Homework 4 (due Feb. 19)

MATH 110 | Introduction to Number Theory | Winter 2023

Problem 1. Consider the *Fibonacci numbers*, define recursively by

$$F_0 = 0, F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2} \quad \text{for all} \quad n \geq 2;$$

so the first few terms are

$$0, 1, 1, 2, 3, 5, 8, 13, \cdots .$$

For all $n \geq 2$, define the rational number r_n by the fraction $\dfrac{F_n}{F_{n-1}}$; so the first few terms are

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \cdots .$$

(a) (5 pts) Prove that for all $n \geq 4$, we have $r_n = r_{n-1} \vee r_{n-2}$.

(b) (5 pts) Prove that the sequence r_n converges (to a real number).

(c) (5 pts) Prove that r_n converges to the *golden ratio*:

$$\phi = \frac{1+\sqrt{5}}{2}.$$

For this problem, you can use any result that you may have seen in your Calculus classes.

Problem 2. (a) (5 pts) What are **all** the possible natural representatives of $n^2 \pmod 8$, where n is an integer.

(b) (5 pts) Determine whether the following equation is solvable in $\mathbb{Q}$ or not? If it is, **find** such a solution $(x, y, z) \in \mathbb{Q}^3$, if not, **prove it**.

$$x^2 + y^2 + z^2 = 2023$$

Hint. First translate it into a Diophantine equation asking solutions in $\mathbb{Z}$. Then consider the congruence modulo 8.

Problem 3. Let p be any prime number and let a and b be any two integers.

(a) (5 pts) **Prove that** if $a \equiv b \pmod p$, then $a^p \equiv b^p \pmod{p^2}$.

(b) (5 pts) **Prove that** if $a \equiv b \pmod p$, then $a^{p^2} \equiv b^{p^2} \pmod{p^3}$.

(c) (Optional, up to 5 extra pts) **Prove that** if $a \equiv b \pmod p$, then $a^{p^k} \equiv b^{p^k} \pmod{p^{k+1}}$ for all positive integer k.

Problem 4 (5 pts)**.** Suppose we have $5x \equiv 11 \pmod{37}$ and $11y \equiv 5 \pmod{37}$. Prove that y is a multiplicative inverse of x modulo 37.

Hint. You need the cancelling property. Note that to cancel a factor, you need to first show it is invertible.

Problem 5 (10 pts)**.** Consider the recursive sequence given by

$$a_0 = 3, \quad a_n = 3^{a_{n-1}}, \quad \text{for all} \quad n \geq 1$$

That is, $a_0 = 3,\ a_1 = 3^3,\ a_2 = 3^{3^3}, \ldots$. What is the last digit of a_{2022}?

Remark. Be aware that $3^{3^3} \neq (3^3)^3$.