Distribution of primes

Although there are infinitely many prime numbers, the number of primes below a given bound is finite.

Definition 2.5.1

The *prime counting function* $\pi(x)$ takes a positive real number as input and outputs the number of primes no larger than x.

Prime counting

Definition 2.5.1

The *prime counting function* $\pi(x)$ takes a positive real number as input and outputs the number of primes no larger than x.

Example 2.5.2

- $\pi(\frac{3}{2}) = 0$ since there is no prime $\leqslant \frac{3}{2}$.
- $\pi(3\sqrt{5}) = \pi(6) = 3$ 2, 3, 5
- $\pi(24) = 8$ 2, 3, 5, 7, 11, 13, 17, 23

Question (Open problem)

Can we have an asymptotic formula for $\pi(x)$? Namely, can we find a simple function $f(x)$ such that $\pi(x) \sim f(x)$, i.e.

$$\lim_{x \to \infty} \frac{\pi(x)}{f(x)} = 1?$$

Furhtermore, can we bound the "error" $|\pi(x) - f(x)|$ in terms of x?

Theorem 2.5.3 (Prime number theorem)

Let $\log(\cdot)$ *be the natural logarithm. Then we have* $\pi(x) \sim \frac{x}{\log(x)}$.

First conjectured by Adrien-Marie Legendre (1797 or 1798) and Carl Friedrich Gauss (1792 or 1793). Studied by Pafnuty Chebyshev (1848 and 1850) and Bernhard Riemann (1859). Finally proved by Jacques Hadamard and Charles Jean de la Vallée Poussin (1896) through a study of the Riemann zeta function $\zeta(s)$. After that, several different proofs of it were found.

We know that $\pi(1 \text{ million}) = 78498$ while $\frac{1 \text{ million}}{\log(1 \text{ million})} \approx 72382$. You may find that the error is a bit large.

A much better approximation is given by the *logarithmic integral*:

$$\mathrm{Li}(x) := \int_2^x \frac{\mathrm{d}t}{\log t} \qquad (x \geqslant 2).$$

Indeed, we have $\mathrm{Li}(1 \text{ million}) \approx 78627$. The error is smaller than $\frac{\sqrt{1 \text{ million}} \log(1 \text{ million})}{8\pi} \approx 550$.

One important consequence of the *Riemann hypothesis* is that

Corollary 2.5.4 (Lowell Schoenfeld, 1976)

If Riemann hypothesis is true, then for all $x \geqslant 2657$,

$$|\pi(x) - \mathrm{li}(x)| \leqslant \frac{\sqrt{x}\log(x)}{8\pi}.$$

Here $\mathrm{li}(x)$ only differ from $\mathrm{Li}(x)$ by a small constant $\mathrm{li}(2) = 1.045\cdots$.

Another direction is to consider the gaps between primes.

The first fact is: the gap between primes can be arbitrarily large.

Another direction is to consider the gaps between primes.

The first fact is: the gap between primes can be arbitrarily large.

Then it is natural to ask: how small could the gap be? Note that every pair of consecutive integers must contain an even number. Hence, the only pair of primes having gap 1 is $(2, 3)$.

Another direction is to consider the gaps between primes.

The first fact is: the gap between primes can be arbitrarily large.

Then it is natural to ask: how small could the gap be? Note that every pair of consecutive integers must contain an even number. Hence, the only pair of primes having gap 1 is $(2, 3)$.

How about gap 2?

Definition 2.5.5

Two primes p, q are called *twin primes* if $|p - q| = 2$.

Question (Open problem)

Are there infinitely many twin primes?

The best results so far are:

Theorem 2.5.6

There infinitely many pairs of primes (p, q) such that

$|p - q| \leqslant 70$ *million* — *(Yitang Zhang, 2013)*

$|p - q| \leqslant 600$ — *(James Maynard, 2013)*

$|p - q| \leqslant 246$ — *(D. H. J. Polymath, 2014)*