Tate Cohomology

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§ I The homotopical story

I.1 Definition. Let G be a finite discrete group. Denote by $\mathbf{B}G$ its *delooping*. A **G-module** in a (stable ∞-)category $\mathcal{C}$ is a (∞-)functor from $\mathbf{B}G$ to $\mathcal{C}$.

I.2 Definition. Let X be a G-module. Its **homotopy invariant** X^{hG} is the homotopy limit of the functor X. Its **homotopy coinvariant** X_{hG} is the homotopy colimit of the functor X.

I.3 Definition. There is a canonical morphism

$$N_G\colon X_{hG} \longrightarrow X^{hG}.$$

Its homotopy cofiber is called the **Tate construction** and denoted by X^{tG}.

I.4 Definition. The **group cohomology** $H^\bullet(G,X)$ is the cohomology of X^{hG}. The **group homology** $H_\bullet(G,X)$ is the cohomology of X_{hG} (after sign swaping of degrees). The **Tate cohomology** $\hat{H}^\bullet(G,X)$ is the cohomology of X^{hG}.

From the above defintions, one immediately get

1. $\hat{H}^n(G,X) = H^n(G,X)$ if $n > 0$;
2. $\hat{H}^0(G,X) = \operatorname{Coker}\left(H_0(G,X) \xrightarrow{N_G} H^0(G,X)\right)$;
3. $\hat{H}^{-1}(G,X) = \operatorname{Ker}\left(H_0(G,X) \xrightarrow{N_G} H^0(G,X)\right)$;
4. $\hat{H}^n(G,X) = H_{-(n+1)}(G,X)$ if $n < -1$;

The norm map

This subsection explain how the norm map is determined universally. Who not care can skip this subsection.

I.5 Definition. Let $f\colon S \to T$ be a map between Kan complexes. Then we have the *pullback functor*

$$f^*\colon \operatorname{Fun}(T,\mathcal{C}) \longrightarrow \operatorname{Fun}(S,\mathcal{C})$$

and its left and right adjoints

$$f_!, f_*\colon \operatorname{Fun}(S,\mathcal{C}) \longrightarrow \operatorname{Fun}(T,\mathcal{C}).$$

Then the **norm map**

$$N_f\colon f_! \longrightarrow f_*$$

should be obtained by adjunction from a canonical map

$$\mathrm{id} \longrightarrow f^*f_*.$$

This map should come from the equivalence

$$f^*f_* \xrightarrow{\sim} p_*q^*$$

from the commutative diagram

$$\begin{tikzcd} S \times_T S \arrow[r, "q"] \arrow[d, "p"] & S \arrow[d, "f"] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

and the canonical map

$$\mathrm{id} \longrightarrow p_*q^*$$

from the composition

$$p^* \longrightarrow \delta_*\delta^*p^* \simeq \delta_* \xrightarrow{N_\delta^{-1}} \delta_! \simeq \delta_!\delta^*q^* \longrightarrow q^*,$$

where $\delta\colon S \to S \times_T S$ is the diagonal map.

Of course this requires the existence of $N_\delta\colon \delta_! \to \delta_*$ and furthermore it is an equivalence.

1. If $f\colon S \to T$ is (-1)-*truncated*, i.e. all its fibers are either empty or conctractible. Then δ is an equivalence and hence N_δ exists tautologically. Since $\mathcal{C}$ is stable, the resulting norm map N_f is an equivalence.

2. If $f\colon S \to T$ is 0-*truncated*, then δ is (-1)-truncated and we thus get the norm map N_f. If furthermore f has *finite fibers*, then N_f is an equivalence.

3. Now, let $f\colon S \to T$ be 0-*truncated*. It is a *relative finite groupoid* if its fiber have finitely many connected components and each component is a delooping of a finite group. In this case, δ is 0-truncated and has finite fibers, hence we can construct N_f.

§ II The homological story

In this section, one can take $\mathcal{A}$ to be the category **Ab** *for simplicity.*

II.1 Definition. Let G be a finite discrete group. Denote by $\mathcal{G}$ the groupoid with only one object and its automorphism group is G. A **G-module** in an abelian category $\mathcal{A}$ is a functor from $\mathcal{G}$ to $\mathcal{A}$.

Exercise 1. Show that $\mathcal{G}$ is the *delooping* of G in the sense that G (viewed as a finite discrete category) is equivalent to the comma category of the functors

$$\mathbf{1} \longrightarrow \mathcal{G} \longleftarrow \mathbf{1}$$

where $\mathbf{1}$ denote the category with only one object and one morphism and the functor send this morphism to the identity of G.

Exercise 2. Verify that this definition covers the usual definition of G-modules or representations of G.

II.2 Definition. Let $\mathcal{I}$ be a finite category. Then $\mathcal{I}$-indexed diagrams, i.e. functors from $\mathcal{I}$ to $\mathcal{A}$ form an abelian category $\mathcal{A}^{\mathcal{I}}$. Then take limits form a left exact functor $\varprojlim\colon \mathcal{A}^{\mathcal{I}} \to \mathcal{A}$ since it is right adjoint to the *pullback functor* $\mathcal{A} \to \mathcal{A}^{\mathcal{I}}$. Then the **homotopy limit functor** $\mathbf{R}\varprojlim$ is the right derived functor of $\varprojlim$. Similarly, take colimits form a right exact functor $\varinjlim\colon \mathcal{A}^{\mathcal{I}} \to \mathcal{A}$. Then the **homotopy colimit functor** $\mathbf{L}\varinjlim$ is the left derived functor of $\varinjlim$.

Let X be a G-module. Its **homotopy invariant** X^{hG} is the homotopy limit of the functor X, i.e. $\mathbf{R}\varprojlim X$. Its **homotopy coinvariant** X_{hG} is the homotopy colimit of the functor X, i.e. $\mathbf{L}\varinjlim X$.

Exercise 3. The limit $\varprojlim X$ is called he **invariant** X^G and the colimit $\varinjlim X$ is called the **coinvariant** X_G. Describe them in concrete.

II.3 Definition. In a triangular category, any morphism $f\colon X \to Y$ can be completed into a triangle $X \to Y \to C(f) \to X[1]$. The object $C(f)$ is called the **mapping cone** of f.

Let X be a G-module in $\mathcal{A}$. Then there is a canonical morphism in $\mathcal{D}(\mathcal{A})$

$$N_G\colon X_{hG} \longrightarrow X^{hG}.$$

Its mapping cone is called the **Tate construction** and denoted by X^{tG}.

Exercise 4. The **norm map** $N_G\colon X_G \to X^G$ can be defined by

$$\bar{x} \longmapsto \sum_{\sigma\in G} \sigma x.$$

Its cokernel is called the **norm residue group**.

Using this norm map to construct the norm map $N_G\colon X_{hG} \to X^{hG}$.

II.4 Definition. Let X be a G-module. The **group cohomology** $H^\bullet(G,X)$ is the cohomology of X^{hG} (viewed as a cochain complex). The **group homology** $H_\bullet(G,X)$ is the homology of X_{hG} (viewed as a chain complex). The **Tate cohomology** $\hat{H}^\bullet(G,X)$ is the cohomology of X^{hG} (viewed as a cochain complex).

Exercise 5. Using the triangle $X_{hG} \to X^{hG} \to X^{tG} \to X_{tG}[1]$ and the construction of norm map, show that

1. $\hat{H}^n(G,X) = H^n(G,X)$ if $n > 0$;
2. $\hat{H}^0(G,X) = \operatorname{Coker}\left(H_0(G,X) \xrightarrow{N_G} H^0(G,X)\right)$;
3. $\hat{H}^{-1}(G,X) = \operatorname{Ker}\left(H_0(G,X) \xrightarrow{N_G} H^0(G,X)\right)$;
4. $\hat{H}^n(G,X) = H_{-(n+1)}(G,X)$ if $n < -1$;

Resolutions

This subsection deals with computation using resolutions.

II.5 Definition. Let S be a simplical object in $\mathcal{A}$. Then its **Moore complex** $C(S)_\bullet$ is defined by

$$C(S)_n = S_n, \qquad \partial_n = \sum_{i=0}^{n} (-1)^i d_i.$$

Besides, its **normalized Moore complex** $N(S)_\bullet$ is defined by

$$N(S)_n = S_n^{\mathrm{nd}}, \qquad \partial_n = \sum_{i=0}^{n} (-1)^i \delta_{\mathrm{nd}} d_i,$$

where S_n^{nd} denotes the nondegenerate part of S_n and δ_{nd} kills degenerate simplexes and leaves the nondegenerate ones.

Exercise 6. Show that $N(S)_\bullet$ is a quotient of $C(S)_\bullet$ and the two complexes are homotopy equivalent.

Remark. The functor $N(\,\cdot\,)_\bullet$ induces a Quillen equivalence between the model category of simplicial objects in $\mathcal{A}$ (endowed with Quillen model structure) and the model category of non-negative chain complexes in $\mathcal{A}$ (endowed with projective model structure). This is called the **Dold-Kan correspondence**.

II.6 Definition. Let S be a simplicial set and R a ring. Then there is a standard way to get a simplicial object $R[S]$ in the category of R-modules:

$$R[S]_\bullet := S_\bullet \pitchfork R.$$

Here $S_\bullet \pitchfork R$ is the *power* operation: for I a set and M an R-module, it represents the functor $\operatorname{Map}(I, \operatorname{Hom}_R(-,M))$. This simplical object is called the **simplicial free R-module generated by S**.

Exercise 7. Using Dold-Kan correspondence or directly show that if the simplicial set S is 0-*truncated*, i.e. all its homotopy groups vanish except π_0, then the complex $N(R[S])_\bullet$ is a 0-*complex*, i.e. its homology groups vanish except at degree 0. In this way, one obtains a free resolution of $R[S_0]$.

Exercise 8. Describe the simplicial set $\mathbf{B}G$ by computing the nerve of the groupoid $\mathcal{G}$ and then describe the complex $N(\mathbb{Z}[\mathbf{B}G])_\bullet$.

Remark. Unfortunately, the simplicial set $\mathbf{B}G$ is NOT 0-truncated: its π_1 is G.

II.7 Definition. The **ordinal sum** of two linearly ordered sets I and J is the disjoint union of them with the extra order that for any $i \in I$ and $j \in J$, $i < j$. We denote it by $I \boxplus J$. For instance,

$$\{0 < 1\} \boxplus \{a < b\} = \{0 < 1 < a < b\}.$$

The ordinal sum induces a tensor product on the category Δ_a of finite ordinals, also denoted by $\boxplus$. In particular, we have $[n] \boxplus [m] = [n+m+1]$. Let ι_0 and ι_1 denote the natural inclusions $[n] \hookrightarrow [n] \boxplus [m]$ and $[m] \hookrightarrow [n] \boxplus [m]$.

Using *Day convolution*, this monoidal structure on Δ_a induces a monoidal structure $\star$ on $\mathbf{PSh}(\Delta_a)$, i.e. the category of *augmented simplicial sets*. The category of simplicial sets can be embedded into it by put $S_{-1} = \mathrm{pt}$ (the *trivial augmentation*) for all simplicial set S. Then, the monoidal strucutre induces an operation of two simplicial sets, called the **join** of them.

Exercise 9. The *Day convolution* in the above case can be written as

$$S \star T \colon [n] \longmapsto \int^{[i],[j] \in \Delta_a} \mathrm{Map}\left([n], [i] \boxplus [j]\right) \times S_i \times S_j.$$

Use this formula, get a concrete description of $\Delta^0 \star S$.

II.8 Definition. Let S be a simplicial set. The **cone above S** is the simplicial set $\Delta^0 \star S$. The functor $S \mapsto \Delta^0 \star S$ has a right adjoint Dec^0. The simplicial set $\mathrm{Dec}^0(S)$ is called the **décalage** of S.

From the natural inclusion $[0] \hookrightarrow [0] \boxplus [n]$, we get an augmentation map $\epsilon \colon \mathrm{Dec}^0(S) \to S_0$. Conversely, from the unique map $[0] \boxplus [n] \to [0]$, we get a section $r \colon S_0 \to \mathrm{Dec}^0(S)$.

Note that, the maps $[0] \hookrightarrow [0] \boxplus [n]$ and $[0] \boxplus [n] \to [0]$ induce simplicial maps $\Delta^0 \hookrightarrow \Delta^{n+1}$ and $\Delta^{n+1} \to \Delta^0$, and the later realizes the former as a deformation retract. Using this observation, it follows that r realizes ϵ as a deformation retract.

Exercise 10. Give a concrete description of $\mathrm{Dec}^0(S)$ and the maps ϵ and r, in terms of the combinatorial data of S. Show that r realizes ϵ as a deformation retract.

Exercise 11. Compute $\mathrm{Dec}^0(\mathbf{B}G)$.

Remark. Now, we have a free resolution

$$N(\mathbb{Z}[\mathrm{Dec}^0\,(\mathbf{B}G)]) \longrightarrow \mathbb{Z}.$$

Furthermore, this is a projective resolution of G-modules. Indeed, the canonical map $\mathrm{Dec}^0\,(\mathbf{B}G) \to \mathbf{B}G$ is a Kan fibration, hence $N(\mathbb{Z}[\mathrm{Dec}^0\,(\mathbf{B}G)])$ is projective in the category $\mathbf{Ab}^G$.

Exercise 12. Using above resolution, give an explicit formula for Tate cohomology groups $\hat{H}^n(G, A)$ where A is a G-module in $\mathbf{Ab}$.

Exercise 13. Let $\mathbf{S}G$ denote the simplicial set

$$[n] \longmapsto [n] \pitchfork G.$$

1. Show that it is a décalage of another simplicial set and then is contractible.
2. Construct a Kan fibration $\mathbf{S}G \to \mathbf{B}G$. Then we get a projective resolution of G-modules
$$N(\mathbb{Z}[\mathbf{S}G]) \longrightarrow \mathbb{Z}.$$
3. Using above resolution, give another explicit formula for Tate cohomology groups $\hat{H}^n(G, A)$ where A is a G-module in $\mathbf{Ab}$.

The norm map

This subsection explain how the norm map comes from nothing.

Exercise 14. Let $f\colon \mathbf{B}G \to [0]$ be the unique map. It can be viewed as the nerv of the unique functor $F\colon \mathcal{G} \to \mathbf{1}$. Compute the diagonal of f by computing the comma construction of the functors F and F.

Exercise 15. Let $d\colon \mathbf{B}G \to \mathbf{B}G \times_{[0]} \mathbf{B}G$ be the diagonal in previous exercise. Let $D\colon \mathcal{G} \to (F \downarrow F)$ be the corresponding functor. Show that d is 0-truncated and has finite fiber by showing that the comma category of the functors D and $\mathbf{1} \to (F \downarrow F)$ is equivalent to G viewed as a finite discrete category.

Exercise 16. Let $f\colon I \to [0]$ be the unique map from a finite discrete set to $[0]$. One can also view it as a functor between discrete categories. Compute the norm map N_f.

Exercise 17. Let $f\colon X \to Y$ be a 0-truncated map with finite fibers, one can consider a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that the comma category of the functors F and $\mathbf{1} \to \mathcal{D}$ is equivalent to a finite discrete category I. Using the result from previous exercise, compute the norm map N_f.

Exercise 18. Let $f\colon \mathbf{B}G \to [0]$ be the unique map. Using results from previous exercises, compute the norm map N_f.