Homework 4 (due Oct. 30)

MATH 110 | Introduction to Number Theory | Fall 2022

Problem 1. Recall that an *integer polynomial* is an expression of the form $P(T) = c_d T^d + \cdots + c_1 T + c_0$, where each c_i is an integer.

(a) (5 pts) **Find** a nonzero integer polynomial $P(T)$ that has $\sqrt{3} + \sqrt[3]{5}$ as a root.

(b) (5 pts) **Prove that** $\sqrt{3} + \sqrt[3]{5}$ is irrational using 1.(a).

Problem 2. By evaluating the Taylor series for the exponential function:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

at $x = 1$, we get the formula

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots .$$

In this problem, you will prove that e is irrational.

(a) (5 pts) Let $s_n := \sum_{k=0}^{n} \frac{1}{k!}$, the n-th partial sum of above series. **Show that**

$$0 \leq e - s_n \leq \frac{1}{n} \cdot \frac{1}{n!}.$$

(b) (5 pts) Assume e is rational, and say a/b is the reduced fraction representing e. Apply the previous result to $n = b$ and arrive at a contradiction.

Problem 3. Consider the *Fibonacci numbers*, define recursively by

$$F_0 = 0, F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2} \quad \text{for all} \quad n \geq 2;$$

so the first few terms are

$$0, 1, 1, 2, 3, 5, 8, 13, \cdots .$$

For all $n \geq 2$, define the rational number r_n by the fraction $\dfrac{F_n}{F_{n-1}}$; so the first few terms are

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \cdots .$$

(a) (5 pts) Prove that for all $n \geq 4$, we have $r_n = r_{n-1} \vee r_{n-2}$.

(b) (5 pts) Prove that the sequence r_n converges (to a real number).

(c) (5 pts) Prove that r_n converges to the *golden ratio*:

$$\phi = \frac{1+\sqrt{5}}{2}.$$

For this problem, you can use any result that you may have seen in your Calculus classes.

Problem 4. This problem exhibits the phenomenon that square roots of different integers are most likely $\mathbb{Q}$-linearly independent.

(a) (3 pts) **Show that** the only pair (a, b) of rational numbers such that $a + b\sqrt{2} = 0$ is $(0, 0)$. (In terms of linear algebra, 1 and $\sqrt{2}$ are $\mathbb{Q}$-linearly independent.)

(b) (3 pts) **Show that** there exist no rational numbers a and b such that

$$a + b\sqrt{2} = \sqrt{3}.$$

Hint. Start with squaring the purported equation.

(c) (4 pts) **Show that** there exist no rational numbers a, b and c such that

$$a + b\sqrt{2} + c\sqrt{3} = \sqrt{6}.$$

Hint. What is the inverse of $\sqrt{2} - c$?

(d) (5 pts) **Show that** there exist no rational numbers a, b and c such that

$$a + b\sqrt{2} + c\sqrt{3} = \sqrt{5}.$$

($*$e) (Optional, up to 5 extra pts) **Show that** there exist no rational numbers a, b, c and d such that

$$a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} = \sqrt{5}.$$