Homework 6 (due Nov. 15)

MATH 110 | Introduction to Number Theory | Fall 2022

Problem 1 (10 pts)**.** Consider the recursive sequence given by

$$a_0 = 3, \quad a_n = 3^{a_{n-1}}, \quad \text{for all} \quad n \geq 1$$

That is, $a_0 = 3,\ a_1 = 3^3,\ a_2 = 3^{3^3}, \ldots$. What is the last digit of a_{2022}?

Remark. Be aware that $3^{3^3} \neq (3^3)^3$.

Problem 2 (10 pts)**.** Suppose p is an odd prime and q is a prime divisor of $2^p - 1$. Prove that $q \equiv 1 \pmod{2p}$.

Problem 3 (10 pts)**.** Prove that if p is any prime and a and b are any nonzero integers such that $a \equiv b \pmod{p^2 - p}$, then $a^a \equiv b^b \pmod{p}$.

Problem 4 (10 pts)**.** Give an example to show that $\mathbb{Z}/20[T]$ has no *unique prime factorization property.*

Remark. Refer to Problem 4 in HW 2 for the related notions. Note that, to show your example fails the *unique prime factorization property*, you need to show your factors are *prime* (in the context of polynomials, irreducible), and not associated to either other (that is, not different by a nonzero constant factor).

Problem 5. Let p be a prime number.

(a) (5 pts) Let $f(T)$ be a polynomial modulo p of degree 2 or 3. Prove that $f(T)$ is irreducible if and only if $f(T)$ has no roots modulo p.

Hint. Prove the contrapositive, looking at the degrees of the divisors of $f(T)$.

(b) (5 pts) Count the number of monic polynomials modulo p of degree d.

(c) (5 pts) Count the number of monic irreducible polynomials modulo p of degree 2.

(d) (5 pts) Count the number of monic irreducible polynomials modulo p of degree 3.

($*$e) (Optional, up to 5 extra pts) Count the number of monic irreducible polynomials modulo p of degree 4.