Homework 1

MATH 110 | Introduction to Number Theory | Summer 2023

Whenever you use a result or claim a statement, provide a **justification** or a **proof**, unless it has been covered in the class. In the later case, provide a **citation** (such as "by the *2-out-of-3 principle*" or "by Coro. 0.31 in the textbook").

You are encouraged to *discuss* the problems with your peers. However, you must write the homework **by yourself** using your words and **acknowledge your collaborators**.

Problem 1. Let $a_1, \cdots, a_n$ be n integers. We will use the notation $\gcd\limits_{1 \leqslant i \leqslant n} a_i$ to denote the *greatest common divisor of* $a_1, \cdots, a_n$ and the notation $\operatorname*{lcm}\limits_{1 \leqslant i \leqslant n} a_i$ to denote the *least common multiple of* $a_1, \cdots, a_n$.

Mimicking the proof of the attached proposition, show that:

For any matrix $(a_{ij})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m}$ of integers, we have

$$\operatorname*{lcm}_{1 \leqslant i \leqslant n} \gcd_{1 \leqslant j \leqslant m} a_{ij} \mid \gcd_{1 \leqslant j \leqslant m} \operatorname*{lcm}_{1 \leqslant i \leqslant n} a_{ij}.$$

Hint. What facts are used in the proof?

Proposition. *Let $(x_{ij})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m}$ be a matrix of real numbers, then we have*

$$\max_{1 \leqslant i \leqslant n} \min_{1 \leqslant j \leqslant m} x_{ij} \leqslant \min_{1 \leqslant j \leqslant m} \max_{1 \leqslant i \leqslant n} x_{ij}.$$

Proof. Define $f(i)$ $(1 \leqslant i \leqslant n)$ to be $\min\limits_{1 \leqslant j \leqslant m} x_{ij}$. Then we have

$$f(i) \leqslant x_{ij} \quad \text{for all} \quad 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m.$$

Therefore, we have

$$\max_{1 \leqslant i \leqslant n} f(i) \leqslant \max_{1 \leqslant i \leqslant n} x_{ij} \quad \text{for all} \quad 1 \leqslant j \leqslant m.$$

In particular, we have

$$\max_{1 \leqslant i \leqslant n} f(i) \leqslant \min_{1 \leqslant j \leqslant m} \max_{1 \leqslant i \leqslant n} x_{ij}$$

as desired. □

Problem 2. This problem is a 3-varibales analogy of the material covered in lectures.

(a) Prove that there exists no integer solution (x, y, z) to the equation

$$18x - 27y + 39z = 4.$$

(b) Find **an** integer solution (x, y, z) to the equation $18x - 27y + 39z = 6$.

(c) Find **all** the integer solutions (x, y, z) to the equation $18x - 27y + 39z = 6$. Your answer should give explicit formulae for x, y, z in terms of two free independent integer parameters m and n.

`Read Chapter 2 to finish the following problems.`

Problem 3. Let a, b and n be positive integers. `Prove` that

(a) $\gcd(a^n, b^n) = \gcd(a, b)^n$ and $\operatorname{lcm}(a^n, b^n) = \operatorname{lcm}(a, b)^n$;

(b) $\gcd(a \cdot n, b \cdot n) = \gcd(a, b) \cdot n$ and $\operatorname{lcm}(a \cdot n, b \cdot n) = \operatorname{lcm}(a, b) \cdot n$;

Problem 4. Let n be any positive integer. `Prove` that there exists a positive integer k (depending on n) such that the following list of n consecutive integers:

$$k, k+1, \cdots, k+n-1$$

contains *no* prime number at all.

Hint. Use the factorial (but $k = n!$ is NOT the correct answer, start from this and try to see what are missing). You also need the *2-out-of-3* property of division.