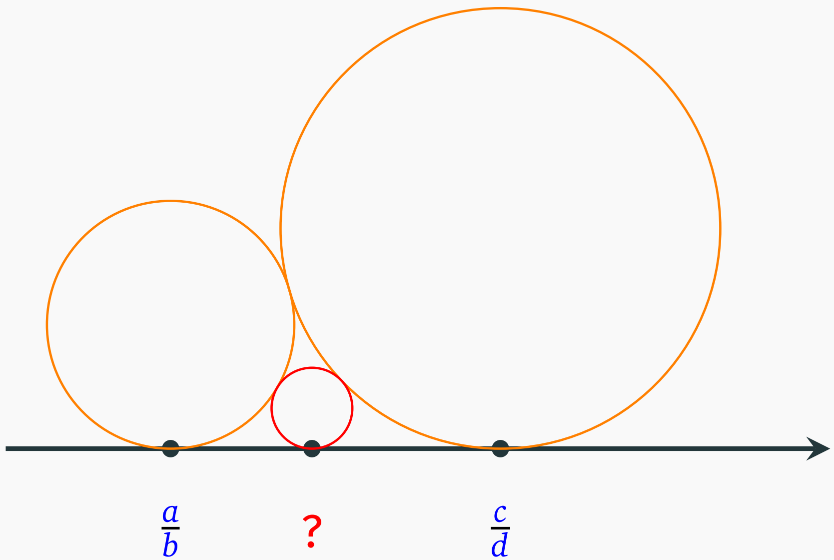
Mediant

Question

Given two Ford circle tangent to each other. Find a third one tangent to both of them.



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- C_1 and C_2 are tangent to each other if and only if $\frac{a}{b} \heartsuit \frac{c}{d}$;

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- C_1 and C_2 are tangent to each other if and only if $\frac{a}{b} \heartsuit \frac{c}{d}$;
- C is tangent to C_1 if and only if $\frac{x}{y} \heartsuit \frac{a}{b}$;

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- $\frac{x}{y}$ is between $\frac{a}{b}$ and $\frac{c}{d}$;
- C_1 and C_2 are tangent to each other if and only if $\frac{a}{b} \heartsuit \frac{c}{d}$;
- C is tangent to C_1 if and only if $\frac{x}{y} \heartsuit \frac{a}{b}$;
- C is tangent to C_2 if and only if $\frac{x}{y} \heartsuit \frac{c}{d}$;

Spell out the relations $\frac{a}{b} \heartsuit \frac{c}{d}$, $\frac{x}{y} \heartsuit \frac{a}{b}$, and $\frac{x}{y} \heartsuit \frac{c}{d}$, we get a system of equations with unknown $x.y$.

$$\begin{cases} |ad - bc| = 1, \\ |xb - ya| = 1, \\ |xd - yc| = 1. \end{cases}$$

One can solve this system and get

$$\text{either} \quad \frac{x}{y} = \frac{a-c}{b-d} \qquad \text{or} \quad \frac{x}{y} = \frac{a+c}{b+d}.$$

Since $\frac{x}{y}$ is between $\frac{a}{b}$ and $\frac{c}{d}$, we must have $\frac{x}{y} = \frac{a+c}{b+d}$.

We thus introduce the following notion:

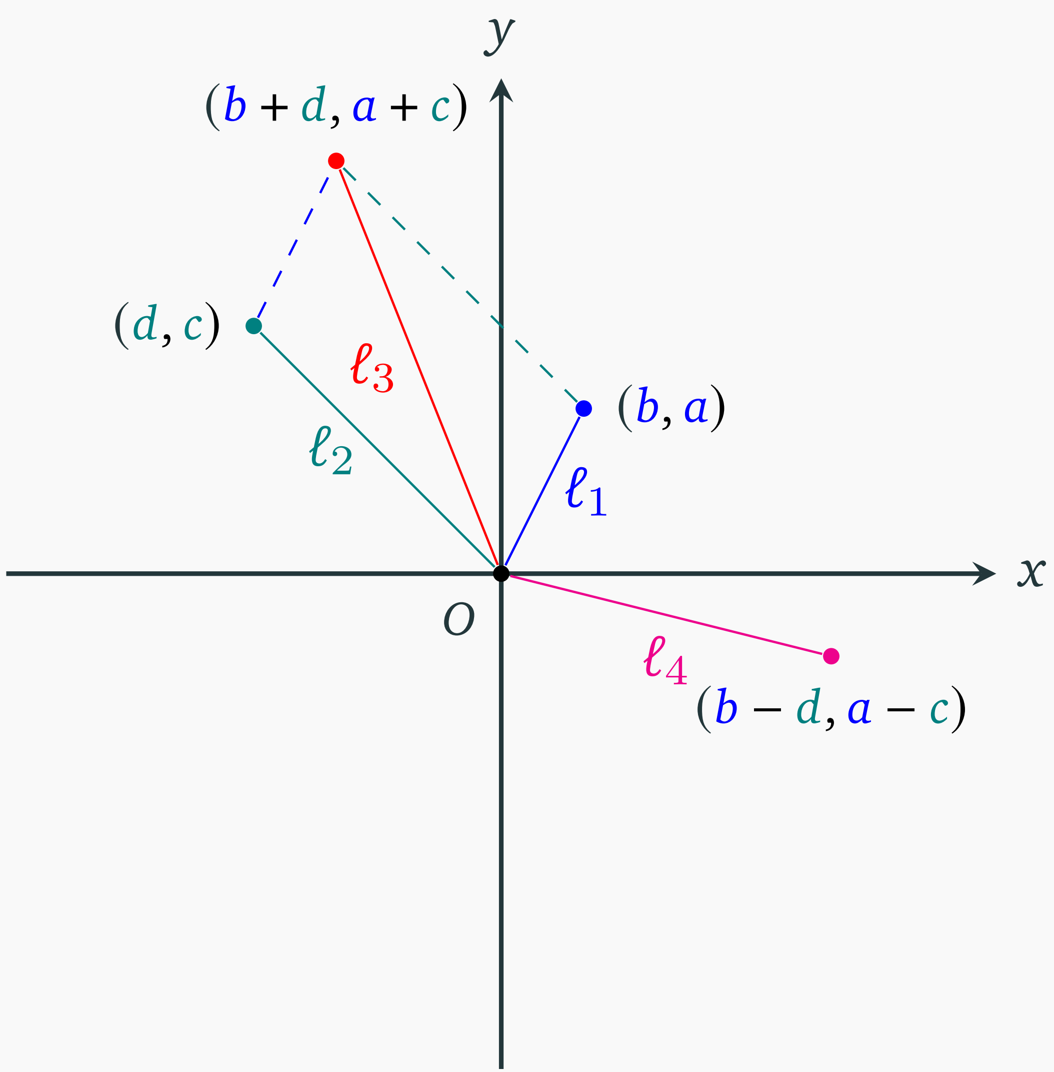
Definition 3.5.1

Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, the *mediant* of them is the fraction

$$\frac{a}{b} \vee \frac{c}{d} := \frac{a+c}{b+d}.$$

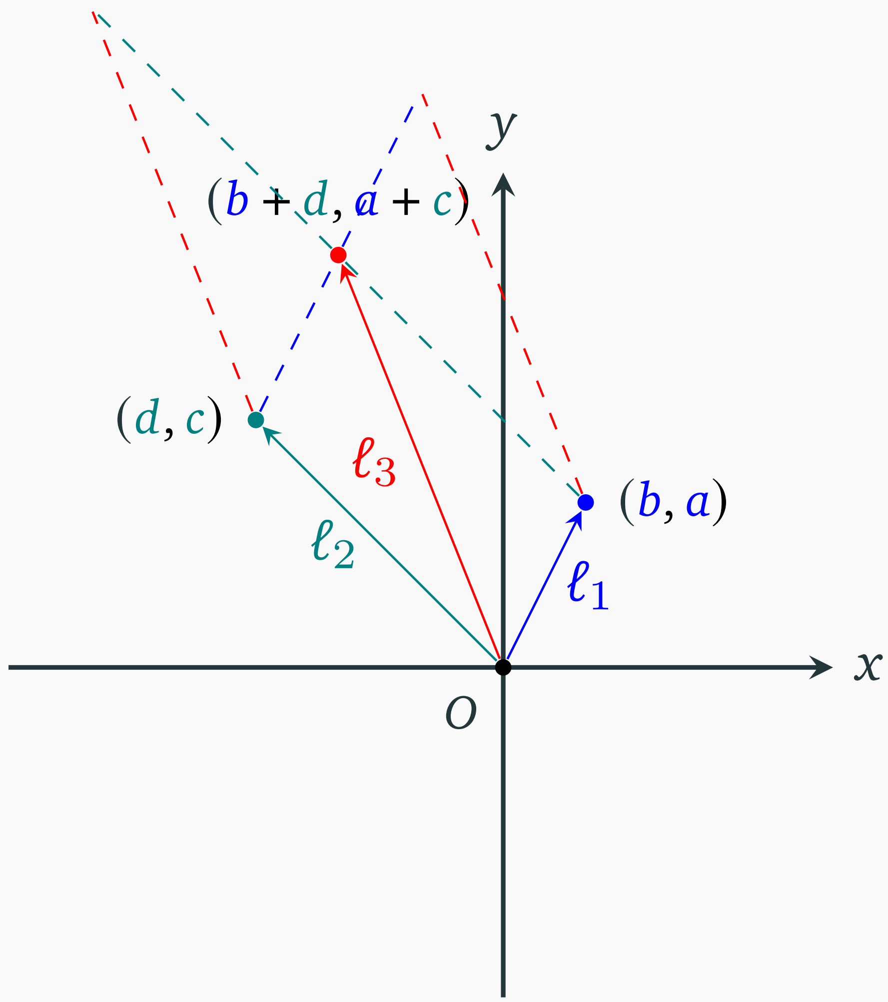
N.B. this is an operation on *fractions*!

Geometric interpretation of mediant



- $\frac{a}{b}$ is the *slope* of the line segment ℓ_1;
- $\frac{c}{d}$ is the *slope* of the line segment ℓ_2;
- $\frac{a+c}{b+d}$ is the *slope* of the line segment ℓ_3;
- $\frac{a-c}{b-d}$ is the *slope* of the line segment ℓ_4.

$\frac{a}{b} \vee \frac{c}{d}$ is between $\frac{a}{b}$ and $\frac{c}{d}$ $\leftrightsquigarrow$ ℓ_3 is between ℓ_1 and ℓ_2.



- Recall that the area of the rectangle forming by vectors u and v is $\|\mathrm{u}\times\mathrm{v}\|$.
- $\frac{a}{b}\heartsuit\frac{c}{d} \leftrightsquigarrow \|\ell_1 \times \ell_2\| = 1$;
- Then we find that the area $\|\ell_1 \times \ell_3\|$ has to be also 1;
- Likewise, the area $\|\ell_1 \times \ell_3\|$ has to be also 1.

Hence, $\frac{a}{b} \vee \frac{c}{d}$ kisses both $\frac{a}{b}$ and $\frac{c}{d}$.

We thus proved the following lemma.

Lemma 3.5.2

If $\frac{a}{b} \heartsuit \frac{c}{d}$, then their mediant $\frac{a}{b} \vee \frac{c}{d}$ kisses both of them.

N.B. By Bézout's identity, we see that the mediant of two kissing reduced fractions must be reduced.

Hence, $\vee$ is rather an operation of (kissing) rational numbers.

In geometric words, if two Ford circles are tangent to each other, then the Ford circle atop their mediant is tangent to both of them.