Homework 2

MATH 110 | Introduction to Number Theory | Summer 2023

Whenever you use a result or claim a statement, provide a **justification** or a **proof**, unless it has been covered in the class. In the later case, provide a **citation** (such as "by the *2-out-of-3 principle*" or "by Coro. 0.31 in the textbook").

You are encouraged to *discuss* the problems with your peers. However, you must write the homework **by yourself** using your words and **acknowledge your collaborators**.

Problem 1. Prove that if n is a positive integer, and $\sigma_0(n)$ is prime then n is a power of a prime number.

Problem 2 (Mersenne, 1644)**.** Describe all circumstances under which $\sigma_1(n)$ is odd.

Hint. Consider the prime factorization of n.

Problem 3. Recall that an *integer polynomial* is an expression of the form

$$P(T) = c_d T^d + \cdots + c_1 T + c_0,$$

where each c_i is an integer.

(a) **Find** a nonzero integer polynomial $P(T)$ that has $\sqrt{3} + \sqrt[3]{5}$ as a root.

(b) **Prove that** $\sqrt{3} + \sqrt[3]{5}$ is irrational using 3.(a).

Problem 4. By evaluating the Taylor series for the exponential function:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

at $x = 1$, we get the formula

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots .$$

In this problem, you will prove that e is *irrational*.

(a) Let $s_n := \sum_{k=0}^{n} \frac{1}{k!}$, the n-th partial sum of above series. **Show that**

$$0 \leq e - s_n \leq \frac{1}{n} \cdot \frac{1}{n!}.$$

(b) Assume e is rational, and say a/b is the reduced fraction representing e. Apply the previous result to $n = b$ and arrive at a contradiction.

Problem 5. Consider the *Fibonacci numbers*, define recursively by

$$F_0 = 0, F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2} \quad \text{for all} \quad n \geq 2;$$

so the first few terms are

$$0, 1, 1, 2, 3, 5, 8, 13, \cdots .$$

For all $n \geq 2$, define the rational number r_n by the fraction $\dfrac{F_n}{F_{n-1}}$; so the first few terms are

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \cdots .$$

(a) Prove that for all $n \geq 4$, we have $r_n = r_{n-1} \vee r_{n-2}$.

(b) Prove that the sequence r_n converges (to a real number).

(c) Prove that r_n converges to the *golden ratio*:

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

For this problem, you can use any result that you may have seen in your Calculus classes.