Twisted restricted conformal blocks of vertex operator algebras

Math-Physics Joint Seminar

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Tongji University

Conformal blocks

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Borcherds 1986; Frenkel, Lepowsky, and Meurman 1988; Beilinson, Feigin, and Mazur 1991; Dong and Lepowsky 1993; Frenkel, Huang, and Lepowsky 1993; Lepowsky and Li 2004; Zhu 1994, 1996; Frenkel and Ben-Zvi 2004; Nagatomo and Tsuchiya 2005

Definition: A *vertex operator algebra* (VOA) is a graded vector space $V = \bigoplus_{k=0}^{\infty} V_k$ with a *vacuum vector* $\mathbf{1} \in V_0$, a *Virasoro element* $\omega \in V_2$, and a *vertex operator*

$$Y(-,z)\colon V \longrightarrow \mathrm{End}(V)[\![z^{\pm 1}]\!],$$

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that is compatible with the V-module structures.

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Given an n-tuple of V-modules, one constructs (and studies) a **vector bundle** $\mathbb{V}$ with a **projectively flat connection** ∇ on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable pointed curves.

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> DEFINITION: Given a stable n-pointed curve $(C, \mathsf{p}_\bullet)$ and an n-tuple of V-modules $M^1, \cdots, M^n$, a *conformal block* associated to these data is a linear functional
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> $$M^1 \otimes \cdots \otimes M^n \longrightarrow \mathbb{C}$$
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THEOREM (Frenkel & Ben-Zvi 04', Nagatomo & Tsuchiya 05'): *The vector space of conformal blocks associated to*

$$(\mathbb{P}^1, \infty, 1, 0, (M^3)', M^1, M^2)$$

is isomorphic to the space of intertwining operators among M^1, M^2, and M^3.

THEOREM (Damiolini, Gibney, and Tarasca 21'–24'): *Given:*

- V *a rational,* C_2*-cofinite VOA with* $V_0 = \mathbb{C}\mathbf{1}$*, and*
- $M^\bullet$ *an* n*-tuple of f.g. admissible* V*-modules,*

$\rightsquigarrow$ *a vector bundle* $[M^\bullet]$ *of finite rank with a projectively flat connection* ∇ *on* $\overline{\mathcal{M}}_{g,n}$*, whose dual bundle classifies the conformal blocks.*

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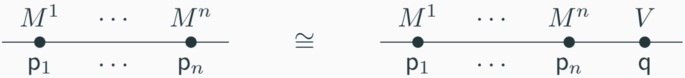

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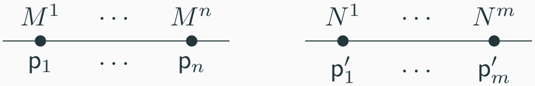

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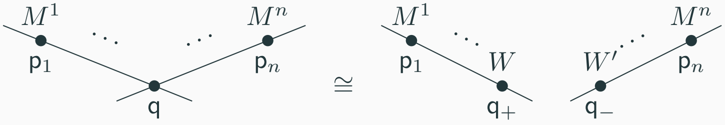

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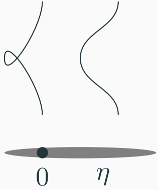
4. *Formal smoothing* $[M^\bullet[\![q]\!]]_{(C,\mathsf{p}_\bullet)} \cong [M^\bullet]_{(C_0,\mathsf{p}_\bullet(0))}[\![q]\!]$



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Frenkel, Lepowsky, and Meurman 1988; Dong 1994; Xu 1995; Dong, Li, and Mason 1998; Barron, Dong, and Mason 2002; Dong and Xu 2006; Huang 2010; Dong, Ren, and Xu 2017; Huang 2018; Huang and Yang 2019...

DEFINITION: Let g be an automorphism of V having order T. An *(admissible) g-twisted module* of a vertex operator algebra V is a graded vector space $M = \bigoplus_{k \in \frac{1}{T}\mathbb{Z}} M(k)$ with a *vertex operator*

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> Definition: Given a stable G-cover $(\mathfrak{X}, \mathsf{p}_\bullet)$ and an n-tuple of twisted V-modules $M^1, \cdots, M^n$, a *twisted conformal block* associated to these data is a linear functional
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Remark. $\mathfrak{X}$ is a stack, points of it are equipped with cyclic stabilizers. For each point p, the sector is given by a g-twisted V-module, where g generates its stabilizer.

DEFINITION: A *stable G-cover* $(\mathfrak{X}, \mathsf{p}_\bullet)$ is

- a stacky curve $\mathfrak{X}$,
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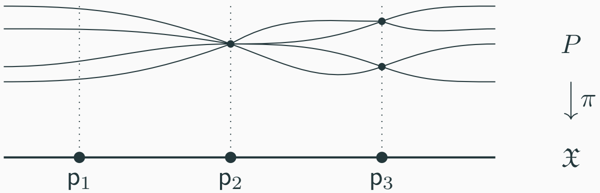

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The latter is $\cong$ to the Lie algebra $L_g(V)$ of g-twisted vertex modes, where g is a generator of the stabilizer of p.

Twisted fusion rules

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> *Note: $[C_w/\mu_T]$ is totally ramified.*

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Proposal: *The vector space of twisted intertwining operators among M^1, M^2, and M^3 is given by the space of twisted conformal blocks associated to the data*

$$(\mathbb{P}^1_{g_3^{-1}, g_1, g_2}, \infty, 1, 0, (M^3)', M^1, M^2),$$

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Q: How to compute the twisted fusion rules? More generally, how to compute the dimension of a space of twisted conformal blocks?

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> NOTATION: For any $m_\bullet \in (\mathbb{Z} \cup \{\infty\})^n$, denote
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Remark. $(\mathcal{V}^G(\Delta_{m_\bullet}), \nabla) = (\mathcal{V}(\pi^*\Delta_{m_\bullet} - \mathfrak{R}), \nabla)^G$.

Example

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Hence, $\mathcal{L}_{(\mathfrak{X},\mathsf{p}_\bullet)}(V)_{\leqslant m_\bullet}$ is spanned by

$$\left\{ [u \otimes z^{\frac{r}{T}+a}(z-1)^b\,\mathrm{d}z] \;\middle|\; \begin{array}{c} u \in V_k \cap V^r \\ a \geqslant k - 1 - \frac{r}{T} - m_0 \\ b \geqslant k - 1 - \frac{r}{T} - m_1 \\ a + b \leqslant k - 1 - \frac{r}{T} + m_\infty \end{array} \right\}$$

Restricted data

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> Notation: $\mathcal{L}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_{\leqslant m} := \mathcal{L}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_{\leqslant \infty \sqcup m}$

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LEMMA: *Suppose M is a (twisted) V-module attached to some q_j. Then, $\mathcal{L}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_{<0} (:= \bigcup_{m<0} \mathcal{L}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_{\leqslant m})$ acts trivially on its bottom level $M(0)$. Hence, $M(0)$ is an $\mathcal{L}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_0$-module.*

Restricted data

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> Lemma: *Suppose M is a (twisted) V-module attached to some q_j. Then, $\mathcal{L}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_{<0} (:= \bigcup_{m<0} \mathcal{L}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_{\leqslant m})$ acts trivially on its bottom level $M(0)$. Hence, $M(0)$ is an $\mathcal{L}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_0$-module. Conversely, any $\mathcal{L}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_0$-module (and any $A_g(V)$-module) can be viewed as an $\mathcal{L}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_{\leqslant 0}$-module.*

Restricted conformal blocks

Definition: Given a stable G-cover $(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)$ and a pair of families of (twisted) V-modules $M^\bullet$ and $A_{g_\diamond}(V)$-modules $U^\diamond$, a *twisted restricted conformal block* associated to these data is a linear functional

$$M^1 \otimes \cdots \otimes M^m \otimes U^1 \otimes \cdots \otimes U^n \longrightarrow \mathbb{C}$$

invariant under the action of the Lie subalgebra $\mathcal{L}_{(\mathfrak{X},\mathsf{p}_\bullet)}(V)_{\leqslant 0}$.

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EXAMPLE: Given an $(m+n)$-tuple of (twisted) V-module $M^\bullet \sqcup N^\diamond$,

$$\mathrm{Conf}(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond, M^\bullet \sqcup N^\diamond) \qquad \mathrm{ResConf}(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond, M^\bullet \sqcup N^\diamond(0)).$$

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Theorem (G., Liu, Zhu 24'): *If $N^\diamond$ are lowest-weight modules, then π is injective. If $N^\diamond$ are further the generalized Verma modules of their bottom levels, then π is an isomorphism.*

Example

Example (G., Liiu, and Zhu 23'): Let g be a VOA-automorphism of V of order T and consider the data:

- $\mathfrak{X} = [\mathbb{P}^1/\mu_T]$, $\mathsf{p}_\bullet = (1)$, and $\mathsf{q}_\diamond = (0, \infty)$;
- attach a untwisted V-module M^1 at 1;
- attach a left $A_g(V)$-module U^2 at 0;
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Then, the space of twisted restricted conformal block is the dual of $U^2 \otimes M^1 \otimes U^3/J$, where J has an explicit description. From which, the space is the dual of

$$U^3 \otimes_{A_g(V)} A_g(M^1) \otimes_{A_g(V)} U^2.$$

Twisted Zhu's algebra and coherence

Basic set-up: $(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)$.

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Lemma: *$\mathcal{L}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_{\leqslant\star}$ form an exhaustive, separated, and split filtration on $\mathcal{L}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)$.*

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> Lemma: $\mathcal{L}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_{\leqslant \star}$ *form an exhaustive, separated, and split filtration on* $\mathcal{L}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)$.

> Lemma (Damiolini, Gibney, and Krashen 23'): *From a split-filtered associative algebra* $(U_{\leqslant \star}, U_\star)$*, one ends up with a complete semi-normed split-filtered associative algebra* $(\widehat{U}^{\mathsf{f}}_{\leqslant \star}, \widehat{U}^{\mathsf{g}}_{\star})$*. In such a pair, any homogeneous ideal* $I \lhd \widehat{U}^{\mathsf{g}}_{\star}$ *induces a pair of closed ideals* $(\overline{I}^{\mathsf{f}}, \overline{I}^{\mathsf{g}})$*, and we obtain another pair of semi-normed split-filtered associative algebra* $(\widehat{U}^{\mathsf{f}}_{\leqslant \star}/\overline{I}^{\mathsf{f}}, \widehat{U}^{\mathsf{g}}_{\star}/\overline{I}^{\mathsf{g}})$.

LEMMA (Damiolini, Gibney, and Krashen 23'): *From a split-filtered associative algebra $(U_{\leqslant\star}, U_\star)$, one ends up with a complete semi-normed split-filtered associative algebra $(\widehat{U}^{\mathsf{f}}_{\leqslant\star}, \widehat{U}^{\mathsf{g}}_\star)$. In such a pair, any homogeneous ideal $I \triangleleft \widehat{U}^{\mathsf{g}}_\star$ induces a pair of closed ideals $(\overline{I}^{\mathsf{f}}, \overline{I}^{\mathsf{g}})$, and we obtain another pair of semi-normed split-filtered associative algebra $(\widehat{U}^{\mathsf{f}}_{\leqslant\star}/\overline{I}^{\mathsf{f}}, \widehat{U}^{\mathsf{g}}_\star/\overline{I}^{\mathsf{g}})$.*

DEFINITION: Take $U_{\leqslant\star} = \mathsf{U}\left(\mathcal{L}_{(\mathfrak{X},\mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_{\leqslant\star}\right)$ and I to be generated by the Jacobi relations. The resulted semi-normed split-filtered associative algebra $\mathcal{U}_{(\mathfrak{X},\mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)$ is called the *(twisted) universal enveloping algebra of V w.r.t. the data $(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)$.*

Definition: The *(twisted) Zhu's algebra of V w.r.t. the data $(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)$.* is

$$\mathcal{A}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V) := \mathcal{U}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_0 / \mathbf{N}^1.$$

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LEMMA: *Suppose $\mathfrak{X}$ is smooth and $\mathsf{q}_\diamond$ contains all the stacky points. Then*

$$\mathcal{A}_{(\mathfrak{X}, \mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V) \cong \bigotimes_{j \in \diamond} \mathcal{A}_{\mathsf{q}_j}(V) \cong \bigotimes_{j \in \diamond} A_{g_j}(V).$$

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(M^1)_{\mathcal{L}_{([\mathbb{P}^1/\mu_T],(1)\sqcup(0,\infty))}(V)_{<0}} &\otimes_{\mathcal{A}_{([\mathbb{P}^1/\mu_T],(1)\sqcup(0,\infty))}(V)} (U^2 \otimes U^3) \\
&\cong A^{\mathrm{op}}(M^1) \otimes_{A_g(V)\otimes A_{g^{-1}}(V)} (U^2 \otimes U^3) \\
&= U^3 \otimes_{A_g(V)} A(M^1) \otimes_{A_g(V)} U^2.
\end{aligned}
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THEOREM (G., Liu, Zhu 24'): *Suppose $M^\bullet \sqcup N^\diamond$ is an $(m+n)$-tuple of (twisted) V-module generated by their finite-dimensional bottom levels. Suppose $\mathsf{q}_\diamond$ contains all the stacky points. Then, the space of coinvariants $(M^\bullet \otimes N^\diamond)_{\mathcal{L}_{(\mathfrak{X},\mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)}$ is finite-dimensional.*

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Proof. We have sujective map

$$(M^\bullet \otimes N^\diamond(0))_{\mathcal{L}_{(\mathfrak{X},\mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_{\leqslant 0}} \twoheadrightarrow (M^\bullet \otimes N^\diamond)_{\mathcal{L}_{(\mathfrak{X},\mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)}.$$

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We also have

$$L.H.S. \cong (M^\bullet)_{\mathcal{L}_{(\mathfrak{X},\mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_{<0}} \otimes_{\mathcal{A}_{(\mathfrak{X},\mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)} N^\diamond(0).$$

$\square$

Coherence of coinvariants

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Note that

$$(M^\bullet)_{\mathcal{L}_{(\mathfrak{X},\mathsf{p}_\bullet \sqcup \mathsf{q}_\diamond)}(V)_{<0}} = (M^\bullet)_{\mathcal{L}_{(\mathfrak{X},\mathsf{p}_\bullet)}(V)} = (M^\bullet)_{\mathcal{L}_{(P,\widetilde{\mathsf{p}}_\bullet)}(V)}.$$

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The coherence of $(M^\bullet)_{\mathcal{L}_{(P,\widetilde{\mathsf{p}}_\bullet)}(V)}$ has been proven by (Damiolini, Gibney, and Tarasca 21'–24'). □

Thank you!