p-adic representations and simplicial distance in Bruhat-Tits buildings

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p-adic representations and stable lattices

***p*-adic representation**

$$\rho\colon G \longrightarrow \mathrm{GL}_K(V),$$

- V is a finite-dimensional vector space over
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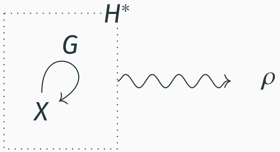

p-ADIC REPRESENTATIONS

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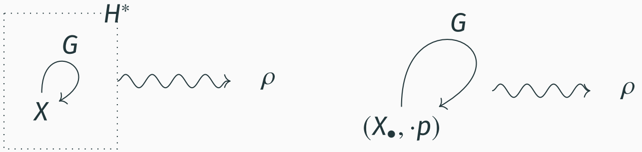

Notations:

- K = a non-Archimedean local field;
- val = the valuation on K (assuming $\mathrm{val}(K^\times) = \mathbb{Z}$);
- K° = the ring of integers $\{x \in K \mid \mathrm{val}(x) \geqslant 0\}$;
- ϖ = a uniformizer, namely ϖK° is the maximal ideal of K°;
- κ = the residue field $K^\circ/\varpi K^\circ$;
- q = the cardinality of κ.

- ***lattice*** = f. g. K°-submodule L of V spanning the entire V.

Stable lattices

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- ***stable lattice*** = lattice stable under the action of $\rho(G)$.
- A stable lattice $L \rightsquigarrow$ a modular representation on $L \otimes_{K^\circ} \kappa$.
    - $L \otimes_{K^\circ} \kappa$ depends on the choice of L,
    - its semisimplification doesn't.

  The later is called the ***mod ϖ reduction*** V_κ of V.

Question

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Compare with:

$$Cl(F) := \{\text{fractional ideas in } F\}/_{\text{homothety}}.$$

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- ~~algebra of Galois modules~~ geometry of Bruhat-Tits buildings

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- lattices in $V \longleftrightarrow$ vertices (0-simplicies) $\mathscr{X}(V)$
- $S(\rho)$ = fixed-point set of $\rho(G)$ convex + simplicial
- $S(\rho)^0 \longleftrightarrow$ set of vertices in $S(\rho)$

Theorem (Junecue, 2021)

(i) *$S(\rho)$ is compact $\iff$ $h(\rho)$ is finite $\iff$ ρ is irreducible.*

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(iii) *$h(\rho \otimes_K E)$ is a polynomial of $[E : K]$ for any totally ramified extension E/K $\iff$ ρ has* ***regular reduction****.*

What if replace $\mathrm{GL}_K(V)$ by other groups?

Example

V has a non-degenerate symplectic (resp. orthogonal) form and the action of G respects it. Then ρ lands in $\mathrm{Sp}(V)$ (resp. $\mathrm{O}(V)$).

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Consider ***stable (almost) self-dual lattices***

In general, we can consider group homomorphisms to the groups of K-rational points of ***reductive groups***:

$$\rho\colon G \longrightarrow \mathcal{G}(K).$$

Expect: the study of fix-point set of $\rho(G)$ in the Bruhat-Tits building of $\mathcal{G}$ can aid the research of ρ.

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- $\mathcal{G}$ reductive group $\rightsquigarrow$ $\mathscr{B}(\mathcal{G})$ ***Bruhat-Tits building***
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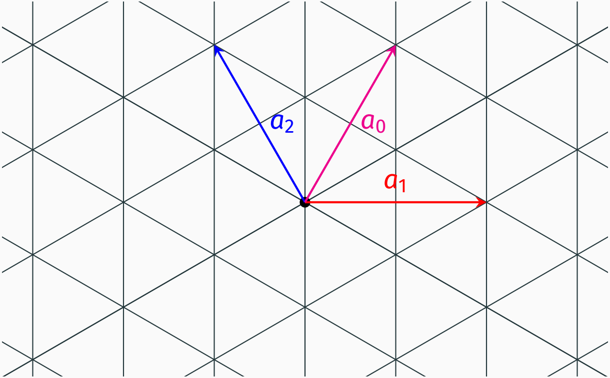
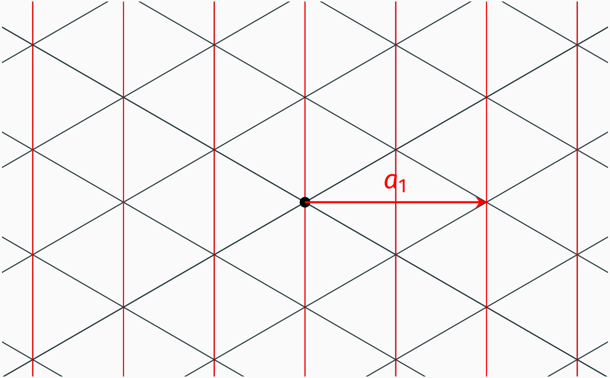
- $\mathcal{T}$ maximal split torus of $\mathcal{G}$ $\rightsquigarrow$ $\mathscr{A}(\mathcal{T})$ ***affine apartment***
  depends on ***root datum*** + $\mathrm{val}(\,\cdot\,)$
- building = gluing apartments cleverly



A_2



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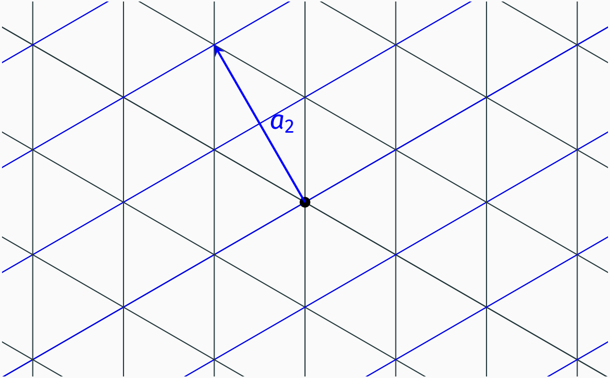

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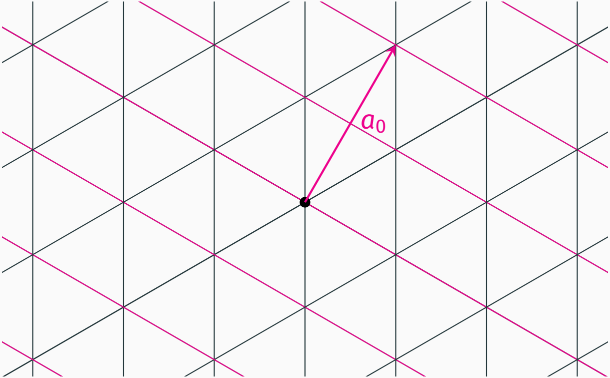

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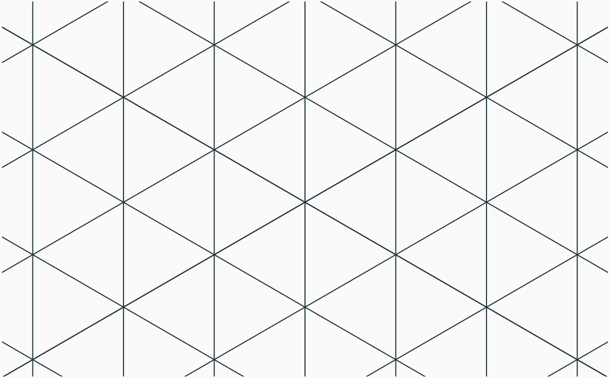
Examples of apartments



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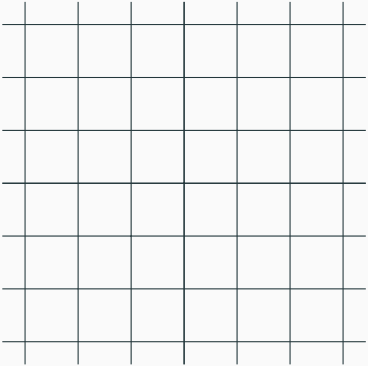
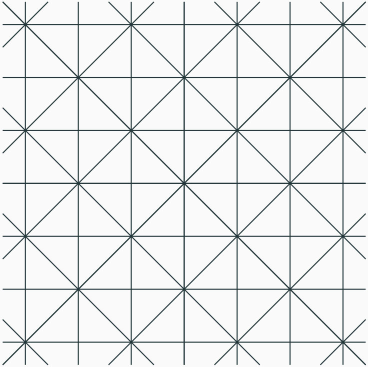
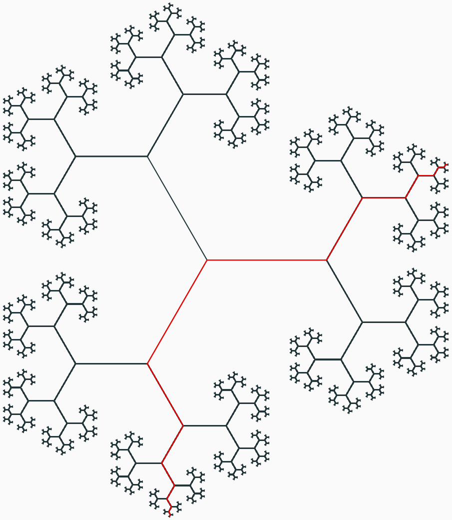


$A_1 \times A_1$



C_2



The Bruhat-Tits tree (building of $\mathrm{GL}_K(2)$) with an apartment specified by red color.

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- We require ρ ***type-preserving*** isometries (***precompact*** representations)
- $S(\rho)$ is convex + simplicial
- $S(\rho)$ is compact $\iff h(\rho)$ is finite
- (Prasad and Yu, 2002) if $\rho(G)$ contains no ***p-torsions***, then $S(\rho)$ is a Bruhat-Tits building.

We may ask the following questions

Question

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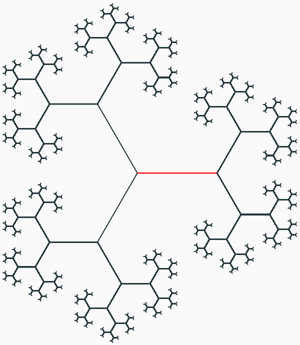
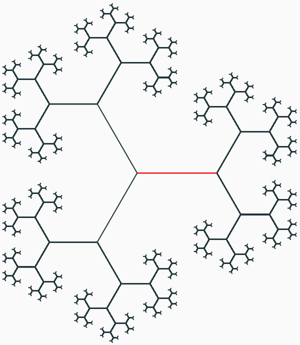
Question

- *How does $S(\rho)$ change along totally ramified extensions?*
- *Can we have a concrete description of $S(\rho)$?*
- *Given an interesting convex simplicial subset of the building, can we realize it as $S(\rho)$ for some ρ?*



The behavior of a “regular” fixed-point set.



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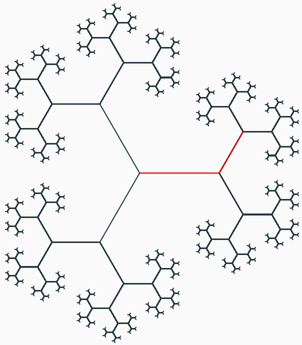
Fixed-point sets



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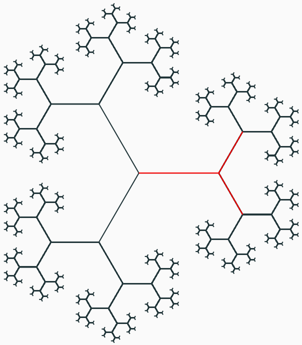

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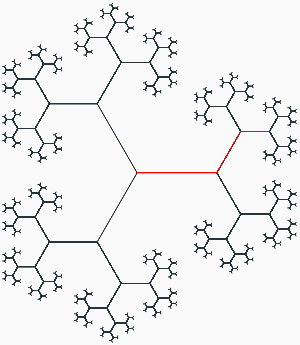

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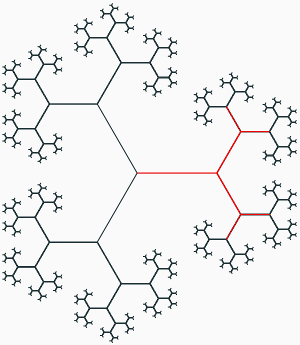

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Simplicial distance

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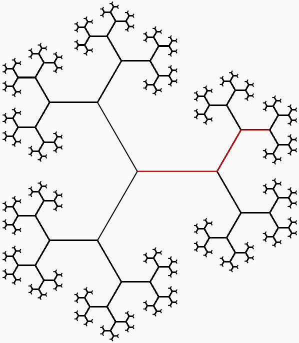
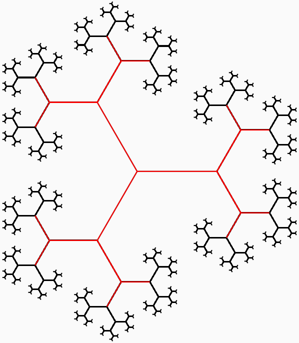
- $\rightsquigarrow$ ***simplicial balls***, ***simplicial volume***, ***simplicial curvature*** etc.



A path of length 3.



Simplicial ball of radius 3.

Questions on simplicial distance

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- *Are view simplicial balls fixed-point sets?*
- *How does simplicial balls grow?*

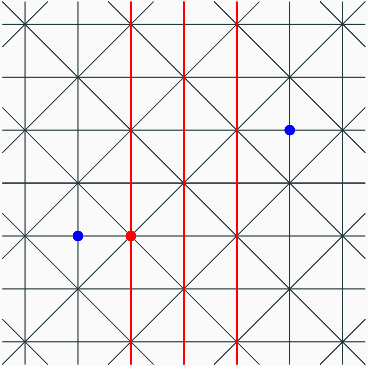
Φ the root system: *irreducible* and *classical* (i.e. A_n, B_n, C_n, or D_n)

Theorem (G., 2022)

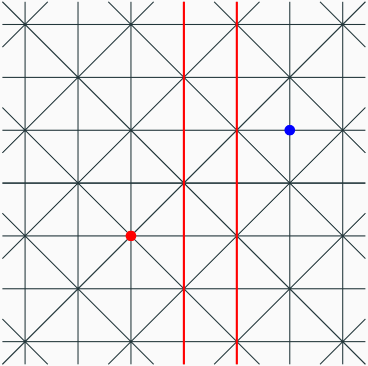
$d(x, y) - 1$ = maximum of numbers of walls between them; in particular, the simplicial distance from the origin o to x is

$$d(x, o) = \lceil a_0(x - o) \rceil,$$

where a_0 = highest root rel. Weyl chamber covering x.

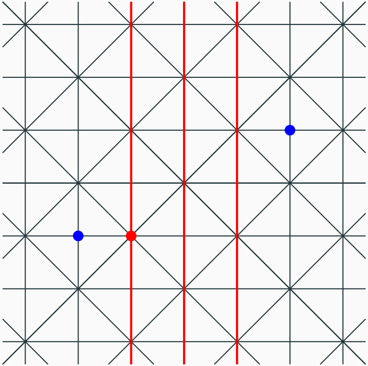


The two blues are separated by 3 walls and have distance 4.



The blue is separated by 2 walls from the red and has distance 3.

Simplicial balls in Classical type buildings



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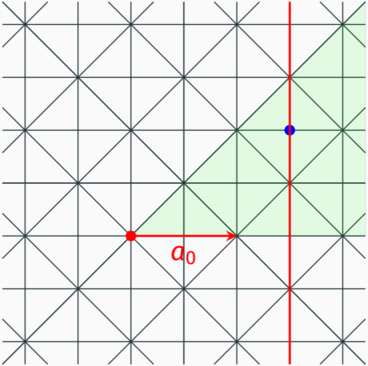
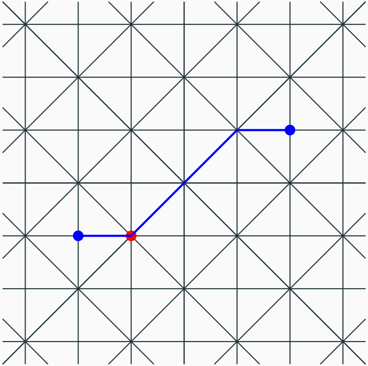
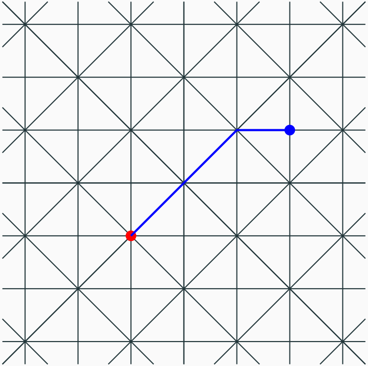


The blue has value 3 under a_0 and is of distance 3 from the red.



A path of length 4.



A path of length 3.

Corollary

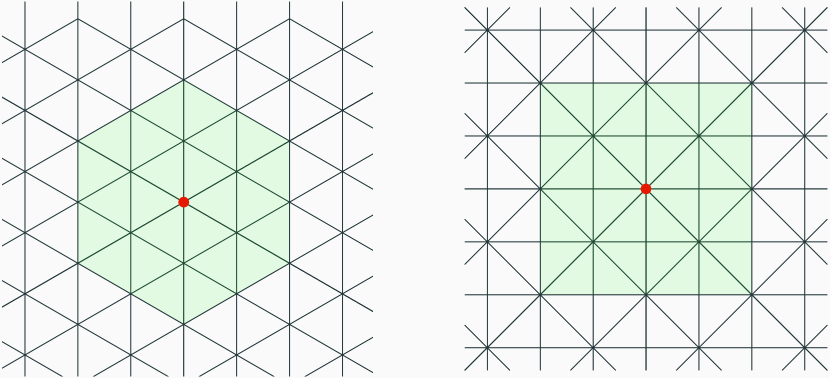
If $\Phi = A_n$ or C_n, then $d(x,y) \leqslant r \iff \exists L \in x, L' \in y$ such that

$$L \supset L' \supset \varpi^r L.$$

When $\Phi = B_n$ or D_n, the simplicial distance could be shorter due to the oriflamme construction.

Simplicial balls

Simplicial balls



Simplicial balls of radius 2.

$B(x, r)$ = simplicial ball of radius r at x

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- (Moy and Prasad, 1996) ***Moy-Prasad subgroups*** $(P_{x,r})_{r \geqslant 0}$ generalizing principal congruence subgroups
- (Yu, 2015) the machinery of ***concave functions***

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(Yu, 2015)

- ***concave function*** f = function on $\widetilde{\Phi} := \Phi \cup \{0\}$ s.t.

$$\forall a, b \in \widetilde{\Phi}, a+b \in \widetilde{\Phi} \implies f(a) + f(b) \geqslant f(a+b).$$

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$$\forall a, b \in \widetilde{\Phi}, a + b \in \widetilde{\Phi} \implies f(a) + f(b) \geqslant f(a+b).$$

- $\rightsquigarrow$ $\mathfrak{G}_f$ s.t. $P_f = \mathfrak{G}_f(K^\circ)$ is bounded

Examples

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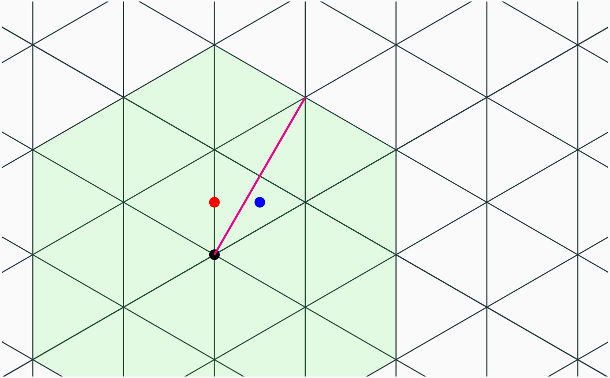
- If $r \geqslant 0$, the shifting $f + r$ is again a concave function.
- In general, $P_f \subseteq P_{f'} \iff f \geqslant f'$.
- In particular, $(P_{f_x+r})_{r \geqslant 0}$ are the *Moy-Prasad subgroups*.

Examples



$$\cdots \subset P_{\bullet,2} \subset P_{\diagup} \subset P_{\bullet} \subset P_{\bullet} \subset P_{\bullet}.$$

Simplicial volume

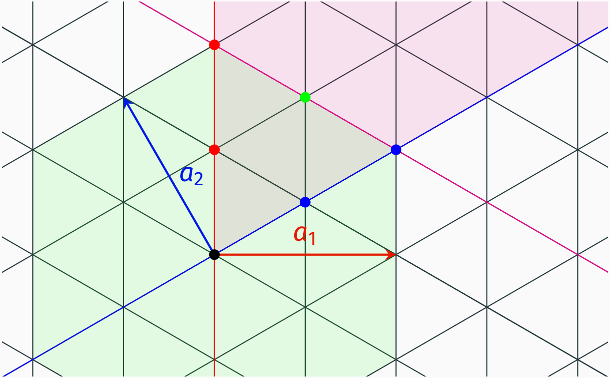
$\mathrm{SV}(r) := |B(o, r)|$ (***simplicial volume***)

Theorem (G., 2022)

$$\mathrm{SV}(r) = \sum_{I \subset \Delta} \frac{\mathscr{P}_{\Phi;I}(q)}{q^{\deg(\mathscr{P}_{\Phi;I})}} \sum_{x \in B(r,C,I)} \prod_{a(x)>0} q^{\lceil a(x) \rceil}$$

- $\mathscr{P}_{\Phi;I}$ = **Poincaré polynomial**
- C = *Weyl chamber*
- $B(r, C, I) = B(o, r) \cap \mathrm{inn}(\bigcap_{a \in I} \ker(a))$

The index set $B(r, C, I)$



$B(2, \blacksquare, a_1) = \{\bullet, \bullet\}$, $\quad$ $B(2, \blacksquare, a_2) = \{\bullet, \bullet\}$,

$B(2, \blacksquare, \Delta) = \{\bullet\}$, $\quad$ $B(2, \blacksquare, \emptyset) = \{\bullet\}$.

The formula

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- We have a concrete description of vertices.
  (However, they DO NOT form a lattice.)
- With such a description, we can describe the index set $B(r, C, I)$ in terms of linear inequalities.

The index set

If $\mathscr{B}$ is of type A_n and $I = \Delta \setminus \{\ell_1, \cdots, \ell_t\}$:

$$B(r, C, I) = \left\{o + c_1\omega_{\ell_1} + \cdots + c_t\omega_{\ell_t} \,\middle|\, c_i \in \mathbb{Z}_{>0}, c_1 + \cdots + c_t \leqslant r\right\},$$

where ω_i are the fundamental coweights.

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$$B(r, C, I) = \left\{o + c_1\omega'_{\ell_1} + \cdots + c_t\omega'_{\ell_t} \,\middle|\, c_i \in \mathbb{Z}_{>0}, c_1 + \cdots + c_t \leqslant r\right\},$$

where $\omega'_i = h_i^{-1}\omega_i$ with $a_0 = \sum_i h_i a_i$.

The index set

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If $\mathscr{B}$ is of type B_n or D_n, then the description is complicated.

The index set

$$\begin{tikzcd}
 & o + \frac{1}{2}\,\mathcal{P}^\vee \arrow[dl, dash] \arrow[dr, dash] & \Xi \arrow[d, dash] \\
\mathcal{V} \arrow[d, dash] & & \Xi \cap o + \frac{1}{2}\,\mathcal{P}^\vee \\
\mathcal{V}_\dagger & &
\end{tikzcd}$$

Vertices in B_n building

$$\begin{tikzcd}
& \mathcal{X}^{00} \cup \mathcal{X}^{10} \cup \mathcal{X}^{01} \cup \mathcal{X}^{11} & \Xi^{00} \cup \Xi^{10} \cup \Xi^{01} \cup \Xi^{11} \\
{\color{red}\mathcal{V}} \arrow[ur, dash] & & (\mathcal{X}^{00} \cap \Xi^{00}) \cup (\mathcal{X}^{10} \cap \Xi^{10}) \cup (\mathcal{X}^{01} \cap \Xi^{01}) \cup (\mathcal{X}^{11} \cap \Xi^{11}) \arrow[ul, dash] \arrow[u, dash] \\
\mathcal{V}_{\dagger} \arrow[u, dash] & &
\end{tikzcd}$$

Vertices in D_n building

Asymptotic estimation

Theorem (G., 2022)

We have the following asymptotic dominant relation as $r \to \infty$:

$$\mathrm{SV}(r) \asymp r^{\epsilon(n)} q^{\pi(n) r},$$

where $\epsilon(n)$ and $\pi(n)$ are in the following table.

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where $\epsilon(n)$ and $\pi(n)$ are in the following table.

X_n	$\epsilon(n)$	$\pi(n)$
A_n (*n is odd*)	0	$(\frac{n+1}{2})^2$
A_n (*n is even*)	1	$\frac{n}{2}(\frac{n}{2}+1)$
B_n $(n=3)$	0	5
B_n $(n \geqslant 4)$	0	$\frac{n^2}{2}$
C_n $(n \geqslant 2)$	0	$\frac{n(n+1)}{2}$
D_n $(n=4)$	2	6
D_n $(n \geqslant 5)$	1	$\frac{n(n-1)}{2}$

Theorem (G., 2022)

Leading coefficients have the following rationality properties:

1. *If $\mathscr{B}$ is of type A_n, C_n, B_3, or D_4, then we have*

$$\mathsf{SV}(r) \sim c(n) \cdot r^{\epsilon(n)} q^{\pi(n) r},$$

where $c(n)$ is a positive number that is a rational function of q.

Leading coefficients

Theorem (G., 2022)

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   where $c(n)$ is a positive number that is a rational function of q.

2. *If $\mathscr{B}$ is of type B_n $(n \geqslant 4)$ or D_n $(n \geqslant 5)$, then we have*

$$\mathrm{SV}(2r) \sim c_0(n) \cdot r^{\epsilon(n)} q^{2\pi(n)r},$$
$$\mathrm{SV}(2r+1) \sim c_1(n) \cdot r^{\epsilon(n)} q^{2\pi(n)r},$$

   where $c_0(n)$ and $c_1(n)$ are positive numbers that are rational functions of q.

Remarks

- *Exceptional types* 

REMARKS

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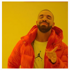
- *Non-irreducible buildings* 

Remarks

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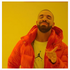
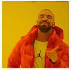
- *Non-irreducible buildings* 
- *non-split types* 

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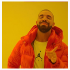
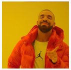
- *Non-irreducible buildings* 
- *non-split types* 
- ***concave functions*** and ***fixed-point sets*** (conjectured)

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