Dirichlet's approximation theorem

What we have?

- *Ford circle*: a circle of diameter $\frac{1}{b^2}$ atop the rational point $\frac{a}{b}$.
- *Kissing fractions* ($\frac{a}{b} \heartsuit \frac{c}{d}$): $\left|\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right| = |ad - bc| = 1$.
- *Mediant*: $\frac{a}{b} \vee \frac{c}{d} := \frac{a+c}{b+d}$.
- *Farey sequence*: repeatedly taking mediants, containing all reduced fractions.

Theorem 3.7.1 (Dirichlet, 1840)

Let α be an irrational number, Then there are infinitely many fractions $\frac{a}{b}$ such that

$$\left|\alpha - \frac{a}{b}\right| \leqslant \frac{1}{2b^2}.$$

To prove Dirichlet's approximation theorem, it is sufficient to show that a vertical line atop an irrational point crosses infinitely many Ford circles.

Lemma 3.7.2

The following process generates all reduced fractions (in geometric words, all Ford circles):

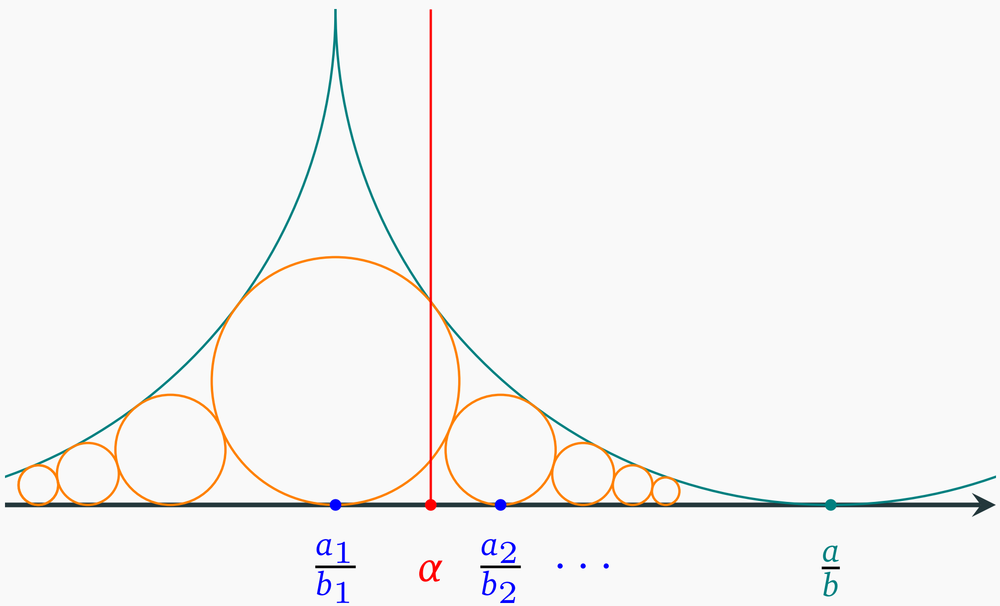
1. *Start with integers, namely fractions of the form* $\frac{n}{1}$ *(in geometric words, Ford circles atop integer points).*
2. *Whenever you have two kissing fractions* $\frac{a}{b}$ *and* $\frac{c}{d}$*, generate their mediant* $\frac{a}{b} \vee \frac{c}{d}$ *(in geometric words, whenever you have two Ford circles tangent to each other, generate the third one atop the mediant).*

Proof of the theorem

Proof. We prove the theorem using the recursive process in the Farey sequence.

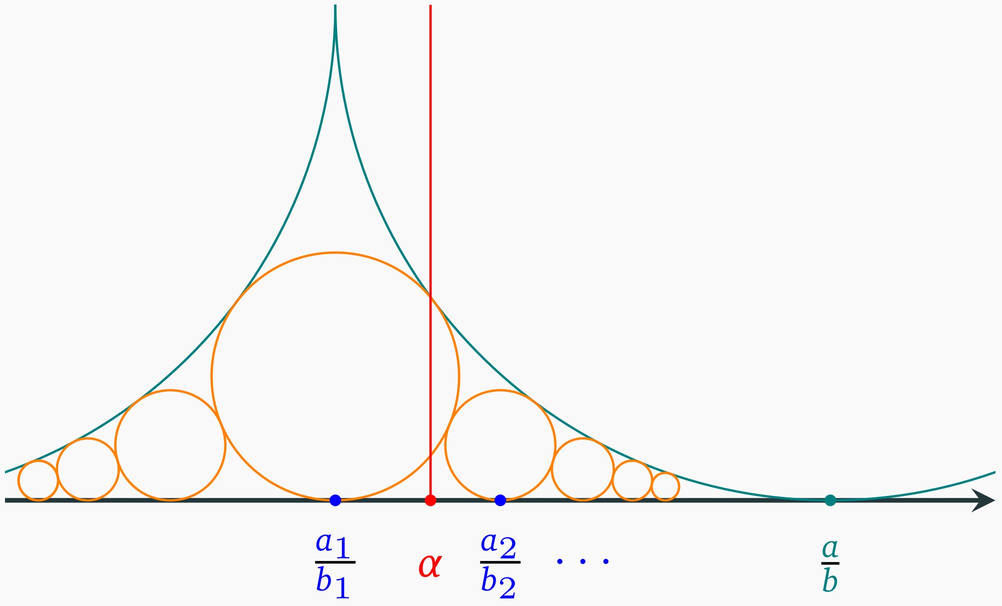
- At the base step, the vertical line $x = \alpha$ must cross one of the Ford circles atop some $\frac{n}{1}$ since α is irrational.
- Whenever the vertical line $x = \alpha$ crosses a Ford circle (saying, atop $\frac{a}{b}$) and falls into the mesh triangle below it, then it must cross another Ford circle inside the mesh triangle.
- The process will go on forever as the Farey sequence and thus produce infinitely many Ford circles crossed by the line $x = \alpha$.

Proof of the theorem



The proof boils down to show the following: Suppose the vertical line $x = \alpha$ crosses the Ford circle atop $\frac{a}{b}$, then it also crosses a Ford circle inside the mesh triangle below.

Proof of the theorem



Suppose the mesh triangle is given by the Ford circles atop $\frac{a}{b}$ and $\frac{c}{d}$. Then we know that α must leave between $\frac{a}{b}$ and $\frac{c}{d}$ since the vertical line $x = \alpha$ crosses the mesh triangle. We may assume $\frac{a}{b} > \alpha > \frac{c}{d}$.

Consider the following sequence of fractions:

$$\frac{a_0}{b_0} := \frac{c}{d}, \frac{a_1}{b_1} := \frac{a}{b} \vee \frac{c}{d}, \cdots, \frac{a_n}{b_n} := \frac{a}{b} \vee \frac{a_{n-1}}{b_{n-1}}, \cdots$$

Then the Ford circle atop each $\frac{a_n}{b_n}$ $(n > 0)$ is tangent to the one atop $\frac{a}{b}$ and all of them leave inside the mesh triangle.

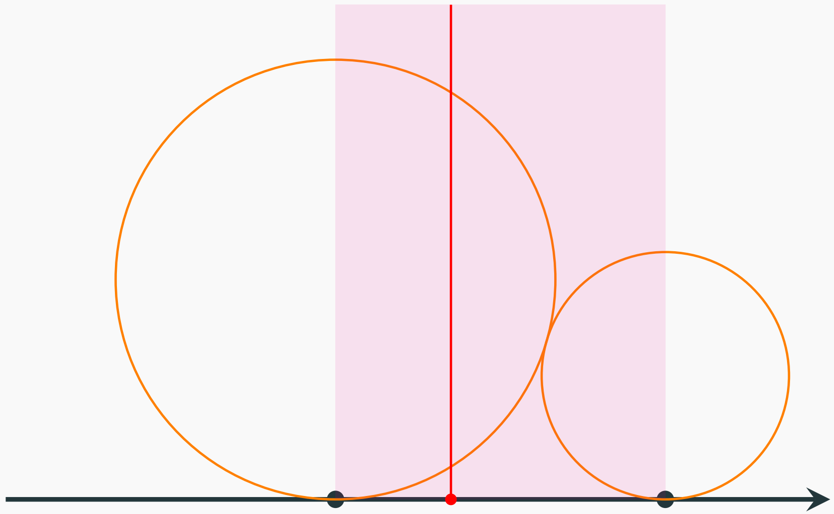
Note that

$$\frac{a_n}{b_n} = \frac{a \cdot n + c}{b \cdot n + d}.$$

Hence, the sequence of rational numbers $(\frac{a_n}{b_n})_{n \in \mathbb{Z}}$ is monotonously increasing and has the limit $\frac{a}{b}$. Then, since $\frac{a}{b} > \alpha > \frac{c}{d}$, there must be a positive integer n such that

$$\frac{a_n}{b_n} > \alpha > \frac{a_{n-1}}{b_{n-1}}.$$

Namely, the vertical line $x = \alpha$ crosses the strip between $\frac{a_n}{b_n}$ and $\frac{a_{n-1}}{b_{n-1}}$.



But notice that $\frac{a_n}{b_n} \heartsuit \frac{a_{n-1}}{b_{n-1}}$. Namely, the Ford circles atop $\frac{a_n}{b_n}$ and $\frac{a_{n-1}}{b_{n-1}}$ are tangent to each other. Hence, to cross the strip between $\frac{a_n}{b_n}$ and $\frac{a_{n-1}}{b_{n-1}}$, the vertical line $x = \alpha$ must cross one of the two Ford circles! Thus, we find a Ford circle inside the initial mesh triangle and is crossed by the line $x = \alpha$ as desired.