Quiz for this time:

Find a polynomial $P(T)$ with integer coefficients s.t.

$$P\left(\sqrt{2+\sqrt{3}}\right) = 0.$$

Let α be $\sqrt{2+\sqrt{3}}$. Then we have

$$\alpha = \sqrt{2+\sqrt{3}}$$

$$\alpha^2 = 2+\sqrt{3}$$

$$\alpha^2 - 2 = \sqrt{3}$$

$$(\alpha^2-2)^2 = 3$$

$$\alpha^4 - 4\cdot\alpha^2 + 1 = 0$$

$$P(T) = T^4 - 4\cdot T^2 + 1.$$

Diophantine Approximation.

Measure how close is a real number to rationals.

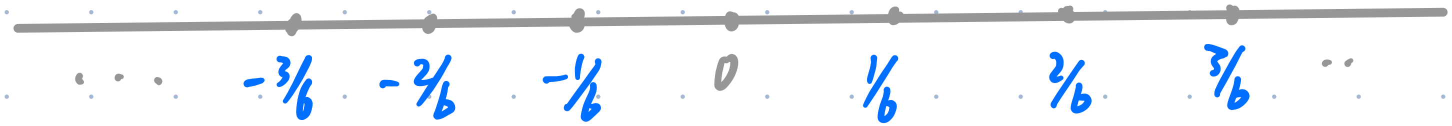
Prop. If α is a real number and b is a positive integer, then there is an integer a s.t.

$\pi = 3.1415\cdots$

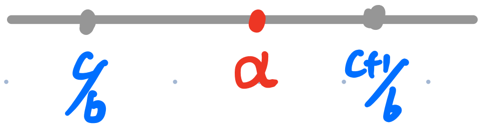
$\frac{3}{1}$ v.s. $\frac{31}{10}$

$$\left|\alpha - \frac{a}{b}\right| \leq \frac{1}{2b}$$

Pf: Plot $\frac{1}{b}\mathbb{Z}$:



Then, say



choose the one (of $\frac{c}{b}$ and $\frac{c+1}{b}$) closer to α to be $\frac{a}{b}$.

Then we have $\left|\alpha - \frac{a}{b}\right| \leq \frac{1}{2}\cdot$ length of $\left[\frac{c}{b}, \frac{c+1}{b}\right] = \frac{1}{2b}$ $\square$

Sometimes, we have far better approximation:

e.g. $\pi = 3.1415926\cdots$

- $\frac{a}{b} = 3.14 = \frac{157}{50}$; $\left|\pi - \frac{a}{b}\right| \approx 0.00159$ compare $\frac{1}{b^2} = 0.01$ (~16%)

- $\frac{a}{b} = \frac{22}{7}$; $\left|\pi - \frac{a}{b}\right| \approx 0.0013$. Compare $\frac{1}{b^2} \approx 0.02$ (~2%)

Relation to Transcendental theory:

If we can approximate $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ by rational numbers too well then α is likely to be transcendental.

e.g. $e = \underbrace{1 + \frac{1}{1!} + \frac{1}{2!} + \cdots +}_{\text{finite partial sum gives rational}} \frac{1}{n!} + \cdots$

$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ decrease too fast!

(Liouville, 1840s)

If α is irrational & algebraic of degree n (i.e. $\exists$ irr poly $P(T)$ of deg n s.t. $P(\alpha)=0$)

then there is a real number $C>0$, s.t.

$$\left|\alpha - \frac{a}{b}\right| > \frac{C}{b^n} \quad \text{for ALL } a, b \in \mathbb{Z},\ b>0$$

(Thue-Siegel-Roth 1900s ~ 1950s)

If α is irrational & algebraic and $\varepsilon>0$ a small positive real number.

then there is a real number $C>0$, s.t.

$$\left|\alpha - \frac{a}{b}\right| > \frac{C}{b^{2+\varepsilon}} \quad \text{for ALL } a, b \in \mathbb{Z},\ b>0$$

Dirichlet's Approximation Theorem (1840)

If α is irrational, then there are INFINITELY many reduced fractions $\frac{a}{b}$ s.t.

$$\left|\alpha - \frac{a}{b}\right| \leqslant \frac{1}{2b^2}$$

Remark (WARNING)

The theorem $\not\Rightarrow$ for every $b>0$, there is such $\frac{a}{b}$.

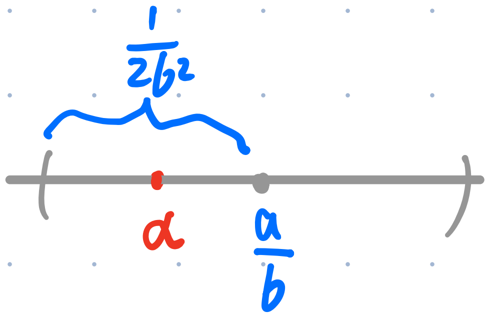
e.g.: $\pi = 3.1415926\cdots$

- $b=1$ works: $\left|\pi - \frac{3}{1}\right| \approx 0.14 < \frac{1}{2}$
- $b=2$ NOT WORK: the closest one is $\frac{7}{2}$, but even for it,

$$\left|\pi - \frac{7}{2}\right| \approx 0.35 > \frac{1}{2\cdot 2^2} = 0.125.$$

How to prove the theorem?

Interpretate $\left|\alpha - \frac{a}{b}\right| \leqslant \frac{1}{2b^2}$ in terms of geometry.



So WANT TO SHOW there are ∞ many $\frac{a}{b}$ s.t. α is within distance $\leqslant \frac{1}{2b^2}$ from it.

Compare to "$\left|\alpha - \frac{a}{b}\right| \leqslant \frac{1}{2b}$", where we used plot of $\frac{1}{b}\mathbb{Z}$, the fractions $\frac{*}{b}$ are separated from each other with distance at $\frac{1}{b}$.

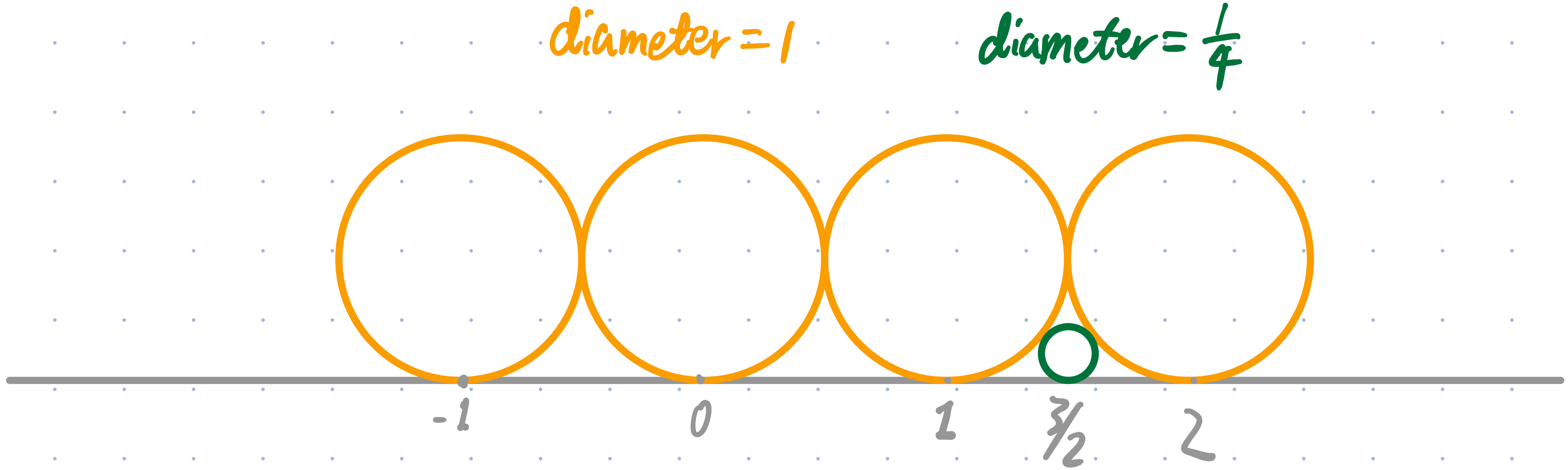
Idea: Cover axis by intervals with center $\frac{a}{b}$ and diameter $\frac{1}{b^2}$.

(shadow of circles with center $\frac{a}{b}$ and diameter $\frac{1}{b^2}$.)

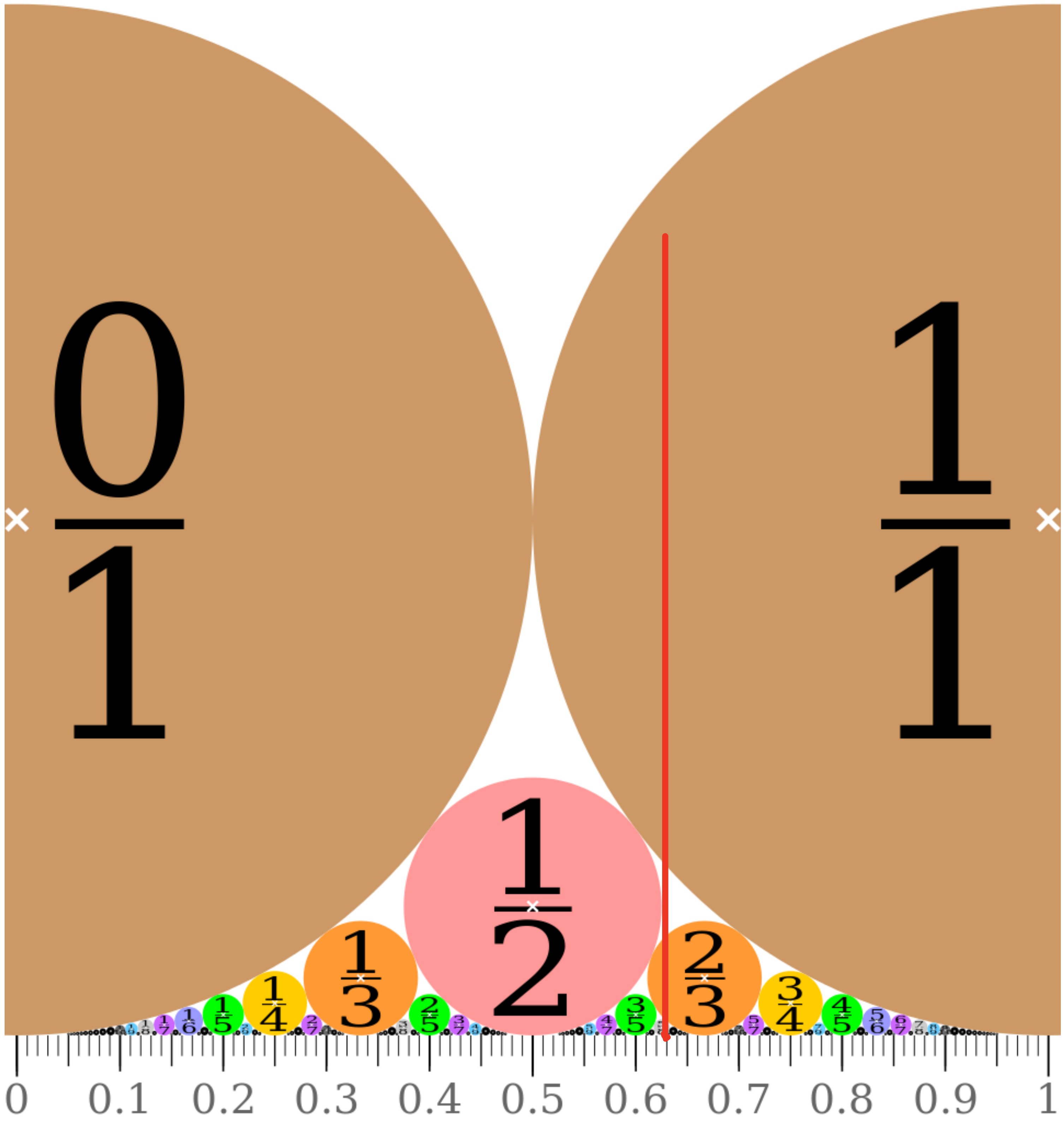
Defn. (Ford circle) (Lester Ford, 1938)

Atop each reduced fraction $\frac{a}{b}$ a circle of diameter $\frac{1}{b^2}$.

(Integers a are treated as $\frac{a}{1}$)



So, to prove Dirichlet's approximation theorem, it suffices to show that α is under the shadow of INFINITELY many Ford circles.



Ford Circles

between $\frac{0}{1}$ & $\frac{1}{1}$.

line above α

cross Ford circle

atop $\frac{a}{b}$

$\iff$

$\left|\alpha - \frac{a}{b}\right| \leq \frac{1}{2b^2}$

radius of

Ford circle

When are two Ford circles tangent to each other?

Defn. Two fractions $\frac{a}{b}$ & $\frac{c}{d}$ kiss each other if

$$\det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \pm 1.$$

$$(ad - bc = \pm 1)$$

Notation: $\frac{a}{b} \heartsuit \frac{c}{d}$

Rmk: $\frac{a}{b} \heartsuit \frac{c}{d} \Rightarrow ax + by = \pm 1$ has integer solution $(d, -c)$

$$\Rightarrow GCD(a, b) \mid \pm 1$$

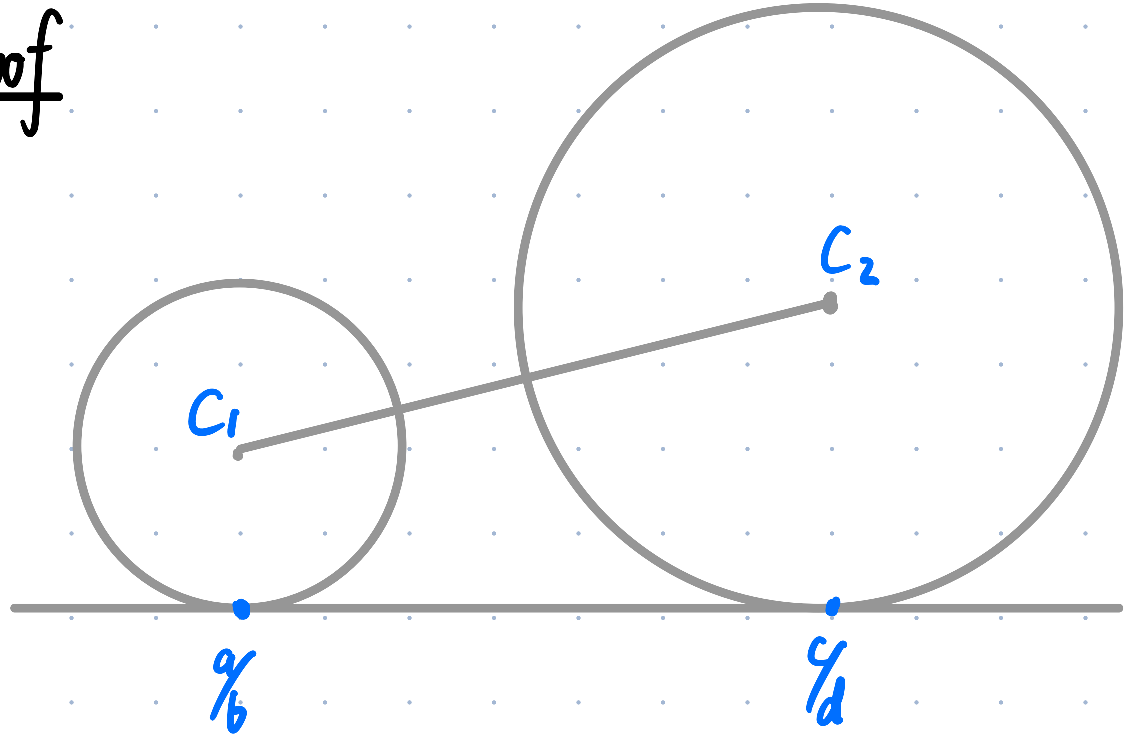
$$\Rightarrow GCD(a, b) = 1.$$

So $\heartsuit$ is rather a relation of rational numbers:

Two rational number kiss each other if so do their reduced fractions.

Theorem. $\frac{a}{b} \heartsuit \frac{c}{d}$ if and only if the Ford circles atop $\frac{a}{b}$ & $\frac{c}{d}$ are tangent to each other.

Proof



Let C_1 & C_2 be the centers of the Ford circles atop $\frac{a}{b}$ & $\frac{c}{d}$. Then:

$$C_1 : \left(\frac{a}{b}, \frac{1}{2b^2}\right)$$

$$C_2 : \left(\underset{x}{\frac{c}{d}}, \underset{y}{\frac{1}{2d^2}}\right)$$

By distance formula,

$$\overline{C_1 \cdot C_2} = \sqrt{\left(\frac{a}{b} - \frac{c}{d}\right)^2 + \left(\frac{1}{2b^2} - \frac{1}{2d^2}\right)^2}$$

The Ford circles are tangent to each other if and only if

$$\overline{C_1 \cdot C_2} = \text{sum of radius} = \frac{1}{2b^2} + \frac{1}{2d^2}$$

Combinate them, we say the Ford circles are tangent to each other

$$\Longleftrightarrow \left(\frac{a}{b} - \frac{c}{d}\right)^2 + \left(\frac{1}{2b^2} - \frac{1}{2d^2}\right)^2 = \left(\frac{1}{2b^2} + \frac{1}{2d^2}\right)^2$$

$$\Longleftrightarrow \left(\frac{a}{b} - \frac{c}{d}\right)^2 = \left(\frac{1}{2b^2} + \frac{1}{2d^2}\right)^2 - \left(\frac{1}{2b^2} - \frac{1}{2d^2}\right)^2$$

$$= \frac{1}{b^2} \cdot \frac{1}{d^2}$$

$X^2 - Y^2 = (X+Y)\cdot(X-Y)$

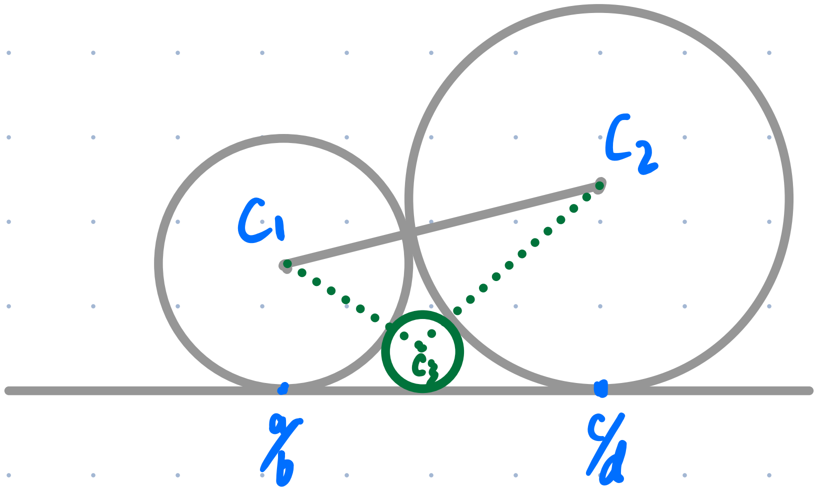
$$\Longleftrightarrow (ad - bc)^2 = 1.$$

$$\Longleftrightarrow \frac{a}{b} \heartsuit \frac{c}{d}.$$

Exercise

Ford Circles have No overlaps. Why?

Suppose $\frac{a}{b}$ ♡ $\frac{c}{d}$, namely we have two Ford circles tangent



$C_1 : \left(\frac{a}{b}, \frac{1}{2b^2} \right)$

$C_2 : \left(\frac{c}{d}, \frac{1}{2d^2} \right)$

Looks like there is a Ford circle between them and tangent to them.

Indeed, suppose $C_3 : (x, y)$. Then we have :

C_3 is between C_1 & C_2 : $\frac{a}{b} < x < \frac{c}{d}$ ①

C_3 is tangent to C_1 : $\overline{C_3 C_1} = r_3 + r_1 = y + \frac{1}{2b^2}$ ②

C_3 is tangent to C_2 : $\overline{C_3 C_2} = r_3 + r_2 = y + \frac{1}{2d^2}$ ③

② : $$\left(x-\frac{a}{b}\right)^2+\left(y-\frac{1}{2b^2}\right)^2=\left(y+\frac{1}{2b^2}\right)^2$$

$$\Leftrightarrow \quad \left(x-\frac{a}{b}\right)^2 = 2\cdot y\cdot\frac{1}{b^2} \qquad (2')$$

③ : $$\left(x-\frac{c}{d}\right)^2+\left(y-\frac{1}{2d^2}\right)^2=\left(y+\frac{1}{2d^2}\right)^2$$

$$\Leftrightarrow \quad \left(x-\frac{c}{d}\right)^2 = 2\cdot y\cdot\frac{1}{d^2} \qquad (3')$$

$b^2(2')-d^2(3')$: $$(bx-a)^2-(dx-c)^2=0$$

$$\Leftrightarrow \quad (bx-a)^2=(dx-c)^2$$

$$\Leftrightarrow \quad x=\frac{a-c}{b-d} \text{ or } \frac{a+c}{b+d}$$

But $\frac{a-c}{b-d}$ is Not between $\frac{a}{b}$ & $\frac{c}{d}$. $\Rightarrow\Leftarrow$ ①

(one can verify it directly but let's wait)

What about $x = \frac{a+c}{b+d}$? One can verify it is between $\frac{a}{b}$ & $\frac{c}{d}$

But let's wait.

Plug in (2') or (3'); $y = \frac{1}{2(b+d)^2}$ looks like a Ford circle!

But is $\frac{a+c}{b+d}$ a reduced fraction? Yes, bec it kisses $\frac{a}{b}$ & $\frac{c}{d}$

$b>0,\ d>0 \Rightarrow b+d>0$, GCD($a+c$, $b+d$) = 0 by prop of kiss.

Summary: If two Ford circles are tangent to each other,

(If two reduced fractions kiss each other)

then there is a third one between them and tangent to them.

(then there is a third one between them and kiss them)