p-adic representations and simplicial balls in Bruhat-Tits buildings

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p-ADIC REPRESENTATIONS AND STABLE LATTICES

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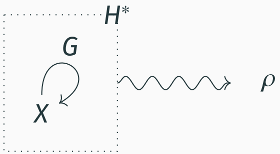

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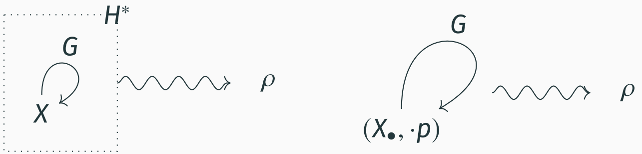

Notations:

- K a non-Archimedean local field;
- val the valuation on K;
- K° the ring of integers $\{x \in K \mid \mathrm{val}(x) \geqslant 0\}$;
- $K^{\circ\circ}$ the maximal ideal $\{x \in K \mid \mathrm{val}(x) > 0\}$;
- ϖ a uniformizer, namely $K^{\circ\circ} = \varpi K^\circ$;
- κ the residue field $K^\circ / K^{\circ\circ}$;
- q the cardinality of κ.

Stable lattices

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- Given a stable lattice L, one can obtain a representative over the residue filed κ by reduction:

$$L \longrightarrow L \otimes_{K^\circ} \kappa.$$

Note that, although $L \otimes_{K^\circ} \kappa$ depends on the choice of L, its semisimplification (the ***mod ϖ reduction*** V_κ of V) is not.

Question

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1. A p-adic representation (V, ρ) has a stable lattice if and only if it is ***precompact***: the image $\rho(G)$ has compact closure in $\mathrm{GL}_K(V)$.
2. Being stable is a property that is maintained through ***homothety***: two lattices L and L' are ***homothetic*** if there is some $x \in K^\times$ such that
$$L' = xL.$$

Therefore, it is reasonable to consider the set

$$S(\rho)^0 := \{\text{stable lattices of } \rho\}/_{\text{homothety}}$$

Its cardinality $h(\rho) = |S(\rho)^0|$ is called the ***class number*** of ρ.

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One can compare above definition with the following one of ***ideal class group*** of a number field F:

$$Cl(F) := \{\text{fractional ideas in } F\}/_{\text{homothety}}.$$

Climb the tower

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But there is one fundamental difficulty: unlike the fractional ideal classes, there is no natural composition law for homothety classes of stable lattices making $S(\rho)^0$ a group, a fortiori a Galois module.

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But there is one fundamental difficulty: unlike the fractional ideal classes, there is no natural composition law for homothety classes of stable lattices making $S(\rho)^0$ a group, a fortiori a Galois module.

One solution is to replace the algebra of Galois modules with the geometry of convex simplicial sets (in Bruhat-Tits buildings).

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Any lattice L defines a norm

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Then the set $S(\rho)^0$ can be geometrically identified with the set of vertices in the fixed-point set $S(\rho)$ of $\rho(G)$ in $\mathscr{X}(V)$.

Theorem (Junecue, 2021)

(i) *$S(\rho)$ is a convex and simplicial subset of $\mathscr{X}(V)$.*

(ii) *Its set of vertices is $S(\rho)^0$.*

(iii) *$S(\rho)$ is compact if and only if $h(\rho)$ is finite if and only if ρ is irreducible.*

(iv) *Maximal simplices in $S(\rho)$ has dimension $r(\rho) - 1$ where $r(\rho)$ is **reduction rank**, namely the number of irreducible components in the reduction V_κ.*

(v) *$h(\rho \otimes_K E)$ is a polynomial of $[E : K]$ for any totally ramified extension E/K if and only if ρ has **regular reduction**.*

It is often the case that a p-adic representation $\rho \colon G \to \mathrm{GL}_K(V)$ actually lands in a nice subgroup of $\mathrm{GL}_K(V)$.

Example

The vector space V has a non-degenerate symplectic (resp. orthogonal) form and the action of G respects this form. Then ρ lands in $\mathrm{Sp}(V)$ (resp. $\mathrm{O}(V)$).

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Example

The vector space V has a non-degenerate symplectic (resp. orthogonal) form and the action of G respects this form. Then ρ lands in $\mathrm{Sp}(V)$ (resp. $\mathrm{O}(V)$).

We may also consider the notion of stable lattices respecting the form (e.g. ***self-dual lattices*** and ***almost self-dual lattices***).

For the relation between such lattices with the Bruhat-Tits building of classical groups, see (Garrett, 1997).

In general, we can consider group homomorphisms to the groups of K-rational points of ***reductive groups***:

$$\rho\colon G \longrightarrow \mathcal{G}(K).$$

We can also expect that the study of fix-point set of $\rho(G)$ in the Bruhat-Tits building of $\mathcal{G}$ can aid the research of ρ.

What are Bruhat-Tits buildings?

Start with the datum of a reductive group $\mathcal{G}$, the Bruhat-Tits theory produces a metric space $\mathscr{B}(\mathcal{G})$ equipped with a simplicial structure and a family of simplicial subcomplex (called ***apartments***). This structure is called the ***Bruhat-Tits building*** of $\mathcal{G}$. Then a subgroup of $\mathcal{G}(K)$ acts nicely on $\mathscr{B}(\mathcal{G})$.

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Any maximal split torus $\mathcal{T}$ of $\mathcal{G}$ gives an apartment, which is a Euclidean affine space equipped with the simplicial structure induced from the ***root datum*** of $(\mathcal{G}, \mathcal{T})$ and the valuation $\mathrm{val}(\,\cdot\,)$.

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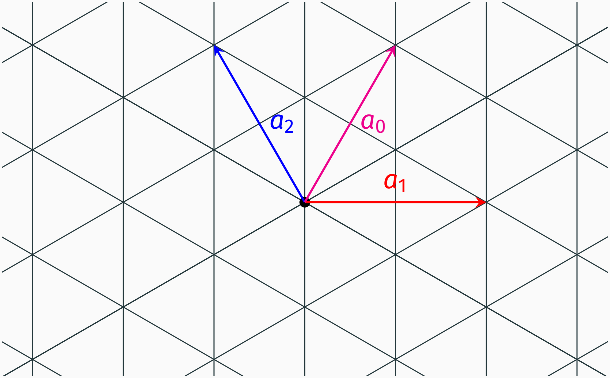
And the building is obtained by gluing those apartments cleverly.

Examples of apartments



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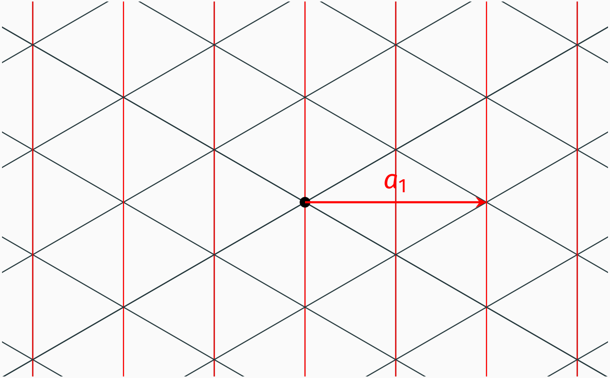
Examples of apartments



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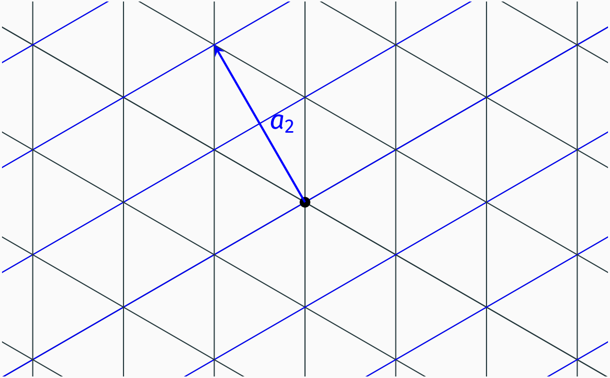

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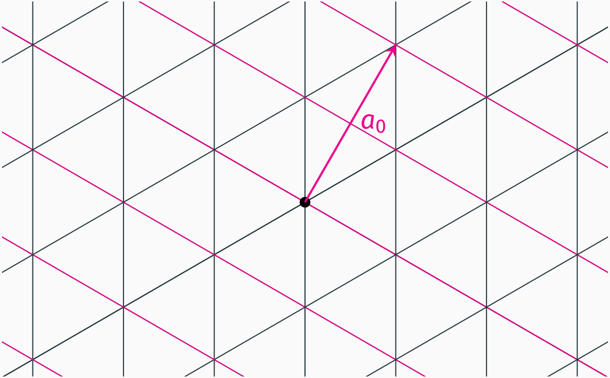

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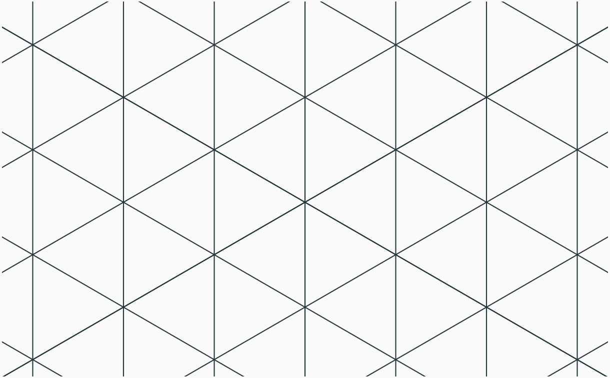

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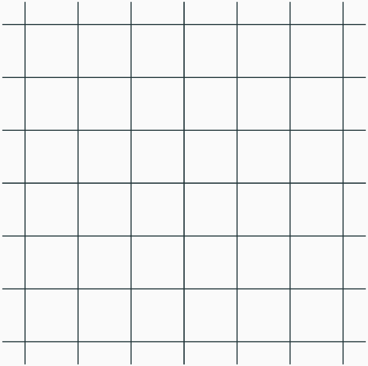
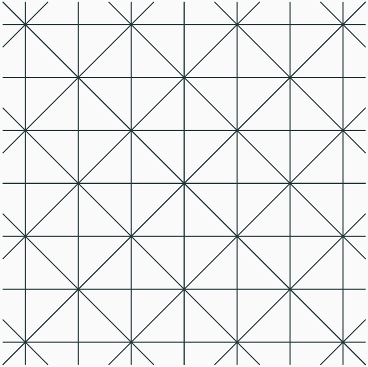


$A_1 \times A_1$



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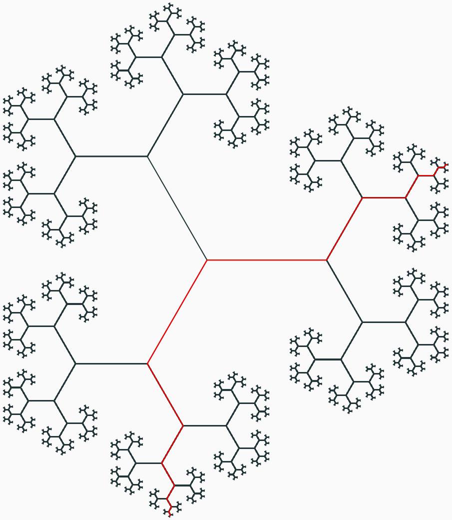

The Bruhat-Tits tree (building of $\mathrm{GL}_K(2)$) with an apartment specified by red color.

Question

What can we say about fixed-point sets in a Bruhat-Tits building?

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Fixed-point sets

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- A *reasonable* fixed-point set must be a convex simplicial subset of the building. (***precompact*** representations)
- The fixed-point set is compact if and only if the class number $h(\rho)$ is finite. ("***irreducible***" representations)
- (Prasad and Yu, 2002) if $\rho(G)$ contains no ***p*-torsions**, then the fixed-point set is a Bruhat-Tits building. (So we care about "irreducible" precompact representations containing p-torsions.)

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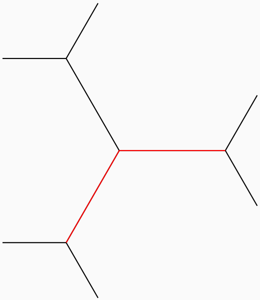
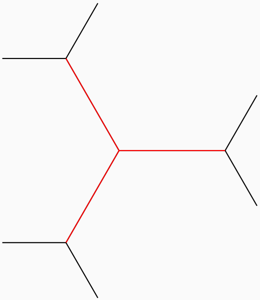
- *How does $S(\rho)$ change along totally ramified extensions?*
- *Can we have a concrete description of $S(\rho)$?*
- *Given an interesting convex simplicial subset of the building, can we make it a fixed-point set for some ρ?*

Fixed-point sets

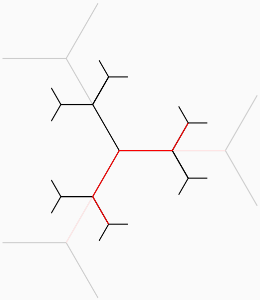


The behavior of a "regular" fixed-point set.



The behavior of an "irregular" fixed-point set.

Fixed-point sets



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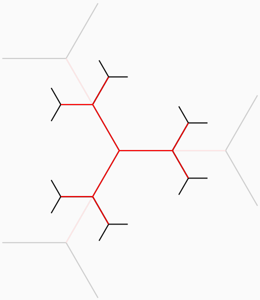

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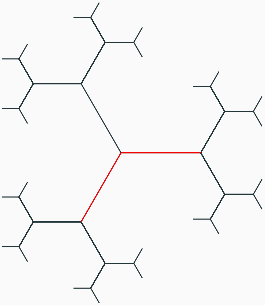

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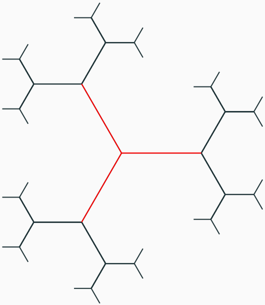

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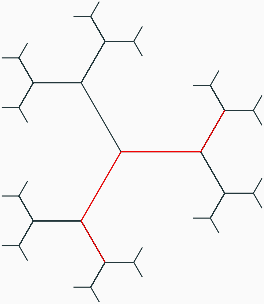

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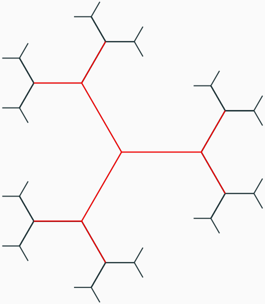

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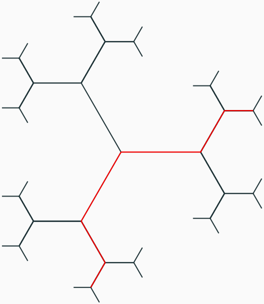

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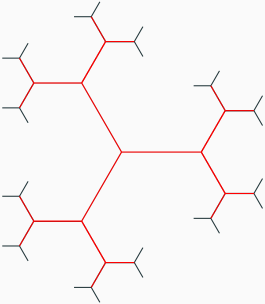


The behavior of a "irregular" fixed-point set.

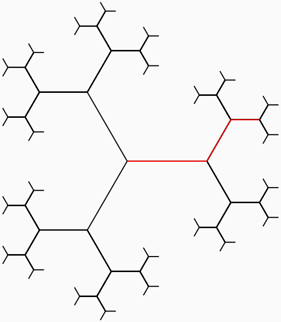
Why Simplicial balls?

Simplicial distance and simplicial balls

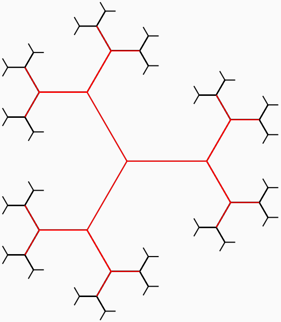
Since we care about the vertices, we can consider Bruhat-Tits buildings in terms of ***incident geometry***. One can find in (Garrett, 1997) how to understand the Bruhat-Tits building as the incident geometry of lattices and containment.

In incident geometry, we naturally have the notions of ***simplicial distance*** and ***simplicial balls***.

Simplicial distance and simplicial balls



A path of length 3.



Simplicial ball of radius 3.

Simplicial distance and simplicial balls

Question

- *Can we have a concrete description of simplicial balls?*

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- *Can we view simplicial balls as fixed-point sets?*

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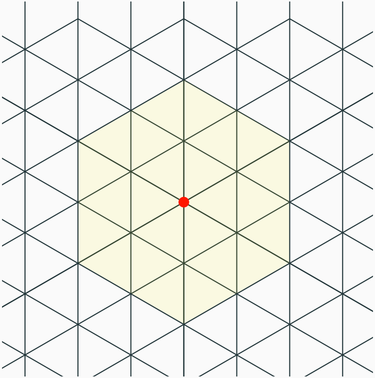
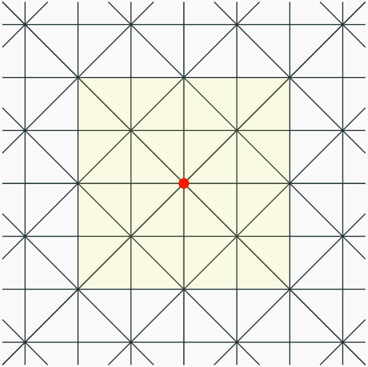
- *Can we view simplicial balls as fixed-point sets?*
- *How does simplicial balls grow?*

Simplicial distance and simplicial balls





Simplicial balls of radius 2.

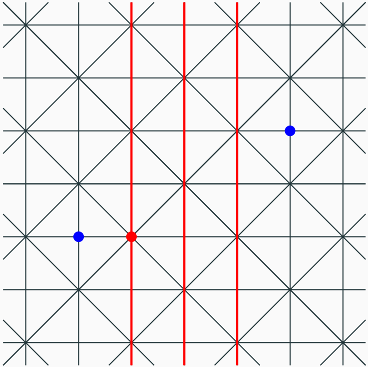
Theorem (G., 2022)

If the root system Φ of the building is irreducible and classical (i.e. of the type A_n, B_n, C_n, or D_n), then we have:

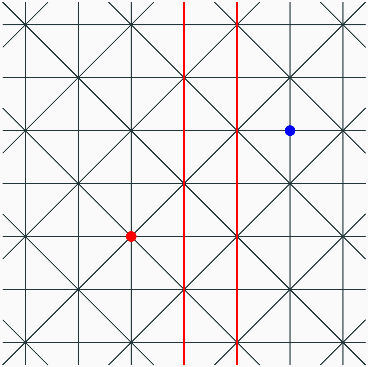
(i) *the simplicial distance between two vertices = the maximum of numbers of walls between them +1;*
in particular, if we normalize the valuation so that the valuation group $\mathrm{val}(K^\times) = \mathbb{Z}$, then for any vertex x, the simplicial distance from the origin o to x is

$$d(x, o) = \lceil a_0(x - o) \rceil,$$

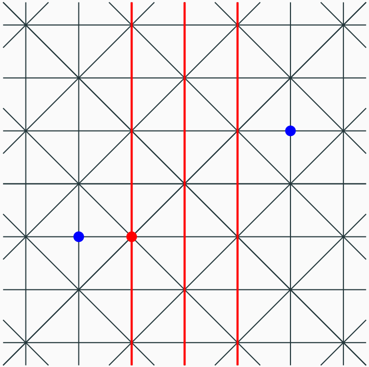
where a_0 is the highest root relative to the Weyl chamber attached to o and covering x.



The two blues are separated by 3 walls and have distance 4.



The blue is separated by 2 walls from the red and has distance 3.



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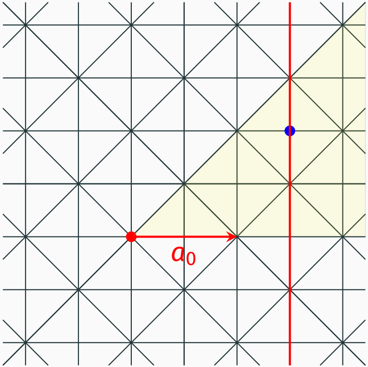
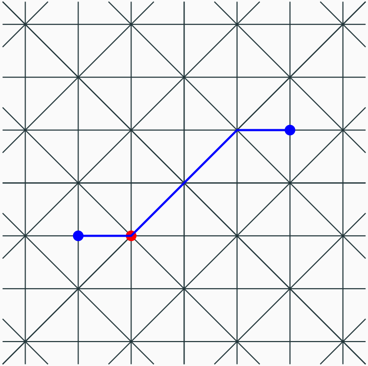
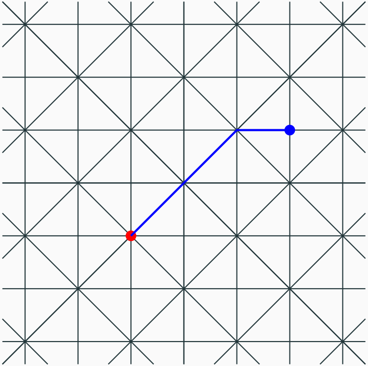


The blue has value 3 under a_0 and is of distance 3 from the red.



A path of length 4.



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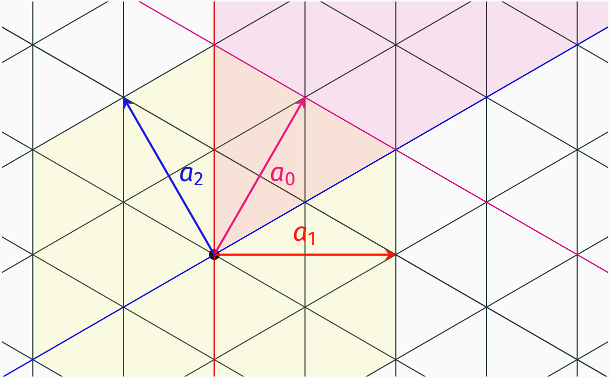
Weyl chamber and simplicial ball



Theorem (G., 2022)

(ii) *the simplicial ball $B(x, r)$ of radius r at x is the fixed-point set of the Moy-Prasad subgroup $P_{x,r}$ (assuming the valuation is normalized). In particular, $B(x, 1)$ is the fixed-point set of the pro-unipotent radical of P_x.*

In (Moy and Prasad, 1996), a filtration $(P_{x,r})_{r\geqslant 0}$ of the stabilizer P_x of x is defined. They are called the ***Moy-Prasad filtrations***. It has been generalized to the machinery of ***concave functions*** in (Yu, 2015).

What are concave functions (Moy-Prasad filtrations)?

The ***parahoric subgroup*** P_x of $\mathcal{G}(K)$ attached to a point $x \in \mathscr{B}(\mathcal{G})$ is (roughly) the stabilizer of x (and hence the simplex containing x). By Bruhat-Tits theory, there is a (connected) smooth model $\mathfrak{G}_x$ of $\mathcal{G}$ such that $\mathfrak{G}_x(K^\circ) = P_x$.

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The machinery of (Yu, 2015) extends such constructions to ***concave functions***. A ***concave function*** is a function on $\widetilde{\Phi} := \Phi \cup \{0\}$ such that

$$\forall a, b \in \widetilde{\Phi}, a + b \in \widetilde{\Phi} \implies f(a) + f(b) \geqslant f(a+b).$$

Any concave function f defines a (connected) smooth model $\mathfrak{G}_f$ of $\mathcal{G}$ and $P_f = \mathfrak{G}_f(K^\circ)$ is a bounded subgroup of $\mathcal{G}(K)$.

Examples

- Any point $x \in \mathscr{B}(\mathcal{G})$ defines a concave function $f_x \colon a \mapsto -a(x)$, and we have $P_{f_x} = P_x$.

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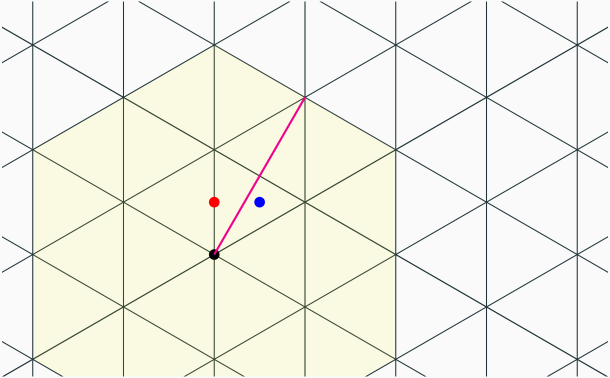
- For any $r \geqslant 0$, the shifting $f + r$ of a concave function f is again a concave function.
- In general, $P_f \subseteq P_{f'}$ corresponds to $f \geqslant f'$.

- Any point $x \in \mathscr{B}(\mathcal{G})$ defines a concave function $f_x\colon a \mapsto -a(x)$, and we have $P_{f_x} = P_x$.
- For any $r \geqslant 0$, the shifting $f + r$ of a concave function f is again a concave function.
- In general, $P_f \subseteq P_{f'}$ corresponds to $f \geqslant f'$.
- In particular, $(P_{f_x+r})_{r\geqslant 0}$ forms the *Moy-Prasad filtration* of P_x.

Examples



$$\cdots \subset P_{\bullet,2} \subset P_{\diagup} \subset P_{\bullet} \subset P_{\bullet} \subset P_{\bullet}.$$

Simplicial volume

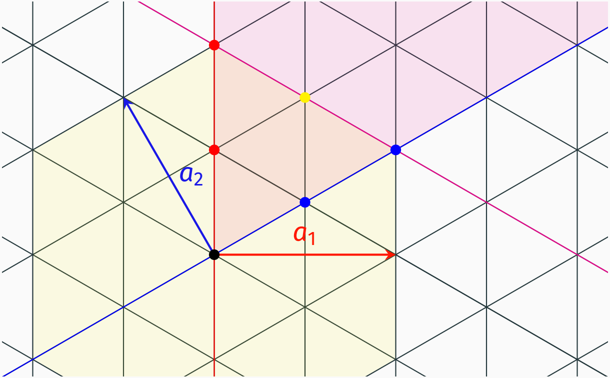
We're interested in the number of vertices in the simplicial ball $B(o, r)$. This gives us a function $\mathrm{SV}(r)$, called the ***simplicial volume*** in the building $\mathscr{B}$.

Theorem (G., 2022)

(iii) *the simplicial volume can be computed by the formula:*

$$\mathrm{SV}(r) = \sum_{I \subset \Delta} \frac{\mathscr{P}_{\Phi;I}(q)}{q^{\deg(\mathscr{P}_{\Phi;I})}} \sum_{x \in B(r,C,I)} \prod_{a(x)>0} q^{\lceil a(x) \rceil},$$

where $\mathscr{P}_{\Phi;I}$ *is the* **Poincaré polynomial**, *C is a Weyl chamber attached to o, and $B(r, C, I)$ is the intersection of $B(o, r)$ and C_I, the inner of* $\bigcap_{a \in I} \ker(a)$.



$$B(2, \blacksquare, a_1) = \{\bullet, \bullet\}, \qquad B(2, \blacksquare, a_2) = \{\bullet, \bullet\},$$

$$B(2, \blacksquare, \Delta) = \{\bullet\}, \qquad B(2, \blacksquare, \emptyset) = \{\bullet\}.$$

The formula

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- We have a concrete description of vertices in C using the ***fundamental coweights*** as a basis.
  (However, they DO NOT form a lattice.)

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- We have a concrete description of vertices in C using the ***fundamental coweights*** as a basis.
  (However, they DO NOT form a lattice.)
- With such a description, we can describe the index set $B(r, C, I)$ in terms of linear inequalities.

If $\mathscr{B}$ is of type A_n and $I = \Delta \setminus \{\ell_1, \cdots, \ell_t\}$:

$$B(r, C, I) = \left\{o + c_1\omega_{\ell_1} + \cdots + c_t\omega_{\ell_t} \,\middle|\, c_i \in \mathbb{Z}_{>0}, c_1 + \cdots + c_t \leqslant r\right\},$$

where ω_i are the fundamental coweights.

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where $\omega'_i = h_i^{-1}\omega_i$ with $a_0 = \sum_i h_i a_i$.

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If $\mathscr{B}$ is of type B_n or D_n, then the description is complicated.

$$
\begin{tikzcd}
& o + \frac{1}{2}\,\mathcal{P}^\vee \arrow[dl, dash] \arrow[dr, dash] & \Xi \arrow[d, dash] \\
\color{red}{\mathcal{V}} \arrow[d, dash] & & \Xi \cap o + \frac{1}{2}\,\mathcal{P}^\vee \\
\mathcal{V}_\dagger & &
\end{tikzcd}
$$

Vertices in B_n building

$$\begin{tikzcd}[column sep=small]
 & \mathcal{X}^{00} \cup \mathcal{X}^{10} \cup \mathcal{X}^{01} \cup \mathcal{X}^{11} \arrow[dl, no head] \arrow[dr, no head] & & \Xi^{00} \cup \Xi^{10} \cup \Xi^{01} \cup \Xi^{11} \arrow[dl, no head] \\
{\color{red}\mathcal{V}} \arrow[d, no head] & & (\mathcal{X}^{00} \cap \Xi^{00}) \cup (\mathcal{X}^{10} \cap \Xi^{10}) \cup (\mathcal{X}^{01} \cap \Xi^{01}) \cup (\mathcal{X}^{11} \cap \Xi^{11}) & \\
\mathcal{V}_{\dagger} & & &
\end{tikzcd}$$

Vertices in D_n building

Theorem (G., 2022)

(iv) *We have the following asymptotic dominant relation as $r \to \infty$:*

$$\mathrm{SV}(r) \asymp r^{\epsilon(n)} q^{\pi(n) r},$$

where $\epsilon(n)$ and $\pi(n)$ are in the following table.

Theorem (G., 2022)

(iv) *We have the following asymptotic dominant relation as $r \to \infty$:*

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where $\epsilon(n)$ and $\pi(n)$ are in the following table.

X_n	$\epsilon(n)$	$\pi(n)$
A_n (n is odd)	0	$(\frac{n+1}{2})^2$
A_n (n is even)	1	$\frac{n}{2}(\frac{n}{2}+1)$
B_n ($n = 3$)	0	5
B_n ($n \geqslant 4$)	0	$\frac{n^2}{2}$
C_n ($n \geqslant 2$)	0	$\frac{n(n+1)}{2}$
D_n ($n = 4$)	2	6
D_n ($n \geqslant 5$)	1	$\frac{n(n-1)}{2}$

Theorem (G., 2022)

(v) *Leading coefficients have the following rationality properties:*

1. *If $\mathscr{B}$ is of type A_n, C_n, B_3, or D_4, then we have*

$$\mathrm{SV}(r) \sim c(n) \cdot r^{\epsilon(n)} q^{\pi(n) r},$$

where $c(n)$ is a positive number that is a rational function of q.

Theorem (G., 2022)

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   where $c(n)$ is a positive number that is a rational function of q.

2. *If $\mathscr{B}$ is of type B_n $(n \geqslant 4)$ or D_n $(n \geqslant 5)$, then we have*

$$\mathrm{SV}(2r) \sim c_0(n) \cdot r^{\epsilon(n)} q^{2\pi(n) r},$$
$$\mathrm{SV}(2r+1) \sim c_1(n) \cdot r^{\epsilon(n)} q^{2\pi(n) r},$$

   where $c_0(n)$ and $c_1(n)$ are positive numbers that are rational functions of q.

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- Similar results are expected for non-split types, which requires a careful study of the non-reduced root systems.
- It seems that concave functions are good tools to study fixed-point sets in Bruhat-Tits buildings.

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