

APPLY TO CRYPTOGRAPHY

We may use the difficulty of discrete logarithms to encrypt communication.

Question (Public key system, Diffie-Helman key exchange)

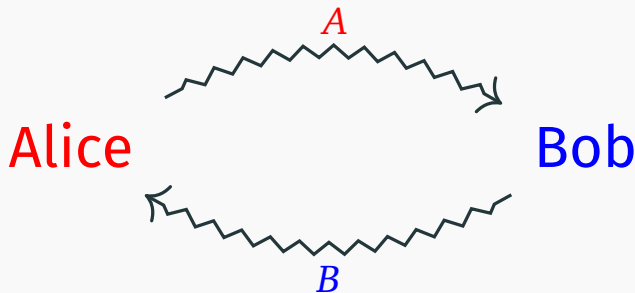
Alice wants to encrypt a message so that only *Bob* can decrypt it, not *Eve*.



1. **Alice** chooses a large ($\sim 2^{2048}$) prime p such that $\varphi(\varphi(p))$ also has a large prime factor, and finds a primitive root g modulo p . Publishes (p, g) , which is the *public key*.

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2. **Alice** chooses a *private key* a and computes $A := g^a \pmod{p}$.
Bob chooses a *private key* b and computes $B := g^b \pmod{p}$.



Then they exchange A and B (through any channel, probably intercepted by **Eve**).

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4. Now **Alice** and **Bob** can encrypt their communication using the secret key S .
5. **Eve** may know (p, g, A, B) . Can **Eve** find out what S is? This is very hard since finding a (resp. b) from A (resp. B) is difficult.

Some remarks:

- A *Sophie Germain prime* is a prime q such that $p := 2q + 1$ is also a prime. Note that $\varphi(p) = 2q$. Hence, when q is large, p would be a safe prime for the public key system.

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- The primality testing is fast, so generating a public key wouldn't cost too much time.
- **Alice** needs to compute $g^a \pmod{p}$ and $B^a \pmod{p}$, while **Bob** needs to compute $g^b \pmod{p}$ and $A^b \pmod{p}$. These are modular exponential problems, and we can solve them effectively using binary exponentiation algorithms.

Example 4.8.1

Alice wants to encrypt communication with Bob using Diffie-Helman key exchange. Suppose the public key is $(467, 2)$.

If the private keys of Alice and Bob are $a = 22$ and $b = 33$ respectively. What are A , B and the secret key S ?

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modulo 467	2	4	16	256	156	52	369	264	113

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$$2. B \equiv g^b \pmod{p} \equiv 2^{33} = 2^{1+32} \equiv 2 \cdot 52 \equiv 104 \pmod{467},$$

$$3. S \equiv A^b \equiv B^a \equiv g^{ab} \pmod{p} \\ \equiv 2^{22 \cdot 33} \equiv 2^{260} = 2^{4+256} \equiv 16 \cdot 113 \equiv 30 \pmod{467}.$$