p-adic representations and simplicial distance in Bruhat-Tits buildings

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Suppose we have a group representation

$$D_8 = \langle r, s \mid r^4 = s^2 = (rs)^2 = 1 \rangle$$
$$r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Entries $\in \mathbb{Q} \hookrightarrow \mathbb{Q}_p \rightsquigarrow$ ***p-adic representation***.

- ***lattice*** = f. g. $\mathbb{Z}_p$-submodule L of V spanning the entire V.
  E.g. $L_0 = \mathbb{Z}_p\binom{1}{0} + \mathbb{Z}_p\binom{0}{1}$ is a lattice in $\mathbb{Q}_p^2$.

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- A stable lattice $L \rightsquigarrow$ a modular representation on L/pL.
  E.g. $r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_p), s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_p)$.
  Irreducible if $p \neq 2$

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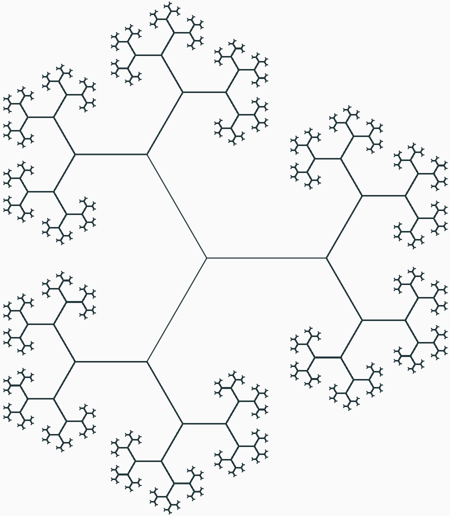
- If $p \neq 2$, $S(\rho)^0 = \{L_0\}$.
- If $p = 2$, $S(\rho)^0 = \{L_0, L(1)\}$, where

$$L(1) := \mathbb{Z}_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\mathbb{Z}_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Bruhat-Tits tree

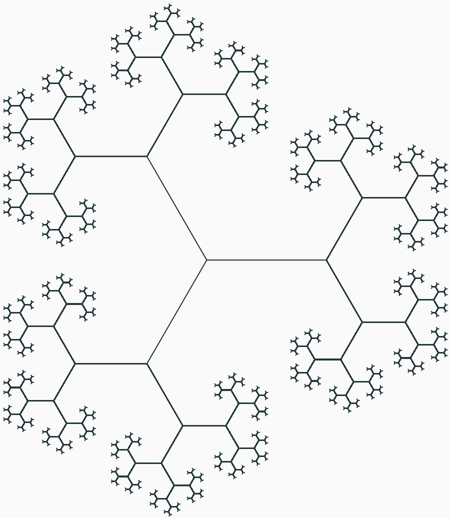
- vertices := homothety classes of lattices



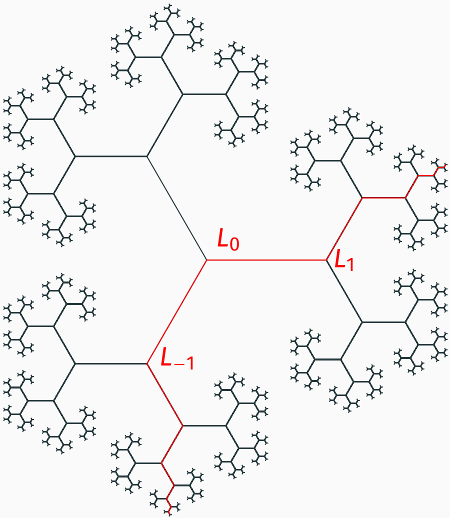
Bruhat-Tits tree

- vertices :=
  homothety classes of lattices
- x, y are adjacent :=
  $\exists L \in x, L' \in y$ such that

$$L \supset L' \supset pL.$$



Take $[L_0]$ as the origin.
$\rightsquigarrow L_k := \mathbb{Z}_p \binom{1}{0} + p^k \mathbb{Z}_p \binom{0}{1} \quad (k \in \mathbb{Z})$
Each $[L_k]$, $[L_{k+1}]$ are adjacent.



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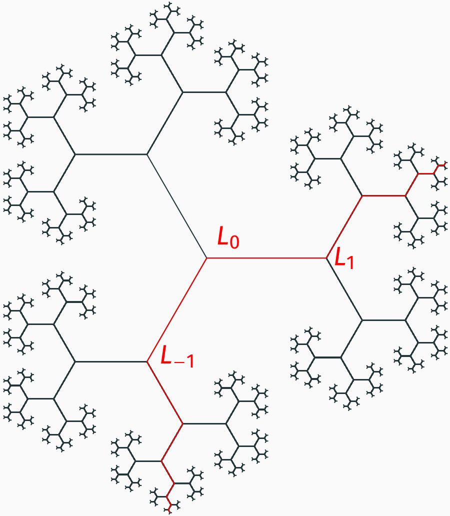
Each $[L_k], [L_{k+1}]$ are adjacent.

$[L]$ and $[L']$ are in the same ***apartment*** means
$\exists$ basis e_1, e_2 such that

$$L = \mathbb{Z}_p e_1 + \mathbb{Z}_p e_2,$$
$$L' = \mathbb{Z}_p e_1 + p^k\mathbb{Z}_p e_2.$$



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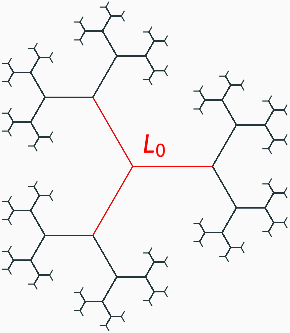

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- $L(u) := \mathbb{Z}_p \binom{1}{u} + p\mathbb{Z}_p \binom{0}{1}$. ($u \in \mathbb{Z}_p$)
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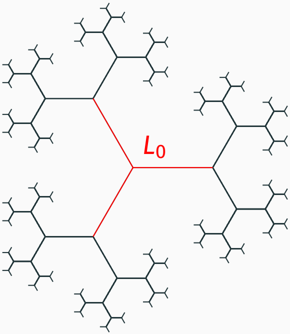

Adjacent vertices in Bruhat-Tits tree

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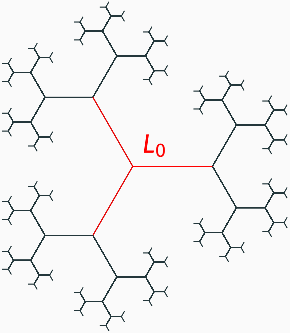

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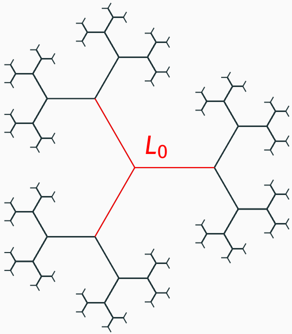
- closed points in $\mathbb{P}^1_{\mathbb{F}_p}$, and



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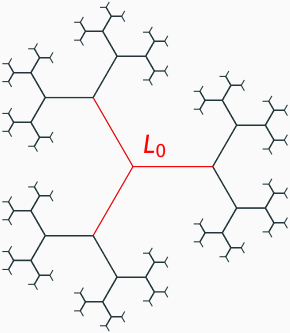
- vertices in the ***Tits building*** of $\mathrm{GL}_2(\mathbb{Q}_p)$.

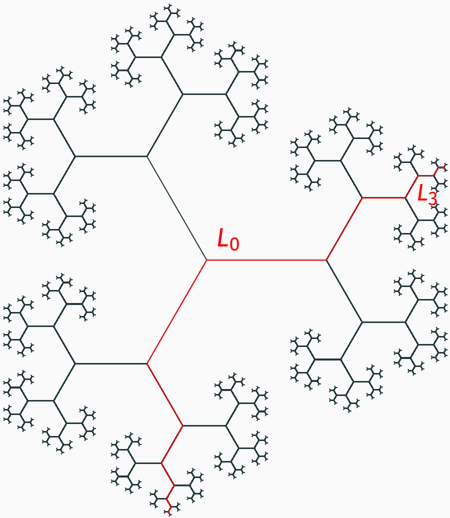


Suppose e_1, e_2 is a basis of L_0 and

$$L_r := \mathbb{Z}_p e_1 + p^r \mathbb{Z}_p e_2$$

Then $d([L_r], [L_0]) = r$:

$$L_0 - L_1 - \cdots - L_r$$

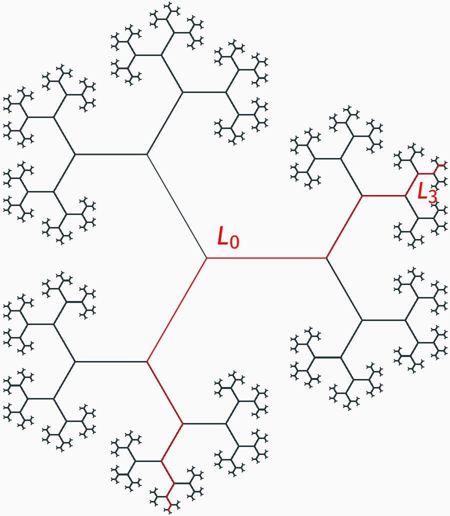


Conversely, if

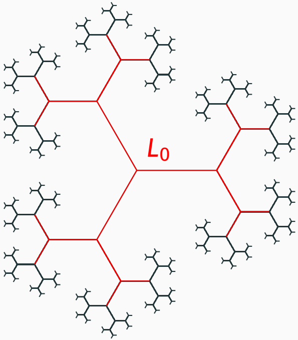
$$d(x, [L_0]) = r,$$

then there is a basis e_1, e_2 of L_0 and $L \in x$ such that

$$L = \mathbb{Z}_p e_1 + p^r \mathbb{Z}_p e_2$$



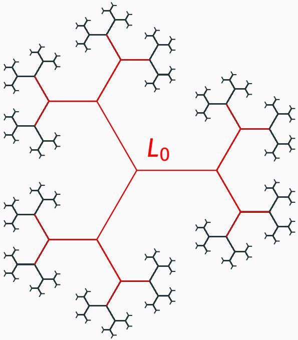
Identify $GL_2(\mathbb{Z}_p)$ with $\mathrm{Aut}(L_0)$ via a basis



Identify $\mathrm{GL}_2(\mathbb{Z}_p)$ with $\mathrm{Aut}(L_0)$ via a basis
Then by the previous, the fixed-point subset of the ***principal congruence subgroup***

$$P_r := I_2 + p^r \,\mathrm{Mat}_{2\times 2}(\mathbb{Z}_p)$$

is the simplicial ball of radius r.



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- $S(\rho)$ is *connected*:
  Indeed, if $[L_0], [L_r] \in S(\rho)^0$, then for any $0 < i < r$ and $g \in \rho(G)$,

  $$g.e_1 \in L_r \subset L_i, \qquad g.p^i e_2 \in p^i L_0 \subset L_i.$$

  Hence, $g.L_i = L_i$.

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- $S(\rho)$ is *connected*:
- $S(\rho)$ is in some $B(r)$:
  Since $\mathbb{Z}_p[\rho(G)]$ contain some P_r.

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- Therefore, $S(\rho)^0 = \{[L_0]\}$.

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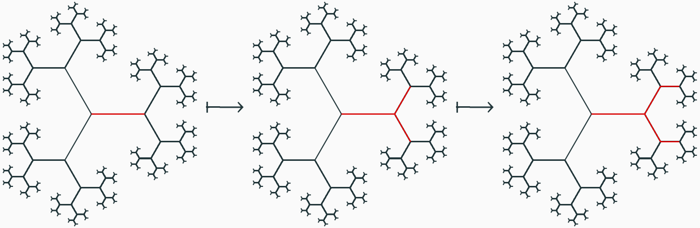

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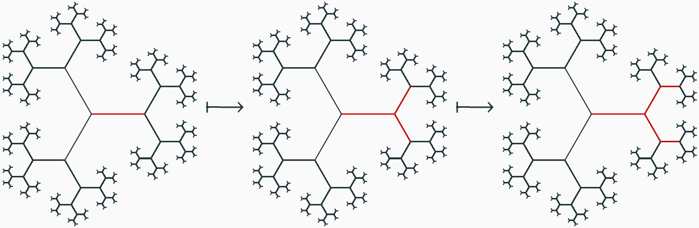

$$\lim_{K/\mathbb{Q}_2 \text{t.r.}} \frac{\log_2 \left| S(\rho \otimes_{Q_2} K)^0 \right|}{[K : \mathbb{Q}_2]} = \frac{1}{2}.$$