Introduction to Number Theory

Math 110 | Winter 2023

Xu Gao

February 3, 2023

What we have seen last lecture

- ***Diophantine approximation***: approximate irrational numbers by rational numbers.
- ***Dirichlet's approximation theorem***: $\left|\alpha - \frac{a}{b}\right| \leqslant \frac{1}{2b^2}$.
- ***Ford circle***: a circle of diameter $\frac{1}{b^2}$ atop the rational point $\frac{a}{b}$.
- ***Kissing fractions*** ($\frac{a}{b} \heartsuit \frac{c}{d}$): $\left|\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right| = |ad - bc| = 1$.

Today's topics

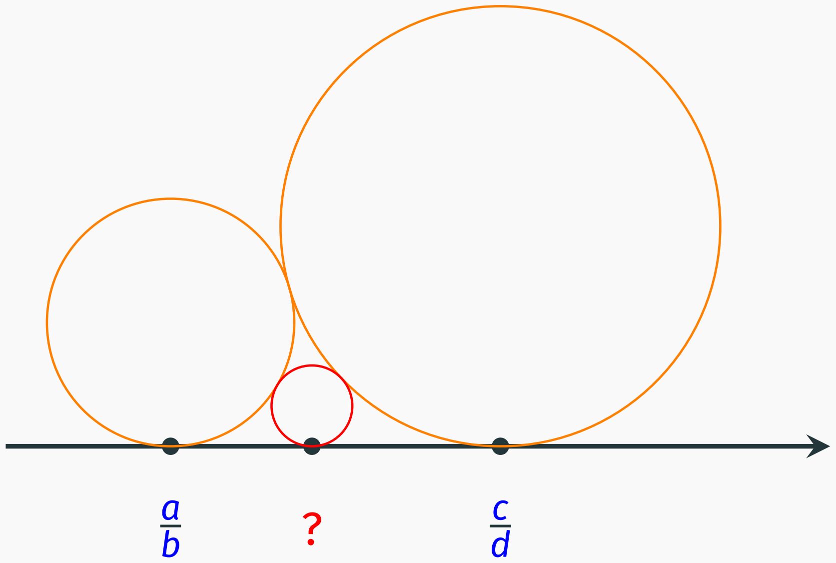
- Ford circles and kissing fractions
- Mediant
- Farey sequence
- Continue proving Dirichlet's approximation theorem

Mediant

Mediant

Question

Given two Ford circle tangent to each other. Find a third one tangent to both of them.



Mediant

To answer this question, let's suppose the two Ford circles C_1 and C_2 are atop rational points $\frac{a}{b}$ and $\frac{c}{d}$ respectively.

Let C be a Ford circle atop $\frac{x}{y}$ between C_1 and C_2. Then we have

Mediant

To answer this question, let's suppose the two Ford circles C_1 and C_2 are atop rational points $\frac{a}{b}$ and $\frac{c}{d}$ respectively.

Let C be a Ford circle atop $\frac{x}{y}$ between C_1 and C_2. Then we have

- $\frac{x}{y}$ is between $\frac{a}{b}$ and $\frac{c}{d}$;
- C_1 and C_2 are tangent to each other, namely $\frac{a}{b} \heartsuit \frac{c}{d}$;
- C is tangent to C_1 if and only if $\frac{x}{y} \heartsuit \frac{a}{b}$;
- C is tangent to C_2 if and only if $\frac{x}{y} \heartsuit \frac{c}{d}$;

Mediant

Translate the relations $\frac{a}{b} \heartsuit \frac{c}{d}$, $\frac{x}{y} \heartsuit \frac{a}{b}$, and $\frac{x}{y} \heartsuit \frac{c}{d}$ into identities, we get a system of equations with unknown $x.y$.

$$\begin{cases} |ad - bc| = 1, \\ |xb - ya| = 1, \\ |xd - yc| = 1. \end{cases}$$

One can solve this system and get

$$\text{either} \quad \frac{x}{y} = \frac{a-c}{b-d} \qquad \text{or} \quad \frac{x}{y} = \frac{a+c}{b+d}.$$

Considering that $\frac{x}{y}$ is between $\frac{a}{b}$ and $\frac{c}{d}$, we must have $\frac{x}{y} = \frac{a+c}{b+d}$.

Mediant

We thus introduce the following notion:

Definition 10.1

Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, the ***mediant*** of them is the fraction

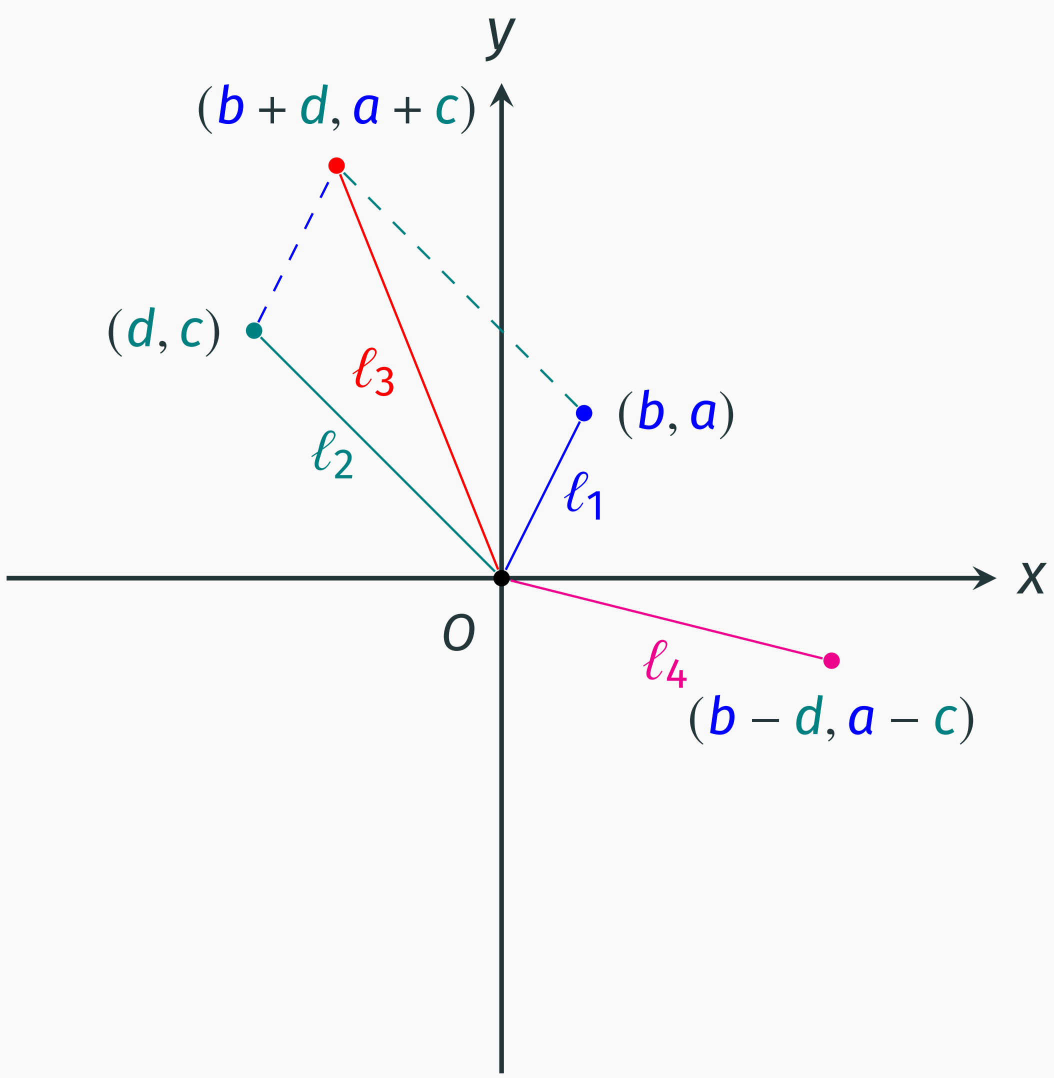
$$\frac{a}{b} \vee \frac{c}{d} := \frac{a+c}{b+d}.$$

N.B. this is an operation on *fractions*!

$\frac{1}{2} \vee \frac{2}{3} = \underline{\frac{3}{5}}$ $\qquad$ $\frac{2}{4} \vee \frac{2}{3} = \underline{\frac{4}{7}}$

$\frac{3}{5} \neq \frac{4}{7}$

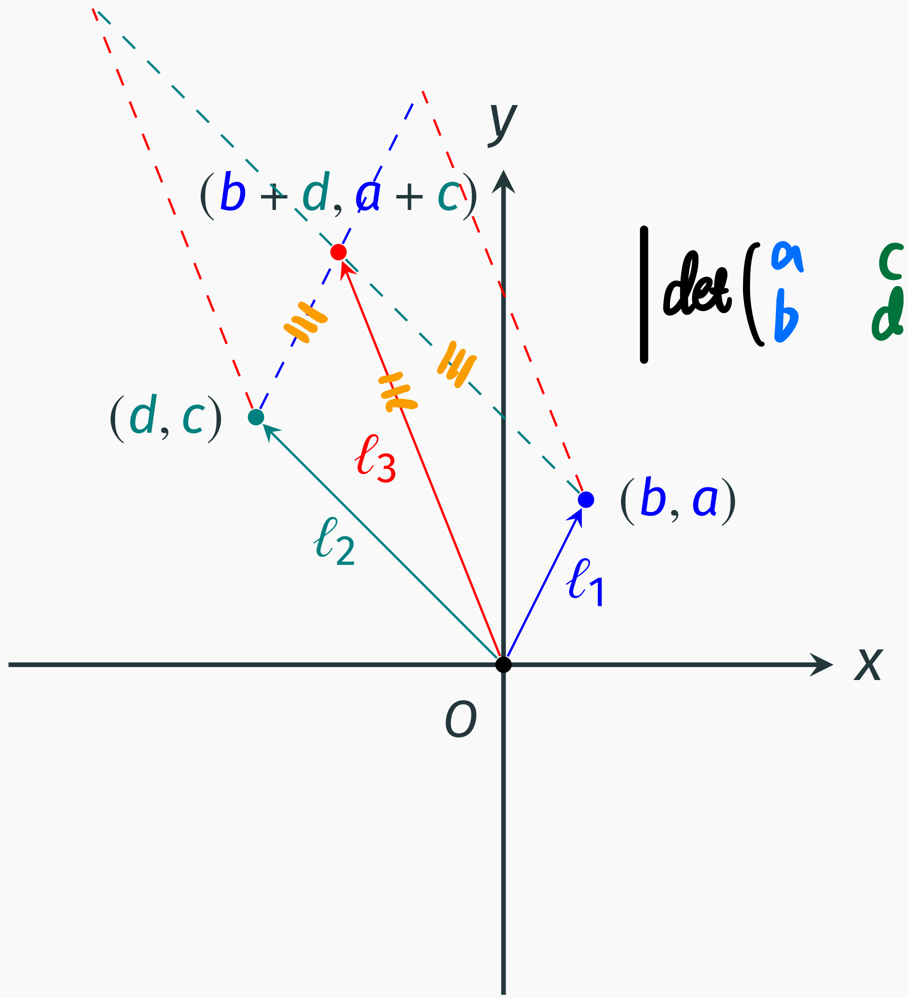
Geometric interpretation of mediant



- $\frac{a}{b}$ is the *slope* of the line segment ℓ_1;
- $\frac{c}{d}$ is the *slope* of the line segment ℓ_2;
- $\frac{a+c}{b+d}$ is the *slope* of the line segment ℓ_3;
- $\frac{a-c}{b-d}$ is the *slope* of the line segment ℓ_4.

$\frac{a}{b} \vee \frac{c}{d}$ is between $\frac{a}{b}$ and $\frac{c}{d}$
$\leftrightsquigarrow$ ℓ_3 is between ℓ_1 and ℓ_2.

Geometric interpretation of mediant



$\left|\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right| = 1$

- Recall that the area of the rectangle forming by vectors u and v is $\|\mathrm{u}\times\mathrm{v}\|$.
- $\frac{a}{b}\heartsuit\frac{c}{d} \leftrightsquigarrow \|\ell_1\times\ell_2\| = 1$;
- Then we find that the area $\|\ell_1\times\ell_3\|$ has to be also 1; $\frac{a}{b}\heartsuit$ med
- Likewise, the area $\|\ell_2\times\ell_3\|$ has to be also 1. $\frac{c}{d}\heartsuit$ med

Hence, $\frac{a}{b}\vee\frac{c}{d}$ kisses both $\frac{a}{b}$ and $\frac{c}{d}$.

We thus proved the following lemma.

Lemma 10.2

If $\frac{a}{b} \heartsuit \frac{c}{d}$, then their mediant $\frac{a}{b} \vee \frac{c}{d}$ kisses both of them.

N.B. By Bézout's identity, we see that the mediant of two kissing reduced fractions must be reduced.

Hence, $\vee$ is rather an operation of (kissing) rational numbers.

$$\underbrace{\left(\frac{a}{b} \vee \frac{c}{d}\right)}_{= \frac{x}{y}} \heartsuit \frac{a}{b}$$

$$ay - bx = \pm 1 \underset{\text{By Bézout}}{\Rightarrow} x, y \text{ coprime}$$

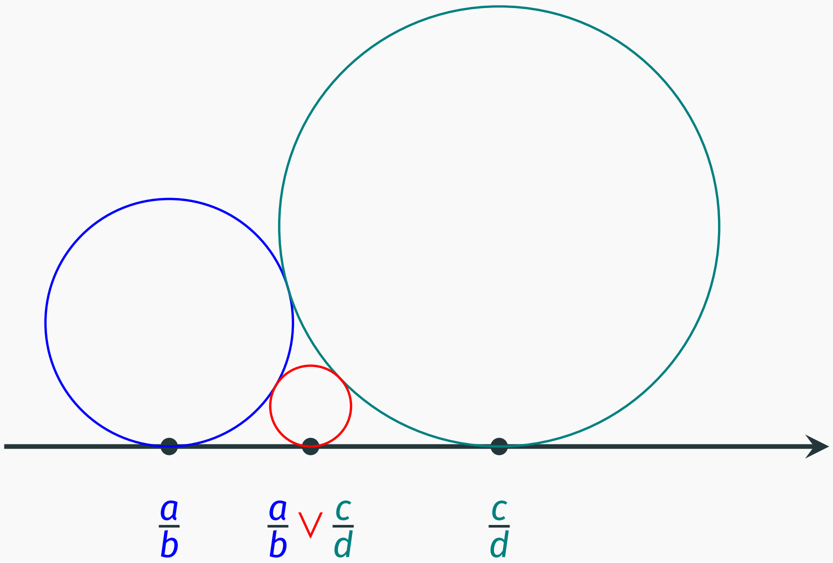
reduced fraction

$\updownarrow$

rational numbers

Mediant

In geometric words, if two Ford circles are tangent to each other, then the Ford circle atop their mediant is tangent to both of them.



Farey sequence

Theorem 10.3

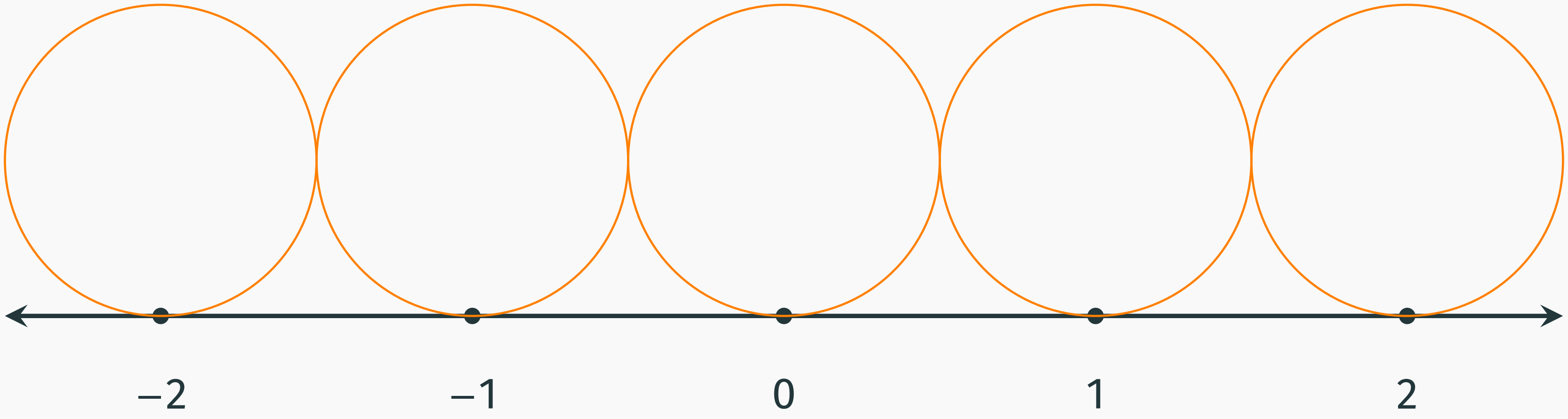
The following process generates all reduced fractions (in geometric words, all Ford circles):

1. *Start with integers, namely fractions of the form* $\frac{n}{1}$ *(in geometric words, Ford circles atop integer points).*
2. *Whenever you have two kissing fractions* $\frac{a}{b}$ *and* $\frac{c}{d}$*, generate their* **mediant** $\frac{a}{b} \vee \frac{c}{d}$ *(in geometric words, whenever you have two Ford circles tangent to each other, generate the third one by lemma 10.2).*

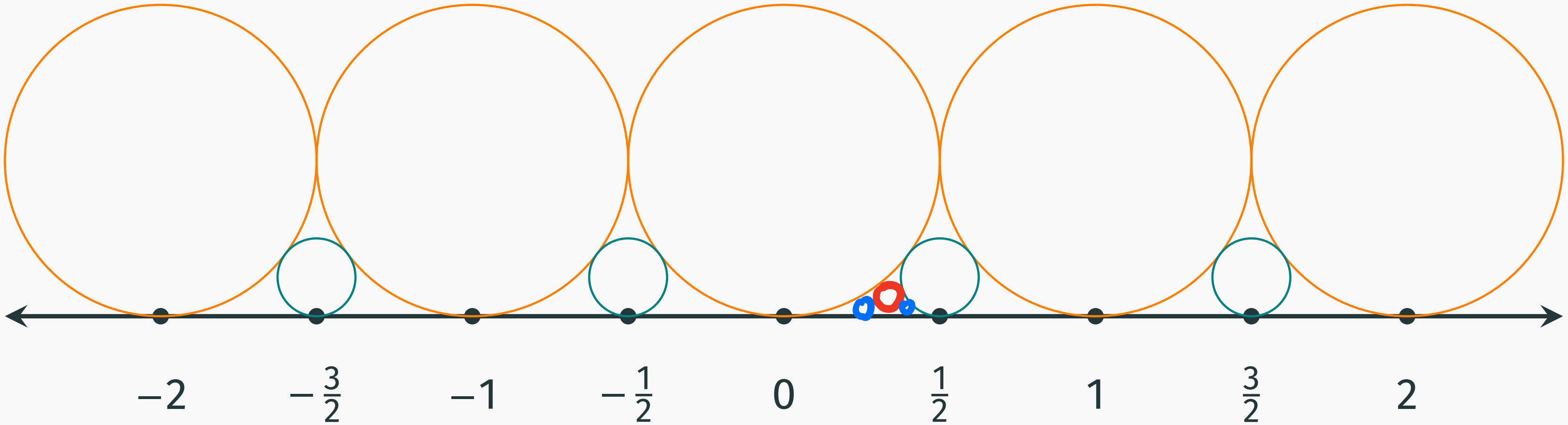
The produced sequence is called the ***Farey sequence***.

Farey sequence

The base case:



Next step:



Proof. We need to show: any reduced fraction can be found in the process. We do this by induction on the denominator b.

First, the base case is clear, any integer appears in the base step.

Now, assume any reduced fraction with denominator less than B can be found in the process. For any reduced fraction of the form $\frac{A}{B}$, we'll show that it can be obtained as the mediant of two kissing fractions, hence appear in the process.

with less denominators

Lemma 10.4

Let $\frac{A}{B}$ be a reduced fraction. Then fractions kissing it are

$$x \qquad y$$

$$\left\{ \frac{x_+ + A \cdot n}{y_+ + B \cdot n}, \frac{x_- + A \cdot n}{y_- + B \cdot n}, \;\middle|\; n \in \mathbb{Z} \right\},$$

where (x_+, y_+) and (x_-, y_-) are specific solutions of the linear Diophantine equations

$$(-B) \cdot x + (A) \cdot y = 1 \qquad \text{and} \qquad (-B) \cdot x + (A) \cdot y = -1$$

respectively.

Proof. A fraction $\frac{x}{y}$ kisses $\frac{A}{B}$ whenever $(-B) \cdot x + (A) \cdot y$ equals 1 or -1. Then the lemma follows from theorem 3.5 (General solutions of linear Diophantine equations). □

Proof of the theorem

$-B\,x_+ + A\,y_+ = 1$

Let's back to the proof of the theorem.

As $B > 1$, we cannot have $B \mid y_+$ (by 2-out-of-3 principle).

Hence the point y_+ must be inside one of above intervals. In other words, there is a (unique) integer n_+ such that $t \cdot B < y_+ < (t+1) \cdot B$

$$0 < y_+ + B \cdot n_+ < B.$$

Similarly, there is a (unique) integer n_- such that

$$0 < y_- + B \cdot n_- < B.$$

Proof of the theorem

Let's set $a = x_+ + A \cdot n_+$, $b = y_+ + B \cdot n_+$, $c = x_- + A \cdot n_-$, and $d = y_- + B \cdot n_-$. Then $\frac{a}{b}$ and $\frac{c}{d}$ are reduced fractions with denominators less than B.

Note that, by their definitions, we have

$$(-B) \cdot a + (A) \cdot b = 1 \qquad \text{and} \qquad (-B) \cdot c + (A) \cdot d = -1.$$

Add them together, we get $(-B) \cdot (a + c) + (A) \cdot (b + d) = 0$. Hence,

$$\frac{A}{B} = \frac{a}{b} \vee \frac{c}{d}.$$

$$\frac{A}{B} = \frac{a+c}{b+d}$$

Lastly, note that

$$ad - bc = a(B - b) - b(A - a) = aB - bA = -1.$$

Hence, we have $\frac{a}{b} \heartsuit \frac{c}{d}$. This finishes the proof.

After Class Work

- Be aware of the translations between the algebraic language of ***reduced fractions*** and the geometric language of ***Ford circles***.
- Be aware of the translations between the ***mediant*** of fractions and the ***sum*** of vectors. Notice how to interpret the relation $\heartsuit$ in terms of the area of rectangle.
- Be aware how we reduce the problem of finding fractions kissing the given $\frac{A}{B}$ to linear Diophantine equations.
- Be aware of the trick of placing the questioned number on the number line divided by multiples of a fixed B.
- We'll finish the proof of Dirichlet's approximation theorem as well as the rest of Chapter 3 in next lecture.
- Please read Chapter 4 by yourself. We will not cover it in the lectures.