Part III

Rational and Algebraic Numbers

Rational numbers

Definition 3.1.1

A *fraction* is an expression of the form $\frac{a}{b}$, where a, b are integers and $b \neq 0$. A *rational number* is a number which can be expressed as a fraction.

Example 3.1.2

$\frac{5}{3}$ and $\frac{15}{9}$ are two distinct fractions, but they express the same rational number. "$\frac{5}{3} = \frac{15}{9}$".

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A *fraction* $\frac{a}{b}$ is *reduced* if a, b are coprime and $b > 0$.

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$\frac{-5}{3}$ is reduced, $\frac{5}{-3}$ is not reduced, and $\frac{-15}{9}$ is not reduced.

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Proof. Let's assume our rational number is expressed as $\frac{a}{b}$. Since $\frac{a}{b} = \frac{-a}{-b}$, we may assume $b > 0$. Let $c = \frac{a}{\gcd(a,b)}$ and $d = \frac{b}{\gcd(a,b)}$. Then $\gcd(c, d) = 1$ and we have $\frac{a}{b} = \frac{c}{d}$.

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$\frac{c}{d} = \frac{c'}{d'}$

Now, suppose $\frac{c'}{d'}$ is another reduced fraction such that $\frac{a}{b} = \frac{c'}{d'}$. Then we have $c'd = cd'$. Hence, $d \mid cd'$ and $d' \mid c'd$. Since $\gcd(c,d) = 1$ and $\gcd(c',d') = 1$, we have $d \mid d'$ and $d' \mid d$. Since both d, d' are positive, by the antisymmetry of divisibility, $d = d'$. Then $c = c'$ and thus $\frac{c}{d}$ and $\frac{c'}{d'}$ are the same fraction. $\square$

We can extend prime factorization from to rational numbers.

Theorem 3.1.5 (Prime factorization)

Let α be a positive rational number.

1. *(existence) α admits a prime factorization, i.e. there exist integers e_p for each prime p such that*

$$\alpha = \prod_{p \text{ is prime}} p^{e_p}$$

2. *(uniqueness) Suppose α admits another prime factorization, say*

$$\alpha = \prod_{p \text{ is prime}} p^{f_p}.$$

Then, for every prime p, we have $e_p = f_p$.

Proof. (*existence*) Let $\frac{a}{b}$ be any fraction expressing α. We may assume a, b are positive. Then by the fundamental theorem of arithmetic,

$$a = \prod_{p \text{ is prime}} p^{v_p(a)}, \qquad b = \prod_{p \text{ is prime}} p^{v_p(b)}.$$

Proof. (*existence*) Let $\frac{a}{b}$ be any fraction expressing α. We may assume a, b are positive. Then by the fundamental theorem of arithmetic,

$$a = \prod_{p \text{ is prime}} p^{v_p(a)}, \qquad b = \prod_{p \text{ is prime}} p^{v_p(b)}.$$

Hence, $\alpha = \frac{a}{b} = \frac{\prod_{p \text{ is prime}} p^{v_p(a)}}{\prod_{p \text{ is prime}} p^{v_p(b)}} = \prod_{p \text{ is prime}} p^{v_p(a)-v_p(b)}.$

$$\frac{p^{v_p(a)}}{p^{v_p(b)}} = p^{v_p(a)-v_p(b)}$$

Note that the integer $v_p(a) - v_p(b)$ does not depend on the choice of the fraction $\frac{a}{b}$. We will denote this integer by $v_p(\alpha)$.

$$\frac{a}{b} = \frac{c}{d} \qquad v_p(a) + v_p(d) = v_p(c) + v_p(b)$$

$$v_p(a) - v_p(b) = v_p(c) - v_p(d)$$

(*uniqueness*) Suppose $\alpha = \prod_{p \text{ is prime}} p^{f_p}$. Let

$$c = \prod_{p \text{ is prime}, f_p > 0} p^{f_p}, \qquad d = \prod_{p \text{ is prime}, f_p < 0} p^{-f_p}.$$

Then $\frac{c}{d}$ is a reduced fraction expressing α. Note that we always have $v_p(c) - v_p(d) = f_p$. Hence, $f_p = v_p(\alpha)$. $\square$

Example

Example 3.1.6

Find the reduced fraction expression of the following rational number and give its prime factorization:

$$-1.56$$

$$\frac{-156}{100} = \frac{-39}{25}$$

$$39 = 3 \times 13$$

$$25 = 5^2$$

$$-1.56 = \frac{-156}{100} = \frac{-39}{25} = -3 \cdot 5^{-2} \cdot 13.$$

$$-1.56 = \frac{-39}{25} = -3 \times 5^{-2} \times 13$$