p-adic representations and simplicial distance in Bruhat-Tits buildings

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p-adic representation

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- *p-adic representation* $= \rho : G \longrightarrow \mathrm{GL}_K(V)$ with K a p-adic field

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- *stable lattice* $=$ lattice stable under the action of $\rho(G)$

  Recall that: lattice = f. g. K°-submodule L of V spanning the entire V.

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- p-adic representation $\rightsquigarrow$ stable lattice $\rightsquigarrow$ mod ϖ reduction

$$(\rho, V) \rightsquigarrow L \rightsquigarrow L \otimes_{K^\circ} \kappa \rightsquigarrow V_\kappa$$

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  $\Longrightarrow$ bridge between ordinary and modular representations

Stable lattices

Geometric study of $$\rho\colon G \longrightarrow \mathrm{GL}_K(V)$$

stable lattices $\approx$ stable vertices in the *Bruhat-Tits buildings* of $\mathrm{GL}_K(V)$

Stable lattices

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$$\begin{aligned}\text{stable lattices} &\approx \text{stable vertices in the } \textit{Bruhat-Tits buildings} \text{ of } \mathrm{GL}_K(V)\\ &= \text{vertices in the fixed-point set } S(\rho)\end{aligned}$$

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Bruhat-Tits buildings?

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- $\mathcal{G}$ reductive group $\rightsquigarrow$ $\mathscr{B}(\mathcal{G})$ *Bruhat-Tits building*
  metric space + polysimplicial complex

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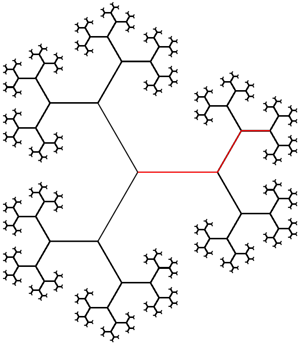
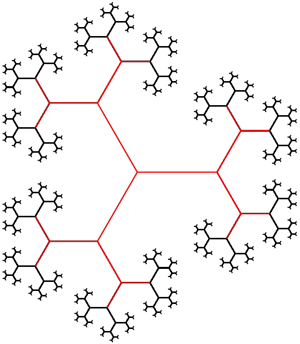
  $\rightsquigarrow$ incident geometry of vertices
  $\rightsquigarrow$ *simplicial distance* (=distance by incident relations)

Simplicial distance



A path of length 3.



Simplicial ball of radius 3.

Results

1. Characterizations of simplicial distance.

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3. Exponential polynomials and simplicial volume.

Simplicial distance in classical type

Φ, the root system, is *irreducible* and *classical* (i.e. A_n, B_n, C_n, or D_n)

Theorem (G., 2022)

$d(x, y) - 1$ = maximum of numbers of walls between them; in particular, the simplicial distance from the origin o to x is

$$d(x, o) = \lceil a_0(x - o) \rceil,$$

where a_0 = highest root rel. Weyl chamber covering x.

Simplicial distance in classical type

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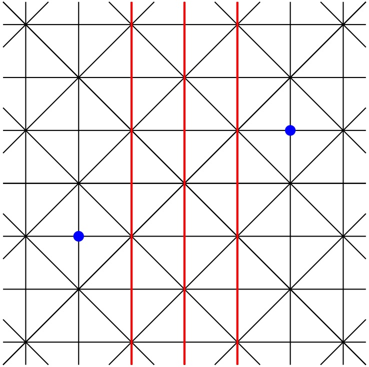
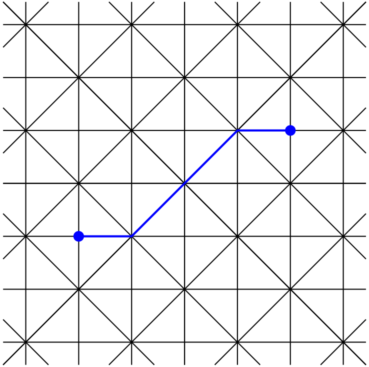


The two blues are separated by 3 walls and have distance 4.



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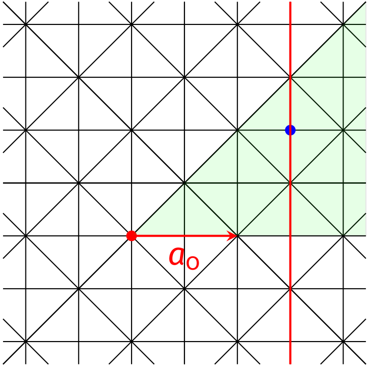
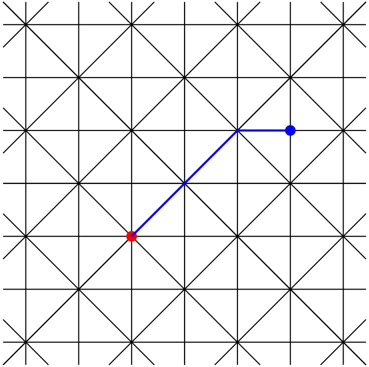
Simplicial distance in classical type



The blue has value 3 under a_0 and is of distance 3 from the red.



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Simplicial distance in classical type

Corollary

If $\Phi = A_n$ *or* C_n*, then* $d(x,y) \leqslant r \iff \exists L \in x, L' \in y$ *such that*

$$L \supset L' \supset \varpi^r L.$$

When $\Phi = B_n$ or D_n, the simplicial distance could be shorter due to the oriflamme construction.

Simplicial balls

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Theorem (G., 2022)

The simplicial ball $B(x, r)$ of radius r at x = fixed-point set of the Moy-Prasad subgroup $P_{x,r}$.

- *Moy-Prasad subgroups* $(P_{x,r})_{r\geqslant 0}$ (Moy-Prasad, 96) generalizing principal congruence subgroups

Moy-Prasad subgroups

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- *concave function* f = function on $\widetilde{\Phi} := \Phi \cup \{0\}$ s.t.

$$\forall a, b \in \widetilde{\Phi}, a + b \in \widetilde{\Phi} \implies f(a) + f(b) \geqslant f(a+b).$$

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- $x \in \mathscr{B}(\mathcal{G}) \rightsquigarrow f_x \colon a \mapsto -a(x) \rightsquigarrow$ *parahoric subgroups* $P_x \approx$ stabilizer of x.

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- $f_x + r$ $(r \geqslant 0) \rightsquigarrow$ *Moy-Prasad subgroups* $P_{x,r}$.

Simplicial volume formula

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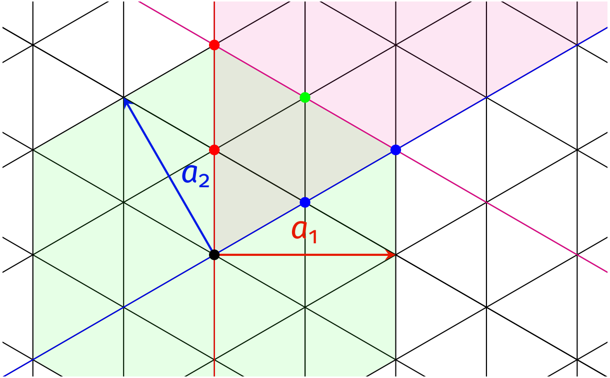
Theorem (G., 2022)

$$\mathrm{SV}(r)(:=|B(o,r)|)=\sum_{I\subset\Delta}\frac{\mathscr{P}_{\Phi;I}(q)}{q^{\deg(\mathscr{P}_{\Phi;I})}}\sum_{x\in B(r,C,I)}\prod_{a(x)>0}q^{\lceil a(x)\rceil}$$

- $\mathscr{P}_{\Phi;I}$ = *Poincaré polynomial*
- C = *Weyl chamber*
- $B(r,C,I)=B(o,r)\cap\mathrm{inn}(\bigcap_{a\in I}\ker(a))$

The index set $B(r, C, I)$



$$B(2, \blacksquare, a_1) = \{\bullet, \bullet\}, \quad B(2, \blacksquare, a_2) = \{\bullet, \bullet\}, \quad B(2, \blacksquare, \Delta) = \{\bullet\}, \quad B(2, \blacksquare, \emptyset) = \{\bullet\}.$$

Asymptotic estimation

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Theorem (G., 2022)

We have the following asymptotic dominant relation as $r \to \infty$:

$$\mathrm{SV}(r) \asymp r^{\epsilon(n)} q^{\pi(n) r},$$

where $\epsilon(n)$ and $\pi(n)$ are in the following table.

	A_n (n odd)	A_n (n even)	B_3	B_n ($n \geqslant 4$)	C_n ($n \geqslant 2$)	D_4	D_n ($n \geqslant 5$)
$\epsilon(n)$	0	1	0	0	0	2	1
$\pi(n)$	$(\frac{n+1}{2})^2$	$\frac{n}{2}(\frac{n}{2}+1)$	5	$\frac{n^2}{2}$	$\frac{n(n+1)}{2}$	6	$\frac{n(n-1)}{2}$

Leading coefficients

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Theorem (G., 2022)

Leading coefficients have the following rationality properties:
If $\mathscr{B}$ is of type A_n, C_n, B_3, or D_4, then we have

$$\mathrm{SV}(r) \sim c(n) \cdot r^{\epsilon(n)} q^{\pi(n) r},$$

otherwise

$$\mathrm{SV}(2r) \sim c_0(n) \cdot r^{\epsilon(n)} q^{2\pi(n) r},$$

$$\mathrm{SV}(2r+1) \sim c_1(n) \cdot r^{\epsilon(n)} q^{2\pi(n) r},$$

where $c(n)$, $c_0(n)$ and $c_1(n)$ are positive numbers that are rational functions of q.

Further directions

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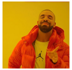
- *Non-irreducible buildings* 

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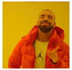
- *Non-irreducible buildings* 
- *non-split types* 

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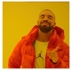
- *Non-irreducible buildings* 
- *non-split types* 
- *Exceptional types* 

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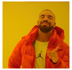
- *Non-irreducible buildings* 
- *non-split types* 
- *Exceptional types* 
- *concave functions* and *fixed-point sets* (conjectured)