Review on Bruhat-Tits buildings

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Abstract

The purpose of this note is to explain the theory of Bruhat-Tits buildings (resp. Tits buildings) for split reductive groups over local fields (resp. finite field). It is intended to be part of my thesis.

In 1950s-1960s, Jacques Tits [Tit74] introduced the notion of *buildings* to provide a uniform geometric framework for understanding semisimple Lie groups and later semisimple algebraic groups (and more generally, reductive groups) over arbitrary fields. Tits' buildings are polysimplicial complexes having nice symmetries so that reductive groups can act nicely on them. Later, François Bruhat and Jacques Tits develop the theory for reductive groups over non-Archimedean fields [BT72, BT84a, BT84b, BT87], giving refined structures on the buildings respecting the valuation. During the same period, they shift the view for a building from merely a polysimplicial complex to a complete metric space with non-positive curvature realizing it geometrically. The fruitful geometric/combinatorial nature of Bruhat-Tits buildings suggests them as non-Archimedean analogues of Riemannian symmetric spaces for real Lie groups.

We refer [RTW15, §3] for a short review on Bruhat-Tits theory, [Tit79, Yu09] for more systematic surveys, [Rou09] for a survey of general theory of Euclidean buildings and [Mil17, Spr98, SGA3] for reductive groups.

Contents

Part I
General theory of Buildings

§ 1 Projective geometry over $\mathbb{F}_q$ and $\mathbb{F}_1$

The notion of buildings does not come from nothing. Back to 1950s, Jacques Tits noticed the following interesting phenomenons [Tit57].

1.1. Let $\mathbb{PF}_q^n$ be the projective space associated to the vector space $\mathbb{F}_q^n$. Then its cardinality (or equivalently, the number of one-dimensional subspaces of $\mathbb{F}_q^n$) can be presented by the *quantum number* $[n]_q := \sum_{i=0}^{n-1} q^i$. If we passing to the limit $q \to 1$, then we get n, the number of coordinate labels $\{1, 2, \cdots, n\}$. Realizing how we count the cardinality of $\mathbb{PF}_q^n$ using the coordinates, we can view the set $P_n = \{1, 2, \cdots, n\}$ as the analogy of $\mathbb{PF}_q^n$ over the imaginary "prime field of characteristic one[1]" $\mathbb{F}_1$.

More generally, we can count points, lines, planes, ... in $\mathbb{PF}_q^n$. They correspond to points of the Grassmannians $\mathrm{Gr}\big(1, \mathbb{F}_q^n\big), \mathrm{Gr}\big(2, \mathbb{F}_q^n\big), \mathrm{Gr}\big(3, \mathbb{F}_q^n\big), \ldots$. In general, the *Grassmannian* $\mathrm{Gr}\big(k, \mathbb{F}_q^n\big)$ consists of subspaces of $\mathbb{F}_q^n$ having dimension k and its cardinality can be presented by the *quantum binomial* $\begin{bmatrix} n \\ k \end{bmatrix}_q$ (see 1.4). If we passing to the limit $q \to 1$, then we get $\binom{n}{k}$, which is the number of k-subsets of P_n.

1.2. The aboves can be organized into incidence geometry: namely the combinatorial gadget describing which proper subspace belongs to which. For the $\mathbb{F}_q$-side, a nontrivial proper subspace of $\mathbb{F}_q^n$ is *of color* k if it is k-dimensional and two such subspaces are said to be *incident* if one of them belongs to another properly. In this way, we organize nontrivial proper subspaces of $\mathbb{F}_q^n$ into a colored simplicial complex $\mathscr{B}(n, q)$, in which a k-simplex is a *flag*

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{k+1} \subsetneq \mathbb{F}_q^n$$

of subspaces of $\mathbb{F}_q^n$. For the $\mathbb{F}_1$-side, a nonempty proper subset of P_n is *of color* k if it has cardinality k and two such subsets are said to be *incident* if one of them belongs to another properly. In this way, we organize nonempty proper subsets of P_n into a colored simplicial complex $\mathscr{B}(n, 1)$, in which a k-simplex is a *flag*

$$\emptyset \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_{k+1} \subsetneq P_n$$

of subsets of P_n.

The two sides are related as follows. Fix a basis $\mathcal{B}$ of $\mathbb{F}_q^n$ (for example, the standard basis). Then to take a nontrivial proper subspace V of $\mathbb{F}_q^n$ having a basis which is part of $\mathcal{B}$ is amount to take a nonempty proper subset I of $\mathcal{B}$ (which is in bijection to P_n) and V is k-dimensional if and only if I has cardinality k. Moreover, to take a flag respecting the basis $\mathcal{B}$ in the sense that each V_i has a basis being part of $\mathcal{B}$ is amount to take a flag of nonempty proper subsets of $\mathcal{B}$.

[1] Namely the addition collapses. For an introduction, see [Lor18] especially §1.1.

However, different choices of bases may give the same subcomplex: for instance, when the two bases are different by a diagonal matrix. To avoid this, it is better to keep in the region of projective geometry. So instead of fix a basis, we fix a *frame*, that is a n-set of points $\{x_1, \cdots, x_n\}$ in $\mathbb{PF}_q^n$ in general position (namely, they do not belong to a common hyperplane), or equivalently, a n-set of one-dimensional subspaces $\{\lambda_1, \cdots, \lambda_n\}$ of $\mathbb{F}_q^n$ spanning $\mathbb{F}_q^n$. Then different choices of frames do give different subcomplexes of $\mathscr{B}(n, q)$.

In this way, we associate to each frame $\mathbf{\Lambda}$ a subcomplex $\mathscr{A}(\mathbf{\Lambda})$ of $\mathscr{B}(n, q)$ isomorphic to $\mathscr{B}(n, 1)$ and the complex $\mathscr{B}(n, q)$ is the union of them. This is the prototype of *buildings* and *apartments*.

1.3. There is a natural action of $G = \mathrm{GL}(\mathbb{F}_q^n)$, the *general linear group* (but essentially, it is the action of $\mathrm{PGL}(\mathbb{F}_q^n)$, the *projective linear group*) on $\mathscr{B}(n, q)$. This action comes from the action of $\mathrm{PGL}(\mathbb{F}_q^n)$ on $\mathbb{PF}_q^n$ and hence on each Grassmannian $\mathrm{Gr}(k, \mathbb{F}_q^n)$.

Fix a frame $\mathbf{\Lambda}$ (for example, the one given by the standard basis), then the stabilizer of the subcomplex $\mathscr{A}(\mathbf{\Lambda})$ is precisely the stabilizer of the frame itself. Let's denote it by $N(\mathbf{\Lambda})$ (in our example, it is the group of *monomial matrices*, i.e. matrices that have precisely one non-zero entry in each row and each column). The fixator of $\mathbf{\Lambda}$ acts trivially on $\mathscr{A}(\mathbf{\Lambda})$. Let's denote it by $Z(\mathbf{\Lambda})$ (in our example, it is the group of diagonal matrices). The quotient group $W(\mathbf{\Lambda}) := N(\mathbf{\Lambda})/Z(\mathbf{\Lambda})$ is called the *Weyl group* associated to $\mathbf{\Lambda}$. Then one finds that $W(\mathbf{\Lambda}) \cong \mathfrak{S}_n$, the *symmetric group*, which acts naturally on P_n and hence on $\mathscr{B}(n, 1)$ exactly as how $W(\mathbf{\Lambda})$ acts on $\mathscr{A}(\mathbf{\Lambda})$.

1.4. Let's consider the maximal simplices in $\mathscr{B}(n, q)$. From the description in 1.2, we see that a maximal simplex is nothing but a *complete flag*

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq \mathbb{F}_q^n$$

of subspaces of $\mathbb{F}_q^n$. Using an induction argument, it is not difficult to see that the number of complete flags is presented by the *quantum factorial* $[n]_q! := \prod_{i=1}^n [i]_q$. The quantum factorials are related to quantum binomials by the formula

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

This can be seen by picking the k-dimensional subspace V_k from a complete flag, breaking it into a complete flag of V_k and a complete flag of $\mathbb{F}_q^n/V_k$.

The maximal simplices in $\mathscr{B}(n, 1)$ are *complete flags*

$$\emptyset \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_{n-1} \subsetneq P_n$$

of subsets of P_n. There are $n!$ such complete flags. The number $n!$ is precisely the $q \to 1$ limit of $[n]_q!$.

The stabilizer of a complete flag is called a *Borel subgroup* of G. Note that the action of G on complete flags are transitive. Hence the number of complete flags is the index of a Borel subgroup in G.

Let's take the standard basis $\mathcal{B} = \{e_1, \cdots, e_n\}$ of $\mathbb{F}_q^n$ and set $V_k = \bigoplus_{i=1}^k \mathbb{F}_q e_i$. Then we get a complete flag and its stabilizer B is precisely the group of invertible upper triangular matrices. Hence B has order $q^{\binom{n}{2}}(q-1)^n$ and thus the order of G is $q^{\binom{n}{2}}(q-1)^n[n]_q!$.

1.5. We summarize aboves as follows.

(a) On the $\mathbb{F}_q$-side, we have the "building" $\mathscr{B}(n,q)$, which is the union of "apartments" $\mathscr{A}(\mathbf{\Lambda})$, one for each frame $\mathbf{\Lambda}$, and the number of them is

$$\frac{\#G}{\#N(\mathbf{\Lambda})} = \frac{\#B \cdot \#\{\text{complete flags}\}}{\#Z(\mathbf{\Lambda}) \cdot \#\mathfrak{S}_n} = \frac{q^{\binom{n}{2}}(q-1)^n[n]_q!}{(q-1)^n n!} = \frac{q^{\binom{n}{2}}[n]_q!}{n!}.$$

Each "apartments" $\mathscr{A}(\mathbf{\Lambda})$ is isomorphic to $\mathscr{B}(n,1)$, the one on the $\mathbb{F}_1$-side. Hence the "building" $\mathscr{B}(n,q)$ can be seen as so many copies of $\mathscr{B}(n,1)$ gluing together. By passing to the limit $q \to 1$, this quantity gives 1, coinciding with the number of "apartments" in $\mathscr{B}(n,1)$.

(b) The quantum factorial $[n]_q!$ counts the maximal simplices in $\mathscr{B}(n,q)$, which becomes $n!$, the number of maximal simplices in $\mathscr{B}(n,1)$ by taking the limit $q \to 1$.

(c) The quantum binomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$ counts the vertices of color k in $\mathscr{B}(n,q)$, which becomes $\binom{n}{k}$, the number of vertices of color k in $\mathscr{B}(n,1)$ by taking the limit $q \to 1$.

(d) There are more combinatorial quantities in $\mathscr{B}(n,q)$ becomes one for $\mathscr{B}(n,1)$ by taking the limit $q \to 1$.

1.6. Tits's observations are not limited for $\mathrm{PGL}(\mathbb{F}_q^n)$. In fact, he did for all semisimple groups over $\mathbb{F}_q$. Of course, there is no $\mathbb{F}_1$-geometry back to Tits' time, but it seems the above observations inspire him to develope the theory of buildings with the following principal:

> Buildings are multifold apartments and apartments are $q \to 1$ limit case of buildings, which can be thought as forgetting the additive arithmetic of the base field.

§ 2 Spherical Buildings

Before moving on, we now give a formal definition of polysimplicial complexes.

2.1 Definition. An *(abstract) simplicial complex* is a nonempty poset $\mathcal{S}$ (whose members are called *simplices*) satisfying

S1. any two simplices σ, τ have an infimum $\sigma \cap \tau$;

So there is a unique smallest element in $\mathcal{S}$, called the *empty simplex*, denoted by $\emptyset$.

S2. for each simplex σ the poset $\mathcal{S}_{\leqslant\sigma}$ of simplices smaller than σ (they are called *faces* of σ) form a *Boolean lattice of rank* r, namely isomorphic to the power set of a r-set, for some finite r. In this case, we see σ is of *dimension* $r-1$ and is a $(r-1)$-*simplex.*

The *dimension* of $\mathcal{S}$ is the supremum of dimensions of its simplices. The minimal nonempty simplices are of dimension 0 and are thus called *vertices.* Let $\mathcal{V}$ denote the set of vertices. Then $\mathcal{S}$ can be identified with a poset of nonempty subsets of $\mathcal{V}$.

A *morphism* between simplicial complexes is a map preserving infima, suprema and the empty simplex $\emptyset$. Note that this implies that such a morphism is determined by vertices. So equivalently, such a morphism is a map between vertices extending to a monotone preserving simplices. A morphism is said to *fix a simplex σ pointwise* if it induces an isomorphism on $\mathcal{S}_{\leqslant\sigma}$.

A *polysimplicial complex* is a cartesian product of simplicial complexes (in the category of posets) and morphisms between polysimplicial complexes are therefore defined.

One can verify that $\mathscr{B}(n,q)$ and $\mathscr{B}(n,1)$ are simplicial complexes.

2.2. Let's analyse how the "apartments" $\mathscr{A}(\mathbf{\Lambda})$ are glued into the "building" $\mathscr{B}(n,q)$.

(a) *For any two simplices F, F' in $\mathscr{B}(n,q)$, there is an "apartment" $\mathscr{A}(\mathbf{\Lambda})$ containing both of them.*

Proof. We may assume F, F' are maximal, i.e. being complete flags:

$$\begin{aligned} F\colon 0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = \mathbb{F}_q^n, \\ F'\colon 0 = V_0' \subsetneq V_1' \subsetneq \cdots \subsetneq V_{n-1}' \subsetneq V_n' = \mathbb{F}_q^n. \end{aligned}$$

Then we may view them as composition series for $\mathbb{F}_q^n$. Therefore by *Jordan-Hölder Theorem*, there is a permutation π of $P_n = \{1, 2, \cdots, n\}$ such that whenever $j = \pi(i)$, we have isomorphisms

$$\frac{V_i}{V_{i-1}} \xleftarrow{\sim} \frac{V_i \cap V_j'}{(V_{i-1}\cap V_j') + (V_i \cap V_{j-1}')} \xrightarrow{\sim} \frac{V_j'}{V_{j-1}'}$$

induced from inclusions. Let λ_i be the one-dimensional subspace of $V_i \cap V_j'$ whose image in above quotients are non-trivial. Then $\mathbf{\Lambda} = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is a frame with $\mathscr{A}(\mathbf{\Lambda})$ containing both F and F'. □

(b) *If $\mathscr{A}(\mathbf{\Lambda})$ and $\mathscr{A}(\mathbf{\Lambda}')$ are two "apartments" containing both F and F', then there is an isomorphism between them fixing F and F' pointwise.*

Proof. Again, we may assume F, F' are maximal and let V_i, V_i', λ_i be as above. Then $i \mapsto \lambda_i$ induces an isomorphism $\phi_{\mathbf{\Lambda}}\colon \mathscr{B}(n,1) \to \mathscr{A}(\mathbf{\Lambda})$. The inverse of it can be described by vertices as

$$\psi_{\mathbf{\Lambda}}\colon U \mapsto \{i \in P_n \mid U \cap V_{i-1} \neq U \cap V_i\}.$$

Similarly we have isomorphism $\phi_{\mathbf{\Lambda}'} : \mathscr{B}(n,1) \to \mathscr{A}(\mathbf{\Lambda}')$ and its inverse $\psi_{\mathbf{\Lambda}'}$. Note that the morphisms $\psi_{\mathbf{\Lambda}}$ (and similarly $\psi_{\mathbf{\Lambda}'}$) is determined by the complete flag F, we conclude that $\psi_{\mathbf{\Lambda}}$ and $\psi_{\mathbf{\Lambda}'}$ coincide on the intersection of $\mathscr{A}(\mathbf{\Lambda})$ and $\mathscr{A}(\mathbf{\Lambda}')$. Then $\phi_{\mathbf{\Lambda}'} \circ \psi_{\mathbf{\Lambda}}$ is an isomorphism between $\mathscr{A}(\mathbf{\Lambda})$ and $\mathscr{A}(\mathbf{\Lambda}')$ fixing F and F' pointwise. □

Then the buildings can be defined as follows.

2.3 Definition. A *building* is a polysimplicial complex $\mathscr{B}$ equipped with a family $\mathcal{A}$ of subcomplexes of $\mathscr{B}$, whose members are called *apartments*, such that the following axioms are satisfied.

B0. Each apartment $A \in \mathcal{A}$ is isomorphic to an (abstract) apartment $\mathscr{A}$.

B1. For any two simplices F, F', there is an apartment A containing them.

B2. If A, A' are two apartments containing both F and F', then there is an isomorphism between A and A' fixing F and F' pointwise.

Of course, one has to define what is an apartment to make this definition work.

2.4. Let's analyse what does the "apartment" $\mathscr{B}(n,1)$ look like.

(a) *All maximal simplices have the same dimension.*

Proof. This is clear, they are precisely the $(n-1)$-subsets of P_n. □

(b) *Any two maximal simplices C, C' are connected by a sequence $(C_0, C_1, \cdots, C_s)$ with $C_0 = C$ and $C_s = C'$ such that for each i, $C_{i-1} \cap C_i$ has codimension 1 in both C_{i-1} and C_i.*

Proof. Note that a maximal simplex in $\mathscr{B}(n,1)$ is complete flag, hence a sequence $(i_1, i_2, \cdots, i_{n-1})$, which can be identified with an ordering of P_n. Hence any two such simplices are different by a permutation $\pi \in \mathfrak{S}_n$. But any permutation can be written as the composition of transpositions while two sequences different by a transposition meets in a sequence with one term being removed. □

In general, a polysimplicial complex has above properties is called a *chamber complex* and its maximal simplices are called *chambers*. A one-codimensional face of a chamber is called a *panel*. A sequence $(C_0, C_1, \cdots, C_s)$ connecting two chambers by panels is called a *gallery*. Note that any Boolean lattice is a chamber complex with a unique chamber: its maximal element. A *chamber map* between chamber complexes is a morphism mapping chambers to chambers.

(c) *There is a coloring, namely a chamber map from the complex to a Boolean lattice.*

Proof. It suffices to define colors for vertices. Then the color of a simplex would be the set of the colors of its vertices. For instance, one can define the *color* of a vertex as its cardinality as in 1.2. □

In general, a chamber complex has this property is said to be *colorable*. It is worth to notice that any two colorings are different by an isomorphism of Boolean lattices (in other words, up to a permutation of the colors of vertices).

(d) *The Weyl group acts transitively on the simplices of the same color.*

Proof. Two simplices $F = (I_i)$ and $F' = (I'_i)$ are of the same color means two things: first, they have the same number of entries; second, each pair of entries (I_i, I'_i) have the same cardinality. This is precisely the condition when there is a permutation $\pi \in \mathfrak{S}_n$ interchanging them. □

(e) *Fix a chamber C, then the stabilizers of its panels are all of order* 2 *and their generators s_j form a generating system S of the Weyl group with generating relations of the form $(s_i s_j)^{m_{ij}} = 1$.*

Proof. As in (b), a chamber C is a sequence $(i_1, i_2, \cdots, i_{n-1})$. Let i_n be the complement of this sequence in P_n. Then for each panel obtained from C by deleting i_j, let s_j be the transposition (i_j, i_n). Then this panel's stabilizer is precisely $\{1, s_j\}$ and one can verify the system $S = \{s_1, \cdots, s_{n-1}\}$ satisfies the requirement. □

Note that it follows from this property that the stabilizer of a face of C is generated by those s_j with j not a color of its vertex. Furthermore, the complex $\mathscr{B}(n, 1)$ can be built from the pair (W, S) of the Weyl group $W = \mathfrak{S}_n$ and the system $S = (s_j)$ of generators in (e). In deed, any face of the chamber C corresponds to the subset I of S generating its stabilizer and any simplex is translated to such a face by an element of W, unique up to the stabilizer $\langle I \rangle$. Therefore, the simplices in $\mathscr{B}(n, 1)$ can be identified with the cosets $w\langle I \rangle$ with $w \in W$ and $I \subseteq S$.

2.5 Definition. A *Coxeter system* is a pair (W, S) of a group W and a system of its generators $S = \{s_1, s_2, \cdots, s_n\}$ such that all s_i are of order 2 and the generating relations for S are of the form $(s_i s_j)^{m_{ij}} = 1$. Its *Coxeter complex* $\Sigma(W, S)$ is the polysimplicial complex defined as the complex of cosets of the form $w\langle I \rangle$ with $w \in W$ and $I \subseteq S$, where the order is given by reverse inclusion.

Then 2.4 shows that $\mathscr{B}(n, 1)$ is isomorphic to the Coxeter complex $\Sigma(\mathfrak{S}_n, S)$, where S can be chosen to be any generating system of transpositions, for instance $S = \{(1, n), (2, n), \cdots, (n-1, n)\}$.

A *morphism* between Coxeter systems (W, S) and (W', S') is a group homomorphism $f \colon W \to W'$ such that $f(S) \subseteq S'$. Therefore a Coxeter system (W, S) is a *product* of subsystems $(W_i, S_i)_{1 \leqslant i \leqslant m}$ if we have a group decomposition $W = W_1 \times \cdots \times W_m$

and a set decomposition $S = S_1 \cup \cdots \cup S_m$. A Coxeter system is *irreducible* if it can not be decomposed into proper subsystems.

One can see that morphisms between Coxeter systems induce morphisms between their Coxeter complexes and such a functor is compatible with the decompositions. In particular, Coxeter complex of an irreducible Coxeter system is simplicial.

Now, we can complete Definition 2.3 by define an *apartment* being a polysimplicial complex isomorphic to the Coxeter complex of some Coxeter system. The complex $\Sigma(W, S)$ is finite if and only if W is finite. In this case, it is said to be *spherical*. A building whose apartments are isomorphic to a spherical Coxeter complex is called a *spherical building*.

We have seen that $\mathscr{B}(n, q)$ is such a building: its apartments are isomorphic to the Coxeter complex $\Sigma(\mathfrak{S}_n, S)$. This is not an accident. In fact, any reductive group over arbitrary field would give rise to such a building. They are called *Tits buildings*. We refer [Bou02, chap.IV] for the theory of Coxeter systems and [Tit74] for a treatment of Tits buildings in the language of Coxeter complexes.

§ 3 Euclidean apartments

Although buildings can be defined and studied in a pure combinatorial way, it would be more intuitive and convenient if we can also realizing them geometrically.

3.1. One way to visualize the Coxeter complex $\Sigma(\mathfrak{S}_n, S)$ is as follows. The group $\mathfrak{S}_n$ acts faithfully on $\mathbb{R}^n$ as permutations of the coordinates under the standard basis. For any transposition (i, j), its fixed points is the hyperplane $\{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i = x_j\}$ and it thus acts as the reflection respect to this hyperplane. Therefore the group $\mathfrak{S}_n$ can be determined by the reflections/hyperplanes defined by the transpositions. Moreover, the hyperplanes partition $\mathbb{R}^n$ into pieces of various dimensions with an obvious order relation: one such a piece belongs to the closure of another. This gives rise to a complex isomorphic to $\Sigma(\mathfrak{S}_n, S)$. The system S can be obtained as the reflections respect to a chamber.

With this example in mind, we can obtain the following definition.

3.2 Definition. A *(Euclidean) apartment* $\mathscr{A}$ is a Euclidean affine space $\mathbb{A}$ equipped with a reflection group W (called its *Weyl group*) on it.

Let $\mathbb{A}$ be a Euclidean affine space. We use ${}^v\mathbb{A}$ to denote its associated vector space. For an affine transformation f on $\mathbb{A}$, we use ${}^vf$ to denote its *vectorial part*. For an affine subspace X of $\mathbb{A}$, we use ${}^vX$ to denote its *direction*.

A *reflection* on $\mathbb{A}$ is an affine isometry whose fixed points form a hyperplane. Any hyperplane H is associated with a reflection r_H with respect to it.

A *reflection group* W is a group of affine isometries generated by reflections and such that its vectorial part ${}^vW$ is finite. W is said to be *irreducible* if ${}^vW$ acts irreducibly on ${}^v\mathbb{A}$ and is said to be *essential* if ${}^vW$ acts essentially on ${}^v\mathbb{A}$ (that is, there is no non-trivial fixed point). An apartment is said to be *irreducible* (resp. *essential*, *trivial*, etc.) if its reflection group is so.

3.3. A *morphism* between apartments $(\mathbb{A}, W)$ and $(\mathbb{A}', W')$ is a continuous affine map $f\colon \mathbb{A} \to \mathbb{A}'$ with a group homomorphism $\phi\colon W \to W'$ such that $\phi(w).f(x) = f(w.x)$ for all $w \in W$ and $x \in \mathbb{A}$. Therefore an apartment $(\mathbb{A}, W)$ is said to be a *product* of apartments $(\mathbb{A}_i, W_i)_{1 \leqslant i \leqslant m}$ if we have an orthogonal decomposition $\mathbb{A} = \mathbb{A}_1 \times \cdots \times \mathbb{A}_m$ and a group decomposition $W = W_1 \times \cdots \times W_m$ such that each W_i acts trivially on the orthogonal complement of $\mathbb{A}_i$.

Any apartment $\mathscr{A}$ admits a decomposition [Bou02, chap.V, §3, no.8]

$$\mathscr{A} = \mathscr{A}_0 \times \mathscr{A}_1 \times \cdots \times \mathscr{A}_m,$$

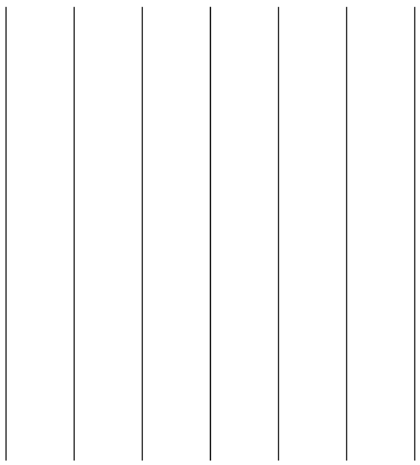
where $\mathscr{A}_0$ is trivial and each $\mathscr{A}_i$ (for $1 \leqslant i \leqslant m$) is irreducible.

3.4. Let $\mathscr{A}$ be an apartment with an essential irreducible reflection group W. Let $T = \ker(W \to {}^vW)$ be the translation group. There are three possibilities [Rou09, 3.3]:

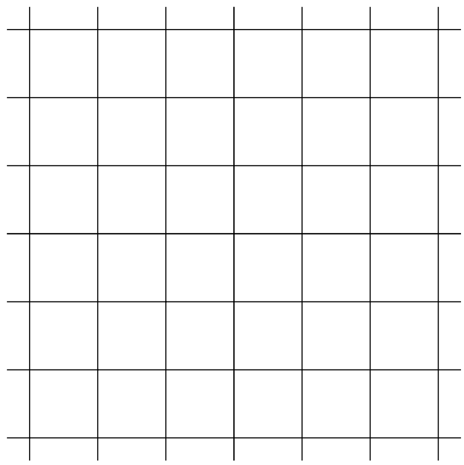
1. If T is trivial, then $\mathscr{A}$ is said to be *spherical*.
2. If T is a lattice in ${}^v\mathbb{A}$, then $\mathscr{A}$ is said to be *discrete affine*.
3. If T is dense in ${}^v\mathbb{A}$, then $\mathscr{A}$ is said to be *dense affine*.

Throughout this note, all apartments are assumed to be *discrete*, namely no irreducible component is dense affine.

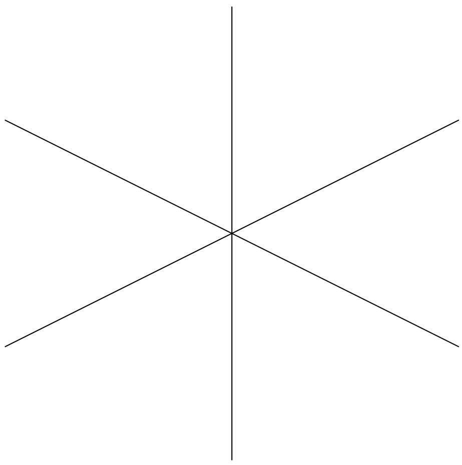
3.5 Example. Before moving on, we give pictures showing some examples, where the first one labelled $\widetilde{A_1} \times \mathbf{0}$ is a *non-essential* apartment while others are essential, the second one labelled $\widetilde{A_1} \times \widetilde{A_1}$ is a *non-irreducible* apartment while others are irreducible, the third one labelled A_2 is *spherical* while others are *affine*, the last one labelled $\widetilde{C_2}$ has *non-special* vertices while others do not.



$\widetilde{A_1} \times \mathbf{0}$



$\widetilde{A_1} \times \widetilde{A_1}$



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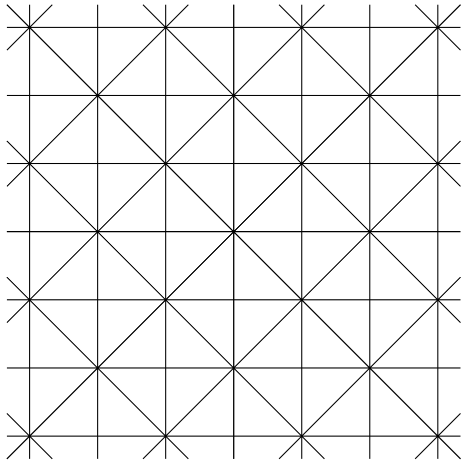


$\widetilde{C_2}$

3.6. Let $\mathscr{A} = (\mathbb{A}, W)$ be an apartment.

The hyperplanes of fixed points of reflections in W are called the *walls* in $\mathscr{A}$. The set $\mathcal{H}$ of walls is stable under W and completely determines it.

A *half-apartment* (also called an *affine root* in [BT72, 1.3.3]) is a closed half-space α of $\mathbb{A}$ bounded by a wall $\partial\alpha$, called its *wall*.

A *facet* in $\mathscr{A}$ is an equivalence class in $\mathbb{A}$ for the relation "x and y are contained in the same half-apartments". A facet F is an open convex subset of in the affine subspace (called the *support* of F) it spans.

The set $\mathcal{F}$ of facets admits an order: a facet F is said to be a *face* of another F', denoted by $F \leqslant F'$, if F is contained in the closure of F'. Such an order gives rise to a polysimplicial complex. To see this, first notice that facets in an apartment are compatible with its decomposition into irreducible components. Hence we may assume our apartment $\mathscr{A}$ is irreducible and essential. Then this can be seen from the fact that any triangulation of a topological space gives rise to a simplicial complex (indeed, this is where the notion comes from). When $\mathscr{A}$ is discrete affine, its facets already triangulate the ambient space. When $\mathscr{A}$ is spherical, its facets triangulate the unit sphere. This is why it is called spherical.

3.7. The maximal facets are called *chambers* (or *alcoves*). They are the connected components of the complement of the union of all walls in $\mathbb{A}$. The Weyl group W acts simply transitively on the set $\mathcal{C}$ of chambers [Bou02, chap.V, §3, no.2, th.1].

Let C be a chamber. Then its closure $\overline{C}$ is a fundamental domain of W in $\mathbb{A}$ [Bou02, chap.V, §3, no.3, th.2] and is the intersection of some half-apartments, whose walls are called the *walls* of C. Equivalently, the walls of C are the supports of panels of it, where a *panel* means a maximal proper face of C. Moreover, W is generated by the set S of reflections with respect to the walls of C and the pair (W, S) is a Coxeter system [Bou02, chap.V, §3, no.2, th.1]. The projection of C onto an irreducible component $\mathscr{A}_i$ is again a chamber in it and induces an irreducible Coxeter system (W_i, S_i). Then (W, S) is the product of them. In other words, decomposition of the pair of $(\mathscr{A}, C)$

of an apartment and a chamber is compatible with the decomposition of the Coxeter system (W, S) it defines.

A *type function* on $\mathscr{A}$ is a morphism τ from the complex $\mathcal{F}$ of facets to a Boolean lattice, which maps chambers to the maximal element and is W*-stable* in the sense that for any facet F and any $w \in W$, $\tau(F) = \tau(w.F)$. The image of this function is denoted by $\mathcal{T}$ and its members are called *types*. This notion is essentially the same as a *coloring* as in 2.4(c) plus 2.4(d). They differs in practice: for a coloring, the target Boolean lattice is viewed as a power set $\mathscr{P}(\mathfrak{I})$ with its usual order $\subseteq$, while for a type function, we use the reverse order $\supseteq$. In other words, a face of type I is of color $\neg I := \mathfrak{I} \setminus I$.

Since any facet is transformed by W to a unique face of C, the type function τ is completely determined by the types of its panels, for which we may viewed as an indexing of S. Indeed, let I be a type, then the set C_I of points $x \in \overline{C}$ such that the reflections $s \in S$ fixing x are indexed by I is a face of C of type I and its stabilizer is the subgroup W_I of W generated by the reflections indexed by I [Bou02, chap.V, §3, no.3, prop.1]. Then $\tau(F) = I$ if and only if F is transformed to C_I.

3.8. A reflection group W is said to be *linear* if it fixes a point. This is the case if and only if W is finite [Bou02, chap.V, §3, no.9]. If this is the case, we can identify W with its vectorial part ${}^vW$ by choosing the fixed point to be the origin of $\mathbb{A}$.

Conversely, the vectorial part ${}^vW$ of the Weyl group W can be viewed as a linear relfection group on ${}^v\mathbb{A}$. The spherical apartment ${}^v\mathscr{A} = ({}^v\mathbb{A}, {}^vW)$ obtained in this way is called the *vectorial apartment* of $\mathscr{A}$. The walls (resp. facets, chambers) in ${}^v\mathscr{A}$ are called the *vectorial walls* (resp. *vectorial facets*, *vectorial chambers*) and the set of them is denoted by ${}^v\mathcal{H}$ (resp. ${}^v\mathcal{F}$, ${}^v\mathcal{C}$). Note that the vectorial walls are precisely the directions of walls in $\mathscr{A}$.

3.9. Let x be a point in $\mathscr{A}$. The stabilizer W_x of x is a linear reflection group whose vectorial part ${}^vW_x$ is a subgroup of ${}^vW$. The apartment $\mathscr{A}_x = (\mathbb{A}, W_x)$ is called the *spherical apartment at* x. The walls in $\mathscr{A}_x$ are precisely the walls in $\mathscr{A}$ passing through x and the set of them is denoted by $\mathcal{H}_x$. The facets (resp. chambers) in $\mathscr{A}_x$ are called the *vectorial facets with base point* x (resp. *vectorial chambers with base point* x) and the set of them is denoted by $\mathcal{F}_x$ (resp. $\mathcal{C}_x$).

A point $x \in \mathbb{A}$ is said to be *special* if the spherical apartment $\mathscr{A}_x$ is isomorphic to ${}^v\mathscr{A}$, or equivalently, the set $\mathcal{H}_x$ is a complete set of representatives of ${}^v\mathcal{H}$. This can happen only if x belongs to a minimal facet.

3.10. The minimal facets are called *vertices*. The set of vertices is denoted by $\mathcal{V}$. When the apartment is essential, they are points. From now on, all apartments are assumed to be essential unless otherwise specified[2].

Under this assumption, every special point is a vertex. Furthermore, any special vertex is an extremal point of the closure of some chamber. Conversely, any chamber admits a special point as an extremal point of its closure [Bou02, chap.V, §3, no.10, prop.11's cor.]. However, not all extremal points, hence not all vertices are special (see $\widetilde{C}_2$ in Example 3.5 for an example).

[2]This means we will only focus on *reduced* buildings, rather than *extended* buildings.

§ 4 Root systems

Before moving on to the definition of buildings, let's look at some examples of Euclidean apartments arising from root systems (as well as root data). They are the key examples in the study of reductive groups.

4.1. Let V be a Euclidean vector space and V^* its dual space. For any $a \in V^* \setminus \{0\}$, let r_a be the reflection with respect to the hyperplane $H_a := \mathrm{Ker}(a)$ and $a^\vee$ the vector orthogonal to H_a satisfying $a(a^\vee) = 2$. So for any $x \in V$, we have $r_a(x) = x - a(x)a^\vee$. Note that r_a also induces a reflection on V^*, namely $f \mapsto f - f(a^\vee)a$. A finite spanning subset $\Phi \subseteq V^* \setminus \{0\}$ is called a *root system* on V if

RS1. for any $a \in \Phi$, $r_a(\Phi) = \Phi$;

RS2. for any $a, b \in \Phi$, $a(b^\vee) \in \mathbb{Z}$;

and is *reduced* if

RS3. for any $a \in \Phi$, $\mathbb{R}a \cap \Phi = \{\pm a\}$.

From now on, all root systems are assumed to be reduced[3].

Elements of Φ are called *roots* in Φ. For a root $a \in \Phi$, the vector $a^\vee$ is called its *coroot*. A subset $\Psi \subseteq \Phi$ is called a *subroot system* if for any $a \in \Psi$, $r_a(\Psi) = \Psi$, and is said to be *closed* if for any $a, b \in \Psi$ such that $a + b$ is a root, $a + b \in \Psi$.

Any root system Φ admits a *Weyl group* ${}^vW(\Phi)$, that is the reflection group of V generated by r_a for $a \in \Phi$. It is a linear reflection group with walls H_a for $a \in \Phi$. In this way, we get a spherical apartment ${}^v\mathscr{A}(\Phi) := (V, {}^vW(\Phi))$. Note that not all spherical apartments arise in this way (see [Bou02, chap.VI, §2, no.5, prop.9]) and non-isomorphic root systems may have isomorphic Weyl groups (for instance root systems of types B_n and C_n).

Root systems can arise from root data.

4.2 Definition. A *(reduced) root datum*[4] $\mathcal{R}$ is a quadruple $(\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ in which

- X and $\mathsf{X}^\vee$ are free $\mathbb{Z}$-modules of finite rank in duality by a pairing

$$\langle \cdot\,, \cdot \rangle \colon \mathsf{X} \times \mathsf{X}^\vee \to \mathbb{Z},$$

- Φ and $\Phi^\vee$ are finite subsets of $\mathsf{X} \setminus \{0\}$ and $\mathsf{X}^\vee \setminus \{0\}$ respectively, in bijection by a correspondence $a \leftrightarrow a^\vee$,

satisfying

RD1. for any $a \in \Phi$, $\langle a, a^\vee \rangle = 2$;

[3]This means we will only focus on split reductive groups.
[4]in the sense of [SGA3, XXI, 1.1.1].

RD2. for any $a \in \Phi$, the "reflection" $r_a\colon x \mapsto x - \langle x, a^\vee \rangle a$ preserves Φ and the "reflection" $r_a\colon y \mapsto y - \langle a, y \rangle a^\vee$ preserves $\Phi^\vee$;

RD3. for any $a \in \Phi$, $\mathbb{Z}a \cap \Phi = \{\pm a\}$.

Note that we do not distinguish the two kinds of "reflections" in symbols since they form isomorphic finite groups of automorphisms on X and $\mathsf{X}^\vee$ respectively and therefore it is better to view them as two representations of a same finite group ${}^vW(\mathcal{R})$. This group is called the *Weyl group* of the root datum.

4.3. If $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ is a root datum, then its Weyl group acts on the real vector space $\mathsf{X}^\vee_\mathbb{R} := \mathsf{X}^\vee \otimes \mathbb{R}$ and there is a unique inner product on it invariant under the action. Let V be the subspace of $\mathsf{X}^\vee_\mathbb{R}$ spanned by $\Phi^\vee$. Then Φ is a (reduced) root system on the Euclidean vector space V.

In general, V is not the entire $\mathsf{X}^\vee_\mathbb{R}$. When it is, we say $\mathcal{R}$ is *semisimple*. So the apartment associated to root systems can also be viewed as the apartment associated to semisimple root data. As for the non-semisimple ones, they give rise to non-essential apartments and hence are ignored in this note.

4.4. A *lattice* L in a $\mathbb{R}$-vector space V is a finitely generated $\mathbb{Z}$-submodule of V spanning V. Its *dual lattice* L^* is the lattice in the dual space V^* consisting of those functionals $f \in V^*$ such that $f(L) \subseteq \mathbb{Z}$.

Given a root system Φ on a Euclidean vector space V, there are four lattices:

$\mathcal{Q}$ the *root lattice*, which is the lattice in V generated by the roots;

$\mathcal{Q}^\vee$ the *coroot lattice*, which is the lattice in V^* generated by the coroots;

$\mathcal{P}$ the *weight lattice*, which is the dual lattice of $\mathcal{Q}^\vee$ in V;

$\mathcal{P}^\vee$ the *coweight lattice*, which is the dual lattice of $\mathcal{Q}$ in V^*.

Suppose the root system Φ is given by a semisimple root data $\mathcal{R}$. Then one can see that X is a lattice in V between $\mathcal{Q}$ and $\mathcal{P}$.

4.5. A root system Φ is said to be *irreducible* if it cannot be written as the union of two proper subsets such that they are orthogonal to each other. A root system Φ is irreducible if and only if so is its Weyl group ${}^vW(\Phi)$ [Bou02, chap.VI, §1, no.2, prop.5's cor.]. Any root system decomposes into disjoint union of irreducible ones and such a decomposition is compatible with the decomposition of Weyl groups and hence of apartments [Bou02, chap.VI, §1, no.2, prop.6 and 7].

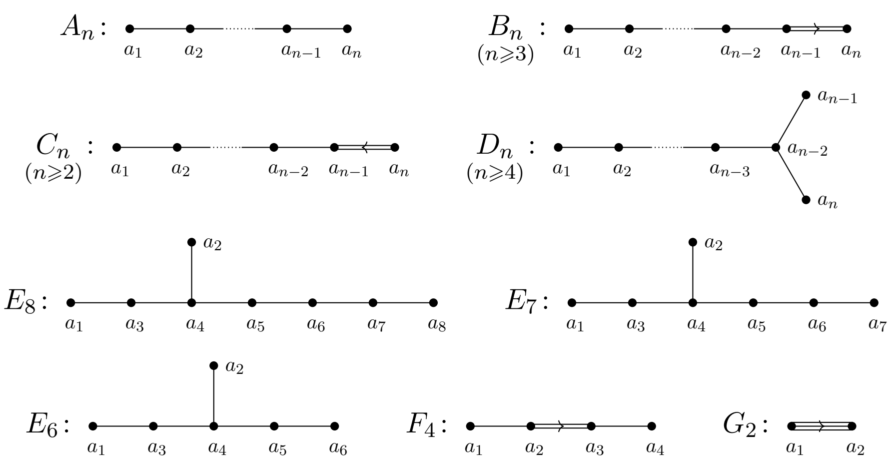
4.6. Let Φ be a root system. Then there is a closed subset Φ^+ of Φ such that for any $a \in \Phi$, either $a \in \Phi^+$ or $-a \in \Phi^+$. This set is called the set of *positive roots*. Once such a set is chosen, elements in the set $\Phi^- := -\Phi^+$ are called *negative roots*. A positive root is called a *simple root* if it cannot be written as the sum of two positive roots. The set Δ of simple roots form a *basis* of Φ in the sense that any root is a $\mathbb{Z}$-linear combination of simple roots and its coefficients are either all non-negative or all non-positive [Bou02, chap.VI, §1, no.6, th.3]. The cardinality of the set Δ is called the *rank* of Φ and is independent of the choice of Δ. Indeed, it equals $\dim(V)$.

Let Δ be a basis of Φ. Then the set ${}^v\mathscr{C} = \{x \in V \mid \forall a \in \Delta, a(x) > 0\}$ is a vectorial chamber, called the *Weyl chamber* associated to Δ [Bou02, chap.VI, §1, no.5, th.2]. Conversely, let ${}^v\mathscr{C}$ be a vectorial chamber. Then for any $x \in {}^v\mathscr{C}$, the sets $\Phi^+ = \{a \in \Phi \mid a(x) > 0\}$ and $\Phi^- = \{a \in \Phi \mid a(x) < 0\}$ form a partition of Φ into positive and negative roots and are independent of the choice of x. Then one can obtained a basis Δ by taking the simple roots. But there is a more direct description: they are the roots defining walls of ${}^v\mathscr{C}$ and point inside. As vectorial chambers are Weyl chambers associated to some choice of basis, we call them *Weyl chambers* to specify that they are chambers in the spherical apartment ${}^v\mathscr{A}(\Phi)$.

4.7. The relation between simple roots and types is the following. First, the Weyl group ${}^vW$ is generated by r_a for $a \in \Delta$ as they are the roots defining walls of ${}^v\mathscr{C}$ and point inside. Therefore a type $I \in \mathcal{T}$ corresponds to a subset of Δ. From now on, we do not distinguish them. Then the face of ${}^v\mathscr{C}$ corresponding to I is the set ${}^v\mathscr{C}_I = \{x \in V \mid \forall a \in I, a(x) = 0, \forall a \in \Delta \setminus I, a(x) > 0\}$. Let Φ_I be the subroot system of Φ generated by I, then W_I is the Weyl group of it. The set $\Psi = \Phi_I \cup \Phi^+$ has the property that $\Psi \cup (-\Psi) = \Phi$ and is closed. Such kind of subsets of Φ are said to be *parabolic*. Given a parabolic subset Ψ of Φ containing Φ^+, then the simple roots in $\Psi \cap (-\Psi) \cap \Phi^+$ gives the type I. See [Bou02, chap.VI, §1, no.7].

4.8. Given a basis Δ of a root system Φ, its *Dynkin diagram* is defined as follows. The vertices are simple roots of Φ and the number of edges between two vertices is $4\cos^2(\theta)$ if the angle between then is θ. Furthermore, these edges are decorated with arrows pointing from longer root to shorter root. It turns out that, up to graph isomorphisms, the Dynkin diagram is independent of the choice of the basis Δ.

From above description, we see that Φ is irreducible if and only if its Dynkin diagram is connected. The Dynkin diagrams of irreducible root systems are classified as follows [Bou02, chap.VI, §4, no.2, th.3], where the subscription n in the notation X_n denote the rank of it.



A spherical apartment is said to be *of type* X_n if it is isomorphic to ${}^v\!\mathscr{A}(\Phi)$ for an irreducible root system Φ of type X_n.

4.9. Let $\mathbb{A}$ be an affine space underlying V with a specified point o. For any $a \in V^*$ and $k \in \mathbb{R}$, denote the affine function $x \mapsto a(x-o)+k$ on V by $a+k$ and denote the closed half-space $\{x \in \mathbb{A} \mid (a+k)(x) \geqslant 0\}$ by α_{a+k}. For each $a \in \Phi$, let Γ_a be a nonempty subset of $\mathbb{R}$. The affine function $a+k$ is called an *affine root* if $a \in \Phi$ and $k \in \Gamma_a$. Let Σ denote the set of closed half-spaces α_{a+k} with $a+k$ an affine root. Then $a+k \mapsto \alpha_{a+k}$ gives rise to a bijection between the set of affine roots and Σ. For this reason, we will not distinguish the affine root $a+k$ and the closed half-space α_{a+k} and will call Σ the *affine root system*[5]. The roots are vectorial part of affine roots. Hence we denote Φ by ${}^v\Sigma$ and call it the *vectorial root system* of Σ.

For $\alpha = \alpha_{a+k}$ an affine root, let ${}^v\alpha$ denote its vectorial part a, let $\partial\alpha$ denote its boundary $\{x \in \mathbb{A} \mid (a+k)(x) = 0\}$, let r_α denote the reflection with respect to $\partial\alpha$, let α^* denote the other affine root sharing the same boundary with α, that is $\overline{\mathbb{A} \setminus \alpha}$, and let α_+ denote the intersection of affine roots containing a neighborhood of α.

4.10. Let Σ be an affine root system on a Euclidean affine space $\mathbb{A}$, its *affine Weyl group* $W(\Sigma)$ is the reflection group on $\mathbb{A}$ generated by r_α for all $\alpha \in \Sigma$. In this way, we obtain an apartment $\mathscr{A}(\Sigma) := (\mathbb{A}, W(\Sigma))$ with ${}^v\!\mathscr{A}({}^v\Sigma)$ being its vectorial apartment. Suppose all Γ_a are the same discrete subgroup $\Gamma \neq 0$ of $\mathbb{R}$, then the walls in the apartment $\mathscr{A}(\Sigma)$ are precisely the boundaries $\partial\alpha$ with $\alpha \in \Sigma$ [Bou02, chap.VI, §2, no.1, prop.2]. For x a point in the apartment $\mathscr{A}(\Sigma)$, let Σ_x be the set of affine roots α such that $x \in \partial\alpha$ and let ${}^v\Sigma_x$ be its vectorial part. Then direct computation shows ${}^v\Sigma_x$ is a closed subroot system of ${}^v\Sigma$. Hence the spherical apartment $\mathscr{A}_x$ at x can be identified with ${}^v\!\mathscr{A}({}^v\Sigma_x)$. Also note that Σ_x can be identified with ${}^v\Sigma_x$ by $\alpha \mapsto {}^v\alpha$. In particular, the roots can be identified with the affine roots in Σ_o.

4.11. Notations as before. Suppose $\Phi = {}^v\Sigma$ is irreducible. Let ${}^v\!C$ be a Weyl chamber of Φ and Δ the simple roots it defines. Then there is a unique root a_0 such that $\|a_0\| \geqslant \|a\|$ for all root a [Bou02, chap.VI, §1, no.8, prop.25]. This a_0 is called the *highest root* with respect to Δ or ${}^v\!C$. The set $C = (o + {}^v\!C) \setminus \alpha^*_{-a_0+}$ is a chamber in $\mathscr{A}(\Phi)$ [Bou02, chap.VI, §2, no.2, prop.5] and is calld the *fundamental alcove* for Δ.

Let $\widetilde{\Delta}$ denote the set of affine roots α defining walls of C, which means $C \subseteq \alpha$ and $\partial\alpha$ is a wall of C. Then it consists of the simple roots and the affine root $\alpha_0 = \alpha_{-a_0+}$. Such a set $\widetilde{\Delta}$ is a *basis* of Σ in the sense that any affine root is a $\mathbb{Z}$-linear combination of its elements and the coefficients are either all non-negative or all non-positive.

Conversely, let C be a chamber in $\mathscr{A}(\Phi)$ and x a special vertex which is also an extremal point of $\overline{C}$. The affine roots defining walls of C form a basis $\widetilde{\Delta}$ of the affine root system $\widetilde{\Phi}$. Among these affine roots, those vanishing at x give rise to a basis Δ of the root system Φ by taking their vectorial parts and the rest one gives rise to the highest root with respect to Δ by taking the negation of its vectorial part. Since

[5] Note that, there is a notion called *affine root system*, defined in a similar way as root system, but for affine spaces. In this note, this terminiology only refers to those arise from (reduced) root systems.

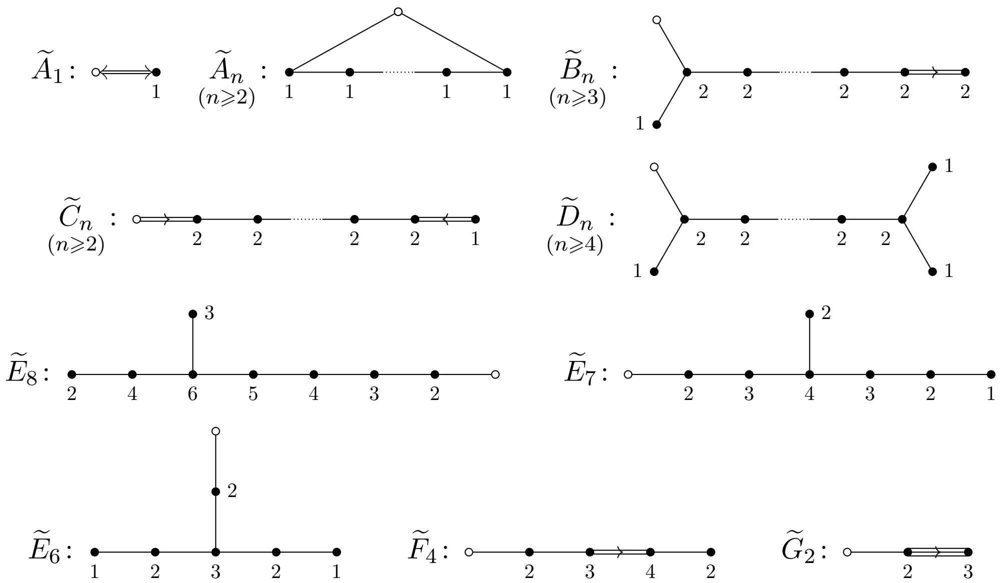
chambers in $\mathscr{A}(\Phi)$ are fundamental alcoves for some basis, we call them *alcoves* to avoid confusion with Weyl chambers.

4.12. The types are introduced as follows. The affine Weyl group $W(\Sigma)$ is generated by r_α for $\alpha \in \widetilde{\Delta}$ as they are the affine roots defining walls of C. Therefore a type $I \in \mathcal{T}$ corresponds to a proper subset of $\widetilde{\Delta}$. From now on, we do not distinguish them. Then the face of C corresponding to I is the set

$$C_I = \overline{C} \cap \Big(\bigcap_{\alpha \in I} \partial\alpha\Big) \setminus \Big(\bigcup_{\alpha \in \widetilde{\Delta}\setminus I} \partial\alpha\Big).$$

4.13. Let Σ be an irreducible affine root system with $\widetilde{\Delta}$ a basis. Then the *extended Dynkin diagram* of it is defined similarly to Dynkin diagram except in the case of $\widetilde{A}_1$, where there is an left-right double arrow between the two vertices.

The followings are extended Dynkin diagrams of all irreducible affine root systems [Bou02, chap.VI, §4, no.3, prop.4], where the notation $\widetilde{X}_n$ indicates this affine root system arises from the root system of type X_n.



Also note that these Dynkin diagrams are decorated in the following way: the part consists of bold vertices is an ordinary Dynkin diagram and its vertices present the simple roots a_i $(1 \leqslant i \leqslant n)$, then the extra hollow vertex presents the (affine root α_0 defined by the) highest root a_0 and each simple root a_i is labelled by its coefficient h_i in the expression

$$a_0 = \sum_{i=1}^{n} h_i a_i$$

presenting the highest root a_0 as $\mathbb{Z}$-linear combination of them.

A discrete affine apartment is said to be *of type* $\widetilde{X}_n$ if it is isomorphic to $\mathscr{A}(\Sigma)$ for an irreducible affine root system Σ of type $\widetilde{X}_n$.

§ 5 Euclidean buddings

It's time to give the definition of Euclidean buildings.

5.1 Definition. A *(Euclidean) building* is a set $\mathscr{B}$ equipped with a polysimplicial complex $\mathcal{F}$, whose members are subsets of $\mathscr{B}$ and are called *facets*, and a family $\mathcal{A}$ of subsets of $\mathscr{B}$, whose members are called *apartment*, such that the following axioms are satisfied.

- **EB0.** For each apartment $A \in \mathcal{A}$, there is an (abstract) apartment $\mathscr{A}$ together with a bijection between them, exchanging the complex $\mathcal{F}_A$ of facets contained in A and the complex of facets in $\mathscr{A}$.

Note that, this allows us to view each apartments in $\mathscr{B}$ as Euclidean affine spaces and hence it makes sense to talk about isometries between them.

- **EB1.** For any two facets F, F', there is an apartment A containing them.

- **EB2.** If A, A' are two apartments containing both F and F', then there is an isomorphism between A and A' fixing F and F' pointwise.

Here an isomorphism between A and A' is an isometry between them exchanging the posets $\mathcal{F}_A$ and $\mathcal{F}_{A'}$.

Note that, from the definition, all apartments $A \in \mathcal{A}$ are isomorphic to an abstract one $\mathscr{A}$. Then $\mathscr{B}$ is said to be *of type* $\mathscr{A}$ and is said to be *spherical* (resp. *discrete affine*, etc.) if so is $\mathscr{A}$. The Weyl group W of $\mathscr{A}$ is also called the *Weyl group* of $\mathscr{B}$.

Remark. One can see that the combinatorial information of $\mathscr{B}$ is encoded in the polysimplicial complex $\mathcal{F}$ and hence is completely determined by it up to a choice of the family $\mathcal{A}$. One can compare the axioms **EB0.–EB2.** with **B0.–B2.**.

Remark. The notions of *walls*, *chambers*, *vertices* and *types* in a building is defined similarly as in an apartment and we will use the same notations as there. Furthermore, there is a *type function* $\tau\colon \mathcal{F} \to \mathcal{T}$ extending the type function on an apartment to the entire building uniquely.

Remark. We have assumed that apartments are essential. In particular, the buildings in Bruhat-Tits theory used in this note are the *reduced buildings*, rather than *extended buildings*. However, this is harmless as we focus more on the polysimplicial structure and we do want the vertices being points.

5.2. An *morphism* between buildings $\mathscr{B}$ and $\mathscr{B}'$ is a continuous map inducing a *chamber map* between $\mathcal{F}$ and $\mathcal{F}'$ and maps apartments in apartments. Then an *automorphism* of a building is an isometry transforming a facet (resp. apartment) in a facet (resp. apartment). Any building can be decomposed into a product of a trivial building with irreducible essential buildings, similarly as in 3.3. However, there is no guarantee that such a decomposition gives a good corresponding on the family $\mathcal{A}$.

5.3. An automorphism is said to be *type-preserving* if it leaves the type function τ invariant. For instance, any $w \in W$ is such an automorphism. A group G of automorphisms is said to be *strongly transitive* if it acts transitively on the pairs (C, A) where C is a chamber in the apartment A. This is the case if and only if G acts transitively on apartments and in any apartment A, the following conditions for a pair of chambers C, C' in A are equivalent:

1. C and C' are conjugated by the Weyl group W;
2. C and C' are conjugated by the stabilizer $N_G(A)$ of A in G;
3. C and C' are conjugated by G.

When a group G of automorphisms is strongly transitive and type-preserving, we have

$$W \cong N_G(A)/C_G(A),$$

where $C_G(A)$ is the fixator of an apartment A in G.

5.4. Let G be a strongly transitive and type-preserving group of automorphisms and F be a facet in an apartment A. The stabilizer (which is also the fixator)

$$G_F := N_G(F) = C_G(F)$$

of F is called a *parabolic subgroup* of G. The parabolic group G_F acts transitively on the apartments containing F. Indeed, one can deduce this from the fact that G_F acts transitively on chambers containing F since G is strongly transitive. Here the former is due to that G acts transitively on chambers and is type-preserving.

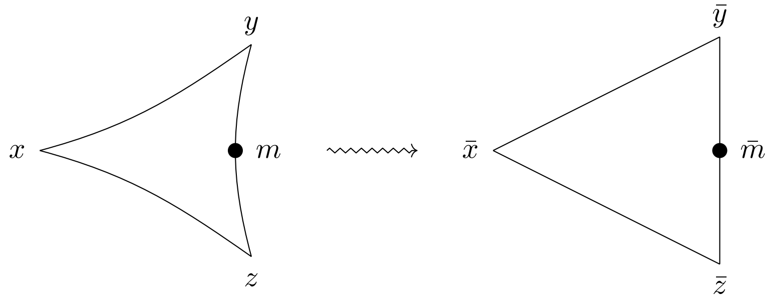
Moreover, we have the *Bruhat decomposition* [Rou09, 6.9]

$$G = G_F.N_G(A).G_F.$$

In particular, if $F = C$ is a chamber, then

$$G = \bigsqcup_{w \in W} G_C w G_C.$$

5.5. The apartments are Euclidean affine spaces, hence have metrics. Those metrics are compatible in the sense that the agree on any overlap, hence are glued into a metric $d(-,-)$ on the entire building $\mathscr{B}$ in a consistent way. Then $\mathscr{B}$ equipped with this metric is a complete metric space having the *CAT(0)-property* [Rou09, 6.5], which means that geodesic triangles in $\mathscr{B}$ are at least as thin as in a Euclidean plane: saying x, y, z are three points in $\mathscr{B}$ forming a geodesic triangle and $\bar{x}, \bar{y}, \bar{z}$ are three points in a Euclidean plane having the same pointwise distance as x, y, z, then for any point m in the geodesic segment $[x, y]$ in the triangle and $\bar{m}$ the corresponding point in the segment $[\bar{x}, \bar{y}]$ (namely, $d(\bar{x}, \bar{m}) = d(x, m)$), then $d(z, m) \leqslant d(\bar{z}, \bar{m})$.



Consequences of this property include: the geodesic segments between points are unique [Rou09, 6.6]; any group of isometries stabilizing a nonempty bounded subset has a fixed point [Rou09, 7.1]; the distance from a point to a nonempty closed convex subset is achieved by a unique point [Rou09, 7.3]. For more on this, see [Rou09, §6,§7].

5.6. A *bornology* on a set X is a collection $\mathcal{B}$ of subsets of X such that it covers X and is stable under inclusion and finite unions. Once such a bornology is chosen, its members are called *bounded subsets* of X. For instance, any metric space has a canonical bornology induced by its metric. Another example is any topological space, where the bornology consists of all relatively compact subsets. A *morphism* between bornological sets is a map preserving the bornologies.

A *bornological group* is a group G equipped with a bornology on it stable under multiplication. For instance, let G be an isometry group on a metric space X, then there is a canonical bornology whose members are subsets M such that the set $M.x$ is bounded in X for some $x \in X$.

Let $\varphi\colon G' \to G$ be a group homomorphism and G a bornological group. Then we can canonically pullback the bornology on G to G': a subset of G' is bounded when its image is bounded in G.

So we can talk about *bounded subgroups* of a group G acting on the building $\mathscr{B}$ regardless its own topology or bornology. But if G is topological or bornological, it makes sense to ask if its bornology is the same as the pullback one. We point out that this is the case when G acts continuously on $\mathscr{B}$.

Part II
Buildings for reductive groups

Tits' building theory was applied to study the structure of reductive groups over arbitrary field, a family of linear algebraic groups play important roles in mathematics. Throughout this part, we fix a ground field K and an algebraic closure K^a (resp. separable closure K^s) of it.

§ 6 Algebraic groups

We first recall some basic notions on algebraic groups.

6.1 Definition. By an *algebraic group* (defined over K), we mean a group object in the category $\mathbf{Sch}_K$ of schemes of finite type over K.

The definition implies that algebraic groups are in particular group-valued functors from the category $\mathbf{Alg}_K$ of finitely generated K-algebras.

An algebraic group is *affine* (resp. *smooth*, *connected*, etc.) if so is its underlying scheme. In particular, affine algebraic groups are precisely representable group-valued functors.

We will use bold letters like G to denote algebraic groups defined over K. For any K-algebra R, the group scheme obtained by base change $\mathsf{G} \otimes_K R$ is denoted by G_R and the group of R-points is denoted by $\mathsf{G}(R)$ (but if we used notations with parenthesis, e.g. $\mathsf{GL}(V)$, to denote an aglebraic group, then its group of R-points is denoted by padding R into the parenthesis as the last parameter, e.g. $\mathsf{GL}(V, R)$). Moreover, $\mathsf{G}(K)$ is simply denoted by G and $\mathsf{G}_R(R) \cong \mathsf{G}(R)$ is simply denoted by G_R. We also use $g \in \mathsf{G}$ to mean that g is a R-point of G for some K-algebra R.

6.2. Let G be an algebraic group. Its *neutral component* G° is the largest connected subgroup of G. Its *component group* $\pi_0(\mathsf{G})$ is the universal étale scheme under G. Then there is an exact sequence [Mil17, 2.37]:

$$1 \longrightarrow \mathsf{G}^\circ \longrightarrow \mathsf{G} \longrightarrow \pi_0(\mathsf{G}) \longrightarrow 1.$$

The above formations are compatible with field extensions and products.

The following conditions on an algebraic group G are equivalent [Mil17, 1.36]:

(a) G is irreducible;

(b) G is connected;

(c) G is geometrically connected;

(d) $\pi_0(\mathsf{G})$ equals the trivial group 1.

6.3 Example. Here we give some algebraic groups presented as functors.

(a) The functor $R \rightsquigarrow (R,+)$ mapping a K-algebra to its underlying abelian group defines an algebraic group $\mathbb{G}_a$, called the *additive group*.

(b) The functor $R \rightsquigarrow (R^\times, \times)$ mapping a K-algebra to its unit group defines an algebraic group $\mathbb{G}_m$, called the *multiplicative group*.

(c) The functor $R \rightsquigarrow \{r \in R \mid r^n = 1\}$ mapping a K-algebra to its set of n-th roots of unity defines an algebraic group μ_n, called the *group of n-th roots of unity*.

(d) Let G be a finite group. The constant functor $R \rightsquigarrow G$ is not a scheme, but its cosheafification $R \rightsquigarrow \mathrm{Map}(\pi_0(R), G)$, where $\pi_0(R)$ is the set of connected components of $\mathrm{Spec}(R)$, defines an algebraic group $\underline{G}$. Such an algebraic group is called a *constant algebraic group*.

(e) Let V be a finite dimensional vector space over K, then the functor $R \rightsquigarrow V_R := V \otimes_K R$ defines an algebraic group $\mathbb{W}(V)$, called the *additive group of V*. Any choice of basis of V gives rise to an isomorphism from this group to a product of copies of $\mathbb{G}_a$.

(f) The functor mapping a K-algebra R to the additive group of $m \times n$ matrices with entries in R defines an algebraic group $\mathsf{M}_{m\times n}$.

(g) Let V be a finite dimensional vector space over K, then the functor $R \rightsquigarrow \mathrm{End}(V_R)$ defines an algebraic group End_V. When V is of dimension n, any choice of basis of V gives an isomorphism from this group to $\mathsf{M}_{n\times n}$.

(h) The functor mapping a K-algebra R to the group of invertible $n \times n$ matrices with entries in R defines an algebraic group GL_n, called the *general linear group*.

(i) Let V be a finite dimensional vector space over K, then the functor $R \rightsquigarrow \mathrm{Aut}(V_R)$ defines an algebraic group $\mathsf{GL}(V)$, called the *general linear group of V*. When V is of dimension n, any choice of basis of V gives rise to an isomorphism from this group to GL_n.

(j) The functor $R \rightsquigarrow \{A \in \mathsf{GL}_n(R) \mid \det(A) = 1\}$ mapping a K-algebra R to the group of invertible $n \times n$ matrices with entries in R and determinant 1 defines an algebraic subgroup SL_n of GL_n, called the *special linear group*.

(k) Let V be a finite dimensional vector space over K, then the functor $R \rightsquigarrow \{A \in \mathsf{GL}(V,R) \mid \det(A) = 1\}$ mapping a K-algebra R to the group of R-automorphisms of V_R having determinant 1 defines an algebraic subgroup $\mathsf{SL}(V)$ of $\mathsf{GL}(V)$, called the *special linear group of V*.

(l) The functor $R \rightsquigarrow \{(a_{ij}) \in \mathsf{GL}_n(R) \mid a_{ij} = 0 \text{ for } i > j\}$ mapping a K-algebra R to the group of upper triangular invertible $n \times n$ matrices with entries in R defines an algebraic subgroup B_n of GL_n.

(m) The functor $R \rightsquigarrow \{(a_{ij}) \in \mathsf{GL}_n(R) | a_{ij} = 0 \text{ for } i > j \text{ and } a_{ij} = 1 \text{ if } i = j\}$ mapping a K-algebra R to the group of upper triangular invertible $n \times n$ matrices with entries in R and diagonal entries 1 defines an algebraic subgroup U_n of B_n.

(n) The functor $R \rightsquigarrow \{\mathrm{diag}(t_1, \cdots, t_n) \in \mathsf{GL}_n \mid t_1, \cdots, t_n \in R\}$ mapping a K-algebra R to the group of invertible diagonal $n \times n$ matrices with entries in R defines an algebraic subgroup D_n of B_n. Note that $\mathsf{D}_n \cong \mathbb{G}_m^n$.

(o) The functor $R \rightsquigarrow \left\{A \in \mathsf{GL}_{2n}(R) \,\middle|\, {}^tAJ_{2n}A = J_{2n}\right\}$, where $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and I_n is the identity matrix, defines an algebraic subgroup Sp_{2n} of GL_{2n}, called the *symplectic group*.

(p) Let V be a finite dimensional vector space over K and $\mathfrak{b}$ be a *symplectic form* on it, then the functor $R \rightsquigarrow \mathrm{Aut}(V_R, \mathfrak{b})$ mapping a K-algebra R to the group of R-automorphisms of V_R leaving the form $\mathfrak{b}$ invariant defines an algebraic subgroup $\mathsf{Sp}(V)$ of $\mathsf{GL}(V)$, called the *symplectic group of* V.

(q) The functor $R \rightsquigarrow \left\{A \in \mathsf{GL}_{2n}(R) \,\middle|\, {}^tAJ_{2n}A = \lambda J_{2n} \text{ for some } \lambda \in R^\times\right\}$ defines an algebraic subgroup GSp_{2n} of GL_{2n}, called the *symplectic similitude group*.

(r) Let V be a finite dimensional vector space over K and $\mathfrak{b}$ be a *symplectic form* on it, then the functor $R \rightsquigarrow \{\varphi \in \mathsf{GL}(V, R) \mid \mathfrak{b} \circ \varphi = \lambda \mathfrak{b} \text{ for some } \lambda \in R^\times\}$ defines an algebraic subgroup $\mathsf{GSp}(V)$ of $\mathsf{GL}(V)$, called the *symplectic similitude group of* V.

(s) Let V be a finite dimensional vector space over K and $\mathfrak{q}$ be a *quadratic form* on it, then the functor $R \rightsquigarrow \mathrm{Aut}(V_R, \mathfrak{q})$ mapping a K-algebra R to the group of R-automorphisms of V_R leaving the form $\mathfrak{q}$ invariant defines an algebraic subgroup $\mathsf{O}(V, \mathfrak{q})$ of $\mathsf{GL}(V)$, called the *orthogonal group of* $(V, \mathfrak{q})$.

(t) The neutral component of $\mathsf{O}(V, \mathfrak{q})$ is denoted by $\mathsf{SO}(V, \mathfrak{q})$ and called the *special orthogonal group of* $(V, \mathfrak{q})$. When the characteristic of K is not 2, it equals the intersection of $\mathsf{O}(V, \mathfrak{q})$ and $\mathsf{SL}(V)$.

(u) Let $\mathfrak{q}$ be the quadratic form

$$(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) \longmapsto x_1y_1 + x_2y_2 + \cdots + x_ny_n,$$

then the algebraic group $\mathsf{O}\big(K^{2n}, \mathfrak{q}\big)$ (resp. $\mathsf{SO}\big(K^{2n}, \mathfrak{q}\big)$) is simply denoted by O_{2n} (resp. SO_{2n}) and called the *orthogonal group* (resp. *special orthogonal group*).

(v) Let $\mathfrak{q}$ be the quadratic form

$$(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n, z) \longmapsto x_1y_1 + x_2y_2 + \cdots + x_ny_n + z^2,$$

then the algebraic group $\mathsf{O}\big(K^{2n+1}, \mathfrak{q}\big)$ (resp. $\mathsf{SO}\big(K^{2n+1}, \mathfrak{q}\big)$) is simply denoted by O_{2n+1} (resp. SO_{2n+1}) and called the *orthogonal group* (resp. *special orthogonal group*).

(w) Let V be a finite dimensional vector space over K and $\mathfrak{q}$ be a *quadratic form* on it, then the functor $R \rightsquigarrow \{\varphi \in \mathsf{GL}(V) \mid \mathfrak{q} \circ \varphi = \lambda\mathfrak{q} \text{ for some } \lambda \in R^\times\}$ defines an algebraic subgroup $\mathsf{GO}(V, \mathfrak{q})$ of $\mathsf{GL}(V)$, called the *orthogonal similitude group of* $(V, \mathfrak{q})$.

(x) Let $\mathfrak{q}$ be the quadratic form

$$(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) \longmapsto x_1y_1 + x_2y_2 + \cdots + x_ny_n,$$

then the algebraic group $\mathsf{GO}\big(K^{2n}, \mathfrak{q}\big)$ is simply denoted by GO_{2n} and called the *orthogonal similitude group*.

(y) Let $\mathfrak{q}$ be the quadratic form

$$(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n, z) \longmapsto x_1y_1 + x_2y_2 + \cdots + x_ny_n + z^2,$$

then the algebraic group $\mathsf{GO}\big(K^{2n+1}, \mathfrak{q}\big)$ is simply denoted by GO_{2n+1} and called the *orthogonal similitude group*.

All above functors are representable. Hence above algebraic groups are affine.

6.4. A *representation* of an algebraic group G is a homomorphism of group-valued functors $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$, where V is a vector space over K and $\mathsf{GL}(V)$ is the functor $R \rightsquigarrow \mathrm{Aut}(V_R)$. When V is finite dimensional, this is a homomorphism of algebraic groups. Such a representation is *faithful* if ρ is injective.

An algebraic group is *linear* if it admits a finite dimensional faithful representation. It turns out that [Mil17, 1.43 and 4.10]:

$$\text{affine algebraic group} = \text{linear algebraic group}.$$

6.5. Let G be an algebraic group over a perfect field K. Then for any $g \in G$, we have *Jordan–Chevalley decomposition* [Mil17, 9.18]: there exist unique elements$g_s, g_u \in G$ such that

$$g = g_sg_u = g_ug_s,$$

and for any representation $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$, $\rho(g_s)$ is semisimple and $\rho(g_u)$ is unipotent. An element $g \in G$ is said to be *semisimple* (reps. *unipotent*) if $g = g_s$ (resp. $g = g_u$).

An algebraic group G is *unipotent* if its every finite dimensional representation $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$ is *unipotent*, namely there exists a G-stable flag

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_m = V,$$

such that G acts trivially on each factor V_i/V_{i-1}. A smooth algebraic group G is unipotent if and only if all elements of $\mathsf{G}(K^a)$ are unipotent [Mil17, 14.12].

6.6. Let G be an algebraic group. Its *Lie algebra* $\mathrm{Lie}(\mathsf{G})$ is the tangent space at the identity equipped with the natural bracket on it. The action of G on itself by conjugations induces a representation $\mathrm{Ad}\colon \mathsf{G} \to \mathsf{GL}(\mathfrak{g})$ of G on the vertor space $\mathfrak{g} := \mathrm{Lie}(\mathsf{G})$, called the *adjoint representation* of G.

An algebraic group is a *vector group* if it is isomorphic to a product of copies of $\mathbb{G}_a$. Let V be a finite dimensional vector space over K, then the algebraic group $\mathbb{W}(V)$ is a vector group.

The above constructions give rise to an equivalence of categories between vector groups and finite dimensional vector spaces over K [Mil17, 10.9]. Moreover, when K is of characteristic zero and G is a unipotent group over it, there is an isomorphism of schemes (and of algebraic groups if G is further commutative)

$$\exp\colon \mathbb{W}(\mathrm{Lie}(\mathsf{G})) \longrightarrow \mathsf{G},$$

called the *exponential map* [Mil17, 14.32].

6.7. An algebraic group is a *torus* if it becomes isomorphic to a product of copies of $\mathbb{G}_m$ over a finite separable extension of K. A torus over K is *split* if it is already isomorphic to a product of copies of $\mathbb{G}_m$ over K.

A *character* of an algebraic group G is a homomorphism $\chi\colon \mathsf{G} \to \mathbb{G}_m$. The group of characters is denoted by $\mathsf{X}(\mathsf{G})$. The *character group* of G is the group $\mathsf{X}^*(\mathsf{G}) := \mathrm{Hom}(\mathsf{G}_{K^s}, \mathbb{G}_{m,K^s})$ of characters defined over K^s.

An algebraic group G is *diagonalizable* if it is of the form $\mathsf{D}(M)$ for some finitely generated abelian group M, where $\mathsf{D}(M)$ is defined by the functor $R \rightsquigarrow \mathrm{Hom}_K(M, R^\times)$. If this is the case, then $\mathsf{G} = \mathsf{D}(\mathsf{X}(\mathsf{G}))$. This terminology is justified by the following fact: an algebraic group is diagonalizable if and only if its every representation is diagonalizable [Mil17, 12.12]. An algebraic group G is of *multiplicative type* if it becomes diagonalizable over a finite separable extension over K.

A *cocharacter* of an algebraic group G is a homomorphism $\lambda\colon \mathbb{G}_m \to \mathsf{G}$. The *cocharacter group* of G is the abelian group $\mathsf{X}_*(\mathsf{G}) := \mathrm{Hom}(\mathbb{G}_{m,K^s}, \mathsf{G}_{K^s})$ of cocharacters defined over K^s.

6.8 Example. Let $\mathsf{G} = \mathsf{D}_n$.

(a) Characters of D_n are of the form $\chi_1^{a_1} \cdots \chi_n^{a_n}$ for some $(a_1 \cdots, a_n) \in \mathbb{Z}^n$, where

$$\chi_i\colon \mathrm{diag}(t_1, \cdots, t_n) \longmapsto t_i.$$

(b) Cocharacters of D_n are of the form

$$t \longmapsto \mathrm{diag}(t^{a_1}, \cdots, t^{a_n})$$

for some $(a_1 \cdots, a_n) \in \mathbb{Z}^n$.

Therefore if T is a torus of dimension n, then both its character group $\mathsf{X}^*(\mathsf{T})$ and cocharacter group $\mathsf{X}_*(\mathsf{T})$ are isomorphic to $\mathbb{Z}^n$.

Let χ be a character and λ be a cocharacter of T. Then the composition $\chi \circ \lambda$ is an endomorphism $t \mapsto t^{\langle \chi, \lambda \rangle}$ of $\mathbb{G}_m$, which can be identified with the integer $\langle \chi, \lambda \rangle \in \mathbb{Z}$. In this way, we get a perfect pairing of $\mathbb{Z}$-modules

$$\langle\,\cdot\,,\,\cdot\,\rangle\colon \mathsf{X}^*(\mathsf{T}) \times \mathsf{X}_*(\mathsf{T}) \to \mathbb{Z}.$$

making $\mathsf{X}^*(\mathsf{T})$ and $\mathsf{X}_*(\mathsf{T})$ in duality.

§ 7 Reductive groups

The notion of reductive groups will be introduced in this section.

7.1. Let G be a smooth connected linear algebraic group.

1. [Mil17, 6.44] There is a largest smooth connected solvable norm subgroup $\mathscr{R}(\mathsf{G})$ of G. It is called the *radical* of G.
2. [Mil17, 6.46] There is a largest smooth connected unipotent norm subgroup $\mathscr{R}_u(\mathsf{G})$ of G. It is called the *unipotent radical* of G.

Since unipotent groups are solvable [Mil17, 14.21], $\mathscr{R}_u(\mathsf{G})$ is a subgroup of $\mathscr{R}(\mathsf{G})$.

7.2 Definition. An algebraic group G is *reductive* (resp. *semisimple*) if its *geometric unipotent radical* $\mathscr{R}_u(\mathsf{G}_{K^a})$ (resp. *geometric radical* $\mathscr{R}(\mathsf{G}_{K^a})$) is trivial.

The formation of $\mathscr{R}_u(\mathsf{G})$ and $\mathscr{R}(\mathsf{G})$ commute with separable field extensions [Mil17, 19.1 and 19.9]. Hence when K is perfect, G is reductive (resp. semisimple) if and only if $\mathscr{R}_u(\mathsf{G})$ (resp. $\mathscr{R}(\mathsf{G})$) is trivial.

7.3 Example. In Example 6.3,

(a) SL_n, $\mathsf{SL}(V)$, Sp_{2n}, $\mathsf{Sp}(V)$, SO_{2n}, SO_{2n+1} and $\mathsf{SO}(V,\mathfrak{q})$ are semisimple;

(b) GL_n, $\mathsf{GL}(V)$, GSp_{2n}, $\mathsf{GSp}(V)$, GO°_{2n}, GO°_{2n+1} and $\mathsf{GO}^\circ(V,\mathfrak{q})$ are reductive but not semisimple.

(c) B_n is solvable and U_n is its unipotent radical.

(d) Any torus is reductive. Conversely, if G is a solvable reductive group, then since $\mathscr{R}_u(\mathsf{G}_{K^a})$ is trivial, it is a torus by [Mil17, 16.33].

7.4. A homomorphism of smooth connected algebraic groups is said to be an *isogeny* if it is surjective and has finite kernel. An isogeny is *central* if its kernel is contained in the centre and is *multiplicative* if its kernel is of multiplicative type. A multiplicative isogeny is central [Mil17, 12.38] and the converse is true if its domain is reductive (since the centre of a reductive group is of multiplicative type [Mil17, 17.62]).

A smooth connected algebraic group is *simply connected* if every multiplicative isogeny to it is an isomorphism. Let G be a smooth connected algebraic group. A *universal covering* on it is an multiplicative isogeny $\widetilde{\mathsf{G}} \to \mathsf{G}$ with $\widetilde{\mathsf{G}}$ a simply connected one. When the universal covering exists, its kernel is called the *fundamental group* $\pi_1(\mathsf{G})$ of G.

7.5 Example. In Example 7.3(a),

(a) SL_n, $\mathsf{SL}(V)$, Sp_{2n} and $\mathsf{Sp}(V)$ are simply-connected;

(b) SO_{2n}, SO_{2n+1} and $\mathsf{SO}(V,\mathfrak{q})$ are not simply-connected and their fundamental groups are isomorphic to $\mathbb{Z}/2$.

7.6. Let G be a reductive group. Then there are various semisimple groups related to it.

(a)

Then its radical $\mathscr{R}(\mathsf{G})$ is the largest subtorus of $\mathsf{Z}(\mathsf{G})$ and the quotient $\mathsf{G}/\mathscr{R}(\mathsf{G})$ is semisimple [Mil17, 17.62], its derived group $\mathsf{G}^{\mathrm{der}}$ is semisimple [Mil17, 19.21], its centre $\mathsf{Z}(\mathsf{G})$ is of multiplicative type [Mil17, 17.62], and we have the following *deconstruction* [Mil17, 19.d].

$$\begin{tikzcd} \mathsf{Z}(\mathsf{G}^{\mathrm{der}}) \arrow[r, hook] \arrow[d, hook] & \mathsf{G}^{\mathrm{der}} \arrow[d, hook] \\ \mathsf{Z}(\mathsf{G}) \arrow[r, hook] & \mathsf{G} \end{tikzcd}$$

Namely, there is a short exact sequence

$$1 \longrightarrow \mathsf{Z}(\mathsf{G}^{\mathrm{der}}) \longrightarrow \mathsf{G}^{\mathrm{der}} \times \mathsf{Z}(\mathsf{G}) \longrightarrow \mathsf{G} \longrightarrow 1$$

and there are isogenies $\mathsf{Z}(\mathsf{G}) \to \mathsf{G}/\mathsf{G}^{\mathrm{der}}$ and $\mathsf{G}^{\mathrm{der}} \to \mathsf{G}^{\mathrm{ad}}$ with kernel $\mathsf{Z}(\mathsf{G}^{\mathrm{der}})$. Here G^{ad} is the quotient $\mathsf{G}/\mathsf{Z}(\mathsf{G})$, which is an *adjoint group*, which means it is semisimple and has trivial centre.

7.7. Let G be a reductive group. It is *splittable* if it has a split maximal torus. A *split reductive group* is a pair (G, T) of a reductive group and a split maximal torus in it. A *homomorphism* between split reductive groups is a homomorphism of algebraic group preserving the split maximal torus. It turns out that, any two maximal split tori (hence split maximal tori if G is splittable) in G are conjugate by an element of G [Mil17, 25.10], while two (not necessarily split) maximal tori are only conjugate over a finite separable extension [Mil17, 17.87].

Let G be a splittable reductive group. Then its *rank* is the dimension of one (hence any) split maximal torus in it and its *semisimple rank* is the rank of $\mathsf{G}/\mathscr{R}(\mathsf{G})$. Since the centre $\mathsf{Z}(\mathsf{G})$ is contained in every maximal torus [Mil17, 17.61], the semisimple rank of G equals $\mathrm{rank}(\mathsf{G}) - \dim(\mathsf{Z}(\mathsf{G}))$. Consequently, the homomorphism $\mathsf{G}/\mathscr{R}(\mathsf{G}) \to \mathsf{G}^{\mathrm{ad}}$ is an isogeny.

§ 8 Root data and Tits buildings

Let G be a reductive group. Associated to it, there is a spherical building ${}^{v}\mathscr{B}(\mathsf{G})$ equipped with a natural G-action, called its *Tits building*. In this section, Tits buildings will be introduced for splittable reductive groups and we will see that the underlying building only depends on the root system and the ground field.

8.1. Let G be a reductive group. A *parabolic subgroup* of it is a smooth subgroup P such that G/P is a complete variety. A subgroup B of G is *Borel* if it is smooth, connected, solvable, and parabolic. It turns out that a smooth subgroup P is parabolic if and only if P_{K^a} contains a Borel subgroup in G_{K^a} [Mil17, 17.16] and every parabolic subgroup is connected and equal to its own normalizer since this is so over K^a [Mil17, 17.49]. When G has a Borel subgroup, it is said to be *quasi-split*. In this case, Borel subgroups are exactly the minimal parabolic subgroups and maximal connected solvable subgroups [Mil17, 17.19] and any two of them are conjugate by an element of G [Mil17, 25.8]. If the Borel subgroup is furthermore split (as a solvable algebraic group, namely it admits a normal series whose factors are isomorphic to either $\mathbb{G}_a$ or $\mathbb{G}_m$), then G is said to be *split*. It turns out that, G is split if and only if it is splittable [Mil17, 21.64].

Let $\pi\colon \mathsf{G} \to \mathsf{Q}$ be a quotient map and H a smooth subgroup of G. Then if H is parabolic (resp. Borel), so is $\pi(\mathsf{H})$. Moreover, every such subgroup of Q arises in this way [Mil17, 17.20]. This allows us to reduce the study of (the poset of) parabolic subgroups from reductive groups to simply-connected semisimple groups. The *Tits building* of a reductive group is essentially this poset [Tit74, 5.2].

8.2. Let (G, T) be a split reductive group. Since T is diagonalizable, it acts (via the adjoint representation) on $\mathfrak{g} := \mathrm{Lie}(\mathsf{G})$ diagonalizably and we hav decomposition:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{a \in \mathsf{X}^*(\mathsf{T})} \mathfrak{g}_a,$$

where $\mathfrak{t} = \mathfrak{g}^{\mathsf{T}} = \mathrm{Lie}(\mathsf{T})$ [Mil17, 10.34] and $\mathfrak{g}_a$ is the subspace on which T acts through a nontrivial character a. A character a is a *root* if $\mathfrak{g}_a$ is nontrivial. The set of roots is denoted by $\Phi(\mathsf{G}, \mathsf{T})$, called the *root system* of the split reductive group (G, T).

For $a \in \Phi(\mathsf{G}, \mathsf{T})$, there is a unique subgroup U_a (called the *root group*) of G satisfying the following properties [Mil17, 21.11 and 21.19].

1. U_a is normalized by T.

2. U_a has Lie algebra $\mathfrak{g}_a$ and a smooth subgroup of G contains U_a if and only if its Lie algebra contains $\mathfrak{g}_a$.

3. U_a is isomorphic to $\mathbb{G}_a$. In fact, there is a unique isomorphism of of algebraic groups $u_a\colon \mathbb{W}(\mathfrak{g}_a) \to \mathsf{U}_a \subseteq \mathsf{G}$ such that its derivation is the inclusion $\mathfrak{g}_a \subseteq \mathfrak{g}$.

4. T acts on U_a through the character a: for every isomorphism $u\colon \mathbb{G}_a \to \mathsf{U}_a$, we have $tu(x)t^{-1} = u(a(t)x)$ for all $t \in \mathsf{T}$ and $x \in \mathbb{G}_a$.

8.3. Let (G, T) be a split reductive group. Then $\mathsf{N} = \mathsf{N}_{\mathsf{G}}(\mathsf{T})$ acts on T, hence on $\mathsf{X}^*(\mathsf{T}) = \mathsf{X}(\mathsf{T})$ by conjugation. The centralizer $\mathsf{Z}_{\mathsf{G}}(\mathsf{T})$ (which equals T itself [Mil17, 17.84]) acts trivially, hence we have an action of the quotient $\mathsf{N} \,/\, \mathsf{T}$ on $\mathsf{X}^*(\mathsf{T})$. It turns out that, this quotient is the étale group scheme of connected components of N [Mil17, 17.39], and is furthermore constant [Mil17, 21.1], namely it is constant as a functor with value a finite group $W(\mathsf{G}, \mathsf{T}) = N/T$. This finite group is called the *Weyl group* of (G, T).

For $a \in \Phi(\mathsf{G}, \mathsf{T})$, let G_a denote the centralizer of the largest subtorus of $\mathrm{Ker}(a)$. Then we have the following facts [Mil17, 21.11].

1. The pair $(\mathsf{G}_a, \mathsf{T})$ is a split reductive group of semisimple rank 1.

2. The Lie algebra of G_a has decomposition

$$\mathrm{Lie}(\mathsf{G}_a) = \mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$$

and the only rational multiples of a in $\Phi(\mathsf{G}, \mathsf{T})$ are $\pm a$.

3. The Weyl group $W(\mathsf{G}_a, \mathsf{T})$ contains exactly one nontrivial element r_a.

4. There is a unique $a^\vee \in \mathsf{X}_*(\mathsf{T})$ such that $r_a(x) = x - \langle x, a^\vee \rangle a$ for all $x \in \mathsf{X}^*(\mathsf{T})$. Moreover, $\langle a, a^\vee \rangle = 2$.

Let $\Phi^\vee(\mathsf{G}, \mathsf{T})$ denote the set of cocharacters $a^\vee$ for $a \in \Phi(\mathsf{G}, \mathsf{T})$, called the *coroot system*. Then the quadruple $(\mathsf{X}^*(\mathsf{T}), \Phi(\mathsf{G}, \mathsf{T}), \mathsf{X}_*(\mathsf{T}), \Phi^\vee(\mathsf{G}, \mathsf{T}))$ form a root datum $\mathcal{R}(\mathsf{G}, \mathsf{T})$. Let V denote the subspace of $\mathsf{X}_*(\mathsf{T}) \otimes \mathbb{R}$ spanned by $\Phi^\vee(\mathsf{G}, \mathsf{T})$, called the *coroot space*. Then we get a root system $\Phi(\mathsf{G}, \mathsf{T})$ on the Euclidean vector space V and hence a spherical apartment ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T})$ on which the Weyl group $W(\mathsf{G}, \mathsf{T})$ acts as its reflection group.

8.4. Let (G, T) be a split reductive group and we fix the following notations in the rest of this section.

N is the normalizer of T.

X^* is the character group of T.

X_* is the cocharacter group of T.

Φ is the root system associated to the pair (G, T).

$\Phi^\vee$ is the coroot system.

U_a is the root subgroup associated to $a \in \Phi$.

V is the coroot space.

W is the Weyl group.

${}^v\mathscr{A}$ is the spherical apartment ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T})$.

First, note that the followings sets are equipollent and acted on simply transitively by W.

1. The set of Borel subgroups B of G containing T.

2. The set of Weyl chambers ${}^v\mathscr{C}$ in ${}^v\mathscr{A}$.

3. The set of systems of positive roots Φ^+ in Φ.

4. The set of bases Δ of Φ.

Indeed, if a system of positive roots Φ^+ is given, then B is generated by T and U_a for all $a \in \Phi^+$ and if a Borel subgroup B containing T is given, then the set of roots a whose Lie algebra $\mathfrak{g}_a$ is contained in the Lie algebra of B forms a system of positive roots Φ^+ [Mil17, 21.d].

More general, after choosing one element of the above equipollent sets. We have the following equipollent sets.

1. The set of parabolic subgroups P of G containing B.
2. The set of faces ${}^vF$ of the Weyl chambers ${}^v\mathscr{C}$.
3. The set of parabolic subset Ψ of positive roots Φ^+ in Φ.
4. The set of types I.

Indeed, if a parabolic subset Ψ is given, then P is generated by T and U_a for all $a \in \Psi$ and if a parabolic subgroup P containing B is given, then the set of roots a whose Lie algebra $\mathfrak{g}_a$ is contained in the Lie algebra of P forms a parabolic subset Ψ [Mil17, 21.i].

Let I be a type and P_I the parabolic subgroup corresponding to it. Then the unipotnet radical of P_I is generated by U_a for all $a \in \Phi^+ \backslash \Psi$ and the reductive quotient of P_I is isomorphic to the centralizer L_I of the largest subtorus of $\bigcap_{a \in I} \mathrm{Ker}(a)$ [Mil17, 21.91]. This reductive group is called the *Levi subgroup* associated to I and $(\mathsf{L}_I, \mathsf{T})$ is a split reductive group with root datum $(\mathsf{X}^*, \Phi_I, \mathsf{X}_*, \Phi_I^\vee)$ and Weyl group W_I [Mil17, 21.90].

8.5 Theorem ([Rou09, §10; Tit74, §5])**.** *Notations as above. There is a unique (up to unique isomorphism) G-set ${}^v\mathscr{B}(\mathsf{G})$ containing V and satisfying the followings.*

1. ${}^v\mathscr{B}(\mathsf{G}) = \bigcup_{g \in G} g.V$*;*
2. *N stabilizes V and acts on it through W;*
3. *the fixator of $\alpha_{a+0} := \{x \in V \mid a(x) \geqslant 0\}$ is $T \cdot U_a$.*

Then ${}^v\mathscr{B}(\mathsf{G})$ is a building of type ${}^v\mathscr{A}$, called the *Tits building* of G. In fact, since N stabilizes V and preserves its apartment structure, each $g.V$ is endowed with such a structure and moreover they agree on intersections.

Remark. Apartments in ${}^v\mathscr{B}(\mathsf{G})$ are one-one corresponding to split maximal tori. In fact, each $g.V$ endowed with its apartment structure is precisely the apartment ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T}^g)$.

The action of G on ${}^v\mathscr{B}(\mathsf{G})$ is strongly transitive and type-preserving. It is also worth to mention that ${}^v\mathscr{B}(\mathsf{G})$ is further a $\mathrm{Aut}(\mathsf{G})$-set. Indeed, if φ is an automorphism of G, then $\varphi(\mathsf{T})$ is also a split maximal torus and the pushforward along φ defines a homomorphism from ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T})$ to ${}^v\mathscr{A}(\mathsf{G}, \varphi(\mathsf{T}))$.

8.6. Let $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ be a root datum. Let X_0 denote the submodule $\{x \in \mathsf{X} \mid \langle x, \Phi^\vee \rangle = 0\}$ and let

- $\mathsf{X}' = \mathsf{X} / \mathsf{X}_0$,

- $\mathsf{X}'^\vee$ be the submodule of $\mathsf{X}^\vee$ dual to X' through $\langle \cdot, \cdot \rangle$,
- Φ' be the image of Φ in X' and $\Phi'^\vee = \Phi^\vee$.

Then $\mathcal{R}' = (\mathsf{X}', \Phi', \mathsf{X}'^\vee, \Phi'^\vee)$ is a semisimple root datum and both $\mathcal{R}$ and $\mathcal{R}'$ give rise to isomorphic root systems, hence isomorphic spherical apartments. This $\mathcal{R}'$ is called the *semisimple quotient* of $\mathcal{R}$.

Let G' be the derived group of G and T' the intersection $\mathsf{T} \cap \mathsf{G}'$. Then $\mathcal{R}(\mathsf{G}', \mathsf{T}')$ is the semisimple quotient of $\mathcal{R}(\mathsf{G}, \mathsf{T})$ [Spr98, 8.1.8]. So ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T})$ can be identified with ${}^v\mathscr{A}(\mathsf{G}', \mathsf{T}')$. Note that the $\mathsf{T} \mapsto \mathsf{T}'$ gives rise to a bijection between the set of maximal κ-split tori in G to the set of maximal κ-split tori in G'. Therefore, from the above we can identify ${}^v\mathscr{B}(\mathsf{G}')$ with ${}^v\mathscr{B}(\mathsf{G})$ and G acts on the former by conjugations on G'.

8.7. Let $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ and $\mathcal{R}' = (\mathsf{X}', \Phi', \mathsf{X}'^\vee, \Phi'^\vee)$ be two root data. An *isogeny of root data* $f \colon \mathcal{R}' \to \mathcal{R}$ (in the sense of [SGA3, XXI,6.2.1] and is called a *central isogeny* in [Mil17, 23.2]) is a linear map $f \colon \mathsf{X}' \to \mathsf{X}$ satisfying

1. it induces a bijection from Φ' to Φ;
2. its transpose ${}^t f \colon \mathsf{X}^\vee \to \mathsf{X}'^\vee$ induces a bijection from $\Phi^\vee$ to $\Phi'^\vee$;
3. f is injective and has finite cokernel.

If $f \colon \mathcal{R}' \to \mathcal{R}$ is an isogeny, then it also induces bijections between bases, systems of posistive roots and Weyl chambers [SGA3, XXI,6.1.3]. As a consequence, it induces an isomorphism of apartments ${}^v\mathscr{A}(\Phi') \cong {}^v\mathscr{A}(\Phi)$.

An *isogeny of split reductive groups* $(\mathsf{G}', \mathsf{T}') \to (\mathsf{G}, \mathsf{T})$ is a homomorphism of split reductive groups such that $\varphi \colon \mathsf{G}' \to \mathsf{G}$ is an isogeny. A homomorphism of split reductive groups φ is a central isogeny if and only if it induces an isogeny of root data $\varphi|_{\mathsf{X}^*(\mathsf{T}')} \colon \mathcal{R}(\mathsf{G}', \mathsf{T}') \to \mathcal{R}(\mathsf{G}, \mathsf{T})$ and all isogenies of root data arise in this way [SGA3, XXII,4.2.11][Mil17, 23.25].

Let $\varphi \colon \mathsf{G}' \to \mathsf{G}$ be a central isogeny between splittable reductive groups. Then $\mathsf{T}' \mapsto \mathsf{T} = \varphi(\mathsf{T}')$ gives rise to a bijection between the set of maximal κ-split tori in G' to the set of maximal κ-split tori in G. Therefore, from the above we see that the induced morphism $\varphi_* \colon {}^v\mathscr{B}(\mathsf{G}') \to {}^v\mathscr{B}(\mathsf{G})$ is an isomorphism of buildings (and G'-sets).

Two splittable semisimple groups are *strictly isogenous* if they have the same simply connected covering group. This is the case if and only if the two semisimple groups have isomorphic root systems. Conversely, any root system arises from a splittable semisimple group (in fact, any root datum arises from a splittable reductive group [Mil17, 23.55]).

8.8. From aboves, we see that the Tits building ${}^v\mathscr{B}(\mathsf{G})$ depends only on the root system Φ and the ground field K and any root system gives rise to such a building. So we can denote this building by ${}^v\mathscr{B}(\Phi, K)$.

§ 9 Valuations on root group data

Given a root group datum on G with a valuation on it, Bruhat and Tits associate a building equipped with natural G-action to these data in [BT72]. This construction will be exposited in this section.

9.1 Definition. Let Φ be a root system and G be a group. A *root group datum*[6] *of type* Φ *on* G is a system $(T, (U_a, M_a)_{a\in\Phi})$, where T is a subgroup of G and for each $a \in \Phi$, U_a is a non-trivial subgroup of G and M_a is a right congruence class modulo T, satisfying the following axioms.

R1. For any $a, b \in \Phi$, the commutator group $[U_a, U_b]$ is contained in the group generated by the U_c for $c = ia + jb \in \Phi$ with $i, j > 0$.

R2. For each $a \in \Phi$, the class M_a satisfies $U_a^* := U_a \setminus \{1\} \subseteq U_a M_a U_a$.

R3. For any $a, b \in \Phi$ and each $m \in M_a$, we have $mU_b m^{-1} \subseteq U_{r_a(b)}$.

R4. If Φ^+ is some (any) positive root system in Φ and if U^+ (resp. U^-) is the subgroup of G generated by the U_a for $a \in \Phi^+$ (resp. $a \in \Phi^-$), then $TU^+ \cap U^- = \{1\}$.

This root group datum is said to be *generating* when G is generated by the subgroups T and U_a for $a \in \Phi$.

9.2. Let $(T, (U_a, M_a)_{a\in\Phi})$ be a root group datum. Then we have the following consequences [BT72, 6.1.2].

1. For each $a \in \Phi$ and any $u \in U_{-a}^*$, there is a unique $m(u) \in M_a$ such that $u \in U_a.m(u).U_a$. If the root group datum is generating, $M_a = m(U_a^*)$.

2. For each $a \in \Phi$, T normalizes U_a and M_a.

3. For each $a \in \Phi$, $M_a = M_a^{-1} = M_{-a}$ and $T \cup M_a$ is a subgroup of G.

4. Let L_a be the subgroup of G generated by U_a, U_{-a} and T. Then $M_a = \{x \in L_a \mid xU_a x^{-1} = U_{-a}$ and

So M_a is completely determined by U_a, U_{-a} and T. Hence we can say $(T, (U_a)_{a\in\Phi})$ is a root group datum without mention M_a.

5. Let N be the subgroup of G generated by T and M_a for all $a \in \Phi$. Then, if Φ is not empty, N is also generated by M_a's and normalizes T. Moreover, there is an epimorphism ${}^v\nu\colon N \to {}^vW(\Phi)$ such that for each $a \in \Phi$ and $n \in N$, we have $nU_a n^{-1} = U_b$ with $b = {}^v\nu(n).a$. In particular, we have ${}^v\nu(M_a) = \{r_a\}$. Also note that $\mathrm{Ker}({}^v\nu) = T$ [BT72, 6.1.11].

9.3 Example ([BT72, 6.1.3c; BT65])**.** Let (G, T) be a split reductive group over K and $(\mathsf{U}_a)_{a\in\Phi}$ the root groups associated to the root system Φ of (G, T). Then $(T, (U_a)_{a\in\Phi})$ forms a generating root group datum on G.

[6]It is called a *(reduced) root datum* in [BT72, 6.1.1].

Note that this fact already will imply Theorem 8.5 using either *Tits system* or similar construction in 9.9.

9.4 Definition. A *valuation* of the root group datum $(G, T, (U_a)_{a\in\Phi})$ is a family $\varphi = (\varphi_a)_{a\in\Phi}$ of functions $\varphi_a\colon U_a \to \mathbb{R}\cup\{\infty\}$ satisfying the following axioms.

V0. For each $a \in \Phi$, the image of φ_a contains at least three elements.

V1. For each $a \in \Phi$ and any $\lambda \in \mathbb{R}\cup\{\infty\}$, the set $U_{a,\lambda} := \varphi_a^{-1}([\lambda,\infty])$ is a subgroup of U_a and $U_{a,\infty} = \{1\}$.

V2. For each $a \in \Phi$ and any $m \in M_a$, the function $u \mapsto \varphi_{-a}(u) - \varphi_a(mum^{-1})$ is constant on U^*_{-a}.

V3. For any pair $a, b \in \Phi$ not proportional and any $\lambda, \mu \in \mathbb{R}\cup\{\infty\}$, the commutator group $[U_{a,\lambda}, U_{b,\mu}]$ is contained in the subgroup generated by $U_{ia+jb,i\lambda+j\mu}$ for all $i, j > 0$ such that $ia + jb \in \Phi$.

V4. For each $a \in \Phi$ and any $u \in U_a$, if $u', u'' \in U_{-a}$ satisfy $m(u) = u'uu''$, then $\varphi_{-a}(u') = \varphi_{-a}(u'') = -\varphi_a(u)$.

For each $a \in \Phi$, let Γ_a denote the set $\varphi_a(U^*_a)$ and for any $k \in \Gamma_a$, let $M_{a,k}$ be the intersection of M_a and $U_{-a}\varphi_a^{-1}(k)U_{-a}$.

9.5. [BT72, 6.2.5] Given a root group datum $(T, (U_a)_{a\in\Phi})$ on G and let φ be a valuation on it. Then for any vector v in the ambient space V of Φ, the family $\psi = (\psi_a)_{a\in\Phi}$ given by $\psi_a\colon u \mapsto \varphi_a(u) + a(v)$ is a valuation and is denoted by $\varphi + v$. The valuations φ and $\psi = \varphi + v$ are said to be *equipollent*. The mapping $(\varphi, v) \mapsto \varphi + v$ defines an action of V on the set of valuations and each equipollent class is an orbit.

Let $\mathbb{A}$ denote the set of valuations equipollent to φ. Then $\mathbb{A}$ is an affine space underlying V and 4.9 applies. For $\alpha = \alpha_{a+k}$ with $a \in \Phi$, $k \in \Gamma_a$, let $U_\alpha = U_{a,k}$ and $U_{\alpha+} = \bigcup_{h>k} U_{a,h}$ (note that $U_{\alpha+} = U_{\alpha_+}$ if Γ_a is discrete). It is clear that the affine root system Σ and the mapping $\alpha \mapsto U_\alpha$ depends only on the equipollent class of φ.

9.6. [BT72, 6.2.5] Let $n \in N$ and $w = \nu(n) \in {}^vW$. Then the family $\psi = (\psi_a)_{a\in\Phi}$ given by $\psi_a\colon u \mapsto \varphi_{w^{-1}.a}(n^{-1}un)$ is a valuation and is denoted by $n.\varphi$. We thus obtain an action of N on the set of valuations such that for any $n \in N$ and $v \in V$, we have $n.(\varphi + v) = n.\varphi + \nu(n).v$.

[BT72, 6.2.10] The action of N stabilizes $\mathbb{A}$ and for any $n \in N$, the map $\nu(n)\colon \varphi \mapsto n.\varphi$ itself is an automorphism of the Euclidean affine space $\mathbb{A}$ whose vectorial part is ${}^v\nu(n)$. For each $a \in \Phi$ and $k \in \Gamma_a$, the image of $M_{a,k}$ under ν is the reflection r_{a+k}. So the automorphism $\nu(n)$ maps affine roots to affine roots and we have $nU_\alpha n^{-1} = U_{\nu(n).\alpha}$. In particular, for $u \in U^*_a$, $\nu(m(u)) = r_{a+\varphi_a(u)}$ [BT72, 6.2.12]. Therefore, the valuation φ is completely determined by the homomorphism $\nu\colon N \to \mathrm{Aut}(\mathbb{A})$.

9.7. [BT72, 6.2.11] Let $T^\circ = \mathrm{Ker}(\nu)$ and $\widehat{W} = \nu(N)$. Let W denote the subgroup of $\widehat{W}$ generated by r_{a+k} with $a \in \Phi$ and $k \in \Gamma_a$. It is a normal subgroup because N permutes

$M_{a,k}$. Let $N' = \nu^{-1}(W)$, $T' = T \cap N'$ and let G' be the subgroup of G generated by N' and the U_a for $a \in \Phi$. Since $M_a \cap N' \neq \varnothing$ for all $a \in \Phi$, we see that $(T', (U_a)_{a\in\Phi})$ is a generating root group datum on G' and for this root group datum, its N is exactly N'.

A valuation φ is *special* if $0 \in \Gamma_a$ for all $a \in \Phi$. If this is the case, then the group W (resp. $\widehat{W}$) can be decomposed as $W = W_\varphi \ltimes \mathrm{Ker}(W \to {}^vW)$ (resp. $\widehat{W} = W_\varphi \ltimes \nu(T)$) [BT72, 6.2.19], where W_φ is the stabilizer of φ.

A valuation φ is *discrete* if Γ_a is a discrete subset of $\mathbb{R}$ for all $a \in \Phi$. If this is the case, then W is the affine Weyl group $W(\Sigma)$ for the affine root system Σ [BT72, 6.2.22].

Suppose Φ is irreducible and φ is discrete and special. Then all Γ_a are the same discrete subgroup Γ of $\mathbb{R}$ [BT72, 6.2.23]. So 4.10 applies and we get an apartment $\mathscr{A}(\Sigma)$. Let $\mathcal{Q}^\vee$ be the *coroot lattice* of Φ, namely the set of $\mathbb{Z}$-linear combinations of coroots, and let $\mathcal{P}^\vee$ be the *coweight lattice* of Φ, namely the set $\{v \in V \mid a(v) \in \mathbb{Z}\}$. Then $\mathrm{Ker}(W \to {}^vW) = \mathcal{Q}^\vee \otimes \Gamma$ and $\nu(T)$ is between $\mathcal{Q}^\vee \otimes \Gamma$ and $\mathcal{P}^\vee \otimes \Gamma$ [BT72, 6.2.20].

9.8. Let Ω be a nonempty subset of $\mathbb{A}$ and let U_Ω denote the subgroup generated by U_α for all affine roots $\alpha \supseteq \Omega$. Define $f_\Omega \colon \Phi \to \mathbb{R} \cup \{\infty\}$ by

$$f_\Omega(a) = \inf\{k \in \mathbb{R} \mid \Omega \subseteq \alpha_{a+k}\}.$$

This is a typical example of *concave functions* on Φ [BT72, 6.4.3]: namely,

1. $f(a) + f(b) \geqslant f(a+b)$ for any pairs $a, b \in \Phi$ such that $a + b \in \Phi$ and
2. $f(a) + f(-a) \geqslant 0$ for any $a \in \Phi$.

For f a concave function on Φ, denote U_f the subgroup generated by $U_{a,f(a)}$ for all $a \in \Phi$ and let $P_f = T^\circ.U_f$, $N_f = N \cap U_f$ and $\Phi_f = \{a \in \Phi \mid f(a) + f(-a) = 0\}$. In particular, for $f = f_\Omega$, U_f coincides with U_Ω and we will denote P_f and Φ_f by P_Ω and Φ_Ω.

The image $\nu(N_{f_\Omega})$ is generated by the reflections r_α for affine roots α such that $\Omega \subseteq \partial\alpha$ and is identified with the Weyl group of Φ_Ω [BT72, 7.1.3]. The preimage of $\nu(N_{f_\Omega})$ is then $N \cap P_\Omega = T^\circ.N_{f_\Omega}$ and denoted by N_Ω. It is the stabilizer (=fixator) of Ω in N' and therefore is contained in the fixator $\widehat{N}_\Omega$ of Ω in N. Let $\widehat{P}_\Omega = \widehat{N}_\Omega.P_\Omega = \widehat{N}_\Omega.U_\Omega$. Then P_Ω and U_Ω are norm subgroups of it.

9.9 Definition. Notations as aboves. The *Bruhat-Tits building of G* (with the root group dataum and the valuation being given) is the quotient set $\mathscr{B}$ of $G \times \mathbb{A}$ under the following equivalent relation [BT72, 7.4.1]:

$$(g, x) \sim (h, y) \iff \exists n \in N : y = \nu(n).x,\ g^{-1}hn \in \widehat{P}_x.$$

Remark. The left multiplication of G on $G \times \mathbb{A}$ is compatible with above equivalent relation, hence $\mathscr{B}$ inherits a G-action. Identifying $\mathbb{A}$ as the subset $\{1\} \times \mathbb{A}$ of $\mathscr{B}$, we have:

1. $\mathscr{B} = \bigcup_{g\in G} g.\mathbb{A}$;

2. each U_α fixes $\alpha \in \Sigma$ pointwise [BT72, 6.4.5];

3. for each nonempty $\Omega \subseteq \mathbb{A}$, its fixator is $\widehat{P}_\Omega$ and it acts transitively on apartments containing Ω [BT72, 6.4.4, 6.4.9];

4. the stabilizer (resp. fixator) of $\mathbb{A}$ is N (resp. T°) [BT72, 6.4.10].

Then one can carry apartment structure on $\mathbb{A}$ to each $g.\mathbb{A}$ and see they agree on intersections [BT72, 7.4.18]. Hence $\mathscr{B}$ is a building of type $\mathscr{A}(\Sigma)$. The action of G on it is strongly transitively by the construction and is not necessarily type-preserving since the affine Weyl group W of $\mathscr{A}(\Sigma)$ is usually not the entire $\widehat{W}$. The subgroup of type-preserving automorphisms is then the group $G' = \nu^{-1}(W)$ introduced in 9.7.

§ 10 Bruhat-Tits buildings

10.1. In the rest, K will be a field equipped with a (trivial or discrete) valuation $\mathrm{val}\colon K \to \Gamma \cup \{\infty\}$. We fix the following associated notations.

$$\begin{aligned} K^\circ &:= \{x \in K \mid \mathrm{val}(x) \geqslant 0\}, \\ (K^\circ)^\times &:= \{x \in K \mid \mathrm{val}(x) = 0\}, \\ K^{\circ\circ} &:= \{x \in K \mid \mathrm{val}(x) > 0\}, \\ \kappa &:= K^\circ / K^{\circ\circ}. \end{aligned}$$

We further assume K is complete with respect to $\mathrm{val}(\,\cdot\,)$ and κ is a finite field with cardinality q and characteristic p.

§ 11 Moy-Prasad filtrations

11.1. Let f be a concave function. Define $f^*\colon \Phi \to \widetilde{\mathbb{R}}$ as follows:

$$f^*(a) = \begin{cases} f(a)+ & \text{if } a \in \Phi_f, \\ f(a) & \text{if } a \notin \Phi_f. \end{cases}$$

Here $\widetilde{\mathbb{R}}$ is the ordered monoid of extended real numbers[7] and for each $k \in \mathbb{R}$, $k+$ is the smalest extended real number larger than k. Then f^* is a concave function in the sense of 9.8 with $\mathbb{R} \cup \{\infty\}$ replaced by $\widetilde{\mathbb{R}}$ [BT72, 6.4.23]. For each $a \in \Phi$ and any $u \in U_{-a,f(-a)}, v \in U_{a,f^*(a)}$ Let T_{f,f^*} denote the subgroup of T° generated by

[7]Foramlly, $\widetilde{\mathbb{R}}$ is the union of $\mathbb{R}$, $\mathbb{R}+ := \{k+ \mid k \in \mathbb{R}\}$ and $\{\infty\}$. The commutative addition on $\mathbb{R}$ is extended as follows: $k + (l+) = (k+) + (l+) = (k + l)+$ for all $k, l \in \mathbb{R}$ and $\lambda + \infty = \infty$ for all $\lambda \in \widetilde{\mathbb{R}}$. The total order on $\mathbb{R}$ is extended as follows: $k < k+ < l$ for all $k, l \in \mathbb{R}$ such that $k < l$ and $\lambda < \infty$ for all $\lambda \neq \infty$.

Part III

Buildings for classical groups

§ 12

§ 13 Norms and buildings

The Bruhat-Tits building associated to a classical group has a concrete interpretation.

13.1. Let V be a K-vector space. A *norm* (defined over K) on V is a map $\alpha\colon V \to \mathbb{R}\cup\{\infty\}$ such that for any $x, y \in V$ and $t \in K$,

1. $\alpha(tx) = \mathrm{val}(t) + \alpha(x)$;
2. $\alpha(x+y) \geqslant \inf\{\alpha(x), \alpha(y)\}$;
3. $\alpha(x) = \infty$ if and only if $x = 0$.

The set of norms on V is denoted by $\mathscr{N}(K,V)$.

If α is a norm, then so is $\alpha + c$ for any $c \in \mathbb{R}$. Such a norm is said to be *homothetic* to α. The set of homothetic classes of norms on V is denoted by $\mathscr{X}(K,V)$.

Let α be a norm and g be an automorphism of V. Then $\alpha \circ g^{-1}$ is also a norm, denoted by $g.\alpha$. In such a way, $\mathrm{GL}(V)$ acts on $\mathscr{N}(K,V)$. Moreover, this action respects homotheties, making $\mathscr{X}(K,V)$ a $\mathrm{GL}(V)$-set.

A family $\mathscr{W}$ of subspaces of V is said to be *splitting* for α if V admits a decomposition $V = \bigoplus_{W\in\mathscr{W}} W$ respecting α in the sense that for any tuple $(x_W)_{W\in\mathscr{W}}$ with $x_W \in W$, we have

$$\alpha(\sum_{W\in\mathscr{W}} x_w) = \inf_{W\in\mathscr{W}} \alpha(x_W).$$

A *frame* in V is a family $\mathscr{L}$ of lines (i.e. one-dimensional subspaces) such that V admits a decomposition $V = \bigoplus_{L\in\mathscr{L}} L$. Given a frame $\mathscr{L}$, the set $\{\alpha \in \mathscr{N}(K,V) \mid \mathscr{L} \text{ is splitting for } \alpha\}$ is invariant under homotheties and its homothetic quotient is denoted by $\mathscr{A}(\mathscr{L})$.

13.2 Theorem. *Let V be a K-vector space. Then the action of $\mathrm{GL}(V)$ on $\mathscr{X}(K,V)$ gives rise to a Bruhat-Tits building strcuture with each apartment being $\mathscr{A}(\mathscr{L})$ for some frame $\mathscr{L}$. Moreover, $\mathscr{X}(K,V) \cong \mathscr{B}(\mathsf{GL}(V))$.*

1. Let $\mathsf{N}(\mathscr{L})$ be the stabilizer of $\mathscr{L}$, that is the algebraic subgroup of $\mathsf{GL}(V)$ representing the functor

$$R \rightsquigarrow \{g \in \mathrm{GL}(V_R) \mid \forall L \in \mathscr{L}, \exists L' \in \mathscr{L} : g.L \subseteq L'\}.$$

Then its K-points $N(\mathscr{L})$ is the stabilizer of $\mathscr{A}(\mathscr{L})$.

2. Any element of $N(\mathscr{L})$ permutes $\mathscr{L}$. We thus get a surjection $N(\mathscr{L}) \twoheadrightarrow \mathfrak{S}_n$. Let D_n denote its kernel. Then D_n is precisely those automorphisms diagonalized by $\mathscr{L}$. Hence $D_n \cong \mathbb{R}^n$.

3. Therefore $\mathscr{A}(\mathscr{L})$ is the apartment $\mathscr{A}(D_n)$.

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