Review on Bruhat-Tits buildings

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Abstract

The purpose of this note is to explain the theory of Bruhat-Tits buildings (resp. Tits buildings) for split reductive groups over local fields (resp. finite field). It is intended to be part of my thesis.

In 1950s-1960s, Jacques Tits [Tit74, Bourbaki] introduced the notion of *buildings* as a uniform geometric framework to study semisimple Lie groups and later semisimple algebraic groups (and more generally, reductive groups) over arbitrary fields. Tits' buildings are polysimplicial complexes with nice symmetries so that reductive groups can act nicely on them. Later, François Bruhat and Jacques Tits [BT-I, BT-II, BT84, BT87] develop the theory for reductive groups over non-Archimedean fields by giving refined structures on the buildings respecting the valuation. During the same period, they shift the view of a building from merely a polysimplicial complex to a geometric object, namely a complete metric space with non-positive curvature and is equipped with a polysimplicial complex of subsets in it. The fruitful geometric/combinatorial nature of Bruhat-Tits buildings suggests them as non-Archimedean analogues of Riemannian symmetric spaces for real Lie groups.

We refer to [RTW15, §3] for a short review on Bruhat-Tits theory, [Tit79, Yu09] for more systematic surveys, [Rou09] for a survey of general theory of Euclidean buildings, and [Mil17, SGA3] for reductive groups.

Contents

§ 1 General theory of Buildings

1.1 Projective geometry over $\mathbb{F}_q$ and $\mathbb{F}_1$

Back in 1950s, Jacques Tits noticed the following interesting phenomenons [Tit57].

1.1.1. Let $\mathbb{PF}_q^n$ be the projective space associated to the vector space $\mathbb{F}_q^n$. Then its cardinality (or equivalently, the number of one-dimensional subspaces of $\mathbb{F}_q^n$) can be presented by the *quantum number* $[n]_q := \sum_{i=0}^{n-1} q^i$. If we pass to the limit $q \longrightarrow 1$, then we get n, the number of coordinate labels $\{1, 2, \cdots, n\}$. Recalling how we count the cardinality of $\mathbb{PF}_q^n$ using the coordinates, we can view the set $P_n = \{1, 2, \cdots, n\}$ as the analogue of $\mathbb{PF}_q^n$ over the imaginary "prime field of characteristic one[1]" $\mathbb{F}_1$.

More generally, we can count points, lines, planes, ... in $\mathbb{PF}_q^n$. They correspond to points of the Grassmannians $\mathrm{Gr}\left(1, \mathbb{F}_q^n\right), \mathrm{Gr}\left(2, \mathbb{F}_q^n\right), \mathrm{Gr}\left(3, \mathbb{F}_q^n\right), \ldots$. In general, the *Grassmannian* $\mathrm{Gr}\left(k, \mathbb{F}_q^n\right)$ consists of subspaces of $\mathbb{F}_q^n$ having dimension k and its cardinality can be presented by the *quantum binomial* $\begin{bmatrix} n \\ k \end{bmatrix}_q$ (see 1.1.4). If we pass to the limit $q \longrightarrow 1$, then we get $\binom{n}{k}$, which is the number of k-subsets of P_n.

1.1.2. The above can be organized into incidence geometry: namely the combinatorial gadget describing which proper subspace belongs to which. On the $\mathbb{F}_q$-side, a nontrivial proper subspace of $\mathbb{F}_q^n$ is *of color k* if it is k-dimensional and two such subspaces are said to be *incident* if one of them belongs to another properly. In this way, we organize nontrivial proper subspaces of $\mathbb{F}_q^n$ into a colored simplicial complex $\mathscr{B}(n, q)$, in which a k-simplex is a *flag*

$$\mathbb{F}_q^n = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_{k+1} \supsetneq 0$$

of subspaces of $\mathbb{F}_q^n$. On the $\mathbb{F}_1$-side, a nonempty proper subset of P_n is *of color k* if it has cardinality k and two such subsets are said to be *incident* if one of them belongs to

[1]Namely the addition collapses. For an introduction, see [Lor18] especially §1.1.

another properly. In this way, we organize nonempty proper subsets of P_n into a colored simplicial complex $\mathscr{B}(n,1)$, in which a k-simplex is a *flag*

$$P_n = I_0 \supsetneq I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_{k+1} \supsetneq \emptyset$$

of subsets of P_n.

The two sides are related as follows. Fix a basis $\mathcal{B}$ of $\mathbb{F}_q^n$ (for example, the standard basis). Then to take a nontrivial proper subspace V of $\mathbb{F}_q^n$ having a basis which is part of $\mathcal{B}$ amounts to taking a nonempty proper subset I of $\mathcal{B}$ (which is in bijection to P_n) and V is k-dimensional if and only if I has cardinality k. Moreover, to take a flag respecting the basis $\mathcal{B}$ in the sense that each V_i has a basis being part of $\mathcal{B}$ amounts to taking a flag of nonempty proper subsets of $\mathcal{B}$.

However, different choices of bases may give the same subcomplex: for instance, when the two bases are different by a diagonal matrix. To avoid this, it is better to keep in the region of projective geometry. So instead of fixing a basis, we fix a *frame* $\mathbf{\Lambda}$, that is an n-set of points $\{x_1, x_2, \cdots, x_n\}$ in $\mathbb{PF}_q^n$ in general position (namely, they do not belong to a common hyperplane), or equivalently, an n-set of one-dimensional subspaces $\{\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \cdots, \boldsymbol{\lambda}_n\}$ of $\mathbb{F}_q^n$ spanning $\mathbb{F}_q^n$. Then different choices of frames do give different subcomplexes of $\mathscr{B}(n,q)$.

In this way, we associate to each frame $\mathbf{\Lambda}$ a subcomplex $\mathscr{A}(\mathbf{\Lambda})$ of $\mathscr{B}(n,q)$ isomorphic to $\mathscr{B}(n,1)$ and the complex $\mathscr{B}(n,q)$ is the union of them. They are the prototypes of *buildings* and *apartments*.

1.1.3. There is a natural action of $G = \mathrm{GL}\left(\mathbb{F}_q^n\right)$, the *general linear group* (but essentially, it is the action of $\mathrm{PGL}\left(\mathbb{F}_q^n\right)$, the *projective linear group*) on $\mathscr{B}(n,q)$. This action comes from the action of $\mathrm{PGL}\left(\mathbb{F}_q^n\right)$ on $\mathbb{PF}_q^n$ and hence on each Grassmannian $\mathrm{Gr}\left(k, \mathbb{F}_q^n\right)$.

Fix a frame $\mathbf{\Lambda}$ (for example, the one given by the standard basis), then the stabilizer of the subcomplex $\mathscr{A}(\mathbf{\Lambda})$ is precisely the stabilizer of the frame itself. Let's denote it by $N(\mathbf{\Lambda})$ (in our example of standard basis, it is the group of *monomial matrices*, i.e. matrices that have precisely one nonzero entry in each row and each column). The fixator of $\mathbf{\Lambda}$ acts trivially on $\mathscr{A}(\mathbf{\Lambda})$. Let's denote it by $Z(\mathbf{\Lambda})$ (in our example of standard basis, it is the group of diagonal matrices). The quotient group $W(\mathbf{\Lambda}) := N(\mathbf{\Lambda})/Z(\mathbf{\Lambda})$ is called the *Weyl group* associated to $\mathbf{\Lambda}$. Then one finds that $W(\mathbf{\Lambda}) \cong \mathfrak{S}_n$, the *symmetric group*, which acts naturally on P_n and hence on $\mathscr{B}(n, 1)$ exactly as $W(\mathbf{\Lambda})$ acts on $\mathscr{A}(\mathbf{\Lambda})$.

1.1.4. Let's consider the maximal simplices in $\mathscr{B}(n, q)$. From the description in 1.1.2, we see that a maximal simplex is nothing but a *complete flag*

$$\mathbb{F}_q^n = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_{n-1} \supsetneq 0$$

of subspaces of $\mathbb{F}_q^n$. Using an induction argument, it is not difficult to see that the number of complete flags is presented by the *quantum factorial* $[n]_q! := \prod_{i=1}^n [i]_q$. The quantum factorials are related to quantum binomials by the formula

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

This can be seen by picking the k-dimensional subspace V_{n-k} from a complete flag, breaking it into a complete flag of V_{n-k} and a complete flag of $\mathbb{F}_q^n/V_{n-k}$.

The maximal simplices in $\mathscr{B}(n, 1)$ are *complete flags*

$$P_n = I_0 \supsetneq I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_{n-1} \supsetneq \emptyset$$

of subsets of P_n. There are $n!$ such complete flags. The number $n!$ is precisely the $q \to 1$ limit of $[n]_q!$.

The stabilizer of a complete flag is called a *Borel subgroup* of G. Note that the action of G on complete flags is transitive. Hence the number of complete flags is the index of a Borel subgroup in G.

Let's take the standard basis $\mathcal{B} = \{e_1, \cdots, e_n\}$ of $\mathbb{F}_q^n$ and let $V_k = \bigoplus_{i=1}^{n-k} \mathbb{F}_q e_i$. Then we get a complete flag whose stabilizer B is precisely the group of invertible upper triangular matrices. Straightforward computation shows that B has order $q^{\binom{n}{2}}(q-1)^n$ and thus G has order $q^{\binom{n}{2}}(q-1)^n [n]_q!$.

1.1.5. We summarize above as follows.

(i) On the $\mathbb{F}_q$-side, we have the "building" $\mathscr{B}(n, q)$, which is the union of "apartments" $\mathscr{A}(\mathbf{\Lambda})$, one for each frame $\mathbf{\Lambda}$, and the number of them is

$$\frac{\#G}{\#N(\mathbf{\Lambda})} = \frac{\#B \cdot \#\{\text{complete flags}\}}{\#Z(\mathbf{\Lambda}) \cdot \# \mathfrak{S}_n} = \frac{q^{\binom{n}{2}}(q-1)^n [n]_q!}{(q-1)^n n!} = \frac{q^{\binom{n}{2}}[n]_q!}{n!}.$$

Each "apartment" $\mathscr{A}(\mathbf{\Lambda})$ is isomorphic to $\mathscr{B}(n, 1)$, the one on the $\mathbb{F}_1$-side. Hence the "building" $\mathscr{B}(n, q)$ can be seen as so many copies of $\mathscr{B}(n, 1)$ glued together. By passing to the limit $q \to 1$, this quantity gives 1, coinciding with the number of "apartments" in $\mathscr{B}(n, 1)$.

(ii) The quantum factorial $[n]_q!$ counts the maximal simplices in $\mathscr{B}(n, q)$, which becomes $n!$, the number of maximal simplices in $\mathscr{B}(n, 1)$ by taking the limit $q \to 1$.

(iii) The quantum binomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$ counts the vertices of color k in $\mathscr{B}(n, q)$, which becomes $\binom{n}{k}$, the number of vertices of color k in $\mathscr{B}(n, 1)$ by taking the limit $q \to 1$.

(iv) There are more combinatorial quantities in $\mathscr{B}(n, q)$ becoming ones for $\mathscr{B}(n, 1)$ by taking the limit $q \to 1$.

1.1.6. Tits's observations are not limited to $\mathrm{PGL}\left(\mathbb{F}_q^n\right)$. In fact, he did for all semisimple groups over $\mathbb{F}_q$. Of course, there was no $\mathbb{F}_1$-geometry back in Tits' time, but it seems the above observations inspired him to develope the theory of buildings with the following principle:

> Buildings are multifold apartments and apartments are $q \to 1$ limit case of buildings, which can be thought as forgetting the additive arithmetic of the base field.

1.2 Abstract Buildings

Before moving on, we now give a formal definition of polysimplicial complexes.

Definition 1.2.1. An *(abstract) simplicial complex* is a nonempty poset $\mathcal{S}$ (whose members are called *simplices*) satisfying

S1. any two simplices σ, τ have an infimum $\sigma \cap \tau$;

So there is a unique smallest element in $\mathcal{S}$, called the *empty simplex*, denoted by $\emptyset$.

S2. for each simplex σ the poset $\mathcal{S}_{\leqslant\sigma}$ of simplices smaller than σ (they are called *faces* of σ) form a *Boolean lattice of rank r*, namely isomorphic to the power set of a r-set, for some finite r. In this case, we see σ is of *dimension* $r-1$ and is a $(r-1)$*-simplex*.

The *dimension* of $\mathcal{S}$ is the supremum of dimensions of its simplices. The minimal nonempty simplices are of dimension 0 and are thus called *vertices*. Let $\mathcal{V}$ denote the set of vertices. Then $\mathcal{S}$ can be identified with a poset of nonempty subsets of $\mathcal{V}$.

A *morphism* between simplicial complexes is a map preserving infima, suprema and the empty simplex $\emptyset$. Note that this implies that such a morphism is determined by its restriction to vertices. So equivalently, such a morphism is a map between vertices extending to a monotonic map preserving simplices. A morphism $\varphi\colon \mathcal{S} \to \mathcal{S}'$ is said to *fix a simplex* $\sigma \in \mathcal{S} \cap \mathcal{S}'$ *pointwise* if it induces an identity from $\mathcal{S}_{\leqslant\sigma}$ to $\mathcal{S}'_{\leqslant\sigma}$.

A *polysimplicial complex* is a cartesian product of simplicial complexes (in the category of posets) and morphisms between polysimplicial complexes are therefore defined.

One can verify that $\mathscr{B}(n, q)$ and $\mathscr{B}(n, 1)$ are simplicial complexes.

1.2.2. Let's analyse how the "apartments" $\mathscr{A}(\mathbf{\Lambda})$ are glued into the "building" $\mathscr{B}(n, q)$.

(i) *For any two simplices F, F' in $\mathscr{B}(n, q)$, there is an "apartment" $\mathscr{A}(\mathbf{\Lambda})$ containing both of them.*

Proof. We may assume F, F' are maximal, i.e. being complete flags:

$$F: \mathbb{F}_q^n = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_{n-1} \supsetneq 0,$$
$$F': \mathbb{F}_q^n = V'_0 \supsetneq V'_1 \supsetneq V'_2 \supsetneq \cdots \supsetneq V'_{n-1} \supsetneq 0.$$

Then we may view them as composition series for $\mathbb{F}_q^n$. Therefore by the *Jordan-Hölder Theorem*, there is a permutation π of $P_n = \{1, 2, \cdots, n\}$ such that whenever $j = \pi(i)$, we have isomorphisms

$$\frac{V_{n-i}}{V_{n-i+1}} \xleftarrow{\sim} \frac{V_{n-i} \cap V'_{n-j}}{(V_{n-i+1} \cap V'_{n-j}) + (V_{n-i} \cap V'_{n-j+1})} \xrightarrow{\sim} \frac{V'_{n-j}}{V'_{n-j+1}}$$

induced from inclusions. Let $\boldsymbol{\lambda}_i$ be the one-dimensional subspace of $V_{n-i} \cap V'_{n-j}$ whose image in above quotients are non-trivial. Then $\boldsymbol{\Lambda} = \{\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \cdots, \boldsymbol{\lambda}_n\}$ is a frame with $\mathscr{A}(\boldsymbol{\Lambda})$ containing both F and F'. □

(ii) *If $\mathscr{A}(\boldsymbol{\Lambda})$ and $\mathscr{A}(\boldsymbol{\Lambda}')$ are two "apartments" containing both F and F', then there is an isomorphism between them fixing F and F' pointwise.*

Proof. Again, we may assume F, F' are maximal and let $V_i, V'_i, \boldsymbol{\lambda}_i$ be as above. Then $i \mapsto \boldsymbol{\lambda}_i$ induces an isomorphism $\phi_{\boldsymbol{\Lambda}} \colon \mathscr{B}(n, 1) \to \mathscr{A}(\boldsymbol{\Lambda})$. The inverse of it can be described by vertices as

$$\psi_{\boldsymbol{\Lambda}} \colon U \mapsto \{i \in P_n \mid U \cap V_{n-i+1} \neq U \cap V_{n-i}\}.$$

Similarly we have an isomorphism $\phi_{\boldsymbol{\Lambda}'} \colon \mathscr{B}(n, 1) \to \mathscr{A}(\boldsymbol{\Lambda}')$ and its inverse $\psi_{\boldsymbol{\Lambda}'}$. Note that the morphism $\psi_{\boldsymbol{\Lambda}}$ (and similarly $\psi_{\boldsymbol{\Lambda}'}$) is determined by the complete flag F, we conclude that $\psi_{\boldsymbol{\Lambda}}$ and $\psi_{\boldsymbol{\Lambda}'}$ coincide on the intersection of $\mathscr{A}(\boldsymbol{\Lambda})$ and $\mathscr{A}(\boldsymbol{\Lambda}')$. Then $\phi_{\boldsymbol{\Lambda}'} \circ \psi_{\boldsymbol{\Lambda}}$ is an isomorphism between $\mathscr{A}(\boldsymbol{\Lambda})$ and $\mathscr{A}(\boldsymbol{\Lambda}')$ fixing F and F' pointwise. □

Then the buildings can be defined as follows.

Definition 1.2.3. A *(abstract) building* is a polysimplicial complex $\mathscr{B}$ equipped with a family $\mathcal{A}$ of subcomplexes of $\mathscr{B}$, whose members are called *apartments*, such that the following axioms are satisfied.

B0. Each apartment $A \in \mathcal{A}$ is isomorphic to an abstract apartment $\mathscr{A}$.

B1. For any two simplices F, F', there is an apartment A containing them.

B2. If A, A' are two apartments containing both F and F', then there is an isomorphism between A and A' fixing F and F' pointwise.

A *morphism between buildings* is a morphism of the underlying polysimplicial complexes which maps apartments in apartments.

Of course, one has to define what is an apartment to make this definition sense.

1.2.4. Let's analyse what the "apartment" $\mathscr{B}(n, 1)$ looks like.

(i) *All maximal simplices have the same dimension.*

Proof. This is clear, they are precisely the $(n-1)$-subsets of P_n. □

(ii) *Any two maximal simplices C, C' are connected by a sequence $(C_0, C_1, \cdots, C_s)$ with $C_0 = C$ and $C_s = C'$ such that for each i, $C_{i-1} \cap C_i$ has codimension 1 in both C_{i-1} and C_i.*

Proof. Note that a maximal simplex in $\mathscr{B}(n, 1)$ is a complete flag, hence a sequence $(i_1, i_2, \cdots, i_{n-1})$, which can be identified with an ordering of P_n. Hence any two such simplices are different by a permutation $\pi \in \mathfrak{S}_n$. But any permutation can be written as the composition of transpositions while two sequences different by a transposition meet in a sequence with one term being removed. □

In general, a polysimplicial complex which has above properties is called a *chamber complex* and its maximal simplices are called *chambers*. A one-codimensional face of a chamber is called a *panel*. A sequence $(C_0, C_1, \cdots, C_s)$ connecting two chambers by panels is called a *gallery*. Note that any Boolean lattice is a chamber complex with a unique chamber: its maximal element. A *chamber map* between chamber complexes is a morphism mapping chambers to chambers.

(iii) *There is a coloring, namely a chamber map from the complex to a Boolean lattice.*

Proof. It suffices to define colors for vertices. Then the color of a simplex would be the set of the colors of its vertices. For instance, one can define the *color* of a vertex as its cardinality as in 1.1.2. □

In general, a chamber complex which has this property is said to be *colorable*. It is worth noticing that any two colorings are different by an isomorphism of Boolean lattices (in other words, up to a permutation of the colors of vertices).

(iv) *The Weyl group acts transitively on the simplices of the same color.*

Proof. Two simplices $F = (I_i)$ and $F' = (I'_i)$ are of the same color means two things: first, they have the same number of entries; second, each pair of entries (I_i, I'_i) have the same cardinality. This is precisely the condition for the existence of a permutation $\pi \in \mathfrak{S}_n$ interchanging them. □

(v) *Fix a chamber C, then the stabilizers in the Weyl group of its panels are all of order 2 and their generators s_j form a generating system S of the Weyl group with generating relations of the form $(s_i s_j)^{m_{ij}} = 1$.*

Proof. As in (ii), a chamber C is a sequence $(i_1, i_2, \cdots, i_{n-1})$. Let i_n be the complement of this sequence in P_n. Then for each panel obtained from C by deleting i_j, let s_j be the transposition (i_j, i_n). Then this panel's stabilizer is precisely $\{1, s_j\}$ and one can verify the system $S = \{s_1, \cdots, s_{n-1}\}$ satisfies the requirement. □

Note that it follows from property (v) that the stabilizer of a face of C is generated by those s_j with j being not a color of its vertex. Furthermore, the complex $\mathscr{B}(n, 1)$ can be built from the pair (W, S) of the Weyl group $W = \mathfrak{S}_n$ and the system $S = (s_j)$ of generators in (v). Indeed, any face of the chamber C corresponds to the subset I of S generating its stabilizer and any simplex is translated to such a face by an element of W, unique up to the stabilizer $\langle I\rangle$. Therefore, the simplices in $\mathscr{B}(n,1)$ can be identified with the cosets $w\langle I\rangle$ with $w \in W$ and $I \subseteq S$.

Definition 1.2.5. A *Coxeter system* is a pair (W,S) of a group W and a system of its generators $S = \{s_1, s_2, \cdots, s_n\}$ such that all s_i are of order 2 and the generating relations for S are of the form $(s_i s_j)^{m_{ij}} = 1$. Its *Coxeter complex* $\Sigma(W,S)$ is the polysimplicial complex defined as the complex of cosets of the form $w\langle I\rangle$ with $w \in W$ and $I \subseteq S$, where the order is given by reverse inclusion.

Then 1.2.4 shows that $\mathscr{B}(n,1)$ is isomorphic to the Coxeter complex $\Sigma(\mathfrak{S}_n, S)$, where S can be chosen to be any generating system of transpositions, for instance $S = \{(1,n),(2,n),\cdots,(n-1,n)\}$.

A *morphism* between Coxeter systems (W,S) and (W',S') is a group homomorphism $f\colon W \to W'$ such that $f(S) \subseteq S'$. In this category, a Coxeter system (W,S) is a *product* of subsystems $(W_i,S_i)_{1\leqslant i\leqslant m}$ if we have a group decomposition $W = W_1 \times \cdots \times W_m$ and a set decomposition $S = S_1 \sqcup \cdots \sqcup S_m$. A Coxeter system is *irreducible* if it can not be decomposed into proper subsystems.

One can see that morphisms between Coxeter systems induce morphisms between their Coxeter complexes and such a functor is compatible with the decompositions. In particular, a Coxeter complex of an irreducible Coxeter system is simplicial.

Now, we can complete Definition 1.2.3 by defining an *apartment* to be a polysimplicial complex isomorphic to the Coxeter complex of some Coxeter system.

A Coxeter complex $\Sigma(W,S)$ is *finite* if and only if W is finite if and only if all m_{ij} are finite. If this is the case, this Coxeter complex is said to be *spherical*, otherwise it is *affine*. A building is said to be *spherical* (resp. *affine*) if its apartments are isomorphic to a spherical (resp. affine) Coxeter complex.

We have seen that $\mathscr{B}(n,q)$ is such a building: its apartments are isomorphic to the Coxeter complex $\Sigma(\mathfrak{S}_n, S)$. This is not an accident. In fact, any reductive group over an arbitrary field gives rise to such a building. They are called the *Tits buildings*. We refer to [Bourbaki, chap.IV] for the theory of Coxeter systems and [Tit74] for a treatment of Tits buildings in the language of Coxeter complexes.

1.3 Euclidean apartments

Although buildings can be defined and studied in a pure combinatorial way, it would be more intuitive and convenient if we can also define them geometrically.

1.3.1. One way to visualize the Coxeter complex $\Sigma(\mathfrak{S}_n, S)$ is the follows. The group $\mathfrak{S}_n$ acts faithfully on $\mathbb{R}^n$ as permutations of the coordinates. For any transposition (i, j), its set of fixed points is the hyperplane $\left\{(x_0, \cdots, x_{n-1}) \in \mathbb{R}^n \,\middle|\, x_i = x_j\right\}$ and it thus acts as the reflection respect to this hyperplane. Therefore the group $\mathfrak{S}_n$ can be determined by the reflections/hyperplanes defined by the transpositions. Moreover, the hyperplanes partition $\mathbb{R}^n$ into pieces of various dimensions with an obvious order relation: one such a piece belongs to the closure of another. This gives rise to a complex isomorphic to $\Sigma(\mathfrak{S}_n, S)$. The system S can be obtained as the reflections respect to a chamber.

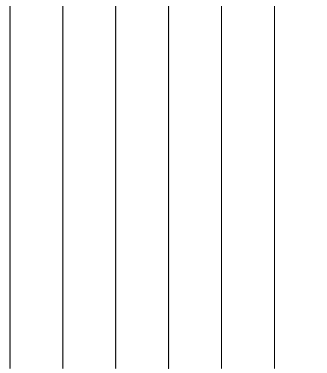
With this example in mind, we make the following definition.

Definition 1.3.2. A *(Euclidean) apartment* $\mathscr{A}$ is a Euclidean affine space $\mathbb{A}$ equipped with a reflection group W (called its *Weyl group*) on it.

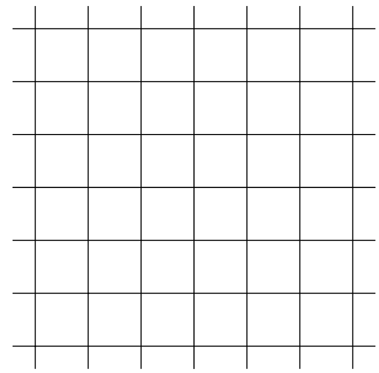
Let $\mathbb{A}$ be a Euclidean affine space. We use ${}^v\mathbb{A}$ to denote its associated vector space. For an affine transformation f on $\mathbb{A}$, we use ${}^v f$ to denote its *vectorial part*. For an affine subspace X of $\mathbb{A}$, we use ${}^v X$ to denote its *direction*.

A *reflection* on $\mathbb{A}$ is an affine isometry whose fixed points form a hyperplane. Any hyperplane H is associated with a reflection r_H with respect to it.

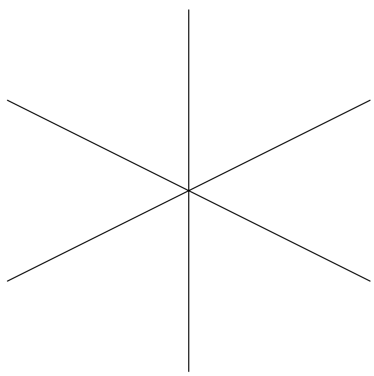
A *reflection group* W is a group of affine isometries generated by reflections and such that its vectorial part ${}^v W$ is finite. W is said to be *irreducible* if ${}^v W$ acts irreducibly on ${}^v\mathbb{A}$ and is said to be *essential* if ${}^v W$ acts essentially on ${}^v\mathbb{A}$ (that is, there is no nonzero fixed point). An apartment is said to be *irreducible* (resp. *essential*, *trivial*, etc.) if its reflection group is so.



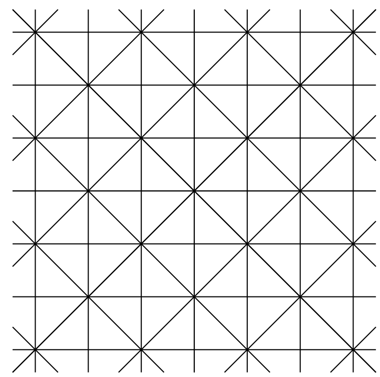
$\widetilde{A_1} \times \mathbf{0}$: a *non-essential* apartment



$\widetilde{A_1} \times \widetilde{A_1}$: a *non-irreducible* apartment



A_2: a *spherical* apartment



$\widetilde{C_2}$: contains *non-special* vertices

Figure 1.3.1: Some examples of apartments

The kernel $T = \ker(W \to {}^vW)$ consists of translations and hence is called the *translation group*. It is then a subgroup of ${}^v\mathbb{A}$. The apartment $\mathscr{A}$ is said to be *spherical* (resp. *affine*, *discrete*) if T is finite (resp. infinite, discrete).

1.3.3. A *morphism* between apartments $(\mathbb{A}, W)$ and $(\mathbb{A}', W')$ is a continuous affine map $f\colon \mathbb{A} \to \mathbb{A}'$ with a group homomorphism $\phi\colon W \to W'$ such that $\phi(w).f(x) = f(w.x)$ for all $w \in W$ and $x \in \mathbb{A}$. In this cateogry, an apartment $(\mathbb{A}, W)$ is said to be a *product* of apartments $(\mathbb{A}_i, W_i)_{1 \leqslant i \leqslant m}$ if we have an orthogonal decomposition $\mathbb{A} = \mathbb{A}_1 \times \cdots \times \mathbb{A}_m$ and a group decomposition $W = W_1 \times \cdots \times W_m$ such that each W_i acts trivially on the orthogonal complement of $\mathbb{A}_i$.

Any apartment $\mathscr{A}$ admits a decomposition [Bourbaki, chap.V, §3, no.8]

$$\mathscr{A} = \mathscr{A}_0 \times \mathscr{A}_1 \times \cdots \times \mathscr{A}_m,$$

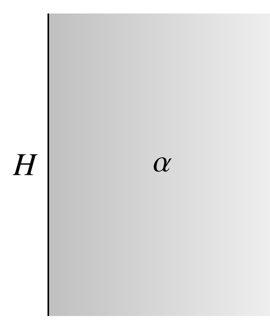
where $\mathscr{A}_0$ is trivial and each $\mathscr{A}_i$ (for $1 \leqslant i \leqslant m$) is irreducible.

Throughout this note, all apartments are assumed to be discrete. This is equivalent to saying that in each irreducible component of it, the translation group T is either finite or a full-rank lattice in ${}^v\mathbb{A}$.

1.3.4. Let $\mathbb{A} = (\mathbb{A}, W)$ be an apartment.

The hyperplanes of fixed points of reflections in W are called the *walls* in $\mathbb{A}$. The set $\mathcal{H}$ of walls is stable under W and completely determines it.

A *half-apartment* (also called an *affine root* in [BT-I, 1.3.3]) is a closed half-space α of $\mathbb{A}$ bounded by a wall $\partial\alpha$, called its *wall*.

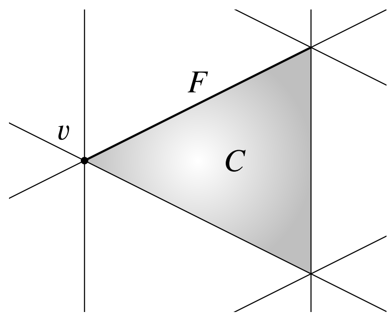


An affine root α and its boundary H

A *facet* in $\mathbb{A}$ is an equivalence class in $\mathbb{A}$ for the relation "x and y are contained in the same half-apartments". A facet F is an open convex subset of the affine subspace (called the *support* of F) that it spans.

The set $\mathcal{F}$ of facets admits an order: a facet F is said to be a *face* of another F', denoted by $F \leqslant F'$, if F is *covered by* F', namely contained in the closure of F'. Such an order gives rise to a polysimplicial complex. To see this, first notice that facets in an apartment are compatible with its decomposition into irreducible components. Hence we may assume our apartment $\mathbb{A}$ is irreducible and essential. Then this can be seen from the fact that any triangulation of a topological space gives rise to a simplicial complex (indeed, this is where the notion comes from). When $\mathbb{A}$ is discrete affine, its facets already triangulate the ambient space. When $\mathbb{A}$ is spherical, its facets triangulate the unit sphere. This is why it is called spherical.

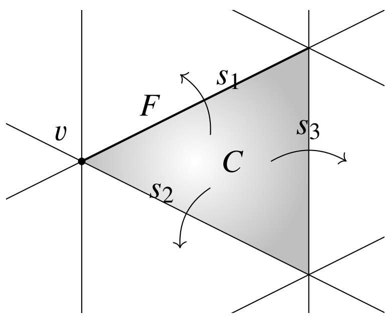
1.3.5. The maximal facets are called *chambers* (or *alcoves*). They are the connected components of the complement of the union of all walls in $\mathbb{A}$. The Weyl group W acts simply transitively on the set $\mathcal{C}$ of chambers [Bourbaki, chap.V, §3, no.2, th.1].



A vertex v, a facet F, and an alcove C

Let C be a chamber. Then its closure $\overline{C}$ is a fundamental domain of W in $\mathbb{A}$ [Bourbaki, chap.V, §3, no.3, th.2] and is the intersection of some half-apartments, whose walls are called the *walls* of C. Equivalently, the walls of C are the supports of panels of it, where a *panel* means a maximal proper face of C. Moreover, W is generated by the set S of reflections with respect to the walls of C and the pair (W,S) is a Coxeter system [Bourbaki, chap.V, §3, no.2, th.1]. The projection of C onto an irreducible component $\mathbb{A}_i$ is again a chamber in it and induces an irreducible Coxeter system (W_i,S_i). Then (W,S) is the product of them. In other words, decomposition of the pair of $(\mathscr{A},C)$ of an apartment and a chamber is compatible with the decomposition of the Coxeter system (W,S) it defines.

A *type function* on $\mathscr{A}$ is a morphism τ from the complex $\mathcal{F}$ of facets to a Boolean lattice, which maps chambers to the maximal element and is W*-stable* in the sense that for any facet F and any $w \in W$, $\tau(F) = \tau(w.F)$. The image of this function is denoted by $\mathcal{T}$ and its members are called *types*. This notion is essentially the same as a *coloring* as in 1.2.4(iii) plus 1.2.4(iv). They differ in one respect: for a coloring, the target Boolean lattice is viewed as a power set $\mathscr{P}(\mathfrak{I})$ with its usual order $\subseteq$, while for a type function, we use the reverse order $\supseteq$. In other words, a face of type I is of color $\neg I := \mathfrak{I} \setminus I$.



The alcove C defines a generating set

$$S = \{s_1, s_2, s_3\},$$

where the reflections s_1, s_2, s_3 are shown in the picture. The facet F in the figure is then a *panel* of C. It has *type* 1 and *color* $\{2, 3\}$. Hence $C_1 = F$. The vertex v in the figure has *type* $\{1, 2\}$ and *color* 3. Hence $C_{\{1,2\}} = v$.

The generating set and the type function associated to C

Since any facet is transformed by W to a unique face of C, the type function τ is completely determined by the types of its panels, which we may view as an indexing of S. Indeed, let I be a type, then the set C_I of points $x \in \overline{C}$ such that the reflections $s \in S$ fixing x are indexed by I is a face of C of type I and its stabilizer is the subgroup W_I of W generated by the reflections indexed by I [Bourbaki, chap.V, §3, no.3, prop.1]. Then $\tau(F) = I$ if and only if F is transformed to C_I.

1.3.6. A reflection group W is said to be *linear* if it fixes a point. This is the case if and only if W is finite [Bourbaki, chap.V, §3, no.9]. If this is the case, we can identify W with its vectorial part ${}^vW$ by choosing the fixed point to be the origin of $\mathbb{A}$.

Conversely, the vectorial part ${}^vW$ of the Weyl group W can be viewed as a linear reflection group on ${}^v\mathbb{A}$. The spherical apartment ${}^v\mathscr{A} = ({}^v\mathbb{A}, {}^vW)$ obtained in this way is called the *vectorial apartment* of $\mathscr{A}$. The walls (resp. facets, chambers) in ${}^v\!\mathscr{A}$ are called the *vectorial walls* (resp. *vectorial facets*, *vectorial chambers*) and the set of them is denoted by ${}^v\mathcal{H}$ (resp. ${}^v\mathcal{F}$, ${}^v\mathcal{C}$). Note that the vectorial walls are precisely the directions of walls in $\mathscr{A}$.

1.3.7. Let x be a point in $\mathscr{A}$. The stabilizer W_x of x is a linear reflection group whose vectorial part ${}^vW_x$ is a subgroup of ${}^vW$. The apartment $\mathscr{A}_x = (\mathbb{A}, W_x)$ is called the *spherical apartment at* x. The walls in $\mathscr{A}_x$ are precisely the walls in $\mathscr{A}$ passing through x and the set of them is denoted by $\mathcal{H}_x$. The facets (resp. chambers) in $\mathscr{A}_x$ are called the *vectorial facets with base point* x (resp. *vectorial chambers with base point* x) and the set of them is denoted by $\mathcal{F}_x$ (resp. $\mathcal{C}_x$).

A point $x \in \mathbb{A}$ is said to be *special* if the spherical apartment $\mathscr{A}_x$ is isomorphic to ${}^v\!\mathscr{A}$, or equivalently, the set $\mathcal{H}_x$ is a complete set of representatives of ${}^v\mathcal{H}$. This can happen only if x belongs to a minimal facet.

1.3.8. The minimal facets are called *vertices*. The set of vertices is denoted by $\mathcal{V}$. When the apartment is essential, they are points. From now on, all apartments are assumed to be essential unless otherwise specified[2].

Under this assumption, every special point is a vertex. Furthermore, any special vertex is an extremal point of the closure of some chamber. Conversely, any chamber admits a special point as an extremal point of its closure [Bourbaki, chap.V, §3, no.10, prop.11's cor.]. However, not all extremal points, hence not all vertices are special (see $\widetilde{C}_2$ in Fig. 1.3.1 for an example).

1.4 Root systems

Before moving on to the definition of buildings, let's look at some examples of Euclidean apartments arising from root systems (as well as root data). They are the key examples used in the study of reductive groups.

[2]This means we will only focus on *reduced* buildings, rather than *extended* buildings.

1.4.1. Let $\mathbb{V}$ be a Euclidean vector space and $\mathbb{V}^*$ its dual space. For any $a \in \mathbb{V}^* \setminus \{0\}$, let r_a be the reflection with respect to the hyperplane $H_a := \mathrm{Ker}(a)$ and $a^\vee$ the vector orthogonal to H_a satisfying $a(a^\vee) = 2$. So for any $\mathbf{v} \in \mathbb{V}$, we have

$$r_a(\mathbf{v}) = \mathbf{v} - a(\mathbf{v})a^\vee.$$

Note that r_a also induces a reflection on $\mathbb{V}^*$, namely $f \mapsto f - f(a^\vee)a$. A finite spanning subset $\Phi \subseteq \mathbb{V}^* \setminus \{0\}$ is called a *root system* on $\mathbb{V}$ if

RS1. for any $a \in \Phi$, $r_a(\Phi) = \Phi$;

RS2. for any $a, b \in \Phi$, $a(b^\vee) \in \mathbb{Z}$;

and is *reduced* if

RS3. for any $a \in \Phi$, $\mathbb{R}a \cap \Phi = \{\pm a\}$.

From now on, all root systems are assumed to be reduced[3].

Elements of Φ are called *roots* in Φ. For a root $a \in \Phi$, the vector $a^\vee$ is called its *coroot*; they form a root system $\Phi^\vee$ on $\mathbb{V}^*$, called the *coroot system*. A subset $\Psi \subseteq \Phi$ is called a *subroot system* if for any $a \in \Psi$, $r_a(\Psi) = \Psi$, and is said to be *closed* if for any $a, b \in \Psi$ such that $a + b$ is a root, $a + b \in \Psi$.

Any root system Φ admits the *Weyl group* ${}^vW(\Phi)$, that is the reflection group of $\mathbb{V}$ generated by r_a for $a \in \Phi$. It is a linear reflection group with walls H_a for $a \in \Phi$. In this way, we get a spherical apartment ${}^v\mathscr{A}(\Phi) := (\mathbb{V}, {}^vW(\Phi))$. Note that not all spherical apartments arise in this way (see [Bourbaki, chap.VI, §2, no.5, prop.9]) and non-isomorphic root systems may have isomorphic Weyl groups (for instance root systems of types B_n and C_n).

1.4.2. A root system Φ is said to be *irreducible* if it cannot be written as the union of two proper subsets such that they are orthogonal to each other. A root system Φ is irreducible if and only if its Weyl group ${}^vW(\Phi)$ is [Bourbaki, chap.VI, §1, no.2,

[3]This means we will only focus on split reductive groups.

prop.5's cor.]. Any root system decomposes into disjoint union of irreducible ones and such a decomposition is compatible with the decomposition of Weyl groups and hence of apartments [Bourbaki, chap.VI, §1, no.2, prop.6 and 7].

1.4.3. Let Φ be a root system. Then there is a closed subset Φ^+ of Φ such that for any $a \in \Phi$, either $a \in \Phi^+$ or $-a \in \Phi^+$. This set is called a *system of positive roots*. Once such a set is chosen, elements in the set $\Phi^- := -\Phi^+$ are called *negative roots*. A positive root is called a *simple root* if it cannot be written as the sum of two positive roots. The set Δ of simple roots form a *basis* of Φ in the sense that any root is a $\mathbb{Z}$-linear combination of simple roots and its coefficients are either all non-negative or all non-positive [Bourbaki, chap.VI, §1, no.6, th.3]. The cardinality of the set Δ is called the *rank* of Φ and is independent of the choice of Δ. Indeed, it equals $\dim \mathbb{V}$.

Let Δ be a basis of Φ. Then the set

$$^{v}C = \{\mathbf{v} \in \mathbb{V} \mid \forall a \in \Delta : a(\mathbf{v}) > 0\}$$

is a vectorial chamber, called the *Weyl chamber* associated to Δ [Bourbaki, chap.VI, §1, no.5, th.2]. Conversely, let ^{v}C be a vectorial chamber. Then for any $\mathbf{v} \in {}^{v}C$, the sets

$$\Phi^+ = \{a \in \Phi \mid a(\mathbf{v}) > 0\} \qquad \text{and} \qquad \Phi^- = \{a \in \Phi \mid a(\mathbf{v}) < 0\}$$

form a partition of Φ into positive and negative roots and are independent of the choice of $\mathbf{v}$. Then one can obtain a basis Δ by taking the simple roots. But there is a more geometric description: they are the roots defining the walls of ^{v}C pointing inside. As vectorial chambers are Weyl chambers associated to some choice of basis, we call them *Weyl chambers* to specify that they are chambers in the spherical apartment $^{v}\mathscr{A}(\Phi)$.

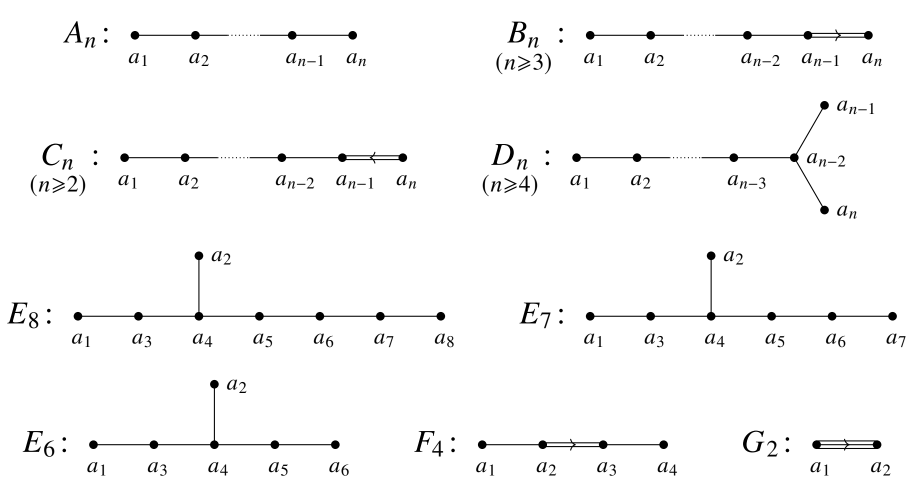
1.4.4. The relation between simple roots and types is the following. First, the Weyl group ^{v}W is generated by r_a for $a \in \Delta$ as they are the roots defining walls of ^{v}C and point inside. Therefore a type $I \in \mathcal{T}$ corresponds to a subset of Δ. From now on, we do not distinguish them. Then the face of ^{v}C corresponding to I is the set

$$^{v}C_I = \{\mathbf{v} \in \mathbb{V} \mid \forall a \in I : a(\mathbf{v}) = 0; \forall a \in \Delta \setminus I : a(\mathbf{v}) > 0\}.$$

Let Φ_I be the subroot system of Φ generated by I, then the stabilizer ${}^vW_I$ is the Weyl group of it. The set $\Psi = \Phi_I \cup \Phi^+$ has the property that $\Psi \cup (-\Psi) = \Phi$ and is closed. Such kind of subsets of Φ are said to be *parabolic*. Given a parabolic subset Ψ of Φ containing Φ^+, then the simple roots in $\Psi \cap (-\Psi) \cap \Phi^+$ gives the type I. See [Bourbaki, chap.VI, §1, no.7].

1.4.5. Given a basis Δ of a root system Φ, its *Dynkin diagram* is defined as follows. The vertices are the simple roots of Φ and the number of edges between two vertices is $4\cos^2(\theta)$ if the angle between them is θ. Furthermore, these edges are decorated with arrows pointing from the longer root to the shorter root. It turns out that, up to graph isomorphisms, the Dynkin diagram is independent of the choice of the basis Δ.

From above description, we see that Φ is irreducible if and only if its Dynkin diagram is connected. The Dynkin diagrams of irreducible root systems are classified as follows [Bourbaki, chap.VI, §4, no.2, th.3], where the subscription n in the notation X_n denotes the rank of it.



Dynkin diagrams of irreducible root systems

A spherical apartment is said to be *of type* X_n if it is isomorphic to ${}^v\mathscr{A}(\Phi)$ for an irreducible root system Φ of type X_n.

1.4.6. Let $\mathbb{A}$ be an affine space such that ${}^v\mathbb{A} = \mathbb{V}$ with a specified point o. For any $a \in \mathbb{V}^*$ and $k \in \mathbb{R}$, denote the affine function $x \mapsto a(x - o) + k$ on $\mathbb{A}$ by $a + k$ and denote the closed half-space $\{x \in \mathbb{A} \mid (a + k)(x) \geqslant 0\}$ by α_{a+k}.

For each $a \in \Phi$, let Γ_a be a prechosen nonempty subset of $\mathbb{R}$. The affine function $a + k$ is called an *affine root* if $a \in \Phi$ and $k \in \Gamma_a$. Let Σ denote the set of closed half-spaces α_{a+k} with $a + k$ an affine root. Then $a + k \mapsto \alpha_{a+k}$ gives rise to a bijection between the set of affine roots and Σ. For this reason, we will not distinguish the affine root $a + k$ and the closed half-space α_{a+k} and will call Σ the *affine root system*[4]. The roots are vectorial part of affine roots. Hence we denote Φ by ${}^v\Sigma$ and call it the *vectorial root system* of Σ.

For $\alpha = \alpha_{a+k}$ an affine root, let ${}^v\alpha$ denote its vectorial part a, let $\partial\alpha$ denote its boundary $\{x \in \mathbb{A} \mid (a + k)(x) = 0\}$, let r_α denote the reflection with respect to $\partial\alpha$, let α^* denote the other affine root sharing the same boundary with α, that is $\overline{\mathbb{A} \setminus \alpha}$, and let α_+ denote the intersection of all the affine roots containing a neighborhood of α.

1.4.7. Let Σ be an affine root system on a Euclidean affine space $\mathbb{A}$, its *affine Weyl group* $W(\Sigma)$ is the reflection group on $\mathbb{A}$ generated by r_α for all $\alpha \in \Sigma$. In this way, we obtain an apartment $\mathscr{A}(\Sigma) := (\mathbb{A}, W(\Sigma))$ with vectorial apartment ${}^v\mathscr{A}({}^v\Sigma)$. Suppose all the subsets Γ_a are taken to be the same discrete subgroup $\Gamma \neq 0$ of $\mathbb{R}$, then the walls in the apartment $\mathscr{A}(\Sigma)$ are precisely the boundaries $\partial\alpha$ with $\alpha \in \Sigma$ [Bourbaki, chap.VI, §2, no.1, prop.2]. For x a point in the apartment $\mathscr{A}(\Sigma)$, let Σ_x be the set of affine roots α such that $x \in \partial\alpha$ and let ${}^v\Sigma_x$ be the set of vectorial parts of affine roots in Σ_x. Then Σ_x can be identified with ${}^v\Sigma_x$ by $\alpha \mapsto {}^v\alpha$. In particular, the roots in ${}^v\Sigma$ can be identified with the affine roots in Σ_o. Note that ${}^v\Sigma_x$ is a closed subroot system of ${}^v\Sigma$. Then the spherical apartment $\mathscr{A}_x$ at x can be identified with ${}^v\mathscr{A}({}^v\Sigma_x)$.

[4]Note that, there is a notion called *affine root system*, defined in a similar way as root system, but for affine spaces. In this note, this terminology only refers to those arise from (reduced) root systems.

1.4.8. Notations as before. Suppose $\Phi = {}^v\Sigma$ is irreducible and all Γ_a are the same discrete subgroup of $\mathbb{R}$. Let ${}^vC$ be a Weyl chamber of Φ and Δ be the set of simple roots it defines. Then there is a unique root a_0 such that $\|a_0\| \geqslant \|a\|$ for all root a [Bourbaki, chap.VI, §1, no.8, prop.25]. This a_0 is called the *highest root* with respect to Δ or ${}^vC$. The set

$$C = (o + {}^vC) \setminus \alpha^*_{-a_0+} = \text{interior of } \Big(\bigcap_{a\in\Delta} \alpha_a\Big) \cap \alpha_{-a_0+}$$

is a chamber in $\mathscr{A}(\Sigma)$ [Bourbaki, chap.VI, §2, no.2, prop.5] and is called the *fundamental alcove* for Δ.

Let $\widetilde{\Delta}$ denote the set of affine roots α defining the walls of C, which means $C \subseteq \alpha$ and $\partial\alpha$ is a wall of C. Then $\widetilde{\Delta}$ consists of the simple roots and the affine root $\alpha_0 = \alpha_{-a_0+}$. Such a set $\widetilde{\Delta}$ is a *basis* of Σ in the sense that any affine root is a $\mathbb{Z}$-linear combination of its elements and the coefficients are either all non-negative or all non-positive.

Conversely, let C be a chamber in $\mathscr{A}(\Sigma)$ and x a special vertex which is also an extremal point of $\overline{C}$. The affine roots defining walls of C form a basis $\widetilde{\Delta}$ of the affine root system Σ. Among these affine roots, those vanishing at x give rise to a basis Δ of the root system Φ by taking their vectorial parts and the rest one gives rise to the highest root with respect to Δ by taking the negation of its vectorial part. Since chambers in $\mathscr{A}(\Sigma)$ are fundamental alcoves for some basis, we call them *alcoves* to avoid confusion with Weyl chambers.

1.4.9. The type function is introduced as follows. The affine Weyl group $W(\Sigma)$ is generated by r_α for $\alpha \in \widetilde{\Delta}$ as they are the affine roots defining walls of C. Therefore a type $I \in \mathcal{T}$ corresponds to a proper subset of $\widetilde{\Delta}$. From now on, we do not distinguish them. Then the face of C corresponding to I is the set

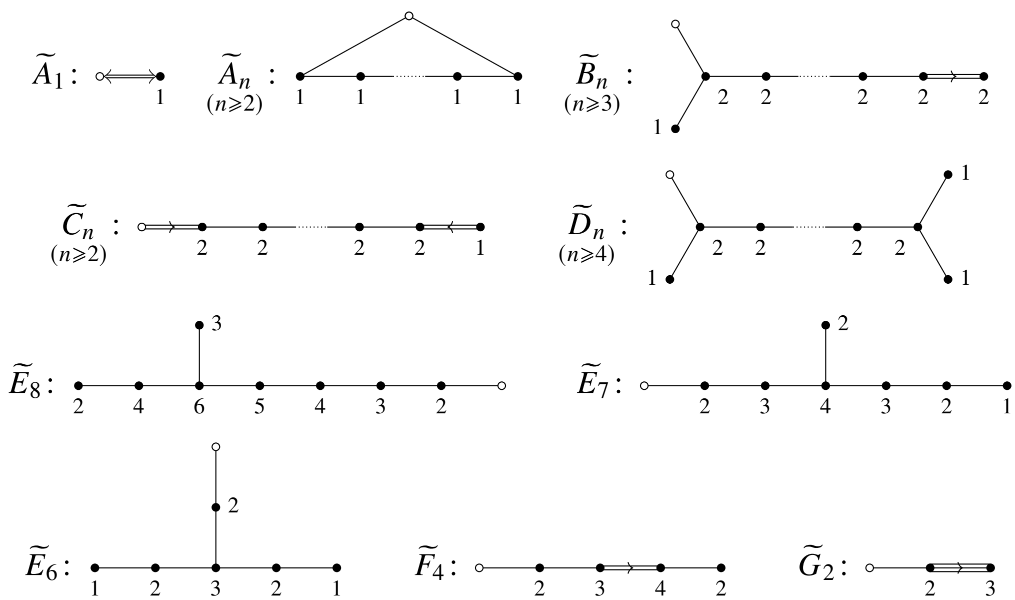
$$C_I = \overline{C} \cap \Big(\bigcap_{\alpha\in I} \partial\alpha\Big) \setminus \Big(\bigcup_{\alpha\in\widetilde{\Delta}\setminus I} \partial\alpha\Big).$$

1.4.10. Let Σ be an irreducible affine root system with $\widetilde{\Delta}$ a basis. Then the *extended Dynkin diagram* of it is defined similarly to Dynkin diagram except in the case of $\widetilde{A}_1$, where there is a left-right double arrow between the two vertices.

The following are the extended Dynkin diagrams of all irreducible affine root systems [Bourbaki, chap.VI, §4, no.3, prop.4], where the notation $\widetilde{X}_n$ indicates the affine root system that arises from the root system of type X_n. Note that the Dynkin diagrams are decorated in the following way: the part consisting of bold vertices is the ordinary Dynkin diagram and its vertices represent the simple roots a_i ($1 \leqslant i \leqslant n$), then the extra hollow vertex presents the (affine root α_0 defined by the) highest root a_0 and each simple root a_i is labelled by its coefficient h_i in the expression

$$a_0 = \sum_{i=1}^{n} h_i a_i$$

representing the highest root a_0 as the $\mathbb{Z}$-linear combination of them.



Extended Dynkin diagrams of irreducible affine root systems

An affine apartment is said to be *of type* $\widetilde{X}_n$ (or *of type* X_n) if it is isomorphic to $\mathscr{A}(\Sigma)$ for an irreducible affine root system Σ of type $\widetilde{X}_n$.

1.5 Root data

Root systems can arise from root data. This subsection focus on root data.

Definition 1.5.1. A *(reduced) root datum*[5] $\mathcal{R}$ is a quadruple $(\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ in which

- X and $\mathsf{X}^\vee$ are free $\mathbb{Z}$-modules of finite rank in duality by a pairing

$$\langle \cdot , \cdot \rangle \colon \mathsf{X} \times \mathsf{X}^\vee \to \mathbb{Z},$$

- Φ and $\Phi^\vee$ are finite subsets of $\mathsf{X} \setminus \{0\}$ and $\mathsf{X}^\vee \setminus \{0\}$ respectively, in bijection by a correspondence $a \leftrightarrow a^\vee$,

satisfying

RD1. for any $a \in \Phi$, $\langle a, a^\vee \rangle = 2$;

RD2. for any $a \in \Phi$, the "reflection" $r_a \colon x \mapsto x - \langle x, a^\vee \rangle a$ preserves Φ and the "reflection" $r_a \colon y \mapsto y - \langle a, y \rangle a^\vee$ preserves $\Phi^\vee$;

RD3. for any $a \in \Phi$, $\mathbb{Z}a \cap \Phi = \{\pm a\}$.

Note that we do not distinguish the two kinds of "reflections" in symbols since they form isomorphic finite groups of automorphisms on X and $\mathsf{X}^\vee$ respectively and therefore it is better to view them as two representations of a same finite group ${}^{v}W(\mathcal{R})$. This group is called the *Weyl group* of the root datum.

1.5.2. If $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ is a root datum, then its Weyl group acts on the real vector space $\mathsf{X}^\vee_{\mathbb{R}} := \mathsf{X}^\vee \otimes \mathbb{R}$ and there is a unique inner product on it invariant under the action. Let $\mathbb{V}$ be the subspace of $\mathsf{X}^\vee_{\mathbb{R}}$ spanned by $\Phi^\vee$, called the *coroot space* of $\mathcal{R}$. Then Φ is a (reduced) root system on the Euclidean vector space $\mathbb{V}$.

In general, $\mathbb{V}$ is not the entire $\mathsf{X}^\vee_{\mathbb{R}}$. When it is, we say $\mathcal{R}$ is *semisimple*. So the apartment associated to root systems can also be viewed as the apartment associated to

[5]in the sense of [SGA3, XXI, 1.1.1].

semisimple root data. As for the non-semisimple ones, they give rise to non-essential apartments and hence are ignored in this note.

The quadruple $\mathcal{R}^\vee = (\mathsf{X}^\vee, \Phi^\vee, \mathsf{X}, \Phi)$ is also a root datum, called the *dual root datum of* $\mathcal{R}$. It is clear that dual root data give rise to coroot systems on the dual spaces.

1.5.3. Let $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ and $\mathcal{R}' = (\mathsf{X}', \Phi', \mathsf{X}'^\vee, \Phi'^\vee)$ be two root data. Then a *morphism* $f\colon \mathcal{R}' \to \mathcal{R}$ between them is a linear map $f\colon \mathsf{X}' \to \mathsf{X}$ inducing a bijection $\Phi \to \Phi'$ and its transpose ${}^t f$ induces a bijection $\Phi'^\vee \to \Phi^\vee$. If f is a morphism of root data, then it also induces bijections between bases, systems of positive roots and Weyl chambers [SGA3, XXI, 6.1.3]. As a consequence, it induces an isomorphism of spherical apartments ${}^v\!\mathscr{A}(\Phi') \cong {}^v\!\mathscr{A}(\Phi)$ (and also an isomorphism of affine apartments $\mathscr{A}(\Sigma') \cong \mathscr{A}(\Sigma)$ if $\Phi = {}^v\Sigma$ and $\Phi' = {}^v\Sigma'$ with covariant choice of Γ_a's).

A morphism of root data $f\colon \mathcal{R}' \to \mathcal{R}$ is an *isogeny*[6] if the linear map f is injective and has finite cokernel $\mathrm{K}(f)$. This $\mathrm{K}(f)$ is also called the *cokernel* of f.

1.5.4. Let $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ be a root datum. Let $\mathsf{X}_0 = \{x \in \mathsf{X} \mid \langle x, \Phi^\vee \rangle = 0\}$ and $\mathsf{X}_0^\vee = \mathsf{X}^\vee / (\mathbb{V} \cap \mathsf{X}^\vee)$. Then X_0 and $\mathsf{X}_0^\vee$ are in duality by the pairing of $\mathcal{R}$ and thus give a trivial root datum $(\mathsf{X}_0, \emptyset, \mathsf{X}_0^\vee, \emptyset)$. It is called the *coradical of* $\mathcal{R}$ and is denoted by $\operatorname{corad}(\mathcal{R})$.

The dual root datum of the coradical of the dual $\mathcal{R}^\vee = (\mathsf{X}^\vee, \Phi^\vee, \mathsf{X}, \Phi)$ is called the *radical of* $\mathcal{R}$ and is denoted by $\operatorname{rad}(\mathcal{R})$. More precisely, let $\mathsf{Y}_0 = \left\{y \in \mathsf{X}^\vee \,\middle|\, \langle \Phi, y \rangle = 0\right\}$ and $\mathsf{Y}_0^\vee = \mathsf{X}/(\mathbb{V}^* \cap \mathsf{X})$, then $\operatorname{rad}(\mathcal{R})$ is the root datum $(\mathsf{Y}_0^\vee, \emptyset, \mathsf{Y}_0, \emptyset)$. It follows that $\mathcal{R}$ is semisimple if and only if $\operatorname{corad}(\mathcal{R}) = 0$ if and only if $\operatorname{rad}(\mathcal{R}) = 0$.

Let $\mathcal{R}^0$ denote the trivial root datum $(\mathsf{X}, \emptyset, \mathsf{X}^\vee, \emptyset)$. Then the inclusion and projection to X induce morphism of root data

$$\operatorname{corad}(\mathcal{R}) \longrightarrow \mathcal{R}^0 \longrightarrow \operatorname{rad}(\mathcal{R}).$$

and the composition $\operatorname{corad}(\mathcal{R}) \to \operatorname{rad}(\mathcal{R})$ is an isogeny [SGA3, XXI, 6.3.4]. Its cokernel

[6]in the sense of [SGA3, XXI, 6.2.1] and is called a *central isogeny* in [Mil17, 23.2]

is denoted by $\mathrm{N}(\mathcal{R})$. Note that there is a pairing:

$$\mathrm{N}(\mathcal{R}) \times \mathrm{N}(\mathcal{R}^\vee) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

1.5.5. A *lattice* $\mathcal{L}$ in a $\mathbb{R}$-vector space $\mathbb{V}$ is a discrete finitely generated $\mathbb{Z}$-submodule of $\mathbb{V}$ spanning $\mathbb{V}$. Its *dual lattice* $\mathcal{L}^*$ is the lattice in the dual space $\mathbb{V}^*$ consisting of those functionals $f \in \mathbb{V}^*$ such that $f(\mathcal{L}) \subseteq \mathbb{Z}$.

Given a root system Φ on a Euclidean vector space $\mathbb{V}$, there are four lattices:

$\mathcal{Q}$	the *root lattice*, which is the lattice in $\mathbb{V}^*$ generated by the roots;
$\mathcal{Q}^\vee$	the *coroot lattice*, which is the lattice in $\mathbb{V}$ generated by the coroots;
$\mathcal{P}$	the *weight lattice*, which is the dual lattice of $\mathcal{Q}^\vee$ in $\mathbb{V}^*$;
$\mathcal{P}^\vee$	the *coweight lattice*, which is the dual lattice of $\mathcal{Q}$ in $\mathbb{V}$.

Suppose the root system Φ is given by a root data $\mathcal{R}$. Then X contains $\mathcal{Q}$. If $\mathcal{R}$ is semisimple, then X is a lattice in $\mathbb{V}^*$ between $\mathcal{Q}$ and $\mathcal{P}$. In this case, the quotient $\mathcal{P}/\mathsf{X}$ is a finite group $\pi_1(\mathcal{R})$, called the *fundamental group of* $\mathcal{R}$; the quotient $\mathsf{X}/\mathcal{Q}$ is a finite group $Z(\mathcal{R})$, called the *centre of* $\mathcal{R}$.

1.5.6. Let $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ be a root datum.

Let Y be a submodule of X containing Φ. Let $i\colon \mathsf{Y} \to \mathsf{X}$ be the inclusion with the transpose ${}^t i\colon \mathsf{X}^\vee \to \mathsf{Y}^\vee$ and let $\Phi_{\mathsf{Y}} = \Phi$, $\Phi^\vee_{\mathsf{Y}} = {}^t i(\Phi^\vee)$. Then $(\mathsf{Y}, \Phi_{\mathsf{Y}}, \mathsf{Y}^\vee, \Phi^\vee_{\mathsf{Y}})$ is a root datum and i is a morphism of root data. It is called *the root datum induced by* $\mathcal{R}$ *on* Y and is denoted by $\mathcal{R}_{\mathsf{Y}}$.

Let $\mathsf{Y}^\vee$ be a submodule of $\mathsf{X}^\vee$ containing $\Phi^\vee$. Then the dual root datum of the root datum induced by $\mathcal{R}^\vee$ on $\mathsf{Y}^\vee$ is called *the root datum coinduced by* $\mathcal{R}$ *on* $\mathsf{Y}^\vee$ and is denoted by $\mathcal{R}^{\mathsf{Y}^\vee}$.

The following are some special cases of above.

$\mathrm{ad}(\mathcal{R})$	the root datum induced by $\mathcal{R}$ on the root lattice $\mathcal{Q}$;

$\mathrm{ss}(\mathcal{R})$	the root datum induced by $\mathcal{R}$ on $\mathbb{V}^* \cap \mathsf{X}$;
$\mathrm{der}(\mathcal{R})$	the root datum coinduced by $\mathcal{R}$ on $\mathbb{V} \cap \mathsf{X}^\vee$;
$\mathrm{sc}(\mathcal{R})$	the root datum coinduced by $\mathcal{R}$ on the coroot lattice $\mathcal{Q}^\vee$.

1.5.7. We have seen various root data constructed from a given one $\mathcal{R}$. They form a diagram of morphisms of root data:

$$\begin{tikzcd}
\mathrm{ad}(\mathcal{R}) \arrow[r] & \mathrm{ss}(\mathcal{R}) \arrow[rr] \arrow[d] \arrow[dr] & & \mathrm{der}(\mathcal{R}) \arrow[r] \arrow[d] & \mathrm{sc}(\mathcal{R}) \\
 & \mathrm{ss}(\mathcal{R}) \times \mathrm{corad}(\mathcal{R}) \arrow[r] & \mathcal{R} \arrow[r] \arrow[ur] & \mathrm{der}(\mathcal{R}) \times \mathrm{rad}(\mathcal{R}) &
\end{tikzcd}$$

Moreover, we have the following propositions [SGA3, XXI, 6.5.5 – 6.5.9].

(i) The horizontal ones are isogenies between root data.

(ii) The diagram is commutative.

(iii) $\mathrm{ad}(\mathcal{R})$ is *adjoint*, namely every isogeny to it is an isomorphism.

(iv) $\mathrm{sc}(\mathcal{R})$ is *simply-connected*, namely every isogeny from it is an isomorphism.

(v) $\mathcal{R}$ is semisimple if and only if the middle triangle consists of isomorphisms.

(vi) If $\mathcal{R}$ is semisimple, its centre $Z(\mathcal{R})$ and fundamental group $\pi_1(\mathcal{R})$ are the cokernels of the isogenies $\mathrm{ad}(\mathcal{R}) \to \mathcal{R}$ and $\mathcal{R} \to \mathrm{sc}(\mathcal{R})$ respectively.

(vii) The cokernels of the isogenies $\mathrm{ss}(\mathcal{R}) \times \mathrm{corad}(\mathcal{R}) \to \mathcal{R}$, $\mathcal{R} \to \mathrm{der}(\mathcal{R}) \times \mathrm{rad}(\mathcal{R})$ and $\mathrm{ss}(\mathcal{R}) \to \mathrm{der}(\mathcal{R})$ are all isomorphic to $\mathrm{N}(\mathcal{R})$.

(viii) $\mathcal{R}$ is the product of a semisimple root datum with a trivial root datum if and only if $\mathrm{N}(\mathcal{R}) = 0$.

(ix) All root data in this diagram have isomorphic root systems and hence isomorphic apartments.

1.6 Euclidean buildings

In this subsection, a geometric definition of buildings (the Euclidean buildings) is given and its properties are further discussed.

Definition 1.6.1. A *(Euclidean) building* is a set $\mathscr{B}$ equipped with a polysimplicial complex $\mathcal{F}$, whose members are subsets of $\mathscr{B}$ and are called *facets*, and a family $\mathcal{A}$ of subsets of $\mathscr{B}$, whose members are called *apartments*, such that the following axioms are satisfied.

EB0. For each apartment $A \in \mathcal{A}$, there is a Euclidean apartment $\mathscr{A}$ together with a bijection between them, exchanging the complex $\mathcal{F}_A$ of facets contained in A and the complex of facets in $\mathscr{A}$.

Note that, this allows us to view each apartments in $\mathscr{B}$ as Euclidean affine spaces and hence it makes sense to talk about isometries between them.

EB1. For any two facets F, F', there is an apartment A containing them.

EB2. If A, A' are two apartments containing both F and F', then there is an isomorphism between A and A' fixing F and F' pointwise.

Here an isomorphism between A and A' is an isometry between them exchanging the posets $\mathcal{F}_A$ and $\mathcal{F}_{A'}$.

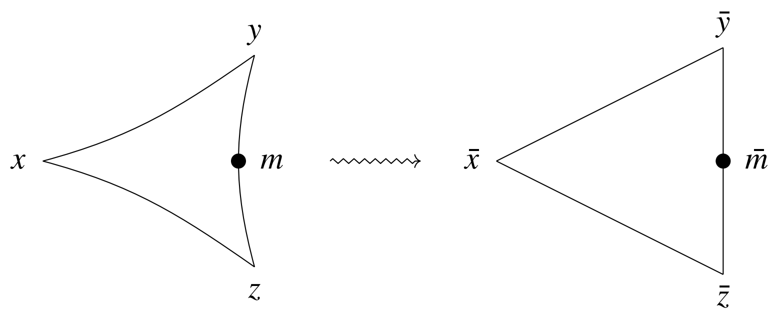
Note that, from the definition, all apartments $A \in \mathcal{A}$ are isomorphic to an abstract one $\mathscr{A}$. Then $\mathscr{B}$ is said to be *of type* $\mathscr{A}$ and is said to be *spherical* (resp. *discrete*, *affine*, etc.) if so is $\mathscr{A}$. The Weyl group W of $\mathscr{A}$ is also called the *Weyl group* of $\mathscr{B}$.

Remark. The notions of *walls*, *chambers*, *vertices* and *types* in a building is defined similarly as in an apartment and we will use the same notations as there. Furthermore, there is a *type function* $\tau\colon \mathcal{F} \to \mathcal{T}$ extending the type function on an apartment to the entire building uniquely.

Remark. We have assumed that apartments are essential. In particular, the buildings in Bruhat-Tits theory used in this note are the *reduced buildings*, rather than *extended buildings*. However, this is harmless as we focus more on the polysimplicial structure and we do want the vertices being points.

Remark. One can see that a discrete building $\mathscr{B}$ is completely determined by its combinatorial information, which is encoded in the polysimplicial complex $\mathcal{F}$ up to a choice of the family $\mathcal{A}$. To see this, one can compare the axioms **EB0.–EB2.** with **B0.–B2.**. Therefore to give a discrete Euclidean building $(\mathscr{B}, \mathcal{A})$ is equivalent to give an abstract building $(\mathcal{F}, \{\mathcal{F}_A\}_{A\in\mathcal{A}})$.

1.6.2. The apartments are Euclidean affine spaces, hence have metrics. Those metrics are compatible in the sense that the agree on any overlap, hence are glued into a metric $d(-,-)$ on the entire building $\mathscr{B}$ in a consistent way. Then $\mathscr{B}$ equipped with this metric is a complete metric space having the *CAT(0)-property* [Rou09, 6.5], which means that geodesic triangles in $\mathscr{B}$ are at least as thin as in a Euclidean plane: saying x, y, z are three points in $\mathscr{B}$ forming a geodesic triangle and $\bar{x}, \bar{y}, \bar{z}$ are three points in a Euclidean plane having the same pointwise distance as x, y, z, then for any point m in the geodesic segment $[x, y]$ in the triangle and $\bar{m}$ the corresponding point in the segment $[\bar{x}, \bar{y}]$ (namely, $d(\bar{x}, \bar{m}) = d(x, m)$), then $d(z, m) \leqslant d(\bar{z}, \bar{m})$.



Consequences of the CAT(0)-property include: the geodesic segments between points are unique [Rou09, 6.6]; any group of isometries stabilizing a nonempty bounded subset has a fixed point [Rou09, 7.1]; the distance from a point to a nonempty closed convex subset is achieved by a unique point [Rou09, 7.3]. For more details, see [Rou09, §6 and 7].

1.6.3. A *morphism* between buildings $\mathscr{B}$ and $\mathscr{B}'$ is a continuous map inducing a *chamber map* between $\mathcal{F}$ and $\mathcal{F}'$ and maps apartments in apartments. Then an *automorphism* of a building is an isometry transforming a facet (resp. apartment) in a facet (resp. apartment). Any building can be decomposed into a product of a trivial building with irreducible ones, similarly as in 1.3.3. However, there is no guarantee that such a decomposition gives a good corresponding decomposition on the family $\mathcal{A}$.

Remark. With above definition of morphisms, we obtain an equivalence of categories between discrete Euclidean buildings and abstract buildings.

1.6.4. An automorphism is said to be *type-preserving* if it leaves the type function τ invariant. For instance, any $w \in W$ is such an automorphism. A group G of automorphisms is said to be *strongly transitive* if it acts transitively on the pairs (C, A) where C is a chamber in the apartment A. This is the case if and only if G acts transitively on apartments and in any apartment A, the following conditions for a pair of chambers C, C' in A are equivalent:

(i) C and C' are conjugated by the Weyl group W;

(ii) C and C' are conjugated by the stabilizer $N_G(A)$ of A in G;

(iii) C and C' are conjugated by G.

When a group G of automorphisms is strongly transitive and type-preserving, we have

$$W \cong N_G(A)/C_G(A),$$

where $C_G(A)$ is the fixator of an apartment A in G.

1.6.5. Let G be a strongly transitive and type-preserving group of automorphisms and F be a facet in an apartment A. The stabilizer (which is also the fixator)

$$G_F := N_G(F) = C_G(F)$$

of F is called a *parabolic subgroup* of G. The parabolic group G_F acts transitively on the apartments containing F. Indeed, one can deduce this from the fact that G_F acts transitively on chambers containing F since G is strongly transitive. Here the former is due to that G acts transitively on chambers and is type-preserving.

Moreover, we have the *Bruhat decomposition* [Rou09, 6.9]

$$G = G_F.N_G(A).G_F.$$

In particular, if $F = C$ is a chamber, then

$$G = \bigsqcup_{w \in W} G_C w G_C.$$

1.6.6. Let x be a point in an affine building $\mathscr{B}$. The the *link of* x is the subcomplex $\mathcal{F}_x$ of the facets covering x. For any apartment A, let $\mathcal{F}_{x,A}$ be the subcomplex $\mathcal{F}_x \cap \mathcal{F}_A$, where $\mathcal{F}_A$ is as in Definition 1.6.1. Then $\mathcal{F}_x$ is an abstract spherical building with the system of apartments $\left\{\mathcal{F}_{x,A}\right\}_{A\in\mathcal{A}}$. To see this, recall that for any vertex x in an affine Euclidean apartment, the facets in the spherical apartment at x can be identified with the facets covering x through a radially shrinking with center x. In this way, we obtain a spherical building $\mathscr{B}_x$, called the *spherical building at* x. Note that the embedding $\mathscr{B}_x \to \mathscr{B}$ is not isometric, only conformal.

1.6.7. A *bornology* on a set X is a collection $\mathcal{B}$ of subsets of X such that it covers X and is stable under inclusion and finite unions. Once such a bornology is chosen, its members are called *bounded subsets* of X. For instance, any metric space has a canonical bornology induced by its metric. Another example is any locally comacpt topological space, where the bornology consists of all relatively compact subsets. A *morphism* between bornological sets is a map preserving the bornologies.

A *bornological group* is a group G equipped with a bornology on it stable under multiplication. For instance, let G be an isometry group on a metric space X, then there is a canonical bornology whose members are subsets M such that the set $M.x$ is bounded in X for some $x \in X$.

Let $\varphi\colon G' \to G$ be a group homomorphism and G a bornological group. Then we can canonically pullback the bornology on G to G': a subset of G' is bounded when its image is bounded in G.

So we can talk about *bounded subgroups* of a group G acting on the building $\mathscr{B}$ regardless its own topology or bornology. But if G is topological or bornological, it makes sense to ask if its bornology is the same as the pullback one. It worth to point out that this is the case when G acts continuously on $\mathscr{B}$.

Let G be a strongly transitive and type-preserving group of automorphisms of $\mathscr{B}$, then for any subgroup H of G, the following conditions are equivalent [Rou09, 7.2].

(i) H is bounded;

(ii) H fixes a point in $\mathscr{B}$;

(iii) H is contained in a parabolic subgroup of G.

In particular, the maximal bounded subgroups of G are the maximal parabolic subgroups and hence the stabilizers of vertices.

In general, even if G is not type-preserving, then maximal bounded subgroups of G are still stabilizers of points, but: *1,* not all such stabilizers are maximal; *2,* not all such stabilizers are stabilizers of vertices.

§ 2 Reductive groups and Tits buildings

Tits' building theory [Tit74, Bourbaki] was applied to study the structure of reductive groups over an arbitrary field, a family of linear algebraic groups which play important roles in mathematics. We refer to [Mil17] for algebraic groups and reductive groups over an arbitrary field and [SGA3] for group schemes and reductive group schemes over general base.

Throughout this section, we fix a ground field K and an algebraic closure K^{a} (resp. separable closure K^{s}) of it.

2.1 Algebraic groups

We first recall some basic notions on algebraic groups.

Definition 2.1.1. By an *algebraic group* [7] (defined over K), we mean a group object in the category of schemes of finite type over K.

An algebraic group is said to be *affine* (resp. *smooth*, *connected*, etc.) if so is its underlying scheme [8]. But it is often useful to have another viewpoint: an algebraic groups is in particular a group-valued functor from the category $\mathbf{Alg}_K$ of finitely generated K-algebras. In particular, affine algebraic groups are precisely the representable group-valued functors.

We will use bold letters like G to denote algebraic groups defined over K. For any K-algebra R, the group scheme obtained by base change $\mathsf{G} \otimes_K R$ is denoted by G_R and the group of R-points is denoted by $\mathsf{G}(R)$ (but if we use notations with parenthesis, e.g.

[7] Generally, a *group scheme* over a base S is a group object in the category of schemes over S. The materials covered in this subsection works over general base, not only over K. Just in case, we write $(\,\cdot\,)/S$ to emphasize the base S.

[8] It worthes to emphasize that the topological terminology such as *connected* talks about the underlying scheme of G, not the underlying set of G (although it may carry a topological structure). For instance, when the ground field K is a local field, the multiplicative group $\mathbb{G}_{\mathrm{m}}$ (see Example 2.1.3) is connected while the topological group $K^{\times}$ is totally-disconnected.

$\mathsf{GL}(V)$, to denote an algebraic group, then its group of R-points is denoted by padding R into the parenthesis as the last parameter, e.g. $\mathsf{GL}(V, R)$). Moreover, $\mathsf{G}(K)$ is simply denoted by G and $\mathsf{G}_R(R) \cong \mathsf{G}(R)$ is simply denoted by G_R. We also write $g \in \mathsf{G}$ to mean that g is an R-point of G for some K-algebra R.

Many group-theoretical constructions apply to algebraic groups. For G an algebraic group and H a subgroup, we use $\mathsf{N}_{\mathsf{G}}(\mathsf{H})$ (resp. $\mathsf{Z}_{\mathsf{G}}(\mathsf{H})$) to denote *normalizer* (resp. *centralizer*) of H in G. In particular, $\mathsf{Z}(\mathsf{G})$ denote the centre of G.

2.1.2. Let G be an algebraic group. Its *neutral component* G° is the largest connected subgroup of G. Its *component group* $\pi_0(\mathsf{G})$ is the universal étale scheme under G. Then there is an exact sequence [Mil17, 2.37]:

$$1 \longrightarrow \mathsf{G}^\circ \longrightarrow \mathsf{G} \longrightarrow \pi_0(\mathsf{G}) \longrightarrow 1.$$

The above formations are compatible with field extensions and products.

The following conditions on an algebraic group G are equivalent [Mil17, 1.36]:

(i) G is irreducible;

(ii) G is connected;

(iii) G is geometrically connected;

(iv) $\pi_0(\mathsf{G})$ equals the trivial group 1.

Example 2.1.3. Here we give some algebraic groups presented as functors.

(a) The functor $R \rightsquigarrow (R, +)$ mapping a K-algebra to its underlying abelian group defines an algebraic group $\mathbb{G}_{\mathrm{a}}$, called the *additive group*.

(b) The functor $R \rightsquigarrow (R^\times, \times)$ mapping a K-algebra to its unit group defines an algebraic group $\mathbb{G}_{\mathrm{m}}$, called the *multiplicative group*.

(c) The functor $R \rightsquigarrow \{r \in R \mid r^n = 1\}$ mapping a K-algebra to its set of n-th roots of unity defines an algebraic group μ_n, called the *group of n-th roots of unity*.

(d) Let G be a finite group. The constant functor $R \rightsquigarrow G$ is not a scheme, but its sheafification $R \rightsquigarrow \mathrm{Map}(\pi_0(R), G)$, where $\pi_0(R)$ is the set of connected components of $\mathrm{Spec}(R)$, defines an algebraic group $\underline{G}$. Such an algebraic group is called a *constant algebraic group*.

(e) Let V be a finite-dimensional vector space over K, then the functor $R \rightsquigarrow V_R := V \otimes_K R$ defines an algebraic group $\mathbb{W}(V)$, called the *additive group of* V. Any choice of basis of V gives rise to an isomorphism from this group to a product of copies of $\mathbb{G}_{\mathrm{a}}$.

(f) The functor mapping a K-algebra R to the additive group of $m \times n$ matrices with entries in R defines an algebraic group $\mathsf{M}_{m\times n}$.

(g) Let V be a finite-dimensional vector space over K, then the functor $R \rightsquigarrow \mathrm{End}(V_R)$ defines an algebraic group $\mathsf{End}(V)$. When V is of dimension n, any choice of basis of V gives an isomorphism from this group to $\mathsf{M}_{n\times n}$.

(h) The functor mapping a K-algebra R to the group of invertible $n \times n$ matrices with entries in R defines an algebraic group GL_n, called the *general linear group*.

(i) Let V be a finite-dimensional vector space over K, then the functor $R \rightsquigarrow \mathrm{Aut}(V_R)$ defines an algebraic group $\mathsf{GL}(V)$, called the *general linear group of* V. When V is of dimension n, any choice of basis of V gives rise to an isomorphism from this group to GL_n.

All above functors are representable. Hence above algebraic groups are affine.

2.1.4. A *representation* of an algebraic group G is a homomorphism of group-valued functors $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$, where V is a vector space over K and $\mathsf{GL}(V)$ is the functor $R \rightsquigarrow \mathrm{Aut}(V_R)$. When V is finite-dimensional, this is a homomorphism of algebraic groups. Such a representation is *faithful* if ρ is injective.

An algebraic group is *linear* if it admits a finite-dimensional faithful representation. Equivalently, an algebraic group is linear if it is isomorphic to an algebraic subgroup of some GL_n. It turns out that [Mil17, 1.43 and 4.10]:

$$\text{affine algebraic group} = \text{linear algebraic group}.$$

Example 2.1.5. Here we give some linear algebraic groups.

(a) The functor $R \rightsquigarrow \{g \in \mathsf{GL}_n(R) \mid \det(g) = 1\}$ mapping a K-algebra R to the group of invertible $n\times n$ matrices with entries in R and determinant 1 defines an algebraic subgroup SL_n of GL_n, called the *special linear group*.

(b) The functor $R \rightsquigarrow \left\{(g_{ij}) \in \mathsf{GL}_n(R) \,\middle|\, g_{ij} = 0 \text{ if } i > j\right\}$ mapping a K-algebra R to the group of upper triangular invertible $n \times n$ matrices with entries in R defines an algebraic subgroup T_n of GL_n.

(c) The functor $R \rightsquigarrow \left\{(g_{ij}) \in \mathsf{GL}_n(R) \,\middle|\, g_{ij} = 0 \text{ if } i > j \text{ and } g_{ij} = 1 \text{ if } i = j\right\}$ mapping a K-algebra R to the group of upper triangular invertible $n \times n$ matrices with entries in R and diagonal entries 1 defines an algebraic subgroup U_n of T_n.

(d) The functor $R \rightsquigarrow \{\mathrm{diag}(t_1, \cdots, t_n) \in \mathsf{GL}_n(R)\}$ mapping a K-algebra R to the group of invertible diagonal $n \times n$ matrices with entries in R defines an algebraic subgroup D_n of T_n. Note that $\mathsf{D}_n \cong \mathbb{G}_{\mathrm{m}}^n$.

(e) The functor $R \rightsquigarrow \{g \in \mathsf{GL}(V, R) \mid \det(g) = 1\}$ mapping a K-algebra R to the group of R-automorphisms of V_R having determinant 1 defines an algebraic subgroup $\mathsf{SL}(V)$ of $\mathsf{GL}(V)$, called the *special linear group of* V.

(f) The quotient of GL_n (resp. $\mathsf{GL}(V)$) by the normal subgroup of scalars is a linear algebraic group. It is denoted by PGL_n (resp. $\mathsf{PGL}(V)$) and is called the *projective linear group*.

2.1.6. An algebraic group G is *unipotent* if every finite-dimensional representation $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$ is *unipotent*, namely there exists a G-stable flag

$$V = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_{m-1} \supsetneq V_m = 0,$$

such that G acts trivially on each factor V_i/V_{i+1}. Equivalently, an algebraic group is unipotent if it is isomorphic to an algebraic subgroup of some U_n.

For any $g \in \mathsf{G}(K^{\mathrm{a}})$, we have *Jordan–Chevalley decomposition* [Mil17, 9.18]: there exist unique elements $g_s, g_u \in \mathsf{G}(K^{\mathrm{a}})$ such that

$$g = g_s g_u = g_u g_s,$$

and for any representation $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$, the linear operator $\rho(g_s)$ is semisimple and $\rho(g_u)$ is unipotent. An element $g \in \mathsf{G}(K^{\mathrm{a}})$ is said to be *semisimple* (reps. *unipotent*) if $g = g_s$ (resp. $g = g_u$). A smooth algebraic group G is unipotent if and only if all elements of $\mathsf{G}(K^{\mathrm{a}})$ are unipotent [Mil17, 14.12].

2.1.7. An algebraic group is a *torus* if it becomes isomorphic to a product of copies of $\mathbb{G}_{\mathrm{m}}$ over some field containing K. A torus over K is *split* if it is already isomorphic to a product of copies of $\mathbb{G}_{\mathrm{m}}$ over K.

An algebraic group G is *diagonalizable* if its every representation is *diagonalizable*, namely it is a sum of one-dimensional representations. Equivalently, an algebraic group is diagonalizable if it is isomorphic to an algebraic subgroup of some D_n.

An algebraic group G is *of multiplicative type* if it becomes diagonalizable over some field containing K. All tori are of multiplicative type. A smooth commutative algebraic group G is of multiplicative type if and only if all elements of $\mathsf{G}(K^{\mathrm{a}})$ are semisimple [Mil17, 12.21].

A *character* of an algebraic group G is a homomorphism $\chi\colon \mathsf{G} \to \mathbb{G}_{\mathrm{m}}$. Let χ and χ' be two characters of G, then the sum $\chi + \chi'$ is defined as

$$(\chi + \chi')(g) = \chi(g) \cdot \chi'(g), \qquad \forall g \in \mathsf{G}.$$

This is again a character and the set of characters is an abelian group, denoted by $\mathsf{X}(\mathsf{G})$. The *character group* of G is the abelian group

$$\mathsf{X}^*(\mathsf{G}) := \mathrm{Hom}\big(\mathsf{G}_{K^{\mathrm{s}}}, \mathbb{G}_{\mathrm{m},K^{\mathrm{s}}}\big).$$

A *cocharacter* of an algebraic group G is a homomorphism $\lambda\colon\ \mathbb{G}_{\mathrm{m}} \to \mathsf{G}$. Suppose G is commutative. Then the sum $\lambda + \lambda'$ of two cocharacters of G is defined as:

$$(\lambda + \lambda')(z) = \lambda(z) \cdot \lambda'(z), \qquad \forall z \in \mathbb{G}_{\mathrm{m}}\,.$$

This is again a cocharacter. The *cocharacter group* of G is then the abelian group

$$\mathsf{X}_*(\mathsf{G}) := \mathrm{Hom}\big(\mathbb{G}_{\mathrm{m},K^{\mathrm{s}}}, \mathsf{G}_{K^{\mathrm{s}}}\big).$$

Example 2.1.8. Let $\mathsf{G} = \mathsf{D}_n$. For each $1 \leqslant i \leqslant n$, define $\chi_i\colon\ \mathsf{D}_n \to \mathbb{G}_{\mathrm{m}}$ as the character

$$\mathrm{diag}(t_1, \cdots, t_n) \mapsto t_i$$

and $\lambda_i\colon\ \mathbb{G}_{\mathrm{m}} \to \mathsf{D}_n$ as the cocharacter

$$t \mapsto \mathrm{diag}(1, \cdots, t, \cdots, 1)$$

with t at the i-th position. Then

(i) characters of D_n are $\mathbb{Z}$-linear combinations of $\chi_1, \cdots, \chi_n$;

(ii) cocharacters of D_n are $\mathbb{Z}$-linear combinations of $\lambda_1, \cdots, \lambda_n$.

Therefore if T is a torus of dimension n, then its character group $\mathsf{X}^*(\mathsf{T})$ (resp. cocharacter group $\mathsf{X}_*(\mathsf{T})$) is isomorphic to $\mathbb{Z}^n$ and furthermore consists of all characters (resp. cocharacters) providing T is split.

Let χ be a character and λ be a cocharacter of T. Then the composition $\chi \circ \lambda$ is an endomorphism $t \mapsto t^{\langle \chi, \lambda \rangle}$ of $\mathbb{G}_{\mathrm{m}}$, which can be identified with the integer $\langle \chi, \lambda \rangle \in \mathbb{Z}$. In this way, we get a perfect pairing of $\mathbb{Z}$-modules

$$\langle\, \cdot\, ,\, \cdot\, \rangle\colon\ \mathsf{X}^*(\mathsf{T}) \times \mathsf{X}_*(\mathsf{T}) \to \mathbb{Z}.$$

making $\mathsf{X}^*(\mathsf{T})$ and $\mathsf{X}_*(\mathsf{T})$ in duality.

Example 2.1.9. Let G be a diagonalizable algebraic group. Then $\mathsf{X}(\mathsf{G})$ is a finitely generated abelian group and (here p is the characteristic of K) [Mil17, 12.5]

(i) G is smooth if and only if $\mathsf{X}(\mathsf{G})$ has no p-torsion;

(ii) G is connected if and only if $\mathsf{X}(\mathsf{G})$ has no torsion other than p-torsions;

(iii) G is smooth and connected if and only if $\mathsf{X}(\mathsf{G})$ is free.

Moreover, the functor $\mathsf{G} \rightsquigarrow \mathsf{X}(\mathsf{G})$ gives a contravariant equivalence from the category of diagonalizable algebraic groups to the category of finitely generated abelian groups [Mil17, 12.9].

More general, the functor $\mathsf{G} \rightsquigarrow \mathsf{X}^*(\mathsf{G})$ gives a contravariant equivalence from the category of algebraic groups of multiplicative type over K to the category of finitely generated $\mathbb{Z}$-modules equipped with a continuous action of the absolute Galois group of K [Mil17, 12.23].

In particular, an algebraic group of multiplicative type is a torus if and only if it is smooth and connected.

2.1.10. An algebraic group G is *trigonalizable* if its every finite-dimensional representation $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$ is *trigonalizable*, namely there exists a G-stable maximal flag

$$V = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_{\dim V-1} \supsetneq V_{\dim V} = 0.$$

Equivalently, an algebraic group is trigonalizable if it is isomorphic to an algebraic subgroup of some T_n [Mil17, 16.2]. All unipotent algebraic groups are trigonalizable.

An algebraic group G is *solvable* if it has a subnormal series

$$\mathsf{G} \supsetneq \mathsf{G}_0 \supsetneq \mathsf{G}_1 \supsetneq \cdots \supsetneq \mathsf{G}_m = 1$$

such that each factor $\mathsf{G}_i/\mathsf{G}_{i+1}$ is commutative. A solvable algebraic group G is *split* if it has a subnormal series (G_i) in which each factor $\mathsf{G}_i/\mathsf{G}_{i+1}$ is isomorphic to either $\mathbb{G}_\mathrm{a}$ or $\mathbb{G}_\mathrm{m}$. Hence split solvable algebraic groups are trigonalizable [Mil17, 16.52].

Any trigonalizable algebraic group G has a subnormal series (G_i) in which G_0 is unipotent, G/G_0 is diagonalizable and each factor $\mathsf{G}_i/\mathsf{G}_{i+1}$ is $(\mathsf{G}/\mathsf{G}_0)$-equivariantly embedded into $\mathbb{G}_\mathrm{a}$ [Mil17, 16.21]. Therefore trigonalizable algebraic groups are solvable. Conversely, every smooth connected solvable algebraic group becomes trigonalizable after some finite filed extension [Mil17, 16.30].

Example 2.1.11. T_n is trigonalizable and hence solvable. It has a normal series

$$\mathsf{T}_n \supsetneq \mathsf{U}_n = \mathsf{U}_n^{(0)} \supsetneq \mathsf{U}_n^{(1)} \supsetneq \cdots \supsetneq \mathsf{U}_n^{(m)} = 1,$$

where $m = \binom{n}{2}$ and for each $0 \leqslant r \leqslant m$,

$$\mathsf{U}_n^{(r)} \colon R \rightsquigarrow \left\{ (u_{ij}) \in \mathsf{U}_n(R) \,\middle|\, u_{ij} = 0 \text{ for } \tfrac{1}{2}(j-i-1)(2n-j+i)+i \leqslant r \right\}.$$

In which, U_n is the largest solvable normal subgroup of T_n (and is in fact smooth and connected), the quotient $\mathsf{U}_n/\mathsf{T}_n$ is isomorphic to D_n and each factor $\mathsf{U}_n^{(r)}/\mathsf{U}_n^{(r+1)}$ is isomorphic to $\mathbb{G}_\mathrm{a}$.

2.1.12 (Cohomology of algebraic groups)**.** Definition 2.1.1 is equivalent to say that an algebraic group is a locally presentable group-valued sheaf on the site K_{fppf} whose underlying category is $\mathbf{Alg}_K^{\mathrm{opp}}$ and is equipped with the *fppf topology*[9]. Let R be an object in this site, its *fppf covering* is a family of K-algebra homomorphisms $R \to R_i$ of finite presentation such that $R \to \prod_i R_i$ is faithfully flat. Therefore a functor F from $\mathbf{Alg}_K$ is a sheaf on K_{fppf} if and only if it satisfies the following [Mil17, 5.65].

(i) (*Local*) For any K-algebras $R_1, \cdots, R_m$,

$$\mathsf{F}(R_1 \times \cdots \times R_m) \cong \mathsf{F}(R_1) \times \cdots \times \mathsf{F}(R_m).$$

[9]The name *fppf* is short of "fidèlement plate de présentation finie", that is, "faithfully flat and of finite presentation" in English. Note that any finitely generated K-algebra R is noetherian, hence all morphisms of finite type in the category $\mathbf{Alg}_K$ are actually of finite presentation. This is not true for general base S and two two sheaf conditions (*Local* and *Decent*) need to be presented in more general form as in [SGA3, IV, 6.3.1].

(ii) (*Descent*) For any faithfully flat K-algebra homomorphism $R' \to R$, the sequence

$$\mathsf{F}(R) \longrightarrow \mathsf{F}(R') \rightrightarrows \mathsf{F}(R' \otimes_R R')$$

is exact, where the homomorphisms $R' \to R' \otimes_R R'$ are $r \mapsto r \otimes 1$ and $r \mapsto 1 \otimes r$ respectively.

Saying a sequence of algebraic groups

$$1 \longrightarrow \mathsf{N} \longrightarrow \mathsf{G} \longrightarrow \mathsf{Q} \longrightarrow 1.$$

is *exact* means N is isomorphic to the kernel of $\mathsf{G} \to \mathsf{Q}$ and $\mathsf{G} \to \mathsf{Q}$ is surjective as a sheaf homomorphism. The later turns out to say that $\mathsf{G} \to \mathsf{Q}$ is faithfully flat [Mil17, 5.43]. When we consider homomorphisms between smooth algebraic groups, this is equivalent to say that $\mathsf{G} \to \mathsf{Q}$ is surjective on closed points [Mil17, 1.71]. Hence to verify a homomorphism between smooth algebraic groups is surjective, it is sufficient to verify on K^{a}-points.

Hence in general we do not have a short exact sequence of the groups of K-points. Instead, there is a long exact sequence:

$$1 \to \mathsf{N}(K) \to \mathsf{G}(K) \to \mathsf{Q}(K) \to \mathrm{H}^1(K, \mathsf{N}) \to \mathrm{H}^1(K, \mathsf{G}) \to \mathrm{H}^1(K, \mathsf{Q}).$$

It turns out that [Mil17, 3.50], if G is a smooth algebraic group, then $\mathrm{H}^1(K, \mathsf{G})$ is canonically isomorphic to the *Galois cohomology* $\mathrm{H}^1(\Gamma, \mathsf{G}(K^{\mathrm{s}}))$ with $\Gamma = \mathsf{Gal}(K^{\mathrm{s}}/K)$. The following are some useful results in Galois cohomology.

(i) (*Hilbert's theorem 90*) If L/K is a Galois extension, then $\mathrm{H}^1(\mathsf{Gal}(L/K), L^\times) = 0$.

(ii) (*Lang's theorem* [Mil17, 17.98]) If G is a smooth connected algebraic group over a finite field K, then $\mathrm{H}^1(K, \mathsf{G}) = 0$.

(iii) ([Mil17, 25.61; BT-III, 4.3]) If G is a simply-connected semisimple group over a local field K, then $\mathrm{H}^1(K, \mathsf{G})$ vanishes.

2.1.13. An algebraic group is *vectorial* if it is isomorphic to a product of copies of $\mathbb{G}_{\mathrm{a}}$. Let V be a finite-dimensional vector space over K, then the algebraic group $\mathbb{W}(V)$ is a vectorial group. A vectorial group is in particular a vector bundle on K_{fppf}.

For V a vector bundle on K_{fppf}, let $\mathsf{V}^{\times}$ denote the open subscheme of V obtained by deleting the zero section. Then the action of $\mathbb{G}_{\mathrm{a}}$ on V induces an action of $\mathbb{G}_{\mathrm{m}}$ on $\mathsf{V}^{\times}$. In particular, if L is a one-dimensional vector space over K, then $\mathbb{W}(L)$ is a line bundle and $\mathbb{W}(L)^{\times}$ is a homogeneous principal $\mathbb{G}_{\mathrm{m}}$-bundle [SGA3, XIX, 4.3-4.4].

2.1.14. For R a K-algebra, its *algebra of dual numbers* is the algebra $R[\epsilon]/(\epsilon^2)$. Let $\mathscr{D}$ denote the functor sending each R to its algebra of dual numbers. For X a K-scheme, the composition $\mathsf{X} \circ \mathscr{D}$ is also a K-scheme, called the *tangent bundle of* X and is denoted by $\mathsf{T}(\mathsf{X})$. For any point x of X, the pullback of $\mathsf{T}(\mathsf{X})$ along $x \hookrightarrow \mathsf{X}$ is called the *tangent space of* X *at* x and is dented by $\mathsf{T}_x(\mathsf{X})$.

Let G be an algebraic group and e be its identity. Then both $\mathsf{T}(\mathsf{G})$ and $\mathsf{T}_e(\mathsf{G})$ are algebraic groups and we have a split short exact sequence [SGA3, II, 3.9.0.2]

$$\begin{tikzcd} 1 \arrow[r] & \mathsf{T}_e(\mathsf{G}) \arrow[r, "i"] & \mathsf{T}(\mathsf{G}) \arrow[r, "\mathrm{pr}"] & \mathsf{G} \arrow[r] \arrow[l, bend left, dotted, "s"] & 1. \end{tikzcd}$$

Let $\varphi\colon \mathsf{G} \to \mathsf{G}'$ be a homomorphism of algebraic groups, then there is a unique morphism of K_{fppf}-vector bundles $\mathrm{d}\varphi\colon \mathsf{T}_e(\mathsf{G}) \to \mathsf{T}_e(\mathsf{G}')$ making the following diagram commute

$$\begin{tikzcd} 1 \arrow[r] & \mathsf{T}_e(\mathsf{G}) \arrow[r] \arrow[d, dotted, "\mathrm{d}\varphi"] & \mathsf{T}(\mathsf{G}) \arrow[r] \arrow[d, "\mathsf{T}(\varphi)"] & \mathsf{G} \arrow[r] \arrow[d, "\varphi"] & 1. \\ 1 \arrow[r] & \mathsf{T}_e(\mathsf{G}') \arrow[r] & \mathsf{T}(\mathsf{G}') \arrow[r] & \mathsf{G}' \arrow[r] & 1. \end{tikzcd}$$

The morphism $\mathrm{d}\varphi$ is called the *differential of* φ. We will not distinguish it from the K-linear map on K-points $\mathrm{d}\varphi(K)\colon \mathsf{T}_e(\mathsf{G}, K) \to \mathsf{T}_e(\mathsf{G}', K)$.

Let $\mathfrak{g}$ denote the vector space $\mathsf{T}_e(\mathsf{G}, K)$, hence $\mathsf{T}_e(\mathsf{G}) = \mathbb{W}(\mathfrak{g})$. Then the action of G on itself by conjugations induces a representation $\mathrm{Ad}\colon \mathsf{G} \to \mathsf{GL}(\mathfrak{g})$ of G on $\mathfrak{g}$: for any $g \in \mathsf{G}$, the endomorphism $\mathrm{Ad}(g)$ is the differential of $\mathrm{inn}(g)$ (conjugated by g). This representation is called the *adjoint representation* of G [SGA3, II, 4.1]. Let

$\mathrm{ad}\colon \mathsf{T}_e(\mathsf{G}) \to \mathsf{End}(\mathfrak{g})$ denote the differential of the adjoint representation and for any $X, Y \in \mathfrak{g}$, define $[X,Y]$ as $\mathrm{ad}(X).Y$. Then this gives rise to a Lie bracket

$$\mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}\colon \qquad X,Y \longmapsto [X,Y].$$

This Lie algebra is called the *Lie algebra of G* and is denoted by $\mathrm{Lie}(\mathsf{G})$.

Example 2.1.15. The Lie algebras of $\mathbb{G}_{\mathrm{m}}$ and $\mathbb{G}_{\mathrm{a}}$ are the trivial Lie algebra K. The Lie algebras of GL_n, SL_n, T_n, U_n and D_n are the Lie algebras $\mathfrak{gl}_n$ of all matrices, $\mathfrak{sl}_n$ of trace zero matrices, $\mathfrak{t}_n$ of all upper triangular matrices, $\mathfrak{u}_n$ of strict upper triangular matrices and $\mathfrak{d}_n$ of all diagonal matrices respectively.

2.1.16. The above constructions give rise to an equivalence of categories between vectorial groups and finite-dimensional vector spaces over K [Mil17, 10.9]. Moreover, when K is of characteristic zero and G is a unipotent group over it, there is an isomorphism of schemes (and of algebraic groups if G is further commutative)

$$\exp\colon \mathsf{T}_e(\mathsf{G}) = \mathbb{W}(\mathrm{Lie}(\mathsf{G})) \longrightarrow \mathsf{G},$$

called the *exponential map* [Mil17, 14.32].

2.2 Reductive groups

Let's introduce the notion of reductive groups.

2.2.1. Let G be a smooth connected linear algebraic group.

(i) [Mil17, 6.44] There is a largest smooth connected solvable norm subgroup $\mathscr{R}(\mathsf{G})$ of G. It is called the *radical* of G.

(ii) [Mil17, 6.46] There is a largest smooth connected unipotent norm subgroup $\mathscr{R}_u(\mathsf{G})$ of G. It is called the *unipotent radical* of G.

Since unipotent groups are solvable, $\mathscr{R}_u(\mathsf{G})$ is a subgroup of $\mathscr{R}(\mathsf{G})$.

Definition 2.2.2. An algebraic group G is *reductive* (resp. *semisimple*) if its *geometric unipotent radical* $\mathscr{R}_u(\mathsf{G}_{K^a})$ (resp. *geometric radical* $\mathscr{R}(\mathsf{G}_{K^a})$) is trivial.

It turns out that the formations of $\mathscr{R}_u(\mathsf{G})$ and $\mathscr{R}(\mathsf{G})$ commute with separable field extensions [Mil17, 19.1 and 19.9]. Hence when K is perfect, G is reductive (resp. semisimple) if and only if $\mathscr{R}_u(\mathsf{G})$ (resp. $\mathscr{R}(\mathsf{G})$) is trivial.

Example 2.2.3. For any finite-dimensional vector space V, $\mathsf{SL}(V)$ is semisimple, while $\mathsf{GL}(V)$ is reductive but not semisimple.

Since any torus becomes a product of copies of $\mathbb{G}_{\mathrm{m}} = \mathsf{GL}_1$ over a finite field extension, it is reductive. Conversely, if G is a solvable reductive group, then since $\mathscr{R}_u(\mathsf{G}_{K^a})$ is trivial, it is a torus by [Mil17, 16.33].

2.2.4. Let G be a reductive group. There are various semisimple groups related to it.

(i) The radical $\mathscr{R}(\mathsf{G})$ is a *central* torus, namely it is contained in the centre $\mathsf{Z}(\mathsf{G})$. Therefore the quotient $\mathsf{G}/\mathsf{Z}(\mathsf{G})$ is semisimple. It is furthermore *adjoint*, namely it is semisimple with trivial centre, and is called the *adjoint group of* G with notation G^{ad}.

(ii) The radical $\mathscr{R}(\mathsf{G})$ turns out to be the largest subtorus of $\mathsf{Z}(\mathsf{G})$ and hence the formation of $\mathscr{R}(\mathsf{G})$ commute with field extensions [Mil17, 19.21]. Therefore the quotient $\mathsf{G}^{\mathrm{ss}} := \mathsf{G}/\mathscr{R}(\mathsf{G})$ is semisimple.

(iii) The derived group $\mathsf{G}^{\mathrm{der}}$ is semisimple [Mil17, 19.21]. Indeed, its geometric radical $\mathscr{R}\left(\mathsf{G}^{\mathrm{der}}_{K^a}\right)$ is normal in G_{K^a} hence $\mathscr{R}\left(\mathsf{G}^{\mathrm{der}}_{K^a}\right) \subseteq \mathscr{R}(\mathsf{G}_{K^a})$ and is central. But $\mathsf{Z}(\mathsf{G}) \cap \mathsf{G}^{\mathrm{der}}$ is finite hence $\mathscr{R}\left(\mathsf{G}^{\mathrm{der}}_{K^a}\right)$ is trivial.

Example 2.2.5. The above semisimple groups associated to $\mathsf{G} = \mathsf{GL}_n$ are the following:

(a) $\mathsf{Z}(\mathsf{GL}_n) \cong \mathbb{G}_{\mathrm{m}}$, hence $\mathsf{G}^{\mathrm{ad}} = \mathsf{PGL}_n$;

(b) $\mathscr{R}(\mathsf{GL}_n) = \mathsf{Z}(\mathsf{GL}_n)$, hence we obtain PGL_n again;

(c) the derived group of GL_n is SL_n.

2.2.6. Let G be a reductive group with $\mathsf{Z}(\mathsf{G})$ its centre, G^{ad} its adjoint group, $\mathsf{G}^{\mathrm{der}}$ its derived group, G^{Ab} its abelianization and let $\mathsf{Z}\left(\mathsf{G}^{\mathrm{der}}\right)$ be the centre of $\mathsf{G}^{\mathrm{der}}$. We have the following *deconstruction of* G [Mil17, 19.25]:

$$\begin{tikzcd}
\mathsf{Z}\left(\mathsf{G}^{\mathrm{der}}\right) \arrow[r, hook] \arrow[d, hook] & \mathsf{G}^{\mathrm{der}} \arrow[d, hook] \arrow[dr, two heads, bend left] & \\
\mathsf{Z}(\mathsf{G}) \arrow[r, hook] \arrow[dr, two heads, bend right] & \mathsf{G} \arrow[r, two heads] \arrow[d, two heads] & \mathsf{G}^{\mathrm{ad}} \\
& \mathsf{G}^{\mathrm{Ab}} &
\end{tikzcd}$$

where the square is bicartesian, namely $\mathsf{Z}\left(\mathsf{G}^{\mathrm{der}}\right) = \mathsf{Z}(\mathsf{G}) \cap \mathsf{G}^{\mathrm{der}}$ and $\mathsf{G} = \mathsf{Z}(\mathsf{G}) \cdot \mathsf{G}^{\mathrm{der}}$, and all rows and columns are exact sequences.

Conversely, suppose we have a triple $(\mathsf{H}, \mathsf{D}, \varphi)$ with H a semisimple algebraic group, D an algebraic group of multiplicative type, and $\varphi\colon \mathsf{Z}(\mathsf{H}) \to \mathsf{D}$ a monomorphism whose cokernel is a torus T. Then the homomorphism

$$\mathsf{Z}(\mathsf{H}) \longrightarrow \mathsf{H} \times \mathsf{D}\colon z \longmapsto (z, \varphi(z)^{-1})$$

is normal and its cokernel, denoted by G, is reductive and with the following deconstruction [Mil17, 19.27]

$$\begin{tikzcd}
\mathsf{Z}(\mathsf{H}) \arrow[r, hook] \arrow[d, hook] & \mathsf{H} \arrow[d, hook] \arrow[dr, two heads, bend left] & \\
\mathsf{D} \arrow[r, hook] \arrow[dr, two heads, bend right] & \mathsf{G} \arrow[r, two heads] \arrow[d, two heads] & \mathsf{H}^{\mathrm{ad}} \\
& \mathsf{T} &
\end{tikzcd}$$

Namely, $\mathsf{Z}(\mathsf{G}) \cong \mathsf{D}, \mathsf{G}^{\mathrm{ad}} \cong \mathsf{H}^{\mathrm{ad}}, \mathsf{G}^{\mathrm{der}} \cong \mathsf{H}$ and $\mathsf{G}^{\mathrm{Ab}} \cong \mathsf{T}$.

More generally, one can start from a triple $(\mathsf{H}, \mathsf{D}, \varphi)$ with φ not necessarily injective. Then we can replace H by the $\mathsf{H}/\mathrm{Ker}(\varphi)$ and everything follows.

2.2.7. Let G be a reductive group with radical $\mathscr{R}(\mathsf{G})$, semisimple quotient G^{ss}, derived group $\mathsf{G}^{\mathrm{der}}$ and abelianization G^{Ab}. Then by [Mil17, 12.46], $\mathsf{G} = \mathscr{R}(\mathsf{G}) \cdot \mathsf{G}^{\mathrm{der}}$ and hence we have another deconstruction of G:

$$\begin{tikzcd}
\mathscr{R}(\mathsf{G}) \cap \mathsf{G}^{\mathrm{der}} \arrow[r, hook] \arrow[d, hook] & \mathsf{G}^{\mathrm{der}} \arrow[d, hook] \arrow[dr, bend left, two heads] & \\
\mathscr{R}(\mathsf{G}) \arrow[r, hook] \arrow[dr, bend right, two heads] & \mathsf{G} \arrow[r, two heads] \arrow[d, two heads] & \mathsf{G}^{\mathrm{ss}} \\
& \mathsf{G}^{\mathrm{Ab}} &
\end{tikzcd}$$

In particular, a reductive group G is a product of a semisimple group and a torus if and only if $\mathscr{R}(\mathsf{G}) \cap \mathsf{G}^{\mathrm{der}} = 1$.

Example 2.2.8. Let $\mathsf{G} = \mathsf{GL}_n$. Then we have the following deconstruction

$$\begin{tikzcd}
\mu_n \arrow[r, hook] \arrow[d, hook] & \mathsf{SL}_n \arrow[d, hook] \arrow[dr, bend left, two heads] & \\
\mathbb{G}_{\mathrm{m}} \arrow[r, hook, "\lambda\mapsto\lambda I_n"] \arrow[dr, bend right, two heads, "\lambda\mapsto\lambda^n"'] & \mathsf{GL}_n \arrow[r, two heads] \arrow[d, two heads, "\det"] & \mathsf{PGL}_n \\
& \mathbb{G}_{\mathrm{m}} &
\end{tikzcd}$$

Conversely, GL_n can be recovered from the triple $(\mathsf{SL}_n, \mathbb{G}_{\mathrm{m}}, \mu_n \hookrightarrow \mathbb{G}_{\mathrm{m}})$.

Similar conclusion applies to $\mathsf{GL}(V)$.

2.2.9. Let G be a reductive group. It is *splittable* if it has a split maximal torus. A *split reductive group* is a pair (G, T) of a reductive group and a split maximal torus in it. A *homomorphism* between split reductive groups is a homomorphism of algebraic group preserving the split maximal torus. It turns out that, any two maximal split tori (hence split maximal tori if G is splittable) in G are conjugate by an element of G [Mil17, 25.10], while two (not necessarily split) maximal tori are only conjugate over a finite separable extension [Mil17, 17.87].

Let G be a splittable reductive group. Then its *rank* is the dimension of one (hence any) split maximal torus in it and its *semisimple rank* is the rank of $\mathsf{G}/\mathscr{R}(\mathsf{G})$. Since the centre $\mathsf{Z}(\mathsf{G})$ is contained in every maximal torus [Mil17, 17.61], the semisimple rank of G equals $\operatorname{rank}(\mathsf{G}) - \dim \mathsf{Z}(\mathsf{G})$.

Example 2.2.10. D_n is a split maximal torus in GL_n and it induces a split maximal torus in PGL_n by modulo the scalars $\mathbb{G}_\mathrm{m}$ and a split maximal torus in SL_n by intersecting with it. Hence GL_n is splittable with rank n and semisimple rank $n-1$.

Example 2.2.11 ([Mil17, 17.89])**.** Let V be a vector space over K of dimension n. Then the conjugacy classes of maximal tori in $\mathsf{GL}(V)$ are one-one corresponding to the isomorphism classes of étale K-algebras of degree n: a maximal torus T gives a decomposition $V = \bigoplus_i V_i$ into simple T-modules and thus finite separable extensions $K_i = \mathrm{End}_{\mathsf{T}}(V_i)$ and an étale K-algebra $A = \prod_i K_i$ of degree n; conversely, as V is a free A-module of rank 1, it decomposes into vector spaces V_i, one-dimensional over K_i, and the A-equivariant automorphisms preserving this decomposition form a maximal torus T such that $\mathsf{T}(K) = A^\times$.

In particular, the only conjugacy class of split maximal tori in $\mathsf{GL}(V)$ corresponds to the étale algebra K^n.

2.2.12. A homomorphism between smooth connected algebraic groups is said to be an *isogeny* if it is surjective and has finite kernel. An *isogeny of split reductive groups* $(\mathsf{G}', \mathsf{T}') \to (\mathsf{G}, \mathsf{T})$ is a homomorphism of split reductive groups such that $\varphi\colon \mathsf{G}' \to \mathsf{G}$ is an isogeny.

An isogeny is *central* if its kernel is central, namely contained in the centre, and is *multiplicative* if its kernel is of multiplicative type. A multiplicative isogeny is central (since every normal multiplicative subgroup of a connected algebraic group is central [Mil17, 12.38]) and the converse is true if its domain is reductive (since the centre of a reductive group is of multiplicative type [Mil17, 17.62]).

Let G be a smooth connected algebraic group. A *universal covering* on it is an multiplicative isogeny $\widetilde{\mathsf{G}} \to \mathsf{G}$ universal in the sense that no other multiplicative isogeny can factor through it. When the universal covering exists, its kernel is called the *fundamental group* $\pi_1(\mathsf{G})$ of G. If this group is trivial, namely, every multiplicative isogeny to G is an isomorphism, then we say G *simply connected*.

Example 2.2.13. In 2.2.6 and 2.2.7, the homomorphisms $\mathsf{G}^{\mathrm{der}} \to \mathsf{G}^{\mathrm{ad}}$, $\mathsf{G}^{\mathrm{der}} \to \mathsf{G}^{\mathrm{ss}}$, $\mathsf{Z}(\mathsf{G}) \to \mathsf{G}^{\mathrm{Ab}}$ and $\mathscr{R}(\mathsf{G}) \to \mathsf{G}^{\mathrm{Ab}}$ are isogenies. In particular, the homomorphism $\mathsf{SL}_n \to \mathsf{PGL}_n$ in Example 2.2.8 is a universal covering (hence $\pi_1(\mathsf{PGL}_n) \cong \mu_n$) and it induces a central isogeny of split reductive groups $(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) \to (\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_{\mathrm{m}})$.

2.3 Root systems and root groups

Given a split reductive group, there is a root system associated to it.

2.3.1 ([BT-II, 1.1.2 and 1.1.3])**.** Let G be an affine algebraic group and T be a split torus in it. Since T is diagonalizable, it acts (via the adjoint representation) on $\mathfrak{g} := \mathrm{Lie}(\mathsf{G})$ diagonalizably and we have a decomposition:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{a \in \mathsf{X}^*(\mathsf{T})} \mathfrak{g}_a,$$

where $\mathfrak{g}_0 = \mathfrak{g}^{\mathsf{T}}$ and $\mathfrak{g}_a$ is the subspace on which T acts through a nontrivial character a. A character a is a *root* if $\mathfrak{g}_a$ is nontrivial. The class of all positive real multiples of a root is called a *radical ray*. The set of all radical rays is denoted by $\Phi(\mathsf{G}, \mathsf{T})$, called the *root system* of the pair (G, T).

If (G, T) is a split reductive group, then $\mathfrak{g}_0 = \mathfrak{t} := \mathrm{Lie}(\mathsf{T})$ [Mil17, 10.34]. Moreover, any radical ray contains exactly one root and $\Phi(\mathsf{G}, \mathsf{T})$ can be further identified with a (reduced) root system, justifying its name.

Example 2.3.2. The pair $(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$ is a split reductive group with Lie algebra the one dimensional vector space K. The adjoint action of $\mathbb{G}_{\mathrm{m}}$ on K is trivial, hence the root system of $(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$ is empty.

Example 2.3.3. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$. The action of D_n on $\mathfrak{gl}_n := \mathrm{Lie}(\mathsf{GL}_n)$ is

$$(\mathrm{diag}(t_1, \cdots, t_n), (g_{ij})_{i,j}) \longmapsto (t_i g_{ij} t_j^{-1})_{i,j}.$$

By Example 2.1.8, the characters of D_n are of the form $c_1\chi_1 + \cdots + c_n\chi_n$. If $(g_{ij})_{i,j}$ is an eigenvector of $c_1\chi_1 + \cdots + c_n\chi_n$, then for any $t_1, \cdots, t_n \in R$, we have

$$\forall i, j : t_i g_{ij} t_j^{-1} = (t_1^{c_1} \cdots t_n^{c_n}) g_{ij}.$$

Therefore: *1,* the Lie algebra $\mathfrak{d}_n$ of D_n consists of all diagonal matrices; *2,* the root system $\Phi(\mathsf{GL}_n, \mathsf{D}_n) = \left\{\chi_i - \chi_j \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$; *3,* for each $a = \chi_i - \chi_j$, the Lie algebra $\mathfrak{g}_a$ is generated by E_{ij}, the matrix with 1 in the ij position and 0 elsewhere.

Example 2.3.4. Let's consider the split reductive group $(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m})$. A character $c_1\chi_1 + \cdots + c_n\chi_n$ of D_n factors through $\mathsf{D}_n/\mathbb{G}_\mathrm{m}$ if and only if $c_1 + \cdots + c_n = 0$. Hence

$$\mathsf{X}^*(\mathsf{D}_n/\mathbb{G}_\mathrm{m}) = \{c_1\chi_1 + \cdots + c_n\chi_n \mid c_1, \cdots, c_n \in \mathbb{Z}, c_1 + \cdots + c_n = 0\}$$

and we see that $\Phi(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m}) = \left\{\chi_i - \chi_j \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$.

The Lie algebra of PGL_n and $\mathsf{D}_n/\mathbb{G}_\mathrm{m}$ are $\mathfrak{pgl}_n := \mathfrak{gl}_n/KI_n$ and $\mathfrak{d}_n/KI_n$. For each $a = \chi_i - \chi_j$, the Lie algebra $\mathfrak{g}_a$ is generated by E_{ij}.

Example 2.3.5. Let's consider the split reductive group $(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$. Then two characters $c_1\chi_1 + \cdots + c_n\chi_n$ and $c'_1\chi_1 + \cdots + c'_n\chi_n$ of D_n may give rise to the same character of $\mathsf{D}_n \cap \mathsf{SL}_n$. This is the case precisely when $c_i - c'_i$ is a constant. Hence

$$\mathsf{X}^*(\mathsf{D}_n \cap \mathsf{SL}_n) = (\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n)/\mathbb{Z}(\chi_1 + \cdots + \chi_n)$$

and we see that $\Phi(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) = \left\{\overline{\chi_i} - \overline{\chi_j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$.

The Lie algebra of SL_n and $\mathsf{D}_n \cap \mathsf{SL}_n$ are $\mathfrak{sl}_n$, consisting of matrices with trace 0, and $\mathfrak{d}_n \cap \mathfrak{sl}_n$. For each $a = \overline{\chi_i} - \overline{\chi_j}$, the Lie algebra $\mathfrak{g}_a$ is generated by E_{ij}.

2.3.6. Let G be an affine algebraic group and T be a split torus in it. Then the normalizer $\mathsf{N} = \mathsf{N}_\mathsf{G}(\mathsf{T})$ acts on T, hence on $\mathsf{X}^*(\mathsf{T})$ by conjugations. The centralizer $\mathsf{Z}_\mathsf{G}(\mathsf{T})$ is the neutral component of N [Mil17, 12.40]. Therefore $\pi_0(\mathsf{N})$ acts on $\mathsf{X}^*(\mathsf{T})$.

If (G, T) is a split reductive group. Then $\mathsf{Z}_\mathsf{G}(\mathsf{T}) = \mathsf{T}$ [Mil17, 17.84] and hence $\pi_0(\mathsf{N}) = \mathsf{N}/\mathsf{T}$, which is constant [Mil17, 21.1]. The finite group N/T is denoted by ${}^v W(\mathsf{G}, \mathsf{T})$ and is called the *Weyl group* of the pair (G, T).

Example 2.3.7. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$. Then N consists of invertible monomial matrices and the regular representation of $\mathfrak{S}_n$ gives a semi-direct product $\mathsf{N} = \mathsf{D}_n \rtimes \underline{\mathfrak{S}_n}$. Hence the Weyl group ${}^vW(\mathsf{GL}_n, \mathsf{D}_n)$ is isomorphic to $\mathfrak{S}_n$.

Similar arguments apply to $(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_{\mathrm{m}})$ and $(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$ and their Weyl groups are ${}^vW(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_{\mathrm{m}}) \cong \mathfrak{S}_n$ and ${}^vW(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) \cong \mathfrak{S}_n$.

2.3.8. Let (G, T) be a split reductive group and $a \in \Phi(\mathsf{G}, \mathsf{T})$ a root of it. Then there is a unique homomorphism $u_a \colon \mathbb{W}(\mathfrak{g}_a) \to \mathsf{G}$ such that its differential $\mathrm{d}u_a$ is the inclusion $\mathfrak{g}_a \hookrightarrow \mathfrak{g}$. Let U_a denote the image of u_a. It is called the *root group* of G and satisfies the following properties [Mil17, 21.11 and 21.19; SGA3, XX, 1.5, XXII, 1.1].

(i) U_a has Lie algebra $\mathfrak{g}_a$ and a smooth subgroup of G contains U_a if and only if its Lie algebra contains $\mathfrak{g}_a$.

(ii) U_a is normalized by T and T acts on U_a through the character a:

$$\mathrm{inn}(\boldsymbol{t}).u_a(X) = u_a(a(\boldsymbol{t})X),$$

for all $\boldsymbol{t} \in \mathsf{T}$ and $X \in \mathbb{W}(\mathfrak{g}_a)$.

(iii) Let L_a be the algebraic subgroup of G generated by U_a, U_{-a} and T, called the *Levi subgroup associated to a*. Then the morphism

$$\mathbb{W}(\mathfrak{g}_{-a}) \times \mathsf{T} \times \mathbb{W}(\mathfrak{g}_a) \longrightarrow \mathsf{L}_a$$

defined by $(Y, \boldsymbol{t}, X) \mapsto u_{-a}(Y) \cdot \boldsymbol{t} \cdot u_a(X)$ is an open immersion.

Moreover, if $m \in \mathsf{N}_{\mathsf{G}}(\mathsf{T})$. Then $b = a \circ \mathrm{inn}(m)$ is a root and we have the following commutative diagram [SGA3, XXII, 1.4].

$$\begin{tikzcd}
\mathbb{W}(\mathfrak{g}_a) \arrow[r, "u_a"] \arrow[d, "\mathrm{Ad}(m)"'] & \mathsf{G} \arrow[d, "\mathrm{inn}(m)"] \\
\mathbb{W}(\mathfrak{g}_b) \arrow[r, "u_b"] & \mathsf{G}
\end{tikzcd}$$

Indeed, both u_a and $\mathrm{inn}(m)^{-1} \circ u_b \circ \mathrm{Ad}(m)$ have the differential $\mathfrak{g}_a \hookrightarrow \mathfrak{g}$.

Remark ([BT-II, 1.1.3 and 1.1.9]). In general, let G be an affine algebraic group and T be a split torus in it. Then a *root subgroup associated to a radical ray* $a \in \Phi(\mathsf{G}, \mathsf{T})$ is the largest connected closed subgroup U_a normalized by T and all characters appearing in the adjoint representation on $\mathrm{Lie}(\mathsf{U}_a)$ belong to a. The notion of Levi subgroups still make sense and we have open immersion

$$\mathsf{U}_a \times \mathsf{T} \times \mathsf{U}_{-a} \longrightarrow \mathsf{L}_a\,.$$

However, U_a is merely split unipotent in general, not necessary vectorial.

Example 2.3.9. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$ and its root $a = \chi_i - \chi_j$. Then U_a is the algebraic group

$$R \rightsquigarrow I_n + RE_{ij}\,.$$

The homomorphism $u_a\colon \mathbb{W}(\mathfrak{g}_a) \to \mathsf{GL}_n$ is

$$xE_{ij} \mapsto I_n + xE_{ij}\,.$$

For any $\boldsymbol{t} = \mathrm{diag}(t_1, \cdots, t_n) \in \mathsf{T}$, we have

$$\mathrm{inn}(\boldsymbol{t}).u_a(xE_{ij}) = I_n + t_i x t_j^{-1} E_{ij} = u_a(t_i t_j^{-1} x E_{ij}) = u_a(a(\boldsymbol{t}) x E_{ij}).$$

2.3.10. Notations as in 2.3.8. Then there is a natural duality on the one-dimensional vectorial groups

$$\mathbb{W}(\mathfrak{g}_a) \times \mathbb{W}(\mathfrak{g}_{-a}) \longrightarrow \mathbb{G}_{\mathrm{a}}\colon \qquad (X, Y) \longmapsto \langle X, Y \rangle.$$

and a unique cocharacter $a^\vee\colon \mathbb{G}_{\mathrm{m}} \to \mathsf{T}$ such that [SGA3, XX, 2.1]:

(i) for any $X \in \mathbb{W}(\mathfrak{g}_a)$ and $Y \in \mathbb{W}(\mathfrak{g}_{-a})$, the product $u_a(X) \cdot u_{-a}(Y)$ lies in the image of the open immersion in Theorem 2.0(iii) if and only if $1 + \langle X, Y \rangle \in \mathbb{G}_{\mathrm{m}}$;

(ii) under these conditions we have the formula

$$u_a(X) \cdot u_{-a}(Y) = u_{-a}((1 + \langle X, Y \rangle)^{-1} Y) \cdot a^\vee(1 + \langle X, Y \rangle) \cdot u_a((1 + \langle X, Y \rangle)^{-1} X);$$

(iii) $\langle a, a^\vee \rangle = 2$.

The above duality induces a pairing of $\mathbb{G}_\mathrm{m}$-bundles [SGA3, XX, 2.6]:

$$\mathbb{W}(\mathfrak{g}_a)^\times \times \mathbb{W}(\mathfrak{g}_{-a})^\times \longrightarrow \mathbb{G}_\mathrm{m} : \qquad (X, Y) \longmapsto XY.$$

Then for any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, there is a unique $X^{-1} \in \mathbb{W}(\mathfrak{g}_{-a})^\times$ such that $XX^{-1} = 1$. This gives rise to an isomorphism $(-)^{-1}$ compatible with the action of $\mathbb{G}_\mathrm{m}$. Then for any $x \in \mathbb{G}_\mathrm{m}$ and $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, we have [SGA3, XX, 2.7]:

$$a^\vee(x) = u_{-a}((x^{-1} - 1)X^{-1})u_a(X)u_{-a}((x-1)X^{-1})u_a(-x^{-1}X).$$

This cocharacter is called the *coroot associated to the root a*.

The root a and its coroot $a^\vee$ induces the following Lie algebra homomorphisms

$$K \xrightarrow{\mathrm{d}a^\vee} \mathfrak{t} \xrightarrow{\mathrm{d}a} K.$$

The vector $H_a := \mathrm{d}a^\vee(1)$ is called the *infinitesimal coroot vector*. Then $H_{-a} = -H_a$ and for any $X \in \mathbb{W}(\mathfrak{g}_a)$, $Y \in \mathbb{W}(\mathfrak{g}_{-a})$ and $H \in \mathbb{W}(\mathfrak{t})$, we have [SGA3, XX, 2.10]:

$$[H, X] = \mathrm{d}a(H)X, \qquad [H, Y] = -\mathrm{d}a(H)Y, \qquad [X, Y] = \langle X, Y \rangle H_a.$$

Hence if $H_a \neq 0$, then for any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, the following define an embedding from the Lie algebra $\mathfrak{sl}_2$:

$$E_{12} \mapsto X, \qquad E_{21} \mapsto X^{-1}, \qquad E_{11} - E_{22} \mapsto H_a.$$

Remark. One can fix such an embedding and hence fix a choice of basis of $\mathfrak{g}_a$ (as well as $\mathfrak{g}_{-a}$). Then one can identify them with K and denote the composition $\mathbb{G}_\mathrm{a} \cong \mathbb{W}(\mathfrak{g}_a) \to \mathsf{G}$ (resp. $\mathbb{G}_\mathrm{a} \cong \mathbb{W}(\mathfrak{g}_a) \to \mathsf{G}$) by u_a (resp. u_{-a}).

Example 2.3.11. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$ and its root $a = \chi_i - \chi_j$. To take the advantage of calculations on GL_2, we can define a homomorphism

ξ_{ij} mapping a 2×2-matrix $M = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathsf{GL}_2(R)$ to the $n\times n$-matrix $\xi_{ij}(M)$ satisfying

$$\xi_{ij}(M).e_k = \begin{cases} xe_i + ze_j & \text{if } k = i, \\ ye_i + we_j & \text{if } k = j, \\ e_k & \text{otherwise,} \end{cases}$$

where $e_1, e_2, \cdots, e_n$ is the standard basis of R^n. Then we have

$$u_a(x) = \xi_{ij}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \qquad \text{and} \qquad xE_{ij} = \mathrm{d}\xi_{ij}\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right).$$

Also note that $\xi_{ji} = \xi_{ij} \circ \text{transpose}$.

Then the duality is

$$\langle xE_{ij}, yE_{ji}\rangle = xy.$$

The coroot $a^\vee$ associated to a is $\lambda_i - \lambda_j$ and one can verify that

$$\begin{aligned}
&u_a(x)\cdot u_{-a}(y) \\
=&\xi_{ij}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\right) \\
=&\xi_{ij}\left(\begin{pmatrix} 1+xy & x \\ y & 1 \end{pmatrix}\right) \\
=&\xi_{ij}\left(\begin{pmatrix} 1 & 0 \\ (1+xy)^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1+xy & 0 \\ 0 & (1+xy)^{-1} \end{pmatrix}\begin{pmatrix} 1 & (1+xy)^{-1} \\ 0 & 1 \end{pmatrix}\right) \\
=&u_{-a}((1+xy)^{-1})\cdot a^\vee(1+xy)\cdot u_a((1+xy)^{-1}).
\end{aligned}$$

The differentials of a and $a^\vee$ are

$$\begin{aligned}
\mathrm{d}a &= \mathrm{d}\chi_i - \mathrm{d}\chi_j :\ \mathrm{diag}(t_1, \cdots, t_n) \mapsto t_i - t_j, \\
\mathrm{d}a^\vee &= \mathrm{d}\lambda_i - \mathrm{d}\lambda_j :\ z \mapsto \xi_{ij}\left(\begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix}\right).
\end{aligned}$$

In particular, the infinitesimal coroot vector associated to a is

$$H_a = \xi_{ij}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right).$$

For any $H = \mathrm{diag}(t_1, \cdots, t_n) \in \mathfrak{t}$, we have

$$[H, xE_{ij}] = \xi_{ij}\left(\begin{pmatrix} 0 & (t_i x - t_j x) \\ 0 & 0 \end{pmatrix}\right) = \mathrm{d}a(H)xE_{ij},$$

$$[H, yE_{ji}] = \xi_{ij}\left(\begin{pmatrix} 0 & 0 \\ (t_j y - t_i y) & 0 \end{pmatrix}\right) = -\mathrm{d}a(H)yE_{ji},$$

$$[xE_{ij}, yE_{ji}] = \xi_{ij}\left(\begin{pmatrix} xy & 0 \\ 0 & -xy \end{pmatrix}\right) = xyH_a.$$

2.3.12. Notations as in 2.3.8 and 2.3.10. Then L_a is the centralizer of the largest subtorus of $\mathrm{Ker}(a)$ and the pair $(\mathsf{L}_a, \mathsf{T})$ is a split reductive group [SGA3, XIX, 1.12 and XXII, 1.1; Mil17, 21.11 and 21.23]. The Lie algebra of it admits a decomposition

$$\mathrm{Lie}(\mathsf{L}_a) = \mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$$

and the Weyl group ${}^{v}W(\mathsf{L}_a, \mathsf{T})$ contains exactly one nontrivial element r_a given by the formula

$$r_a \colon \chi \longmapsto \chi - \langle \chi, a^\vee \rangle a.$$

Moreover, for any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, let

$$m_a(X) = u_a(X) \cdot u_{-a}(-X^{-1}) \cdot u_a(X).$$

Then we have [SGA3, XX, 3.1; Mil17, 20.39]:

(i) $m_a(X) \in \mathsf{N}_{\mathsf{L}_a}(\mathsf{T})$;

(ii) let M_a° be the image of $m_a \colon \mathbb{W}(\mathfrak{g}_a)^\times \to \mathsf{N}_{\mathsf{L}_a}(\mathsf{T})$, then $\mathsf{M}_a = \mathsf{T} \cdot \mathsf{M}_a^\circ$ is a right congruence class modulo T: indeed, for any $z \in \mathbb{G}_{\mathrm{m}}$ and $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, we have

$$m_a(zX) = a^\vee(z) m_a(X);$$

(iii) this right congruence class is precisely r_a: indeed, for any $t \in \mathsf{T}$ and $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, we have

$$\operatorname{inn}(m_a(X)).t = t \cdot a^\vee(a(t))^{-1};$$

(iv) for any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$ and $Y \in \mathbb{W}(\mathfrak{g}_{-a})^\times$, we have

$$m_a(X)m_{-a}(Y) = a^\vee(XY).$$

Example 2.3.13. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$ and its root $a = \chi_i - \chi_j$. Then the algebraic subgroup L_a is

$$R \rightsquigarrow \mathsf{D}_n(R) + RE_{ij} + RE_{ji}.$$

Its Lie algebra is $\mathfrak{d}_n + KE_{ij} + KE_{ji}$, which is precisely $\mathfrak{d}_n \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$.

The normalizer of D_n in L_a is precisely the monomial matrices belonging to L_a. Hence the Weyl group ${}^vW(\mathsf{L}_a, \mathsf{T})$ contains exactly one nontrivial element $r_a = (i, j)$, the permutation of i-th and j-th coordinates.

The map m_a is

$$xE_{ij} \longmapsto \xi_{ij}\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -x^{-1} & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) = \xi_{ij}\left(\begin{pmatrix}0 & x\\ -x^{-1} & 0\end{pmatrix}\right).$$

We have

$$\begin{aligned} m_a(x) &= \xi_{ij}\left(\begin{pmatrix}0 & x\\ -x^{-1} & 0\end{pmatrix}\right) \\ &= \xi_{ij}\left(\begin{pmatrix}x & 0\\ 0 & -x^{-1}\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right) \\ &= a^\vee(x)m_a(1). \end{aligned}$$

The action of $m_a(x)$ on $t = \mathrm{diag}(t_1, \cdots, t_n) \in \mathsf{T}$ is

$$\begin{aligned}
\mathrm{inn}(m_a(x)).t &= \mathrm{inn}\left(\xi_{ij}\left(\begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}\right)\right). \mathrm{diag}(t_1, \cdots, t_n) \\
&= \mathrm{inn}\big(a^\vee(x)\big). \mathrm{inn}\left(\xi_{ij}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\right). \mathrm{diag}(t_1, \cdots, t_n) \\
&= \mathrm{inn}\big(a^\vee(x)\big). \mathrm{diag}\big(t_{\sigma(1)}, \cdots, t_{\sigma(1)}\big) \qquad \text{with } \sigma = (i, j) \\
&= \mathrm{diag}\big(t_{\sigma(1)}, \cdots, t_{\sigma(1)}\big) \qquad \text{with } \sigma = (i, j) \\
&= \mathrm{diag}(t_1, \cdots, t_n)\xi_{ij}\left(\begin{pmatrix} t_i^{-1}t_j & 0 \\ 0 & t_it_j^{-1} \end{pmatrix}\right) \\
&= \mathrm{diag}(t_1, \cdots, t_n)a^\vee(a(t))^{-1}.
\end{aligned}$$

We also have

$$\begin{aligned}
m_a(x) \cdot m_{-a}(y) &= \xi_{ij}\left(\begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & -y^{-1} \\ y & 0 \end{pmatrix}\right) \\
&= \xi_{ij}\left(\begin{pmatrix} xy & 0 \\ 0 & (xy)^{-1} \end{pmatrix}\right) \\
&= a^\vee(xy).
\end{aligned}$$

2.4 Root data

Definition 2.4.1. Let (G, T) be a split reductive group. Then there is a *root datum* $\mathcal{R}(\mathsf{G}, \mathsf{T}) = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ associated to it [SGA3, XXII, 1.14; Mil17, 21.c], where

- the $\mathbb{Z}$-module X is the character group $\mathsf{X}^*(\mathsf{T})$;
- the root system Φ is the root system $\Phi(\mathsf{G}, \mathsf{T})$;
- the dual $\mathbb{Z}$-module $\mathsf{X}^\vee$ is the cocharacter group $\mathsf{X}_*(\mathsf{T})$;
- the coroot system $\Phi^\vee$ is the set $\Phi^\vee(\mathsf{G}, \mathsf{T})$ of coroots $a^\vee$ associated to the roots $a \in \Phi(\mathsf{G}, \mathsf{T})$.

Let $\mathbb{V}$ denote the subspace of $\mathsf{X}_*(\mathsf{T}) \otimes \mathbb{R}$ spanned by $\Phi^\vee(\mathsf{G},\mathsf{T})$ equipped with a ${}^vW(\mathsf{G},\mathsf{T})$-invariant inner product, called the *coroot space*. Then we get a spherical apartment ${}^v\!\mathscr{A}(\mathsf{G},\mathsf{T})$ with underlying Euclidean vector space $\mathbb{V}$ on which the Weyl group ${}^vW(\mathsf{G},\mathsf{T})$ acts as its reflection group.

2.4.2. The *rank* of a root datum $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ is the rank of the $\mathbb{Z}$-module X and its *semisimple rank* is the dimension of the coroot space $\mathbb{V}$. Let (G,T) be a split reductive group. Then the rank (resp. semisimple rank) of the root datum $\mathcal{R}(\mathsf{G},\mathsf{T})$ is the rank (resp. semisimple rank) of G.

Example 2.4.3. Let's consider the split reductive group $(\mathbb{G}_\mathrm{m}, \mathbb{G}_\mathrm{m})$. Then by Example 2.3.2, the root datum $\mathcal{R}(\mathbb{G}_\mathrm{m}, \mathbb{G}_\mathrm{m})$ is $(\mathbb{Z}, \emptyset, \mathbb{Z}, \emptyset)$.

Example 2.4.4. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$. Then by Examples 2.3.3, 2.3.9 and 2.3.11, the root datum $\mathcal{R}(\mathsf{GL}_n, \mathsf{D}_n)$ is

- $\mathsf{X} = \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n$;
- $\Phi = \left\{\chi_i - \chi_j \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$;
- $\mathsf{X}^\vee = \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_n$;
- $\Phi^\vee = \left\{\lambda_i - \lambda_j \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$.

The coroot space is

$$\mathbb{V} = \{c_1\lambda_1 + \cdots + c_n\lambda_n \mid c_1, \cdots, c_n \in \mathbb{R}, c_1 + \cdots + c_n = 0\}.$$

Example 2.4.5. Let's consider the split reductive group $(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m})$. Two characters $c_1\lambda_1 + \cdots + c_n\lambda_n$ and $c'_1\lambda_1 + \cdots + c'_n\lambda_n$ of D_n give rise to the same cocharacter of $\mathsf{D}_n/\mathbb{G}_\mathrm{m}$ precisely when $c_i - c'_i$ is a constant. Hence

$$\mathsf{X}_*(\mathsf{D}_n/\mathbb{G}_\mathrm{m}) = (\mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_n)/\mathbb{Z}(\lambda_1 + \cdots + \lambda_n)$$

and we see that $\Phi^\vee(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m}) = \left\{\overline{\lambda_i} - \overline{\lambda_j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$ with coroot space

$$\mathbb{V} = \left\{c_1\overline{\lambda_1} + \cdots + c_n\overline{\lambda_n} \,\middle|\, c_1, \cdots, c_n \in \mathbb{R}, c_1 + \cdots + c_n = 0\right\}.$$

Then, by Example 2.3.4, the root datum $\mathcal{R}(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_{\mathrm{m}})$ is

- $\mathsf{X} = \{c_1\chi_1 + \cdots + c_n\chi_n \mid c_1, \cdots, c_n \in \mathbb{Z}, c_1 + \cdots + c_n = 0\}$;
- $\Phi = \left\{\chi_i - \chi_j \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$;
- $\mathsf{X}^\vee = (\mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_n)/\mathbb{Z}(\lambda_1 + \cdots + \lambda_n)$;
- $\Phi^\vee = \left\{\overline{\lambda_i} - \overline{\lambda_j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$.

Example 2.4.6. Let's consider the split reductive group $(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$. A cocharacter $c_1\lambda_1 + \cdots + c_n\lambda_n$ of factors through $\mathsf{D}_n \cap \mathsf{SL}_n$ if and only if $c_1 + \cdots + c_n = 0$. Hence

$$\mathsf{X}_*(\mathsf{D}_n \cap \mathsf{SL}_n) = \{c_1\lambda_1 + \cdots + c_n\lambda_n \mid c_1, \cdots, c_n \in \mathbb{Z}, c_1 + \cdots + c_n = 0\}$$

and we see that $\Phi^\vee(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) = \left\{\lambda_i - \lambda_j \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$ with coroot space

$$\mathbb{V} = \{c_1\lambda_1 + \cdots + c_n\lambda_n \mid c_1, \cdots, c_n \in \mathbb{R}, c_1 + \cdots + c_n = 0\}.$$

Then, by Example 2.3.5, the root datum $\mathcal{R}(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$ is

- $\mathsf{X} = (\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n)/\mathbb{Z}(\chi_1 + \cdots + \chi_n)$;
- $\Phi = \left\{\overline{\chi_i} - \overline{\chi_j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$;
- $\mathsf{X}^\vee = \{c_1\lambda_1 + \cdots + c_n\lambda_n \mid c_1, \cdots, c_n \in \mathbb{Z}, c_1 + \cdots + c_n = 0\}$;
- $\Phi^\vee = \left\{\lambda_i - \lambda_j \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$.

2.4.7. Let $\varphi\colon (\mathsf{G}, \mathsf{T}) \to (\mathsf{G}', \mathsf{T}')$ be a homomorphism between split reductive groups. Then it induces a linear map $f = \varphi^*\colon \mathsf{X}^*(\mathsf{T}') \to \mathsf{X}^*(\mathsf{T})$. Then f is a morphism of root data if and only if there is a bijection $u\colon \Phi(\mathsf{G}, \mathsf{T}) \to \Phi(\mathsf{G}', \mathsf{T}')$ such that

$$f(u(a)) = a, \qquad {}^t\!f(a^\vee) = u(a)^\vee.$$

A homomorphism of split reductive groups $\varphi\colon (\mathsf{G}, \mathsf{T}) \to (\mathsf{G}', \mathsf{T}')$ induces an isogeny of root data $\varphi^*\colon \mathcal{R}(\mathsf{G}', \mathsf{T}') \to \mathcal{R}(\mathsf{G}, \mathsf{T})$ if and only if it is a central isogeny. Moreover, all isogenies of root data arise in this way [SGA3, XXII, 4.2.11; Mil17, 23.25].

Example 2.4.8. Let's consider the inclusion

$$\iota\colon (\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \hookrightarrow (\mathsf{GL}_n, \mathsf{D}_n).$$

Then the linear map $f = \iota^*\colon \mathsf{X}^*(\mathsf{GL}_n, \mathsf{D}_n) \to \mathsf{X}^*(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$ is the linear map

$$\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n \longrightarrow \mathbb{Z}$$

mapping χ_i to 1. Then this is not a morphism of root data since it does not induce a bijection on roots.

Example 2.4.9. Let's consider the determinant homomorphism

$$\det\colon (\mathsf{GL}_n, \mathsf{D}_n) \longrightarrow (\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}).$$

Then the linear map $f = \det^*\colon \mathsf{X}^*(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \to \mathsf{X}^*(\mathsf{GL}_n, \mathsf{D}_n)$ is the linear map

$$\mathbb{Z} \longrightarrow \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n$$

mapping 1 to $\chi_1 + \cdots + \chi_n$. Then this is not a morphism of root data since it does not induce a bijection on roots.

Example 2.4.10. Let's consider the isogeny

$$\varphi\colon (\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \longrightarrow (\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})\colon t \mapsto t^n.$$

Then the linear map $f = \varphi^*\colon \mathsf{X}^*(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \to \mathsf{X}^*(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$ is the linear map

$$\mathbb{Z} \longrightarrow \mathbb{Z}\colon 1 \longmapsto n.$$

It is an isogeny of root data with finite cokernel μ_n.

Example 2.4.11. Let's consider the inclusion

$$\iota\colon (\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) \hookrightarrow (\mathsf{GL}_n, \mathsf{D}_n).$$

Then the linear map $f = \iota^*\colon \mathsf{X}^*(\mathsf{GL}_n, \mathsf{D}_n) \to \mathsf{X}^*(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$ is the projection

$$\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n \twoheadrightarrow (\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n)/\mathbb{Z}(\chi_1 + \cdots + \chi_n).$$

This is a morphism of root data but not an isogeny since f is not injective.

Example 2.4.12. Let's consider the quotient map

$$\pi\colon (\mathsf{GL}_n, \mathsf{D}_n) \twoheadrightarrow (\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m}).$$

Then the linear map $f = \pi^*\colon\ \mathsf{X}^*(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m}) \to \mathsf{X}^*(\mathsf{GL}_n, \mathsf{D}_n)$ is the inclusion

$$\{c_1 + \cdots + c_n = 0\} \cap \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n \hookrightarrow \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n.$$

This is a morphism of root data but not an isogeny since f has infinite cokernel.

Example 2.4.13. Let's consider the composition

$$\varphi\colon\quad (\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) \xhookrightarrow{\iota} (\mathsf{GL}_n, \mathsf{D}_n) \xrightarrow{\pi} (\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m})$$

of previous two homomorphisms. Then the linear map

$$f = \varphi^*\colon\ \mathsf{X}^*(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m}) \longrightarrow \mathsf{X}^*(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$$

is the following restriction of the projection π^*

$$\{c_1 + \cdots + c_n = 0\} \cap \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n \longrightarrow (\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n)/\mathbb{Z}(\chi_1 + \cdots + \chi_n)$$

It turns out that this is an isogeny of root data with finite cokernel $\mu_n\overline{\chi_1}$.

2.4.14. Let (G, T) be a split reductive group. Then by 2.2.6 and 2.2.7, we have the following commutative diagram of split reductive groups:

$$\begin{tikzcd}
\mathrm{sc}(\mathsf{G},\mathsf{T}) \arrow[r] & \mathrm{der}(\mathsf{G},\mathsf{T}) \arrow[rr] \arrow[dr] & & \mathrm{ss}(\mathsf{G},\mathsf{T}) \arrow[r] & \mathrm{ad}(\mathsf{G},\mathsf{T}) \\
& & (\mathsf{G},\mathsf{T}) \arrow[ur] \arrow[dr] & & \\
& \mathrm{rad}(\mathsf{G},\mathsf{T}) \arrow[ur] \arrow[rr] & & \mathrm{corad}(\mathsf{G},\mathsf{T}) &
\end{tikzcd}$$

where the horizontal arrows are isogenies, the diagonals are short exact sequences and

$\mathrm{ad}(\mathsf{G}, \mathsf{T})$ — the adjoint group G^{ad} and the image of T in it $(\mathsf{T}/Z(\mathsf{G}))$;

$\mathrm{ss}(\mathsf{G},\mathsf{T})$	the semisimple quotient G^{ss} and the image of T in it $(\mathsf{T}/\mathscr{R}(\mathsf{G}))$;
$\mathrm{der}(\mathsf{G},\mathsf{T})$	the derived group $\mathsf{G}^{\mathrm{der}}$ and the preimage of T in it $(\mathsf{T}\cap\mathsf{G}^{\mathrm{der}})$;
$\mathrm{sc}(\mathsf{G},\mathsf{T})$	the universal covering of all above;
$\mathrm{rad}(\mathsf{G},\mathsf{T})$	the radical $\mathscr{R}(\mathsf{G})$ and the trivial torus 1;
$\mathrm{corad}(\mathsf{G},\mathsf{T})$	the abelianization G^{Ab} and the trivial torus 1.

Moreover, we have [SGA3, XXII, 4.3.7, 6.2.1 and 6.2.3; Mil17, 23.a]

(i) $\mathcal{R}(\mathrm{ad}(\mathsf{G},\mathsf{T})) = \mathrm{ad}(\mathcal{R}(\mathsf{G},\mathsf{T}))$;

(ii) $\mathcal{R}(\mathrm{ss}(\mathsf{G},\mathsf{T})) = \mathrm{ss}(\mathcal{R}(\mathsf{G},\mathsf{T}))$;

(iii) $\mathcal{R}(\mathrm{der}(\mathsf{G},\mathsf{T})) = \mathrm{der}(\mathcal{R}(\mathsf{G},\mathsf{T}))$;

(iv) $\mathcal{R}(\mathrm{sc}(\mathsf{G},\mathsf{T})) = \mathrm{sc}(\mathcal{R}(\mathsf{G},\mathsf{T}))$;

(v) $\mathcal{R}(\mathrm{rad}(\mathsf{G},\mathsf{T})) = \mathrm{rad}(\mathcal{R}(\mathsf{G},\mathsf{T}))$;

(vi) $\mathcal{R}(\mathrm{corad}(\mathsf{G},\mathsf{T})) = \mathrm{corad}(\mathcal{R}(\mathsf{G},\mathsf{T}))$;

(vii) the morphisms between above root data come from the homomorphisms between corresponding split reductive groups.

Example 2.4.15. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$. Then the deconstruction in Example 2.2.8 gives the following isogenies of root data.

$$\mathcal{R}(\mathsf{SL}_n, \mathsf{D}_n\cap\mathsf{SL}_n) \xleftarrow{\textit{Example } 2.4.11} \mathcal{R}(\mathsf{GL}_n, \mathsf{D}_n) \xleftarrow{\textit{Example } 2.4.12} \mathcal{R}(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_{\mathrm{m}})$$
$$\mathcal{R}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \xleftarrow{\textit{Example } 2.4.10} \mathcal{R}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$$

where $\mathcal{R}(\mathsf{SL}_n, \mathsf{D}_n\cap\mathsf{SL}_n)$ is described in Example 2.4.4, $\mathcal{R}(\mathsf{GL}_n, \mathsf{D}_n)$ in Example 2.4.5, $\mathcal{R}(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_{\mathrm{m}})$ in Example 2.4.6, and $\mathcal{R}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$ in Example 2.4.3 respectively.

2.5 Tits buildings

Let G be a reductive group. Associated to it, there is a spherical building ${}^v\mathscr{B}(\mathsf{G})$ equipped with a natural G-action, called its *Tits building*. In this subsection, Tits buildings will be introduced for splittable reductive groups and we will see that the underlying building only depends on the root system and the ground field.

2.5.1. Let G be a reductive group. A *parabolic subgroup* of it is a smooth subgroup P such that G/P is a complete variety. A subgroup T of G is *Borel* if it is smooth, connected, solvable, and parabolic. It turns out that a smooth subgroup P is parabolic if and only if $\mathsf{P}_{K^{\mathrm{a}}}$ contains a Borel subgroup in $\mathsf{G}_{K^{\mathrm{a}}}$ [Mil17, 17.16] and every parabolic subgroup is connected and equal to its own normalizer since this is so over K^{a} [Mil17, 17.49]. When G has a Borel subgroup, it is said to be *quasi-split*. In this case, Borel subgroups are exactly the minimal parabolic subgroups and maximal connected solvable subgroups [Mil17, 17.19] and any two of them are conjugated by an element of G [Mil17, 25.8]. If the Borel subgroup is furthermore split (as a solvable algebraic group, namely it admits a normal series whose factors are isomorphic to either $\mathbb{G}_{\mathrm{a}}$ or $\mathbb{G}_{\mathrm{m}}$), then G is said to be *split*. It turns out that, G is split if and only if it is splittable [Mil17, 21.64].

Let $\pi\colon \mathsf{G} \to \mathsf{Q}$ be a quotient map and H a smooth subgroup of G. Then if H is parabolic (resp. Borel), so is $\pi(\mathsf{H})$. Moreover, every such subgroup of Q arises in this way [Mil17, 17.20]. This allows us to reduce the study of (the poset of) parabolic subgroups from reductive groups to simply-connected semisimple groups. The *Tits building* of a reductive group is essentially this poset [Tit74, 5.2].

2.5.2. Let (G, T) be a split reductive group. Then the following sets are equinumerous and the Weyl group ${}^vW(\mathsf{G}, \mathsf{T})$ acts simply transitively on them.

(i) The set of Borel subgroups B of G containing T.

(ii) The set of Weyl chambers ${}^vC$ in the spherical apartment ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T})$.

(iii) The set of systems of positive roots Φ^+ in the root system $\Phi(\mathsf{G}, \mathsf{T})$.

(iv) The set of bases Δ of $\Phi(\mathsf{G}, \mathsf{T})$.

Indeed, if a system of positive roots Φ^+ is given, then B is generated by T and U_a for all $a \in \Phi^+$ and if a Borel subgroup B containing T is given, then the set of roots a whose Lie algebra $\mathfrak{g}_a$ is contained in the Lie algebra of T forms a system of positive roots Φ^+ [Mil17, 21.d].

More generally, after choosing one element of the above equinumerous sets. We have the following equinumerous sets.

(i) The set of parabolic subgroups P of G containing B.

(ii) The set of faces ${}^vF$ of the Weyl chamber ${}^vC$.

(iii) The set of parabolic subsets Ψ of $\Phi(\mathsf{G}, \mathsf{T})$ containing Φ^+.

(iv) The set of types I on ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T})$.

Indeed, if a parabolic subset Ψ is given, then P is generated by T and U_a for all $a \in \Psi$ and if a parabolic subgroup P containing B is given, then the set of roots a whose Lie algebra $\mathfrak{g}_a$ is contained in the Lie algebra of P forms a parabolic subset Ψ [Mil17, 21.i].

Fix a Borel subgroup B of G containing T. Let I be a type and P_I the parabolic subgroup corresponding to it. Then the unipotent radical of P_I is generated by U_a for all $a \in \Phi^+ \setminus \Psi$ and the reductive quotient of P_I is isomorphic to the centralizer L_I of the largest subtorus contained in $\mathrm{Ker}(a)$ for all $a \in I$ [Mil17, 21.91]. This reductive group is called the *Levi subgroup associated to I* and $(\mathsf{L}_I, \mathsf{T})$ is a split reductive group with root datum $(\mathsf{X}^*, \Phi_I, \mathsf{X}_*, \Phi_I^\vee)$ and Weyl group ${}^vW_I$ [Mil17, 21.90].

Example 2.5.3. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$. Then the subgroup T_n of upper triangular invertible matrices is a Borel subgroup containing D_n. It corresponds to the system of positive roots $\Phi^+ = \left\{\chi_i - \chi_j \,\middle|\, 1 \leqslant i < j \leqslant n\right\}$ with basis $\Delta = \{a_1 = \chi_1 - \chi_2, \cdots, a_{n-1} = \chi_{n-1} - \chi_n\}$. The Weyl chamber corresponding to it is

$${}^vC = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1 > x_2 > \cdots > x_n\}.$$

Let $I = \Delta \setminus \{l_1 = k_1, l_2 = k_1 + k_2, \cdots, l_{t-1} = k_1 + k_2 + \cdots + k_{t-1}\}$ be a type on the apartment, identified with a subset of Δ. Then the parabolic subgroup P_I, its unipotent radical $\mathscr{R}_u(\mathsf{P}_I)$ and the Levi subgroup L_I consist of the matrices of the following forms respectively

$$\begin{pmatrix} A_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & A_t \end{pmatrix}, \qquad \begin{pmatrix} I_{k_1} & * & * \\ 0 & \ddots & * \\ 0 & 0 & I_{k_t} \end{pmatrix}, \qquad \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_t \end{pmatrix},$$

where A_i is a $k_i \times k_i$ matrix. The facet corresponding to them is

$${}^vF = \left\{(x_1, \cdots, x_n) \in \mathbb{R}^n \,\middle|\, x_1 = \cdots = x_{l_1} > x_{l_1+1} \cdots x_{l_{t-1}} > x_{l_{t-1}+1} = \cdots = x_n\right\}.$$

Theorem 2.1 ([Rou09, §10; Tit74, §5])**.** *Let* (G, T) *be a split reductive group with Weyl group* ${}^vW$*, coroot space* $\mathbb{V}$*, normalizer* N *of* T *and the root groups* U_a*. Then there is a unique (up to unique isomorphism) G-set* ${}^v\mathscr{B}(\mathsf{G})$ *containing* $\mathbb{V}$ *and satisfying the following.*

(i) ${}^v\mathscr{B}(\mathsf{G}) = \bigcup_{g \in G} g.\mathbb{V}$*;*

(ii) *N stabilizes $\mathbb{V}$ and acts on it through ${}^vW$;*

(iii) *For every $a \in \Phi$, the fixator of $\alpha_{a+0} := \{\mathbf{v} \in \mathbb{V} \mid a(\mathbf{v}) \geqslant 0\}$ is $T \cdot U_a$.*

Then ${}^v\mathscr{B}(\mathsf{G})$ is a building of type ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T})$. Indeed, since N stabilizes $\mathbb{V}$ and preserves its apartment structure, each $g.\mathbb{V}$ is endowed with such a structure and moreover they agree on intersections.

Definition 2.5.4. The building ${}^v\mathscr{B}(\mathsf{G})$ is called the *Tits building* of G.

Remark. Apartments in ${}^v\mathscr{B}(\mathsf{G})$ are one-one corresponding to split maximal tori. In fact, each $g.\mathbb{V}$ endowed with its apartment structure is precisely the spherical apartment ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T}^g)$.

The action of G on ${}^v\mathscr{B}(\mathsf{G})$ is strongly transitive and type-preserving. It is also worth to mention that ${}^v\mathscr{B}(\mathsf{G})$ is further an $\mathrm{Aut}(\mathsf{G})$-set. Indeed, if φ is an automorphism of

G, then $\varphi(\mathsf{T})$ is also a split maximal torus and the pushforward along φ defines an isomorphism from ${}^v\!\mathscr{A}(\mathsf{G},\mathsf{T})$ to ${}^v\!\mathscr{A}(\mathsf{G},\varphi(\mathsf{T}))$.

Example 2.5.5. The simplicial complex structure on the Tits building of GL_n can be described as in 1.1.2.

2.5.6. Let G be a splittable reductive group. Let φ be a homomorphism in the following sequence.

$$\mathsf{G}^{\mathrm{sc}} \longrightarrow \mathsf{G}^{\mathrm{der}} \longrightarrow \mathsf{G} \longrightarrow \mathsf{G}^{\mathrm{ss}} \longrightarrow \mathsf{G}^{\mathrm{ad}}$$

Then for any split maximal torus T in G, its image or preimage under φ is again a split maximal torus T' and such a corresponding $\mathsf{T} \mapsto \mathsf{T}'$ gives rise to a bijection between the set of maximal tori. Therefore by 1.5.6 and 2.4.14, we see that all above reductive groups have isomorphic Tits buildings.

Conversely, any root datum arises from a splittable reductive group [Mil17, 23.55; SGA3, XXV, 1.2]. Hence we see that the Tits building ${}^v\!\mathscr{B}(\mathsf{G})$ depends only on the root system Φ and the ground field K and any root system gives rise to such a building. So we can denote this building by ${}^v\!\mathscr{B}(\Phi, K)$.

§ 3 Bruhat-Tits buildings

The datum of a split reductive group over a local field gives rise to a root group datum and a valuation on it [BT-II]. Bruhat and Tits [BT-I] introduced an affine building based on such purely group-theoretical data. They also show in [BT-II] that these data come with some extra schematic structures, which turn out to be an important ingredient in the theory of reductive groups over local fields.

In the rest of this note, the ground field K is assumed to be equipped with a discrete valuation $\mathrm{val}(\,\cdot\,)\colon K \to \mathbb{R} \cup \{\infty\}$. Its valuation group $\mathrm{val}(K^\times)$ is denoted by Γ and we fix the following associated notations.

$$\begin{aligned} K^\circ &:= \{x \in K \mid \mathrm{val}(x) \geqslant 0\}, \\ (K^\circ)^\times &:= \{x \in K \mid \mathrm{val}(x) = 0\}, \\ K^{\circ\circ} &:= \{x \in K \mid \mathrm{val}(x) > 0\}, \\ \kappa &:= K^\circ / K^{\circ\circ}. \end{aligned}$$

We also fix a *uniformizer* ϖ, namely a generator of $K^{\circ\circ}$. Let $\gamma = \mathrm{val}(\varpi)$.

3.1 Valuations on root group data

Definition 3.1.1. Let Φ be a root system and G be a group. A *root group datum*[10] *of type Φ in G* is a system $(T, (U_a, M_a)_{a\in\Phi})$, where T is a subgroup of G and for each $a \in \Phi$, U_a is a non-trivial subgroup of G and M_a is a right congruence class modulo T, satisfying the following axioms.

RGD1. For any $a, b \in \Phi$, the commutator group $[U_a, U_b]$ is contained in the group generated by the U_c for $c = ia + jb \in \Phi$ with $i, j > 0$.

RGD2. For each $a \in \Phi$, the class M_a satisfies $U_{-a}^* := U_{-a} \setminus \{1\} \subseteq U_a M_a U_a$.

[10]It is called a *reduced root datum* in [BT-I, 6.1.1]. We only focus on reduced root datum as we focus on split reductive groups. As for general case, see the original papers [BT-I].

RGD3. For any $a, b \in \Phi$ and each $m \in M_a$, we have $\mathrm{inn}(m).U_b \subseteq U_{r_a(b)}$, where r_a is the reflection associated to a.

RGD4. Let Φ^+ be a system of positive roots in Φ and if U^+ (resp. U^-) is the subgroup of G generated by the U_a for $a \in \Phi^+$ (resp. $a \in \Phi^-$), then $TU^+ \cap U^- = \{1\}$.

This root group datum is said to be *generating* when G is generated by the subgroups T and U_a for $a \in \Phi$.

3.1.2. Let $(T, (U_a, M_a)_{a\in\Phi})$ be a root group datum. We have the following consequences of above axioms [BT-I, 6.1.2].

(i) $U_a \neq U_{-a}$ and $U_a M_a U_a \cap N_G(U_a) = \emptyset$.

(ii) For any $u \in U^*_{-a}$, there is a unique triple $(u', m, u'') \in U_a \times G \times U_a$ such that $u = u'mu''$, $\mathrm{inn}(m).U_a = U_{-a}$ and $\mathrm{inn}(m).U_{-a} = U_a$. Moreover, $m \in M_a$ and $u' \neq 1$.

Let $m(-)\colon U^*_{-a} \to M_a$ denote the map $u \mapsto m$ in above and put M°_a being its image.

(iii) T normalizes U_a and M_a.

(iv) $M_a = M_a^{-1} = M_{-a}$ and $T \cup M_a$ is a subgroup of G.

(v) Let L_a be the subgroup of G generated by U_a, U_{-a} and T. Then

$$L_a = U_a M_a U_a \cup TU_a.$$

(vi) $N_G(U_a) \cap L_a = TU_a$ and

$$M_a = \{g \in L_a \mid \mathrm{inn}(g).U_a = U_{-a} \text{ and } \mathrm{inn}(g).U_{-a} = U_a\}.$$

So M_a is completely determined by U_a, U_{-a} and T. Hence we can say $(T, (U_a)_{a\in\Phi})$ is a root group datum without mentioning M_a.

(vii) Let N be the subgroup of G generated by T and M_a for all $a \in \Phi$. Then, if Φ is nonempty, N is already generated by M_a's and normalizes T. Moreover, there is an epimorphism ${}^v\nu\colon N \longrightarrow {}^vW(\Phi)$ such that for each $a \in \Phi$ and $m \in N$, we have $\mathrm{inn}(m).U_a = U_b$ with $b = {}^v\nu(m).a$. In particular, we have ${}^v\nu(M_a) = \{r_a\}$. Also note that $\mathrm{Ker}({}^v\nu) = T$ [BT-I, 6.1.11].

(viii) Suppose Φ is nonempty. Let N° be the subgroup of G generated by M_a° for all $a \in \Phi$ and let $T^\circ = N^\circ \cap T$. Then $(T^\circ, (U_a, M_a^\circ)_{a\in\Phi})$ is a generating root group datum on the subgroup G° of G generated by U_a for all $a \in \Phi$.

Example 3.1.3 ([BT-I, 6.1.3.b; BT65]). Let (G, T) be a split reductive group over K, $(\mathsf{U}_a)_{a\in\Phi}$ be the root groups associated to the root system Φ of (G, T) and $(\mathsf{M}_a)_{a\in\Phi}$ be the right congruence classes in 2.3.12. Then $(T, (U_a, M_a)_{a\in\Phi})$ forms a generating root group datum in G:

RGD1. Let a and b be two roots in Φ. Then for any $i, j > 0$ such that $ia + jb \in \Phi$, there is a linear function [SGA3, XXII, 5.5.4]

$$f_{a,b;i,j}\colon\ \mathfrak{g}_a^{\otimes i} \otimes_K \mathfrak{g}_b^{\otimes j} \longrightarrow \mathfrak{g}_{ia+jb}$$

such that for any $X \in \mathbb{W}(\mathfrak{g}_a)$ and $Y \in \mathbb{W}(\mathfrak{g}_b)$, we have

$$[u_a(X), u_b(Y)] = \prod_{ia+jb\in\Phi} u_{ia+jb}(f_{a,b;i,j}(X^i \otimes Y^j))$$

where the product is taken in any order.

RGD2. Let $a \in \Phi$. Let $\mathsf{U}^*_{-a} = u_{-a}(\mathbb{W}(\mathfrak{g}_{-a})^\times)$, then $\mathsf{U}^*_{-a}(K) = U^*_{-a}$. Taking any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, then we have $X^{-1} \in \mathbb{W}(\mathfrak{g}_a)^\times$ and

$$\begin{aligned} u_{-a}(X^{-1}) &= u_a(X)u_a(-X)u_{-a}(-(-X)^{-1})u_a(-X)u_a(X) \\ &= u_a(X)m_a(-X)u_a(X). \end{aligned}$$

RGD3. This follows from 2.3.8 and Theorem 2.0(iii).

RGD4. There are closed immersions [SGA3, XXII, 5.5.1 and 5.6.5; Mil17, 21.68]

$$\mathsf{T} \times \prod_{a \in \Phi^+} \mathsf{U}_a \longrightarrow \mathsf{G} \qquad \text{and} \qquad \prod_{a \in \Phi^+} \mathsf{U}_a \longrightarrow \mathsf{G},$$

with images $\mathsf{T} \cdot \mathsf{U}_+$ and U_+. Where $\mathsf{T} \cdot \mathsf{U}_+$ is a Borel subgroup of G corresponding to the system of positive roots Φ^+, while U_+ is its unipotent radical and is generated by the root groups U_a for all $a \in \Phi^+$. Similarly we have Borel subgroup $\mathsf{T} \cdot \mathsf{U}_-$ and its unipotent radical U_-. Then the Borel subgroups $\mathsf{T} \cdot \mathsf{U}_+$ and $\mathsf{T} \cdot \mathsf{U}_-$ are *opposite*, namely their intersection is T [SGA3, 5.9.2; Mil17, 21.84]. Therefore $\mathsf{T} \cdot \mathsf{U}_+ \cap \mathsf{U}_-$ is trivial.

Moreover, G is generated by T and the root groups U_a for all $a \in \Phi$ [Mil17, 21.11].

We also verify the corollaries in 3.1.2:

(i) This is clear.

(ii) We have already seen in above discussion that if $u = u_{-a}(X^{-1})$, then the triple $(u_a(X), m_a(-X), u_a(X))$ satisfies the requirements. Suppose (u', m, u'') is another triple, then $m_a(-X) = u_a(-X)u'mu''u_a(-X) \in M_a$ and hence it maps U_a to U_{-a} by conjugate. Then $u_a(-X)u'$ normalizes U_{-a} and $u''u_a(-X)$ normalizes U_a, hence $u' = u_a(X)$ and $u'' = u_a(X)$ by Theorem 3.0(i).

(iii) The first follows from Theorem 2.0(ii) and as for the second: let $\boldsymbol{t} \in \mathsf{T}$ and $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, then we have

$$\begin{aligned}
\operatorname{inn}(\boldsymbol{t}).m_a(X) &= \operatorname{inn}(\boldsymbol{t}).(u_a(X)u_{-a}(-X^{-1})u_a(X)) \\
&= (\operatorname{inn}(\boldsymbol{t}).u_a(X))(\operatorname{inn}(\boldsymbol{t}).u_{-a}(-X^{-1}))(\operatorname{inn}(\boldsymbol{t}).u_a(X)) \\
&= u_a(a(\boldsymbol{t})X) \cdot u_{-a}((-a)(\boldsymbol{t})(-X^{-1})) \cdot u_a(a(\boldsymbol{t})X) \\
&= u_a(a(\boldsymbol{t})X) \cdot u_{-a}(-(a(\boldsymbol{t})X)^{-1}) \cdot u_a(a(\boldsymbol{t})X) \\
&= m_a(a(\boldsymbol{t})X).
\end{aligned}$$

(iv) For any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, we have

$$\begin{aligned} m_a(X)^{-1} &= (u_a(X)u_{-a}(-X^{-1})u_a(X))^{-1} \\ &= u_a(-X)u_{-a}(X^{-1})u_a(-X) \\ &= m_a(-X). \end{aligned}$$

By Theorem 2.0(iv), we also have

$$m_a(X)^{-1} = m_{-a}(X^{-1}).$$

These prove the first part. As for the second: let $X, Y \in \mathbb{W}(\mathfrak{g}_a)^\times$, then we have

$$m_a(X)m_a(Y) = m_a(X)m_{-a}(-Y^{-1}) = a^\vee(\langle X, -Y^{-1}\rangle) \in \mathsf{T}\,.$$

(v) This follows from the Bruhat decomposition [SGA3, XXII, 5.7.4; Mil17, 21.73] for L_a:

$$\mathsf{L}_a = \mathsf{B} \sqcup \mathscr{R}_u(\mathsf{B}) w\, \mathsf{B},$$

where B is the Borel subgroup $\mathsf{T} \cdot \mathsf{U}_a$ of L_a with unipotent radical $\mathscr{R}_u(\mathsf{B}) = \mathsf{U}_a$ and w is the only nontrivial element of ${}^vW(\mathsf{L}_a, \mathsf{T})$, hence M_a.

(vi) The first follows from the normalizer theorem [Mil17, 17.50]. As for the second: suppose $g \in \mathsf{L}_a$ has the property that $\mathrm{inn}(g).\mathsf{U}_a = \mathsf{U}_{-a}$ and $\mathrm{inn}(g).\mathsf{U}_{-a} = \mathsf{U}_a$. Then $g \notin \mathsf{T} \cdot \mathsf{U}_a$ and hence $g \in \mathsf{U}_a \cdot \mathsf{M}_a \cdot \mathsf{U}_a$. But $\mathsf{U}_a \cap \mathsf{N}_{\mathsf{G}}(\mathsf{U}_{-a})$ is trivial. Hence $g \in \mathsf{M}_a$.

(vii) $\mathsf{N} = \mathsf{N}_{\mathsf{G}}(\mathsf{T})$ is generated by M_a for all $a \in \Phi$, the epimorphism ${}^v\nu \colon N \to {}^vW(\Phi)$ is the quotient map $N \to {}^vW$ and the statement follows from 2.3.12.

(viii) By [Mil17, 21.49], $\mathsf{G}^{\mathrm{der}}$ is generated by U_a for all $a \in \Phi$. Then it is clear that $G^\circ = \mathsf{G}^{\mathrm{der}}(K)$, $T^\circ = T \cap G^\circ$ and $N^\circ = N_{G^\circ}(T^\circ)$.

Note that the above facts already will imply Theorem 2.1 using either *Tits system* or similar construction in Definition 3.2.9.

Definition 3.1.4. A *valuation on the root group datum* $(T, (U_a, M_a)_{a\in\Phi})$ is a family $\varphi = (\varphi_a)_{a\in\Phi}$ of functions $\varphi_a : U_a \longrightarrow \mathbb{R} \cup \{\infty\}$ satisfying the following axioms.

V0. For each $a \in \Phi$, the image of φ_a contains at least three elements.

V1. For each $a \in \Phi$ and any $\lambda \in \mathbb{R} \cup \{\infty\}$, the set $U_{a,\lambda} := \varphi_a^{-1}([\lambda, \infty])$ is a subgroup of U_a and $U_{a,\infty} = \{1\}$.

V2. For each $a \in \Phi$ and any $m \in M_a$, the function $u \mapsto \varphi_{-a}(u) - \varphi_a(mum^{-1})$ is constant on U^*_{-a}.

V3. For any pair $a, b \in \Phi$ not proportional and any $\lambda, \mu \in \mathbb{R}\cup\{\infty\}$, the commutator group $[U_{a,\lambda}, U_{b,\mu}]$ is contained in the subgroup generated by $U_{ia+jb,i\lambda+j\mu}$ for all $i, j > 0$ such that $ia + jb \in \Phi$.

V4. For each $a \in \Phi$ and any $u \in U_a$, $u', u'' \in U_{-a}$ such that $u'uu'' \in M_a$, we have $\varphi_{-a}(u') = \varphi_{-a}(u'') = -\varphi_a(u)$.

For each $a \in \Phi$, let Γ_a denote the set $\varphi_a(U_a^*)$ and for any $k \in \Gamma_a$, let $M_{a,k}$ be the intersection of M_a and $U_{-a}\varphi_a^{-1}(k)U_{-a}$.

Example 3.1.5. Let $\mathsf{T} = \mathbb{G}_{\mathrm{m}}^n$. Then (T, T) is a split reductive group with empty root system. Then there is only one way to define a valuation on the root group datum $(T, \emptyset)$, namely the *trivial valuation* $\mathbf{0}$.

Example 3.1.6. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$ over K. Denote $a_{ij} = \chi_i - \chi_j \in \Phi$ and define $\varphi = (\varphi_{a_{ij}})_{a_{ij}\in\Phi}$ as

$$\varphi_{a_{ij}}(u_{a_{ij}}(t)) = \mathrm{val}(t).$$

Note that we have

$$\varphi_{a_{ij}}\left(\xi_{ij}\left(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\right)\right) = \mathrm{val}(t), \qquad \varphi_{-a_{ij}}\left(\xi_{ij}\left(\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}\right)\right) = \mathrm{val}(t).$$

Then φ is a valuation on the root group datum $(D_n, (U_{a_{ij}}, M_{a_{ij}})_{a_{ij}\in\Phi})$ with $\Gamma_{a_{ij}} = \Gamma$:

V0. This clear as val is nontrivial.

V1. For any $\lambda \in \mathbb{R} \cup \{\infty\}$, we have

$$U_{a_{ij},\lambda} = \left\{ \xi_{ij}\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) \in U_{a_{ij}} \,\middle|\, \operatorname{val}(t) \geqslant \lambda \right\}.$$

Then $U_{a_{ij},\infty} = \{I_n\}$ and for any $x, y \in K$ with $\operatorname{val}(x), \operatorname{val}(y) \geqslant \lambda$, we have

$$u_{a_{ij}}(x) \cdot u_{a_{ij}}(y)^{-1} = u_{a_{ij}}(x - y),$$

and its valuation is $\operatorname{val}(x - y) \geqslant \min\{\operatorname{val}(x), \operatorname{val}(y)\} \geqslant \lambda$.

V2. For any $x, y, z \in K^{\times}$ and

$$m = \xi_{ij}\left(\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\right) \in M_{a_{ij}}, \qquad u = \xi_{ij}\left(\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\right) \in U^{*}_{-a_{ij}},$$

we have

$$mum^{-1} = \xi_{ij}\left(\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} 0 & y^{-1} \\ x^{-1} & 0 \end{pmatrix}\right) = \xi_{ij}\left(\begin{pmatrix} 1 & xzy^{-1} \\ 0 & 1 \end{pmatrix}\right).$$

Therefore

$$\begin{aligned} \varphi_{-a_{ij}}(u) - \varphi_{a_{ij}}(mum^{-1}) &= \varphi_{-a_{ij}}\left(u_{-a_{ij}}(z)\right) - \varphi_{a_{ij}}\left(u_{a_{ij}}(xzy^{-1})\right) \\ &= \operatorname{val}(z) - \operatorname{val}\left(xzy^{-1}\right) \\ &= -\operatorname{val}(x) + \operatorname{val}(y), \end{aligned}$$

independently of u.

V3. We need the following *commutator formula*:

$$[u_{a_{ij}}(x), u_{a_{kl}}(y)] = \begin{cases} u_{a_{il}}(xy) & \text{if } i \neq l, k = j, \\ u_{a_{kj}}(-xy) & \text{if } i = l, k \neq j, \\ I_n & \text{if } i \neq l, k \neq j. \end{cases}$$

From which, we see that if $\varphi_{a_{ij}}(u) \geqslant \lambda$ and $\varphi_{a_{kl}}(v) \geqslant \mu$, then either $[u, v] = I_n$ or $a_{ij} + a_{kl} \in \Phi$ and $\varphi_{a_{ij}+a_{kl}}([u, v]) \geqslant \lambda + \mu$.

V4. For any $x, y, z \in K$ and $u = u_{a_{ij}}(x)$, $u' = u_{-a_{ij}}(y)$, $u'' = u_{-a_{ij}}(z)$, we have

$$u'uu'' = \xi_{ij}\left(\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\right) = \xi_{ij}\left(\begin{pmatrix} 1+xz & x \\ y+z+xyz & 1+xy \end{pmatrix}\right).$$

If $u'uu'' \in M_{a_{ij}}$, then we must have

$$1 + xz = 1 + xy = 0, \qquad \text{and} \qquad -x^{-1} = y + z + xyz.$$

Hence we have $y = z = -x^{-1}$ and thus $\varphi_{-a_{ij}}(u') = \varphi_{-a_{ij}}(u'') = -\varphi_{a_{ij}}(u)$.

Now, let $\mathrm{val}(x) = \lambda$. Then we see that

$$M_{a_{ij},\lambda} = \left\{ \xi_{ij}\left(\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}\right) \middle| \, t \in K, \mathrm{val}(t) = \lambda \right\}.$$

Example 3.1.7. Notations as in Example 3.1.3. For any $a \in \Phi$, define

$$\varphi_a\left(u_a(t)\right) = \mathrm{val}(t).$$

Then $\varphi = (\varphi_a)_{a\in\Phi}$ is a valuation on the root group datum $(T, (U_a, M_a)_{a\in\Phi})$ with $\Gamma_a = \Gamma$. The proof is basically the same as in Example 3.1.6 plus the following facts:

(i) By Theorem 2.0(ii), any $m \in M_a$ can be written as $m = \boldsymbol{t}m_a(1)$ for some $\boldsymbol{t} \in T$. Hence for any $x \in K^\times$ and $u = u_{-a}(x)$, we have

$$\begin{aligned} \varphi_{-a}(u) - \varphi_a(mum^{-1}) &= \varphi_{-a}(u_{-a}(x)) - \varphi_a\left(\mathrm{inn}(\boldsymbol{t}m_a(1)).u_{-a}(x)\right) \\ &= \varphi_{-a}(u_{-a}(x)) - \varphi_a\left(\mathrm{inn}(\boldsymbol{t}).u_a(-x)\right) \\ &= \varphi_{-a}(u_{-a}(x)) - \varphi_a\left(u_a(-a(\boldsymbol{t})x)\right) \\ &= -\mathrm{val}(a(\boldsymbol{t})). \end{aligned}$$

(ii) For any $x, y \in K$, we have [SGA3, XXII, 5.5.2]

$$[u_a(x), u_b(y)] = \prod_{ia+jb\in\Phi} u_{ia+jb}(C_{a,b;i,j}x^iy^j),$$

where $C_{a,b;i,j} \in K$ is a constant.

3.1.8. Given a root group datum $(T, (U_a)_{a\in\Phi})$ in G and let φ be a valuation on it. Then for any vector $\mathbf{v}$ in the ambient space $\mathbb{V}$ of Φ, the family $\psi = (\psi_a)_{a\in\Phi}$ given by $\psi_a\colon u \mapsto \varphi_a(u) + a(\mathbf{v})$ is a valuation [BT-I, 6.2.5] and is denoted by $\varphi + \mathbf{v}$. The valuations φ and $\psi = \varphi + \mathbf{v}$ are said to be *equipollent*. The mapping $(\varphi, \mathbf{v}) \mapsto \varphi + \mathbf{v}$ defines an action of $\mathbb{V}$ on the set of valuations and each equipollent class is an orbit.

Let $\mathbb{A}$ denote the set of valuations equipollent to φ. Then $\mathbb{A}$ is an affine space with ${}^{v}\mathbb{A} = \mathbb{V}$ and 1.4.6 applies. For $\alpha = \alpha_{a+k}$ with $a \in \Phi$, $k \in \Gamma_a$, let $U_\alpha = U_{a,k}$ and $U_{\alpha+} = \bigcup_{h>k} U_{a,h}$ (note that $U_{\alpha+} = U_{\alpha_+}$ if Γ_a is discrete). It is clear that the affine root system Σ and the mapping $\alpha \mapsto U_\alpha$ depends only on the equipollent class of φ.

Example 3.1.9. Continue Example 3.1.6. The coroot space is (by Example 2.4.4)

$$\mathbb{V} = \{c_1\lambda_1 + \cdots + c_n\lambda_n \mid c_1, \cdots, c_n \in \mathbb{R}, c_1 + \cdots + c_n = 0\}.$$

Then for any $\mathbf{v} = c_1\lambda_1 + \cdots + c_n\lambda_n$, the valuation $\varphi + \mathbf{v}$ is given by

$$u_{a_{ij}}(x) \longmapsto \mathrm{val}(x) + a_{ij}(\mathbf{v}) = \mathrm{val}(x) + c_i - c_j.$$

Let $k \in \Gamma_a$, then $\varphi + \mathbf{v} \in \alpha_{a+k}$ if and only if $c_i - c_j + k \geqslant 0$.

Example 3.1.10. Notations as in Example 3.1.7. A valuation $\psi = (\psi_a)_{a\in\Phi}$ is said to be *compatible with* $\mathrm{val}(\,\cdot\,)$ if for all $u \in U_a$ and $\boldsymbol{t} \in T$,

$$\psi_a\left(\boldsymbol{t}u\boldsymbol{t}^{-1}\right) = \psi_a(u) + \mathrm{val}(a(\boldsymbol{t})).$$

It turns out that [BT-II, 4.2.9]: a valuation ψ is compatible with $\mathrm{val}(\,\cdot\,)$ if and only if it is equipollent to φ given in Example 3.1.7. Hence, $\mathbb{A}$ is precisely the set of all valuations compatible with $\mathrm{val}(\,\cdot\,)$.

3.1.11. Let $m \in N$ and $w = {}^{v}\nu(m) \in {}^{v}W$. Then the family $\psi = (\psi_a)_{a\in\Phi}$ given by $\psi_a\colon u \mapsto \varphi_{w^{-1}.a}(m^{-1}um)$ is a valuation [BT-I, 6.2.5] and is denoted by $m.\varphi$. We thus obtain an action of N on the set of valuations such that for any $m \in N$ and $\mathbf{v} \in \mathbb{V}$, we have $m.(\varphi + \mathbf{v}) = m.\varphi + {}^{v}\nu(m).\mathbf{v}$. Moreover [BT-I, 6.2.10]:

(i) The action of N stabilizes $\mathbb{A}$ and for any $m \in N$, the map $\nu(m)\colon \varphi \mapsto m.\varphi$ is an automorphism of the Euclidean affine space $\mathbb{A}$ whose vectorial part is ${}^v\nu(m)$.

(ii) For each $a \in \Phi$ and $k \in \Gamma_a$, the image of $M_{a,k}$ under ν is the reflection r_{a+k}.

(iii) The automorphism $\nu(m)$ maps affine roots to affine roots. For any $\alpha \in \Sigma$, we have $mU_\alpha m^{-1} = U_{\nu(m).\alpha}$

In particular, for $u \in U_a^*$, $\nu(m(u)) = r_{a+\varphi_a(u)}$ [BT-I, 6.2.12]. Therefore, the valuation φ is completely determined by the homomorphism $\nu\colon N \longrightarrow \mathrm{Aut}(\mathbb{A})$.

Example 3.1.12. Continue Examples 3.1.6 and 3.1.9.

(i) The normalizer N is the group of monomial matrices in $\mathsf{GL}_n(K)$. Any $m \in N$ can be written as

$$m = \sum_{k=1}^{n} x_k E_{\sigma(k)k},$$

where $\sigma \in \mathfrak{S}_n$ is a permutation such that $w = {}^v\nu(m)$ is identified with σ through ${}^vW \cong \mathfrak{S}_n$. Then, for any $u = u_{a_{ij}}(t) \in U_{a_{ij}}$, we have

$$\begin{aligned} m^{-1}um &= \left(\sum_{k=1}^{n} x^{-1}_{\sigma^{-1}(k)} E_{\sigma^{-1}(k)k}\right)\left(I_n + tE_{ij}\right)\left(\sum_{k=1}^{n} x_k E_{\sigma(k)k}\right) \\ &= I_n + x^{-1}_{\sigma^{-1}(i)} t x_{\sigma^{-1}(j)} E_{\sigma^{-1}(i)\sigma^{-1}(j)} \\ &= u_{a_{\sigma^{-1}(i)\sigma^{-1}(j)}}\left(x^{-1}_{\sigma^{-1}(i)} t x_{\sigma^{-1}(j)}\right) \in U_{a_{\sigma^{-1}(i)\sigma^{-1}(j)}} = U_{w^{-1}.a_{ij}}. \end{aligned}$$

Hence the valuation $m.\varphi$ is given by

$$(m.\varphi)_{a_{ij}}\colon u = u_{a_{ij}}(t) \longmapsto \varphi_{w^{-1}.a_{ij}}(m^{-1}um) = \mathrm{val}\left(x^{-1}_{\sigma^{-1}(i)} t x_{\sigma^{-1}(j)}\right).$$

From above computations, it is also clear that $mU_\alpha m^{-1} = U_{\nu(m).\alpha}$ holds for any affine root α.

(ii) For any $\mathbf{v} \in \mathbb{V}$, one can verify that

$$\begin{aligned}(m.(\varphi + \mathbf{v}))_{a_{ij}}(u) &= (\varphi + \mathbf{v})_{w^{-1}.a_{ij}}(m^{-1}um) \\ &= \varphi_{w^{-1}.a_{ij}}(m^{-1}um) + (w^{-1}.a_{ij})(\mathbf{v}) \\ &= (m.\varphi)_{a_{ij}}(u) + a_{ij}(w.\mathbf{v}) \\ &= (m.\varphi + w.\mathbf{v})_{a_{ij}}(u).\end{aligned}$$

Also note that

$$\begin{aligned}(m.\varphi)_{a_{ij}}(u) - \varphi_{a_{ij}}(u) &= \operatorname{val}\left(x^{-1}_{\sigma^{-1}(i)} t x_{\sigma^{-1}(j)}\right) - \operatorname{val}(t) \\ &= \operatorname{val}\left(x^{-1}_{\sigma^{-1}(i)}\right) - \operatorname{val}\left(x^{-1}_{\sigma^{-1}(j)}\right) \\ &= \left\langle a_{ij}, \sum_{k=1}^{n} \operatorname{val}\left(x^{-1}_{\sigma^{-1}(k)}\right)\lambda_k \right\rangle.\end{aligned}$$

Therefore the affine transformation $\nu(m)\colon \varphi \mapsto m.\varphi$ has vectorial part $w = {}^{v}\nu(m)$ and translation part

$$\mathbf{v}_m = \sum_{k=1}^{n} \left(\operatorname{val}\left(x^{-1}_{\sigma^{-1}(k)}\right) + \tfrac{1}{n}\operatorname{val}(\det(m))\right)\lambda_k \in \mathbb{V}.$$

(iii) If $u = u_{a_{ij}}(x) \in U^*_{a_{ij}}$, then

$$m(u) = m_{a_{ij}}(x) = \xi_{ij}\left(\begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}\right).$$

Hence ${}^{v}\nu(m(u)) = (i, j)$ and the translation part is

$$\mathbf{v}_{m(u)} = \operatorname{val}\left(x^{-1}\right)\lambda_i + \operatorname{val}(-x)\lambda_j = -\operatorname{val}(x)a^{\vee}_{ij} = -\varphi_{a_{ij}}(u)a^{\vee}_{ij}.$$

Then we have

$$\begin{aligned}\nu(m(u)).(\varphi + \mathbf{v}) &= \varphi + (i, j).\mathbf{v} - \varphi_{a_{ij}}(u)a^{\vee}_{ij} \\ &= \varphi + \mathbf{v} - a_{ij}(\mathbf{v})a^{\vee}_{ij} - \varphi_{a_{ij}}(u)a^{\vee}_{ij} \\ &= r_{a_{ij}+\varphi_{a_{ij}}(u)}(\varphi + \mathbf{v}).\end{aligned}$$

Example 3.1.13. The condition of being compatible with $\mathrm{val}(\,\cdot\,)$ in Example 3.1.10 can be interpreted as follows. Let $\psi = (\psi_a)_{a\in\Phi}$ be a valuation and $\mathbb{A}$ the set of valuations equipollent to it. For any $\boldsymbol{t} \in T$, the automorphism $\nu(\boldsymbol{t})\colon \mathbb{A} \to \mathbb{A}$ is a translation, denoted by $\mathbf{v}_{\boldsymbol{t}}$. Then, for all $a \in \Phi$ and $u \in U_a$, we have

$$\psi_a\left(\boldsymbol{t}^{-1}u\boldsymbol{t}\right) - \psi_a(u) = \langle a, \mathbf{v}_{\boldsymbol{t}}\rangle.$$

Therefore ψ is compatible with $\mathrm{val}(\,\cdot\,)$ if and only if

$$\langle a, \mathbf{v}_{\boldsymbol{t}}\rangle = -\operatorname{val}(a(\boldsymbol{t}))$$

for all $\boldsymbol{t} \in T$ and $a \in \Phi$.

Now, suppose ψ is compatible with $\mathrm{val}(\,\cdot\,)$. Then the above shows that for all $\chi \in \mathsf{X}_{\mathrm{ss}} := \mathbb{V}^* \cap \mathsf{X}$, we have

$$\langle \chi, \mathbf{v}_{\boldsymbol{t}}\rangle = -\operatorname{val}(\chi(\boldsymbol{t})).$$

This implies that $\mathbf{v}_{\boldsymbol{t}} \in \mathsf{X}_{\mathrm{ss}}^\vee \otimes \Gamma$, where $\mathsf{X}_{\mathrm{ss}}^\vee$ is the dual lattice of X_{ss} and is the cocharacter group of the semisimple quotient $(\mathsf{G}^{\mathrm{ss}}, \mathsf{T}/\mathscr{R}(\mathsf{G}))$.

3.1.14. Let $H = \operatorname{Ker}(\nu)$ and $\widehat{W} = \nu(N)$. Let W denote the subgroup of $\widehat{W}$ generated by r_{a+k} with $a \in \Phi$ and $k \in \Gamma_a$. It is a normal subgroup because N permutes $M_{a,k}$. Let $N' = \nu^{-1}(W)$. It is usually not the entire N. We say the root group datum (together with the valuation φ) is *simply-connected* when $N' = N$. Let $T' = T \cap N'$ and let G' be the subgroup of G generated by N' and the U_a for $a \in \Phi$. Since $M_a \cap N' \neq \emptyset$ for all $a \in \Phi$, we see that $(T', (U_a)_{a\in\Phi})$ is a simply-connected generating root group datum in G' [BT-I, 6.2.11]. Recall that (Theorem 3.0(viii)) N° is generated by M_a° for all $a \in \Phi$, hence $N^\circ \subseteq N'$ and therefore the generating root group datum $(T^\circ, (U_a)_{a\in\Phi})$ on G° is also simply-connected.

The valuation φ is said to be *special* if $0 \in \Gamma_a$ for all $a \in \Phi$. If this is the case, then the group W (resp. $\widehat{W}$) can be decomposed as $W = W_\varphi \ltimes \operatorname{Ker}\big(W \to {}^{v}W\big)$ (resp. $\widehat{W} = W_\varphi \ltimes \nu(T)$) [BT-I, 6.2.19], where W_φ is the stabilizer of φ.

The valuation φ is said to be *discrete* if Γ_a is a discrete subset of $\mathbb{R}$ for all $a \in \Phi$. If this is the case, then W is the affine Weyl group $W(\Sigma)$ for the affine root system Σ [BT-I, 6.2.22].

Suppose Φ is irreducible and φ is discrete and special. This is the case we most focus on. Then all Γ_a are the same discrete subgroup Γ of $\mathbb{R}$ [BT-I, 6.2.23]. So 1.4.7 applies and we get an apartment $\mathscr{A}(\Sigma)$. Then $\mathrm{Ker}\big(W \to {}^{v}W\big) = \mathcal{Q}^\vee \otimes \Gamma$ and $\nu(T)$ is between $\mathcal{Q}^\vee \otimes \Gamma$ and $\mathcal{P}^\vee \otimes \Gamma$ [BT-I, 6.2.20].

Example 3.1.15. Continue Example 3.1.5. Since $\mathbb{A} = \{\mathbf{0}\}$, we must have $N' = N = T$. On the other side, $N^\circ = \{1\}$ gives a smaller simply-connected root group datum. The trivial valuation $\mathbf{0}$ on them is both special and discrete.

Example 3.1.16. Continue Examples 3.1.6, 3.1.9 and 3.1.12. Let $m \in N$ with related notations as before. Then for $m \in \mathrm{Ker}(\nu)$, one must have both ${}^{v}\nu(m) = \mathrm{id}$ and $\mathbf{v}_m = 0$. Hence m is diagonal and for all $1 \leqslant k \leqslant n$, $\mathrm{val}(x_k) = \frac{1}{n}\,\mathrm{val}(\det(m))$. Therefore

$$H = \{\mathrm{diag}(x_1, \cdots, x_n) \in \mathsf{D}_n(K) \mid \mathrm{val}(x_1) = \cdots = \mathrm{val}(x_n)\}.$$

It is clear from the computations above that the translation group $\nu(T)$ is $\mathsf{X}_{\mathrm{ss}}^\vee \otimes \Gamma$. On the other hand, the translation group of W is clearly $\mathcal{Q}^\vee \otimes \Gamma$, which has index n in the previous one.

Let $m \in N$ with related notations as before. Then for $m \in N'$, one must have $\mathbf{v}_m \in \mathcal{Q}^\vee \otimes \Gamma$, which is equivalent to say that for all $1 \leqslant k \leqslant n$, $\mathrm{val}\big(x_k^{-1}\big) + \frac{1}{n}\,\mathrm{val}(\det(m)) \in \Gamma$, which is the case if and only if $\frac{1}{n}\,\mathrm{val}(\det(m)) \in \Gamma$. Therefore

$$N' = \{m \in N \mid \mathrm{val}(\det(m)) \in n\Gamma\}.$$

Therefore we have

$$G' = \{g \in \mathsf{GL}_n(K) \mid \mathrm{val}(\det(g)) \in n\Gamma\}.$$

It is worth to mention that there is a group

$$\mathsf{GL}_n(K)^1 := \{g \in \mathsf{GL}_n(K) \mid \mathrm{val}(\det(g)) = 0\},$$

between G' and $G^\circ = \mathsf{SL}_n(K)$. Hence for this group, the generating root group datum $(D_n \cap \mathsf{GL}_n(K)^1, (U_{a_{ij}})_{a_{ij}\in\Phi})$ is simply-connected.

Example 3.1.17. Continue Examples 3.1.7, 3.1.10 and 3.1.13. Let $\mathbb{A}$ be the affine space of all valuations compatible with $\operatorname{val}(\,\cdot\,)$. For any $m \in N$, the automorphism $\nu(m)$ is trivial if and only if $m \in T$ (hence ${}^{v}\nu(m) = \mathrm{id}$) and the translation vector $\mathbf{v}_m = 0$. Therefore

$$H = \{\boldsymbol{t} \in T \mid \operatorname{val}(\chi(\boldsymbol{t})) = 0 \text{ for all } \chi \in \mathsf{X}_{\mathrm{ss}}\}.$$

It follows from Example 3.1.13 that $\nu(T) \subseteq \mathsf{X}_{\mathrm{ss}}^\vee \otimes \Gamma$. Conversely, for any $\lambda \in \mathsf{X}_{\mathrm{ss}}^\vee$ and $t \in K^\times$, we have

$$\langle \chi, \mathbf{v}_{\lambda(t)} \rangle = -\operatorname{val}(\chi(\lambda(t))) = -\operatorname{val}(t)\langle \chi, \lambda \rangle.$$

Hence $\nu(T) \subseteq \mathsf{X}_{\mathrm{ss}}^\vee \otimes \Gamma$. On the other hand, the translation group of W is clearly $\mathcal{Q}^\vee \otimes \Gamma$.

One can replace $T = \mathsf{T}(K)$ by a suitable subgroup and obtain a different root group datum. The discussions on valuations still hold. It is also worth to mention that [BT-II, 4.2.16]: there is a group

$$G^1 := \{g \in \mathsf{G}(K) \mid \operatorname{val}(\chi[g]) = 0 \text{ for all } \chi \in \mathsf{X}(\mathsf{G})\},$$

between G' and $G^\circ = \mathsf{G}^{\mathrm{der}}$. Hence for this group, the generating root group datum $(T \cap G^1, (U_a)_{a\in\Phi})$ is simply-connected.

3.2 Bruhat-Tits building

Given a root group datum $(T, (U_a, M_a)_{a\in\Phi})$ in G with a valuation $\varphi = (\varphi_a)_{a\in\Phi}$ on it, Bruhat and Tits [BT-I] associate an affine building equipped with natural G-action to these data.

3.2.1. Let Ω be a nonempty subset of $\mathbb{A}$ and let U_Ω denote the subgroup generated by U_α for all affine roots $\alpha \supseteq \Omega$. Then the image of $N \cap U_\Omega$ under $\nu\colon N \to \widehat{W}$ is generated by the reflections r_α for affine roots α such that $\Omega \subseteq \partial\alpha$ and is identified with the

Weyl group of $\Phi_\Omega := \left\{a \in \Phi \,\middle|\, \exists \alpha, {}^v\alpha = a, \partial\alpha \supseteq \Omega\right\}$ [BT-I, 7.1.3]. Let N_Ω denote its preimage and let $P_\Omega = H \cdot U_\Omega$. Then

$$N_\Omega = N \cap P_\Omega.$$

Let $\widehat{N}_\Omega$ denote the fixator of Ω in N:

$$\widehat{N}_\Omega := \{n \in N \mid \nu(n).x = x \text{ for all } x \in \Omega\}.$$

Then $\widehat{N}_\Omega$ contains N_Ω and normalizes P_Ω. Hence

$$\widehat{P}_\Omega := \widehat{N}_\Omega \cdot P_\Omega = \widehat{N}_\Omega \cdot U_\Omega$$

is a group having P_Ω and U_Ω as its normal subgroups. By 3.1.11, for any $n \in N$, we have

$$\operatorname{inn}(n).P_\Omega = P_{\nu(n).\Omega} \qquad \text{and} \qquad \operatorname{inn}(n).\widehat{P}_\Omega = \widehat{P}_{\nu(n).\Omega}.$$

Note that the map $\Omega \mapsto U_\Omega$ (resp. Φ_Ω, N_Ω, P_Ω, $\widehat{N}_\Omega$, $\widehat{P}_\Omega$) reverses the order of inclusions.

Remark. For $x \in \mathbb{A}$ a point, $\widehat{N}_x = \nu^{-1}(\widehat{W}_x)$. Hence if the generating root group datum is simply-connected, we have $\widehat{N}_x = N_x$ and hence $\widehat{P}_x = P_x$.

Example 3.2.2. Continue Examples 3.1.6, 3.1.9, 3.1.12 and 3.1.16.

First consider $\Omega = \alpha_{a_{ij}+k} \in \Sigma$. Then $U_\Omega = U_{a_{ij},k}$, $\Phi_\Omega = \emptyset$ and

$$P_\Omega = H \cdot U_{a_{ij},k} = \left\{ \operatorname{diag}(x_1, \cdots, x_n) + tE_{ij} \in \mathsf{GL}_n(K) \,\middle|\, \begin{array}{l} \operatorname{val}(x_1) = \cdots = \operatorname{val}(x_n), \\ \operatorname{val}(t) - \operatorname{val}(x_i) \geqslant k \end{array} \right\}.$$

In particular, $N_\Omega = H$. Note that we also have $\widehat{N}_\Omega = H$ since Ω contains an open in $\mathbb{A}$. Therefore $\widehat{P}_\Omega = P_\Omega$.

Next, consider $x = \varphi + \mathbf{v} \in \mathbb{A}$. Then U_x is generated by $U_{a_{ij},-a_{ij}(\mathbf{v})}$ for all $a_{ij} \in \Phi$ and $\Phi_x = \left\{a_{ij} \in \Phi \,\middle|\, a_{ij}(\mathbf{v}) \in \Gamma\right\}$. Then W_x is generated by $r_{a,-a(\mathbf{v})}$ for all $a \in \Phi_x$ but $\widehat{W}_x$ may be larger in general: it contains $r_{a,-a(\mathbf{v})}$ even when $a(\mathbf{v}) \notin \Gamma$. Now, suppose x is special. Then $\widehat{W}_x = W_x \cong {}^vW$ and $\widehat{P}_x = P_x$ is generated by $H \cdot U_{a_{ij},-a_{ij}(\mathbf{v})}$ for all $a_{ij} \in \Phi$.

Example 3.2.3. In the above example, if we insteadly use the root group datum $(D_n \cap \mathsf{GL}_n(K)^1, (U_{a_{ij}})_{a_{ij}\in\Phi})$. Then for $\Omega = \alpha_{a_{ij}+k}$, we have

$$\widehat{P}_\Omega = P_\Omega = \left\{ \mathrm{diag}(x_1, \cdots, x_n) + tE_{ij} \in \mathsf{GL}_n(K) \;\middle|\; \begin{array}{l} \mathrm{val}(x_1) = \cdots = \mathrm{val}(x_n) = 0, \\ \mathrm{val}(t) - \mathrm{val}(x_i) \geqslant k \end{array} \right\}.$$

To see what are $\widehat{P}_\Omega$ and P_Ω in general, we need more knowledge on such subgroups.

Proposition 3.2.4 ([BT-I, 7.1.11])**.** *Let Ω be a nonempty subset of $\mathbb{A}$. Then*

$$\widehat{P}_\Omega = \bigcap_{x\in\Omega} \widehat{P}_x.$$

So in particular, $\widehat{P}_\Omega \cap \widehat{P}_{\Omega'} = \widehat{P}_{\Omega\cup\Omega'}$.

3.2.5. Let Ω be a nonempty subset of $\mathbb{A}$. The *enclosure* $\mathrm{cl}(\Omega)$ of Ω is the intersection of all affine roots α containing Ω. It turns out that [BT-I, 7.1.2 and 7.1.9]

$$U_{\mathrm{cl}(\Omega)} = U_\Omega, \qquad \widehat{N}_{\mathrm{cl}(\Omega)} = \widehat{N}_\Omega \qquad \text{and} \qquad \widehat{P}_{\mathrm{cl}(\Omega)} = \widehat{P}_\Omega.$$

From its definition, we see that $\mathrm{cl}(\Omega)$ must be a disjoint union of some facets in $\mathbb{A}$. With Proposition 3.2.4, we conclude that the groups $\widehat{P}_\Omega$ are all of the form

$$\bigcap_F \widehat{P}_F,$$

where F ranges over all facets in $\mathbb{A}$ such that $F \subseteq \overline{\mathrm{cl}(\Omega)}$.

Hence to understand the subgroups $\widehat{P}_\Omega$, it suffices to understand those $\widehat{P}_F$.

3.2.6. Let ${}^vC$ be a Weyl chamber in $\mathbb{A}$ and $\Phi^+_{{}^vC}$ (resp. $\Phi^-_{{}^vC}$) the system of positive (resp. negative) roots in Φ defined by ${}^vC$. Let $U^+_{{}^vC}$ (resp. $U^-_{{}^vC}$) the subgroup of G generated by the U_a for $a \in \Phi^+_{{}^vC}$ (resp. $a \in \Phi^-_{{}^vC}$). Then for any $x \in \mathbb{A}$, we have $U_{x\pm{}^vC} \subseteq U^\pm_{{}^vC}$ and $\widehat{N}_{x\pm{}^vC} = N_{x\pm{}^vC} = H$. As a consequence, $\widehat{P}_{x+{}^vC} = P_{x+{}^vC}$. Denote it by $B_{x,{}^vC}$.

In general, let Ω be a nonempty subset of $\mathbb{A}$. Then we have [BT-I, 7.1.4]

$$P_\Omega \cap U^\pm_{{}^vC} = U_{\Omega\pm{}^vC} \qquad \text{and} \qquad P_\Omega = N_\Omega \cdot U_{\Omega+{}^vC} \cdot U_{\Omega-{}^vC}.$$

As a consequence, we have [BT-I, 7.1.8]

$$\widehat{P}_\Omega \cap U^\pm_{{}^vC} = U_{\Omega\pm{}^vC}, \qquad \widehat{P}_\Omega \cap N = \widehat{N}_\Omega \qquad \text{and} \qquad \widehat{P}_\Omega = \widehat{N}_\Omega \cdot U_{\Omega+{}^vC} \cdot U_{\Omega-{}^vC}.$$

Theorem 3.1 (Bruhat decomposition [BT-I, 7.3.4]). *Let ${}^{v}C$ and ${}^{v}C'$ be two Weyl chambers and x, x' be two points in $\mathbb{A}$.*

(i) *We have*

$$G = B_{x,{}^{v}C} \cdot N \cdot B_{x',{}^{v}C'}.$$

(ii) *More precisely, the canonical map from N to the set of double cosets induces a bijection from $\widehat{W} \cong N/H$ to $B_{x,{}^{v}C} \backslash G / B_{x',{}^{v}C'}$.*

Example 3.2.7. Continue Examples 3.1.6, 3.1.9, 3.1.12, 3.1.16 and 3.2.2. Let Ω be a nonempty subset of $\mathbb{A}$. We claim that [11] (with convention that $a_{ii} = 0$)

$$\widehat{P}_\Omega = \left\{ g = (g_{ij})_{i,j} \in \mathsf{GL}_n(K) \;\middle|\; \forall i, j : \mathrm{val}\big(g_{ij}\big) - \tfrac{1}{n}\,\mathrm{val}(\det(g)) \geqslant -\inf_{x \in \Omega} a_{ij}(x) \right\}.$$

Proof. Denote the right hand side by L_Ω. Then it is clear that

$$L_\Omega = \bigcap_{x \in \Omega} L_x.$$

Therefore it suffices to show $\widehat{P}_x = L_x$.

First, we have $H \subseteq L_x$ and for any $a_{ij} \in \Phi$,

$$L_x \cap U_{a_{ij}} = U_{a_{ij}, -a_{ij}(x)} = P_x \cap U_{a_{ij}}.$$

Therefore, $P_x \subseteq L_x$. Let ${}^{v}C$ be any Weyl chamber, then we have $B_{x,{}^{v}C} \subseteq P_x \subseteq L_x$. Hence by Theorem 3.1, we have

$$L_x = B_{x,{}^{v}C} \cdot (L_x \cap N) \cdot B_{x,{}^{v}C}.$$

Therefore it suffices to show $L_x \cap N = \widehat{N}_x$.

If $m = \sum_{k=1}^{n} x_k E_{\sigma(k)k} \in L_x \cap N$ with $\sigma = {}^{v}\nu(m)$, then we have

$$\mathrm{val}\Big(x_{\sigma^{-1}(k)}\Big) - \frac{1}{n}\,\mathrm{val}(\det(m)) \geqslant -a_{k\sigma^{-1}(k)}(x) \qquad (1 \leqslant k \leqslant n).$$

[11] Slightly different from [BT-I, 10.2.9] due to different conventions on the root group datum

This implies that

$$\sum_{k=1}^{n} \mathrm{val}(x_k) - \mathrm{val}(\det(m)) \geqslant 0,$$

which should be an equality. Therefore for all $1 \leqslant k \leqslant n$, we have

$$\mathrm{val}\left(x_{\sigma^{-1}(k)}^{-1}\right) + \frac{1}{n}\mathrm{val}(\det(m)) = a_{k\sigma^{-1}(k)}(x).$$

Then, by Example 3.1.12, we have (writing x as $\varphi + \mathbf{v}$)

$$m.x - x = \sigma.\mathbf{v} - \mathbf{v} + \sum_{k=1}^{n} a_{k\sigma^{-1}(k)}(x)\lambda_k.$$

Note that

$$\sigma.\mathbf{v} - \mathbf{v} = \sum_{k=1}^{n} \langle a_{\sigma^{-1}(k)k}, \mathbf{v}\rangle \lambda_k.$$

Therefore we have

$$m.x - x = \sum_{k=1}^{n} \left(a_{\sigma^{-1}(k)k}(x) + a_{k\sigma^{-1}(k)}(x)\right) \lambda_k = 0.$$

This shows $L_x \cap N \subseteq \widehat{N}_x$.

Conversely, if $m = \sum_{k=1}^{n} x_k E_{\sigma(k)k} \in \widehat{N}_x$ with $\sigma = {}^{v}\nu(m)$, then $m.x = x$. Which, by similar argument as above, implies

$$\mathrm{val}\left(x_{\sigma^{-1}(k)}^{-1}\right) + \frac{1}{n}\mathrm{val}(\det(m)) = a_{k\sigma^{-1}(k)}(x).$$

Since other entries of m are 0, the inequality holds trivially. Therefore $m \in L_x$. □

Example 3.2.8. Consider the root group datum $(D_n \cap \mathsf{GL}_n(K)^1, (U_{a_{ij}})_{a_{ij}\in\Phi})$. Similar argument as in Example 3.2.7 shows that (note that it is simply connected)

$$P_\Omega = \left\{(g_{ij})_{i,j} \in \mathsf{GL}_n(K)^1 \;\middle|\; \forall i,j : \mathrm{val}\left(g_{ij}\right) \geqslant -\inf_{x\in\Omega}\left(\chi_i(x) - \chi_j(x)\right)\right\}.$$

In particular, if we take Ω to be the origin $o = \varphi$, then we have

$$P_o = \left\{(g_{ij})_{i,j} \in \mathsf{GL}_n(K)^1 \;\middle|\; \forall i,j : \mathrm{val}\left(g_{ij}\right) \geqslant 0\right\} = \mathsf{GL}_n(K^\circ).$$

More generally, a special point $x = \varphi + \mathbf{v}$ defines a K°-submodule

$$L = \{(x_1, \cdots, x_n) \in K^n \mid \mathrm{val}(x_i) + \chi_i(\mathbf{v}) \geqslant 0\}$$

of K^n and we have $P_x = \left\{g \in \mathsf{GL}_n(K)^1 \,\middle|\, g.L = L\right\}$.

Definition 3.2.9. The *Bruhat-Tits building* of a valuation φ on a root group datum $(T, (U_a)_{a\in\Phi})$ in G is the quotient set $\mathscr{B}(\varphi)$ of $G \times \mathbb{A}$ under the following equivalent relation [BT-I, 7.4.1]:

$$(g, x) \sim (h, y) \iff \exists n \in N : y = \nu(n).x,\ g^{-1}hn \in \widehat{P}_x.$$

We will simply denote this set by $\mathscr{B}$ if there is no ambiguity.

Remark. Let (G, T) be a split reductive group over K. By Example 3.1.10, there is essentially only one (up to equipollence) reasonable way to define a valuation on the *standard root group datum* [12] given in Example 3.1.3. Therefore there is a unique affine building $\mathscr{B}(\mathsf{G})$ associated to it. It is called the *Bruhat-Tits building* of G.

Then following the same argument in 2.5.6, we see that the Bruhat-Tits building depends only on the root system Φ and the ground field K.

3.2.10. The left multiplication of G on the product $G \times \mathbb{A}$ is compatible with above equivalent relation, hence $\mathscr{B}$ inherits a G-action. Identifying $\mathbb{A}$ with the subset $\{1\} \times \mathbb{A}$ of $\mathscr{B}$, we have:

(i) $\mathscr{B} = \bigcup_{g\in G} g.\mathbb{A}$;

(ii) each U_α fixes $\alpha \in \Sigma$ pointwise [BT-I, 7.4.5];

(iii) for each nonempty $\Omega \subseteq \mathbb{A}$, its fixator is $\widehat{P}_\Omega$ and it acts transitively on apartments containing Ω [BT-I, 7.4.4, 7.4.9];

(iv) the stabilizer (resp. fixator) of $\mathbb{A}$ is N (resp. H) [BT-I, 7.4.10].

[12] as long as on root group data deduced from the standard one such as $(T \cap G^1, (U_a)_{a\in\Phi})$.

Then one can carry the apartment structure on $\mathbb{A}$ to each $g.\mathbb{A}$ and see that they agree on the intersections [BT-I, 7.4.18]. Hence $\mathscr{B}$ is a building of type $\mathscr{A}(\Sigma)$. The action of G on it is strongly transitively by the construction but is not necessarily type-preserving since the affine Weyl group W of $\mathscr{A}(\Sigma)$ is usually not the entire $\widehat{W}$. The subgroup of type-preserving automorphisms is then the group $G' = \nu^{-1}(W)$ introduced in 3.1.14.

3.2.11. Let $\lambda\colon \Phi \to \mathbb{R}_{>0}$ be a function, constant on each irreducible component, and let $\mathbf{v} \in \mathbb{V}$. Then the family $u \mapsto \lambda(a)\varphi_a(u) + a(\mathbf{v})$ defines a valuation [BT-I, 6.2.5] which is denoted by $\lambda\varphi + \mathbf{v}$. A valuation ψ is said to be *equivalent to* φ if $\psi = \lambda\varphi + \mathbf{v}$ for some λ and $\mathbf{v}$. If this is the case, then there is a unique G-equivalent map $i\colon \mathscr{B}(\varphi) \to \mathscr{B}(\psi)$ such that its restriction to $\mathbb{A}$ is an *affinié* from $\mathbb{A} = \varphi + \mathbb{V}$ to $\psi + \mathbb{V}$ with homothetic ratio λ [BT-I, 7.4.3].

3.2.12. Let Φ_1 be a closed subroot system of Φ, N_1° be the subgroup generated by M_a° for all $a \in \Phi_1$ and let $T_1^\circ = N_1^\circ \cap T$. In addition, let T_1 be a subgroup of T containing T_1° and let G_1 be the subgroup of G generated by U_a for all $a \in \Phi_1$ and T_1. Then $(T_1, (U_a, M_a^\circ.T_1)_{a\in\Phi_1})$ is a generating root group datum on G_1 and φ induces a valuation on it. Let $\mathscr{B}_1$ be the Bruhat-Tits building associated to these data. Then the underlying set of $\mathscr{B}_1$ is canonically identified with the quotient of the subset $G_1.\mathbb{A}$ (as a G_1-set) of $\mathscr{B}$ by the intersection of the kernels of all $a \in \Phi_1$ [BT-I, 7.6.4]. The image of $\mathbb{A}$ in $\mathscr{B}_1$ is denoted by $\mathbb{A}_1$.

3.2.13. The *bornology defined by* φ is the bornology $\mathcal{B}(\varphi)$ on G induced from the action of G on the building $\mathscr{B}(\varphi)$ as in 1.6.7. It is the smallest bornology on G containing the bornology on N induced from the action of N on $\mathbb{A}$, the subgroups $U_{a,k}$ for all a and k and is compatible with the group law [BT-I, 8.1.4, 8.1.8]. This bornology makes G a bornological group and in which each $U_{a,k}$ is bounded while each U_a is not. Note that the subgroups $\widehat{P}_x$ are bounded but not necessarily maximal bounded. We refer last two paragraphs of 1.6.7 for a discussion of its bounded subgroups.

Let ψ be another valuation. Then the following are equivalent [BT-I, 8.1.10]

(i) ψ is equivalent to φ;

(ii) $\mathcal{B}(\psi) = \mathcal{B}(\varphi)$;

(iii) $\mathcal{B}(\psi)$ and $\mathcal{B}(\varphi)$ agrees on each U_a;

(iv) $\mathcal{B}(\psi)$ and $\mathcal{B}(\varphi)$ agrees on N.

Example 3.2.14. Continue Example 3.2.7. Before moving on, note that all discussions before apply if we replace $G = \mathsf{GL}_n(K)$ by a subgroup G_1 obtained as in 3.2.12. So we simply let G denote either $\mathsf{GL}_n(K)$ or such a subgroup.

The bornology $\mathcal{B}$ on G defined by φ can be described as follows: a subset M is bounded when the set

$$\left\{\mathrm{val}\big(g_{ij}\big) - \tfrac{1}{n}\,\mathrm{val}(\det(g)) \,\middle|\, g = (g_{ij}) \in M, 1 \leqslant i, j \leqslant n\right\}$$

is bounded from below. Indeed, it suffices to verify on N: for any $M \subseteq N$, the above set is $\{-\chi_i(v_m) \mid m \in M, 1 \leqslant i \leqslant n\}$, which is bounded from below if and only if it is bounded (since $\chi_1 + \cdots + \chi_n$ vanishes on $\mathbb{V}$) if and only if $M.\varphi$ is bounded.

3.3 Concave functions

One important ingredient in Bruhat-Tits theory is the theory of various subgroups associated to concave functions. They are refinements of parabolic subgroups and generalizations of $\widehat{P}_*$ and P_* in previous subsection.

3.3.1 ([BT-I, 6.4.1])**.** Let's first introduce the *ordered monoid of extended real numbers* $\widetilde{\mathbb{R}}$. Formally, $\widetilde{\mathbb{R}}$ is the union of

$$\mathbb{R}, \qquad \mathbb{R}+ := \{k+ \mid k \in \mathbb{R}\} \qquad \text{and} \qquad \{\infty\}$$

The commutative addition on $\mathbb{R}$ is extended to $\widetilde{\mathbb{R}}$ as follows:

- for all $k, l \in \mathbb{R}$, $k + (l+) = (k+) + (l+) = (k + l)+$;
- for all $\lambda \in \widetilde{\mathbb{R}}$, $\lambda + \infty = \infty$.

The total order on $\mathbb{R}$ is extended to $\widetilde{\mathbb{R}}$ as follows:

- for all $k, l \in \mathbb{R}$ such that $k < l$, $k < k+ < l$;
- for all $\lambda \in \widetilde{\mathbb{R}}$ such that $\lambda \neq \infty$, $\lambda < \infty$.

Whenever we have a filtration $\{F_k\}_{k\in\mathbb{R}}$ (for instance, the filtration $\{U_{a,k}\}_{k\in\mathbb{R}}$ of a root subgroup U_a in 3.1.4), we can extend it to $\{F_\lambda\}_{\lambda\in\widetilde{\mathbb{R}}}$ by defining

$$F_\lambda = \bigcup_{k\in\mathbb{R}, k\geqslant\lambda} F_k, \qquad F_\infty = \bigcap_{k\in\mathbb{R}} F_k.$$

We say $k \in \mathbb{R}$ is a *jump* of the filtration if $F_{k+} \neq F_k$. In our most usage of the filtrations, the jumps are elements of Γ. For any $\lambda \in \widetilde{\mathbb{R}}$, we use the notation $\lceil\lambda\rceil$ to denote the smallest $k \in \Gamma$ such that $\lambda \leqslant k$.

Definition 3.3.2 ([BT-I, 6.4.3; BT-II, 4.5.3])**.** Let Φ be a root system and denote $\widetilde{\Phi} = \Phi \cup \{0\}$. A *concave function on* $\widetilde{\Phi}$ is a function $f: \widetilde{\Phi} \to \widetilde{\mathbb{R}}$ such that

C. for any finite family (a_i) in $\widetilde{\Phi}$ such that $\sum_i a_i \in \widetilde{\Phi}$, we have

$$\sum_i f(a_i) \geqslant f(\sum_i a_i).$$

Note that the axiom is equivalent to the following:

C1. for any roots $a, b \in \Phi$ such that $a + b \in \Phi$, we have $f(a) + f(b) \geqslant f(a + b)$;

C2. for any root $a \in \Phi$, we have $f(a) + f(-a) \geqslant f(0)$;

C3. $f(0) \geqslant 0$.

A concave function f on $\widetilde{\Phi}$ is said to be a *concave function on* Φ if $f(0) = 0$ and $f(\Phi) \subseteq \mathbb{R}$. Equivalently, a concave function f on Φ is a function $f: \Phi \to \mathbb{R}$ satisfying **C1.** and **C2.**.

3.3.3. Let f be a concave function on $\widetilde{\Phi}$. We use U_f to denote the subgroup generated by $U_{a,f(a)}$ for all $a \in \Phi$. Given a choice of positive roots Φ^+ of Φ, we denote the intersection $U_f \cap U^+$ (resp. $U_f \cap U^-$) by U_f^+ (resp. U_f^-). Then we have the following facts [BT-I, 6.4.9].

(i) $U_f \cap U_a = U_{a,f(a)}$ for any $a \in \Phi$;

(ii) The homomorphisms

$$\prod_{a\in\Phi^+} U_{a,f(a)} \to U_f^+ \qquad \text{and} \qquad \prod_{a\in\Phi^-} U_{a,f(a)} \to U_f^-$$

are bijective regardless of the order of factors.

We refer to [BT-I, 6.4.38] for the condition of a *good filtration* $\{H_k\}_{k\geqslant 0}$ on H, under the name *prolongement de la valuation*. Note that one of the requirement is

$$H_{[0]} \subseteq H_0 \subseteq H,$$

where $H_{[0]}$ is the subgroup of H generated by $U_{a,k} \cup U_{-a,-k}$ [BT-I, 6.4.14].

We fix a *good filtration* H_k on H. Let P_f denote the subgroup $H_{f(0)} \cdot U_f$, then we have the following multiplication map:

$$\prod_{a\in\Phi^+} U_{a,f(a)} \times H_{f(0)} \times \prod_{a\in\Phi^-} U_{a,f(a)} \longrightarrow P_f. \tag{3.3.1}$$

It is injective in general and moreover bijective if $f(0) > 0$ [BT-I, 6.4.48].

Example 3.3.4 ([BT-I, 6.4.2; BT-II, 4.6.26])**.** Let Ω be a nonempty subset of $\mathbb{A}$. Define $f_\Omega\colon \Phi \to \mathbb{R} \cup \{\infty\}$ by

$$f_\Omega(a) = \inf\{k \in \mathbb{R} \mid \Omega \subseteq \alpha_{a+k}\}.$$

Then f_Ω is a concave function on Φ. We then have $U_{f_\Omega} = U_\Omega$ and the group P_{f_Ω} is a subgroup of P_Ω in general. The group P_{f_F} with F a facet in $\mathbb{A}$ is called a *parahoric subgroup*, but this terminology usually restricts to a specific choice of H_0.

Note that $f_\Omega \neq f_{\mathrm{cl}(\Omega)}$ in general, while $U_\Omega = U_{\mathrm{cl}(\Omega)}$ and $P_\Omega = P_{\mathrm{cl}(\Omega)}$.

Example 3.3.5. To make above more clear, let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$ and refer Examples 3.1.6, 3.1.9, 3.1.12, 3.1.16, 3.2.2, 3.2.7 and 3.2.14. Then we have

$$U_{a_{ij},k} = \left\{ \xi_{ij}\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) \in U_{a_{ij}} \;\middle|\; \mathrm{val}(t) \geqslant k \right\}.$$

$$H = \{\mathrm{diag}(x_1, \cdots, x_n) \in \mathsf{GL}_n \mid \mathrm{val}(x_1) = \cdots = \mathrm{val}(x_n)\},$$

$$H_{[0]} = \{\mathrm{diag}(x_1, \cdots, x_n) \in \mathsf{SL}_n \mid \mathrm{val}(x_1) = \cdots = \mathrm{val}(x_n) = 0\}.$$

We will also consider

$$H^\circ := \{\mathrm{diag}(x_1, \cdots, x_n) \in \mathsf{GL}_n \mid \mathrm{val}(x_1) = \cdots = \mathrm{val}(x_n) = 0\}.$$

Now, let $\Omega = \alpha_{a_{ij}+k}$. Then we have

$$U_\Omega = U_{\alpha_{a_{ij},k}} = U_{\alpha_{a_{ij},\lceil k \rceil}} = U_{\alpha_{a_{ij}+\lceil k \rceil}} = U_{\mathrm{cl}(\Omega)},$$

$$P_\Omega = \left\{ \mathrm{diag}(x_1, \cdots, x_n) + tE_{ij} \in \mathsf{GL}_n(K) \;\middle|\; \begin{matrix} \mathrm{val}(x_1) = \cdots = \mathrm{val}(x_n), \\ \mathrm{val}(t) - \mathrm{val}(x_i) \geqslant k \end{matrix} \right\}.$$

On the other hand, the concave functions f_Ω, $f_{\mathrm{cl}(\Omega)}$ are

$$f_\Omega(a) = \begin{cases} k & \text{if } a = a_{ij}, \\ \infty & \text{if } a \neq a_{ij}. \end{cases} \qquad f_{\mathrm{cl}(\Omega)}(a) = \begin{cases} \lceil k \rceil & \text{if } a = a_{ij}, \\ \infty & \text{if } a \neq a_{ij}. \end{cases}$$

Therefore $f_\Omega \neq f_{\mathrm{cl}(\Omega)}$ in general, while $U_{f_\Omega} = U_{a_{ij},k} = U_\Omega$. It is then clear that $P_{f_\Omega} = P_\Omega$ if we take $H_0 = H$. But for other choices, namely $H = H^\circ$ or $H_{[0]}$, we have

$$P_{f_\Omega} = \left\{ \boldsymbol{h} + tE_{ij} \in \mathsf{GL}_n(K) \;\middle|\; \boldsymbol{h} \in H, \mathrm{val}(t) \geqslant k \right\}.$$

Example 3.3.6. Continue Example 3.3.5 with a general nonempty subset Ω. First note that $H \cap \mathsf{GL}_n(K)^1 = H^\circ$. Hence using Example 3.2.8, we have

$$\begin{aligned} P_{f_\Omega} &= \left\{ (g_{ij})_{i,j} \in G \;\middle|\; \forall i,j : \mathrm{val}(g_{ij}) \geqslant -\inf_{x\in\Omega} \left(\chi_i(x) - \chi_j(x)\right) \right\} \\ &= \left\{ (g_{ij})_{i,j} \in G \;\middle|\; \forall i,j : \mathrm{val}(g_{ij}) \geqslant f_\Omega\left(\chi_i(x) - \chi_j(x)\right) \right\}, \end{aligned}$$

where $G = \mathsf{GL}_n(K)^1$ if we take $H_0 = H^\circ$ and $G = \mathsf{SL}_n(K)$ if we take $H_0 = H_{[0]}$.

3.3.7 ([BT-I, 6.4.10; BT-II, 4.5.2, 4.6.12])**.** Let f be a concave function on Φ. Define f' as follows:

$$f'(a) := \inf\{k \in \Gamma_a \mid k \geqslant f(a)\}.$$

Then f' is also a concave function on Φ, called the *optimization* of f. If $f' = f$, we say f is *optimal*. Note that under the assumption that $\Gamma_a = \Gamma$ for all $a \in \Phi$, we have $f'(a) = \lceil f(a) \rceil$.

The set of roots $a \in \Phi$ such that $f'(a) + f'(-a) = 0$ is denoted by Φ_f, called the *root system associated to f*.

Remark. Note that for any $a \in \Phi$, we have

$$f_\Omega(a) + f_\Omega(-a) = -\inf_{x\in\Omega} a(x) - \inf_{x\in\Omega}(-a(x)) = \sup_{x\in\Omega} a(x) - \inf_{x\in\Omega} a(x) \geqslant 0.$$

The equality holds if and only if $a(x)$ is a constant for $x \in \Omega$. Note that this constant can be outside of Γ, hence the condition merely says that Ω is contained in a hyperplane parallel to the wall $\partial\alpha_{a+0}$. On the other side, from the definition of optimizes, $a \in \Phi_{f_\Omega}$ can be interpreted as $\Omega \subseteq \partial\alpha$ for some affine root α with vectorial part a. Therefore, $\Phi_{f_\Omega} = \Phi_\Omega$. Another way to see this is use the observation that $f'_\Omega = f_{\mathrm{cl}(\Omega)}$.

3.3.8 ([BT-I, 6.4.23; BT-II, 4.6.9])**.** Let f be a concave function on Φ. Define $f^* : \widetilde{\Phi} \to \widetilde{\mathbb{R}}$ as follows:

$$f^*(a) := \begin{cases} f(a) & \text{if } f(a) + f(-a) > 0, \\ f(a)+ & \text{if } f(a) + f(-a) = 0. \end{cases}$$

Then f^* is a concave function.

Let $\overline{G}_f$ denote the quotient P_f/P_{f^*} and let $\overline{U}_{f;a}$ (resp. $\overline{T}_f$) be the image of $U_{a,f(a)}$ (resp. $H_{f(0)}$) in $\overline{G}_f$. Then $(\overline{T}_f, (\overline{U}_{f;a})_{a\in\Phi_f})$ is a generating root group datum of type Φ_f on $\overline{G}_f$.

Example 3.3.9. In Example 3.3.5, we have $f^*_{\alpha_{a_{ij}+k}} = f_{\alpha_{a_{ij}+k}}$. Hence $\overline{G}_{f_{\alpha_{a_{ij}+k}}}$ is the trivial group.

Example 3.3.10. Assume $\Omega = \mathrm{cl}(\Omega)$ in Example 3.3.6. Hence f_Ω is optimal. Then

$$f^*_\Omega(a) = \begin{cases} f(a) & \text{if } a \notin \Phi_\Omega, \\ f(a)+ & \text{if } a \in \Phi_\Omega. \end{cases}$$

Take H_0 to be H°. Then the group $P_{f^*_\Omega}$ can be computed using (3.3.1):

$$P_{f^*_\Omega} = I_n + \left\{ (g_{ij})_{i,j} \in \mathsf{GL}_n(K^\circ) \,\middle|\, \forall i, j : \mathrm{val}\big(g_{ij}\big) \geqslant f^*_\Omega\left(\chi_i(x) - \chi_j(x)\right)\right\}.$$

In particular, if we take Ω to be the origin o, then we have

$$P_{f^*_o} = I_n + \varpi\, \mathsf{M}_{n\times n}(K^\circ).$$

Therefore $P_{f_o}/P_{f^*_o}$ is nothing other than $\mathsf{GL}_n(\kappa)$.

Now suppose Ω contains o, then for any $a \in \Phi$, either $f_\Omega(a) = 0$ or $f_\Omega(-a) = 0$. Then $\Psi_\Omega := \{a \in \Phi \mid f_\Omega(a) = 0\}$ is a parabolic subset. We can thus choose a system of positive roots Φ^+ such that $\Phi^+ \subseteq \Psi_\Omega$. Hence we may assume $a_{ij}(x) \geqslant 0$ for all $1 \leqslant i < j \leqslant n$. Then we have (idenitfied as subgroups of $\mathsf{GL}_n(\kappa)$):

$$\begin{aligned} P_{f_\Omega}/(P_{f^*_o} \cap P_{f_\Omega}) &= \left\{ (g_{ij})_{i,j} \in \mathsf{GL}_n(\kappa) \,\middle|\, \forall i, j : a_{ij} \notin \Psi_\Omega \implies g_{ij} = 0\right\}, \\ P_{f_\Omega}/P_{f^*_\Omega} &= \left\{ (g_{ij})_{i,j} \in \mathsf{GL}_n(\kappa) \,\middle|\, \forall i, j : a_{ij} \notin \Phi_\Omega \implies g_{ij} = 0\right\}. \end{aligned}$$

Note that: through the above identification, $P_{f_\Omega}/(P_{f^*_o} \cap P_{f_\Omega})$ is (the group of κ-points of) a parabolic subgroup P_{I_Ω} of $\mathsf{GL}_{n,\kappa}$ and $P_{f_\Omega}/P_{f^*_\Omega}$ is (the group of κ-points of) its Levi subgroup L_{I_Ω}, where the type I_Ω is defined by the parabolic subset Ψ_Ω.

3.4 Smooth models associated to concave functions

In this subsection, we will take (G, T) to be a split reductive group and $G = \mathsf{G}(K)$. The second part [BT-II] of Bruhat-Tits theory says that there are more algebraic-geometric structures on its Bruhat-Tits building. We follow [Yu15] to state such result and deduce some properties which will be used later. We emphasis that since we focus on split reductive group only, a lot of difficulties vanish. However, we still keep the general statement unless we turn to specific examples.

3.4.1. Let $\mathfrak{T}^{\mathrm{NR}}$ denote the *Néron-Raynaud model* [CY01, 3.1] of the torus T, namely the neutral component of the standard *lft Néron model* [BLR90, 10.1.1]. In our case, since T is split, its Néron-Raynaud model $\mathfrak{T}^{\mathrm{NR}}$ can be characterized as the connected smooth model such that $\mathfrak{T}^{\mathrm{NR}}(K^{\circ})$ equals the subgroup [BLR90, 10.1.5]

$$H^{\circ} := \{\boldsymbol{t} \in \mathsf{T}(K) \mid \mathrm{val}(\chi(\boldsymbol{t})) = 0, \text{ for all } \chi \in \mathsf{X}(\mathsf{T})\}.$$

The *Moy-Prasad filtration* $\{\mathsf{T}(K)_k\}_{k \geqslant 0}$ [Yu15, 4.2; MP96, 3.2] is defined as:

$$\mathsf{T}(K)_k := \{\boldsymbol{t} \in H^{\circ} \mid \mathrm{val}(\chi(\boldsymbol{t}) - 1) \geqslant k, \text{ for all } \chi \in \mathsf{X}(\mathsf{T})\}.$$

It defines a good filtration $H_k := \mathsf{T}(K)_k$ on H. There is also a *Moy-Prasad filtration* $\{\mathfrak{t}_k\}_{k \geqslant 0}$ of the Lie algebra $\mathfrak{t} = \mathrm{Lie}(\mathsf{T})$:

$$\mathfrak{t}_k := \{T \in \mathfrak{t} \mid \mathrm{val}(\mathrm{d}\chi(T)) \geqslant k, \text{ for all } \chi \in \mathsf{X}(\mathsf{T})\}.$$

There is a family $\{\mathfrak{T}_k\}_{k \geqslant 0}$ of connected smooth models of T such that [Yu15, §4]:

(i) $\mathfrak{T}_k(K^{\circ}) = \mathsf{T}(K)_k$;

(ii) the special fiber $(\mathfrak{T}_k)_{\kappa}$ is unipotent for all $k > 0$;

(iii) the congruence subgroup

$$\Gamma(\varpi^m, \mathfrak{T}_k) := \mathrm{Ker}(\mathfrak{T}_k(K^{\circ}) \to \mathfrak{T}_k(K^{\circ}/\varpi^m))$$

equals $\mathsf{T}(K)_{k+m\gamma}$ for all $m \geqslant 0$;

(iv) the Lie algebra of $\mathfrak{T}_k$ equals $\mathfrak{t}_k$.

Remark. The scheme $\mathfrak{T}_k$ is constructed as follows [Yu15, 4.5]. First, consider the *higher unit group*

$$\mathbb{G}_{\mathrm{m}}(K^{\circ})_k := \{1 + t \in \mathbb{G}_{\mathrm{m}}(K^{\circ}) \mid \mathrm{val}(t) \geqslant k\}.$$

It admits a smooth model of $\mathbb{G}_{\mathrm{m}}/K$ via a *dilatation* [BLR90, §3.2] in the Néron-Raynaud model $\mathbb{G}_{\mathrm{m}}/K^\circ$.

$$(\mathbb{G}_{\mathrm{m}}/K^\circ)^{(k)} := \mathrm{Spec}\left(K^\circ\left[\frac{X-1}{\varpi^{\frac{\lceil k\rceil}{\gamma}}}, \frac{X^{-1}-1}{\varpi^{\frac{\lceil k\rceil}{\gamma}}}\right]\right).$$

Then $\mathfrak{T}_k$ can be obtained by extension the isomorphism

$$(\mathbb{G}_{\mathrm{m}}(K^\circ)_k)^n \xrightarrow{\sim} \mathsf{T}(K)_k$$

to above smooth model.

Remark. The good filtration H_k can be taken fairly general, depending on which model of T to use. See [BT-II, §4.4] for discussion on models $\mathfrak{T}_0$ of T and [Yu15, §4 and §5] for discussion on the filtrations H_k and $\mathfrak{T}_k$.

3.4.2 ([Yu15, 6.2; BT-II, §4.3])**.** For each root subgroup U_a, the filtration $\{U_{a,k}\}_{k\in\mathbb{R}}$ extends to a family $\{\mathfrak{U}_{a,k}\}_{k\in\mathbb{R}}$ of connected smooth models of U_a such that:

(i) $\mathfrak{U}_{a,k}(K^\circ) = U_{a,k}$;

(ii) the special fiber $(\mathfrak{U}_{a,k})_\kappa$ is unipotent for all k;

(iii) the congruence subgroup

$$\Gamma(\varpi^m, \mathfrak{U}_{a,k}) := \mathrm{Ker}\big(\mathfrak{U}_{a,k}(K^\circ) \to \mathfrak{U}_{a,k}(K^\circ/\varpi^m)\big)$$

equals $U_{a,k+m\gamma}$ for all $m \geqslant 0$;

(iv) the Lie algebras $\mathfrak{u}_{a,k}$ of $\mathfrak{U}_{a,k}$ form a filtration on the Lie algebra $\mathfrak{u}_a$ of U_a.

Remark. With out assumption, the scheme $\mathfrak{U}_{a,k}$ can be obtained by extending the isomorphism of one-dimensional free K°-modules [13]

$$K_k := \{x \in K \mid \mathrm{val}(x) \geqslant k\} \xrightarrow{u_a} U_{a,k}$$

[13]This is not the case in general where U_a is merely split unipotent, not necessary vectorial. However, the scheme $\mathfrak{U}_{a,k}$ is still constructed explicitly. See [BT-II, §4.3] for details.

to an isomorphism of vectorial K°-group schemes

$$u_a \colon \mathbb{W}_{K^\circ}(K_k) \xrightarrow{\sim} \mathfrak{U}_{a,k}\ .$$

In particular, we have

$$\mathfrak{U}_{a,k}(K^\circ/\varpi^m) \cong K_{\geqslant k} \otimes K^\circ/\varpi^m \tag{3.4.1}$$

compatible with the filtrations. In particular, $\mathfrak{U}_{a,k}(K^\circ/\varpi^m)$ is a m-dimensional vector space over κ.

At this stage, we have group schemes $\mathfrak{T}_k$ and $\mathfrak{U}_{a,k}$ ($a \in \Phi$). Such a datum is basically Bruhat-Tits' *schematic root group datum* [BT-II, 3.1.1].

The main theorem of the *schematic Bruhat-Tits theory* is

Theorem 3.2 ([Yu15, 8.3; BT-II, §4.6])**.** *Fix a choice of* $(\mathfrak{T}_k, (\mathfrak{U}_{a,k})_{a\in\Phi})$. *For a concave function* f *on* $\widetilde{\Phi}$, *there is a connected smooth model* $\mathfrak{G}_f$ *of* G *such that* $\mathfrak{G}_f(K^\circ) = P_f$. *Moreover:*

(i) *The schematic closure of* T *in* $\mathfrak{G}_f$ *is* $\mathfrak{T}_{f(0)}$.

(ii) *For each* $a \in \Phi$, *the schematic closure of* U_a *in* $\mathfrak{G}_f$ *is* $\mathfrak{U}_{a,f(a)}$.

(iii) *The multiplication morphism (the products can be taken in any order)*

$$\prod_{a\in\Phi_f^+} \mathfrak{U}_{a,f(a)} \cdot \mathfrak{T}_{f(0)} \cdot \prod_{a\in\Phi_f^-} \mathfrak{U}_{a,f(a)} \longrightarrow \mathfrak{G}_f$$

is an open immersion. If $f(0) > 0$, *it induces an isomorphism on special fibers.*

Definition 3.4.3 ([BT-II, 5.2.6])**.** Let $(\mathfrak{T}_k, (\mathfrak{U}_{a,k})_{a\in\Phi})$ be as in 3.4.1 and 3.4.2 and $f = f_F$ for some facet F. The group P_{f_F} is called the *parahoric subgroup* of G. When F is an alcove, it is called a *Iwahori subgroup*. A parahoric subgroup P_{f_F} is also called the *connected stablizer* of F in the sense that $\mathfrak{G}_f$ is a connected group scheme and P_{f_F} is the largest subgroup of $\widehat{P}_F$ having this property.

Example 3.4.4. In Examples 3.3.6 and 3.3.10, the smooth model $\mathfrak{G}_{f_o}$ can be taken as

$$\mathfrak{G}_{f_o}: R \longmapsto \mathsf{GL}_n(R).$$

Indeed, in this case, we have $\mathfrak{T}_{f(0)}(R) = \mathsf{D}_n(R)$ and

$$\mathfrak{U}_{a_{ij}, f_o(a_{ij})}(R) = \left\{ I_n + rE_{ij} \,\middle|\, r \in R \right\}.$$

Remark. More generally, let (G, T) be a split reductive group and x be a special point in its Bruhat-Tits building. Then the smooth model $\mathfrak{G}_{f_x}$ is the bare bone of the *Chevally group scheme* [BT-II, §3.2 and 4.6.15; SGA3, XXV]: it says that for the reduced root datum $\mathcal{R}(\mathsf{G}, \mathsf{T})$, there is a smooth affine group scheme $\mathfrak{G}$ over $\mathbb{Z}$ such that for any field F, $\mathfrak{G}_F$ is a split reductive group with root datum $\mathcal{R}(\mathsf{G}, \mathsf{T})$ over F.

3.4.5. Let f be a concave function on Φ. Let $\overline{\mathsf{R}}_f$ and $\overline{\mathfrak{G}}_f$ denote the unipotent radical and the reductive quotient of $(\mathfrak{G}_f)_\kappa$ respectively. Since κ is a finite field, we have $\overline{\mathfrak{G}}_f(\kappa) = \mathfrak{G}_f(\kappa)/\overline{\mathsf{R}}_f(\kappa)$. Note that, by Theorem 3.2, we have [BT-II, 4.6.4]

(i) $(\mathfrak{T}_0)_\kappa$ is the centralizer of it self in $(\mathfrak{G}_f)_\kappa$.

(ii) Φ is the root system of the pair $\big((\mathfrak{G}_f)_\kappa, (\mathfrak{T}_0)_\kappa\big)$ and for any $a \in \Phi$, $(\mathfrak{U}_{a,f(a)})_\kappa$ is the root subgroup associated to it.

Let f^* be defined as in 3.3.8. Using the filtrations in 3.4.1 and 3.4.2, we have:

(iii) The unipotent radiacl of $(\mathfrak{T}_0)_\kappa$ is the image of $(\mathfrak{T}_{0+})_\kappa$ in it.

(iv) [BT-II, 4.6.10] The intersection of the unipotent radiacl $\overline{\mathsf{R}}_f$ and the root subgroup $(\mathfrak{U}_{a,f(a)})_\kappa$ is the image of $(\mathfrak{U}_{a,f^*(a)})_\kappa$ in $(\mathfrak{G}_f)_\kappa$.

Remark. However, in our case, what are in the unipotent radiacl is clear: the congruence property in 3.4.1 implies that $\mathfrak{T}_{0+}$ maps to 0 in $(\mathfrak{T}_0)_\kappa$; then 3.4.2 plus the fact that $\mathfrak{U}_{a,f(a)}$ is one-dimensional vectorial group imply that the intersection $\overline{\mathsf{R}}_f \cap (\mathfrak{U}_{a,f(a)})_\kappa$ is either trivial or the entire $(\mathfrak{U}_{a,f(a)})_\kappa$.

(v) [BT-II, 1.1.11] The multiplication morphism

$$\prod_{a\in\Phi_f^+}\Big(\overline{\mathsf{R}}_f\cap(\mathfrak{U}_{a,f(a)})_\kappa\Big)\cdot\mathscr{R}_u((\mathfrak{T}_0)_\kappa)\cdot\prod_{a\in\Phi_f^-}\Big(\overline{\mathsf{R}}_f\cap(\mathfrak{U}_{a,f(a)})_\kappa\Big)\longrightarrow\overline{\mathsf{R}}_f$$

is an isomorphism.

Therefore we have exact sequence

$$P_{f^*}\hookrightarrow\mathfrak{G}_f(K^\circ)\twoheadrightarrow\overline{\mathfrak{G}}_f(\kappa).$$

Moreover, let $\overline{\mathfrak{T}}_f$ (resp. $\overline{\mathfrak{U}}_{f;a}$) denote the image of $\mathfrak{T}_0$ (resp. $\mathfrak{U}_{a,f(a)}$) in $\overline{\mathfrak{G}}_f$. Then, $(\overline{\mathfrak{G}}_f,\overline{\mathfrak{T}}_f)$ is a split reductive group with root system Φ_f and root subgroups $(\overline{\mathfrak{U}}_{f;a})_{a\in\Phi_f}$.

Remark. Note that P_{f^*} is a *pro-unipotent group* in the following sense. First, we have a projective system of groups

$$\cdots\twoheadrightarrow\mathfrak{G}_f\big(K^\circ/\varpi^{i+1}\big)\twoheadrightarrow\mathfrak{G}_f\big(K^\circ/\varpi^{i}\big)\twoheadrightarrow\cdots\twoheadrightarrow\mathfrak{G}_f(\kappa).$$

Then, by the theory of Greenberg functors, we have a projective system of algebraic groups over κ:

$$\cdots\twoheadrightarrow\mathscr{F}_{K^\circ/\varpi^{i+1}}\big(\mathfrak{G}_f\big)\twoheadrightarrow\mathscr{F}_{K^\circ/\varpi^{i}}\big(\mathfrak{G}_f\big)\twoheadrightarrow\cdots\twoheadrightarrow(\mathfrak{G}_f)_\kappa.$$

It induces a projective system of their unipotent radicals and for each $\mathscr{F}_{K^\circ/\varpi^{i}}\big(\mathfrak{G}_f\big)$, its unipotent radical is precisely the preimage of $\overline{\mathsf{R}}_f$. Therefore P_{f^*} is the limit of a projective system of groups of κ-points of unipotent algebraic groups over κ.

3.4.6. Let f,g be two concave functions on Φ with $g\geqslant f$. Then $P_g\subseteq P_f$ extends to a morphism of group schemes [BT-II, 6.4.24]

$$\mathfrak{G}_g\longrightarrow\mathfrak{G}_f\,.$$

Since $g^*\geqslant f^*$, the image of

$$\mathscr{R}_u\big((\mathfrak{G}_g)_\kappa\big)\subset(\mathfrak{G}_g)_\kappa\longrightarrow(\mathfrak{G}_f)_\kappa$$

is contained in $\mathscr{R}_u\big((\mathfrak{G}_f)_\kappa\big)$ by 3.4.5.

Now, suppose for any $a \in \Phi$, either $f(a) = g(a)$ or $f(-a) = g(-a)$. Then

$$\Psi_{f,g} := \{a \in \Phi \mid f(a) = g(a)\}$$

is a parabolic subset. The image of $(\mathfrak{U}_{a,g(a)})_\kappa$ in $(\mathfrak{G}_f)_\kappa$ is either the entire $(\mathfrak{U}_{a,f(a)})_\kappa$ if $a \in \Psi_{f,g}$ or contained in $\mathscr{R}_u\big((\mathfrak{G}_f)_\kappa\big)$ if $a \notin \Psi_{f,g}$. So the image of $(\mathfrak{G}_g)_\kappa$ in $\overline{\mathfrak{G}}_f$ is generated by $\overline{\mathsf{T}}$ and $\overline{\mathsf{U}}_a$ for all $a \in \Psi$. This shows that the image $\overline{\mathsf{P}}_{f,g}$ is a parabolic subgroup of $\overline{\mathfrak{G}}_f$ with parabolic subset $\Psi_{f,g}$.

Example 3.4.7. Let Ω be a set in an apartment containing a special point x. Then we have $f_\Omega \geqslant f_x$ and for any $a \in \Phi$, either $f_\Omega(a) = f_x(a)$ or $f_\Omega(-a) = f_x(-a)$. Hence above applies and we get a parabolic subgroup of $\overline{\mathfrak{G}}_{f_x}$.

List of Notations

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