§ 1 Review on Bruhat-Tits buildings

We refer [Rou09] for a survey of general theory of Euclidean buildings, [Mil17; Spr98; SGA3] for the theory of reductive groups, [Tit74] for the theory of Tits buildings, [RTW15] for a short review on Bruhat-Tits theory, [Tit79] together with [Yu09] a survey of Bruhat-Tits theory and [BT72; BT84a; BT84b; BT87] for the origin of the thoery.

Contents

1.1 Apartments and buildings

In this subsection, some basic notions and facts on apartments and buildings will be exposited.

1.1.1 Definition. An *apartment* $\mathscr{A}$ is an Euclidean affine space $\mathbb{A}$ equipped with a reflection group W on it.

Let $\mathbb{A}$ be an Euclidean affine space and ${}^v\mathbb{A}$ its associated vector space. The *vectorial part* of an affine transformation f on $\mathbb{A}$ is denoted by ${}^vf$. The *direction* of an affine subspace X is denoted by ${}^vX$.

A *reflection* on $\mathbb{A}$ is an affine isometry whose fix points form a hyperplane. Any hyperplane H is associated with a reflection r_H with respect to it.

A *reflection group* W is a group of affine isometries generated by reflections and such that its vectorial part ${}^vW$ is finite. W is said to be *irreducible* if ${}^vW$ acts irreducibly on ${}^v\mathbb{A}$ and is said to be *essential* if ${}^vW$ acts essentially on ${}^v\mathbb{A}$ (that is, there is no non-trivial fixed point). An apartment is said to be *irreducible* (resp. *essential*, *trivial*, etc.) if its reflection group is so.

1.1.2. A *morphism* bewteen apartments $(\mathbb{A}, W)$ and $(\mathbb{A}', W')$ is a continuous affine map $f\colon \mathbb{A} \to \mathbb{A}'$ with a group homomorphism $\phi\colon W \to W'$ such that $\phi(w).f(x) = f(w.x)$ for all $w \in W$ and $x \in \mathbb{A}$. Therefore an apartment $(\mathbb{A}, W)$ is said to be a *product* of apartments $(\mathbb{A}_1, W_1)$ and $(\mathbb{A}_2, W_2)$ if $\mathbb{A}$ is the orthogonal direct product of $\mathbb{A}_1$ and $\mathbb{A}_2$ and W admits a decomposition $W = W_1 \times W_2$ such that W_i acts only on $\mathbb{A}_i$.

Any apartment $\mathscr{A}$ admits a decomposition [Bou02, chap.V, §3, no.8]

$$\mathscr{A} = \mathscr{A}_0 \times \mathscr{A}_1 \times \cdots \times \mathscr{A}_m,$$

where $\mathscr{A}_0$ is trivial and each $\mathscr{A}_i$ (for $1 \leqslant i \leqslant m$) is irreducible.

1.1.3. Let $\mathscr{A}$ be an apartment with an essential irreducible reflection group W. Let $T = \ker(W \to {}^vW)$ be the translation group. There are three possibilities [Rou09, 3.3]:

1. If T is trivial, then $\mathscr{A}$ is said to be *spherical*.
2. If T is a lattice in ${}^v\mathbb{A}$, then $\mathscr{A}$ is said to be *discrete affine*.
3. If T is dense in ${}^v\mathbb{A}$, then $\mathscr{A}$ is said to be *dense affine*.

Throughout this draft, all apartments are assumed to be *discrete*, that is no irreducible component is dense affine.

1.1.4. Let $\mathscr{A} = (\mathbb{A}, W)$ be an apartment. A *wall* of W is the hyperplane of fixed points of a reflection in W. The set $\mathcal{H}$ of walls is stable under W and completely determines it.

A *half-apartment* (also called an *affine root* in [BT72, 1.3.3]) is a closed half-space α of $\mathbb{A}$ bounded by a wall $\partial\alpha$, called its wall.

A *facet* in $\mathscr{A}$ is an equivalence class in the complement of fixed points of W in $\mathbb{A}$ for the relation "x and y are contained in the same half-apartments". A facet F is an open convex subset of in the affine subspace (called the *support* of F) it spans.

The set $\mathcal{F}$ of facets admits an order: a facet F is said to be a *face* of another F', denoted by $F \leqslant F'$, if F is contained in the closure of F'. Such an order gives rise to a polysimplicial complex, see 1.1.9.

1.1.5. The maximal facets are called *chambers* (or *alcoves*). They are the connected components of the complement of the union of all walls in $\mathbb{A}$. Note that the reflection group W acts simply transitively on the set $\mathcal{C}$ of chambers [Bou02, chap.V, §3, no.2, th.1].

Let C be a chamber. Then its closure $\overline{C}$ is a fundamental domain of W in $\mathbb{A}$ [Bou02, chap.V, §3, no.3, th.2] and is the intersection of some half-apartments, whose walls are called the *walls* of C. Moreover, W is generated by the set S of reflections with respect to walls of C and the pair (W, S) is a Coxeter system [Bou02, chap.V, §3, no.2, th.1].

The projection of C on an irreducible component $\mathscr{A}_i$ is a chamber in it and induces an irreducible Coxeter system (W_i, S_i). Then (W, S) is the product of them: that is $W = W_1 \times \cdots \times W_m$ and $S = S_1 \cup \cdots \cup S_m$.

Fix a set $\mathfrak{I}$ of numbers having the same cardinality with S. Then a *type function* on $\mathscr{A}$ is an antitone map τ from the poset $\mathcal{F}$ of facets to the power set of $\mathfrak{I}$ such that for any facet F and any $w \in W$, $\tau(F) = \tau(w.F)$. The image of this function is denoted by $\mathcal{T}$ and its members are called *types*. Since any facet is transformed by W to a unique face of C, τ is determined completely by how $\mathfrak{I}$ numberings S. Indeed, let I be a type, then the set C_I of points $x \in \overline{C}$ such that the reflections $s \in S$ fixing x are indexed by I, is a face of C whose stabilizer is the subgroup W_I of W generated by the reflections indexed by I [Bou02, chap.V, §3, no.3, prop.1]. Then $\tau(F) = I$ if the facet F is transformed to C_I.

1.1.6. A reflection group W is said to be *linear* if it fixes a point. This is the case if and only if W is finite [Bou02, chap.V, §3, no.9]. If so, we can identify W with its vectorial part ${}^vW$.

Conversely, ${}^vW$ can be viewed as a linear relfection group on ${}^v\mathbb{A}$. The spherical apartment ${}^v\mathscr{A} = ({}^v\mathbb{A}, {}^vW)$ is called the *vectorial apartment* of $\mathscr{A}$ and we have the following associated notions.

${}^v\mathcal{H}$ is the set of *vectorial walls*, i.e. walls of ${}^vW$. They are precisely the directions of walls in $\mathscr{A}$;

${}^v\mathcal{F}$ is the set of *vectorial facets*, i.e. facets in ${}^v\mathscr{A}$.

${}^v\mathcal{C}$ is the set of *vectorial chambers*, i.e. chambers in ${}^v\mathscr{A}$.

1.1.7. The minimal facets are called *vertices*. The set of vertices is denoted by $\mathcal{V}$. When the action of W is essential, they are points. From now on, the reflection groups are always assumed to be essential unless otherwise specified.

A point $x \in \mathbb{A}$ is said to be *special* if the set $\mathcal{H}_x$ of walls passing through x is a complete set of representatives of ${}^v\mathcal{H}$. Every special point is a vertex and is an extremal point of the closure of some chamber, and conversely, any chamber admits a special point as an extremal point of its closure [Bou02, chap.V, §3, no.10, prop.11's cor.]. However, not all extremal points, hence not all vertices are special.

1.1.8. Let x be a point in $\mathscr{A}$. The stabilizer W_x of x is a linear reflection group whose vectorial part ${}^vW_x$ is a subgroup of ${}^vW$. The apartment $\mathscr{A}_x = (\mathbb{A}, W_x)$ is called the *spherical apartment at x* and we have the following associated notations:

$\mathcal{H}_x$ is the set of walls in $\mathscr{A}_x$. They are walls in $\mathscr{A}$ passing through x.

$\mathcal{F}_x$ is the set of facets in $\mathscr{A}_x$. It is a subset of $x + {}^v\mathcal{F}$ and its elements are called *vectorial facets with base point x*.

$\mathcal{C}_x$ is the set of chambers in $\mathscr{A}_x$. It is a subset of $x + {}^v\mathcal{C}$ and its elements are called *vectorial chambers with base point x*.

1.1.9. An *abstract simplicial complex* $\mathcal{S}$ with vertices $\mathcal{V}$ is a poset (whose members are called *simplices*) of finite nonempty subsets of $\mathcal{V}$ such that:

1. if σ is a simplex and τ is a nonempty subset of it, then τ is also a simplex (called a *face* of σ);
2. every singleton is a simplex.

A simplex having $k+1$ elements is called a *k-simplex.* The set of k-simplices is denoted by $\mathcal{S}_k$. The *dimension* of $\mathcal{S}$ is the supremum of dimensions of its simplices.

A *simplicial complex* is a poset isomorphic to an abstract one. The facets in an irreducible apartment give rise to an example. For a discrete affine one, its facets triangulate the ambient space hence form a simplicial complex. For a spherical one, its facets triangulate the unit sphere hence form a simplicial complex. This is why they are called spherical apartments.

A *polysimplicial complex* is a Cartesian product of simplicial complexes. The facets in an apartment are compatible with its decomposition into irreducible components hence give rise to a polysimplicial complex.

1.1.10 Definition. A *building* is a set $\mathscr{B}$ equipped with a polysimplicial complex $\mathcal{F}$, whose members are subsets of $\mathscr{B}$ and are called *facets*, and a family $\mathcal{A}$ of subsets of $\mathscr{B}$, whose members are called *apartments*, such that the following axioms are satisfied.

B0. For each $A \in \mathcal{A}$, there is an (abstract) apartment $\mathscr{A}$ together with a bijection between them, exchanging the poset $\mathcal{F}_A$ of facets contained in A and the poset of facets in $\mathscr{A}$.

Note that, this means we can view each A as an affine Euclidean space and hence it makes sense to talk about its metric and the closure of its subset.

B1. For any two facets, there is an apartment containing them.

B2. For any two apartments, their intersection is a union of facets.

B3. For any $F, F' \in \mathcal{F}$ such that $F, F' \subseteq A \cap A'$ for some $A, A' \in \mathcal{A}$, there is an isomorphism bewteen A and A' pointwisely fixing $\overline{F}$ and $\overline{F'}$.

Here an isomorphism bewteen A and A' is an isometry between them exchanging the posets $\mathcal{F}_A$ and $\mathcal{F}_{A'}$.

Note that, from the definition, all apartments $A \in \mathcal{A}$ are isomorphic to an abstract one $\mathscr{A}$. Then $\mathscr{B}$ is said to be *of type* $\mathscr{A}$ and is said to be *spherical* (resp. *discrete affine*, etc.) if so is $\mathscr{A}$.

Remark. We have assumed the reflection is essential. In particular, the buildings in Bruhat-Tits theory used in this draft are the *reduced buildings*, rather than *extended buildings.* However, this is harmless as we focus more on the polysimplicial structure and want the ambient space being reduced.

Remark. One can see that the combinatorial information of $\mathscr{B}$ is encoded in the polysimplicial complex $\mathcal{F}$ and hence is determined completely by it up to a choice of the family $\mathcal{A}$. In this since, a building is essentially a polysimplicial complex (together with a family of subcomplexes called apartments) satisfying axioms **B1.–B3.** in polysimplicial complex version.

Remark. The notions of *chambers*, *vertices* and *types* in a building is defined similarly as in an apartment and we will use the same notations as there. Moreover, there is a *type function* $\tau\colon \mathcal{F} \to \mathcal{T}$ extending the type function on an apartment to the entire building uniquely.

1.1.11. [Rou09, 6.5] One can extend the metric on an apartment to the entire building $\mathscr{B}$ in a consistent way. Let $d(-,-)$ denote this metric, then $\mathscr{B}$ equipped with it is a complete metric space having the *CAT(0)-property*, which means that geodesic triangles in $\mathscr{B}$ are at least as thin as in Euclidean planes: saying x, y, z are three points in $\mathscr{B}$ forming a geodesic triangle and $\bar{x}, \bar{y}, \bar{z}$ are three points in an Euclidean plane having the same pointwise distance as x, y, z, then for any point m in the geodesic segment $[x, y]$ in the triangle and $\bar{m}$ the corresponding point in the segment $[\bar{x}, \bar{y}]$ (namely, $d(\bar{x}, \bar{m}) = d(x, m)$), then $d(z, m) \leqslant d(\bar{z}, \bar{m})$. Consequences of this property include: the geodesic segments between points are unique [Rou09, 6.6]; any group of isometries stabilizing a nonempty bounded subset has a fixed point [Rou09, 7.1]; the distance from a point to a nonempty closed convex subset is achieved by a unique point [Rou09, 7.2].

1.1.12. An *morphism* between buildings $\mathscr{B}$ and $\mathscr{B}'$ is a continuous map inducing a *chamber map* between $\mathcal{F}$ and $\mathcal{F}'$, that is a monotone map mapping chambers to chambers, and mapping apartments in apartments. Then an *automorphism* of a building is an isometry transforming a facet (resp. apartment) in a facet (resp. apartment).

A group of automorphisms is said to be *strongly transitive* if it acts transitively on the pairs (C, A) where C is a chamber in the apartment A. An automorphism is said to be *type-preserving* if it leaves the type function τ invariant. For instance, any $w \in W$ is such an automorphism.

[Rou09, 6.8 and 6.9] Let G be a strongly transitive and type-preserving group of automorphisms. Then $W \cong N_G(A)/C_G(A)$ where $N_G(A)$ (resp. $C_G(A)$) is the stabilizer (resp. fixator) of an apartment A in G. The stabilizers (which are also fixators) $G_F := N_G(F) = C_G(F)$ of facets are called *parabolic subgroups* of G. Let F be a facet in an apartment A. Then the parabolic group G_F is transitive on the apartments containing F and we have the *Bruhat decomposition* $G = G_F N_G(A) G_F$. Moreover, if $F = C$ is a chamber, then

$$G = \bigsqcup_{w \in W} G_C w G_C.$$

1.2 Root systems

In this subsection, spherical apartments arising from root systems (as well as root data) will be exposited.

1.2.1. Let V be an Euclidean vector space and V^* its dual space. For any $a \in V^* \setminus \{0\}$, let r_a be the reflection with respect to the hyperplane $H_a := \mathrm{Ker}\,(a)$ and $a^\vee$ the vector orthogonal to H_a and satisfying $a(a^\vee) = 2$. So for $x \in V$, we have $r_a(x) = x - a(x)a^\vee$. Note that r_a also induces a reflection on V^*, namely $f \mapsto f - f(a^\vee)a$. A finite spanning subset $\Phi \subseteq V^* \setminus \{0\}$ is called a *(reduced) root system* on V if

1. for any $a \in \Phi$, $r_a(\Phi) = \Phi$;
2. for any $a, b \in \Phi$, $a(b^\vee) \in \mathbb{Z}$;
3. for any $a \in \Phi$, $\mathbb{R}a \cap \Phi = \{\pm a\}$.

Elements of Φ are called *roots* in Φ. For a root $a \in \Phi$, the vector $a^\vee$ is called its *coroot*. A subset $\Psi \subseteq \Phi$ is called a *subroot system* if for any $a \in \Psi$, $r_a(\Psi) = \Psi$, and is said to be *closed* if for any $a, b \in \Psi$ such that $a + b$ is a root, $a + b \in \Psi$.

Any root system Φ admits a *Weyl group* ${}^vW(\Phi)$, that is the reflection group of V generated by r_a for $a \in \Phi$. It is a linear reflection group with walls H_a for $a \in \Phi$. In this way, we get a spherical apartment ${}^v\mathscr{A}\,(\Phi) := (V, {}^vW(\Phi))$. Note that not all spherical apartments arise in this way and non-isomorphic root systems may have isomorphic Weyl groups.

Root systems can arise from root data.

1.2.2 Definition. A *(reduced) root datum*[1] $\mathcal{R}$ is a quadruple $(\mathrm{X}, \Phi, \mathrm{X}^\vee, \Phi^\vee)$ in which

- X and $\mathrm{X}^\vee$ are free $\mathbb{Z}$-modules of finite rank in duality by a pairing
$$\langle\,\cdot\,,\,\cdot\,\rangle\colon\ \mathrm{X} \times \mathrm{X}^\vee \to \mathbb{Z},$$
- Φ and $\Phi^\vee$ are finite subsets of $\mathrm{X} \setminus \{0\}$ and $\mathrm{X}^\vee \setminus \{0\}$ respectively, in bijection by a correspondence $a \leftrightarrow a^\vee$,

satisfying

1. for any $a \in \Phi$, $\langle a, a^\vee \rangle = 2$;
2. for any $a \in \Phi$, the "reflection" $r_a\colon x \mapsto x - \langle x, a^\vee \rangle a$ preserves Φ and the "reflection" $r_a\colon y \mapsto y - \langle a, y \rangle a^\vee$ preserves $\Phi^\vee$;
3. for any $a \in \Phi$, $\mathbb{Z}a \cap \Phi = \{\pm a\}$.

[1] in the sense of [SGA3, XXI,1.1.1].

Note that we do not distinguish the two kinds of "reflections" in symbols since they form isomorphic finite groups of automorphisms on X and $\mathrm{X}^\vee$ respectively and therefore it is better to view them as two representations of a same finite group $W(\mathcal{R})$. This group is called the *Weyl group* of the root datum.

1.2.3. If $\mathcal{R} = (\mathrm{X}, \Phi, \mathrm{X}^\vee, \Phi^\vee)$ is a root datum, then its Weyl group acts on the real vector space $\mathrm{X}^\vee_\mathbb{R} := \mathrm{X}^\vee \otimes \mathbb{R}$ and there is a unique inner product on it invariant under the action. Let V be the subspace of $\mathrm{X}^\vee_\mathbb{R}$ spanned by $\Phi^\vee$. Then Φ is a (reduced) root system on the Euclidean vector space V.

In general, V is not the entire $\mathrm{X}^\vee_\mathbb{R}$. It is when $\mathcal{R}$ is *semisimple*. So the apartment associated to root systems can also be viewed as the apartment associated to semisimple root data. As for the non-semisimple ones, they give rise to non-essential apartments and hence are ignored in this draft.

1.2.4. A root system Φ is said to be *irreducible* if it cannot be written as the union of two proper subsets such that they are orthogonal to each other. A root system Φ is irreducible if and only if so is its Weyl group ${}^vW(\Phi)$ [Bou02, chap.VI, §1, no.2, prop.5's cor.]. Any root system decomposes into disjoint union of irreducible ones and such a decomposition is compatible with the decomposition of Weyl groups and hence apartments [Bou02, chap.VI, §1, no.2, prop.6 and 7].

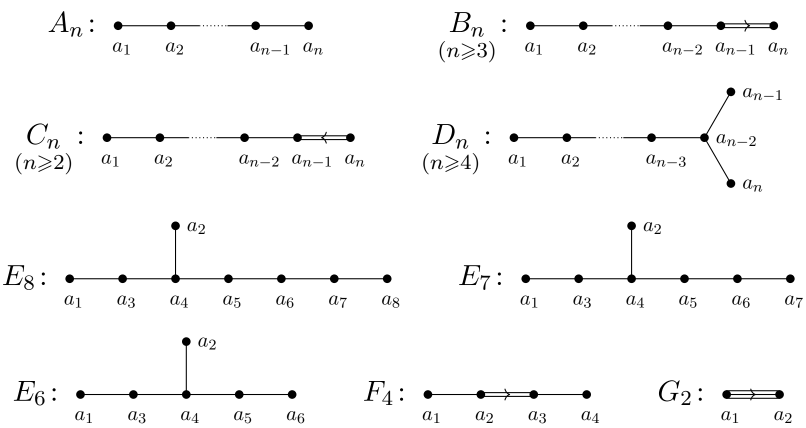
1.2.5. Let Φ be a root system. Then there is a closed subset Φ^+ of Φ such that for any $a \in \Phi$, either $a \in \Phi^+$ or $-a \in \Phi^+$. This set is called the set of *positive roots*. Once such a set is choosen, elements in the set $\Phi^- := -\Phi^+$ are called *negative roots*. A positive root is called a *simple root* if it cannot be written as the sum of two positive roots. The set Δ of simple roots form a *basis* of Φ in the sense that any root is a $\mathbb{Z}$-linear combination of simple roots and the coefficients are either all non-negative or all non-positive [Bou02, chap.VI, §1, no.6, th.3]. The size of the set Δ is called the *rank* of Φ.

Let Δ be a basis of Φ. Then the set ${}^v\!C = \{x \in V \mid \forall a \in \Delta, a(x) > 0\}$ is a vectorial chamber, called the *Weyl chamber* associated to Δ [Bou02, chap.VI, §1, no.5, th.2]. Conversely, let ${}^v\!C$ be a vectorial chamber. Then for any $x \in {}^v\!C$, the sets $\Phi^+ = \{a \in \Phi \mid a(x) > 0\}$ and $\Phi^- = \{a \in \Phi \mid a(x) < 0\}$ form a partition of Φ into positive and negative roots and are independent of the choice of x. Then one can obtained a basis Δ by taking the simple roots. But there is a more direct description: they are the roots defining walls of ${}^v\!C$ and point inside. As vectorial chambers are Weyl chambers associated to some choice of basis, we call them *Weyl chambers* to specify that they are chambers in the spherical apartment ${}^v\!\mathscr{A}(\Phi)$.

1.2.6. The relation between simple roots and types are as follows. First, the Weyl group ${}^vW$ is generated by r_a for $a \in \Delta$ as they are the roots defining walls of ${}^v\!C$ and point inside. Therefore a type $I \in \mathcal{T}$ corresponds to a subset of Δ. From now on, we do not distinguish them. Then the face of ${}^v\!C$ corresponding to I is the set ${}^v\!C_I = \{x \in V \mid \forall a \in I, a(x) = 0, \forall a \in \Delta \setminus I, a(x) > 0\}$. Let Φ_I be the subroot system of Φ generated by I, then W_I is the Weyl group of it. The set $\Psi = \Phi_I \cup \Phi^+$ has the property that $\Psi \cup (-\Psi) = \Phi$ and is closed. Such kind of subsets of Φ are said to be *parabolic*. Given a parabolic subset Ψ of Φ containing Φ^+, then the simple roots in $\Psi \cap (-\Psi) \cap \Phi^+$ gives the type I. See [Bou02, chap.VI, §1, no.7].

1.2.7. Given a basis Δ of a root system Φ, its *Dynkin diagram* is defined as follows. The vertices are simple roots of Φ and the number of edges between two vertices is $4\cos^2(\theta)$ if the angle between then is θ. Furthermore, these edges are decorated with arrows pointing from longer root to shorter root. It turns out that, up to graph isomorphisms, the Dynkin diagram is independent of the choice of the basis Δ.

From above description, we see that Φ is irreducible if and only if its Dynkin diagram is connected. The Dynkin diagrams of irreducible root systems are classified as follows [Bou02, chap.VI, §4, no.2, th.3], where the subscription n in the notation X_n denote the rank of it.



A spherical apartment is said to be *of type* X_n if it is isomorphic to ${}^v\!\mathscr{A}(\Phi)$ for an irreducible root system Φ of type X_n.

1.2.8. Let $\mathbb{A}$ be an affine space underlying V with a specified point o. For any $a \in V^*$ and $k \in \mathbb{R}$, denote the affine function $x \mapsto a(x-o)+k$ on V by $a+k$ and denote the closed half-space $\{x \in \mathbb{A} \mid (a+k)(x) \geqslant 0\}$ by α_{a+k}. For each $a \in \Phi$, let Γ_a be a nonempty subset of $\mathbb{R}$. The affine function $a+k$ is called an *affine root* if $a \in \Phi$ and $k \in \Gamma_a$. Let Σ denote the set of closed half-spaces α_{a+k} with $a+k$ an affine root. Then $a+k \mapsto \alpha_{a+k}$ gives rise to a bijection between the set of affine roots and Σ. For this reason, we will not distinguish the affine root $a+k$ and the closed half-space α_{a+k} and will call Σ the *affine root system*[2]. The roots are vectorial part of affine roots. Hence we denote Φ by ${}^v\Sigma$ and call it the *vectorial root system* of Σ.

[2]Note that, there is a notion called *affine root system*, defined in a similar way as root

For $\alpha = \alpha_{a+k}$ an affine root, let ${}^v\alpha$ denote its vectorial part a, let $\partial\alpha$ denote its boundary $\{x \in \mathbb{A} \mid (a+k)(x) = 0\}$, let r_α denote the reflection with respect to $\partial\alpha$, let α^* denote the other affine root sharing the same boundary with α, that is $\overline{\mathbb{A} \setminus \alpha}$, and let α_+ denote the intersection of affine roots containing a neighborhood of α. When α_+ is an affine root hence so is $(\alpha^*)_+$, let $\alpha_- = ((\alpha^*)_+)^*$.

1.2.9. Let Σ be an affine root system on an Euclidean affine space $\mathbb{A}$, its *affine Weyl group* $W(\Sigma)$ is the reflection group on $\mathbb{A}$ generated by r_α for all $\alpha \in \Sigma$. In this way, we obtain an apartment $\mathscr{A}(\Sigma) := (\mathbb{A}, W(\Sigma))$ with ${}^v\mathscr{A}({}^v\Sigma)$ being its vectorial apartment. Suppose all Γ_a are the same discrete subgroup $\Gamma \neq 0$ of $\mathbb{R}$, then the walls in the apartment $\mathscr{A}(\Sigma)$ are precisely the boundaries $\partial\alpha$ with $\alpha \in \Sigma$ [Bou02, chap.VI, §2, no.1, prop.2]. For x a point in the apartment $\mathscr{A}(\Sigma)$, let Σ_x be the set of affine roots α such that $x \in \partial\alpha$ and let ${}^v\Sigma_x$ be its vectorial part. Then direct computation shows ${}^v\Sigma_x$ is a closed subroot system of ${}^v\Sigma$. Then the spherical apartment $\mathscr{A}_x$ at x can be identified with ${}^v\mathscr{A}({}^v\Sigma_x)$. Also note that Σ_x can be identified with ${}^v\Sigma_x$ by $\alpha \mapsto {}^v\alpha$. In particular, the roots can be identified with the affine roots in Σ_o.

1.2.10. Notations as before. Suppose $\Phi = {}^v\Sigma$ is irreducible. Let ${}^vC$ be a Weyl chamber of Φ and Δ the simple roots it defines. Then there is a unique root a_0 such that $\|a_0\| \geqslant \|a\|$ for all root a [Bou02, chap.VI, §1, no.8, prop.25]. This a_0 is called the *highest root* with respect to Δ or ${}^vC$. The set $C = (o + {}^vC) \setminus \alpha_{a_0+0-}$ is a chamber in $\mathscr{A}(\Phi)$ [Bou02, chap.VI, §2, no.2, prop.5] and is calld the *fundamental alcove* associated to Δ.

Let $\widetilde{\Delta}$ denote the set of affine roots α defining walls of C, which means $C \subseteq \alpha$ and $\partial\alpha$ is a wall of C. Then it consists of the simple roots and an affine root $\alpha_0 = (\alpha^*_{a_0+0})_+$. Such a set $\widetilde{\Delta}$ is a *basis* of Σ in the sense that any affine root is a $\mathbb{Z}$-linear combination of its elements and the coefficients are either all non-negative or all non-positive.

Conversely, let C be a chamber in $\mathscr{A}(\Phi)$ and x a special vertex which is also an extremal point of $\overline{C}$. The affine roots defining walls of C form a basis $\widetilde{\Delta}$ of the affine root system $\widetilde{\Phi}$. Among these affine roots, those vanishing at x give rise to a basis Δ of the root system Φ by taking their vectorial parts and the rest one gives rise to the highest root with respect to Δ by taking the negation of its vectorial part. Since chambers in $\mathscr{A}(\Phi)$ are fundamental alcoves associated to some choice of basis, we call them *alcoves* to avoid confusion with Weyl chambers.

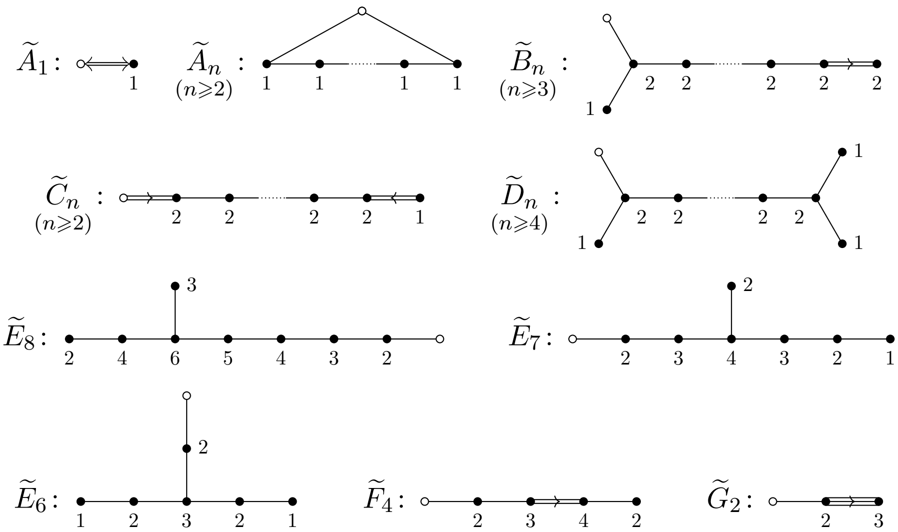
1.2.11. The types are introduced as follows. First, the affine Weyl group $W(\Sigma)$ is generated by r_α for $\alpha \in \widetilde{\Delta}$ as they are the affine roots defining walls of C. Therefore a type $I \in \mathcal{T}$ corresponds to a proper subset of $\widetilde{\Delta}$. From

system, but for affine spaces. In this draft, this terminiology only refers to those arise from (reduced) root systems.

now on, we do not distinguish them. Then the face of C corresponding to I is the set $C_I = \overline{C} \cap (\bigcap_{\alpha \in I} \partial\alpha) \setminus (\bigcup_{\alpha \in \widetilde{\Delta} \setminus I} \partial\alpha)$.

1.2.12. Let Σ be an irreducible affine root system with $\widetilde{\Delta}$ a basis. Then the *extended Dynkin diagram* of it is defined similarly to Dynkin diagram except in the case of $\widetilde{A}_1$, where there is an left-right double arrow between the two vertices.

The followings are extended Dynkin diagrams of all irreducible affine root systems [Bou02, chap.VI, §4, no.3, prop.4], where the notation $\widetilde{X}_n$ indicates this affine root system arises from the root system of type X_n.



Also note that these Dynkin diagrams are decorated in the following way: the part consists of bold vertices is an ordinary Dynkin diagram and its vertices present the simple roots a_i ($1 \leqslant i \leqslant n$), then the extra hollow vertex presents the (affine root α_0 defined by the) highest root a_0 and each simple root a_i is labelled by its coefficient h_i in the expression

$$a_0 = \sum_{i=1}^{n} h_i a_i$$

presenting the highest root a_0 as $\mathbb{Z}$-linear combination of them.

A discrete affine apartment is said to be *of type* $\widetilde{X}_n$ if it is isomorphic to $\mathscr{A}(\Sigma)$ for an irreducible affine root system Σ of type $\widetilde{X}_n$.

1.3 Reductive groups

Tits buildings and Bruhat-Tits buildings are defined for reductive groups, a family of algebraic groups which contains classical groups. They will be introduced in this subsection. We fix a ground field K and an algebraic closure K^a (resp. separable closure K^s) of it.

1.3.1. By an *algebraic group* (defined over K), we mean a group object in the category of schemes of finite type over K. An algebraic group is *affine* (resp. *smooth*, *connected*) if so is its underlying scheme and is *linear* if it admits a finite dimensional faithful representation. It turns out that affine algebraic group = linear algebraic group.

Let G be an algebraic group defined over K. For a general K-algebra R, the group scheme obtained by base change $\mathsf{G} \otimes_K R$ is denoted by G_R and the group of R-points is denoted by $\mathsf{G}(R)$. Moreover, $\mathsf{G}(K)$ is simply denoted by G and $\mathsf{G}_R(R) \cong \mathsf{G}(R)$ is simply denoted by G_R. We also use $g \in \mathsf{G}$ to mean that g is a R-point of G for some R.

Let G be an algebraic group. Then its *neutral component* G° is the largest connected subgroup of G. Its *derived group* $\mathsf{G}^{\mathrm{der}}$ is the smallest norm subgroup of G such that the quotient group is commutaive. Its *Lie algebra* $\mathrm{Lie}(\mathsf{G})$ is the tangent space at the identity. The action of G on itself by conjugation defines a representation $\mathrm{Ad}\colon \mathsf{G} \to \mathsf{GL}_{\mathfrak{g}}$ of G on the vertor space $\mathfrak{g} := \mathrm{Lie}(\mathsf{G})$, called the *adjoint representation* of G.

1.3.2. The functor $R \rightsquigarrow R^\times$ mapping a K-algebra to its unit group defines an algebraic group $\mathbb{G}_m$, called the *multiplicative group*.

An algebraic group is a *torus* if it becomes isomorphic to a product of copies of $\mathbb{G}_m$ over a finite separable extension of K. A torus over K is *split* if it is already isomorphic to a product of copies of $\mathbb{G}_m$ over K.

A *character* of an algebraic group G is a homomorphism $\chi\colon \mathsf{G} \to \mathbb{G}_m$. The group of characters is denoted by $\mathrm{X}(\mathsf{G})$. The *character group* of G is the group $\mathrm{X}^*(\mathsf{G}) := \mathrm{Hom}(\mathsf{G}_{K^s}, \mathbb{G}_{m,K^s})$ of characters defined over K^s.

An algebraic group G is *diagonalizable* if it is of the form $\mathsf{D}(M)$ for some finitely generated abelian group M, where $\mathsf{D}(M)$ is defined by the functor $R \rightsquigarrow \mathrm{Hom}_K(M, R^\times)$. If this is the case, then $\mathsf{G} = \mathsf{D}(\mathrm{X}(\mathsf{G}))$. This terminology is justified by the following fact: an algebraic group is diagonalizable if and only if its every representation is diagonalizable [Mil17, 12.12]. An algebraic group G is of *multiplicative type* if it becomes diagonalizable over a finite separable extension over K.

A *cocharacter* of an algebraic group G is a homomorphism $\lambda\colon \mathbb{G}_m \to \mathsf{G}$. The *cocharacter group* of G is the abelian group $\mathrm{X}_*(\mathsf{G}) := \mathrm{Hom}(\mathbb{G}_{m,K^s}, \mathsf{G}_{K^s})$ of cocharacters defined over K^s.

Let χ be a character and λ be a cocharacter. Then the composition $\chi \circ \lambda$ is an endomorphism of $\mathbb{G}_m$, which can be identified with an integer. In this way, we get a pairing $\langle\,\cdot\,,\,\cdot\,\rangle\colon\ \mathrm{X}^*(\mathsf{G}) \times \mathrm{X}_*(\mathsf{G}) \to \mathbb{Z}$.

1.3.3. Let G be an algebraic group. Then an element $g \in G_{K^a}$ is said to be *unipotent* if it acts as unipotent matrix for all finite dimensional representations. The algebraic group G is *unipotent* if all elements of G_{K^a} are unipotent.

Let G be a smooth connected linear algebraic group. Then it contains a largest smooth connected unipotent norm subgroup $\mathscr{R}_u(\mathsf{G})$, called its *unipo-*

tent radical, and a largest smooth connected solvable norm subgroup $\mathscr{R}(\mathsf{G})$, called its *radical* [Mil17, 6.46]. Note that $\mathscr{R}_u(\mathsf{G})$ is a subgroup of $\mathscr{R}(\mathsf{G})$. The algebraic group G is *reductive* (resp. *semisimple*) if its *geometric unipotent radical* $\mathscr{R}_u(\mathsf{G}_{K^a})$ (resp. *geometric radical* $\mathscr{R}(\mathsf{G}_{K^a})$) is trivial. The formation of $\mathscr{R}_u(\mathsf{G})$ and $\mathscr{R}(\mathsf{G})$ commute with separable field extensions [Mil17, 19.1 and 19.9]. Hence when K is perfect, G is reductive (resp. semisimple) if and only if $\mathscr{R}_u(\mathsf{G})$ (resp. $\mathscr{R}(\mathsf{G})$) is trivial.

1.3.4. The functor $R \rightsquigarrow R$ mapping a K-algebra to its underlying abelian group defines an algebraic group $\mathbb{G}_a$, called the *additive group*. An algebraic group is a *vector group* if it is isomorphic to a product of copies of $\mathbb{G}_a$. Let V be a vector space over K, then the functor $R \rightsquigarrow V_R \colon V \otimes_K R$ defines a vector group $\mathbb{W}(V)$. This construction together with the Lie algebra construction gvies rise to an equivalence of categories between vector groups and finite dimensional vector spaces over K [Mil17, 10.9]. Moreover, when K is of characteristic zero and G is a unipotent group over it, there is an isomorphism of schemes (and of algebraic groups if G is commutative)

$$\exp\colon \mathbb{W}(\mathrm{Lie}(\mathsf{G})) \longrightarrow \mathsf{G},$$

called the *exponential map* [Mil17, 14.32].

1.3.5. A homomorphism of smooth connected algebraic groups is said to be an *isogeny* if it is surjective and has finite kernel. An isogeny is *central* if its kernel is contained in the centre and is *multiplicative* if its kernel is of multiplicative type. A multiplicative isogeny is central [Mil17, 12.38] and the converse is true if its domain is reductive (since the centre of a reductive group is of multiplicative type).

A smooth connected algebraic group is *simply connected* if every multiplicative isogeny to it is an isomorphism. Let G be a smooth connected algebraic group. A *universal covering* on it is an multiplicative isogeny $\widetilde{\mathsf{G}} \to \mathsf{G}$ with $\widetilde{\mathsf{G}}$ a simply connected one. When the universal covering exists, its kernel is called the *fundamental group* $\pi_1(\mathsf{G})$ of G.

1.3.6. Let G be a reductive group. Then its radical $\mathscr{R}(\mathsf{G})$ is the largest subtorus of $\mathsf{Z}(\mathsf{G})$ and the quotient $\mathsf{G}/\mathscr{R}(\mathsf{G})$ is semisimple [Mil17, 17.62], its derived group $\mathsf{G}^{\mathrm{der}}$ is semisimple [Mil17, 19.21], its centre $\mathsf{Z}(\mathsf{G})$ is of multiplicative type [Mil17, 17.62], and we have the following *deconstruction* [Mil17, 19.d].

$$\begin{tikzcd} \mathsf{Z}\left(\mathsf{G}^{\mathrm{der}}\right) \arrow[r, hook] \arrow[d, hook] & \mathsf{G}^{\mathrm{der}} \arrow[d, hook] \\ \mathsf{Z}(\mathsf{G}) \arrow[r, hook] & \mathsf{G} \end{tikzcd}$$

Namely, there is a short exact sequence

$$1 \longrightarrow \mathsf{Z}\left(\mathsf{G}^{\mathrm{der}}\right) \longrightarrow \mathsf{G}^{\mathrm{der}} \times \mathsf{Z}(\mathsf{G}) \longrightarrow \mathsf{G} \longrightarrow 1$$

and there are isogenies $\mathsf{Z}(\mathsf{G}) \to \mathsf{G} / \mathsf{G}^{\mathrm{der}}$ and $\mathsf{G}^{\mathrm{der}} \to \mathsf{G}^{\mathrm{ad}}$ with kernel $\mathsf{Z}\left(\mathsf{G}^{\mathrm{der}}\right)$. Here G^{ad} is the quotient $\mathsf{G}/\mathsf{Z}(\mathsf{G})$, which is an *adjoint group*, which means it is semisimple and has trivial centre.

1.3.7. Let G be a reductive group. It is *splittable* if it has a split maximal torus. A *split reductive group* is a pair (G, T) of a reductive group and a split maximal torus in it. A *homomorphism* between split reductive groups is a homomorphism of algebraic group preserving the split maximal torus. It turns out that, any two maximal split tori (hence split maximal tori if G is splittable) in G are conjugate by an element of G [Mil17, 25.10], while two (not necessarily split) maximal tori are only conjugate over a finite separable extension [Mil17, 17.87].

Let G be a splittable reductive group. Then its *rank* is the dimension of one (hence any) split maximal torus in it and its *semisimple rank* is the rank of $\mathsf{G} / \mathscr{R}(\mathsf{G})$. Since the centre $\mathsf{Z}(\mathsf{G})$ is contained in every maximal torus [Mil17, 17.61], the semisimple rank of G equals $\operatorname{rank}(\mathsf{G}) - \dim(\mathsf{Z}(\mathsf{G}))$. Consequently, the homomorphism $\mathsf{G} / \mathscr{R}(\mathsf{G}) \to \mathsf{G}^{\mathrm{ad}}$ is an isogeny.

1.4 Root data and Tits buildings

Let G be a reductive group. Associated to it, there is a spherical building ${}^{v}\mathscr{B}(\mathsf{G})$ equipped with a natural G-action, called its *Tits building*. In this subsection, Tits buildings will be introduced for splittable reductive groups and we will see that the underlying building only depends on the root system and the ground field.

1.4.1. Let G be a reductive group. A *parabolic subgroup* of it is a smooth subgroup P such that G / P is a complete variety. A subgroup B of G is *Borel* if it is smooth, connected, solvable, and parabolic. It turns out that a smooth subgroup P is parabolic if and only if P_{K^a} contains a Borel subgroup in G_{K^a} [Mil17, 17.16] and every parabolic subgroup is connected and equal to its own normalizer since this is so over K^a [Mil17, 17.49]. When G has a Borel subgroup, it is said to be *quasi-split*. In this case, Borel subgroups are exactly the minimal parabolic subgroups and maximal connected solvable subgroups [Mil17, 17.19] and any two of them are conjugate by an element of G [Mil17, 25.8]. If the Borel subgroup is furthermore split (as a solvable algebraic group, namely it admits a normal series whose factors are isomorphic to either $\mathbb{G}_a$ or $\mathbb{G}_m$), then G is said to be *split*. It turns out that, G is split if and only if it is splittable [Mil17, 21.64].

Let $\pi\colon \mathsf{G} \to \mathsf{Q}$ be a quotient map and H a smooth subgroup of G. Then if H is parabolic (resp. Borel), so is $\pi(\mathsf{H})$. Moreover, every such subgroup of Q arises in this way [Mil17, 17.20]. This allows us to reduce the study of (the poset of) parabolic subgroups from reductive groups to simply-connected semisimple groups. The *Tits building* of a reductive group is essentially this poset [Tit74, 5.2].

1.4.2. Let (G, T) be a split reductive group. Since T is diagonalizable, it acts (via the adjoint representation) on $\mathfrak{g} := \mathrm{Lie}(\mathsf{G})$ diagonalizably and we hav decomposition:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{a \in \mathrm{X}^*(\mathsf{T})} \mathfrak{g}_a,$$

where $\mathfrak{t} = \mathfrak{g}^{\mathsf{T}} = \mathrm{Lie}(\mathsf{T})$ [Mil17, 10.34] and $\mathfrak{g}_a$ is the subspace on which T acts through a nontrivial character a. A character a is a *root* if $\mathfrak{g}_a$ is nontrivial. The set of roots is denoted by $\Phi(\mathsf{G}, \mathsf{T})$, called the *root system* of the split reductive group (G, T).

For $a \in \Phi(\mathsf{G}, \mathsf{T})$, there is a unique subgroup U_a (called the *root group*) of G satisfying the following properties [Mil17, 21.11 and 21.19].

1. U_a is normalized by T.

2. U_a has Lie algebra $\mathfrak{g}_a$ and a smooth subgroup of G contains U_a if and only if its Lie algebra contains $\mathfrak{g}_a$.

3. U_a is isomorphic to $\mathbb{G}_a$. In fact, there is a unique isomorphism of of algebraic groups $u_a \colon \mathbb{W}(\mathfrak{g}_a) \to \mathsf{U}_a \subseteq \mathsf{G}$ such that its derivation is the inclusion $\mathfrak{g}_a \subseteq \mathfrak{g}$.

4. T acts on U_a through the character a: for every isomorphism $u \colon \mathbb{G}_a \to \mathsf{U}_a$, we have $tu(x)t^{-1} = u(a(t)x)$ for all $t \in \mathsf{T}$ and $x \in \mathbb{G}_a$.

1.4.3. Let (G, T) be a split reductive group. Then $\mathsf{N} = \mathsf{N}_{\mathsf{G}}(\mathsf{T})$ acts on T, hence on $\mathrm{X}^*(\mathsf{T}) = \mathrm{X}(\mathsf{T})$ by conjugation. The centralizer $\mathsf{Z}_{\mathsf{G}}(\mathsf{T})$ (which equals T itself [Mil17, 17.84]) acts trivially, hence we have an action of the quotient N / T on $\mathrm{X}^*(\mathsf{T})$. It turns out that, this quotient is the étale group scheme of connected components of N [Mil17, 17.39], and is furthermore constant [Mil17, 21.1], namely it is constant as a functor with value a finite group $W(\mathsf{G}, \mathsf{T}) = N/T$. This finite group is called the *Weyl group* of (G, T).

For $a \in \Phi(\mathsf{G}, \mathsf{T})$, let G_a denote the centralizer of the largest subtorus of $\mathrm{Ker}(a)$. Then we have the following facts [Mil17, 21.11].

1. The pair $(\mathsf{G}_a, \mathsf{T})$ is a split reductive group of semisimple rank 1.

2. The Lie algebra of G_a has decomposition

   $$\mathrm{Lie}(\mathsf{G}_a) = \mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$$

   and the only rational multiples of a in $\Phi(\mathsf{G}, \mathsf{T})$ are $\pm a$.

3. The Weyl group $W(\mathsf{G}_a, \mathsf{T})$ contains exactly one nontrivial element r_a.

4. There is a unique $a^\vee \in \mathrm{X}_*(\mathsf{T})$ such that $r_a(x) = x - \langle x, a^\vee \rangle a$ for all $x \in \mathrm{X}^*(\mathsf{T})$. Moreover, $\langle a, a^\vee \rangle = 2$.

Let $\Phi^\vee(\mathsf{G},\mathsf{T})$ denote the set of cocharacters $a^\vee$ for $a \in \Phi(\mathsf{G},\mathsf{T})$, called the *coroot system*. Then the quadruple $(\mathrm{X}^*(\mathsf{T}), \Phi(\mathsf{G},\mathsf{T}), \mathrm{X}_*(\mathsf{T}), \Phi^\vee(\mathsf{G},\mathsf{T}))$ form a root datum $\mathcal{R}(\mathsf{G},\mathsf{T})$. Let V denote the subspace of $\mathrm{X}_*(\mathsf{T}) \otimes \mathbb{R}$ spanned by $\Phi^\vee(\mathsf{G},\mathsf{T})$, called the *coroot space*. Then we get a root system $\Phi(\mathsf{G},\mathsf{T})$ on the Euclidean vector space V and hence a spherical apartment ${}^v\!\mathscr{A}(\mathsf{G},\mathsf{T})$ on which the Weyl group $W(\mathsf{G},\mathsf{T})$ acts as its reflection group.

1.4.4. Let (G,T) be a split reductive group and we fix the following notations in the rest of this section.

N is the normalizer of T.

X^* is the character group of T.

X_* is the cocharacter group of T.

Φ is the root system associated to the pair (G,T).

$\Phi^\vee$ is the coroot system.

U_a is the root subgroup associated to $a \in \Phi$.

V is the coroot space.

W is the Weyl group.

${}^v\!\mathscr{A}$ is the spherical apartment ${}^v\!\mathscr{A}(\mathsf{G},\mathsf{T})$.

First, note that the followings sets are equipollent and acted on simply transitively by W.

1. The set of Borel subgroups B of G containing T.
2. The set of Weyl chambers ${}^v C$ in ${}^v\!\mathscr{A}$.
3. The set of systems of positive roots Φ^+ in Φ.
4. The set of bases Δ of Φ.

Indeed, if a system of positive roots Φ^+ is given, then B is generated by T and U_a for all $a \in \Phi^+$ and if a Borel subgroup B containing T is given, then the set of roots a whose Lie algebra $\mathfrak{g}_a$ is contained in the Lie algebra of B forms a system of positive roots Φ^+ [Mil17, 21.d].

More general, after choosing one element of the above equipollent sets. We have the following equipollent sets.

1. The set of parabolic subgroups P of G containing B.
2. The set of faces ${}^v F$ of the Weyl chambers ${}^v C$.
3. The set of parabolic subset Ψ of positive roots Φ^+ in Φ.

4. The set of types I.

Indeed, if a parabolic subset Ψ is given, then P is generated by T and U_a for all $a \in \Psi$ and if a parabolic subgroup P containing B is given, then the set of roots a whose Lie algebra $\mathfrak{g}_a$ is contained in the Lie algebra of P forms a parabolic subset Ψ [Mil17, 21.i].

Let I be a type and P_I the parabolic subgroup corresponding to it. Then the unipotnet radical of P_I is generated by U_a for all $a \in \Phi^+ \setminus \Psi$ and the reductive quotient of P_I is isomorphic to the centralizer L_I of the largest subtorus of $\bigcap\limits_{a\in I} \operatorname{Ker}(a)$ [Mil17, 21.91]. This reductive group is called the *Levi subgroup* associated to I and $(\mathsf{L}_I, \mathsf{T})$ is a split reductive group with root datum $(\mathrm{X}^*, \Phi_I, \mathrm{X}_*, \Phi_I^\vee)$ and Weyl group W_I [Mil17, 21.90].

1.4.5 Theorem ([Rou09, §10; Tit74, §5])**.** *Notations as above. There is a unique (up to unique isomorphism) G-set ${}^v\mathscr{B}(\mathsf{G})$ containing V and satisfying the followings.*

1. ${}^v\mathscr{B}(\mathsf{G}) = \bigcup_{g\in G} g.V$*;*
2. *N stabilizes V and acts on it through W;*
3. *the fixator of $\alpha_{a+0} := \{x \in V \mid a(x) \geqslant 0\}$ is $T \cdot U_a$.*

Then ${}^v\mathscr{B}(\mathsf{G})$ is a building of type ${}^v\mathscr{A}$, called the *Tits building* of G. In fact, since N stabilizes V and preserves its apartment structure, each $g.V$ is endowed with such a structure and moreover they agree on intersections.

Remark. Apartments in ${}^v\mathscr{B}(\mathsf{G})$ are one-one corresponding to split maximal tori. In fact, each $g.V$ endowed with its apartment structure is precisely the apartment ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T}^g)$.

The action of G on ${}^v\mathscr{B}(\mathsf{G})$ is strongly transitive and type-preserving. It is also worth to mention that ${}^v\mathscr{B}(\mathsf{G})$ is further a $\operatorname{Aut}(\mathsf{G})$-set. Indeed, if φ is an automorphism of G, then $\varphi(\mathsf{T})$ is also a split maximal torus and the push-forward along φ defines a homomorphism from ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T})$ to ${}^v\mathscr{A}(\mathsf{G}, \varphi(\mathsf{T}))$.

1.4.6. Let $\mathcal{R} = (\mathrm{X}, \Phi, \mathrm{X}^\vee, \Phi^\vee)$ be a root datum. Let X_0 denote the submodule $\{x \in \mathrm{X} \mid \langle x, \Phi^\vee\rangle = 0\}$ and let

- $\mathrm{X}' = \mathrm{X} / \mathrm{X}_0$,
- $\mathrm{X}'^\vee$ be the submodule of $\mathrm{X}^\vee$ dual to X' through $\langle\cdot\,,\cdot\rangle$,
- Φ' be the image of Φ in X' and $\Phi'^\vee = \Phi^\vee$.

Then $\mathcal{R}' = (\mathrm{X}', \Phi', \mathrm{X}'^\vee, \Phi'^\vee)$ is a semisimple root datum and both $\mathcal{R}$ and $\mathcal{R}'$ give rise to isomorphic root systems, hence isomorphic spherical apartments. This $\mathcal{R}'$ is called the *semisimple quotient* of $\mathcal{R}$.

Let G' be the derived group of G and T' the intersection $\mathsf{T} \cap \mathsf{G}'$. Then $\mathcal{R}\left(\mathsf{G}', \mathsf{T}'\right)$ is the semisimple quotient of $\mathcal{R}(\mathsf{G}, \mathsf{T})$ [Spr98, 8.1.8]. So ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T})$ can be identified with ${}^v\mathscr{A}\left(\mathsf{G}', \mathsf{T}'\right)$. Note that the $\mathsf{T} \mapsto \mathsf{T}'$ gives rise to a bijection between the set of maximal κ-split tori in G to the set of maximal κ-split tori in G'. Therefore, from the above we can identify ${}^v\mathscr{B}\left(\mathsf{G}'\right)$ with ${}^v\mathscr{B}(\mathsf{G})$ and G acts on the former by conjugations on G'.

1.4.7. Let $\mathcal{R} = (\mathrm{X}, \Phi, \mathrm{X}^\vee, \Phi^\vee)$ and $\mathcal{R}' = (\mathrm{X}', \Phi', \mathrm{X}'^\vee, \Phi'^\vee)$ be two root data. An *isogeny of root data* $f\colon \mathcal{R}' \to \mathcal{R}$ (in the sense of [SGA3, XXI,6.2.1] and is called a *central isogeny* in [Mil17, 23.2]) is a linear map $f\colon\ \mathrm{X}' \to \mathrm{X}$ satisfying

1. it induces a bijection from Φ' to Φ;
2. its transpose ${}^t f\colon\ \mathrm{X}^\vee \to \mathrm{X}'^\vee$ induces a bijection from $\Phi^\vee$ to $\Phi'^\vee$;
3. f is injective and has finite cokernel.

If $f\colon \mathcal{R}' \to \mathcal{R}$ is an isogeny, then it also induces bijections between bases, systems of posistive roots and Weyl chambers [SGA3, XXI,6.1.3]. As a consequence, it induces an isomorphism of apartments ${}^v\mathscr{A}(\Phi') \cong {}^v\mathscr{A}(\Phi)$.

An *isogeny of split reductive groups* $(\mathsf{G}', \mathsf{T}') \to (\mathsf{G}, \mathsf{T})$ is a homomorphism of split reductive groups such that $\varphi\colon\ \mathsf{G}' \to \mathsf{G}$ is an isogeny. A homomorphism of split reductive groups φ is a central isogeny if and only if it induces an isogeny of root data $\varphi|_{\mathrm{X}^*(\mathsf{T}')}\colon\ \mathcal{R}\left(\mathsf{G}', \mathsf{T}'\right) \to \mathcal{R}(\mathsf{G}, \mathsf{T})$ and all isogenies of root data arise in this way [SGA3, XXII,4.2.11; Mil17, 23.25].

Let $\varphi\colon\ \mathsf{G}' \to \mathsf{G}$ be a central isogeny between splittable reductive groups. Then $\mathsf{T}' \mapsto \mathsf{T} = \varphi(\mathsf{T}')$ gives rise to a bijection between the set of maximal κ-split tori in G' to the set of maximal κ-split tori in G. Therefore, from the above we see that the induced morphism $\varphi_*\colon\ {}^v\mathscr{B}\left(\mathsf{G}'\right) \to {}^v\mathscr{B}(\mathsf{G})$ is an isomorphism of buildings (and G'-sets).

Two splittable semisimple groups are *strictly isogenous* if they have the same simply connected covering group. This is the case if and only if the two semisimple groups have isomorphic root systems. Conversely, any root system arises from a splittable semisimple group (in fact, any root datum arises from a splittable reductive group [Mil17, 23.55]).

1.4.8. From aboves, we see that the Tits building ${}^v\mathscr{B}(\mathsf{G})$ depends only on the root system Φ and the ground field K and any root system gives rise to such a building. So we can denote this building by ${}^v\mathscr{B}(\Phi, K)$.

1.5 Valuations on root group data

Given a root group datum on G with a valuation on it, Bruhat and Tits associate a building equipped with natural G-action to these data in [BT72]. This construction will be exposited in this subsection.

1.5.1 Definition. Let Φ be a root system and G be a group. A *root group datum*[3] *of type* Φ *on* G is a system $(T, (U_a, M_a)_{a\in\Phi})$, where T is a subgroup of G and for each $a \in \Phi$, U_a is a non-trivial subgroup of G and M_a is a right congruence class modulo T, satisfying the following axioms.

R1. For any $a, b \in \Phi$, the commutator group $[U_a, U_b]$ is contained in the group generated by the U_c for $c = ia + jb \in \Phi$ with $i, j > 0$.

R2. For each $a \in \Phi$, the class M_a satisfies $U_a^* := U_a \setminus \{1\} \subseteq U_a M_a U_a$.

R3. For any $a, b \in \Phi$ and each $m \in M_a$, we have $mU_b m^{-1} \subseteq U_{r_a(b)}$.

R4. If Φ^+ is some (any) positive root system in Φ and if U^+ (resp. U^-) is the subgroup of G generated by the U_a for $a \in \Phi^+$ (resp. $a \in \Phi^-$), then $TU^+ \cap U^- = \{1\}$.

This root group datum is said to be *generating* when G is generated by the subgroups T and U_a for $a \in \Phi$.

1.5.2. Let $(T, (U_a, M_a)_{a\in\Phi})$ be a root group datum. Then we have the following consequences [BT72, 6.1.2].

1. For each $a \in \Phi$ and any $u \in U_{-a}^*$, there is a unique $m(u) \in M_a$ such that $u \in U_a.m(u).U_a$. If the root group datum is generating, $M_a = m(U_a^*)$.
2. For each $a \in \Phi$, T normalizes U_a and M_a.
3. For each $a \in \Phi$, $M_a = M_a^{-1} = M_{-a}$ and $T \cup M_a$ is a subgroup of G.
4. Let L_a be the subgroup of G generated by U_a, U_{-a} and T. Then $M_a = \{x \in L_a \mid xU_a x^{-1} = U_{-a} \text{ and } xU_{-a}x^{-1} = U_a\}$.

So M_a is completely determined by U_a, U_{-a} and T. Hence we can say $(T, (U_a)_{a\in\Phi})$ is a root group datum without mention M_a.

5. Let N be the subgroup of G generated by T and M_a for all $a \in \Phi$. Then, if Φ is not empty, N is also generated by M_a's and normalizes T. Moreover, there is an epimorphism ${}^v\nu\colon N \to {}^vW(\Phi)$ such that for each $a \in \Phi$ and $n \in N$, we have $nU_a n^{-1} = U_b$ with $b = {}^v\nu(n).a$. In particular, we have ${}^v\nu(M_a) = \{r_a\}$. Also note that $\mathrm{Ker}\,({}^v\nu) = T$ [BT72, 6.1.11].

1.5.3 Example ([BT72, 6.1.3c; BT65])**.** Let (G, T) be a split reductive group over K and $(\mathsf{U}_a)_{a\in\Phi}$ the root groups associated to the root system Φ of (G, T). Then $(T, (U_a)_{a\in\Phi})$ forms a generating root group datum on G.

[3] It is called a *(reduced) root datum* in [BT72, 6.1.1].

Note that this fact already will imply Theorem 1.4.5 using either *Tits system* or similar construction in 1.5.9.

1.5.4 Definition. A *valuation* of the root group datum $(G, T, (U_a)_{a\in\Phi})$ is a family $\varphi = (\varphi_a)_{a\in\Phi}$ of functions $\varphi_a\colon U_a \to \mathbb{R}\cup\{\infty\}$ satisfying the following axioms.

V0. For each $a \in \Phi$, the image of φ_a contains at least three elements.

V1. For each $a \in \Phi$ and any $\lambda \in \mathbb{R}\cup\{\infty\}$, the set $U_{a,\lambda} := \varphi_a^{-1}([\lambda, \infty])$ is a subgroup of U_a and $U_{a,\infty} = \{1\}$.

V2. For each $a \in \Phi$ and any $m \in M_a$, the function $u \mapsto \varphi_{-a}(u) - \varphi_a(mum^{-1})$ is constant on U^*_{-a}.

V3. For any pair $a, b \in \Phi$ not proportional and any $\lambda, \mu \in \mathbb{R}\cup\{\infty\}$, the commutator group $[U_{a,\lambda}, U_{b,\mu}]$ is contained in the subgroup generated by $U_{ia+jb,i\lambda+j\mu}$ for all $i, j > 0$ such that $ia + jb \in \Phi$.

V4. For each $a \in \Phi$ and any $u \in U_a$, if $u', u'' \in U_{-a}$ satisfy $m(u) = u'uu''$, then $\varphi_{-a}(u') = \varphi_{-a}(u'') = -\varphi_a(u)$.

For each $a \in \Phi$, let Γ_a denote the set $\varphi_a(U^*_a)$ and for any $k \in \Gamma_a$, let $M_{a,k}$ be the intersection of M_a and $U_{-a}\varphi_a^{-1}(k)U_{-a}$.

1.5.5. [BT72, 6.2.5] Given a root group datum $(T, (U_a)_{a\in\Phi})$ on G and let φ be a valuation on it. Then for any vector v in the ambient space V of Φ, the family $\psi = (\psi_a)_{a\in\Phi}$ given by $\psi_a\colon u \mapsto \varphi_a(u) + a(v)$ is a valuation and is denoted by $\varphi+v$. The valuations φ and $\psi = \varphi+v$ are said to be *equipollent*. The mapping $(\varphi, v) \mapsto \varphi + v$ defines an action of V on the set of valuations and each equipollent class is an orbit.

Let $\mathbb{A}$ denote the set of valuations equipollent to φ. Then $\mathbb{A}$ is an affine space underlying V and 1.2.8 applies. For $\alpha = \alpha_{a+k}$ with $a \in \Phi$, $k \in \Gamma_a$, let $U_\alpha = U_{a,k}$ and $U_{\alpha+} = \bigcup_{h>k} U_{a,h}$ (note that $U_{\alpha+} = U_{\alpha_+}$ if Γ_a is discrete). It is clear that the affine root system Σ and the mapping $\alpha \mapsto U_\alpha$ depends only on the equipollent class of φ.

1.5.6. [BT72, 6.2.5] Let $n \in N$ and $w = \nu(n) \in {}^vW$. Then the family $\psi = (\psi_a)_{a\in\Phi}$ given by $\psi_a\colon u \mapsto \varphi_{w^{-1}.a}(n^{-1}un)$ is a valuation and is denoted by $n.\varphi$. We thus obtain an action of N on the set of valuations such that for any $n \in N$ and $v \in V$, we have $n.(\varphi + v) = n.\varphi + \nu(n).v$.

[BT72, 6.2.10] The action of N stabilizes $\mathbb{A}$ and for any $n \in N$, the map $\nu(n)\colon \varphi \mapsto n.\varphi$ itself is an automorphism of the Euclidean affine space $\mathbb{A}$ whose vectorial part is ${}^v\nu(n)$. For each $a \in \Phi$ and $k \in \Gamma_a$, the image of $M_{a,k}$ under ν is the reflection r_{a+k}. So the automorphism $\nu(n)$ maps affine roots to affine roots and we have $nU_\alpha n^{-1} = U_{\nu(n).\alpha}$. In particular, for $u \in U^*_a$, $\nu(m(u)) = r_{a+\varphi_a(u)}$ [BT72, 6.2.12]. Therefore, the valuation φ is completely determined by the homomorphism $\nu\colon N \to \mathrm{Aut}\,(\mathbb{A})$.

1.5.7. [BT72, 6.2.11] Let $T^\circ = \mathrm{Ker}\,(\nu)$ and $\widehat{W} = \nu(N)$. Let W denote the subgroup of $\widehat{W}$ generated by r_{a+k} with $a \in \Phi$ and $k \in \Gamma_a$. It is a normal subgroup because N permutes $M_{a,k}$. Let $N' = \nu^{-1}(W)$, $T' = T \cap N'$ and let G' be the subgroup of G generated by N' and the U_a for $a \in \Phi$. Since $M_a \cap N' \neq \varnothing$ for all $a \in \Phi$, we see that $(T', (U_a)_{a\in\Phi})$ is a generating root group datum on G' and for this root group datum, its N is exactly N'.

A valuation φ is *special* if $0 \in \Gamma_a$ for all $a \in \Phi$. If this is the case, then the group W (resp. $\widehat{W}$) can be decomposed as $W = W_\varphi \ltimes \mathrm{Ker}\,(W \to {}^vW)$ (resp. $\widehat{W} = W_\varphi \ltimes \nu(T)$) [BT72, 6.2.19], where W_φ is the stabilizer of φ.

A valuation φ is *discrete* if Γ_a is a discrete subset of $\mathbb{R}$ for all $a \in \Phi$. If this is the case, then W is the affine Weyl group $W(\Sigma)$ for the affine root system Σ [BT72, 6.2.22].

Suppose Φ is irreducible and φ is discrete and special. Then all Γ_a are the same discrete subgroup Γ of $\mathbb{R}$ [BT72, 6.2.23]. So 1.2.9 applies and we get an apartment $\mathscr{A}\,(\Sigma)$. Let $\mathcal{Q}^\vee$ be the *coroot lattice* of Φ, namely the set of $\mathbb{Z}$-linear combinations of coroots, and let $\mathcal{P}^\vee$ be the *coweight lattice* of Φ, namely the set $\{v \in V \mid a(v) \in \mathbb{Z}\}$. Then $\mathrm{Ker}\,(W \to {}^vW) = \mathcal{Q}^\vee \otimes \Gamma$ and $\nu(T)$ is between $\mathcal{Q}^\vee \otimes \Gamma$ and $\mathcal{P}^\vee \otimes \Gamma$ [BT72, 6.2.20].

1.5.8. Let Ω be a nonempty subset of $\mathbb{A}$ and let U_Ω denote the subgroup generated by U_α for all affine roots $\alpha \supseteq \Omega$. Define $f_\Omega \colon \Phi \to \mathbb{R} \cup \{\infty\}$ by

$$f_\Omega(a) = \inf\{k \in \mathbb{R} \mid \Omega \subseteq \alpha_{a+k}\}.$$

This is a typical example of *concave functions* on Φ [BT72, 6.4.3]: namely,

1. $f(a) + f(b) \geqslant f(a+b)$ for any pairs $a, b \in \Phi$ such that $a + b \in \Phi$ and
2. $f(a) + f(-a) \geqslant 0$ for any $a \in \Phi$.

For f a concave function on Φ, denote U_f the subgroup generated by $U_{a,f(a)}$ for all $a \in \Phi$ and let $P_f = T^\circ.U_f$, $N_f = N \cap U_f$ and $\Phi_f = \{a \in \Phi \mid f(a) + f(-a) = 0\}$. In particular, for $f = f_\Omega$, U_f coincides with U_Ω and we will denote P_f and Φ_f by P_Ω and Φ_Ω.

The image $\nu(N_{f_\Omega})$ is generated by the reflections r_α for affine roots α such that $\Omega \subseteq \partial\alpha$ and is identified with the Weyl group of Φ_Ω [BT72, 7.1.3]. The preimage of $\nu(N_{f_\Omega})$ is then $N \cap P_\Omega = T^\circ.N_{f_\Omega}$ and denoted by N_Ω. It is the stabilizer (=fixator) of Ω in N' and therefore is contained in the fixator $\widehat{N}_\Omega$ of Ω in N. Let $\widehat{P}_\Omega = \widehat{N}_\Omega.P_\Omega = \widehat{N}_\Omega.U_\Omega$. Then P_Ω and U_Ω are norm subgroups of it.

1.5.9 Definition. Notations as aboves. The *Bruhat-Ttis building of G* (with the root group dataum and the valuation being given) is the quotient set $\mathscr{B}$ of $G \times \mathbb{A}$ under the following equivalent relation [BT72, 7.4.1]:

$$(g, x) \sim (h, y) \iff \exists n \in N : y = \nu(n).x,\ g^{-1}hn \in \widehat{P}_x.$$

Remark. The left multiplication of G on $G \times \mathbb{A}$ is compatible with above equivalent relation, hence $\mathscr{B}$ inherits a G-action. Identifying $\mathbb{A}$ as the subset $\{1\} \times \mathbb{A}$ of $\mathscr{B}$, we have:

1. $\mathscr{B} = \bigcup_{g \in G} g.\mathbb{A}$;
2. each U_α fixes $\alpha \in \Sigma$ pointwisely [BT72, 6.4.5];
3. for each nonempty $\Omega \subseteq \mathbb{A}$, its fixator is $\widehat{P}_\Omega$ and it acts transitively on apartments containing Ω [BT72, 6.4.4, 6.4.9];
4. the stabilizer (resp. fixator) of $\mathbb{A}$ is N (resp. T°) [BT72, 6.4.10].

Then one can carry apartment structure on $\mathbb{A}$ to each $g.\mathbb{A}$ and see they agree on intersections [BT72, 7.4.18]. Hence $\mathscr{B}$ is a building of type $\mathscr{A}(\Sigma)$. The action of G on it is strongly transitively by the construction and is not necessarily type-preserving since the affine Weyl group W of $\mathscr{A}(\Sigma)$ is usually not the entire $\widehat{W}$. The subgroup of type-preserving automorphisms is then the group $G' = \nu^{-1}(W)$ introduced in 1.5.7.

1.6 Bruhat-Tits buildings

1.6.1. In the rest, K will be a field equipped with a (trivial or discrete) valuation $\mathrm{val}\colon K \to \Gamma \cup \{\infty\}$. We fix the following associated notations.

$$\begin{aligned} K^\circ &:= \{x \in K \mid \mathrm{val}(x) \geqslant 0\}, \\ (K^\circ)^\times &:= \{x \in K \mid \mathrm{val}(x) = 0\}, \\ K^{\circ\circ} &:= \{x \in K \mid \mathrm{val}(x) > 0\}, \\ \kappa &:= K^\circ / K^{\circ\circ}. \end{aligned}$$

We further assume K is complete with respect to $\mathrm{val}(\,\cdot\,)$ and κ is a finite field with cardinality q and characteristic p.

1.7 Moy-Prasad filtrations

1.7.1. Let f be a concave function. Define $f^*\colon \Phi \to \widetilde{\mathbb{R}}$ as follows:

$$f^*(a) = \begin{cases} f(a)+ & \text{if } a \in \Phi_f, \\ f(a) & \text{if } a \notin \Phi_f. \end{cases}$$

Here $\widetilde{\mathbb{R}}$ is the ordered monoid of extended real numbers[4] and for each $k \in \mathbb{R}$, $k+$ is the smalest extended real number larger than k. Then f^* is a concave

[4] Foramlly, $\widetilde{\mathbb{R}}$ is the union of $\mathbb{R}$, $\mathbb{R}+ := \{k+ \mid k \in \mathbb{R}\}$ and $\{\infty\}$. The commutative addition on $\mathbb{R}$ is extended as follows: $k + (l+) = (k+) + (l+) = (k+l)+$ for all $k, l \in \mathbb{R}$ and $\lambda + \infty = \infty$ for all $\lambda \in \widetilde{\mathbb{R}}$. The total order on $\mathbb{R}$ is extended as follows: $k < k+ < l$ for all $k, l \in \mathbb{R}$ such that $k < l$ and $\lambda < \infty$ for all $\lambda \neq \infty$.

function in the sense of 1.5.8 with $\mathbb{R} \cup \{\infty\}$ replaced by $\widetilde{\mathbb{R}}$ [BT72, 6.4.23]. For each $a \in \Phi$ and any $u \in U_{-a,f(-a)}, v \in U_{a,f^*(a)}$ Let T_{f,f^*} denote the subgroup of T° generated by

References

[Bou02] N. Bourbaki. *Lie groups and Lie algebras. Chapters 4–6.* Elements of Mathematics (Berlin). Translated from the 1968 French original by Andrew Pressley. Springer-Verlag, Berlin, 2002.

[BT65] A. Borel and J. Tits. "Groupes réductifs". In: *Publications Mathématiques de l'IHÉS* 27 (1965), pp. 55–151.

[BT72] F. Bruhat and J. Tits. "Groupes réductifs sur un corps local : I. Données radicielles valuées". In: *Publications Mathématiques de l'IHÉS* 41 (1972), pp. 5–251.

[BT84a] F. Bruhat and J. Tits. "Groupes réductifs sur un corps local : II. Schémas en groupes. Existence d'une donnée radicielle valuée". In: *Publications Mathématiques de l'IHÉS* 60 (1984), pp. 5–184.

[BT84b] F. Bruhat and J. Tits. "Schémas en groupes et immeubles des groupes classiques sur un corps local". In: *Bull. Soc. Math. France* 112.2 (1984), pp. 259–301.

[BT87] F. Bruhat and J. Tits. "Schémas en groupes et immeubles des groupes classiques sur un corps local. II. Groupes unitaires". In: *Bull. Soc. Math. France* 115.2 (1987), pp. 141–195.

[Mil17] J. S. Milne. *Algebraic groups.* Vol. 170. Cambridge Studies in Advanced Mathematics. The theory of group schemes of finite type over a field. Cambridge University Press, Cambridge, 2017.

[Rou09] G. Rousseau. "Euclidean buildings". In: *Géométries à courbure négative ou nulle, groupes discrets et rigidités.* Vol. 18. Sémin. Congr. Soc. Math. France, Paris, 2009, pp. 77–116.

[RTW15] B. Rémy, A. Thuillier, and A. Werner. "Bruhat-Tits buildings and analytic geometry". In: *Berkovich spaces and applications.* Vol. 2119. Lecture Notes in Math. Springer, Cham, 2015, pp. 141–202.

[SGA3] M. Demazure and A. Grothendieck. *Schémas en Groupes I, II, III, Séminaire de Géométrie Algébrique 3.* Vol. 151,152,153. Lecture Notes in Math. Springer-Verlag, Berlin, 1970.

[Spr98] T. A. Springer. *Linear algebraic groups.* Second. Vol. 9. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1998, pp. xiv+334.

[Tit74] J. Tits. *Buildings of spherical type and finite BN-pairs.* Lecture Notes in Mathematics, Vol. 386. Springer-Verlag, Berlin-New York, 1974.

[Tit79] J. Tits. "Reductive groups over local fields". In: *Automorphic forms, representations and L-functions, Part 1*. Vol. 33. Proc. Sympos. Pure Math. AMS, 1979, pp. 29–69.

[Yu09] J.-K. Yu. "Bruhat-Tits theory and buildings". In: *Ottawa lectures on admissible representations of reductive p-adic groups*. Vol. 26. Fields Inst. Monogr. AMS, 2009, pp. 53–77.