Homework 5 (due Feb. 26)

MATH 110 | Introduction to Number Theory | Winter 2023

Problem 1. Recall the definitions of the *Möbius function* μ (in HW 3) and the *Dirichlet convolution* [Lecture Note, Definition 16.5]. Then finish the followings.

(a) (3 pts) Let δ_1 be the function defined as follows:

$$\delta_1(n) := \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if otherwise.} \end{cases}$$

Prove that $\delta_1 \star f = f \star \delta_1 = f$ for any arithmetic function f. (In other words, δ_1 is the *identity* for the binary operation $\star$.)

(b) (3 pts) Let $\mathbf{1}$ be the constant function mapping any positive integer to 1. **Prove that** $\mu \star \mathbf{1} = \delta_1$. (In other words, μ is a *unit* for the binary operation $\star$.)

(c) (3 pts) **Prove that** $f \star g = g \star f$ for any arithmetic functions f and g. (In other words, the binary operation $\star$ is *commutative*.)

Hint. Using [Lecture Note, Exercise 7.1]. More precisely, show that $d \mapsto \frac{n}{d}$ is a bijection from $\mathscr{D}(n)$ to itself and then use this bijection to show $f \star g = g \star f$.

(d) (6 pts) **Prove that** $(f \star g) \star h = f \star (g \star h)$ for any arithmetic functions f, g, and h. (In other words, the binary operation $\star$ is *associative*.)

Hint. Using [Lecture Note, Exercise 7.2]. More precisely, define $f \star g \star h$ as follows:

$$(f \star g \star h)(n) := \sum_{abc=n} f(a)g(b)h(c),$$

where the summation is taken over the set $\mathscr{D}_3(n) := \{(a,b,c) \in \mathscr{D}(n)^3 \mid abc = n\}$. Show that each of $(f \star g) \star h$ and $f \star (g \star h)$ is equal to $f \star g \star h$ using a bijective map from its summation index set to $\mathscr{D}_3(n)$.

(At this stage, we see that the set of arithmetic functions equipped with the binary operation $\star$ and the element δ_1 forms a *commutative monoid*.)

(e) (5 pts) Suppose f and g are two multiplicative functions. **Prove that** $f \star g$ is a multiplicative function.

Hint. For any copirme pairs (m, n), use the bijection $\Phi\colon \mathscr{D}(m) \times \mathscr{D}(n) \to \mathscr{D}(mn)$ in [Lecture Note, Theorem 7.1] to do the substitutions.

(Hence, the subset of *multiplicative* functions forms a *submonoid*.)

(f) (5 pts) Say an arithmetic function f is invertible under the operation $\star$ if there is another arithmetic function g such that $f \star g = \delta_1$. **Prove that** f is invertible under the operation $\star$ if and only if $f(1) \neq 0$.

Hint. Spell out the equality $(f \star g)(n) = \delta_1(n)$ and solve out $f(n)$.

Problem 2 (10 pts)**.** Suppose p is an odd prime and q is a prime divisor of $2^p - 1$. **Prove that** $q \equiv 1 \pmod{2p}$.

Hint. First show that p is the length of a multiplicative dynamic. Then use [Lecture Note, Theorem 13.11].

Problem 3 (10 pts)**.** A *Sophie Germain prime* is a prime number p such that $2p + 1$ is also a prime. For example, $p = 2, 3, 5$ are Sophie Germain primes, but $p = 7$ is not (since $15 = 2 \cdot 7 + 1$ is not a prime).
Prove that if p is a Sophie Germain prime, then $2p + 1$ is a divisor either of $2^p - 1$ or of $2^p + 1$, but not of both.

Hint. Use Fermat's little theorem and the primality of q.

References

[Lecture Note] *Lecture notes of Math 110*, Xu Gao.