General sigma functions

We generalize $\sigma_0(\cdot)$ to the following functions ($k \in \mathbb{Z}$):

$$\sigma_k(n) := \sum_{d \in \mathscr{D}(n)} d^k.$$

Theorem 2.7.1

Each $\sigma_k(\cdot)$ is a multiplicative function.

Proof. Let m, n be two coprime integers. Then we have

$$
\begin{aligned}
\sigma_k(mn) = \sum_{c \in \mathscr{D}(mn)} c^k &\overset{\spadesuit}{=} \sum_{(a,b) \in \mathscr{D}(m) \times \mathscr{D}(n)} (ab)^k \\
&= \left(\sum_{a \in \mathscr{D}(m)} a^k \right) \cdot \left(\sum_{b \in \mathscr{D}(n)} b^k \right) \\
&= \sigma_k(m) \sigma_k(n)
\end{aligned}
$$

$\square$

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&= \sigma_k(m)\sigma_k(n)
\end{aligned}$$

□

Let me explain what happened at ♠: we have changed the expression from the previous one using the *bijection*

$$\Phi\colon \; \mathscr{D}(m) \times \mathscr{D}(n) \to \mathscr{D}(mn) \colon \; (a, b) \mapsto ab.$$

Let's generalize corollary 2.6.5 to $\sigma_k(\cdot)$.

Theorem 2.7.2

Let n be a positive integer and $k \in \mathbb{Z}$. We have

$$\sigma_k(n) = \prod_{p \text{ is prime}} \frac{p^{k(v_p(n)+1)} - 1}{p^k - 1}.$$

Let's generalize corollary 2.6.5 to $\sigma_k(\cdot)$.

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Clearly, it suffices to show for all prime p and natural number e that

$$\sigma_k(p^e) = \frac{p^{k(e+1)} - 1}{p^k - 1}.$$

We first introduce a lemma.

Lemma 2.7.3

If x is a real number other than 1 and e is a natural number, then

$$\sum_{i=0}^{e} x^i := 1 + x + x^2 + \cdots + x^e = \frac{x^{e+1} - 1}{x - 1}.$$

Proof.

$$\begin{aligned} \text{Let} \qquad S &= 1 + x + x^2 + \cdots + x^e, \\ \text{then} \qquad xS &= \phantom{1 + {}} x + x^2 + \cdots + x^e + x^{e+1}. \\ \text{Hence,} \quad (x-1)S &= x^{e+1} - 1. \end{aligned}$$

Since $x \neq 1$, we can divide both sides by $x - 1$. $\square$

Proof

Now we can prove the theorem 2.7.2.

Proof. Let p be a prime and e a natural number. Then since $\mathscr{D}(p^e) = \{1, p, \cdots, p^e\}$, we have

$$\sigma_k(p^e) = \sum_{i=0}^{e} (p^i)^k = \sum_{i=0}^{e} (\underbrace{p^k}_{x})^i = \frac{p^{k(e+1)} - 1}{p^k - 1}.$$

Here, in the last step, we applied lemma 2.7.3 to $x = p^k$. $\square$

Example 2.7.4

Compute $\sigma_3(12)$.

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Compute $\sigma_3(12)$.

First note that $12 = 2^2 \cdot 3$. Hence, $\sigma_3(12) = \sigma_3(2^2)\sigma_3(3)$.

By theorem 2.7.2, we have

$$\sigma_3(2^2) = \frac{2^{3\cdot(2+1)} - 1}{2^3 - 1} = \frac{2^9 - 1}{2^3 - 1} = 73$$

$$\sigma_3(3) = \frac{3^{3\cdot(1+1)} - 1}{3^3 - 1} = \frac{3^6 - 1}{3^3 - 1} = 28$$

Hence, $\sigma_3(12) = 73 \cdot 28 = 2044$.