr{D})_l$ and $(\mathscr{M} \otimes_{\mathscr{O}} F_p\mathscr{D})_r$.*

Proof: For any $m \in \mathscr{M}$, $P, Q \in \mathscr{D}$, we have

$$\iota((m \otimes P)Q) = \iota(m \otimes PQ) = (m \otimes 1).PQ = \iota(m \otimes P).Q.$$

On the other hand, for any $\theta \in \Theta$, we have

$$\begin{aligned} (m \otimes P).\theta &= (m.\theta) \otimes P - m \otimes (\theta P) \\ &= ((m.\theta) \otimes 1)P - (m \otimes \theta)P \\ &= ((m \otimes 1).\theta)P = \iota(m \otimes \theta)P \end{aligned}$$

Since $\mathscr{D}$ is generated by $\mathscr{O}$ and Θ, the above implies

$$(m \otimes P).Q = \iota(m \otimes Q)P.$$

Then we have

$$\begin{aligned}\iota((m \otimes P).\theta) &= \iota((m.\theta) \otimes P - m \otimes (\theta P)) \\ &= ((m.\theta) \otimes 1).P - (m \otimes 1).\theta P \\ &= (m \otimes \theta).P = \iota(m \otimes P)\theta.\end{aligned}$$

This shows ι exchanges the two right $\mathscr{D}$-module structures. It is clear that ι is the identity on the submodule $\mathscr{M} \otimes 1$. Then, we have

$$\iota\iota(m \otimes P) = \iota\iota((m \otimes 1)P) = \iota(\iota(m \otimes 1).P) = \iota\iota(m \otimes 1)P = m \otimes P.$$

This shows ι is an involution. The uniqueness is clear.

(iii) is clear since by Lemma 1.1.2, for any $f \in \mathscr{O}$ and $P \in \mathscr{D}$, fP and Pf are in the same filtrations. □

1.5 Left-right transformation

Let $\mathscr{M}$ be a left $\mathscr{D}$-module, we have seen (Proposition 1.4.2.(v)) that $\omega \otimes_{\mathscr{O}} \mathscr{M}$ is a right $\mathscr{D}$-module. Conversely, let $\mathscr{N}$ be a right $\mathscr{D}$-module, $\mathscr{H}om_{\mathscr{O}}(\omega, \mathscr{N})$ is a left $\mathscr{D}$-module. In this way, we get two functors, one from $\mathcal{M}^l(\mathscr{D})$ to $\mathcal{M}^r(\mathscr{D})$ and another goes conversely. It turns out that they are inverse to each other.

1.5.1 Theorem *The functors $\omega \otimes_{\mathscr{O}} -$ and $\mathscr{H}om_{\mathscr{O}}(\omega, -)$ form a pair of adjoint equivalences between $\mathcal{M}^l(\mathscr{D})$ and $\mathcal{M}^r(\mathscr{D})$.*

To prove the theorem, first notice that ω is an invertible $\mathscr{O}$-module. Indeed, the canonical evaluation morphism

$$\mathscr{H}om_{\mathscr{O}}(\omega, \mathscr{O}) \otimes_{\mathscr{O}} \omega \longrightarrow \mathscr{O}$$

is an isomorphism. To see this, one only needs to choose a local coordinate system (z^i), and then the morphism is bijective since it admits an inverse

$$f \longmapsto \varphi \otimes (f dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m)$$

where φ maps $f dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m$ to f. For this reason, we use ω^{-1} to denote its inverse sheaf $\mathscr{H}om_{\mathscr{O}}(\omega, \mathscr{O})$.

In general, we have

1.5.2 Lemma *Let $\mathscr{L}$ be an invertible $\mathscr{O}$-module (with inverse sheaf $\mathscr{L}^{-1}$) and $\mathscr{R}$ an $\mathscr{O}$-algebra. Then $\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}$ is also an $\mathscr{O}$-algebra. Moreover, $\mathscr{L} \otimes_{\mathscr{O}} -$ defines an equivalence of categories from the category $\mathcal{M}(\mathscr{R})$ of left $\mathscr{R}$-modules to the category $\mathcal{M}(\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1})$ of left $\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}$-modules.*

PROOF: The product is given by the composition

$$(\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}) \otimes_{\mathscr{O}} (\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}) \xrightarrow{\langle,\rangle} \mathscr{L} \otimes_{\mathscr{O}} (\mathscr{R} \otimes_{\mathscr{O}} \mathscr{R}) \otimes_{\mathscr{O}} \mathscr{L}^{-1} \longrightarrow \mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}$$

where the first comes from the pairing $\mathscr{L}^{-1} \otimes_{\mathscr{O}} \mathscr{L} \to \mathscr{O}$ and the second comes from the product of $\mathscr{R}$.

Let $\mathscr{M}$ be a left $\mathscr{R}$-module, then the left $\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}$-module structure on $\mathscr{L} \otimes_{\mathscr{O}} \mathscr{M}$ is given by

$$(\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}) \otimes_{\mathscr{O}} (\mathscr{L} \otimes_{\mathscr{O}} \mathscr{M}) \xrightarrow{\langle,\rangle} \mathscr{L} \otimes_{\mathscr{O}} (\mathscr{R} \otimes_{\mathscr{O}} \mathscr{M}) \longrightarrow \mathscr{L} \otimes_{\mathscr{O}} \mathscr{M}$$

where the first comes from the pairing and the second comes from the left $\mathscr{R}$-module structure on $\mathscr{M}$. Then it is clear $\mathscr{L} \otimes_{\mathscr{O}} -$ defines a functor from $\mathcal{M}(\mathscr{R})$ to $\mathcal{M}(\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1})$.

Conversely, let $\mathscr{N}$ be a left $\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}$-module, then we define the left $\mathscr{R}$-module structure on $\mathscr{L}^{-1} \otimes_{\mathscr{O}} \mathscr{N}$ is given by

$$\mathscr{R} \otimes_{\mathscr{O}} (\mathscr{L}^{-1} \otimes_{\mathscr{O}} \mathscr{N}) \longrightarrow (\mathscr{L}^{-1} \otimes_{\mathscr{O}} \mathscr{L}) \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1} \otimes_{\mathscr{O}} \mathscr{N} \longrightarrow \mathscr{L}^{-1} \otimes_{\mathscr{O}} \mathscr{N}$$

where the first comes from the inverse of the pairing and the second comes from the left $\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1}$-module structure on $\mathscr{N}$. Therefore $\mathscr{L}^{-1} \otimes_{\mathscr{O}} -$ defines a functor from $\mathcal{M}(\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{-1})$ to $\mathcal{M}(\mathscr{R})$.

Then it is easy to verify those functors are inverse to each other hence equivalence of categories. □

1.5.3 Lemma *Let $\mathscr{L}$ be an invertible $\mathscr{O}$-module (with inverse sheaf $\mathscr{L}^{-1}$). Then $\mathscr{L} \otimes_{\mathscr{O}} -$ and $\mathscr{L}^{-1} \otimes_{\mathscr{O}} -$ are adjoint to each other.*

PROOF: In fact, they are inverse to each other, *a fortiori* adjoint. □

1.5.4 Lemma *Let $\mathscr{L}$ be an invertible $\mathscr{O}$-module (with inverse sheaf $\mathscr{L}^{-1}$). Then we have natural isomorphisms*

$$\mathscr{L}^{-1} \otimes_{\mathscr{O}} - \cong \mathscr{H}om_{\mathscr{O}}(\mathscr{L}, -),$$
$$\mathscr{L} \otimes_{\mathscr{O}} - \cong \mathscr{H}om_{\mathscr{O}}(\mathscr{L}^{-1}, -).$$

PROOF: Because the adjoint functor of a given functor is unique up to a unique natural isomorphism. □

In our case, we additionally have

1.5.5 Lemma *There is an isomorphism of $\mathscr{O}$-algebras*

$$\mathscr{D}^{\mathrm{op}} \longrightarrow \omega \otimes_{\mathscr{O}} \mathscr{D} \otimes_{\mathscr{O}} \omega^{-1}.$$

PROOF: First, the right $\mathscr{D}$-module structure on ω defines a homomorphism

$$\mathscr{D}^{\mathrm{op}} \longrightarrow \mathscr{E}nd(\omega).$$

On the other hand, the homomorphism

$$\omega \otimes_{\mathscr{O}} \mathscr{D} \otimes_{\mathscr{O}} \omega^{-1} \longrightarrow \omega \otimes_{\mathscr{O}} \mathscr{E}nd(\mathscr{O}) \otimes_{\mathscr{O}} \omega^{-1} \longrightarrow \mathscr{E}nd(\omega).$$

where the first comes from the embedding $\mathscr{D} \to \mathscr{E}nd(\mathscr{O})$ and the second comes from the composition

$$\mathscr{H}om(\mathscr{O}, \omega) \otimes \mathscr{H}om(\mathscr{O}, \mathscr{O}) \otimes \mathscr{H}om(\omega, \mathscr{O}) \longrightarrow \mathscr{H}om(\omega, \omega).$$

Then it is not difficult to verify the above two homomorphisms are into $\mathscr{E}nd(\omega)$. Then, to show there is a desired isomorphism, it suffices to show their image in $\mathscr{E}nd(\omega)$ coincide. But this is clear since both of $\mathscr{D}^{\mathrm{op}}$ and $\omega \otimes_{\mathscr{O}} \mathscr{D} \otimes_{\mathscr{O}} \omega^{-1}$ are generated by the images of $\mathscr{O}$ and Θ in them. $\square$

Now, combining the previous lemmas, we proves Theorem 1.5.1.

From now on, for a left $\mathscr{D}$-module $\mathscr{M}$, we will use $\mathscr{M}^r$ to denote the right $\mathscr{D}$-module $\omega \otimes_{\mathscr{O}} \mathscr{M}$; for a right $\mathscr{D}$-module $\mathscr{N}$, we will use $\mathscr{N}^l$ to denote the right $\mathscr{D}$-module $\mathscr{H}om_{\mathscr{O}}(\omega, \mathscr{N})$. We call such corresponding ***left-right transformation***.

On a local coordinate chart, one can choose a volume form dV. Then we locally have an isomorphism

$$\omega \xrightarrow{\cong} \mathscr{O}: \quad f dV \longmapsto f. \tag{1.5.6}$$

Then the *left-right transformation* has a local expression as follows.

1.5.7 Proposition *Let (z^i) be a local coordinate system. Fix a volume form*

$$dV = A dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m,$$

then after the identification $\omega \otimes_{\mathscr{O}} \mathscr{D} \otimes_{\mathscr{O}} \omega^{-1} \cong \mathscr{D}$, the isomorphism in Lemma 1.5.5 can be expressed as

$$(P = \sum_{\lambda} f_\lambda \partial^\lambda) \longmapsto P^{\sim} = \sum_{\lambda} \frac{1}{A}(-\partial)^\lambda \circ (f_\lambda A).$$

In particularly, for $\mathscr{M}$ a left $\mathscr{D}$-module (resp. right $\mathscr{D}$-module), the left action (resp. right action) of P on $\mathscr{M}$ is the same as the right action (resp. left action) of $P^{\sim}$ on $\omega \otimes_{\mathscr{O}} \mathscr{M}$ (resp. $\mathscr{H}om_{\mathscr{O}}(\omega, \mathscr{M})$) being identified with $\mathscr{M}$ through (1.5.6).

PROOF: First note that, the right $\mathscr{D}$-module structure on ω corresponds each differential operator

$$P = \sum_{\lambda} f_\lambda \partial^\lambda$$

to the endomorphism

$$gdz^1 \wedge dz^2 \wedge \cdots \wedge dz^m \longmapsto \sum_{\lambda} (-\mathcal{L}_\partial)^\lambda (f_\lambda g dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m).$$

Then through the identification (1.5.6), the above endomorphism becomes the composition

$$\begin{aligned} g \longmapsto & gdV = gAdz^1 \wedge dz^2 \wedge \cdots \wedge dz^m \\ \longmapsto & \sum_{\lambda} (-\mathcal{L}_\partial)^\lambda (f_\lambda g A dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m) \\ & = \sum_{\lambda} (-\partial)^\lambda (f_\lambda g A) dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m \\ & = \left(\sum_{\lambda} \frac{1}{A} (-\partial)^\lambda (f_\lambda g A) \right) dV \\ \longmapsto & \sum_{\lambda} \frac{1}{A} (-\partial)^\lambda (f_\lambda g A). \end{aligned}$$

This shows the desired expression. □

1.5.8 Proposition *Let $\mathscr{M}^l, \mathscr{N}^l$ be two left $\mathscr{D}$-modules and $\mathscr{M}^r, \mathscr{N}^r$ their corresponding right $\mathscr{D}$-modules. Then there is a canonical isomorphism of right $\mathscr{D}$-modules.*

$$\mathscr{M}^r \otimes_{\mathscr{O}} \mathscr{N}^l \cong \mathscr{N}^r \otimes_{\mathscr{O}} \mathscr{M}^l.$$

PROOF: This isomorphism is given by

$$(\omega \otimes m) \otimes n \longmapsto (\omega \otimes n) \otimes m.$$

To show it is a homomorphism of right $\mathscr{D}$-modules, it suffices to show it commutes with the right action of Θ. Indeed, for any $\theta \in \Theta$, we have

$$\begin{aligned} ((\omega \otimes m) \otimes n).\theta &= (\omega \otimes m).\theta \otimes n - (\omega \otimes m) \otimes (\theta.n) \\ &= (\omega.\theta \otimes m) \otimes n - (\omega \otimes (\theta.m)) \otimes n - (\omega \otimes m) \otimes (\theta.n) \\ &\mapsto (\omega.\theta \otimes n) \otimes m - (\omega \otimes n) \otimes (\theta.m) - (\omega \otimes (\theta.n)) \otimes m \\ &= (\omega \otimes n).\theta \otimes m - (\omega \otimes n) \otimes (\theta.m) \\ &= ((\omega \otimes n) \otimes m).\theta. \end{aligned}$$

Hence the claim follows. □

1.6 ¶The sheaf of principal parts

In this subsection, we deal with the notion of sheaves of principal parts. This notion has been well-studied in [TCGA] and [Kan77].

Let M be a complex manifold. Then we have the following canonical morphisms:

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\mathrm{pr}_i} M$$

where M is the diagonal morphism and pr_i is the projection to i-th factor. Let $\mathscr{J}_\Delta$ be the kernel of the canonical homomorphism $\Delta^\sharp \colon \Delta^{-1}(\mathscr{O}_{M\times M}) \to \mathscr{O}_M$. Let $M_\Delta^{(p)}$ be the locally ringed space whose underlying topological space is M and whose structure sheaf is $\Delta^{-1}(\mathscr{O}_{M\times M})/\mathscr{J}_\Delta^{p+1}$. Then we have an inductive system of locally ringed spaces

$$M = M_\Delta^{(0)} \longrightarrow M_\Delta^{(1)} \longrightarrow M_\Delta^{(2)} \longrightarrow \cdots$$

over $M \times M$ such that $\Delta \colon M \to M \times M$ factors through each structure morphism $\Delta^{(p)} \colon M_\Delta^{(p)} \to M \times M$. Therefore, $M_\Delta^{(p)}$ is called the ***p*-th infinitesimal neighborhood** of M with respect to Δ.

Note that we have morphisms

$$M_\Delta^{(p)} \xrightarrow{\Delta^{(p)}} M \times M \xrightarrow{\mathrm{pr}_i} M$$

where $(\Delta^{(p)})^\sharp$ is the quotient homomorphism. Let pr_i^* be the homomorphism $(\mathrm{pr}_i \circ \Delta^{(p)})^\sharp \colon \mathscr{O}_M \to \mathscr{O}_{M_\Delta^{(p)}}$. Then each pr_i^* gives $\mathscr{O}_{M_\Delta^{(p)}}$ an *augmented $\mathscr{O}_M$-algebra* structure. Then, the ***sheaf of principal parts*** of M is sheaf $\mathscr{O}_{M_\Delta^{(p)}}$ equipped with the $\mathscr{O}_M$-algebra structure from pr_1^*, denoted by $\mathscr{P}_M^{(p)}$. From now on, we will identify $\mathscr{O}_M$ with its image in $\mathscr{P}_M^{(p)}$. On the other hand, $d^{(p)} := \mathrm{pr}_2^* \colon \mathscr{O}_M \to \mathscr{P}_M^{(p)}$ is called the ***universal differential operator of order*** p. For any section f of $\mathscr{O}_M$, the section $df = d^{(1)}(f) - f$ of $\mathscr{P}_M^{(1)}$ is called the ***(holomorphic) differential*** of f.

1.6.1 Lemma *As $\mathscr{O}_M$-algebras, $\mathscr{P}_M^{(p)} \cong \mathscr{O}_M \oplus \mathscr{J}_\Delta/\mathscr{J}_\Delta^{p+1}$.*

Proof: This follows from the fact that the augmented $\mathscr{O}_M$-algebra structure on $\mathscr{P}_M^{(p)}$ makes the following exact sequence split:

$$0 \longrightarrow \mathscr{J}_\Delta/\mathscr{J}_\Delta^{p+1} \longrightarrow \mathscr{P}_M^{(p)} \longrightarrow \mathscr{O}_M \longrightarrow 0 \qquad \square$$

In particular, we can see that for any section f of $\mathscr{O}_M$, the section df belongs to $\mathscr{J}_\Delta/\mathscr{J}_\Delta^2$. In this sense, we call $\mathscr{J}_\Delta/\mathscr{J}_\Delta^2$ the ***sheaf of (holomorphic) differentials***. Later, we sill show that it is isomorphic to the sheaf of holomorphic 1-forms.

Note that the ideal $\mathscr{J}_\Delta$ gives $\Delta^{-1}(\mathscr{O}_{M\times M})$ a $\mathscr{J}_\Delta$-adic filtration, hence we have the associated graded algebra

$$\mathscr{G}r_\bullet(\mathscr{P}_M) := \bigoplus_{p\geqslant 0} \mathscr{J}_\Delta^p/\mathscr{J}_\Delta^{p+1}. \tag{1.6.2}$$

Since $\mathscr{G}r_0(\mathscr{P}_M) = \Delta^{-1}(\mathscr{O}_{M\times M})/\mathscr{J}_\Delta \cong \mathscr{O}_M$, we see that $\mathscr{G}r_\bullet(\mathscr{P}_M)$ is a graded $\mathscr{O}_M$-algebra. Moreover, this $\mathscr{O}_M$-algebra structure coincides with those from pr_1^* and pr_2^*. Then the $\mathscr{O}_M$-linear multiplication of $\mathscr{G}r_\bullet(\mathscr{P}_M)$ induces a surjective homomorphism of graded $\mathscr{O}_M$-algebras

$$\mathbb{S}^\bullet_{\mathscr{O}_M}(\mathscr{G}r_1(\mathscr{P}_M)) \longrightarrow \mathscr{G}r_\bullet(\mathscr{P}_M).$$

We will show

1.6.3 Theorem *Each $\mathscr{P}_M^{(p)}$ is a locally free $\mathscr{O}_M$-module of finite rank.*

PROOF: It suffices to show that $\mathscr{P}_{M,x}^{(p)}$ is a free $\mathscr{O}_{M,x}$-module of finite rank for any point $x \in M$. By choosing a local coordinate (z,w) of $M\times M$ we reduce to the case where x is the origin $(0,0) \in \mathbb{C}^m \times \mathbb{C}^m$.

Let $f(z,w)$ be a germ of holomorphic functions at $(0,0) \in \mathbb{C}^m \times \mathbb{C}^m$. Then since we have invertible holomorphic linear transformation $(z,w) \mapsto (z, w-z)$, it can be uniquely written as

$$f(z,w) = \sum_{\lambda\in\mathbb{N}^m} f_\lambda(z)(w-z)^\lambda,$$

where $f_\lambda(z)$ are germs of holomorphic functions at $0 \in \mathbb{C}^m$. In this way, we obtain an injective homomorphism of $\mathscr{O}_{M,x}$-algebras

$$\mathscr{O}_{M\times M,(x,x)} \longrightarrow \mathscr{O}_{M,x}[[w-z]],$$

where $\mathscr{O}_{M,x}[[w-z]]$ denotes the formal power series ring over $\mathscr{O}_{M,x}$. We identify $\mathscr{O}_{M\times M,(x,x)}$ with its image. Note that $\Delta_x^\sharp$ maps $f(z,w)$ to $f(z,z)$. Then we have

$$\mathscr{J}_{\Delta,x} = \{f(z,w); f_0(z) = 0\} \subset \left\{ \sum_{|\lambda|\geqslant 1} f_\lambda(z)(w-z)^\lambda \right\}.$$

Consequently, we have

$$\mathscr{J}_{\Delta,x}^p = \{f(z,w); f_\lambda(z) = 0, \forall|\lambda| < p\} \subset \left\{ \sum_{|\lambda|\geqslant p} f_\lambda(z)(w-z)^\lambda \right\}.$$

Therefore

$$\mathscr{P}_{M,x}^{(p)} \cong \{f(z,w); f_\lambda(z) = 0, \forall|\lambda| > p\} = \left\{ \sum_{|\lambda|\leqslant p} f_\lambda(z)(w-z)^\lambda \right\},$$

which is a free $\mathscr{O}_{M,x}$-module with the finite basis

$$\{(w-z)^\lambda; |\lambda| \leqslant p\}. \qquad \square$$

From the above proof, it is clear that

1.6.4 Corollary *The homomorphism* (1.6.2) *is an isomorphism.*

1.6.5 Corollary $\mathscr{G}r_1(\mathscr{P}_M)$ *is isomorphic to the sheaf of holomorphic* 1*-forms.*

PROOF: From previous reasoning, we see that being restricted to a coordinate chart, $\mathscr{G}r_1(\mathscr{P}_M)$ has a basis $\{w^i - z^i; 1 \leqslant i \leqslant m\}$. Then, the map $w^i - z^i \mapsto dz^i$ gives an isomorphism to Ω^1_M which is compatible with the transition maps. Hence the conclusion. $\square$

Note that, after identify $\mathscr{G}r_1(\mathscr{P}_M)$ with Ω^1_M, the morphism $d\colon \mathscr{O}_M \to \Omega^1_M$ has the following explicit expression in local coordinate:

$$f \longmapsto \sum_{i=1}^m \frac{\partial f}{\partial z^i} dz^i.$$

One can see it coincides with the usual differential.

1.6.6 Theorem *For each p, we have*

$$\mathscr{H}om_{\mathscr{O}_M}(\mathscr{P}^{(p)}_M, \mathscr{O}_M) \cong \mathscr{D}iff^p_M.$$

PROOF: We may assume we are working on a coordinate chart. Then $d^{(p)}\colon \mathscr{O}_M \to \mathscr{P}^{(p)}_M$ maps each f to

$$d^{(p)}(f) = \sum_{|\lambda| \leqslant p} \frac{\partial^\lambda(f)(z)}{\lambda!}(w-z)^\lambda.$$

Then it is straightforward to verify that $d^{(p)}$ is a differential operator of order p from $\mathscr{O}_M$ to $\mathscr{P}^{(p)}_M$. Consequently, for each homomorphism of $\mathscr{O}_M$-modules $P\colon \mathscr{P}^{(p)}_M \to \mathscr{O}_M$, the composition $P \circ d^{(p)}$ is a differential operator of order p on $\mathscr{O}_M$. This gives the desired homomorphism.

It remains to show it is bijective. To show this, we construct its inverse as follows: for any P a differential operator of order p on $\mathscr{O}_M$, let $\widetilde{P}\colon \mathscr{P}^{(p)}_M \to \mathscr{O}_M$ be defined by

$$\widetilde{P}(w^\lambda) = P(z^\lambda)$$

(note that $\{w^\lambda; |\lambda| \leqslant p\}$ is also a basis of $\mathscr{P}_M^{(p)}$). Then, we have

$$\begin{aligned}
\widetilde{P}(d^{(p)}(f)) &= \widetilde{P}(\sum_{|\lambda|\leqslant p} \frac{\partial^\lambda(f)(z)}{\lambda!}(w-z)^\lambda) \\
&= \sum_{|\lambda|\leqslant p} \frac{\partial^\lambda(f)(z)}{\lambda!} \sum_{\mu\leqslant\lambda} \binom{\lambda}{\mu}(-z)^{\lambda-\mu}\widetilde{P}(w^\mu) \\
&= \sum_{|\lambda|\leqslant p} \frac{\partial^\lambda(f)(z)}{\lambda!} \sum_{\mu\leqslant\lambda} \binom{\lambda}{\mu}(-z)^{\lambda-\mu}P(z^\mu) \\
&= \sum_{|\lambda|\leqslant p} \frac{\partial^\lambda(f)(z)}{\lambda!}[P, z^\lambda](1).
\end{aligned}$$

If we know that $\mathscr{D}\!\mathit{iff}_M$ is generated by $\mathscr{D}\!\mathit{iff}_M^1$, then we only need to verify the last line equals $P(f)$ for $P = \partial^\mu$ where $|\mu| = p$. Hence the conclusion.□

The previous theorem can also be obtained from the following one.

1.6.7 Theorem *For each p, we have*

$$\mathscr{H}\!om_{\mathscr{O}_M}(\mathscr{D}\!\mathit{iff}_M^p, \mathscr{O}_M) \cong \mathscr{P}_M^{(p)}.$$

Proof: We may assume we are working on a coordinate chart. For any $\varphi\colon \mathscr{D}\!\mathit{iff}_M^p \to \mathscr{O}_M$, we define the corresponding section in $\mathscr{P}_M^{(p)}$ as

$$\sum_{|\lambda|\leqslant p} \varphi(\partial^\lambda)(w-z)^\lambda.$$

Then, the bijectivity follows from the fact that $\{\partial^\lambda; |\lambda| \leqslant p\}$ form a basis of the free $\mathscr{O}_M$-module $\mathscr{D}\!\mathit{iff}_M^p$. □

Remark Note that the product $M \times M$ is took in the category of complex manifolds. If one takes products in the category of locally ringed spaces, then one will have

$$\Delta^{-1}(\mathscr{O}_{M\times M}) = \mathscr{O}_M \otimes_{\mathbb{C}} \mathscr{O}_M$$

and the result sheaves, denoted by $\mathscr{P}_{M/\mathbb{C}}^{(p)}$ instead of $\mathscr{P}_M^{(p)}$, is not locally free. However, one can see that

$$\mathscr{H}\!om_{\mathscr{O}_M}(\mathscr{P}_{M/\mathbb{C}}^{(p)}, \mathscr{O}_M) \cong \mathscr{D}\!\mathit{iff}_M^p$$

by the recursive definition. Therefore $\mathscr{P}_{M/\mathbb{C}}^{(p)}$ and $\mathscr{P}_M^{(p)}$ have the same dual $\mathscr{O}_M$-modules, hence the later is the double dual of the first.

1.7 The symplectic structure of the cotangent bundle

Note that since $[F_p\mathscr{D}, F_q\mathscr{D}] \subset F_{p+q-1}\mathscr{D}$, for any differential operators P and Q, we have

$$\operatorname{ord}([P,Q]) \leqslant \operatorname{ord}(P) + \operatorname{ord}(Q) - 1.$$

Therefore $(P,Q) \mapsto \sigma([P,Q])$ defines homomorphisms

$$F_p\mathscr{D} \otimes F_q\mathscr{D} \longrightarrow \mathscr{H}om(\mathbb{S}^{p+q-1}(\mathscr{O}), \mathscr{O})$$

factorizing through $\mathrm{gr}_{p+q-1}(\mathscr{D})$ and annihilating $F_{p-1}\mathscr{D} \otimes F_q\mathscr{D}$ as well as $F_p\mathscr{D} \otimes F_{q-1}\mathscr{D}$. Therefore they induces homomorphisms

$$\mathrm{gr}_p(\mathscr{D}) \otimes \mathrm{gr}_q(\mathscr{D}) \longrightarrow \mathrm{gr}_{p+q-1}(\mathscr{D}).$$

In this way, we obtain a graded binary operation of degree -1

$$\{-,-\}\colon\ \mathrm{gr}_\bullet(\mathscr{D}) \otimes \mathrm{gr}_\bullet(\mathscr{D}) \longrightarrow \mathrm{gr}_\bullet(\mathscr{D}) \tag{1.7.1}$$

satisfying

$$\{\sigma(P), \sigma(Q)\} = \sigma([P,Q])$$

for arbitrary differential operators P and Q. Note that it has the following properties

(a) $\{-,-\}$ makes $\mathrm{gr}(\mathscr{D})$ a Lie algebra.

(b) For any sections f, g, h of $\mathrm{gr}(\mathscr{D})$, we have the *Leibniz rule*:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

In this way, we get a *Coisson algebra* (i.e. *commutative Poisson algebra*) structure on $\mathrm{gr}(\mathscr{D})$.

Remark A ***Poisson algebra*** is an associative algebra A equipped with a binary bracket $\{-,-\}$ such that

(P1) $\{-,-\}$ makes A a Lie algebra.

(P2) For any $f, g, h \in A$, we have the *Leibniz rule*:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

A ***Coisson algebra*** is a Poisson algebra whose underlying associative algebra is commutative.

The Coisson structure can be obtained in another way: the *symplectic structure* of the *cotangent bundle* T^*M.

Like the *tangent bundle* is the vector bundle associated to the locally free $\mathscr{O}_M$-module Θ_M, the cotangent bundle is the vector bundle $\pi\colon T^*M \to M$

associated to the locally free $\mathscr{O}_M$-module Ω^1_M. Let $(z^i)_{1\leqslant i\leqslant m}$ be a local coordinate system of M on a chart U. Since $(dz^i)_{1\leqslant i\leqslant m}$ is a basis of Ω^1_M on U, we see that T^*M has a local coordinate system $(z^i;\xi^j)_{1\leqslant i,j\leqslant m}$ on chart $\pi^{-1}(U)$ such that: any local section $\omega=\sum_{i=1}^m f_i dz^i$ of the sheaf Ω^1_M on U is corresponding to the following section of the projection π on U:

$$z=(z^1,z^2,\cdots,z^m)\longmapsto (z,\omega):=(z^1,z^2,\cdots,z^m;f_1,f_2,\cdots,f_m)$$

written under this coordinate system.

Note that, for any morphism $\varphi\colon N\to M$, we have a canonical homomorphism $\varphi^*\Omega^1_M\to\Omega_N$, hence a morphism $N\times_M T^*M\to T^*N$. Taking φ to be $\pi\colon T^*M\to M$, we obtained a morphism

$$\pi_\pi\colon T^*M\times_M T^*M\to T^*T^*M.$$

Then we get a section of the projection $\pi\colon T^*T^*M\to T^*M$ by compositing above with the diagonal morphism $T^*M\to T^*M\times_M T^*M$. In other words, we obtain a canonical 1-form α_M on T^*M, called the ***tautological form***. Let $\omega_M=-d\alpha_M$. Note that, under a local coordinate system $(z^i;\xi^j)_{1\leqslant i,j\leqslant m}$, these forms can be written as

$$\alpha_M=\sum_{i=1}^m \xi^i dz^i,\qquad \omega_M=\sum_{i=1}^m dz^i\wedge d\xi^i. \tag{1.7.2}$$

Then it is clear that ω_M is a *symplectic form* on T^*M, called the ***canonical symplectic form***.

Remark (A little bit sympletic geometry) On a manifold of even dimension $2m$, a ***symplectic form*** is a closed 2-form σ such that $\omega^m=\omega\wedge\omega\wedge\cdots\wedge\omega$ vanishes nowhere (hence is a volume form).

On a vector space, a ***symplectic structure*** is a skew-symmetric non-degenerate bilinear form. A closed 2-form is symplectic if and only if it induces a symplectic strucutre on the tangent space at every point.

Given a ***symplectic manifold***, that is a manifold M with a symplectic form ω, we can obtain a Poisson algebra structure on $\mathscr{O}_M$ as follows. First note that, at each point x, the skew-symmetric non-degenerate bilinear form ω_x gives a linear isomorphism from T_xM to T^*_xM. Consequently, we have a linear isomorphism

$$H\colon\Theta_M\to\Omega^1_M.$$

Let f be a section of $\mathscr{O}_M$, then its ***Hamiltonian vector field*** is the unique vector field H_f such that

$$H_f=H(df).$$

For any two sections of $\mathscr{O}_M$, their *Poisson bracket* is

$$\{f,g\}:=H_f(g).$$

Then, one can verify that this does define a Coisson algebra structure on $\mathscr{O}_M$ and that

$$[H_f, H_g] = H_{\{f,g\}}.$$

Or, one can deduce aboves by verifying that

$$\{f, g\} = \omega_M(H_g, H_f).$$

Now, we have a sympletic manifold (T^*M, ω_M). Under a local coordinate system $(z^i; \xi^j)_{1\leqslant i,j\leqslant m}$, we have

$$H_{z^i} = -\partial_{\xi^i}, \qquad H_{\xi^i} = \partial_{z^i}.$$

Therefore the Poisson bracket on T^*M can be written as

$$\{f, g\} = \sum_{i=1}^{m}(\frac{\partial f}{\partial \xi^i}\frac{\partial g}{\partial z^i} - \frac{\partial f}{\partial z^i}\frac{\partial g}{\partial \xi^i}). \tag{1.7.3}$$

To see how the above constructions relate to (1.7.1), consider follows. First, Θ_M *can be identified with functions on* T^*M *which are linear in the fibers.* Indeed, given a section s of $\pi\colon T^*M \to M$ with ω_s the corresponding 1-form on M, then a vector field θ acts on a point $y \in s(M)$ as evaluating the function $\langle \omega_s, \theta\rangle$ at $\pi(y)$. Conversely, if f is a function on T^*M which is linear in the fibers, then for each 1-form ω_s with corresponding section s of π, we define the action of f on ω_s as

$$(f.\omega_s)(x) = f(s(x)).$$

The linear in fibers property implies that such action is $\mathscr{O}_M$-linear, hence defines a derivation on M. Furthermore, we obtain a monomorphism

$$\mathbb{S}_{\mathscr{O}_M}(\Theta_M) \longrightarrow \pi_*\mathscr{O}_{T^*M} \tag{1.7.4}$$

and hence can identify the first with its image in the later, which precisely consists of the sections which are *polynomials in the fibers*, i.e. sections of the polynomial algebra $\mathscr{O}_M[\xi^1, \cdots, \xi^m]$. Then, since (1.1.5) is a graded isomorphism, we obtain a homomorphism of $\mathscr{O}_M$-modules

$$\mathscr{D}_M \longrightarrow \mathrm{gr}_\bullet(\mathscr{D}_M) \cong \mathbb{S}_{\mathscr{O}_M}(\Theta_M) \longrightarrow \pi_*\mathscr{O}_{T^*M}.$$

Now, the Coisson algebra structure (1.7.1) on $\mathrm{gr}_\bullet(\mathscr{D}_M)$ is precisely that pullback from one on the symplectic manofold T^*M.

Remark One can also obtain above constructions as follows. Let φ be a section of $\mathscr{H}om(\mathbb{S}^p(\mathscr{O}_M), \mathscr{O}_M)$, define its action on a point $y \in s(M)$ as the function

$$y \longmapsto \frac{1}{p!}\varphi(\ell_\xi(y), \ell_\xi(y), \cdots, \ell_\xi(y))(\pi(y)),$$

where $\ell_\xi \colon T^*M \to \mathscr{O}_M$ acts on y as

$$\ell_\xi(y) := \sum_{i=1}^{m} \xi^i(y) z^i.$$

Above construction applies to those from differential operators of order p, hence induces a homomorphism of $\mathscr{O}_M$-modules

$$F_p\mathscr{D}_M \longrightarrow \mathscr{H}om(\mathbb{S}^p(\mathscr{O}_M), \mathscr{O}_M) \longrightarrow \pi_* \mathscr{O}_{T^*M}.$$

The image of a differential operator P is also called the ***p-th symbol*** and denoted by $\sigma_p(P)$. Then we also have the notion of ***principal symbol*** and the notation $\sigma(P)$. (We have already defined variants of the notion of *symbols of differential operators*, they are closely related to each other and share the same terminology and notations. However, ambiguity can be avoided if where those symbols take values are clear.)

It is straightforward to show that if $\varphi = \sigma(\theta_1\theta_2 \cdots \theta_p)$, where $\theta_1, \theta_2, \cdots, \theta_p$ are vector fields on M, then

$$\varphi.y = \theta_1(\ell_\xi(y))\theta_2(\ell_\xi(y)) \cdots \theta_p(\ell_\xi(y)), \quad \forall y \in T^*M.$$

It is also clear that for any vector field θ on M,

$$\theta(\ell_\xi(y)) = \theta.y, \quad \forall y \in T^*M,$$

where the action of θ on y is defined as before. In this way, we obtain a sequence of homomorphisms of $\mathscr{O}_M$-modules

$$\mathbb{S}^p_{\mathscr{O}_M}(\Theta_M) \longrightarrow \mathrm{gr}_p(\mathscr{D}_M) \longrightarrow \mathscr{H}om_{\mathscr{O}_M}(\mathbb{S}^p(\mathscr{O}_M), \mathscr{O}_M) \longrightarrow \pi_* \mathscr{O}_{T^*M}$$

whose composition agree with p-th component of (1.7.4). Note that this gives another way to show (1.1.5) is injective even without knowing $\mathscr{D}_M$ is generated by $\mathscr{O}_M \oplus \Theta_M$ and applies to singular case (i.e. complex analytic spaces which are not manifolds).

Any way, we can conclude that

1.7.5 Proposition *We have isomorphisms of graded $\mathscr{O}_M$-algebras*

$$\mathrm{gr}_p(\mathscr{D}_M) \cong \mathbb{S}^p_{\mathscr{O}_M}(\Theta_M) \cong \mathscr{O}_M[\xi^1, \cdots, \xi^m].$$

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