Farey sequence

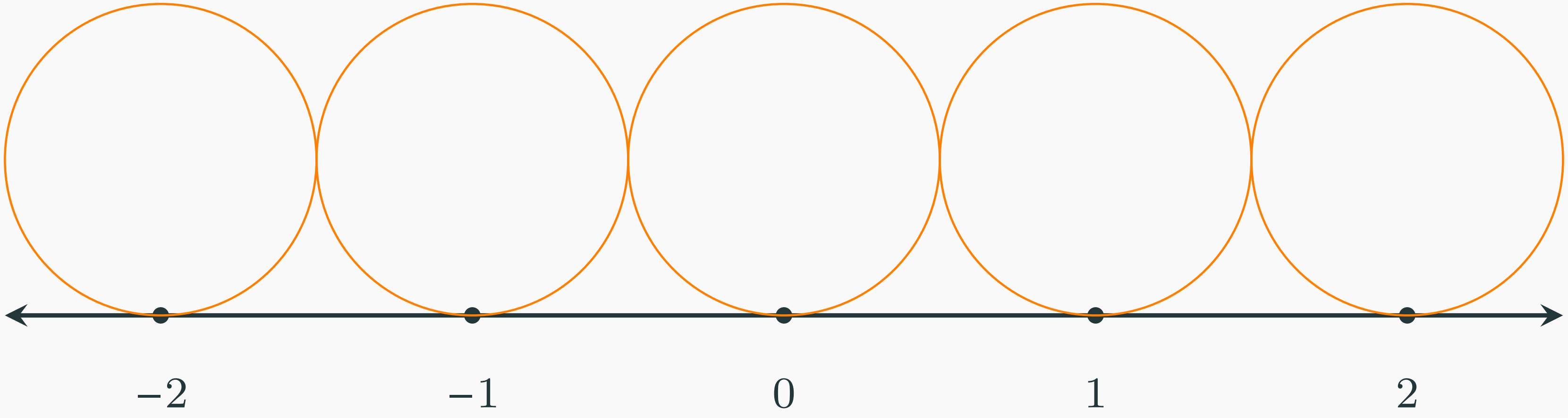
Theorem 3.6.1

The following process generates all reduced fractions (in geometric words, all Ford circles):

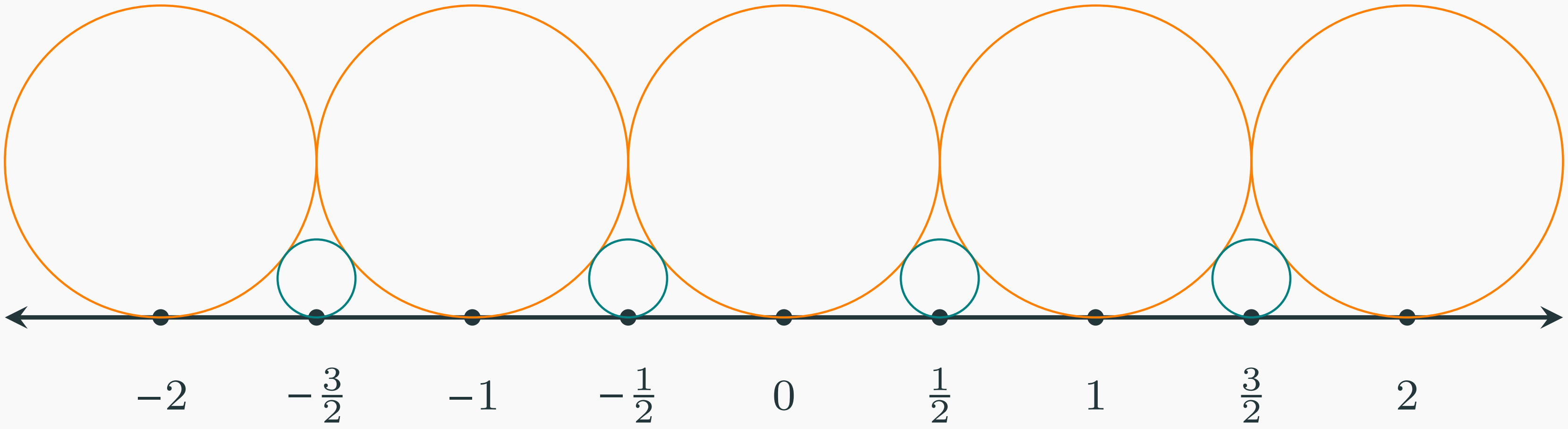
1. *Start with integers, namely fractions of the form* $\frac{n}{1}$ *(in geometric words, Ford circles atop integer points).*
2. *Whenever you have two kissing fractions* $\frac{a}{b}$ *and* $\frac{c}{d}$*, generate their mediant* $\frac{a}{b} \vee \frac{c}{d}$ *(in geometric words, whenever you have two Ford circles tangent to each other, generate the third one by lemma 3.5.2).*

The produced sequence is called the *Farey sequence*.

The base case:



Next step:



Proof. We need to show: any reduced fraction can be found in the process. We do this by induction on the denominator b.

First, the base case is clear, any integer appears in the base step.

Now, assume any reduced fraction with denominator less than B can be found in the process. For any reduced fraction of the form $\frac{A}{B}$, we'll show that it can be obtained as the mediant of two kissing fractions, hence appear in the process.

Lemma 3.6.2

Let $\frac{A}{B}$ be a reduced fraction. Then fractions kissing it are

$$\left\{ \frac{x_+ + A \cdot n}{y_+ + B \cdot n}, \frac{x_- + A \cdot n}{y_- + B \cdot n}, \;\middle|\; n \in \mathbb{Z} \right\},$$

where (x_+, y_+) and (x_-, y_-) are specific solutions of the linear Diophantine equations

$\frac{x}{y}$ ♡ $\frac{A}{B} \Leftrightarrow |Ay - Bx| = 1$

$$(-B) \cdot x + (A) \cdot y = 1 \qquad \text{and} \qquad (-B) \cdot x + (A) \cdot y = -1$$

respectively.

Proof. A fraction $\frac{x}{y}$ kisses $\frac{A}{B}$ whenever $(-B) \cdot x + (A) \cdot y$ equals 1 or -1. Then the lemma follows from theorem 1.4.2 (General solutions of linear Diophantine equations). $\square$

Proof of the theorem

Let's back to the proof of the theorem. $|Ay - Bx| = 1$

As $B > 1$, we cannot have $B \mid y_+$ (by 2-out-of-3 principle).

Hence the point y_+ must be inside one of above intervals. In other words, there is a (unique) integer n_+ such that

$$0 < y_+ + B \cdot n_+ < B.$$

Similarly, there is a (unique) integer n_- such that

$$0 < y_- + B \cdot n_- < B.$$

Let's set $a = x_+ + A \cdot n_+$, $b = y_+ + B \cdot n_+$, $c = x_- + A \cdot n_-$, and $d = y_- + B \cdot n_-$. Then $\frac{a}{b}$ and $\frac{c}{d}$ are reduced fractions with denominators less than B.

Note that, by their definitions, we have

$$(-B) \cdot a + (A) \cdot b = 1 \qquad \text{and} \qquad (-B) \cdot c + (A) \cdot d = -1.$$

Add them together, we get $(-B) \cdot (a + c) + (A) \cdot (b + d) = 0$. Hence,

$$\frac{A}{B} = \frac{a}{b} \vee \frac{c}{d}.$$

Lastly, note that

$$ad - bc = a(B - b) - b(A - a) = aB - bA = -1.$$

Hence, we have $\frac{a}{b} \heartsuit \frac{c}{d}$. This finishes the proof.