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Research Statement

Background

Let G be a group and p a prime number. A *p-adic representation* of G is a group homomorphism

$$\rho\colon G \longrightarrow \mathrm{GL}_K(V),$$

from G to the group of K-linear automorphisms of a finite-dimensional vector space V over a local field K with residue characteristic p. Such representations are widely used in number theory. It is often the case that one has a group G (for instance, the Galois group of a number field) acting on some object X (for example, a variety) equipped with a cohomology theory (for instance, the l-adic cohomology, the crystalline cohomology, etc.). Then a p-adic representation of G arises when we want to study the cohomology of X.

Given a p-adic representation $\rho\colon G \to \mathrm{GL}_K(V)$, a *stable lattice* in V is a $\mathscr{O}_K$-lattice in V that is stable under the action of G. The notion of stable lattices is a bridge between ordinary representations and modular representations. Such a lattice exists exactly when ρ is *precompact*. That is, the image of ρ has compact closure. If L is a stable lattice, then so is xL for any $x \in K^\times$. Such two lattices L and xL are said to be *homothetic*. Therefore, it is reasonable to consider the set $S(\rho)^0$ of homothety classes of stable lattices. The cardinality $h(\rho) = |S(\rho)^0|$ is called the *class number* of ρ. However, it isn't easy to compute $h(\rho)$ directly. Instead, my thesis advisor Junecue Suh studied the growth of $h(\rho)$ along totally ramified extensions, introduced new invariants, and produced new results in representation theory. See [1, 2].

Suh's work is inspired by Iwasawa's theory of the class numbers of ideal class groups in $\mathbb{Z}_p$-extensions. However, there is a fundamental difference: the set $S(\rho)^0$ of homothety classes of stable lattices has no natural composition law. Hence, the method of Galois modules does not apply. Instead, his work relies on the geometry of the *(reduced) Bruhat-Tits building of the general linear group*. There are two geometries on a building: the *metric* geometry inheriting from Euclidean affine spaces, and the *incident* geometry induced by the simplicial structure. Both of them play important roles.

It is often the case that a p-adic representation $\rho\colon G \to \mathrm{GL}_K(V)$ actually lands in a nice subgroup of $\mathrm{GL}_K(V)$. For instance, one obtains this representation from a cohomology theory equipped with a cup product that induces a bilinear form on V, and the action of G respects it. As a consequence, the image of G lands in $\mathrm{Sp}(V)$ (for a symplectic form) or $\mathrm{O}(V)$ (for a symmetric form). It is then natural to ask if Suh's results can be generalized to respect the extra features of those nice subgroups.

There is a class of algebraic groups called *reductive groups* containing some of the most important groups in mathematics. In particular, the groups mentioned above can be viewed as groups of K-rational points of certain reductive groups. The Bruhat-Tits theory asserts any reductive group over a local field admits a geometric object called its Bruhat-Tits building. For the general linear group, its Bruhat-Tits building consists of homothety classes of norms on the underlying vector space. For the symplectic group, its Bruhat-Tits building consists of self-dual norms. For the orthogonal group, its Bruhat-Tits building consists of maximinorante norms.

So the general setup is: given a group homomorphism $\rho\colon G \to \mathsf{H}(K)$ with H a reductive group and $\mathsf{H}(K)$ its group of K-rational points, one has an action of G on the associated Bruhat-Tits building $\mathscr{B}$. Let $S(\rho)$ be the set of fixed points of G in $\mathscr{B}$, which turns out to be a simplicial subset. One impression from Suh's work is that $S(\rho)$ is an interesting object and tells us much about the representation. In particular, I'm interested in its behavior along totally ramified extensions E/K. Then it is natural to ask the following geometric question.

Question 1. *When does the image of $S(\rho)$ in $\mathscr{B}_E$, the Bruhat-Tits building of H_E (the extension of scales of H), coincide with $S(\rho \otimes_K E)$?*

Unlike in the Bruhat-Tits building of the general linear group, the vertices in a general Bruhat-Tits building can be *non-special*. Namely, there are not enough *walls* (i.e., hyperplanes which are part of the simplicial structure) passing through it. Hence, there is more than one possible generalization of the *class number* $h(\rho)$, and they are closely related. For any such a generalization, we can ask the following numerical question.

Question 2. *When is the class number $h(\rho)$ finite? How does the class number grow along totally ramified extensions?*

On the other hand, back in [3], Gopal Prasad and Jiu-Kang Yu have shown that if a finite group Γ acts on a reductive group H over a local field whose residual characteristic does not divide the order of Γ, then the Bruhat-Tits building $\mathscr{B}(H)$ is naturally isomorphic to the fixed-point set $\mathscr{B}(H)^{\Gamma}$. In particular, the fix-point set $S(\rho)$ is either a single point or a building. From this perspective, the study of finite, non-singleton $S(\rho)$ can be viewed as a complementary story of Prasad-Yu's work: instead of considering finite groups whose order is coprime to the residual characteristic, we care more about those divided by the residual characteristic; instead of tamely-ramified extensions, we consider totally ramified extensions, especially the wild ones.

Current Research

In a simplicial complex, two vertices (0-simplices) are *adjacent* if there is an edge (1-simplex) connecting them. A path between two vertices x and y is a sequence of adjacent vertices $(z_i)_{0\leqslant i\leqslant s}$ with $z_0 = x$ and $z_s = y$. The number s is called the *length* of this path. The *simplicial distance* $d(x,y)$ between two vertices x and y is the minimum length of a path between them. Then the *simplicial ball* (with center x and radius r) $B(x,r)$ is the simplicial subset whose vertices are those within simplicial distance r from x. The number of vertices in $B(x,r)$ is called its *simplicial volume*.

The simplicial distance is a natural notion in incident geometry. In the case of Bruhat-Tits buildings, vertices are (roughly) corresponding to (certain) lattices, and the adjacency of vertices amounts to the containment of lattices. See [4]. In particular, the simplicial distance measures how different two lattices are.

Simplicial balls are essential ingredients in Suh's work: simplicial balls with radius one (which are called *tangent cones* by Suh) are the central object to study if ρ is regular, and general simplicial balls are used as coverings of $S(\rho)$ if ρ is irregular. Hence, it is natural to ask:

Question 3. *Can we have an explicit description of simplicial balls? How do they change along totally ramified extensions?*

In a Bruhat-Tits building, any vertex is adjacent to a *special vertex*. Consequently, the study of simplicial balls with a special center covers most needs of the study of simplicial balls. In what follows, o is a special vertex chosen to be the origin of the ambient Euclidean affine space (which is called an *apartment* in the building). The simplicial ball with center o and radius r is denoted by $B(r)$, and the number of vertices in it is denoted by $\mathrm{SV}(r)$. The quantity $\mathrm{SV}(r)$ does not depend on the choice of o, and the function $\mathrm{SV}(\,\cdot\,)$ is called the *simplicial volume function* in the building.

The first observation is: $B(r)$ is stable under the action of G_o, the stabilizer of o under the action of a nice automorphism group G of the building. Hence, to count the vertices in $B(r)$ amounts to describing the fundamental domain of $B(r)$ under G_o and compute the index of the stabilizer of $\{o,x\}$ in G_o. Following this idea, with the help of the theory of concave functions, I prove the following formula:

$$(\star) \qquad \mathrm{SV}(r) = \sum_{I\subseteq\Delta} \frac{\mathscr{P}_{\Phi;I}(q)}{q^{\deg(\mathscr{P}_{\Phi;I})}} \sum_{x\in B(r,{}^{v}C,I)} \prod_{a(x)>0} q^{\lceil a(x)\rceil},$$

where

- $\lceil\,\cdot\,\rceil$ is the ceiling function,
- Δ is a basis of the root system Φ,
- $\mathscr{P}_{\Phi;I}$ is the Poincaré polynomial associated to the pair (Φ, I),
- ${}^{v}C$ is a Weyl chamber of Φ,
- and the index sets $B(r,{}^{v}C,I)$ (resp. $\partial(r,{}^{v}C,I)$) consists of the vertices in $\overline{o+{}^{v}C}$ having type I with simplicial distance at most r (resp. exactly r) from o.

To apply this formula, one still needs an explicit description of the index set $B(r,{}^{v}C,I)$. This amounts to asking:

Question 4. *Can we have an explicit description of simplicial balls in terms of the root system?*

The general answer is still unclear due to the tricky combinatorics involved in this question. In my research [5], the following characterization is found.

Theorem 1. *If the root system Φ of the building is irreducible and classical (i.e. of the type A_n, B_n, C_n, or D_n), then for any vertex x, we have:*

$$d(x,o)\leqslant r \iff a_0(x-o)\leqslant r,$$

where a_0 is the highest root relative to the Weyl chamber covering both o and x.

It turns out that the simplicial distance is compatible with the decomposition of the building into irreducible ones. Consequently, the above theorem allows us to obtain an explicit of the index set $B(r, {}^vC, I)$ (and hence, the simplicial balls with a special center) in a Bruhat-Tits building of *split classical type*.

Using the formula ($\star$) and the theorem 1, one can see that the simplicial volume function is computed by a multi-summation of an exponential of q, the residual cardinality, where the exponent is a linear function of the summation index plus some parity functions. In my research [5], I systematically studied the behavior of such summations. Then I obtain the following results:

Theorem 2. *Let $\mathscr{B}$ be an irreducible Bruhat-Tits building of split classical type over a local field K with residue cardinality q. Then the simplicial volume function $\mathrm{SV}(\,\cdot\,)$ in it has the following asymptotic relation:*

$$\mathrm{SV}(r) \asymp r^{\varepsilon(n)} q^{\pi(n) r},$$

where $\varepsilon(n)$ and $\pi(n)$ are given in the following table.

Split type of $\mathscr{B}$	$\varepsilon(n)$	$\pi(n)$
A_n *(n is odd)*	0	$(\frac{n+1}{2})^2$
A_n *(n is even)*	1	$\frac{n}{2}(\frac{n}{2}+1)$
B_n *($n = 3$)*	0	5
B_n *($n \geqslant 4$)*	0	$\frac{n^2}{2}$
C_n *($n \geqslant 2$)*	0	$\frac{n(n+1)}{2}$
D_n *($n = 4$)*	2	6
D_n *($n \geqslant 5$)*	1	$\frac{n(n-1)}{2}$

Moreover, the leading coefficients have the following rationality properties.

1. *Suppose $\mathscr{B}$ is of split type A_n, C_n, B_3, or D_4. Then the simplicial volume $\mathrm{SV}(\,\cdot\,)$ in it has the following asymptotic growth as $r \to \infty$:*

$$\mathrm{SV}(r) \sim C(n) \cdot r^{\varepsilon(n)} q^{\pi(n) r},$$

where $C(n)$ is a positive number that is a rational function of q.

2. *Suppose $\mathscr{B}$ is of split type B_n ($n \geqslant 4$) or D_n ($n \geqslant 5$). Then the simplicial volume $\mathrm{SV}(\,\cdot\,)$ in it has the following asymptotic growth as $r \to \infty$:*

$$\mathrm{SV}(2r) \sim C_0(n) \cdot r^{\varepsilon(n)} q^{2\pi(n) r},$$
$$\mathrm{SV}(2r+1) \sim C_1(n) \cdot r^{\varepsilon(n)} q^{2\pi(n) r},$$

where $C_0(n)$ and $C_1(n)$ are positive numbers that are rational functions of q.

It is worth mentioning that the simplicial balls with a special center have a good pattern along a totally ramified extension E/K: the image of $B(r)$ in the building $\mathscr{B}_E$ over E is again a simplicial ball with special center and its radius is $r \cdot e$, where e is the ramification index. Hence, my results answer the question 3 for Bruhat-Tits building of *split classical type*.

Furthermore, the simplicial ball $B(r)$ can be viewed as the fix-point set $S(\rho)$ under the action of the r-th principal congruence subgroup of G_o. (In the case of $\mathrm{GL}_n(K)$-building, they are precisely the principal congruence subgroup of $\mathrm{SL}_n(\mathscr{O}_K)$.) Hence, my results answer the question 2 in a specific but important case.

Further Research Plan

One ongoing work is to extend [5] to non-split reductive groups, which relies on careful study of the non-reduced root systems. Results similar to theorems 1 and 2 are expected. Based on my current results and expected ones in non-split case, it is possible to answer questions 1 and 2 for classical groups. On the other hand, to extend my results to exceptional groups, a different characterization of simplicial distance is critical since theorem 1 fails even for the root system G_2.

In addition to the questions 1–4, there are other possible directions from my results. For instance, theorem 1 indeed gives an explicit description of simplicial balls with a special center in Bruhat-Tits buildings of split classical type. Consequently, constructions related to those balls may be done more concretely and such applications are expected. The formula ($\star$) relies on the fact that the simplicial ball is stable under the action of G_o. But there are other sets having this property. Indeed, similar formulas can be deduced for them and then my work on multi-summations can be used to study their growth once the fundamental domain is clear. One impression is that such sets grow exponentially and can be viewed as the extremely opposite of the regular representations. Besides, multi-summations of exponentials appears in many computations, and their arithmetic properties are worth to study in deep. My work can be viewed as an attempt in this direction.

Reference

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