Homework 7 (due Mar. 12)

MATH 110 | Introduction to Number Theory | Winter 2023

Problem 1 (20 pts)**. Find all** natural representatives x modulo 63 such that

$$x^2 \equiv 22 \pmod{63}.$$

Hint. Use *Chinese Remainder theorem* [Lecture Note, Lecture 19].

Problem 2 (20 pts)**. Find all** roots of the polynomial $x^3 + x + 1$ modulo 27. Write your answer in natural representatives modulo 27.

Hint. Use *Hensel's lifting* [Lecture Note, Lecture 20].

Problem 3. In what follows, we fix a prime number p. For n an integer, recall that $v_p(n)$ is the exponent of p appearing in the prime factorization of n. Namely, $p^{v_p(n)} \mid n$, while $p^{v_p(n)+1} \nmid n$. Extend this definition to nonzero fractions as follows:

$$v_p(\frac{n}{m}) := v_p(n) - v_p(m).$$

(a) (2 pts) **Show that**, if the two fractions $\frac{n}{m}$ and $\frac{n'}{m'}$ represent the same rational number, then $v_p(\frac{n}{m}) = v_p(\frac{n'}{m'})$.

Hence, we obtain a function $v_p \colon \mathbb{Q}^\times \to \mathbb{Z}$. (Recall that $\mathbb{Q}^\times$ consists of nonzero rational numbers). The ***p*-adic norm** of a rational number x is defined to be

$$|x|_p := \begin{cases} p^{-v_p(x)} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

For example,

$$\left|\frac{24}{25}\right|_2 = \frac{1}{8}, \qquad \left|\frac{24}{25}\right|_3 = \frac{1}{3}, \qquad \left|\frac{24}{25}\right|_5 = 25.$$

(b) (3 pts) **Prove that** $|-x|_p = |x|_p$, and $|xy|_p = |x|_p |y|_p$.

(c) (5 pts) **Prove** the *ultrametric triangle inequality*

$$|x + y|_p \leq \max\Big\{|x|_p, |y|_p\Big\}.$$

Remark. Note that $\max\Big\{|x|_p, |y|_p\Big\} \leqslant |x|_p + |y|_p$. Hence, the ultrametric triangle inequality implies the usual triangle inequality. The previous two says that $|\cdot|_p$ can be viewed as analogy of the usual Euclidean norm of vectors, or the absolute value of real numbers.

References

[Lecture Note] *Lecture notes of Math 110*, Xu Gao.