Algebraic Geometry Note Series

Henselian local rings

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Abstract

This note is about many characterizations of henselian local rings.

Contents

§ 1 Hensel's lemma for polynomials

Let $(R, \mathfrak{m}, \kappa)$ be a local ring (with its maximal ideal and residue field). When passing from R to κ, we denote the image by putting a line above.

The first set of conditions are about polynomials, clearly motivated by the usual Hensel's lemma.

- **P**1 For any polynomial $f \in R[T]$ and any simple root $\alpha_0 \in \kappa$ of $\overline{f}$, there exists a root $\alpha \in R$ of f such that $\alpha_0 = \bar{\alpha}$.
- **P**2 For any polynomial $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ over κ with g_0, h_0 coprime, there exists a factorization $f = gh$ over R such that $g_0 = \bar{g}$ and $h_0 = \overline{h}$.
- **P**3 For any polynomial $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ over κ with g_0, h_0 coprime, there exists a factorization $f = gh$ over R such that $g_0 = \bar{g}$, $h_0 = \overline{h}$ and moreover $\deg(g) = \deg(g_0)$.

Clearly, we have

$$\mathbf{P}3 \implies \mathbf{P}2, \mathbf{P}1.$$

1.1 Lemma. *Assuming **P1**, the root α is necessarily simple and is unique.*

Proof. It is clear that α is a simple root. Suppose β is another root such that $\alpha_0 = \overline{\beta}$. Then $0 = f(\alpha) - f(\beta) = f'(\alpha)(\alpha - \beta) + C(\alpha - \beta)^2$ for some $C \in R$. Since α is a simple root, $f'(\alpha)$ is a unit. Then we get $(\alpha - \beta)(1 + f'(\alpha)^{-1}C(\alpha - \beta)) = 0$. By assumption, $\alpha - \beta \in \mathfrak{m}$, hence $1 + f'(\alpha)^{-1}C(\alpha - \beta)$ is a unit. Then we have $\alpha - \beta = 0$. $\square$

Currently, we only consider the following weak versions.

- **P**'1 For any monic polynomial $f \in R[T]$ and any simple root $\alpha_0 \in \kappa$ of $\overline{f}$, there exists a root $\alpha \in R$ of f such that $\alpha_0 = \bar{\alpha}$.
- **P**'2 For any monic polynomial $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ over κ with g_0, h_0 coprime, there exists a factorization $f = gh$ over R such that $g_0 = \bar{g}$ and $h_0 = \overline{h}$.
- **P**'3 For any monic polynomial $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ over κ with g_0, h_0 coprime, there exists a factorization $f = gh$ over R such that $g_0 = \bar{g}$, $h_0 = \overline{h}$ and moreover $\deg(g) = \deg(g_0)$.
- **P**'4 For any monic polynomial $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ over κ with g_0, h_0 coprime, there exists a factorization $f = gh$ over R such that $g_0 = \bar{g}$, $h_0 = \overline{h}$ and moreover g, h are coprime.

Note that g, h *are coprime* means g, h generate the unit ideal.

1.2 Lemma. ***P'2**, **P'3** and **P'4** are equivalent.*

Proof. The conditions $f = gh$ and g, h *are coprime* implies the following factorization via *Chinese Remainder Theorem*:

$$R[T]/(f) = R[T]/(g) \times R[T]/(h).$$

Passing to the residue field κ, the facts $g_0 = \bar{g}$ and $h_0 = \bar{h}$ show that the above is a lift of the following factorization

$$\kappa[T]/(\bar{f}) = \kappa[T]/(g_0) \times \kappa[T]/(h_0).$$

By Nakayama's lemma, the ranks of the free R-module $R[T]/(g)$ and the free κ-module $\kappa[T]/(g_0)$ are equal. Hence $\deg(g) = \deg(g_0)$.

Conversely, suppose (g, h) is a lift of (g_0, h_0) such that $f = gh$. Consider the ideal $\mathfrak{a}$ of $A = R[T]/(f)$ generated by g, h. Then we have $\bar{\mathfrak{a}} = \bar{A}$ since g_0, h_0 are coprime. Since A is a finite R-module, by Nakayama's lemma, $\mathfrak{a} = A$ as well. This shows g, h are coprime. $\square$

Clearly, **P**3 implies **P'**3 and then implies **P'**1. Therefore, to show the equivalence of previous seven conditions, it is sufficient to show:

$$\mathbf{P'}1 \implies \mathbf{P}3.$$

§ 2 Factorization of finite algebras

The next set of conditions are about finite R-algebras. Recall that a finite R-algebra means an R-algebra which is finite as an R-module, and a finite free R-algebra means an R-algebra which is finitely free as an R-module.

- **F**1 For any monic polynomial $f \in R[T]$, the finite free R-algebra $R[T]/(f)$ is a product of local rings.
- **F**2 Any finite free R-algebra is a product of local rings.
- **F**3 Any finite R-algebra is a product of local rings.

Clearly, we have

$$\mathbf{F}3 \implies \mathbf{F}2 \implies \mathbf{F}1.$$

As for the relation of them with those in first section, we have

2.1 Lemma. *The condition **F1** implies **P'3**.*

Proof. Let f, g_0 and h_0 be as in the hypothesis in **P'**3. Let $A = R[T]/(f)$. Over the field κ, we have

$$\bar{A} = \kappa[T]/(g_0) \times \kappa[T]/(h_0).$$

But by if we already have $A = \prod_i A_i$ with each A_i a local R-algebra, we have also a decomposition $\bar{A} = \prod_i \bar{A_i}$ with each $\bar{A_i}$ an Artinian local ring. Then,

$\kappa[T]/(g_0)$ and $\kappa[T]/(h_0)$ must be some products of those $\overline{A_i}$. We may assume $\kappa[T]/(g_0) = \overline{A_1} \times \cdots \times \overline{A_n}$. Then for

$$B = A_1 \times \cdots \times A_n,$$

we have $\overline{B} = \kappa[T]/(g_0)$.

Observations: B is a direct factor of the finite free R-module A, hence is projective and is therefore free since R is local. By Nakayama's Lemma, its rank equals that of $\overline{B}$, i.e. $\deg(g_0)$. Since $\overline{B}$ is invariant under the action of T, by Nakayama's Lemma, B is also T-invariant. Then the characteristic polynomial of T on B, saying g, gives a lift of g_0 with $\deg(g) = \deg(g_0)$. Now the map $R[t]/(g) \to B$ is surjective and hence an isomorphism by Cayley-Hamilton theorem.

The factorization of the map $R[T] \to B$ through A gives the factorization $f = gh$ for some $h \in R[T]$ and we also have $\overline{h} = h_0$ as desired. ☐

Note that, if A is a finite R-algebra, then it is integral and hence any maximal ideal $\mathfrak{n}$ of A is lying above $\mathfrak{m}$. Localization of A at them gives explicit forms of previous ones.

F'1 For any monic polynomial $f \in R[T]$, the finite free R-algebra $R[T]/(f)$ is a product of its localizations at maximal ideals.

F'2 Any finite free R-algebra is a product of its localizations at maximal ideals.

F'3 Any finite R-algebra is a product of its localizations at maximal ideals.

Note that, if A is a finite R-algebra, then by passing to the residue field κ, we get a finite algebra $\overline{A}$ over κ, which is therefore an Artinian ring and contains only finitely many primes ideals. But we have seen that maximal ideals of A are all lying above $\mathfrak{m}$, hence one-one corresponding to maximal ideals of $\overline{A}$ and are finitely many. Therefore, in previous conditions, we can replace "product" by "finite product".

To show the equivalence of those six conditions, we need to show:

$$\mathbf{F}1 \implies \mathbf{F}'3.$$

§ 3 Connected components

For any ring A, we have canonical covering

$$\coprod_{\mathfrak{n} \in \mathrm{Spm}(A)} \mathrm{Spec}(A_{\mathfrak{n}}) \longrightarrow \mathrm{Spec}(A)$$

since any prime ideal is contained in a maximal one. Note that if $\mathrm{Spm}(A)$ is finite, then the above is equivalent to a monomorphism

$$A \longrightarrow \prod_{\mathfrak{n} \in \mathrm{Spm}(A)} A_{\mathfrak{n}}.$$

In the case of Artinian ring, we have

3.1 Lemma. *If $\overline{A}$ is an Artinian ring. Then it is the product of its localizations at all maximal ideals. Using geometric terminology, this means that* $\operatorname{Spec}(\overline{A})$ *is a finite discrete set.*

Proof. Let $\mathfrak{N}$ be the nilradical of $\overline{A}$. Then the inclusion $\operatorname{Spec}(\overline{A}/\mathfrak{N}) \hookrightarrow \operatorname{Spec}(\overline{A})$ is a homeomoprhism. Note that: all prime ideals of $\overline{A}$ are maximal ideals. Hence by *Chinese Remainder Theorem*, $\overline{A}/\mathfrak{N}$ is isomorphic to the product of residue fields of $\overline{A}$. □

Now, back to our situation, A is a finite R-algebra. We have the following commutative diagram of canonical maps.

$$\begin{tikzcd} \coprod_{\mathfrak{n}} \operatorname{Spec}(\overline{A}_{\mathfrak{n}}) \arrow[r, hook] \arrow[d, equal] & \coprod_{\mathfrak{n}} \operatorname{Spec}(A_{\mathfrak{n}}) \arrow[d] \\ \operatorname{Spec}(\overline{A}) \arrow[r, hook] & \operatorname{Spec}(A) \end{tikzcd}$$

Note that each $\operatorname{Spec}(A_{\mathfrak{n}})$ is connected. Hence it follows that the canonical map

$$\phi \colon \pi_0(\operatorname{Spec}(\overline{A})) \longrightarrow \pi_0(\operatorname{Spec}(A))$$

is surjective by looking at the upper arrow. Moreover, ϕ is bijective if and only if the right vertical covering has no overlaps and hence is an isomorphism.

We therefore translate the conditions **F**'1, **F**'2 and **F**'3 into the followings.

C1 For any monic polynomial $f \in R[T]$ and the R-algebra $A = R[T]/(f)$, the canonical map ϕ is bijective.

C2 For any finite free R-algebra A, the canonical map ϕ is bijective.

C3 For any finite R-algebra A, the canonical map ϕ is bijective.

§ 4 Correspondence of idempotents

The next set of equivalent conditions are about idempotents. Recall that an element a of a ring A is said to be idempotent if $a^2 = a$. Use $\operatorname{Idem}(A)$ to denote the set of idempotents in A.

For any finite R-algebra A, the canonical map $A \to \overline{A}$ maps idempotents to idempotents. Hence we have a canonical map

$$\Phi \colon \operatorname{Idem}(A) \longrightarrow \operatorname{Idem}(\overline{A}).$$

4.1 Lemma. *If A is a finite R-algebra. Then the canonical map Φ is injective.*

Proof. Suppose a, b are two idempotents in A such that $\bar{a} = \bar{b}$. Note that

$$(a-b)^3 = a^3 - 3a^b + 3ab^2 - b^3 = a - 3ab + 3ab - b = a - b.$$

Hence $(a-b)(1-(a-b)^2) = 0$. Since $a - b \in \mathfrak{m}A \subset \mathfrak{n}$ for any maximal ideal $\mathfrak{n}$ of A, we can conclude that $1 - (a-b)^2$ is a unit. Hence $a = b$. □

Now the next set of conditions are

I1 For any monic polynomial $f \in R[T]$ and the R-algebra $A = R[T]/(f)$, the canonical map Φ is bijective.

I2 For any finite free R-algebra A, the canonical map Φ is bijective.

I3 For any finite R-algebra A, the canonical map Φ is bijective.

We have the following geometric interpretation of the set $\mathrm{Idem}(A)$.

4.2 Lemma. *Let A be a ring. Then the map $a \mapsto D(a)$ gives a one-one correspondence between* $\mathrm{Idem}(A)$ *and the set of clopen subsets of* $\mathrm{Spec}(A)$*.*

Proof. Because $D(1-a)$ is precisely the complement of $D(a)$. □

Now, we have translated the canonical map Φ in geometric terminology:

$$\Phi\colon \{\text{clopen subsets of } \mathrm{Spec}(A)\} \longrightarrow \{\text{clopen subsets of } \mathrm{Spec}(\overline{A})\}.$$

Look back to the canonical canonical covering

$$\coprod_{\mathfrak{n}\in \mathrm{Spm}(A)} \mathrm{Spec}(A_{\mathfrak{n}}) \longrightarrow \mathrm{Spec}(A).$$

When $\mathrm{Spm}(A)$ is finite, its cardinal gives an upper bound of the number of connected components of $\mathrm{Spec}(A)$. In particular, this number is finite. Note that, if this is the case, connected components are precisely the minimal clopen subsets. Therefore the set of clopen subset is precisely the power set of the set of connected components and hence Φ is preciesly ϕ^*. Consequently, Φ is bijective if and only if ϕ is bijective.

4.3 Lemma. *The condition **C1** implies **I3**.*

Proof. Let A be a finite R-algebra. For any $\bar{e} \in \mathrm{Idem}(\overline{A})$, we need to find a preimage $e \in \Phi^{-1}(\bar{e})$.

Let $a \in A$ be an arbitrary lift of $\bar{e}$ and let $A' = R[a]$. The canonical inclusion $A' \subset A$ induces a canonical map

$$\mathrm{Spec}(\overline{A}) \longrightarrow \mathrm{Spec}(\overline{A'}).$$

Moreover, the preimage of $D(\bar{a})$ under it is precisely $D(\bar{e})$. Note that $\mathrm{Spec}(\overline{A'})$ is discrete. Hence $D(\bar{a})$ is clopen and there is an idempotent $\bar{e}'$ in $\overline{A'}$ such that its image in $\overline{A}$ is $\bar{e}$. Now, it suffices to find a preimage of $\bar{e}'$ under Φ.

Therefore, we may assume $A = R[a]$. Let $f \in R[T]$ be the minimal polynomial of a. Then A is a quotient of the R-algebra $R[T]/(f)$. Consequently, $\mathrm{Spec}(A)$ is a closed subscheme of $\mathrm{Spec}(R[T]/(f))$. Looking at the canonical covering of them, we are done. □

§ 5 Étale morphisms

We begin with the following lemma.

5.1 Lemma. *Let A be a finite free R-algebra. Then the functor* $\mathrm{Idem}(A \otimes_R -)$ *is represented by an étale R-algebra R_A.*

Proof. Let $e_1, e \cdots, e_n$ be a basis of A. Suppose the multiplication of A are given by

$$e_i e_j = \sum_{k=1}^{n} c_{ijk} e_k, \qquad \forall i, j = 1, \cdots, n.$$

Then an element $a = \sum a_i e_i$ of A is idempotent if and only if

$$\sum_{i,j=1}^{n} c_{ijk} a_i a_j - a_k = 0, \qquad \forall k = 1, \cdots, n.$$

Let P_k be the polynomial $\sum_{i,j=1}^{n} c_{ijk} T_i T_j - T_k$. Then for any R-algebra A', the idempotents of $A \otimes_R A'$ are precisely the A'-solutions of the system of polynomials $\{P_1, \cdots, P_n\}$. Therefore the functor $\mathrm{Idem}(A \otimes_R -)$ is represented by the R-algebra $R_A = R[T_1, \cdots, T_n]/(f_1, \cdots, f_n)$.

It remains to show R_A is étale over R. Since $R \to R_A$ is of finite presentation, it suffices to show it is foramlly étale: for any R-algebra A' and any square-zero ideal I of A', the canonical map

$$\mathrm{Hom}_R(R_A, A') \longrightarrow \mathrm{Hom}_R(R_A, A'/I)$$

is bijective. But this map is identified with

$$\mathrm{Idem}(A \otimes_R A') \longrightarrow \mathrm{Idem}(A \otimes_R A'/I).$$

Hence it suffices to show the canonical map

$$\pi_0(\mathrm{Spec}(A \otimes_R A'/I)) \longrightarrow \pi_0(\mathrm{Spec}(A \otimes_R A'))$$

is bijective. Indeed, we have

$$\mathrm{Spec}(A \otimes_R A'/I) \cong \mathrm{Spec}(A \otimes_R A')$$

since I is nilpotent. □

From now on, let $S = \mathrm{Spec}(R)$ and s be its closed point. Then $s = \mathrm{Spec}(\kappa)$ in particular. From the above lemma, we know that $\mathrm{Idem}(A) = \mathrm{Hom}_R(R_A, R)$. In geometric terminology, the idempotents are identified with sections of the étale morphism

$$\mathrm{Spec}(R_A) \longrightarrow S.$$

On the otherhand, we have $\mathrm{Idem}(\overline{A}) = \mathrm{Hom}_R(R_A, \kappa)$. Hence, the idempotents of $\overline{A}$ are identified with κ-points of $\mathrm{Spec}(R_A)$ lying above s.

This motivates the next set of conditions.

E1 For any étale morphism $g\colon X \to S$ and any point κ-point x of X lying above s, there is a section of g.

$$\begin{tikzcd}
 & X \arrow[d, "g"'] \\
\operatorname{Spec}(\kappa) \arrow[ur, "x"] \arrow[r, "s"] & S \arrow[u, dotted, bend right]
\end{tikzcd}$$

E2 For any étale morphism $g\colon X \to S$, any section of g_s induces one of g.

$$\begin{tikzcd}
X_s \arrow[r] \arrow[d, "g_s"] & X \arrow[d, "g"'] \\
\operatorname{Spec}(\kappa) \arrow[u, bend left] \arrow[r, "s"] & S \arrow[u, dotted, bend right]
\end{tikzcd}$$

5.2 Lemma. *The conditions **E1** and **E2** are equivalent.*

Proof. This follows from the universal property of fibred product. □

§ 6 Equivalence of conditions

At this stage, we already have the following implications.

$$\begin{tikzcd}
\mathbf{E}1 \arrow[r, Leftrightarrow] \arrow[dr, Rightarrow] & \mathbf{E}2 & & \\
\mathbf{I}1 \arrow[d, Leftrightarrow] & \mathbf{I}2 \arrow[l, Rightarrow] \arrow[d, shift left, Rightarrow] & \mathbf{I}3 \arrow[l, Rightarrow] \arrow[d, Leftrightarrow] & \\
\mathbf{C}1 \arrow[d, Leftrightarrow] \arrow[urr, Rightarrow] & \mathbf{C}2 \arrow[l, Rightarrow] \arrow[u, shift left, Rightarrow] \arrow[d, Leftrightarrow] & \mathbf{C}3 \arrow[l, Rightarrow] \arrow[d, Leftrightarrow] & \\
\mathbf{F}'1 \arrow[d, Rightarrow] & \mathbf{F}'2 \arrow[l, Rightarrow] \arrow[d, Rightarrow] & \mathbf{F}'3 \arrow[l, Rightarrow] \arrow[d, Rightarrow] & \\
\mathbf{F}1 \arrow[dr, Rightarrow] & \mathbf{F}2 \arrow[l, Rightarrow] & \mathbf{F}3 \arrow[l, Rightarrow] & \\
\mathbf{P}'1 & \mathbf{P}'3 \arrow[l, Rightarrow] \arrow[rr, Rightarrow, bend right] & \mathbf{P}'4 \arrow[l, Rightarrow] & \mathbf{P}'2 \arrow[l, Rightarrow]
\end{tikzcd}$$

It suffices to show

$$\mathbf{P}'1 \implies \mathbf{E}2.$$

Before doing that, let's consider *what is a section of g_s?* Note that since g is étale, so is g_s. First we have

6.1 Lemma. *Any étale algebra over a field κ is a finite product of separable extensions of κ.*

Proof. Let B be an étale κ-algebra. Then $B^a := B \otimes_\kappa \kappa^a$ is étale over κ^a. Suppose B^a is a finite product of copies of κ^a. Then $\mathrm{Spec}(B^a)$ is a finite discrete space and hence so is $\mathrm{Spec}(B)$. Therefore B is a finite product of Artinian local rings. Note that each factor of B^a is a field, hence so is B. More precisely, B is a finite product of algebraic extensions λ_i of κ. But since B is étale over κ, each of those λ_i is étale over κ and hence must be separable over κ.

Therefore we may assume $\kappa = \kappa^a$. Since $\mathrm{Spec}(B)$ is smooth at each closed point, we see that each localization $B_\mathfrak{n}$ at a maximal ideal is a regular local ring. In particular, B is a domain. Since $\Omega^1_{B/\kappa} = 0$, we see that at each closed point $\mathfrak{n}$, the cotangent space $\mathfrak{n}/\mathfrak{n}^2$ vanishes. Therefore $B_\mathfrak{n} = \kappa$ and hence B is a finite product of copies of κ. $\square$

To relate a étale morphism to **P'1**, we need the following lemma.

6.2 Lemma (Chevalley's Theorem)**.** *Let $R \to B$ be an étale local homomorphism. Then there is a monic polynomial $f \in R[T]$ and a maximal ideal $\mathfrak{n}$ of the finite R-algebra $A = R[T]/(f)$ such that $f' \notin \mathfrak{n}$ and that $B \cong A_\mathfrak{n}$.*

Proof. By *Zariski Main Theorem*, we can factor the morphism $\mathrm{Spec}(B) \to S$ into an open immersion $\mathrm{Spec}(B) \hookrightarrow \mathrm{Spec}(A)$ and a finite morphism $\mathrm{Spec}(A) \to S$. Let $\mathfrak{n}$ be the maximal ideal of A corresponding to the one of B. Note that then we have $B \cong A_\mathfrak{n}$.

Then, $A/\mathfrak{n}$ equals the residue field of B, which is a finite separable extension of κ by Lemma 6.1. Then, by *Chinese Remainder Theorem*, we can find an element $a \in A$ contained in all maximal ideals except $\mathfrak{n}$ and its image in $A/\mathfrak{n}$ is a primitive element. Let $\mathfrak{n}' = \mathfrak{n} \cap R[a]$. Then the inclusion $R[a] \hookrightarrow A$ induces an isomorphism $R[a]_{\mathfrak{n}'} \cong A_\mathfrak{n}$. Therefore, we may replace the pair $(A, \mathfrak{n})$ by $(R[a], \mathfrak{n}')$.

Note that $\bar{a} \in \bar{A}$ is a generator. Hence we can find a monic polynomial $f \in R[T]$ such that $\bar{f} \in \kappa[T]$ is a minimal polynomial of $\bar{a}$. Moreover, since $\bar{A}_\mathfrak{n}$ is étale over κ, we have $\bar{f}'(\bar{a}) \notin \mathfrak{n}\bar{A}$ and hence $f'(a) \notin \mathfrak{n}$.

Now, we have a surjective R-homomorphism

$$\varphi\colon R[T]/(f) \longrightarrow A, \qquad T \mapsto a.$$

Look at the composition of S-morphisms

$$\mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(R[T]/(f)) \longrightarrow S.$$

The entire composition is étale at $\mathfrak{n}$ and the right one is étale at $\varphi^{-1}(\mathfrak{n})$ since $f' \notin \varphi^{-1}(\mathfrak{n})$. Therefore the local homomorphism

$$A_\mathfrak{n} \longrightarrow B'_{\mathfrak{n}'}$$

is étale and in particular flat. Then it is faithfully flat and hence injective. But it is automatically surjective. Hence it is an isomorphism and we are done by replace the pair $(A, \mathfrak{n})$ by $(R[T]/(f), \varphi^{-1}(\mathfrak{n}))$. $\square$

Now, we can prove the equivalence.

6.3 Theorem. *The condition* ***P'1*** *implies* ***E2****.*

Proof. Suppose we have a section of g_s and its image in X is x. Then a section of $\mathrm{Spec}(\mathscr{O}_{X,x}) \to S$ gives rise to a section of g. Therefore, we may assume X is local. By Chevalley's Theorem, $X = \mathrm{Spec}(A_{\mathfrak{n}})$ where $A = R[T]/(f)$, f is a monic polynomial and $f' \notin \mathfrak{n}$. Then $X_s = \mathrm{Spec}(\overline{A}_{\mathfrak{n}})$ and a section of g_s gives rise to a simple root of $\overline{f}$ in κ. By **P'1**, it can be lifted to a root $\alpha \in R$ of f. Then $T \mapsto \alpha$ defines a R-homomorphism

$$A_{\mathfrak{n}} = (R[T]/(f))_{\mathfrak{n}} \longrightarrow R$$

and hence a S-morphism

$$S \longrightarrow \mathrm{Spec}(A_{\mathfrak{n}}) = X,$$

i.e. a section of g. $\square$

6.4 Lemma. *Étale morphisms are flat.*

6.5 Lemma. *If a local homomorphism is flat, it is faithfully flat.*

6.6 Lemma. *Flat homomorphisms are injcetive.*