Homework 3 (due Oct. 20)

MATH 110 | Introduction to Number Theory | Fall 2022

Whenever you use a result or claim a statement, provide a **justification** or a **proof**, unless it has been covered in the class. In the later case, provide a **citation** (such as "by the *2-out-of-3* property of *division*" or "by Coro. 0.31 in the textbook").

You are encouraged to *discuss* the problems with your peers. However, you must write the homework **by yourself** using your words and **acknowledge your collaborators**.

Problem 1. For this problem, you may want to review one-variable Calculus

(a) (3 pts) Recall the definition (In this course, $\log = \log_e$ denotes the *natural logarithm*)

$$\mathrm{Li}(x) := \int_2^x \frac{\mathrm{d}t}{\log t} \qquad (x > 2).$$

Question: What is the $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Li}(x)$ of $\mathrm{Li}(x)$?

(b) (5 pts) Two real functions $f(x)$ and $g(x)$ are *asymptotically equal* if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

Prove that: $\mathrm{Li}(x)$ and $\frac{x}{\log x}$ are asymptotically equal.

Problem 2 (5 pts)**.** Let p be a prime number and k, l be two natural numbers. **Show that**

$$\sum_{i=0}^{k} \sigma_i(p^l) = \sum_{i=0}^{l} \sigma_i(p^k).$$

Problem 3 (5 pts)**.** Let n be a positive integer and k a natural number. **Show that**

$$\sigma_k(n) = \sigma_{-k}(n)n^k.$$

Conclude that n is *perfect* if and only if $\sigma_{-1}(n) = 2$.

Problem 4. We say that a positive integer n is **square-free** if n is not divisible by p^2 for any prime number p. (E.g. 15 and 37 are square-free, but 24 and 49 are not.) Consider the arithmetic function μ (named after A.F. Möbius, popularly known for his strip) as follows:

$$\mu(n) := \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ is NOT sqaure-free}, \\ (-1)^t & \text{if } n \text{ is sqaure-free and has exactly } t \text{ prime divisors.} \end{cases}$$

(a) (3 pts) **Compute** $\mu(n)$ for $n = 1, \cdots, 15$.

(b) (4 pts) **Prove that** μ is *multiplicative*. That is, $\mu(ab) = \mu(a)\mu(b)$ whenever a, b are *coprime*.

Hint. Proceed by cases, taking cue from the definition of μ.

Problem 5. Let $f(n)$ and $g(n)$ be two arithmetic functions. Define $(f \star g)(n)$ by the formula

$$(f \star g)(n) := \sum_{d|n} f(d)g(\frac{n}{d}),$$

where the summation is taken over the set $\mathscr{D}(n) := \{d \mid d \text{ is a divisor of } n\}$. The new function $f \star g$ is called the **convolution** of f and g. The idea originates from Fourier analysis.

(a) (4 pts) Let id denote the function mapping each positive integer n to itself. **Compute** the values of $(\mathrm{id} \star \mu)(n)$ for $n = 1, \cdots, 12$.

(b) (2 pts) Let δ_1 be the function defined as follows:

$$\delta_1(n) := \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if otherwise.} \end{cases}$$

Prove that $\delta_1 \star f = f \star \delta_1 = f$ for any arithmetic function f. (In other words, δ_1 is the *identity* for the binary operation $\star$.)

(c) (2 pts) Show that $f \star g = g \star f$ for any arithmetic functions f and g. (In other words, the binary operation $\star$ is *commutative*.)

Hint. Show that $d \mapsto \frac{n}{d}$ is a bijection from $\mathscr{D}(n)$ to itself.

(d) (6 pts) Show that $(f \star g) \star h = f \star (g \star h)$ for any arithmetic functions f, g, and h. (In other words, the binary operation $\star$ is *associative*.)

Hint. Define $f \star g \star h$ as follows:

$$(f \star g \star h)(n) := \sum_{abc=n} f(a)g(b)h(c),$$

where the summation is taken over the set $\mathscr{D}_3(n) := \{(a, b, c) \in \mathscr{D}(n)^3 \mid abc = n\}$. Show that each of $(f \star g) \star h$ and $f \star (g \star h)$ is equal to $f \star g \star h$ using a bijective map from its summation index set to $\mathscr{D}_3(n)$.

(At this stage, we see that the set of arithmetic functions equipped with the binary operation $\star$ and the element δ_1 forms a *commutative monoid*.)

(e) (6 pts) Suppose f and g are two multiplicative functions. **Prove that** $f \star g$ is a multiplicative function.

Hint. For any copirme pairs (m, n), use the bijection $\Phi \colon \mathscr{D}(m) \times \mathscr{D}(n) \to \mathscr{D}(mn)$.

(Hence, the subset of *multiplicative* functions forms a *submonoid*.)