Dirichlet's Approximation Theorem (1840)

If α is irrational, then there are INFINITELY many reduced fractions $\frac{a}{b}$ s.t.

$$\left|\alpha - \frac{a}{b}\right| \leq \frac{1}{2b^2}$$

best exponent is 2

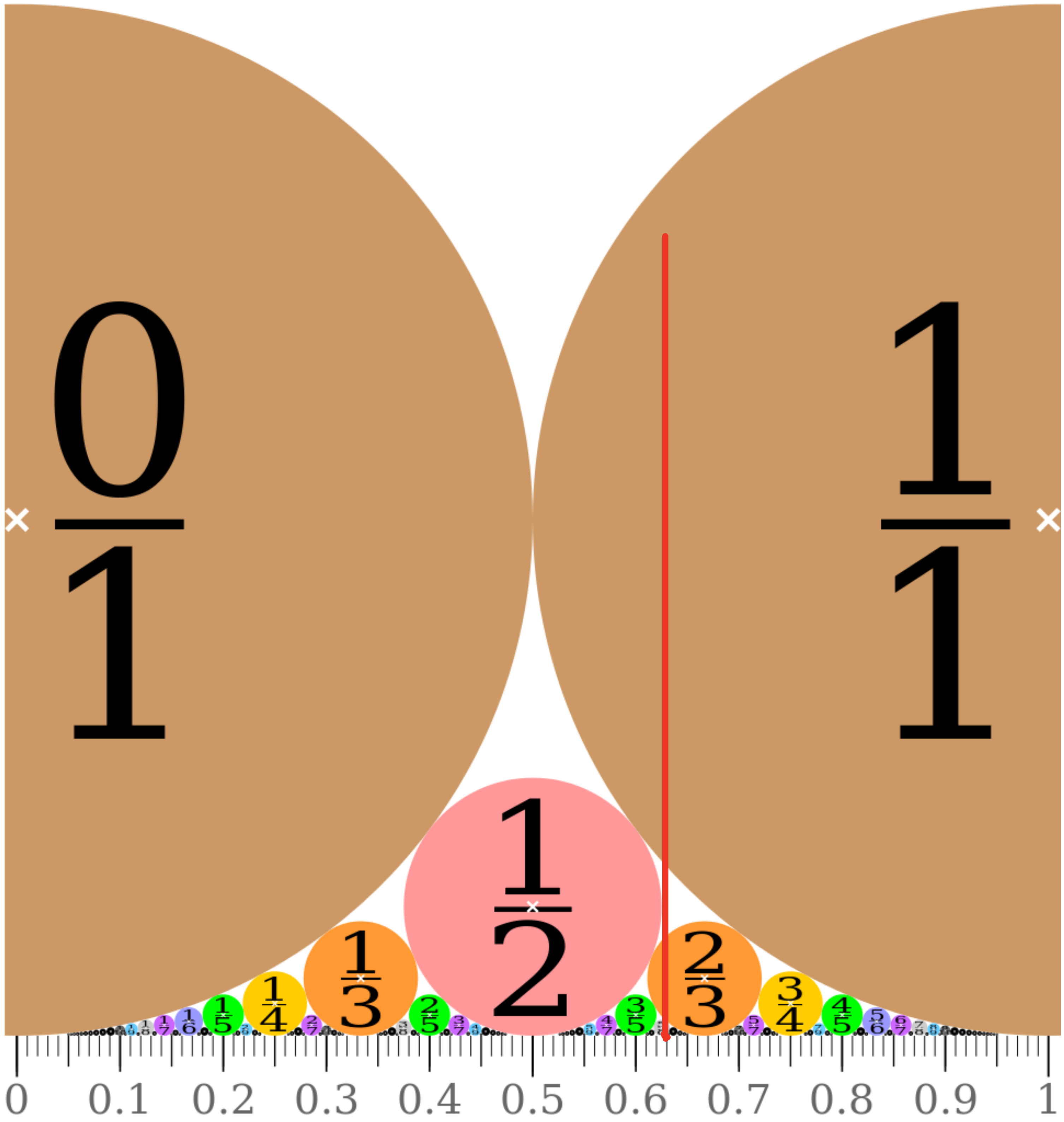
best coefficient is $\sqrt{5}$

Defn. (Ford circle) (Lester Ford, 1938)

Atop each reduced fraction $\frac{a}{b}$ a circle of diameter $\frac{1}{b^2}$.

(Integers a are treated as $\frac{a}{1}$)

Idea: To show that the vertical line atop α crosses INFINITELY many Ford circles.



Ford Circles

between $\frac{0}{1}$ & $\frac{1}{1}$.

line above α

cross Ford circle

atop $\frac{a}{b}$

$\iff$

$$\left|\alpha - \frac{a}{b}\right| \leq \frac{1}{2b^2}$$

$\downarrow$

radius of

Ford circle

- Two Ford circles atop $\frac{a}{b}$ & $\frac{c}{d}$ are tangent to each other

$\Leftrightarrow \frac{a}{b}$ ♡ $\frac{c}{d}$ (Namely, $|ad - bc| = 1$)

- Given such a pair of Ford circles, there is a third one between them and tangent to them. It is atop $\frac{a+c}{b+d}$

Need to show:

1. $\frac{a-c}{b-d}$ is Not between $\frac{a}{b}$ & $\frac{c}{d}$
2. $\frac{a+c}{b+d}$ is between $\frac{a}{b}$ & $\frac{c}{d}$
3. $\frac{a+c}{b+d}$ kisses both $\frac{a}{b}$ & $\frac{c}{d}$
4. $\frac{a+c}{b+d}$ is a reduced fraction.

$|a \cdot (b+d) - b \cdot (a+c)| = 1$

$|c \cdot (b+d) - d \cdot (a+c)| = 1$

WRONG ADDITION of fractions (mediant)

$$\frac{a}{b} + \frac{c}{d} \neq \frac{a+c}{b+d}$$

This is not how fractions been added, but how vectors been added.

$$(a, b) + (c, d) = (a+c, b+d)$$

<u>Defn</u>. Let $\frac{a}{b}$ and $\frac{c}{d}$ be two fractions, then their mediant is

$$\frac{a}{b} \vee \frac{c}{d} := \frac{a+c}{b+d}$$

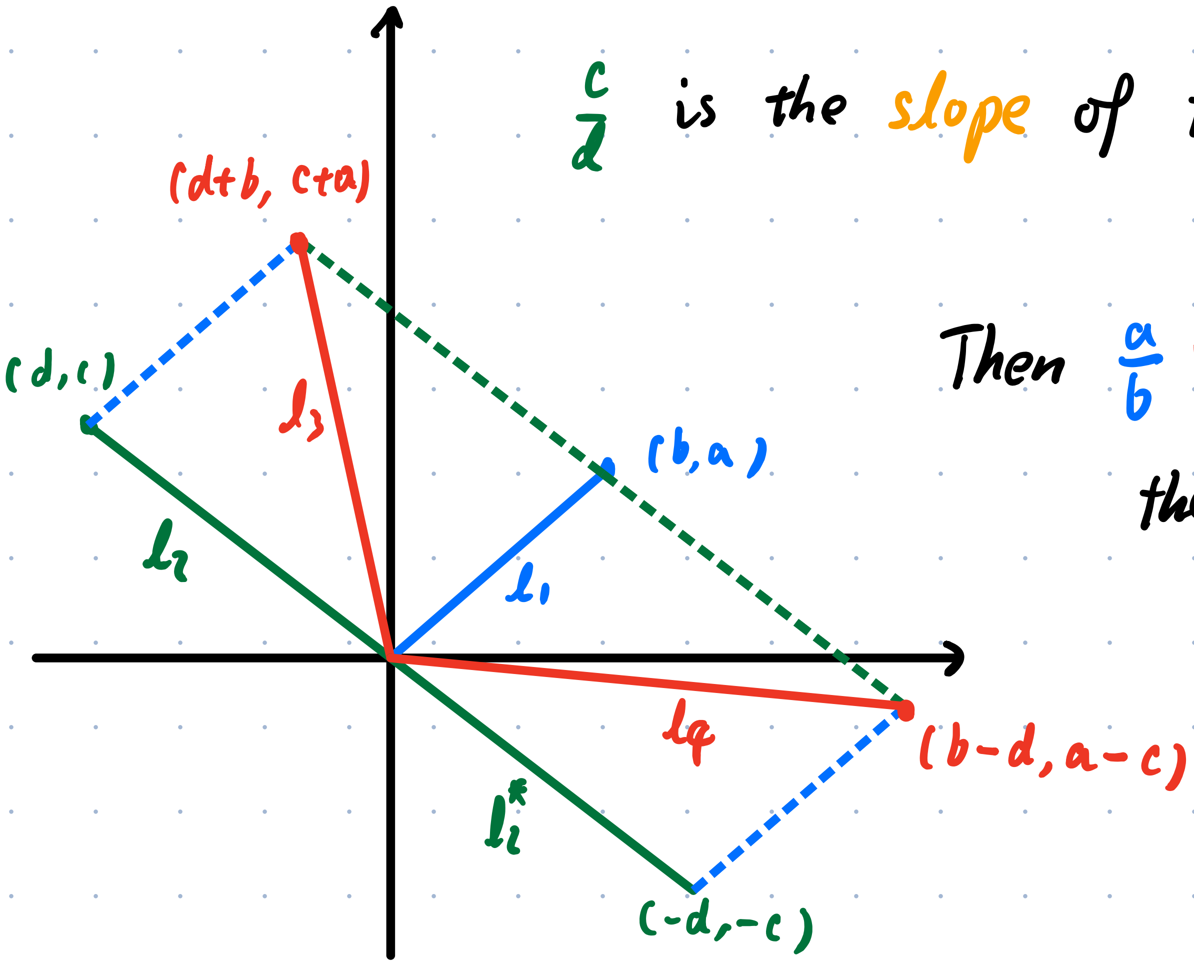
e.g.

$$\left.\begin{aligned} 1 \vee \tfrac{1}{2} &= \tfrac{1}{1} \vee \tfrac{1}{2} = \tfrac{2}{3} \\ 1 \vee \tfrac{2}{4} &= \tfrac{1}{1} \vee \tfrac{2}{4} = \tfrac{3}{5} \end{aligned}\right\} \text{different!}$$

Geometric interpretation :

$\frac{a}{b}$ is the slope of the line segment l_1 (from $(0,0)$ to (b,a))

$\frac{c}{d}$ is the slope of the line segment l_2 (from $(0,0)$ to (d,c))



Then $\frac{a}{b} \vee \frac{c}{d}$ is the slope of the line segment l_3 (from $(0,0)$ to $(d+b, c+a)$)

By the way, $\frac{a-c}{b-d}$ is the slope of the line segmnt l_4

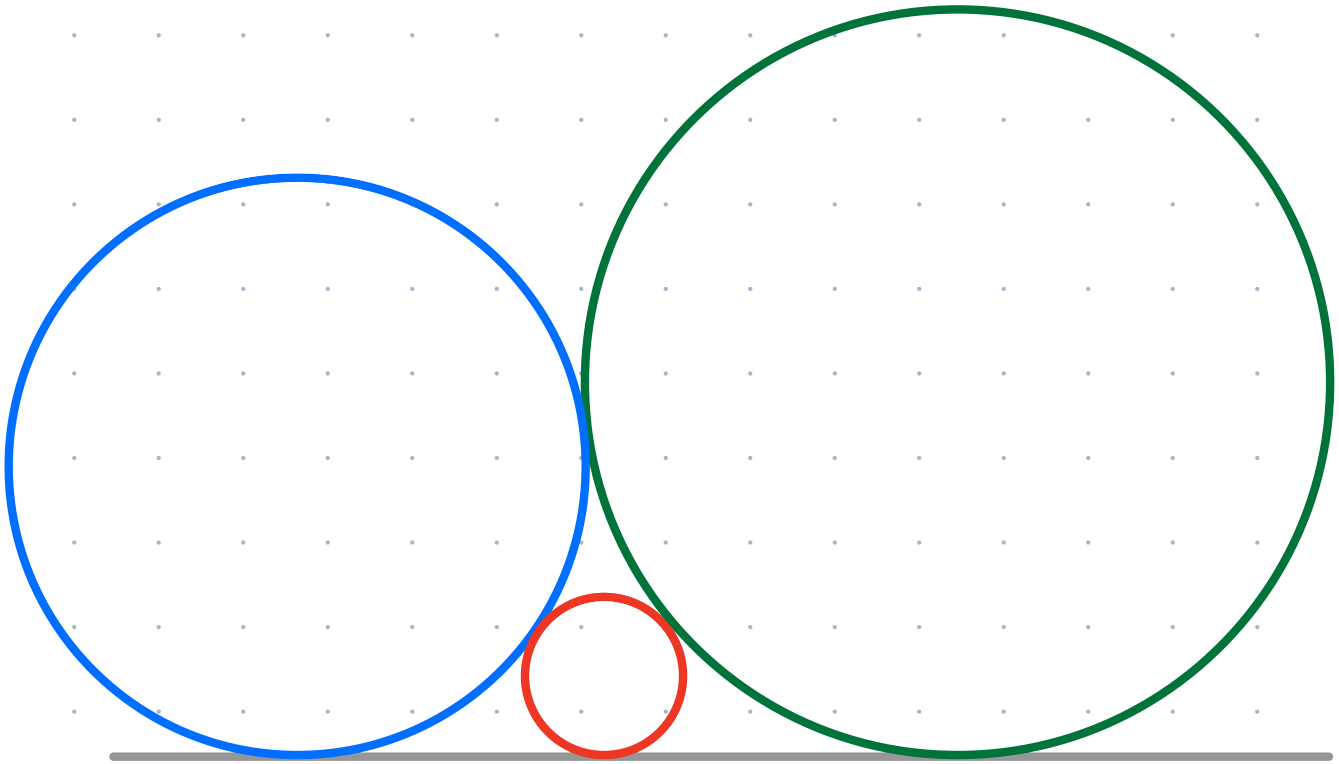
Conclusion : $\frac{a}{b} \vee \frac{c}{d}$ is between $\frac{a}{b}$ and $\frac{c}{d}$.

Coro: If two Ford circles are tangent to each other,

(If two reduced fractions kiss each other)

then there is a third one between them and tangent to them.

(then there is a third one between them and kiss them)

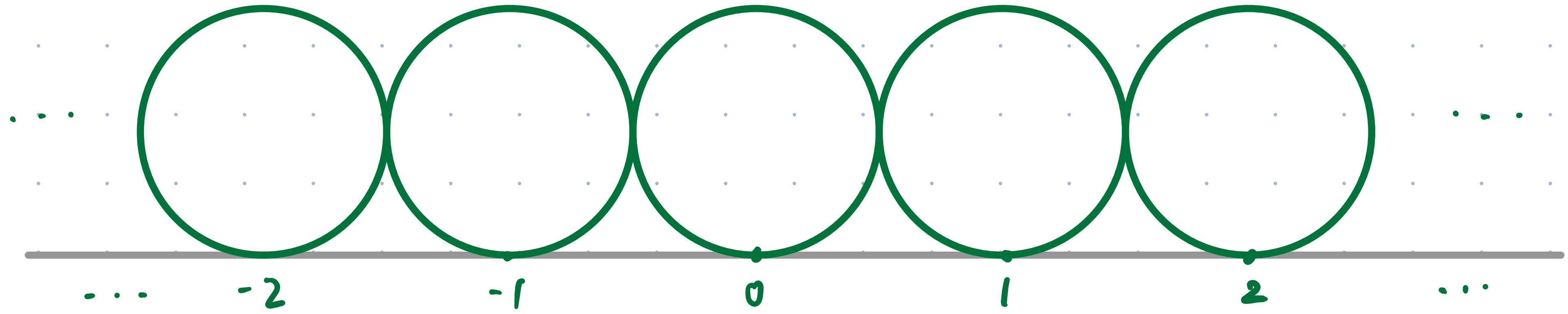


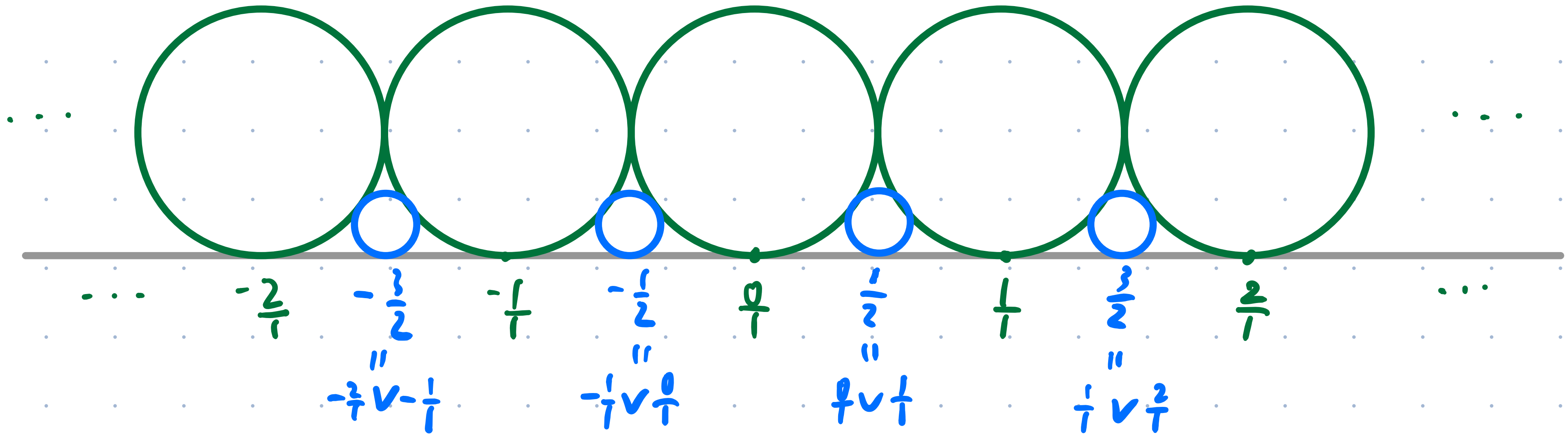
$\frac{a}{b}$ ♡ $\frac{c}{d}$ $\Rightarrow$ $\frac{a}{b} \vee \frac{c}{d}$ kiss both $\frac{a}{b}$ & $\frac{c}{d}$.

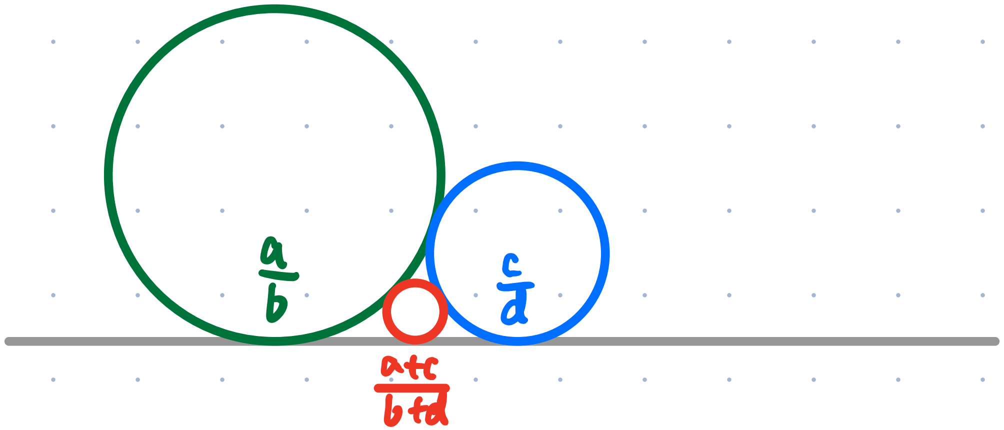
Theorem: (Faray sequence)

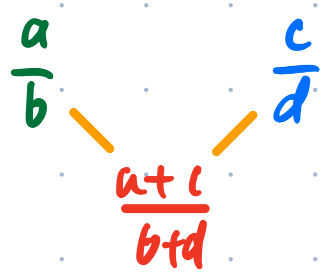
The following process generating ALL
(Ford Circles / reduced fractions)

	Ford Circles	reduced fractions
Base Step:	Ford circles atop integers	Fractions of the form $\frac{n}{1}$
Processing Step:	Whenever you have two tangent Ford circles, form the third one tanget to them from the Coro.	Whenever you have two kissing reduced fractions, form the third one as their mediant.









In other words, we need to show:

Every Ford circle can be found in the process	Every reduced fraction can be found in the process

Idea: prove by induction on denominator $b > 0$

$P(b)$: Every reduced fraction of the form $\frac{*}{b}$ can be found in the process.

- $P(1)$: This is the base step.

- $\bigwedge_{b<B} P(b) \Rightarrow P(B)$

“every reduced fraction of the form $\frac{*}{b}$ with $b<B$ can be found”

$\Rightarrow$ “every reduced fraction of the form $\frac{*}{B}$ can be found”

Idea: Write each $\frac{A}{B}$ as a mediant $\frac{a}{b} \vee \frac{c}{d}$, where

$\frac{a}{b} \heartsuit \frac{c}{d}$ and $b, d < B$

Prop. (Fractions kissing a given one)

Let $\frac{A}{B}$ be a reduced fraction. Then fractions kissing $\frac{A}{B}$ are

$$\left\{ \frac{x_+ + A\cdot n}{y_+ + B\cdot n}, \frac{x_- + A\cdot n}{y_- + B\cdot n} \,\middle|\, n\in\mathbb{Z} \right\}$$

Proof: Say a fraction $\frac{x}{y}$ kisses $\frac{A}{B}$ means $|Ay - Bx| = 1$.

Hence, by the general solution of linear Diophantine equations, we have

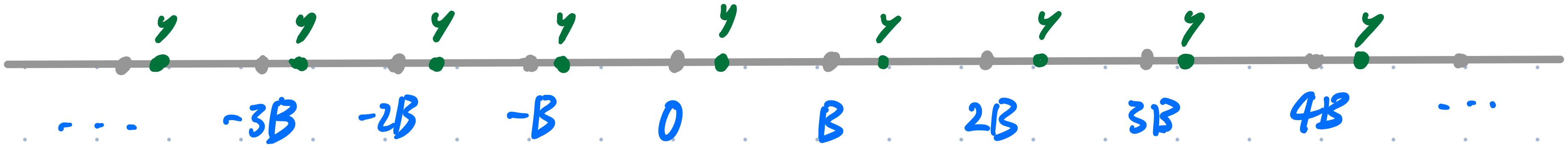
$$\begin{cases} x = x_+ + A\cdot n \\ y = y_+ + B\cdot n \end{cases} \text{ or } \begin{cases} x = x_- + A\cdot n \\ y = y_- + B\cdot n \end{cases}$$

where (x_+, y_+) is a solution to $Ay - Bx = 1$,

and (x_-, y_-) is a solution to $Ay - Bx = -1$.

Note that: If $B > 1$, then we cannot have $B \mid y_+$

Since otherwise $B \mid 1 = A y_+ - B x_+$. (2-out-of-3)



Hence there is exactly one integer n_+ s.t.

$$0 < y_+ + B \cdot n_+ < B.$$

Let us call it (one of them) b and define $a := x_+ + A \cdot n_+$.

Then we have: $\frac{a}{b} \heartsuit \frac{A}{B}$ & $0 < b < B$

Similarly, we can define n_- s.t. $0 < y_- + B \cdot n_- < B$.

Then $c := x_- + A \cdot n_-$ & $d := y_- + B \cdot n_-$.

Note that :

- $\frac{a}{b} \heartsuit \frac{A}{B}$ & $0 < b < B$

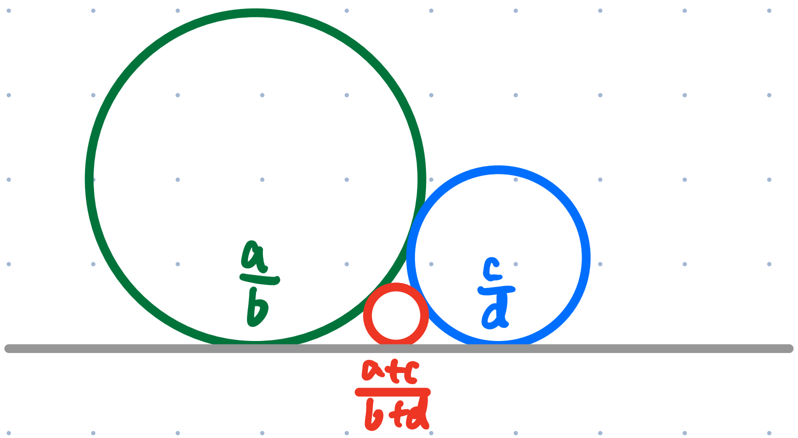
- $\frac{c}{d} \heartsuit \frac{A}{B}$ & $0 < d < B$

- $\frac{a}{b} \vee \frac{c}{d} = ?$

Recall that $Ab - Ba = 1$ and $Ad - Bc = -1$

$$\Rightarrow A \cdot (b+d) - B \cdot (a+c) = 0$$

$$\Rightarrow \frac{A}{B} = \frac{a+c}{b+d} = \frac{a}{b} \vee \frac{c}{d}$$

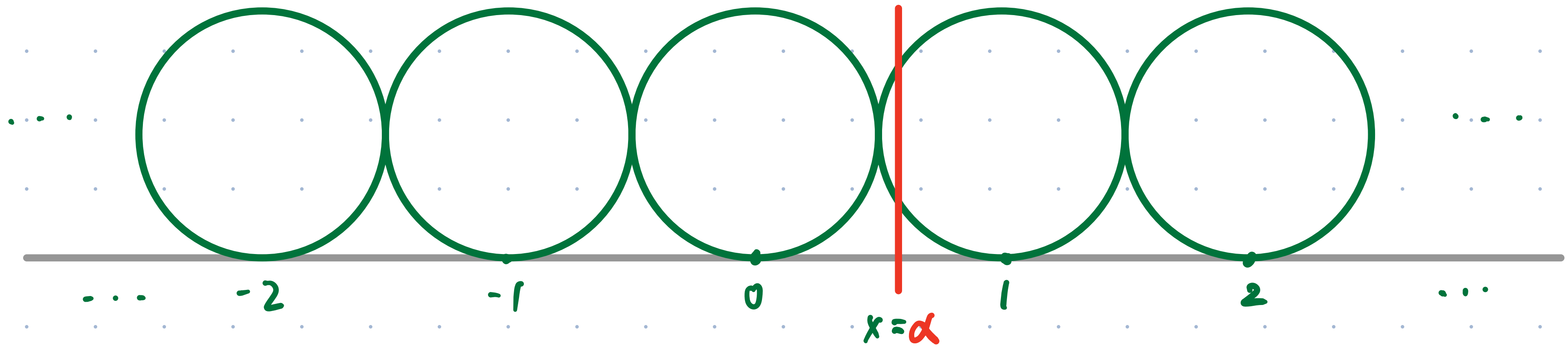


Any Ford circle (atop $\frac{A}{B}$) can be constructed from two Ford circles (atop $\frac{a}{b}$ & $\frac{c}{d}$) in previous steps.

Proof of Dirichlet's Approximation Theorem:

(Idea: To show that the vertical line atop α crosses INFINITELY many Ford circles.)

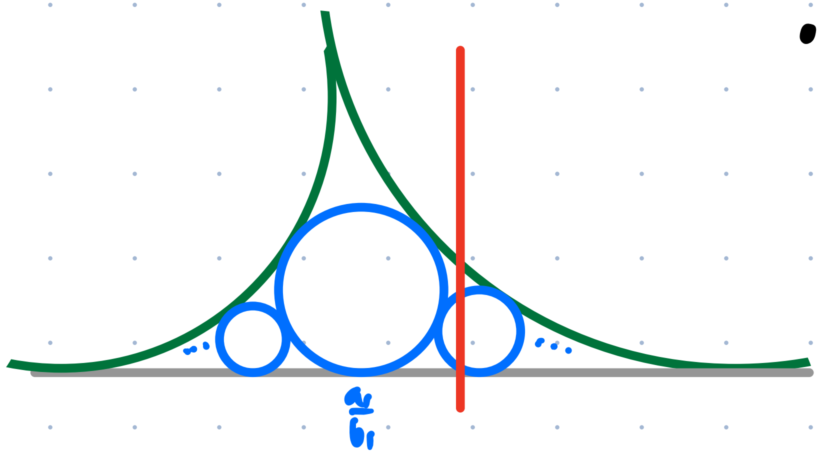
- First, the line must cross one Ford circle in the base step.



(It won't tangent the circles since α is irrational!)

- Then the line falls into a mesh triangle. We claim:

  It has to cross one Ford circle inside the mesh triangle.



- Indeed, we will show:

  Whenever the line crosses a Ford circle and falls into the mesh triangle below, it has to cross one Ford circle in the mesh triangle.

The line crosses the Ford circle $\frac{a}{b}$ and falls in a mesh triangle

means: $\left|\alpha - \frac{a}{b}\right| < \frac{1}{2b^2}$ (We may assume $\alpha < \frac{a}{b}$)

and there is a $\frac{c}{d}$ kissing $\frac{a}{b}$ and $\alpha > \frac{c}{d}$.

Now, consider $\frac{a_1}{b_1} := \frac{a}{b} \vee \frac{c}{d}$. If $\left|\alpha - \frac{a_1}{b_1}\right| < \frac{1}{2b_1^2}$. ✓

Otherwise, we replace $\frac{c}{d}$ by $\frac{a_1}{b_1}$ and repeat above.

Namely, we consider sequence:

$$\frac{a_0}{b_0} = \frac{c}{d},\ \frac{a_1}{b_1} := \frac{a}{b} \vee \frac{c}{d},\ \cdots,\ \frac{a_n}{b_n} := \frac{a}{b} \vee \frac{a_{n-1}}{b_{n-1}}.$$

We have : $\dfrac{a_n}{b_n} = \dfrac{n\cdot a + c}{n\cdot b + d}$ [Note that we have assumed $\frac{c}{d} < \frac{a}{b}$)

Hence, the sequence $\left\{\dfrac{a_n}{b_n}\right\}$ increases and its limit is $\dfrac{a}{b}$.

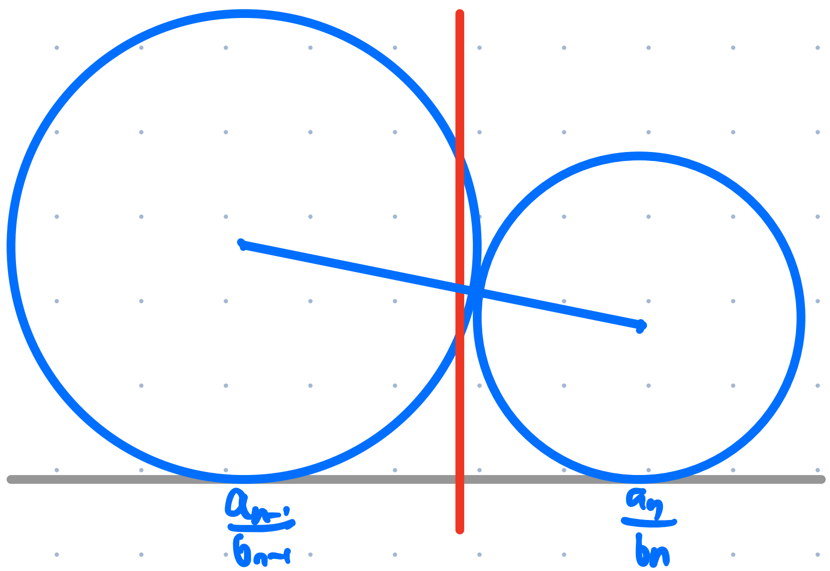
But we have assumed $\alpha < \frac{a}{b}$. Hence there must be some n s.t.

$$\frac{a_{n-1}}{b_{n-1}} < \alpha < \frac{a_n}{b_n}.$$

(CANNOT equal, why?)

But $\dfrac{a_{n-1}}{b_{n-1}} \heartsuit \dfrac{a_n}{b_n}$, namely the two Ford circles atop them are tangent to each other.

Hence, to cross the area between them, the line must cross one of them !



Now, we can finish the proof as follows:

① the line $x=\alpha$ must cross one of the Ford circles in base step

(give one "good" approximation $\frac{n}{1}$)

② Whenever $x=\alpha$ crosses a Ford circle (atop $\frac{a}{b}$) and falls in the mesh triangle below, it has to cross a Ford circle (atop $\frac{a'}{b'}$) inside the mesh triangle. (Hence, $b' > b$)

"good" approximation $\frac{a}{b}$ $\Rightarrow$ "good" approximation $\frac{a'}{b'}$

③ In the landscape of Ford circles, we can repeat ② infinitely many times and resuts infinitely many "good" approximation.

$\boxed{\Sigma}$