Quiz:

Find the reduced fraction representation of the following rational number and give its prime factorization.

"-1.56"

Answer:

$$-1.56 = \frac{-156}{100} = \frac{-39}{25} = -2^0 \cdot 3^1 \cdot 5^{-2} \cdot 7^0 \cdot 11^0 \cdot 13^1$$

Defn. A complex number is irrational if it is NOT rational.

e.g. $\sqrt{2}$ (Pythagorean or Hippasus, ~500 BC)

$\sqrt{2} = \frac{a}{b}$ reduced $\Rightarrow \sqrt{2}\cdot b = a \Rightarrow \underbrace{2\cdot b^2}_{\text{even}} = a^2 \Rightarrow a^2$ is even $\Rightarrow a$ is even

$\Rightarrow 4 \mid a^2$, say $a^2 = 4\cdot c$ $\quad \cancel{2}\cdot b^2 = \cancel{4}_{2}\cdot c \quad b^2 = \underbrace{2\cdot c}_{\text{even}} \Rightarrow b$ is even $\not\Leftarrow$

Prop (Irrationality of roots)

Let $\frac{a}{b}$ be a reduced fraction and $n \geq 2$ an integer.

Then $\sqrt[n]{\frac{a}{b}}$ is a rational number

a root of $b\cdot T^n - a$

$\Leftrightarrow$ both a & b are perfect n-th powers.

(i.e. there are integers c & d such that $a = c^n$ and $b = d^n$).

E.g. $\sqrt{7}$, $\sqrt{8} = 2\sqrt{2}$, $\sqrt[3]{9}$, ……

Pf: ($\Rightarrow$) Suppose $\alpha := \sqrt[n]{\frac{a}{b}}$ is rational and is represented by a reduced fraction $\frac{c}{d}$. Then we have

$$\frac{a}{b} = \alpha^n = \frac{c^n}{d^n}.$$

$$\left.\begin{array}{l}\text{Since } GCD(c,d)=1,\ GCD(c^n,d^n)=1 \\ \text{Since } d>0,\ d^n>0\end{array}\right\} \Rightarrow \frac{c^n}{d^n} \text{ is reduced.}$$

By uniqueness of reduced fraction representation,

$$a = c^n \text{ and } b = d^n$$

($\Leftarrow$) If a & b are perfect n-th powers, saying $a = c^n$ and $b = d^n$, then $\alpha = \sqrt[n]{\frac{a}{b}} = \sqrt[n]{\frac{c^n}{d^n}} = \frac{c}{d}$ hence a rational number.

Rational Root Theorem

Suppose $\frac{a}{b}$ is a reduced fraction representing a (root) zero of a polynomial $p(T) = c_n T^n + \cdots + c_1 T + c_0 \quad (c_i \in \mathbb{Z})$.

Then $a \mid c_0$ and $b \mid c_n$.

e.g. $dT^n - c$

Pf: By assumption,

$$p\left(\frac{a}{b}\right) = c_n \left(\frac{a}{b}\right)^n + \cdots + c_1 \frac{a}{b} + c_0 = 0.$$

Hence, $c_n a^n + c_{n-1} a^{n-1} b + \cdots + c_1 a b^{n-1} + c_0 b^n = 0$

$$\Rightarrow \begin{cases} c_n a^n = -b(c_{n-1} a^{n-1} + \cdots + c_1 a b^{n-2} + c_0 b^{n-1}) \\ c_0 b^n = -a(c_n a^{n-1} + \cdots + c_1 b^{n-1}) \end{cases} \Rightarrow \begin{cases} b \mid c_n a^n \\ a \mid c_0 b^n \end{cases}$$

Since $GCD(a, b) = 1$, we conclude that $a \mid c_0$ and $b \mid c_n$

Defn. Say a complex number α is an algebraic number if it is a root (zero) of a nonzero polynomial $P(T) = c_d T^d + \cdots + c_1 T + c_0$, where $c_i \in \mathbb{Z}$ with at least one of them nonzero.

Otherwise, α is said to be transcendental.

E.g.

- rational numbers are algebraic

  $\alpha = \frac{a}{b}$ is a zero of $P(T) := b \cdot T - a$.

- $\sqrt{2}$ is not rational, but still algebraic.

  More general, $\sqrt[n]{\frac{a}{b}}$ are algebraic

  $\alpha = \sqrt[n]{\frac{a}{b}}$ is a zero of $P(T) := b \cdot T^n - a$

Example: $2\sqrt{2}+\sqrt{3}$ is algebraic

Let $\alpha := 2\sqrt{2}+\sqrt{3}$. Want to find $P(T)$ s.t. $P(\alpha)=0$.

$$\alpha = 2\sqrt{2}+\sqrt{3}$$

$$\alpha-\sqrt{3} = 2\sqrt{2}$$

$$(\alpha-\sqrt{3})^2 = (2\sqrt{2})^2$$

$$\alpha^2-2\sqrt{3}\alpha+3 = 8$$

$$\alpha^2-5 = 2\sqrt{3}\alpha$$

$$(\alpha^2-5)^2 = (2\sqrt{3}\alpha)^2$$

$$\alpha^4-10\alpha^2+25 = 12\alpha^2$$

$\rightsquigarrow$ $\alpha^4-22\alpha^2+25=0$.

Hence α is a zero of $P(T)=T^4-22T^2+25$. ☐

Applications (Example)

- $2\sqrt{2} + \sqrt{3}$ is irrational.

proof: Look at $P(T) = T^4 - 22T^2 + 25$.

If $2\sqrt{2} + \sqrt{3} = \frac{a}{b}$, then by Rational Root Theorem,

we have $a \mid 25$ and $b \mid 1$.

So all possibilities of $\frac{a}{b}$ are ± 1, ± 5, and ± 25.

Check none of them equals $2\sqrt{2} + \sqrt{3}$:

$$2\sqrt{2} + \sqrt{3} = \sqrt{8} + \sqrt{3}$$

$$\left.\begin{array}{l} 2 = \sqrt{4} < \sqrt{8} < \sqrt{9} = 3 \\ 1 = \sqrt{1} < \sqrt{3} < \sqrt{4} = 2 \end{array}\right\} \Rightarrow 3 < 2\sqrt{2} + \sqrt{3} < 5$$

But none of ± 1, ± 5, and ± 25 lies between 3 & 5.

Fact: If α and β are algebraic numbers, then

$$\alpha+\beta, \quad \alpha-\beta, \quad \alpha\cdot\beta, \quad \text{and } \alpha/\beta \ (\text{if } \beta \neq 0)$$

are also algebraic. $\mathbb{Q}^{alg}$ is closed under operations

The proof is beyond the scope of this course and will be taught in Math 111B. (Using Galois Theory)

It is NOT true that sum/difference/product/ratio of transcendental numbers is again transcendental.

e.g. $\underbrace{\pi + (-\pi)}_{\text{transcendental}} = \underset{\text{rational}}{0}$ $\qquad \pi \cdot \frac{1}{\pi} = 1 \qquad \pi - \pi = 0 \quad \ldots\ldots$

Facts about transcendental:

- e and π are transcendental (by Lindemann & Weierstrass)

Coro: We cannot square the circle. That is, using only straight ruler and compass, we cannot construct a square with area π.

- Constructible number = geometric quantity constructed using straight ruler and compass only.

- Thm 3.3: addition, substraction, multiplication, division, and square root suffices to describe constructible numbers.

e.g. rational numbers, square roots, $\sqrt{2+\sqrt{3}}$

In particularly, they are algebraic.

But we don't know if $e+\pi$ is transcendental.

Similarly for $e-\pi$, $e\cdot\pi$, e/π, e^e, π^π, π^e

- e^π is transcendental (Gelfond - Schneider)

- $\ln(2)$, $\ln(3)$, $\cdots$ are transcendental (Lindemann - Weierstrass)

- $\zeta(2) := \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (Basel problem, by Euler 1734) is transcendental

- $\zeta(3) := \sum_{n=1}^{\infty} \frac{1}{n^3}$ is only known to be irrational (Apéry, 1979)

  don't know if it is transcendental.

Very few are known. It is easy to ask very difficult problems in Transcendental number theory.

However, transcendental numbers are WAY more than algebraic ones.

Defn. A set S is countable if there is an injective map $S \longrightarrow \mathbb{N}$.

e.g. $\mathbb{N}$, $\mathbb{Z}$

$+n \rightsquigarrow 2n$

$-n \rightsquigarrow 2n+1$

Facts:

- $\mathbb{Q}$ is countable
- $\mathbb{Q}^{alg} := \{\text{algebraic numbers}\}$ is countable.

(Using "countable union of countable sets is countable")

- But $\mathbb{C}$ is NOT countable.

(Cantor's diagonal argument)

Quiz for next time:

Find a polynomial $P(T)$ with integer coefficients s.t.

$$P\left(\sqrt{2+\sqrt{3}}\right) = 0.$$

- Please prepare the above quiz for next meeting.
- Please read the rest of chapter 3 for next lecture.