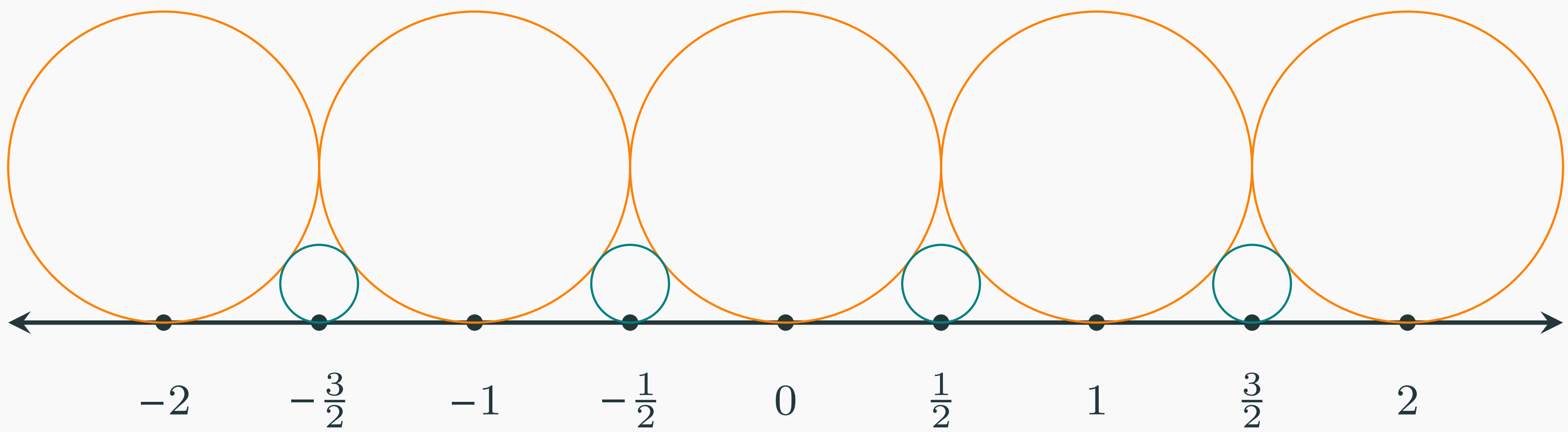
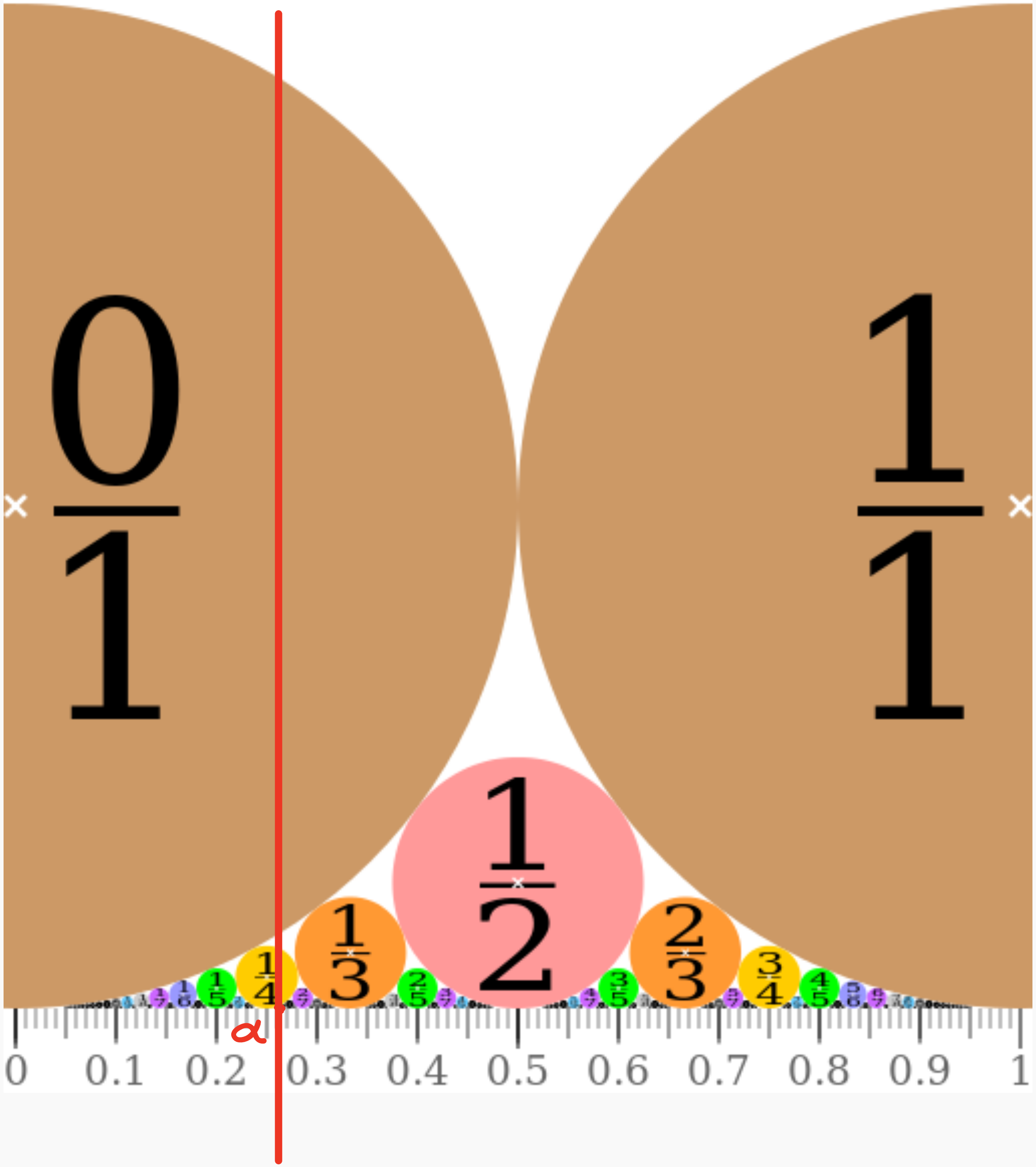
Ford circles

Definition 3.4.1 (Lester Ford, 1938)

A *Ford circle* is a circle of diameter $\frac{1}{b^2}$ atop the rational point on the number line corresponding to the reduced fraction $\frac{a}{b}$. (Integers are expressed as reduced fractions with denominator 1.)





The left shows Ford circles between 0 and 1. Draw a vertical line crossing α, then the inequality $\left|\alpha - \frac{a}{b}\right| \leqslant \frac{1}{2b^2}$ holds whenever the line crosses the Ford circle atop $\frac{a}{b}$.

So to prove Dirichlet's approximation theorem, it is sufficient to show that a vertical line atop an irrational point crosses infinitely many Ford circles.

We will do this through an induction on b. For this purpose, we need a recursive description of Ford circles.

We first note that there are no overlaps between Ford circles: they are either tangent to or separated from each other.

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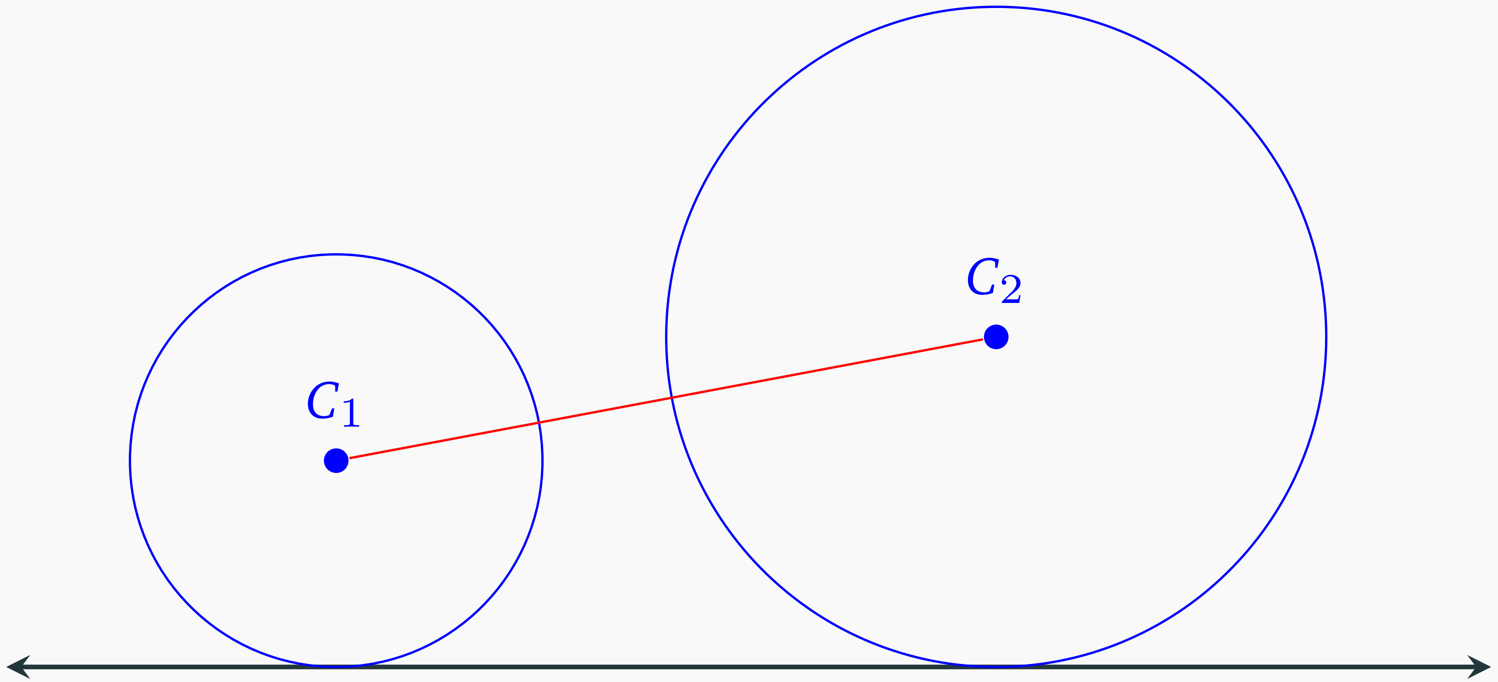
Indeed, let C_1 and C_2 be two Ford circles atop $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ respectively. Then we know that they can be described by the equations

$$(x - \frac{a_1}{b_1})^2 + (y - \frac{1}{2b_1})^2 = \frac{1}{2b_1}, \quad \text{and} \quad (x - \frac{a_2}{b_2})^2 + (y - \frac{1}{2b_2})^2 = \frac{1}{2b_2}$$

repsectively.


C_1
C_2

Therefore, the distance between their centers is

$$\begin{aligned} d(C_1, C_2) &= \sqrt{\left(\frac{a_2}{b_2} - \frac{a_1}{b_1}\right)^2 + \left(\frac{1}{2b_2} - \frac{1}{2b_1}\right)^2} \\ &= \sqrt{\frac{(a_2 b_1 - a_1 b_2)^2}{{b_1}^2 {b_2}^2} + \left(\frac{1}{2b_2} - \frac{1}{2b_1}\right)^2} \\ &\geqslant \sqrt{\frac{1}{{b_1}^2 {b_2}^2} + \left(\frac{1}{2b_2} - \frac{1}{2b_1}\right)^2} \\ &= \frac{1}{2b_1} + \frac{1}{2b_2}. \end{aligned}$$

Kissing fractions

So when do two Ford circles tangent to each other?

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Note that in the previous slide, the quality holds if and only if $|a_2 b_1 - b_2 a_1| = 1$. We thus have the following notion:

Definition 3.4.2

We say two fractions $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ *kiss* each other if

$$\left|\det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\right| = |a_2 b_1 - a_1 b_2| = 1.$$

We will use the notation $\frac{a_1}{b_1} \heartsuit \frac{a_2}{b_2}$ to denote this.

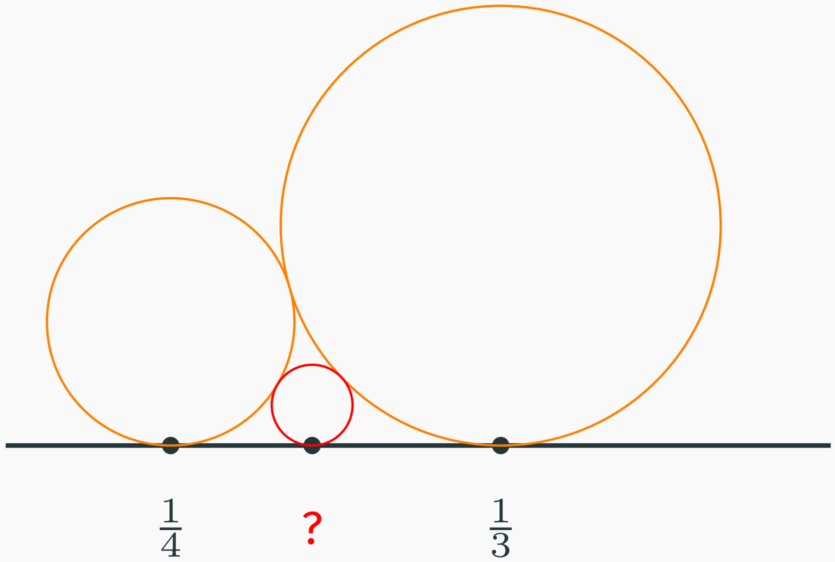
N.B. $\frac{a_1}{b_1} \heartsuit \frac{a_2}{b_2}$ implies that $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are *reduced* fractions (why?). Since any rational number has a unique reduced fraction expression, $\heartsuit$ is rather a relation between rational numbers.

What we have proved can be interpreted as:

Lemma 3.4.3

Two Ford circles are tangent to each other if and only if the fractions they atop kiss each other.

In next lecture, we will show that if you have two Ford circle tangent to each other, then you can find a third one tangent to both of them.



Translating this into fractions, it means if you have a pair of kissing fractions, then there is a third one kisses both of them.