Diophantine approximation

Question

Given a real number α, approximate it by rational numbers.

A typical Diophantine approximation theorem would claim the existence or infinitude of rational numbers r approximating the given real number α within a reasonable bound $f(r)$:

$$|\alpha - r| \leqslant f(r).$$

The first theorem in the field of Diophantine approximation follows from the geometry of number line.

Theorem 3.3.1

Let α be a real number and b be a positive integer. Then there is an integer a such that

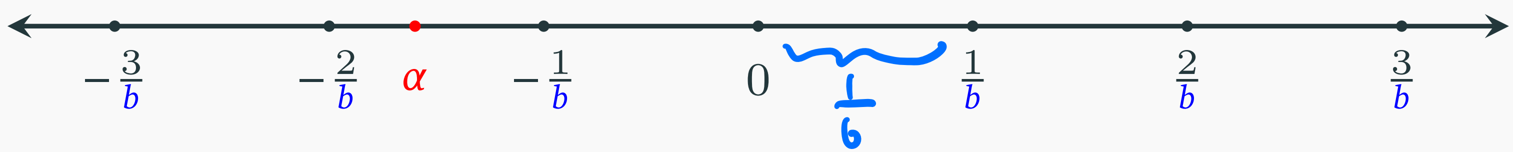
$$\left|\alpha - \frac{a}{b}\right| \leqslant \frac{1}{2b}.$$

E.g. $\pi = 3.1415926\ldots$.

$$\left|\pi - \frac{314}{100}\right| \leqslant 0.0016 \quad < \quad \frac{1}{2\times 100} = \frac{1}{200} = 0.005$$

Proof of the theorem

Proof. Let's first plot $\frac{1}{b}\mathbb{Z}$ on the number line:



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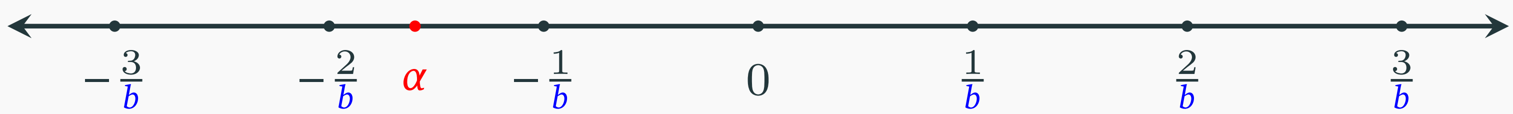

Let's say α is between $\frac{c}{b}$ and $\frac{c+1}{b}$. One of $\frac{c}{b}$ and $\frac{c+1}{b}$ is closer to α than the other. Choose the closer one to be $\frac{a}{b}$. Then we have

$$\left|\alpha - \frac{a}{b}\right| \leqslant \frac{1}{2}\text{length of the interval } [\tfrac{c}{b}, \tfrac{c+1}{b}] = \frac{1}{2b}. \qquad \square$$

Sometimes, we have far better approximation.

Example 3.3.2

$\pi = 3.1415926\ldots.$

- $\frac{a}{b} = 3.14 = \frac{157}{50}$: $\left|\pi - \frac{a}{b}\right| \approx 0.00159$, while $\frac{1}{2b} = 0.01$. ($\sim 16\%$)
- $\frac{a}{b} = \frac{22}{7}$: $\left|\pi - \frac{a}{b}\right| \approx 0.0013$, while $\frac{1}{2b} \approx 0.07$. ($\sim 2\%$)
- $\frac{a}{b} = \frac{355}{113}$: $\left|\pi - \frac{a}{b}\right| \approx 0.00000027$, while $\frac{1}{2b} \approx 0.0044$. ($\sim 0.006\%$)

One motivation to study Diophantine approximation is the following phenomenon.

Guideline

If an irrational number α can be approximated by rational numbers too well, then α is likely to be transcendental.

E.g. $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots$

$$\left| e - \underbrace{\sum_{k=0}^{n} \frac{1}{k!}}_{\frac{\cdots}{n!}} \right| = \sum_{k=n+1}^{\infty} \frac{1}{k!} < \frac{1}{2(n+1)!} \quad \text{v.s.} \quad \frac{1}{2n!}$$

$$\frac{1}{n+1}$$

Theorem 3.3.3 (Liouville, 1840s)

Let α be an irrational algebraic number of degree $\leqslant n$ (which means it is a root of an integer polynomial of degree n). Then there is a constant $C > 0$ such that

$$\left|\alpha - \frac{a}{b}\right| > \frac{C}{b^n} \qquad \text{for all} \qquad a \in \mathbb{Z}, b \in \mathbb{Z}_+.$$

Theorem 3.3.4 (Thue-Siegel-Roth, 1900s–1950s)

Let α be an irrational algebraic number and ε a small positive real number. Then there is a constant $C > 0$ such that

$$\left|\alpha - \frac{a}{b}\right| > \frac{C}{b^{2+\varepsilon}} \qquad \text{for all} \qquad a \in \mathbb{Z}, b \in \mathbb{Z}_{+}.$$

Dirichlet's approximation theorem

Theorem 3.3.5 (Dirichlet, 1840)

Let α be an irrational number, Then there are infinitely many fractions $\frac{a}{b}$ such that

$$\left|\alpha - \frac{a}{b}\right| \leqslant \frac{1}{2b^2}.$$

Theorem 3.3.5 (Dirichlet, 1840)

Let α be an irrational number, Then there are infinitely many fractions $\frac{a}{b}$ such that

$$\left|\alpha - \frac{a}{b}\right| \leqslant \frac{1}{2b^2}.$$

N.B. this theorem doesn't imply that for *all* positive integer b, there is a fraction $\frac{a}{b}$ approximating α with above error bound. (Compare it with theorem 3.3.1)

E.g. for $\pi = 3.1415926\ldots$:

- $b = 1$ works: $\left|\pi - \frac{3}{1}\right| \approx 0.14 < \frac{1}{2}$.
- $b = 2$ doesn't work: $\left|\pi - \frac{6}{2}\right| \approx 0.14 > \frac{1}{2\cdot 2^2} = 0.125$.

Outline of the proof

To prove this theorem, we first interpret the inequality

$$\left|\alpha - \frac{a}{b}\right| \leqslant \frac{1}{2b^2}$$

in terms of geometry:
it means the point α is within distance $\frac{1}{2b^2}$ from the point $\frac{a}{b}$.

Outline of the proof

To prove this theorem, we first interpret the inequality

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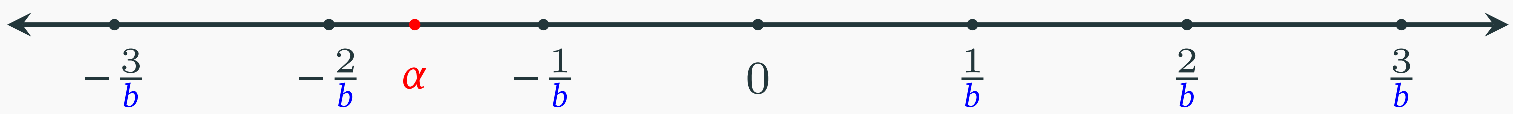
in terms of geometry:
it means the point α is within distance $\frac{1}{2b^2}$ from the point $\frac{a}{b}$.

Recall how we prove Theorem 3.3.1, mark $\frac{1}{b}\mathbb{Z}$ on the number line.



Instead of consider intervals $[\frac{c}{b}, \frac{c+1}{b}]$, we put circles of diameter $\frac{1}{b^2}$ at each $\frac{a}{b}$. So the inequality holds whenever α is covered by one of such circles.