Transcendence of Periods

Xu Gao

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Outline

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- Generalization is worth to consider:

integration of a closed n-form along a n-cycle.

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Definition (Kontsevich and Zagier, 2001)

A ***n*-period** of X is the integration

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- $\mathrm{H}_1(\mathbb{C}^\times, \mathbb{Z})$ is generated by the unit circle.
- Periods of $\mathbb{G}_m$ are multiples of $\oint \frac{\mathrm{d}z}{z} = 2\pi\sqrt{-1}$.

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- The lattice $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ gives rise to the isomorphism

$$E^{\mathrm{an}} \cong \mathbb{C}/\Lambda.$$

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More radically, one can conjecture the following:

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The *n*-periods ($n \geqslant 1$) are either 0 or transcendental.

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- (Lang, 1966) The *Schneider-Lang Theorem*, which unifomizes proofs of above results.

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- (Schneider, 1941) On an abelian variety, not all periods are algebraic.
- (Lang, 1966) The *Schneider-Lang Theorem*, which unifomizes proofs of above results.
- (Wüstholz, 1987, 1989) The *Analytic Subgroup Theorem*, which implies the transcendence and linearly independence of 1-periods of smooth projective varieties.

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1-periods on smooth projective varieties

Let X be a smooth projective variety defined over $\overline{\mathbb{Q}}$.

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Linearly Independence (Wüstholz, 1987)

Let $\mathcal{V}$ be the $\overline{\mathbb{Q}}$-vector space spanned by $1, \pi$ and all 1-periods of X. Then

$$\dim_{\overline{\mathbb{Q}}} \mathcal{V} = 2 + \sum_{i=1}^{k} \frac{2\,(\dim A_i)^2}{[\mathrm{End}_0\, A_i : \mathbb{Q}]},$$

where A_i are simple abelian subvarieties of the Albanese variety of X.

Outline

Periods on Calabi-Yau varieties

Study higher periods on good varieties

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- where the lower periods are ruled out.

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- Hence $\dim_{\overline{\mathbb{Q}}} \mathcal{P} = 1$ implies $\mathrm{H}^{n,0} = \mathrm{H}^{n,0}_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$.
- This is an unusual property since in many case, or with a little bit more assumptions, it would imply that the Hodge decomposition is defined over $\overline{\mathbb{Q}}$.

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Stronger Conjecture

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This result is proven through the *Kuga-Satake correspondence*.

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However, similar construction is not available for higher Hodge structures.

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The remarkable hypersurface

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- whose geometry was studied by Schoen (1985, 1986).

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Conjecture

At least one of the periods of X' is transcendental.

References I

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