STRATIFICATION FOR FINITE GROUPOIDS

XU GAO

Abstract. We apply methods of [2] to solve the (co)stratification problem for finite groupoids.

Contents

1. Representations of finite groupoids

In this section, we introduce the notion of representations of a finite groupoid and construct a Frobenius category of them.

1.1. *Notation.* Let R be a commutative noetherian ring. We adopt the following notations for various cateogries.

$\mathsf{Mod}\,R$**:** the category of R-modules;
$\mathsf{mod}\,R$**:** the category of noetherian R-modules;
$\mathsf{Proj}\,R$**:** the category of projective R-modules;
$\mathsf{Inj}\,R$**:** the category of injective R-modules;
$\mathbf{D}(R)$**:** the derived category of R.

1.2. *Definition.* Let $\mathcal{G}$ be an *essentially finite* groupoid: that is to say, both $\pi_0(\mathcal{G})$ and every hom-set are finite. By an *R-representation* of $\mathcal{G}$, we mean a functor from $\mathcal{G}$ to $\mathsf{Mod}\,R$. We adopt the following notations for various cateogries.

$\mathsf{Mod}\,\mathcal{G}$**:** the category $\mathsf{Fun}(\mathcal{G}, \mathsf{Mod}\,R)$;
$\mathsf{Proj}\,\mathcal{G}$**:** the full subcategory of projective objects in $\mathsf{Mod}\,\mathcal{G}$;
$\mathsf{Mod}(\mathcal{G}; R)$**:** the category $\mathsf{Fun}(\mathcal{G}, \mathsf{Proj}\,R)$;
$\mathsf{mod}(\mathcal{G}; R)$**:** the category $\mathsf{Fun}(\mathcal{G}, \mathsf{Proj}\,R \cap \mathsf{mod}\,R)$.

Given an R-representation F, we use $\mathsf{Add}(F)$ for the full subcategory of $\mathsf{Mod}\,\mathcal{G}$ consisting of direct summands of direct sums of (infinitely many) copies of F and $\mathsf{add}(F)$ the corresponding subcategory where only finite copies are allowed.

1.3. *Remark.* There is an obvious tensor structure on $\mathsf{Mod}\,\mathcal{G}$ induced from the one on $\mathsf{Mod}\,R$. The tensor unit is the constant representation $\underline{R}\colon \mathcal{G} \to \mathsf{Mod}\,R$ mapping all objects to R and all arrows to id_R. However, $\underline{R}$ is not a generator of the category $\mathsf{Mod}\,\mathcal{G}$: since any morphism $\underline{R}^I \to F$ factors through the limit $\lim_{\mathcal{G}} F$, saying $\underline{R}^I \to F$ is an epimorphism forces the canonical morphism $\lim_{\mathcal{G}} F \to F$ being surjective.

Frobenius categories. An exact category is *Frobenius* if it has enough projective and enough injective objects, where the class of projective objects coincides with the class of injective objects.

1.4. *Example.* The exact category $\mathsf{Proj}\,R$ is Frobenius: its projective and injective objects are all the objects.

We mimic [2, Lemma 2.3] to show the following lemmas.

1.5. **Lemma.** *The exact category* $\mathsf{Mod}(\mathcal{G};R)$ *has enough projective objects and enough injective objects.*

Proof. For each object x of $\mathcal{G}$, let $u_x\colon \mathsf{Mod}(\mathcal{G};R) \to \mathsf{Proj}\,R$ be the functor mapping each R-representation F to the R-module $F(x)$. The functor u_x is clearly exact. Hence, its left adjoint u_x^l preserves projectivity and its right adjoint u_x^r preserves injectivity (their existence follows from the *adjoint functor theorem*). Since all the objects of $\mathsf{Proj}\,R$ are both projective and injective, we see that the essential images of the endofunctors $u_x^l u_x$ and $u_x^r u_x$ are projective and injective objects respectively. However, unlike in [2, Lemma 2.3], the counit $\epsilon_x\colon u_x^l u_x \to \mathrm{id}$ needs not to be an epimorphism, and likewise, the unit $\eta_x\colon \mathrm{id} \to u_x^r u_x$ needs not to be a monomorphism.

To resolve this issue, note that there are natural transformations $u_f\colon u_x \to u_y$ for morphisms $f\colon x \to y$ in $\mathcal{G}$. By adjunctions, we also have natural transformations $u_f^l\colon u_y^l \to u_x^l$ and $u_f^r\colon u_y^r \to u_x^r$ respectively. We thus obtain two $\mathcal{G}$-shape diagrams in $\mathsf{End}(\mathsf{Mod}(\mathcal{G};R))$:

$$u^l\colon x \mapsto u_x^l u_x, f \mapsto u_{f^{-1}}^l u_f \quad \text{and} \quad u^r\colon x \mapsto u_x^r u_x, f \mapsto u_{f^{-1}}^r u_f.$$

Consider the functor $[-]\colon \mathcal{G} \to \pi_0(\mathcal{G})$ taking any object x of $\mathcal{G}$ to its isomorphism class $[x]$. We pick a section of $[-]$, saying $\varsigma\colon \pi_0(\mathcal{G}) \to \mathcal{G}$. Then, it induces a functor

$$\varsigma^*\colon \mathsf{Fun}(\mathcal{G}, \mathsf{End}(\mathsf{Mod}(\mathcal{G};R))) \to \mathsf{Fun}(\pi_0(\mathcal{G}), \mathsf{End}(\mathsf{Mod}(\mathcal{G};R))).$$

Now, consider the $\pi_0(\mathcal{G})$-shape diagrams $\varsigma^*(u^l)$ and $\varsigma^*(u^r)$. Since $\pi_0(\mathcal{G})$ is a finite discrete category, taking limits and colimits of shape $\pi_0(\mathcal{G})$ in $\mathsf{End}(\mathsf{Mod}(\mathcal{G};R))$ are precisely taking finite products and coproducts respectively. Hence, the essential images of the endofunctors $\varsigma^l \coloneqq \operatorname{colim} \varsigma^*(u^l)$ and $\varsigma^r \coloneqq \lim \varsigma^*(u^r)$ are projective and injective objects respectively, and they are canonically isomoprhic.

Now, the induced morphisms $\epsilon_\varsigma\colon \varsigma^l \to \mathrm{id}$ and $\eta_\varsigma\colon \mathrm{id} \to \varsigma^r$ provides a projective cover and an injective hull respectively. Indeed, for any object x of $\mathcal{G}$, we can see that $u_x\epsilon_\varsigma = u_x\epsilon_{\varsigma[x]}$ and is thus (since $\varsigma[x] \cong x$) isomorphic to $u_x\epsilon_x$, which is an epimorphism by the adjunction. As $u_x\epsilon_\varsigma$ is an epimorphism for all $x \in \mathcal{G}$, we conclude that ϵ_ς is an epimorphism. Likewise, η_ς is a monomorphism. This finishes the proof. $\square$

1.6. **Proposition.** *We have*

$$\mathsf{Proj}(\mathsf{Mod}(\mathcal{G};R)) = \mathsf{Add}\big(\varsigma^l \underline{R}\big) = \mathsf{Proj}\,\mathcal{G} \quad \textit{and} \quad \mathsf{Inj}(\mathsf{Mod}(\mathcal{G};R)) = \mathsf{Add}(\varsigma^r \underline{R}).$$

In particular, $\mathsf{Mod}(\mathcal{G};R)$ *is a Frobenius category.*

Proof. First, we show that $\varsigma^l \underline{R}$ is a generator of $\mathsf{Mod}\,\mathcal{G}$. Indeed, we have

$$\mathrm{Hom}_{\mathsf{Mod}\,\mathcal{G}}\big(\varsigma^l \underline{R}, -\big) = \prod_{x\in\varsigma\pi_0(\mathcal{G})} \mathrm{Hom}_{\mathsf{Mod}\,\mathcal{G}}\big(u_x^l u_x \underline{R}, -\big) = \prod_{x\in\varsigma\pi_0(\mathcal{G})} \mathrm{Hom}_R(R, u_x(-)),$$

which is conservative. By a similar reasoning, $\varsigma^r \underline{R}$ is a cogenerator of $\mathsf{Mod}\,\mathcal{G}$.

Since $\varsigma^l \underline{R}$ is a generator of $\mathsf{Mod}\,\mathcal{G}$, we must have $\mathsf{Proj}\,\mathcal{G} \subset \mathsf{Add}\big(\varsigma^l \underline{R}\big)$. On the other hand, $\varsigma^l \underline{R}$ itself is projective in both $\mathsf{Mod}(\mathcal{G}; R)$ and $\mathsf{Mod}\,\mathcal{G}$ (since R is projective in both $\mathsf{Proj}\,R$ and $\mathsf{Mod}\,R$). We thus conclude that $\mathsf{Proj}(\mathsf{Mod}(\mathcal{G}; R)) = \mathsf{Add}\big(\varsigma^l \underline{R}\big) = \mathsf{Proj}\,\mathcal{G}$.

Likewise, since $\varsigma^r \underline{R}$ is a cogenerator of $\mathsf{Mod}\,\mathcal{G}$, and hence of $\mathsf{Mod}(\mathcal{G}, R)$, we must have $\mathsf{Inj}(\mathsf{Mod}(\mathcal{G}; R)) \subset \mathsf{Add}(\varsigma^r \underline{R})$. Since $\varsigma^r \underline{R}$ itself is injective in $\mathsf{Mod}(\mathcal{G}, R)$, we conclude that $\mathsf{Inj}(\mathsf{Mod}(\mathcal{G}; R)) = \mathsf{Add}(\varsigma^r \underline{R})$.

Now, since $\varsigma^l \underline{R} \cong \varsigma^r \underline{R}$, we have $\mathsf{Proj}(\mathsf{Mod}(\mathcal{G}; R)) = \mathsf{Inj}(\mathsf{Mod}(\mathcal{G}; R))$. Then, by Lemma 1.5, $\mathsf{Mod}(\mathcal{G}; R)$ is Frobenius. □

1.7. *Notation.* For any R-representation F in $\mathsf{Mod}(\mathcal{G}; R)$, repeating applying the projective cover ϵ_ς yields a projective resolution of F, denoted by $\mathbf{p}_\varsigma F$. Likewise, the injective hull η_ς yields an injective resolution $\mathbf{i}_\varsigma F$.

In what follows, we fix a choice of ς and omit it if there is no ambiguity.

2. The triangulated categories

In this section, following methods in [2, §2], we construct a compactly generated triangulated category $\mathsf{Rep}(\mathcal{G}, R)$ whose compact objects identify with the bounded derived category of the Frobenius category $\mathsf{mod}(\mathcal{G}; R)$.

The stable category. For a Frobenius exact category $\mathcal{A}$, its *stable category* $\mathsf{St}\,\mathcal{A}$ is the category whose objects are the same as $\mathcal{A}$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathsf{St}\,\mathcal{A}}(M, N) \coloneqq \mathrm{Hom}_{\mathcal{A}}(M, N)/\sim,$$

where two morphisms are equivalent if their difference factors through a projective object. This category carries a natural triangulated structure [6, Section 3.3]. In our case, we set

$$\mathsf{stmod}(\mathcal{G}, R) \coloneqq \mathsf{St}(\mathsf{mod}(\mathcal{G}; R)) \quad \text{and} \quad \mathsf{StMod}(\mathcal{G}, R) \coloneqq \mathsf{St}(\mathsf{Mod}(\mathcal{G}; R)).$$

The homotopy category of projectives. Let $\mathcal{A}$ be an abelian category with enough projective objects and $\mathbf{K}(\mathsf{Proj}\,\mathcal{A})$ be homotopy category of complexes of projective objects in $\mathcal{A}$. Then taking projective resolution $X \mapsto \mathbf{p}X$ yields a left adjoint to the canonical functor $\mathbf{q}\colon \mathbf{K}(\mathsf{Proj}\,\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ and identifies the derived category $\mathbf{D}(\mathcal{A})$ with the full subcategory of *K-projective complexes* in $\mathbf{K}(\mathsf{Proj}\,\mathcal{A})$.

2.1. *Definition.* Note that a complex of R-representations of $\mathcal{G}$ can also be viewed as a functor from $\mathcal{G}$ to the category of complexes of R-modules. We say such a complex has *K-projective values* if it, as a functor, lands in the category of K-projective complexes of R-modules. We write $\mathbf{K}(\mathsf{Proj}\,\mathcal{G}, R)$ for the full subcategory of complexes in $\mathbf{K}(\mathsf{Proj}\,\mathcal{G})$ that have K-projective values and $\mathbf{A}\mathrm{c}(\mathsf{Proj}\,\mathcal{G}, R)$ for the full subcategory consisting of acyclic complexes in $\mathbf{K}(\mathsf{Proj}\,\mathcal{G}, R)$.

2.2. *Remark.* If a complex F of R-representations is acyclic and has K-projective values, its each value $F(x)$ is an acyclic K-projective complex of R-modules and is therefore contractible. Hence, F is a complex in $\mathsf{Mod}(\mathcal{G}; R)$.

2.3. *Definition.* Recall (1.7) that, for any R-representation F in $\mathsf{Mod}(\mathcal{G};R)$, we have a projective resolution $\mathbf{p}F$ and an injective resolution $\mathbf{i}F$. Completing the canonical morphism $\mathbf{p}F \to \mathbf{i}F$ to an exact triangle

$$\mathbf{p}F \longrightarrow \mathbf{i}F \longrightarrow \mathbf{t}F \longrightarrow$$

yields an acyclic complex $\mathbf{t}F$ of objectives in $\mathsf{Mod}(\mathcal{G};R)$ satisfying $\mathsf{Z}^0(\mathbf{t}F) = F$, where Z^0 denotes the module of 0-cocycle in the complex. The complex $\mathbf{t}F$ is called the *Tate resolution* of F.

2.4. **Lemma.** *The assignment $\mathsf{Z}^0(-)$ induces an R-linear triangle equivalence*

$$\mathbf{A}\mathrm{c}(\mathsf{Proj}\,\mathcal{G}, R) \xrightarrow{\sim} \mathsf{StMod}(\mathcal{G}, R).$$

Proof. By Remark 2.2, the functor is well-defined. Taking Tate resolution gives its quasi-inverse. □

We write $\mathbf{D}(\mathcal{G})$ for the derived category of the Grothendieck category $\mathsf{Mod}\,\mathcal{G}$.

2.5. **Proposition.** *The canonical functor $\mathbf{q}\colon \mathbf{K}(\mathsf{Proj}\,\mathcal{G}) \to \mathbf{D}(\mathcal{G})$ induces a localisation sequence*

$$\mathbf{A}\mathrm{c}(\mathsf{Proj}\,\mathcal{G}, R) \overset{\mathbf{t}}{\leftrightarrows} \mathbf{K}(\mathsf{Proj}\,\mathcal{G}, R) \overset{\mathbf{p}}{\leftrightarrows} \mathbf{D}(\mathcal{G})$$

Proof. The functorial projective resolution $\mathbf{p}$ defines a left adjoint of the canonical functor $\mathbf{q}$. By the exact triangles $\mathbf{p}X \to X \to \mathbf{t}X \to$, the Tate resolution functor $\mathbf{t}$ gives the left adjoint of the inclusion of $\mathbf{A}\mathrm{c}(\mathsf{Proj}\,\mathcal{G}, R)$. □

Compact generation. Compact objects in $\mathsf{Mod}\,\mathcal{G}$ are those in $\mathsf{mod}(\mathcal{G}, R)$. Let $\mathsf{Mod}^{\mathrm{lf}}(\mathcal{G}, R)$ be the *localising* subcategory of $\mathsf{Mod}(\mathcal{G}, R)$ generated by $\mathsf{mod}(\mathcal{G}, R)$. That is to say, $\mathsf{Mod}^{\mathrm{lf}}(\mathcal{G}, R)$ is a full subcategory of $\mathsf{Mod}(\mathcal{G}, R)$ containing $\mathsf{mod}(\mathcal{G}, R)$, closed under all coproducts, and satisfying the *two-out-of-three property*: for any short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0,$$

if any two of the terms are in the subcategory so is the third. By Proposition 1.6, the exact category $\mathsf{Mod}^{\mathrm{lf}}(\mathcal{G}, R)$ is Frobenius. Set

$$\mathsf{StMod}^{\mathrm{lf}}(\mathcal{G}, R) \coloneqq \mathsf{St}\Big(\mathsf{Mod}^{\mathrm{lf}}(\mathcal{G}; R)\Big).$$

2.6. **Proposition.** *The triangulated category $\mathsf{Stmod}^{\mathrm{lf}}(\mathcal{G}, R)$ is compactly generated and its subcategory of compact objects identifies with the idempotent completion of $\mathsf{stmod}(\mathcal{G}, R)$.*

Proof. The canonical functor from a Frobenius category to its stable category preserves coproducts. Then, the statement follows by [6, Proposition 3.4.15]. □

2.7. **Lemma.** *For any $x \in \mathcal{G}$, the right adjoint u_x^r of u_x preserves coproducts.*

Proof. Note that, for any $F \in \mathsf{Mod}\,\mathcal{G}$ and any $x \in \mathcal{G}$, we have

$$F(x) = \mathrm{Hom}_R(R, u_x F) = \mathrm{Nat}\big(u_x^l R, F\big).$$

Hence, by the Yoneda lemma, $u_x^l R$ is precisely $R[\mathsf{h}_x]$, the composite of the free R-module functor $R[-]$ with the representable functor $\mathsf{h}_x \coloneqq \mathrm{hom}(x, -)$.

To show u_x^r preserves coproducts, it suffices to show each $u_y u_x^r$ ($y \in \mathcal{G}$) preserves coproducts. Indeed, we have

$$\begin{aligned} u_y u_x^r(-) &= \operatorname{Hom}_R(R, u_y u_x^r -) = \operatorname{Hom}_R\big(u_x u_y^l R, -\big) \\ &= \operatorname{Hom}_R(R[\hom(x,y)], -) = (-)^{\hom(x,y)}, \end{aligned}$$

which preserves coproducts since $\hom(x,y)$ is finite. $\square$

2.8. *Notation.* We write $\mathbf{K}^{\mathrm{lf}}(\mathsf{Proj}\,\mathcal{G}, R)$ for the localising subcategory of $\mathbf{K}(\mathsf{Proj}\,\mathcal{G})$ generated by the objects satisfying the following equivalent conditions:

(a) X is compact in $\mathbf{K}(\mathsf{Proj}\,\mathcal{G})$ and has K-projective values;

(b) X fits into an extension $0 \to X'' \to X \to X' \to 0$ such that $X' = \mathbf{i}F$ for some $F \in \mathsf{mod}(\mathcal{G}, R)$ and X'' is perfect[1];

(c) X is in $\mathbf{K}^{+,b}(\mathsf{proj}\,\mathcal{G})$: it is a complex in $\mathsf{proj}\,\mathcal{G} \coloneqq \mathsf{Proj}\,\mathcal{G} \cap \mathsf{mod}(\mathcal{G}, R)$ that is bounded from below (i.e. $X^n = 0$ for $n \ll 0$) and cohomologically bounded from above (i.e $\mathsf{H}^n(X) = 0$ for $n \gg 0$), where the cohomology is defined in the ambient exact category $\mathsf{mod}(\mathcal{G}, R)$.

Proof. (of the equivalence) (a)$\Rightarrow$(c): Suppose X is compact in $\mathbf{K}(\mathsf{Proj}\,\mathcal{G})$ and has K-projective values. Since each u_x^r preserves coproducts, the valuation functor $u_x \colon \mathbf{K}(\mathsf{Proj}\,\mathcal{G}) \to \mathbf{K}(\mathsf{Proj}\,R)$ preserves compactness. Then, each value $X(x)$ is a compact object in $\mathbf{K}(\mathsf{Proj}\,R)$ and, by [7, Proposition 7.6], belongs to $\mathbf{K}^{+,b}(\mathsf{proj}\,R)$. Therefore, X is in $\mathbf{K}^{+,b}(\mathsf{proj}\,\mathcal{G})$.

(b)$\Rightarrow$(a): It suffices to verify that both $X' = \mathbf{i}F$ and X'' are compact and have K-projective values. Clearly, X'' is compact, and the later statement is evident. Since $F \in \mathsf{mod}(\mathcal{G}, R)$ is noetherian, by [5, Lemma 2.1], $X' = \mathbf{i}F$ is compact.

(b)$\Leftrightarrow$(c): For a complex X in $\mathbf{K}^{+,b}(\mathsf{proj}\,\mathcal{G})$, truncating it at degree n for sufficiently large n yields the extension. $\square$

2.9. *Remark.* If F is a complex in $\mathbf{K}^{\mathrm{lf}}(\mathsf{Proj}\,\mathcal{G}, R)$, its each value $F(x)$ is then perfect, since $F(x)$ is both compact and K-projective.

We write $\mathbf{D}^{\mathrm{perf}}(\mathcal{G})$ for the full subcategory $\mathbf{D}(\mathcal{G})$ consisting of perfect complexes.

2.10. **Proposition.** *The triangulated category $\mathbf{K}^{\mathrm{lf}}(\mathsf{Proj}\,\mathcal{G}, R)$ is compactly generated and the canonical functor from it to $\mathbf{D}(\mathcal{G})$ induces a triangle equivalence from its subcategory $\mathbf{K}^{\mathrm{lf}}(\mathsf{Proj}\,\mathcal{G}, R)^c$ of compact objects to $\mathbf{D}^b(\mathsf{mod}(\mathcal{G}, R))$.*

Proof. The compact generation follows from the construction. The equivalence of descriptions of objects in $\mathbf{K}^{+,b}(\mathsf{proj}\,\mathcal{G})$ shows that $\mathbf{K}^{\mathrm{lf}}(\mathsf{Proj}\,\mathcal{G}, R)^c = \mathbf{K}^{+,b}(\mathsf{proj}\,\mathcal{G})$. The canonical functor $\mathbf{K}^{+,b}(\mathsf{proj}\,\mathcal{G}) \to \mathbf{D}^b(\mathsf{mod}(\mathcal{G}, R))$ is a triangle equivalence since $\mathsf{mod}(\mathcal{G}, R)$ has enough injective objects. $\square$

2.11. **Corollary.** *The canonical functor induces a recollement*

$$\mathsf{StMod}^{\mathrm{lf}}(\mathcal{G}, R) \; \overset{\mathbf{t}}{\underset{\twoheadleftarrow}{\overset{\twoheadleftarrow}{\longrightarrow}}} \; \mathbf{K}^{\mathrm{lf}}(\mathsf{Proj}\,\mathcal{G}, R) \; \overset{\mathbf{p}}{\underset{\leftarrowtail}{\overset{\leftarrowtail}{\rightarrowtail}}} \; \mathbf{D}(\mathcal{G})$$

and the pair of left adjoints $(\mathbf{p}, \mathbf{t})$ induces (when restricted to compact objects) a triangle equivalence

$$\mathbf{D}^b(\mathsf{mod}(\mathcal{G}, R)) / \mathbf{D}^{\mathrm{perf}}(\mathcal{G}) \xrightarrow{\sim} \mathsf{stmod}(\mathcal{G}, R).$$

[1]Recall that a complex is *perfect* if it is quasi-isomorphic to a bounded complex of notherian projective objects.

Proof. The functorial injective resolution $\mathbf{i}$ defines a right adjoint of the canonical functor $\mathbf{q}$. This yields the right half of the recollement; the left half is a consequence, where we identify $\mathsf{StMod}^{\mathrm{lf}}(\mathcal{G}, R)$ with $\mathbf{A}\mathrm{c}(\mathsf{Proj}\,\mathcal{G}, R)$ by Lemma 2.4.

The description of compact objects follows from Propositions 2.6 and 2.10. The last statement follows from [6, Proposition 4.4.18]: for any Frobenius category $\mathcal{A}$ with full subcategory of projective objects $\mathcal{P}$, we have $\mathbf{D}^b(\mathcal{A})/\mathbf{D}^b(\mathcal{P}) \simeq \mathsf{St}\,\mathcal{A}$. $\square$

2.12. *Notation.* We write $\mathsf{Rep}(\mathcal{G}, R)$ for the triangulated category $\mathbf{K}^{\mathrm{lf}}(\mathsf{Proj}\,\mathcal{G}, R)$ and $\mathsf{rep}(\mathcal{G}, R)$ for $\mathbf{D}^b(\mathsf{mod}(\mathcal{G}, R))$. Recall that $\mathsf{Rep}(\mathcal{G}, R)^c \simeq \mathsf{rep}(\mathcal{G}, R)$.

Note that, for the *delooping groupoid* $\mathcal{B}G$ of G (the one-object groupoid with G as its only hom-set), our catefories $\mathsf{Rep}(\mathcal{B}G, R)$ and $\mathsf{rep}(\mathcal{B}G, R)$ are the same as the categories $\mathsf{Rep}(G, R)$ and $\mathsf{rep}(G, R)$ in [2]. In what follows, we will drop the letter $\mathcal{B}$ in the noation $\mathcal{B}G$ when it appears in a notation of cateogry such as $\mathsf{Rep}(\mathcal{B}G, R)$.

3. Rigidly-compactly generation

In this section, we show that the tensor structure (Remark 1.3) on the abelian tensor category $\mathsf{Mod}\,\mathcal{G}$ induces one on the triangulated category $\mathsf{Rep}(\mathcal{G}, R)$ making it a rigidly-compactly generated tensor triangulated category.

3.1. **Lemma.** *The tensor structure on* $\mathsf{Mod}\,\mathcal{G}$ *induces one on* $\mathsf{Rep}(\mathcal{G}, R)$*. The tensor unit is given by the injective resolution* $\mathbf{i}\underline{R}$ *of the constant representation* $\underline{R}$*.*

Proof. If F and G are two projective R-representations, then so is $F \otimes G$ by the characterization of projectivity in Proposition 1.6. Hence, the tensor product on $\mathsf{Mod}\,\mathcal{G}$ descends to $\mathbf{K}(\mathsf{Proj}\,\mathcal{G})$. On the other hand, the entire tensor structure restricts to $\mathsf{mod}(\mathcal{G}, R)$, which induces one on $\mathsf{rep}(\mathcal{G}, R) = \mathbf{D}^b(\mathsf{mod}(\mathcal{G}, R))$. Since the $\mathsf{Rep}(\mathcal{G}, R)^c$ is equivalent to $\mathsf{rep}(\mathcal{G}, R)$, where a triangle equivalence is given by the functorial injective resolution $\mathbf{i}$ (cf. Notation 2.8), we conclude that: 1, the tensor product restricts to $\mathsf{Rep}(\mathcal{G}, R)^c$ and induces the derived tensor product on $\mathsf{rep}(\mathcal{G}, R)$; and 2, the injective resolution $\mathbf{i}\underline{R}$ is the tensor unit in $\mathsf{Rep}(\mathcal{G}, R)^c$. Now, since $\mathsf{Rep}(\mathcal{G}, R)$ is compactly generated, we conclude that the tensor structure on $\mathsf{Mod}\,\mathcal{G}$ induces one on $\mathsf{Rep}(\mathcal{G}, R)$ with the tensor unit $\mathbf{i}\underline{R}$. $\square$

Rigidity. A tensor category $\mathcal{T}$ is *closed* if it has *internal Hom objects* $\underline{\mathrm{Hom}}(-,-)$ characterized by the *tensor-Hom adjunction*

$$\mathrm{Hom}_{\mathcal{T}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathcal{T}}(X, \underline{\mathrm{Hom}}(Y, Z)).$$

For any objects $X, Y \in \mathcal{T}$, the above adjunction provides a natural map

$$\underline{\mathrm{Hom}}(X, \mathbf{1}) \otimes Y \longrightarrow \underline{\mathrm{Hom}}(X, Y), \tag{3.2}$$

and X is *rigid* if this map is an isomorphism for all Y. If all objects are rigid, one says that $\mathcal{T}$ is *rigid*.

3.3. *Example.* The tensor category $\mathsf{Mod}\,\mathcal{G}$ is closed, where the internal Hom object $\underline{\mathrm{Hom}}(F, G)$ is given by $x \mapsto \mathrm{Hom}_R(F(x), G(x))$ and $f \mapsto F(f^{-1})^* \circ G(f)_*$. Clearly, a R-representation F is rigid if and only if each R-module $F(x)$ is rigid, namely, finitely generated and projective.

The subcategory $\mathsf{mod}(\mathcal{G}, R)$ is rigid. Indeed, if $F, G \in \mathsf{mod}(\mathcal{G}, R)$, then it is evident that $\underline{\mathrm{Hom}}(F, G) \in \mathsf{mod}(\mathcal{G}, R)$. Now, the natural map Eq. (3.2) becomes

$$\underline{\mathrm{Hom}}(F, \underline{R}) \otimes G \longrightarrow \underline{\mathrm{Hom}}(F, G),$$

which is an isomorphism if and only if the natural maps

$$\mathrm{Hom}_R(F(x), R) \otimes G(x) \longrightarrow \underline{\mathrm{Hom}}(F(x), G(x))$$

are isomorphisms, which is true since the R-modules $F(x)$ are finitely generated and projective.

Let $\mathcal{T}$ be a compactly generated tensor triangulated category. Then, $\mathcal{T}$ admits a closed structure $\underline{\mathrm{Hom}}(-,-)$ since the tensor product preserves coproducts and hence admits a right adjoint. Assume that $\mathbf{1}$ is compact. Then, every rigid object is compact. If the converse is true, then one says that $\mathcal{T}$ is is *rigidly-compactly generated*.

3.4. **Proposition.** *The tensor triangulated category* $\mathsf{Rep}(\mathcal{G}, R)$ *is rigidly-compactly generated.*

Proof. Since $\mathsf{Rep}(\mathcal{G}, R)$ is compactly generated and it has been shown in Lemma 3.1 that $\mathsf{Rep}(\mathcal{G}, R)^c$ is a tensor subcategory, it suffices to show that $\mathsf{Rep}(\mathcal{G}, R)^c$ is rigid. The latter is tensor triangulated equivalent to $\mathbf{D}^b(\mathsf{mod}(\mathcal{G}, R))$, whose rigidity follows from the rigidity of $\mathsf{mod}(\mathcal{G}, R)$, which has been shown in Example 3.3. □

4. Tensor triangulated decomposition

In this section, we show that $\mathsf{Rep}(\mathcal{G}, R)$ is tensor triangulated equivalent to a product of several rigidly-compactly generated tensor triangulated categories.

4.1. *Notation.* For any object x of $\mathcal{G}$, let $\mathcal{G}_x$ denote its automorphism group. Note that, for members in a given isomorphism class $[x] \in \pi_0(\mathcal{G})$, their automorphism groups are conjugated to each other by the involved isomorphism. As in Section 1, we fix a choice ς of representatives of the isomorphism classes in $\pi_0(\mathcal{G})$ and write $\mathcal{G}_{[x]}$ for the automorphism group $\mathcal{G}_{\varsigma[x]}$. Then the skeleton of $\mathcal{G}$, denoted as $\mathsf{sk}\,\mathcal{G}$, can be constructed as follows: its objects are $\pi_0(\mathcal{G})$ and the only hom-sets are $\mathrm{hom}([x],[x]) \coloneqq \mathcal{G}_{[x]}$.

4.2. **Lemma.** *The abelian tensor category* $\mathsf{Mod}\,\mathcal{G}$ *is equivalent to*

$$\prod_{[x]\in\pi_0(\mathcal{G})} \mathsf{Mod}\,\mathcal{G}_{[x]}.$$

Proof. Let $\mathsf{sk}\,\mathcal{G}$ denote the skeleton of $\mathcal{G}$. Then, it is clear that the abelian tensor category $\mathsf{Mod}\,\mathcal{G}$ is tensor equivalent to $\mathsf{Fun}(\mathsf{sk}\,\mathcal{G}, \mathsf{Mod}\,R)$. Note that $\mathsf{ob}(\mathsf{sk}\,\mathcal{G}) = \pi_0(\mathcal{G})$ and the only hom-sets in $\mathsf{sk}\,\mathcal{G}$ are the automorphism groups $\mathcal{G}_{[x]}$. Hence, $\mathsf{sk}\,\mathcal{G}$ is the disjoint union $\coprod_{[x]\in\pi_0(\mathcal{G})} \mathcal{B}\,\mathcal{G}_{[x]}$ of the delooping groupoids of $\mathcal{G}_{[x]}$. The statement then follows. □

Replacing $\mathsf{Mod}(R)$ by $\mathsf{Proj}(R)$ and $\mathsf{Proj}(R) \cap \mathsf{mod}(R)$ in the above argument, we conclude that:

4.3. **Corollary.** *We have the following tensor equivalence:*

$$\mathsf{Mod}(\mathcal{G}, R) \simeq \prod_{[x]\in\pi_0(\mathcal{G})} \mathsf{Mod}\big(\mathcal{G}_{[x]}, R\big) \quad \textit{and} \quad \mathsf{mod}(\mathcal{G}, R) \simeq \prod_{[x]\in\pi_0(\mathcal{G})} \mathsf{mod}\big(\mathcal{G}_{[x]}, R\big).$$

4.4. **Lemma.** *Let* $\{\mathcal{A}_i\}_{i\in I}$ *be a finite family of exact cateogries, and let* $\mathcal{A}$ *be their product category. Then, we have triangle equivalence:*

$$\mathbf{D}(\mathcal{A}) \simeq \prod_{i\in I} \mathbf{D}(\mathcal{A}_i).$$

Proof. The similar statements for the homotopy categories of chain complexes are evident. To show the desired equivalence, note that $\mathbf{D}(-)$ is a localisation of $\mathbf{K}(-)$, which is a 2-colimit in the 2-category of categories, while taking product $(-) \times \mathcal{C}$ with a category $\mathcal{C}$ is a left 2-adjoint and hence preserving 2-colimits. □

4.5. **Proposition.** *We have tensor triangle equivalences:*

$$\mathsf{Rep}(\mathcal{G}, R) \simeq \prod_{[x]\in\pi_0(\mathcal{G})} \mathsf{Rep}\big(\mathcal{G}_{[x]}, R\big) \quad \textit{and} \quad \mathsf{rep}(\mathcal{G}, R) \simeq \prod_{[x]\in\pi_0(\mathcal{G})} \mathsf{rep}\big(\mathcal{G}_{[x]}, R\big).$$

Proof. Recall that $\mathsf{rep}(\mathcal{G}, R) = \mathbf{D}^b(\mathsf{mod}(\mathcal{G}, R))$. Hence, the second triangle equivalence follows from Corollary 4.3 and Lemma 4.4. As for the first one, note that

$$\mathbf{K}(\mathsf{Proj}\,\mathcal{G}) \simeq \prod_{[x]\in\pi_0(\mathcal{G})} \mathbf{K}\big(\mathsf{Proj}\,\mathcal{G}_{[x]}\big).$$

Since $\mathsf{Rep}(\mathcal{G}, R)$ (resp. $\mathsf{Rep}\big(\mathcal{G}_{[x]}, R\big)$) is generated by $\mathsf{Rep}(\mathcal{G}, R)^c \simeq \mathsf{rep}(\mathcal{G}, R)$ (resp. $\mathsf{Rep}\big(\mathcal{G}_{[x]}, R\big) \simeq \mathsf{rep}\big(\mathcal{G}_{[x]}, R\big)$) as a subcategory of $\mathbf{K}(\mathsf{Proj}\,\mathcal{G})$ (resp. $\mathbf{K}\big(\mathsf{Proj}\,\mathcal{G}_{[x]}\big)$), the first triangle equivalence follows from the second one. They are tensor triangle equivalence by discussion of the tensor structures in Section 3. □

Groupoid cohomology. The *groupoid cohomology* of an R-representation F of $\mathcal{G}$ is defined to be

$$\mathsf{H}^*(\mathcal{G}, F) \coloneqq \operatorname{Ext}^*_{\mathcal{G}}(\underline{R}, F).$$

As a corollary of Proposition 4.5, we show that the groupoid cohomology decomposes into group cohomologies.

4.6. **Corollary.** *For any R-representation F of $\mathcal{G}$, the tensor triangle equivalences in Proposition 4.5 induces the following decomposition*

$$\mathsf{H}^*(\mathcal{G}, F) \cong \prod_{[x]\in\pi_0(\mathcal{G})} \mathsf{H}^*\big(\mathcal{G}_{[x]}, F([x])\big),$$

where $F([x])$ is viewed as a $\mathcal{G}_{[x]}$-module in the evident way. In particular, we have the following decomposition of graded commutative R-algebras

$$\mathsf{H}^*(\mathcal{G}, \underline{R}) \cong \prod_{[x]\in\pi_0(\mathcal{G})} \mathsf{H}^*\big(\mathcal{G}_{[x]}, R\big),$$

and the aforementioned decomposition respects their actions on each side.

Proof. The tensor triangle equivalences in Proposition 4.5 provides

$$\operatorname{Hom}^*_{\mathsf{Rep}(\mathcal{G},R)}(X, Y) \cong \prod_{[x]\in\pi_0(\mathcal{G})} \operatorname{Hom}^*_{\mathsf{Rep}\left(\mathcal{G}_{[x]},R\right)}(X([x]), Y([x])).$$

Applying this to $\mathbf{i}\underline{R}$ and F yields the first statement. The second statement follows from the first one plus the naturality. □

5. BIK-(co)stratifications and Balmer spectra

In this section, we recall the notions of (co)stratification in the sense of Benson-Iyengar-Krause [3, 4]. Their basic setup is a rigidly-compactly generated tensor triangulated category $\mathcal{T}$ with a graded-commutative noetherian ring S acting on it. The mechanism there produces a support map

$$\operatorname{supp}_{\mathcal{T}\circlearrowleft S} : \{\text{localising ideals of } \mathcal{T}\} \longrightarrow \Big\{\text{subsets of } \operatorname{Spec}^{\mathrm{h}}(S)\Big\}$$

and a cosupport map

$$\operatorname{cosupp}_{\mathcal{T}\circlearrowleft S}\colon \{\text{colocalising coideals of } \mathcal{T}\} \longrightarrow \Big\{\text{subsets of } \operatorname{Spec}^{\mathrm{h}}(S)\Big\},$$

where $\operatorname{Spec}^{\mathrm{h}}(S)$ denote the *homogeneous spectrum* of S, i.e. the topological space of homogeneous prime ideals of S. Then, saying tensor triangulated category $\mathcal{T}$ is *stratified by* S (resp. *costratified by* S) is amount to say that the map $\operatorname{supp}_{\mathcal{T}\circlearrowleft S}$ (resp. $\operatorname{cosupp}_{\mathcal{T}\circlearrowleft S}$) is bijective.

5.1. *Example.* The stratification and costratification of $\mathsf{Rep}(G, R)$, where G is a finite group scheme and R is a commutative noetherian ring, is demonstrated in [2, Theorem A].

Local (co)homology and (co)support. Recall that, in the Zariski topology on $\operatorname{Spec}^{\mathrm{h}}(S)$, the closed sets are

$$\mathcal{V}(I) \coloneqq \Big\{\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(S) \;\Big|\; \mathfrak{p} \supset I\Big\},$$

where I is any homogeneous ideal of S. On the other hand, For any $\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(S)$, there is a specialization closed subset

$$\mathcal{Z}(\mathfrak{p}) \coloneqq \Big\{\mathfrak{q} \in \operatorname{Spec}^{\mathrm{h}}(S) \;\Big|\; \mathfrak{q} \not\subset \mathfrak{p}\Big\}.$$

Let $\mathcal{W}$ be a specialization closed subsets of $\operatorname{Spec}^{\mathrm{h}}(S)$. Recall that an S-module M is said to be $\mathcal{W}$-*torsion* if $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \notin \mathcal{W}$. Then, an object $X \in \mathcal{T}$ is said to be $\mathcal{W}$-*torsion* if the S-modules $\operatorname{Hom}_{\mathcal{T}}(C, X)$ are $\mathcal{W}$-torsion for all $C \in \mathcal{T}^{c}$. The $\mathcal{W}$-torsion objects form a localising subcategory $\mathcal{T}(\mathcal{W})$. Hence, the inclusion functor admits a right adjoint $\Gamma_{\mathcal{W}}$ fitting into the exact triangle

$$\Gamma_{\mathcal{W}} \longrightarrow \mathrm{id} \longrightarrow L_{\mathcal{W}} \longrightarrow$$

where $L_{\mathcal{W}}$ is the corresponding localisation functor. Both $\Gamma_{\mathcal{W}}$ and $L_{\mathcal{W}}$ admits right adjoints (denoted by $\Lambda^{\mathcal{W}}$ and $V^{\mathcal{W}}$ respectively) fitting into the exact triangle

$$V^{\mathcal{W}} \longrightarrow \mathrm{id} \longrightarrow \Lambda^{\mathcal{W}} \longrightarrow$$

For any $\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(S)$, recall that an S-module M is said to be $\mathfrak{p}$-*local* if the localisation map $M \to M_{\mathfrak{p}}$ is invertible. On the other hand, a $\mathcal{V}(\mathfrak{p})$-torsion module is commonly called a $\mathfrak{p}$-*torsion module*. An object X of $\mathcal{T}$ is said to be $\mathfrak{p}$-*local* (resp. $\mathfrak{p}$-*torsion*) if the S-modules $\operatorname{Hom}_{\mathcal{T}}(C, X)$ are $\mathfrak{p}$-local (resp. $\mathfrak{p}$-torsion) for all $C \in \mathcal{T}^{c}$. Consider the full subcategories

$$\mathcal{T}_{\mathfrak{p}} \coloneqq \{X \in \mathcal{T} \mid X \text{ is } \mathfrak{p}\text{-local}\},$$
$$\Gamma_{\mathfrak{p}}\,\mathcal{T} \coloneqq \{X \in \mathcal{T} \mid X \text{ is } \mathfrak{p}\text{-local and } \mathfrak{p}\text{-torsion}\}.$$

Note that $\Gamma_{\mathfrak{p}}\,\mathcal{T} \subset \mathcal{T}_{\mathfrak{p}} \subset \mathcal{T}$ are localising subcategories. In fact, $\mathcal{T}_{\mathfrak{p}} = L_{\mathcal{Z}(\mathfrak{p})}\,\mathcal{T}$ and $\Gamma_{\mathfrak{p}}\,\mathcal{T} = \Gamma_{\mathcal{V}(\mathfrak{p})}\,\mathcal{T}_{\mathfrak{p}}$ (cf. [3, Corollaries 4.9, 5.10]). The composition

$$\Gamma_{\mathcal{V}(\mathfrak{p})} L_{\mathcal{Z}(\mathfrak{p})}\colon\ \mathcal{T} \longrightarrow \Gamma_{\mathfrak{p}}\,\mathcal{T}$$

is called the *local cohomology functor respect to* $\mathfrak{p}$ and is denoted by $\Gamma_{\mathfrak{p}}$. Then, the *BIK support* of an object X is

$$\operatorname{supp}_{\mathcal{T}\circlearrowleft S}(X) \coloneqq \Big\{\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(S) \;\Big|\; \Gamma_{\mathfrak{p}} X \neq 0\Big\}.$$

An object X of $\mathcal{T}$ is said to be $\mathfrak{p}$-*complete* if the natural map $X \to \Lambda^{\mathcal{V}(\mathfrak{p})}X$ is an isomorphism. Consider the full subcategory

$$\Lambda^{\mathfrak{p}}\,\mathcal{T} \coloneqq \{X \in \mathcal{T} \mid X \text{ is } \mathfrak{p}\text{-local and } \mathfrak{p}\text{-complete}\}.$$

Note that $\Lambda^{\mathfrak{p}}\,\mathcal{T} \subset \mathcal{T}_{\mathfrak{p}} \subset \mathcal{T}$ are colocalising subcategories. In fact, $\mathcal{T}_{\mathfrak{p}} = V^{\mathcal{Z}(\mathfrak{p})}\,\mathcal{T}$ and $\Lambda^{\mathfrak{p}}\,\mathcal{T} = \Lambda^{\mathcal{V}(\mathfrak{p})}\,\mathcal{T}_{\mathfrak{p}}$ (cf. [4, Corollaries 4.8, 4.9]). The composition

$$\Lambda^{\mathcal{V}(\mathfrak{p})}V^{\mathcal{Z}(\mathfrak{p})}\colon\ \mathcal{T} \longrightarrow \Lambda^{\mathfrak{p}}\,\mathcal{T}$$

is called the *local homology functor respect to* $\mathfrak{p}$ and is denoted by $\Lambda^{\mathfrak{p}}$. Then, the *BIK cosupport* of an object X is

$$\operatorname{cosupp}_{\mathcal{T}^{\circlearrowleft S}}(X) \coloneqq \left\{\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(S) \,\middle|\, \Lambda^{\mathfrak{p}}X \neq 0\right\}.$$

5.2. *Definition.* The tensor triangulated category $\mathcal{T}$ is said to be *stratified by S* (resp. *costratified by S*) if for each $\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(S)$, the localising subcategory $\Gamma_{\mathfrak{p}}\,\mathcal{T}$ (resp. the colocalising subcategory $\Lambda^{\mathfrak{p}}\,\mathcal{T}$) is either zero or minimal.

Zariski descent. The notions of BIK-(co)stratification are Zariski-local.

5.3. **Proposition.** *Let $\{\mathcal{T}_i^{\circlearrowleft S_i}\}_{i\in I}$ be a finite faimly of rigidly-compactly generated tensor triangulated categories $\mathcal{T}_i$ equipped with graded-commutative noetherian rings S_i acting on each of them respectively. Let $\mathcal{T} = \prod_{i\in I}\mathcal{T}_i$, $S = \prod_{i\in I} S_i$, and S acts on $\mathcal{T}$ in the evident way. By abuse of notation, we write π_i for the i-th projection of either categories or rings. Then, $\mathcal{T}$ is stratified by S if and only if each $\mathcal{T}_i$ is stratified by S_i. In particular, we have*

$$\operatorname{supp}_{\mathcal{T}^{\circlearrowleft S}} = \bigcup_{i\in I}\pi_i^*\operatorname{supp}_{\mathcal{T}_i^{\circlearrowleft S_i}}\pi_i \quad \textit{and} \quad \operatorname{cosupp}_{\mathcal{T}^{\circlearrowleft S}} = \bigcup_{i\in I}\pi_i^*\operatorname{cosupp}_{\mathcal{T}_i^{\circlearrowleft S_i}}\pi_i.$$

Proof. We first show the identities on supports and cosupports. Indeed, for any object X of each $\mathcal{T}_i$, by [4, Corollary 7.8], we have

$$\pi_i^*\operatorname{supp}_{\mathcal{T}_i^{\circlearrowleft S_i}}(X) \subset \operatorname{supp}_{\mathcal{T}^{\circlearrowleft S}}(X) \quad \text{and} \quad \operatorname{supp}_{\mathcal{T}^{\circlearrowleft S}}(X) \subset \pi_i^*\operatorname{supp}_{\mathcal{T}_i^{\circlearrowleft S_i}}(X).$$

Hence, $\operatorname{supp}_{\mathcal{T}^{\circlearrowleft S}}(X) = \pi_i^*\operatorname{supp}_{\mathcal{T}_i^{\circlearrowleft S_i}}(X)$. Since $\mathrm{id} = \sum_{i\in I}\pi_i$ on $\mathcal{T}$, The identity on supports then follows. The identity on cosupports is shown in a similar way.

From the identities on supports and cosupports, the only if part is evident. Indeed, we have $\Gamma_{\mathfrak{p}}\,\mathcal{T}_i = \pi_i\Gamma_{\mathfrak{p}}\,\mathcal{T}$ and $\Lambda^{\mathfrak{p}}\,\mathcal{T}_i = \Lambda^{\mathfrak{p}}\,\mathcal{T} \cap \mathcal{T}_i$.

As for the if part, first note that for any prime $\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}} S$, we have $\mathfrak{p} = \pi_i^*\pi_i\mathfrak{p}$ for some i. Hence,

$$\mathcal{V}(\mathfrak{p}) = \pi_i^*\,\mathcal{V}(\pi_i\mathfrak{p}) \quad \text{and} \quad \mathcal{Z}(\mathfrak{p}) = \pi_i^*\,\mathcal{Z}(\pi_i\mathfrak{p}) \cup \bigsqcup_{j\neq i}\operatorname{Spec}^{\mathrm{h}} R_j.$$

By [3, Proposition 6.1], this gives us

$$\begin{aligned}\Gamma_{\mathfrak{p}} &= \Gamma_{\mathcal{V}(\mathfrak{p})}L_{\mathcal{Z}(\mathfrak{p})} = \Gamma_{\pi_i^*\,\mathcal{V}(\pi_i\mathfrak{p})}L_{\pi_i^*\,\mathcal{Z}(\pi_i\mathfrak{p})}L_{\bigsqcup_{j\neq i}\operatorname{Spec}^{\mathrm{h}} R_j},\\ \Lambda^{\mathfrak{p}} &= V^{\mathcal{Z}(\mathfrak{p})}\Lambda^{\mathcal{V}(\mathfrak{p})} = V^{\pi_i^*\,\mathcal{Z}(\pi_i\mathfrak{p})}\Lambda^{\pi_i^*\,\mathcal{V}(\pi_i\mathfrak{p})}V^{\bigsqcup_{j\neq i}\operatorname{Spec}^{\mathrm{h}} R_j}.\end{aligned}$$

Note that, since $\operatorname{Spec}^{\mathrm{h}}(R) \setminus \bigsqcup_{j\neq i}\operatorname{Spec}^{\mathrm{h}} R_j = \operatorname{Spec}^{\mathrm{h}}(R_i)$, by [3, Theorem 5.6] and [4, Proposition 4.7], the functors $L_{\bigsqcup_{j\neq i}\operatorname{Spec}^{\mathrm{h}} R_j}$ and $V^{\bigsqcup_{j\neq i}\operatorname{Spec}^{\mathrm{h}} R_j}$ vanish on $\mathcal{T}_j$

whenever $j \neq i$ and acts as the identity on $\mathcal{T}_i$. Therefore, by [4, Theorem 7.7],

$$\Gamma_{\mathfrak{p}} = \Gamma_{\mathcal{V}(\pi_i\mathfrak{p})} L_{\mathcal{Z}(\pi_i\mathfrak{p})} \pi_i = \Gamma_{\pi_i\mathfrak{p}} \pi_i,$$
$$\Lambda^{\mathfrak{p}} = V^{\mathcal{Z}(\pi_i\mathfrak{p})} \Lambda^{\mathcal{V}(\pi_i\mathfrak{p})} \pi_i = \Lambda^{\pi_i\mathfrak{p}} \pi_i.$$

Now, suppose $\mathcal{S}$ is a proper localizing (resp. colocalizing) subcategory of $\Gamma_{\mathfrak{p}}\,\mathcal{T}$ (resp. $\Lambda^{\mathfrak{p}}\,\mathcal{T}$). Then, the above formulas show that $\mathcal{S} = \mathcal{S}_i$ for a proper localizing (resp. colocalizing) subcategory $\mathcal{S}_i$ of $\Gamma_{\pi_i\mathfrak{p}}\,\mathcal{T}_i$ (resp. $\Lambda^{\pi_i\mathfrak{p}}\,\mathcal{T}_i$) . Since $\mathcal{T}_i$ is stratified (resp. costratified) by S_i, we must have $\mathcal{S}_i = 0$. Consequently $\Gamma_{\mathfrak{p}}\,\mathcal{T}$ (resp. $\Lambda^{\mathfrak{p}}\,\mathcal{T}$) is either zero or minimal. Namely, $\mathcal{T}$ is stratified (resp. costratified) by S. $\square$

Balmer spectra. The Balmer spectrum [1] of a tensor triangulated category is a crucial notion in its geometric study.

5.4. *Definition.* For a tensor triangulated category $\mathcal{T}$, its *Balmer spectrum* $\operatorname{Spc}(\mathcal{T})$ is the set of prime tensor ideals of $\mathcal{T}$ with the topology whose closed sets are

$$\mathcal{Z}(\mathcal{S}) \coloneqq \{\mathcal{P} \in \operatorname{Spc}(\mathcal{T}) \mid \mathcal{S} \cap \mathcal{P} = \varnothing\},$$

where $\mathcal{S}$ is any set of objects.

For a rigidly-compactly generated tensor triangulated category $\mathcal{T}$ stratified by a raded-commutative noetherian ring S, restricting the support map to the tensor ideals in the subcategory $\mathcal{T}^c$ of compact objects yields a computation of the Balmer spectrum $\operatorname{Spc}(\mathcal{T}^c)$.

5.5. *Example.* As a consequence of [2, Theorem A], one gets a computation of the Balmer spectrum:

$$\operatorname{Spc}(\mathsf{rep}(G, R)) \cong \operatorname{Spec}^{\mathrm{h}} \mathsf{H}^*(G, R).$$

6. Conclusion

We are now ready to state the main result of this paper.

6.1. **Theorem.** *Let R be a commutative noetherian ring and $\mathcal{G}$ an essentially finite groupoid. The tensor triangulated category $\mathsf{Rep}(\mathcal{G}, R)$ is stratified and costratified by the action of $\mathsf{H}^*(\mathcal{G}, \underline{R})$, with (co)support equal to $\operatorname{Spec}^{\mathrm{h}} \mathsf{H}^*(\mathcal{G}, \underline{R})$. In particular one gets a computation of the Balmer spectrum of the derived category:*

$$\operatorname{Spc}(\mathsf{rep}(\mathcal{G}, R)) \cong \operatorname{Spec}^{\mathrm{h}} \mathsf{H}^*(\mathcal{G}, \underline{R}).$$

Proof. By Proposition 4.5, the rigidly-compactly generated tensor triangulated category $\mathcal{T} \coloneqq \mathsf{Rep}(\mathcal{G}, R)$ decomposes into $\mathcal{T}_{[x]} \coloneqq \mathsf{Rep}\big(\mathcal{G}_{[x]}, R\big)$. By Corollary 4.6, the groupoid cohomology ring $S \coloneqq \mathsf{H}^*(\mathcal{G}, \underline{R})$ decomposes into group cohomology rings $S_{[x]} \coloneqq \mathsf{H}^*\big(\mathcal{G}_{[x]}, R\big)$. Hence, we are in the situation of Proposition 5.3. By Example 5.1, each $\mathcal{T}_{[x]}$ is stratified and costratified by $S_{[x]}$. Therefore, $\mathcal{T}$ is stratified and costratified by S. The computation of the Balmer spectrum then follows. $\square$

References

[1] P. Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005. 11

[2] T. Barthel, D. Benson, S. B. Iyengar, H. Krause, and J. Pevtsova. Lattices over finite group schemes and stratification, 2023. 1, 2, 3, 6, 9, 11

[3] D. Benson, S. B. Iyengar, and H. Krause. Local cohomology and support for triangulated categories. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(4):575–621, 2008. 8, 9, 10

[4] D. J. Benson, S. B. Iyengar, and H. Krause. Colocalizing subcategories and cosupport. *J. Reine Angew. Math.*, 673:161–207, 2012. 8, 10, 11

[5] H. Krause. The stable derived category of a noetherian scheme. *Compos. Math.*, 141(5):1128–1162, 2005. 5

[6] H. Krause. *Homological theory of representations*, volume 195 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2022. 3, 4, 6

[7] A. Neeman. The homotopy category of flat modules, and Grothendieck duality. *Invent. Math.*, 174(2):255–308, 2008. 5