Homework 5 (due Nov. 8)

MATH 110 | Introduction to Number Theory | Fall 2022

Problem 1. (a) (5 pts) What are **all** the possible natural representatives of $n^2 \pmod 8$, where n is an integer.

(b) (5 pts) Determine whether the following equation is solvable in $\mathbb{Q}$ or not? If it is, **find** such a solution $(x, y, z) \in \mathbb{Q}^3$, if not, **prove it**.

$$x^2 + y^2 + z^2 = 2023$$

Hint. First translate it into a Diophantine equation asking solutions in $\mathbb{Z}$. Then consider the congruence modulo 8.

Problem 2. Let p be any prime number and let a and b be any two integers.

(a) (5 pts) Prove that if $a \equiv b \pmod p$, then $a^p \equiv b^p \pmod{p^2}$.

(b) (5 pts) Prove that if $a \equiv b \pmod p$, then $a^{p^2} \equiv b^{p^2} \pmod{p^3}$.

(c) (Optional, up to 5 extra pts) Can you generalise?

Problem 3 (10 pts)**.** Solve the congruences $5x \equiv 11 \pmod{37}$ and $11y \equiv 5 \pmod{37}$.

Problem 4. Consider the following modular dynamical system, which is neither additive nor multiplicative.

(a) (5 pts) Let $X = \mathbb{Z}/13$ and let $f : X \to X$ be given by

$$x \mapsto f(x) := x^2 + 3 \pmod{13}.$$

Draw the complete diagram for the dynamics of f.

(b) (5 pts) Let $A_0 = 0$ and let $A_{n+1} = f(A_n) \pmod{13}$ for all integers $n \geq 0$. What is $A_{2022} \pmod{13}$?

Problem 5 (10 pts)**.** Compute the length of the cycles in the dynamics of $\boxed{\times a \pmod 8}$ for every $a \in \Phi(8)$. Compare the length with $\varphi(8)$ (in the sense of divisibility or equality).