Note on

Rationality Theorems

From Yvette Amice's *Les nombres p-adiques*

Note: Xu Gao

Last update:June 25, 2018

Abstract

Let $f(X) = \sum_{n\geqslant 0} a_n X^n$ be a formal power series with coefficients in a number field K. The *Borel-Dwork Theorem* states a criterion for rationality of f in terms of its image in some valued-complete algebraic closed extension of K.

§ 1 Algebraic Criteria, Determinants

Let $a = (a_n)_{n\geqslant 0}$ be a sequence in K. The *Hankel determinant* of rank n and order k of the sequence a is the determinant

$$D_n^k(a) := \det \begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+k} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+k} & a_{n+k+1} & \cdots & a_{n+2k} \end{pmatrix}.$$

The determinant $D_0^n(a)$, also denoted $D^n(a)$, is called the n-th *Kronecker determinant* of a. Convention: $D_n^{-1}(a) = 1$.

1.1 Proposition *The formal series $f(X) = \sum_{n\geqslant 0} a_n X^n$ is rational if and only if there exists some k such that $D_n^k(a) = 0$ for sufficiently large n.*

Proof: Suppose that f is rational, saying $f = P/Q$ with P a polynomial and $Q(X) = q_k X^k + q_{k-1}X^{k-1} + \cdots + q_0$. Then, for $n > \deg P$ we have

$$\begin{aligned} q_k a_{n-k} + q_{k-1}a_{n-k+1} + \cdots + q_0 a_n &= 0, \\ q_k a_{n-k+1} + q_{k-1}a_{n-k+2} + \cdots + q_0 a_{n+1} &= 0, \\ \cdots \qquad\qquad & \cdots \\ q_k a_n + q_{k-1}a_{n+1} + \cdots + q_0 a_{n+k} &= 0. \end{aligned}$$

Thus, for $n > \deg P$, $D_{n-k}^k(a) = 0$.

Conversely, we need the following lemma:

1.1.1 Lemma (Sylvester's relation for determinants) *Let $A = (a_{ij})_{0\leqslant i,j\leqslant m}$. Let $A_{u,v}$ be the matrix obtained from A by delete the rows with index in u and columns with index in v. Let $D = \det A$, $d = \det A_{0m,0m}$, $M_{i,j} = \det A_{i,j}$ and $C_{i,j} = (-1)^{i+j} M_{i,j}$ then*

$$Dd = M_{0,0}M_{m,m} - M_{0,m}M_{m,0}.$$

PROOF: Let $B = (b_{ij})$ where $b_{i0} = C_{0,i}$, $b_{im} = C_{m,i}$, and $b_{ij} = \delta_{ij}$ when $j \neq 0, m$. Let $D' = \det B$. Then it is clear

$$D' = C_{0,0}C_{m,m} - C_{0,m}C_{m,0} = M_{0,0}M_{m,m} - M_{0,m}M_{m,0}.$$

Note that the 0-th and m-th columns of B are precisely those of the adjugate matrix of A and other columns of B are precisely those of I_{m+1}. Therefore the 0-th and m-th columns of AB are precisely those of DI_{m+1} and other columns of AB are precisely those of A. Thus

$$DD' = \det AB = D^2 d.$$

Since D is not a zero polynomial, by divide D on both sides, we get

$$Dd = D' = M_{0,0}M_{m,m} - M_{0,m}M_{m,0}. \qquad \square$$

Apply this lemma to Hankel determinants, we get

$$D_n^k D_{n+2}^{k-2} = D_{n+2}^{k-1} D_n^{k-1} - (D_{n+1}^{k-1})^2, \quad (k \geqslant 1).$$

If the sequence a is stationary, then f is clearly rational. so we shall assume a is not stationary.

1.1.2 Lemma *Suppose a is not stationary. If there exist k and n_0 such that $D_n^k(a) = 0$ when $n \geqslant n_0$, then there exist h and n_1 such that $D_n^h(a) = 0$ when $n \geqslant n_1$ and $D_n^{h-1}(a) \neq 0$ for $n \geqslant n_1 + 1$.*

PROOF: We shall say that k is *admissible* for a if $D_n^k(a) = 0$ for sufficiently large n. Note that since $D_n^0 = a_n$ and a is not stationary, 0 cannot be admissible for a.

By hypothesis, there exists admissible k. Let h be the least one of them and let n_1 be the least number such that $D_n^h = 0$ for $n \geqslant n_1$. For $n \geqslant n_1 + 1$, if $D_n^{h-1} = 0$ then the equality

$$D_n^h D_{n+2}^{h-2} = D_{n+2}^{h-1} D_n^{h-1} - (D_{n+1}^{h-1})^2$$

implies that $D_{n+1}^{h-1} = 0$. Inductively, $h-1$ is admissible, contradicting the minimality of h. $\square$

Supposing that a is not stationary and let h and n_1 be as defined in the lemma. Consider the following family of linear equations in the variables $Y_0, \cdots, Y_h$:

$$\begin{cases} Y_h a_n + Y_{h-1} a_{n+1} + \cdots + Y_0 a_{n+h} = 0, \\ Y_h a_{n+1} + Y_{h-1} a_{n+2} + \cdots + Y_0 a_{n+h+1} = 0, \\ \qquad \cdots \qquad\qquad \cdots \\ Y_h a_{n+h} + Y_{h-1} a_{n+h+1} + \cdots + Y_0 a_{n+2h} = 0. \end{cases}$$

Note that its determinant is D_n^h. Therefore, for $n \geqslant n_1$, it has nontrivial solutions. On the other hand, its $(h,0)$-minor D_{n+1}^{h-1} is nonzero. Hence there exists exactly one solution such that $Y_h = 1$. Moreover, $D_{n+1}^h = 0$ and $D_{n+2}^{h-1} \neq 0$ guarantee that if $Y_0, \cdots, Y_h$ is a solution for the system for n, then it is a solution for the systems for $n+1$. Therefore, letting

$$Q(X) = X^h + Y_{h-1} X^{h-1} + \cdots + Y_0,$$

we have fQ is a polynomial, and hence f is rational. $\square$

1.2 Corollary *The formal series $f(X) = \sum_{n \geqslant 0} a_n X^n$ is rational if and only if $D^n(a) = 0$ for sufficiently large n.*

PROOF: Let $A_n^k(a) = (a_n, a_{n+1}, \cdots, a_{n+k})$. Then

$$D^n(a) = \det(A_0^n, A_1^n, \cdots, A_n^n).$$

If f is rational, saying $f = P/Q$ with P a polynomial and $Q(X) = q_k X^k + q_{k-1} X^{k-1} + \cdots + q_0$. Then, for $n > \deg P$ we have

$$\begin{gathered} q_k a_{n-k} + q_{k-1} a_{n-k+1} + \cdots + q_0 a_n = 0, \\ q_k a_{n-k+1} + q_{k-1} a_{n-k+2} + \cdots + q_0 a_{n+1} = 0, \\ \cdots \qquad\qquad \cdots \\ q_k a_{2n-k} + q_{k-1} a_{2n-k+1} + \cdots + q_0 a_{2n} = 0. \end{gathered}$$

That is

$$q_k A_{n-k}^n + q_{k-1} A_{n-k+1}^n + \cdots + q_0 A_n^n = 0.$$

Thus $D^n(a) = 0$.

Conversely, suppose that $D^n(a) = 0$ for $n \geqslant n_0$. Then by Sylvester's relation for determinants, we have

$$D_0^{n+1} D_2^{n-1} = D_2^n D_0^n - (D_1^n)^2.$$

Since $D_0^{n+1} = D_0^n = 0$ for $n \geqslant n_0$, we have $D_1^n = 0$ for $n \geqslant n_0$. Suppose we have shown that $D_h^n = 0$ for $n \geqslant n_0$, then

$$D_h^{n+1} D_2^{n-1} = D_{h+2}^n D_h^n - (D_{h+1}^n)^2.$$

implies that $D_{h+1}^n = 0$ for $n \geqslant n_0$. Therefore, for all m, $D_m^n = 0$ for $n \geqslant n_0$. By Proposition 1.1, f is rational. $\square$

§ 2 The Borel-Dwork Theorem

Let C be a valued-complete algebraically closed extension of K.

Definition Let $f(X) = \sum_{n \geqslant 0} a_n X^n$ be a formal series with coefficients in C and $R > 0$. We say that f defines a *meromorphic* function at 0 of radius R if, whenever $r < R$, there exists a polynomial Q_r such that the series $Q_r f$ converges for $|X| \leqslant r$.

2.1 Lemma *Let $f(X) = \sum_{n \geqslant 0} a_n X^n$ be a series with coefficients in C defining a meromorphic function at 0 of radius R. Let $r < R$ and Q be a polynomial of degree s such that Qf converges for $|X| \leqslant r$. Let $k \geqslant s$. Then there exists M such that for $n \geqslant 0$,*

$$|D_n^k| \leqslant \frac{M}{b^{ns} r^{n(k-s+1)}},$$

where b is less than all absolute values of zeros of Q.

Proof: Let $a_n(f)$ denote the coefficients of f and $g = Qf$, where $Q(X) = q_s X^s + \cdots + q_0$. Since f (resp. g) defines a holomorphic function in the disc $|X| \leqslant b$ (resp. $|X| \leqslant r$), by *Cauchy's inequality*, there exist constants L, L' such that

$$|a_n(f)| \leqslant \frac{L}{b^n}, \qquad |a_n(g)| \leqslant \frac{L'}{r^n}.$$

From now on, equip C^{k+1} with the norm $\|X\|$, which is defined for $X = (X_0, \cdots, X_k)$ by

$$\|X\| = (|X_0|^2 + \cdots + |X_k|^2)^{1/2}$$

if C is Archimedean and

$$\|X\| = \max_i |X|$$

if C is non-Archimedean.

Denote $A_n^k(f) = (a_n(f), \cdots, a_{n+k}(f)) \in C^{k+1}$. Then for any n, k,

$$\|A_n^k(f)\| \leqslant \begin{cases} Lb^{-n} \max\{1, b^{-k}\} & \text{if } C \text{ is non-Archimedean,} \\ Lb^{-n}(1 + b^{-2} + \cdots + b^{-2k})^{1/2} & \text{if } C \text{ is Archimedean,} \end{cases}$$

and

$$\|A_n^k(g)\| \leqslant \begin{cases} L'r^{-n} \max\{1, r^{-k}\} & \text{if } C \text{ is non-Archimedean,} \\ L'r^{-n}(1 + r^{-2} + \cdots + r^{-2k})^{1/2} & \text{if } C \text{ is Archimedean.} \end{cases}$$

Therefore, for given k, there exist L_k, L'_k such that

$$\|A_n^k(f)\| \leqslant \frac{L_k}{b^n}, \qquad \|A_n^k(g)\| \leqslant \frac{L'_k}{r^n}.$$

Note that

$$D_n^k = D_n^k(a(f)) = \det(A_n^k(f), A_{n+1}^k(f), \cdots, A_{n+k}^k(f)).$$

For $k, n \geqslant s$, we have

$$\begin{aligned}
q_s a_{n-s}(f) + q_{s-1} a_{n-s+1}(f) + \cdots + q_0 a_n(f) &= a_n(g), \\
q_s a_{n-s+1}(f) + q_{s-1} a_{n-s+2}(f) + \cdots + q_0 a_{n+1}(f) &= a_{n+1}(g), \\
\cdots \qquad\qquad & \cdots \\
q_s a_{n+k-s}(f) + q_{s-1} a_{n+k-s+1}(f) + \cdots + q_0 a_{n+k}(f) &= a_{n+k}(g).
\end{aligned}$$

That is

$$q_s A_{n-s}^k(f) + q_{s-1} A_{n-s+1}^k(f) + \cdots + q_0 A_n^k(f) = A_n^k(g).$$

Therefore, for $k \geqslant s$,

$$D_n^k = \det(A_n^k(f), \cdots, A_{n+s-1}^k(f), A_{n+s}^k(g), \cdots, A_{n+k}^k(g)).$$

We need the following.

2.1.1 Lemma (Hadamard's inequality) *Let $A = (a_{ij})_{0 \leqslant i,j \leqslant m}$, $D = \det A$ and $A_j = (a_{0j}, \cdots, a_{mj}) \in C^{m+1}$. Then we have*

$$|D| \leqslant \|A_0\| \|A_1\| \cdots \|A_m\|.$$

Proof: If one of A_j degenerates, the inequality is trivial. If we replace some A_j by sA_j, then both $|D|$ and $\|A_0\|\|A_1\| \cdots \|A_m\|$ are scaled by s. Therefore, we may assume $\|A_j\| = 1$ for all j.

In the case C is non-Archimedean, the condition means $|a_{ij}| \leqslant 1$ for all i, j. Hence A is a matrix on $\mathcal{O}_C$. Thus $D \in \mathcal{O}_C$ and which means $|D| \leqslant 1$.

In the case C is Archimedean, we can use Gram-Schmidt process to get an orthonormal sequence $U_0, \cdots, U_m$ from $A_0, \cdots, A_m$. Then

$$D = \det(A_0, \cdots, A_m) \leqslant \det(U_0, \cdots, U_m) = 1. \qquad \square$$

Apply above lemma, we get

$$\begin{aligned}
|D_n^k| &\leqslant \|A_n^k(f)\| \cdots \|A_{n+s-1}^k(f)\| \|A_{n+s}^k(g)\| \cdots \|A_{n+k}^k(g)\| \\
&\leqslant \frac{L_k}{b^n} \cdots \frac{L_k}{b^{n+s-1}} \cdot \frac{L'_k}{r^{n+s}} \cdots \frac{L'_k}{r^{n+k}} \\
&= \frac{L_k^s L_k'^{k-s+1}}{b^{(2n+s-1)s/2} r^{(2n+k+s)(k-s+1)/2}}.
\end{aligned}$$

Therefore

$$|D_n^k| \leqslant \frac{M}{b^{ns} r^{n(k-s+1)}}$$

where

$$M = L_k^s L_k'^{k-s+1} b^{-(s-1)s/2} r^{-(k+s)(k-s+1)/2}. \qquad \square$$

2.2 Theorem (Borel-Dwork) *Let K be a number field and $f(X) = \sum_{n \geqslant 0} a_n X^n$ a formal series with coefficients in K. Let Σ the set of places of K and Σ_{f} the set of finite places. The absolute values $|c|_v$ of places v is normalized by*

$$\prod_{v \in \Sigma} |c|_v = 1, \qquad (\forall c \in K^{\times}).$$

If there exists a cofinite subset Σ_0 of Σ_{f} such that

(i) for $v \in \Sigma_0$ and each $n \geqslant 0$, $\log^+ |a_n|_v = 0$;

(ii) for each $v \notin \Sigma_0$, $i_v(f)$ defines a meromorphic function at 0 of radius R_v;

(iii) we have $R = \prod_{v \notin \Sigma_0} R_v > 1$.

Then f is rational.

Proof: The main ideal is to find certain upper bound of $D_n^k(a)$ under all valuations. For $v \in \Sigma_0$, we have

2.2.1 Lemma *If for each $n \geqslant 0$, $\log^+ |a_n|_v = 0$, then $|D_n^k|_v \leqslant 1$.*

Proof: Since $|x|_v \leqslant 1 \iff x \in \mathcal{O}_v$ and D_n^k is a integral polynomial of $a_n, a_{n+1}, \cdots, a_{2n}$. □

For $v \notin \Sigma_0$, we apply Lemma 2.1. Choose r_v for $v \notin \Sigma_0$ such that

$$r_v < R_v \qquad \text{and} \qquad r = \prod_{v \notin \Sigma_0} r_v > 1.$$

Let Q_v be a polynomial of degree s_v such that $Q_v(0) \neq 0$ and $Q_v i_v(f)$ converges for $|X|_v \leqslant r_v$, and let b_v be the less than all absolute values of zeros of Q_v. Let s be the maximum of s_v. Then for $k \geqslant s$, we have constant $M_{v,k}$ such that

$$|D_n^k|_v \leqslant \frac{M_{v,k}}{b_v^{n s_v} r_v^{n(k - s_v + 1)}}.$$

Let $\Delta_v(k) = (b_v^{s_v} r_v^{(k - s_v + 1)})^{-1}$ and $\Delta(k) = \prod_{v \notin \Sigma_0} \Delta_v(k)$. Then

$$\lim_{k \to \infty} (\Delta(k))^{1/k} = r^{-1} < 1.$$

Hence one may find k such that $\Delta(k) < 1$. Fixing such k and taking $\Delta = \Delta(k)$, define $M = \prod_{v \notin \Sigma_0} M_{v,k}$. Then

$$\prod_{v \notin \Sigma_0} |D_n^k|_v \leqslant \prod_{v \notin \Sigma_0} \frac{M_{v,k}}{b_v^{n s_v} r_v^{n(k - s_v + 1)}} = M \Delta^n.$$

If $D_n^k \neq 0$, then we have

$$1 = \prod_{v \in \Sigma} |D_n^k|_v = \prod_{v \in \Sigma_0} |D_n^k|_v \times \prod_{v \notin \Sigma_0} |D_n^k|_v \leqslant M \Delta^n$$

which cannot be true for large n since $\Delta < 1$. □

§ 3 Rationality of zeta functions