Irrational numbers

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Definition 3.2.1

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The discovery of irrational numbers can be traced back to ancient Greece.

Example 3.2.2 (Pythagorean or Hippasus, 500 BC)

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Example 3.2.2 (Pythagorean or Hippasus, 500 BC)

$\sqrt{2}$ is irrational.

Proof. Suppose $\sqrt{2}$ is rational and can be expressed by the reduced fraction $\frac{a}{b}$. Then we have

$$\frac{2}{1} = 2 = \frac{a^2}{b^2}.$$

But since a, b are coprime, the right-hand side is reduced. Hence, by the uniqueness of reduced fraction expression, we must have $2 = a^2$ and $1 = b^2$. But this is impossible: 2 is not a perfect square. □

Theorem 3.2.3 (Irrationality of roots)

Let $\frac{a}{b}$ be a reduced fraction and n is an integer $\geqslant 2$. Then $\sqrt[n]{\frac{a}{b}}$ gives rational values if and only if both a and b are perfect n-th power (i.e. there are integers c, d such that $a = c^n$ and $b = d^n$.)

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Proof. The "if" part is clear. Let's prove the "only if" part. Suppose our number α can be expressed as a reduced fraction $\frac{c}{d}$. Then

$\parallel$

$\sqrt[n]{\frac{a}{b}}$

$$\frac{c^n}{d^n} = \left(\frac{c}{d}\right)^n = \alpha^n = \frac{a}{b}.$$

By the uniqueness of reduced fraction expression, we must have $a = c^n$ and $b = d^n$. $\square$

Another useful result is the following criterion:

Theorem 3.2.4 (Rational root theorem)

Let $\frac{a}{b}$ be a reduced fraction expressing a root of a polynomial

$$P(T) = c_n T^n + \cdots + c_1 T + c_0 \qquad (c_i \in \mathbb{Z}).$$

Then $a \mid c_0$ and $b \mid c_n$.

Theorem 3.2.4 (Rational root theorem)

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$$P(T) = c_n T^n + \cdots + c_1 T + c_0 \qquad (c_i \in \mathbb{Z}).$$

Then $a \mid c_0$ and $b \mid c_n$.

Proof. Substitute $\frac{a}{b}$ into the polynomial,

$$c_n \left(\frac{a}{b}\right)^n + \cdots + c_1 \left(\frac{a}{b}\right) + c_0 = 0.$$

We thus have

$$c_n a^n + c_{n-1} a^{n-1} b + \cdots + c_1 a b^{n-1} + c_0 b^n = 0.$$

Then we must also have $a \mid c_0 b^n$ and $b \mid c_n a^n$. Since a, b are coprime, we have $a \mid c_0$ and $b \mid c_n$. □

Algebraic numbers

Definition 3.2.5

A complex number α is *algebraic* if it is a root of a nonzero integer polynomial. Namely, there are integers $c_0, \cdots, c_n$ such that

$$c_n\alpha^n + \cdots + c_1\alpha + c_0 = 0.$$

Otherwise, we say α is *transcendental*.

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Example 3.2.6

1. Rational numbers are algebraic. Indeed, if $\frac{a}{b}$ is a fraction expressing our rational number α, then α is a root of $bT - a$.
2. n-th roots of rational numbers are algebraic. Indeed, $\sqrt[n]{\frac{a}{b}}$ is a root of $bT^n - a$.

Example 3.2.7

$2\sqrt{2}+\sqrt{3}$ is algebraic.

Proof. Let $\alpha = 2\sqrt{2}+\sqrt{3}$. We want to find an integer polynomial $P(T)$ such that $P(\alpha) = 0$.

$$\alpha = 2\sqrt{2}+\sqrt{3} \qquad \text{our definition}$$

□

Example 3.2.7

$2\sqrt{2} + \sqrt{3}$ is algebraic.

Proof. Let $\alpha = 2\sqrt{2} + \sqrt{3}$. We want to find an integer polynomial $P(T)$ such that $P(\alpha) = 0$.

$$\begin{aligned} \alpha &= 2\sqrt{2} + \sqrt{3} && \text{our definition} \\ \alpha - \sqrt{3} &= 2\sqrt{2} && \text{separate the roots} \end{aligned}$$

$\square$

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$$\begin{aligned} \alpha &= 2\sqrt{2} + \sqrt{3} && \text{our definition} \\ \alpha - \sqrt{3} &= 2\sqrt{2} && \text{separate the roots} \\ \alpha^2 - 2\sqrt{3}\alpha + 3 &= 8 && \text{square both sides} \end{aligned}$$

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\alpha^4 - 10\alpha^2 + 25 &= 12\alpha^2 && \text{square both sides}
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$$
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\alpha - \sqrt{3} &= 2\sqrt{2} && \text{separate the roots}\\
\alpha^2 - 2\sqrt{3}\alpha + 3 &= 8 && \text{square both sides}\\
\alpha^2 - 5 &= 2\sqrt{3}\alpha && \text{separate the roots}\\
\alpha^4 - 10\alpha^2 + 25 &= 12\alpha^2 && \text{square both sides}
\end{aligned}
$$

Therefore, $\alpha^4 - 22\alpha^2 + 25 = 0$. Namely, α is a root of the integer polynomial $T^4 - 22T^2 + 25$. □

Corollary 3.2.8

$2\sqrt{2} + \sqrt{3}$ *is irrational.*

Proof. Suppose for the sake of contradiction that $2\sqrt{2} + \sqrt{3}$ can be expressed by the reduced fraction $\frac{a}{b}$.

Corollary 3.2.8

$2\sqrt{2}+\sqrt{3}$ *is irrational.*

Proof. Suppose for the sake of contradiction that $2\sqrt{2}+\sqrt{3}$ can be expressed by the reduced fraction $\frac{a}{b}$.

Then since it is a root of integer polynomial T^4-22T^2+25, by the *Rational Root Theorem*, we must have $a \mid 25$ and $b \mid 1$. Therefore, the fraction $\frac{a}{b}$ can only be one of the following:

$$\pm 25, \pm 5, \pm 1.$$

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Note that $2<2\sqrt{2}<3$ since $4<8<9$, and that $1<\sqrt{3}<2$ since $1<3<4$. Thus, $3<2\sqrt{2}+\sqrt{3}<5$. But none of above falls in this interval, which is a contradiction. □