Homework 3 (due Feb. 12)

MATH 110 | Introduction to Number Theory | Winter 2023

Whenever you use a result or claim a statement, provide a **justification** or a **proof**, unless it has been covered in the class. In the later case, provide a **citation** (such as "by the *2-out-of-3 principle*" or "by Coro. 0.31 in the textbook").

You are encouraged to *discuss* the problems with your peers. However, you must write the homework **by yourself** using your words and **acknowledge your collaborators**.

Problem 1. For this problem, you may want to review one-variable Calculus

(a) (3 pts) Recall the definition (In this course, $\log = \log_e$ denotes the *natural logarithm*)

$$\mathrm{Li}(x) := \int_2^x \frac{\mathrm{d}t}{\log t} \qquad (x > 2).$$

Question: What is the $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Li}(x)$ of $\mathrm{Li}(x)$?

(b) (5 pts) Two real functions $f(x)$ and $g(x)$ are *asymptotically equal* if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

Prove that: $\mathrm{Li}(x)$ and $\frac{x}{\log x}$ are asymptotically equal.

Problem 2 (5 pts)**.** Let p be a prime number and k, l be two natural numbers. **Show that**

$$\sum_{i=0}^{k} \sigma_i(p^l) = \sum_{i=0}^{l} \sigma_i(p^k).$$

Problem 3 (5 pts)**.** Let n be a positive integer and k a natural number. **Show that**

$$\sigma_k(n) = \sigma_{-k}(n) n^k.$$

Conclude that n is *perfect* if and only if $\sigma_{-1}(n) = 2$.

Problem 4. We say that a positive integer n is **square-free** if n is not divisible by p^2 for any prime number p. (E.g. 15 and 37 are square-free, but 24 and 49 are not.) Consider the arithmetic function μ (named after A.F. Möbius, popularly known for his strip) as follows:

$$\mu(n) := \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ is NOT sqaure-free}, \\ (-1)^t & \text{if } n \text{ is sqaure-free and has exactly } t \text{ prime divisors.} \end{cases}$$

(a) (3 pts) **Compute** $\mu(n)$ for $n = 1, \cdots, 15$.

(b) (4 pts) **Prove that** μ is *multiplicative*. That is, $\mu(ab) = \mu(a)\mu(b)$ whenever a, b are *coprime*.

Hint. Proceed by cases, taking cue from the definition of μ.

Problem 5. Recall that an *integer polynomial* is an expression of the form $P(T) = c_d T^d + \cdots + c_1 T + c_0$, where each c_i is an integer.

(a) (5 pts) **Find** a nonzero integer polynomial $P(T)$ that has $\sqrt{3} + \sqrt[3]{5}$ as a root.
(b) (5 pts) **Prove that** $\sqrt{3} + \sqrt[3]{5}$ is irrational using 5.(a).

Problem 6. By evaluating the Taylor series for the exponential function:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

at $x = 1$, we get the formula

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots .$$

In this problem, you will prove that e is *irrational*.

(a) (5 pts) Let $s_n := \sum_{k=0}^{n} \frac{1}{k!}$, the n-th partial sum of above series. **Show that**

$$0 \leq e - s_n \leq \frac{1}{n} \cdot \frac{1}{n!}.$$

(b) (5 pts) Assume e is rational, and say a/b is the reduced fraction representing e. Apply the previous result to $n = b$ and arrive at a contradiction.