Review on Bruhat-Tits buildings

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Abstract

The purpose of this note is to explain the theory of Bruhat-Tits buildings (resp. Tits buildings) for split reductive groups over local fields (resp. finite field). It is intended to be part of my thesis.

In 1950s-1960s, Jacques Tits [Tit74] introduced the notion of *buildings* to provide a uniform geometric framework for understanding semisimple Lie groups and later semisimple algebraic groups (and more generally, reductive groups) over arbitrary fields. Tits' buildings are polysimplicial complexes having nice symmetries so that reductive groups can act nicely on them. Later, François Bruhat and Jacques Tits develop the theory for reductive groups over non-Archimedean fields [BT72, BT84a, BT84b, BT87a], giving refined structures on the buildings respecting the valuation. During the same period, they shift the view for a building from merely a polysimplicial complex to a complete metric space with non-positive curvature realizing it geometrically. The fruitful geometric/combinatorial nature of Bruhat-Tits buildings suggests them as non-Archimedean analogues of Riemannian symmetric spaces for real Lie groups.

We refer [RTW15, §3] for a short review on Bruhat-Tits theory, [Tit79, Yu09] for more systematic surveys, [Rou09] for a survey of general theory of Euclidean buildings, [Gro02] for classical groups and [Mil17, SGA3] for reductive groups.

Contents

§ 1 General theory of Buildings

1.1 Projective geometry over $\mathbb{F}_q$ and $\mathbb{F}_1$

The notion of buildings does not come from nothing. Back to 1950s, Jacques Tits noticed the following interesting phenomenons [Tit57].

1.1.1. Let $\mathbb{P}\mathbb{F}_q^n$ be the projective space associated to the vector space $\mathbb{F}_q^n$. Then its cardinality (or equivalently, the number of one-dimensional subspaces of $\mathbb{F}_q^n$) can be presented by the *quantum number* $[n]_q := \sum_{i=0}^{n-1} q^i$. If we passing to the limit $q \to 1$, then we get n, the number of coordinate labels $\{1, 2, \cdots, n\}$. Realizing how we count the cardinality of $\mathbb{P}\mathbb{F}_q^n$ using the coordinates, we can view the set $P_n = \{1, 2, \cdots, n\}$ as the analogy of $\mathbb{P}\mathbb{F}_q^n$ over the imaginary "prime field of characteristic one[1]" $\mathbb{F}_1$.

More generally, we can count points, lines, planes, ... in $\mathbb{P}\mathbb{F}_q^n$. They correspond to points of the Grassmannians $\mathrm{Gr}(1, \mathbb{F}_q^n), \mathrm{Gr}(2, \mathbb{F}_q^n), \mathrm{Gr}(3, \mathbb{F}_q^n), \ldots$. In general, the *Grassmannian* $\mathrm{Gr}(k, \mathbb{F}_q^n)$ consists of subspaces of $\mathbb{F}_q^n$ having dimension k and its cardinality can be presented by the *quantum binomial* $\begin{bmatrix} n \\ k \end{bmatrix}_q$ (see 1.1.4). If we passing to the limit $q \to 1$, then we get $\binom{n}{k}$, which is the number of k-subsets of P_n.

1.1.2. The aboves can be organized into incidence geometry: namely the combinatorial gadget describing which proper subspace belongs to which. For the $\mathbb{F}_q$-side, a nontrivial proper subspace of $\mathbb{F}_q^n$ is *of color k* if it is k-dimensional and two such subspaces are said to be *incident* if one of them belongs to another properly. In this way, we organize nontrivial proper subspaces of $\mathbb{F}_q^n$ into a colored simplicial complex $\mathscr{B}(n, q)$, in which a k-simplex is a *flag*

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{k+1} \subsetneq \mathbb{F}_q^n$$

of subspaces of $\mathbb{F}_q^n$. For the $\mathbb{F}_1$-side, a nonempty proper subset of P_n is *of color k* if it has cardinality k and two such subsets are said to be *incident* if one of them belongs to another properly. In this way, we organize nonempty proper subsets of P_n into a colored simplicial complex $\mathscr{B}(n, 1)$, in which a k-simplex is a *flag*

$$\emptyset \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_{k+1} \subsetneq P_n$$

of subsets of P_n.

The two sides are related as follows. Fix a basis $\mathcal{B}$ of $\mathbb{F}_q^n$ (for example, the standard basis). Then to take a nontrivial proper subspace V of $\mathbb{F}_q^n$ having a basis which is part of $\mathcal{B}$ is amount to take a nonempty proper subset I of $\mathcal{B}$ (which is in bijection to P_n) and V is k-dimensional if and only if I has cardinality k. Moreover, to take a flag respecting the basis $\mathcal{B}$ in the sense that each V_i has a basis being part of $\mathcal{B}$ is amount to take a flag of nonempty proper subsets of $\mathcal{B}$.

However, different choices of bases may give the same subcomplex: for instance, when the two bases are different by a diagonal matrix. To avoid this, it is better to keep in the region of projective geometry. So instead of fix a basis, we fix a *frame*,

[1] Namely the addition collapses. For an introduction, see [Lor18] especially §1.1.

that is a n-set of points $\{x_1, \cdots, x_n\}$ in $\mathbb{PF}_q^n$ in general position (namely, they do not belong to a common hyperplane), or equivalently, a n-set of one-dimensional subspaces $\{\lambda_1, \cdots, \lambda_n\}$ of $\mathbb{F}_q^n$ spanning $\mathbb{F}_q^n$. Then different choices of frames do give different subcomplexes of $\mathscr{B}(n,q)$.

In this way, we associate to each frame $\mathbf{\Lambda}$ a subcomplex $\mathscr{A}(\mathbf{\Lambda})$ of $\mathscr{B}(n,q)$ isomorphic to $\mathscr{B}(n,1)$ and the complex $\mathscr{B}(n,q)$ is the union of them. They are the prototypes of *buildings* and *apartments*.

1.1.3. There is a natural action of $G = \mathrm{GL}(\mathbb{F}_q^n)$, the *general linear group* (but essentially, it is the action of $\mathrm{PGL}(\mathbb{F}_q^n)$, the *projective linear group*) on $\mathscr{B}(n,q)$. This action comes from the action of $\mathrm{PGL}(\mathbb{F}_q^n)$ on $\mathbb{PF}_q^n$ and hence on each Grassmannian $\mathrm{Gr}(k, \mathbb{F}_q^n)$.

Fix a frame $\mathbf{\Lambda}$ (for example, the one given by the standard basis), then the stabilizer of the subcomplex $\mathscr{A}(\mathbf{\Lambda})$ is precisely the stabilizer of the frame itself. Let's denote it by $N(\mathbf{\Lambda})$ (in our example, it is the group of *monomial matrices*, i.e. matrices that have precisely one non-zero entry in each row and each column). The fixator of $\mathbf{\Lambda}$ acts trivially on $\mathscr{A}(\mathbf{\Lambda})$. Let's denote it by $Z(\mathbf{\Lambda})$ (in our example, it is the group of diagonal matrices). The quotient group $W(\mathbf{\Lambda}) := N(\mathbf{\Lambda})/Z(\mathbf{\Lambda})$ is called the *Weyl group* associated to $\mathbf{\Lambda}$. Then one finds that $W(\mathbf{\Lambda}) \cong \mathfrak{S}_n$, the *symmetric group*, which acts naturally on P_n and hence on $\mathscr{B}(n,1)$ exactly as how $W(\mathbf{\Lambda})$ acts on $\mathscr{A}(\mathbf{\Lambda})$.

1.1.4. Let's consider the maximal simplices in $\mathscr{B}(n,q)$. From the description in 1.1.2, we see that a maximal simplex is nothing but a *complete flag*

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq \mathbb{F}_q^n$$

of subspaces of $\mathbb{F}_q^n$. Using an induction argument, it is not difficult to see that the number of complete flags is presented by the *quantum factorial* $[n]_q! := \prod_{i=1}^n [i]_q$. The quantum factorials are related to quantum binomials by the formula

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

This can be seen by picking the k-dimensional subspace V_k from a complete flag, breaking it into a complete flag of V_k and a complete flag of $\mathbb{F}_q^n/V_k$.

The maximal simplices in $\mathscr{B}(n,1)$ are *complete flags*

$$\emptyset \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_{n-1} \subsetneq P_n$$

of subsets of P_n. There are $n!$ such complete flags. The number $n!$ is precisely the $q \to 1$ limit of $[n]_q!$.

The stabilizer of a complete flag is called a *Borel subgroup* of G. Note that the action of G on complete flags are transitive. Hence the number of complete flags is the index of a Borel subgroup in G.

Let's take the standard basis $\mathcal{B} = \{e_1, \cdots, e_n\}$ of $\mathbb{F}_q^n$ and set $V_k = \bigoplus_{i=1}^k \mathbb{F}_q e_i$. Then we get a complete flag and its stabilizer B is precisely the group of invertible upper triangular matrices. Hence B has order $q^{\binom{n}{2}}(q-1)^n$ and thus the order of G is $q^{\binom{n}{2}}(q-1)^n[n]_q!$.

1.1.5. We summarize aboves as follows.

(a) On the $\mathbb{F}_q$-side, we have the "building" $\mathscr{B}(n,q)$, which is the union of "apartments" $\mathscr{A}(\mathbf{\Lambda})$, one for each frame $\mathbf{\Lambda}$, and the number of them is

$$\frac{\#G}{\#N(\mathbf{\Lambda})} = \frac{\#B \cdot \#\{\text{complete flags}\}}{\#Z(\mathbf{\Lambda}) \cdot \#\mathfrak{S}_n} = \frac{q^{\binom{n}{2}}(q-1)^n[n]_q!}{(q-1)^n n!} = \frac{q^{\binom{n}{2}}[n]_q!}{n!}.$$

Each "apartments" $\mathscr{A}(\mathbf{\Lambda})$ is isomorphic to $\mathscr{B}(n,1)$, the one on the $\mathbb{F}_1$-side. Hence the "building" $\mathscr{B}(n,q)$ can be seen as so many copies of $\mathscr{B}(n,1)$ gluing together. By passing to the limit $q \to 1$, this quantity gives 1, coinciding with the number of "apartments" in $\mathscr{B}(n,1)$.

(b) The quantum factorial $[n]_q!$ counts the maximal simplices in $\mathscr{B}(n,q)$, which becomes $n!$, the number of maximal simplices in $\mathscr{B}(n,1)$ by taking the limit $q \to 1$.

(c) The quantum binomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$ counts the vertices of color k in $\mathscr{B}(n,q)$, which becomes $\binom{n}{k}$, the number of vertices of color k in $\mathscr{B}(n,1)$ by taking the limit $q \to 1$.

(d) There are more combinatorial quantities in $\mathscr{B}(n,q)$ becomes one for $\mathscr{B}(n,1)$ by taking the limit $q \to 1$.

1.1.6. Tits's observations are not limited for $\mathrm{PGL}(\mathbb{F}_q^n)$. In fact, he did for all semisimple groups over $\mathbb{F}_q$. Of course, there is no $\mathbb{F}_1$-geometry back to Tits' time, but it seems the above observations inspire him to develope the theory of buildings with the following principal:

> Buildings are multifold apartments and apartments are $q \to 1$ limit case of buildings, which can be thought as forgetting the additive arithmetic of the base field.

1.2 Spherical Buildings

Before moving on, we now give a formal definition of polysimplicial complexes.

1.2.1 Definition. An *(abstract) simplicial complex* is a nonempty poset $\mathcal{S}$ (whose members are called *simplices*) satisfying

S1. any two simplices σ, τ have an infimum $\sigma \cap \tau$;

So there is a unique smallest element in $\mathcal{S}$, called the *empty simplex*, denoted by $\emptyset$.

S2. for each simplex σ the poset $\mathcal{S}_{\leqslant \sigma}$ of simplices smaller than σ (they are called *faces* of σ) form a *Boolean lattice of rank* r, namely isomorphic to the power set of a r-set, for some finite r. In this case, we see σ is of *dimension* $r-1$ and is a $(r-1)$-*simplex.*

The *dimension* of $\mathcal{S}$ is the supremum of dimensions of its simplices. The minimal nonempty simplices are of dimension 0 and are thus called *vertices*. Let $\mathcal{V}$ denote the set of vertices. Then $\mathcal{S}$ can be identified with a poset of nonempty subsets of $\mathcal{V}$.

A *morphism* between simplicial complexes is a map preserving infima, suprema and the empty simplex $\emptyset$. Note that this implies that such a morphism is determined by vertices. So equivalently, such a morphism is a map between vertices extending to a monotone preserving simplices. A morphism is said to *fix a simplex σ pointwise* if it induces an isomorphism on $\mathcal{S}_{\leqslant\sigma}$.

A *polysimplicial complex* is a cartesian product of simplicial complexes (in the category of posets) and morphisms between polysimplicial complexes are therefore defined.

One can verify that $\mathscr{B}(n,q)$ and $\mathscr{B}(n,1)$ are simplicial complexes.

1.2.2. Let's analyse how the "apartments" $\mathscr{A}(\mathbf{\Lambda})$ are glued into the "building" $\mathscr{B}(n,q)$.

(a) *For any two simplices F, F' in $\mathscr{B}(n,q)$, there is an "apartment" $\mathscr{A}(\mathbf{\Lambda})$ containing both of them.*

Proof. We may assume F, F' are maximal, i.e. being complete flags:

$$F\colon 0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = \mathbb{F}_q^n,$$
$$F'\colon 0 = V_0' \subsetneq V_1' \subsetneq \cdots \subsetneq V_{n-1}' \subsetneq V_n' = \mathbb{F}_q^n.$$

Then we may view them as composition series for $\mathbb{F}_q^n$. Therefore by *Jordan-Hölder Theorem*, there is a permutation π of $P_n = \{1, 2, \cdots, n\}$ such that whenever $j = \pi(i)$, we have isomorphisms

$$\frac{V_i}{V_{i-1}} \xleftarrow{\sim} \frac{V_i \cap V_j'}{(V_{i-1}\cap V_j') + (V_i \cap V_{j-1}')} \xrightarrow{\sim} \frac{V_j'}{V_{j-1}'}$$

induced from inclusions. Let λ_i be the one-dimensional subspace of $V_i \cap V_j'$ whose image in above quotients are non-trivial. Then $\mathbf{\Lambda} = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is a frame with $\mathscr{A}(\mathbf{\Lambda})$ containing both F and F'. □

(b) *If $\mathscr{A}(\mathbf{\Lambda})$ and $\mathscr{A}(\mathbf{\Lambda}')$ are two "apartments" containing both F and F', then there is an isomorphism between them fixing F and F' pointwise.*

Proof. Again, we may assume F, F' are maximal and let V_i, V_i', λ_i be as above. Then $i \mapsto \lambda_i$ induces an isomorphism $\phi_{\mathbf{\Lambda}}\colon \mathscr{B}(n,1) \to \mathscr{A}(\mathbf{\Lambda})$. The inverse of it can be described by vertices as

$$\psi_{\mathbf{\Lambda}}\colon U \mapsto \{i \in P_n \mid U \cap V_{i-1} \neq U \cap V_i\}.$$

Similarly we have isomorphism $\phi_{\mathbf{\Lambda}'}\colon \mathscr{B}(n,1) \to \mathscr{A}(\mathbf{\Lambda}')$ and its inverse $\psi_{\mathbf{\Lambda}'}$. Note that the morphisms $\psi_{\mathbf{\Lambda}}$ (and similarly $\psi_{\mathbf{\Lambda}'}$) is determined by the complete flag F, we conclude that $\psi_{\mathbf{\Lambda}}$ and $\psi_{\mathbf{\Lambda}'}$ coincide on the intersection of $\mathscr{A}(\mathbf{\Lambda})$ and $\mathscr{A}(\mathbf{\Lambda}')$. Then $\phi_{\mathbf{\Lambda}'} \circ \psi_{\mathbf{\Lambda}}$ is an isomorphism between $\mathscr{A}(\mathbf{\Lambda})$ and $\mathscr{A}(\mathbf{\Lambda}')$ fixing F and F' pointwise. □

Then the buildings can be defined as follows.

1.2.3 Definition. A *building* is a polysimplicial complex $\mathscr{B}$ equipped with a family $\mathcal{A}$ of subcomplexes of $\mathscr{B}$, whose members are called *apartments*, such that the following axioms are satisfied.

B0. Each apartment $A \in \mathcal{A}$ is isomorphic to an (abstract) apartment $\mathscr{A}$.

B1. For any two simplices F, F', there is an apartment A containing them.

B2. If A, A' are two apartments containing both F and F', then there is an isomorphism between A and A' fixing F and F' pointwise.

Of course, one has to define what is an apartment to make this definition work.

1.2.4. Let's analyse what does the "apartment" $\mathscr{B}(n, 1)$ look like.

(a) *All maximal simplices have the same dimension.*

Proof. This is clear, they are precisely the $(n-1)$-subsets of P_n. □

(b) *Any two maximal simplices C, C' are connected by a sequence $(C_0, C_1, \cdots, C_s)$ with $C_0 = C$ and $C_s = C'$ such that for each i, $C_{i-1} \cap C_i$ has codimension 1 in both C_{i-1} and C_i.*

Proof. Note that a maximal simplex in $\mathscr{B}(n, 1)$ is complete flag, hence a sequence $(i_1, i_2, \cdots, i_{n-1})$, which can be identified with an ordering of P_n. Hence any two such simplices are different by a permutation $\pi \in \mathfrak{S}_n$. But any permutation can be written as the composition of transpositions while two sequences different by a transposition meets in a sequence with one term being removed. □

In general, a polysimplicial complex has above properties is called a *chamber complex* and its maximal simplices are called *chambers*. A one-codimensional face of a chamber is called a *panel*. A sequence $(C_0, C_1, \cdots, C_s)$ connecting two chambers by panels is called a *gallery*. Note that any Boolean lattice is a chamber complex with a unique chamber: its maximal element. A *chamber map* between chamber complexes is a morphism mapping chambers to chambers.

(c) *There is a coloring, namely a chamber map from the complex to a Boolean lattice.*

Proof. It suffices to define colors for vertices. Then the color of a simplex would be the set of the colors of its vertices. For instance, one can define the *color* of a vertex as its cardinality as in 1.1.2. □

In general, a chamber complex has this property is said to be *colorable*. It is worth to notice that any two colorings are different by an isomorphism of Boolean lattices (in other words, up to a permutation of the colors of vertices).

(d) *The Weyl group acts transitively on the simplices of the same color.*

Proof. Two simplices $F = (I_i)$ and $F' = (I'_i)$ are of the same color means two things: first, they have the same number of entries; second, each pair of entries (I_i, I'_i) have the same cardinality. This is precisely the condition when there is a permutation $\pi \in \mathfrak{S}_n$ interchanging them. □

(e) *Fix a chamber C, then the stabilizers of its panels are all of order 2 and their generators s_j form a generating system S of the Weyl group with generating relations of the form $(s_i s_j)^{m_{ij}} = 1$.*

Proof. As in (b), a chamber C is a sequence $(i_1, i_2, \cdots, i_{n-1})$. Let i_n be the complement of this sequence in P_n. Then for each panel obtained from C by deleting i_j, let s_j be the transposition (i_j, i_n). Then this panel's stabilizer is precisely $\{1, s_j\}$ and one can verify the system $S = \{s_1, \cdots, s_{n-1}\}$ satisfies the requirement. □

Note that it follows from this property that the stabilizer of a face of C is generated by those s_j with j not a color of its vertex. Furthermore, the complex $\mathscr{B}(n,1)$ can be built from the pair (W,S) of the Weyl group $W = \mathfrak{S}_n$ and the system $S = (s_j)$ of generators in (e). In deed, any face of the chamber C corresponds to the subset I of S generating its stabilizer and any simplex is translated to such a face by an element of W, unique up to the stabilizer $\langle I \rangle$. Therefore, the simplices in $\mathscr{B}(n,1)$ can be identified with the cosets $w\langle I \rangle$ with $w \in W$ and $I \subseteq S$.

1.2.5 Definition. A *Coxeter system* is a pair (W,S) of a group W and a system of its generators $S = \{s_1, s_2, \cdots, s_n\}$ such that all s_i are of order 2 and the generating relations for S are of the form $(s_i s_j)^{m_{ij}} = 1$. Its *Coxeter complex* $\Sigma(W,S)$ is the polysimplicial complex defined as the complex of cosets of the form $w\langle I \rangle$ with $w \in W$ and $I \subseteq S$, where the order is given by reverse inclusion.

Then 1.2.4 shows that $\mathscr{B}(n,1)$ is isomorphic to the Coxeter complex $\Sigma(\mathfrak{S}_n, S)$, where S can be chosen to be any generating system of transpositions, for instance $S = \{(1,n), (2,n), \cdots, (n-1,n)\}$.

A *morphism* between Coxeter systems (W,S) and (W',S') is a group homomorphism $f \colon W \to W'$ such that $f(S) \subseteq S'$. Therefore a Coxeter system (W,S) is a *product* of subsystems $(W_i, S_i)_{1 \leqslant i \leqslant m}$ if we have a group decomposition $W = W_1 \times \cdots \times W_m$ and a set decomposition $S = S_1 \cup \cdots \cup S_m$. A Coxeter system is *irreducible* if it can not be decomposed into proper subsystems.

One can see that morphisms between Coxeter systems induce morphisms between their Coxeter complexes and such a functor is compatible with the decompositions. In particular, Coxeter complex of an irreducible Coxeter system is simplicial.

Now, we can complete Definition 1.2.3 by define an *apartment* being a polysimplicial complex isomorphic to the Coxeter complex of some Coxeter system. The complex

$\Sigma(W, S)$ is finite if and only if W is finite. In this case, it is said to be *spherical*. A building whose apartments are isomorphic to a spherical Coxeter complex is called a *spherical building*.

We have seen that $\mathscr{B}(n,q)$ is such a building: its apartments are isomorphic to the Coxeter complex $\Sigma(\mathfrak{S}_n, S)$. This is not an accident. In fact, any reductive group over arbitrary field would give rise to such a building. They are called *Tits buildings*. We refer [Bou02, chap.IV] for the theory of Coxeter systems and [Tit74] for a treatment of Tits buildings in the language of Coxeter complexes.

1.3 Euclidean apartments

Although buildings can be defined and studied in a pure combinatorial way, it would be more intuitive and convenient if we can also realizing them geometrically.

1.3.1. One way to visualize the Coxeter complex $\Sigma(\mathfrak{S}_n, S)$ is as follows. The group $\mathfrak{S}_n$ acts faithfully on $\mathbb{R}^n$ as permutations of the coordinates under the standard basis. For any transposition (i, j), its fixed points is the hyperplane $\{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i = x_j\}$ and it thus acts as the reflection respect to this hyperplane. Therefore the group $\mathfrak{S}_n$ can be determined by the reflections/hyperplanes defined by the transpositions. Moreover, the hyperplanes partition $\mathbb{R}^n$ into pieces of various dimensions with an obvious order relation: one such a piece belongs to the closure of another. This gives rise to a complex isomorphic to $\Sigma(\mathfrak{S}_n, S)$. The system S can be obtained as the reflections respect to a chamber.

With this example in mind, we can obtain the following definition.

1.3.2 Definition. A *(Euclidean) apartment* $\mathscr{A}$ is a Euclidean affine space $\mathbb{A}$ equipped with a reflection group W (called its *Weyl group*) on it.

Let $\mathbb{A}$ be a Euclidean affine space. We use ${}^v\mathbb{A}$ to denote its associated vector space. For an affine transformation f on $\mathbb{A}$, we use ${}^vf$ to denote its *vectorial part*. For an affine subspace X of $\mathbb{A}$, we use ${}^vX$ to denote its *direction*.

A *reflection* on $\mathbb{A}$ is an affine isometry whose fixed points form a hyperplane. Any hyperplane H is associated with a reflection r_H with respect to it.

A *reflection group* W is a group of affine isometries generated by reflections and such that its vectorial part ${}^vW$ is finite. W is said to be *irreducible* if ${}^vW$ acts irreducibly on ${}^v\mathbb{A}$ and is said to be *essential* if ${}^vW$ acts essentially on ${}^v\mathbb{A}$ (that is, there is no non-trivial fixed point). An apartment is said to be *irreducible* (resp. *essential*, *trivial*, etc.) if its reflection group is so.

1.3.3. A *morphism* between apartments $(\mathbb{A}, W)$ and $(\mathbb{A}', W')$ is a continuous affine map $f : \mathbb{A} \to \mathbb{A}'$ with a group homomorphism $\phi : W \to W'$ such that $\phi(w).f(x) = f(w.x)$ for all $w \in W$ and $x \in \mathbb{A}$. Therefore an apartment $(\mathbb{A}, W)$ is said to be a *product* of apartments $(\mathbb{A}_i, W_i)_{1 \leqslant i \leqslant m}$ if we have an orthogonal decomposition $\mathbb{A} = \mathbb{A}_1 \times \cdots \times \mathbb{A}_m$ and a group decomposition $W = W_1 \times \cdots \times W_m$ such that each W_i acts trivially on the orthogonal complement of $\mathbb{A}_i$.

Any apartment $\mathscr{A}$ admits a decomposition [Bou02, chap.V, §3, no.8]

$$\mathscr{A} = \mathscr{A}_0 \times \mathscr{A}_1 \times \cdots \times \mathscr{A}_m,$$

where $\mathscr{A}_0$ is trivial and each $\mathscr{A}_i$ (for $1 \leqslant i \leqslant m$) is irreducible.

1.3.4. Let $\mathscr{A}$ be an apartment with an essential irreducible reflection group W. Let $T = \ker(W \to {}^vW)$ be the translation group. There are three possibilities [Rou09, 3.3]:

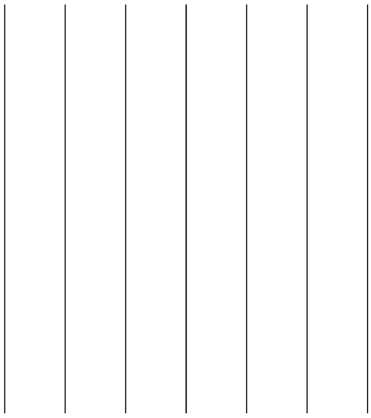
(a) If T is trivial, then $\mathscr{A}$ is said to be *spherical*.

(b) If T is a lattice in ${}^v\mathbb{A}$, then $\mathscr{A}$ is said to be *discrete affine*.

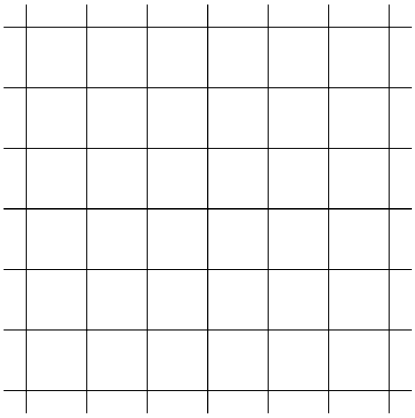
(c) If T is dense in ${}^v\mathbb{A}$, then $\mathscr{A}$ is said to be *dense affine*.

Throughout this note, all apartments are assumed to be *discrete*, namely no irreducible component is dense affine.

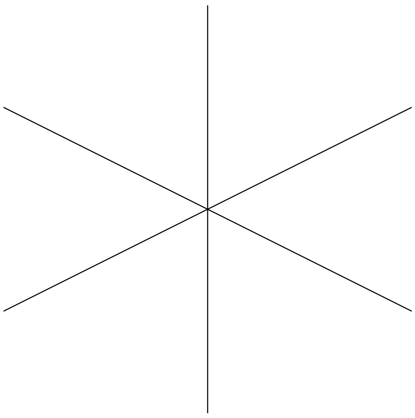
1.3.5 Example. Before moving on, we give pictures showing some examples.



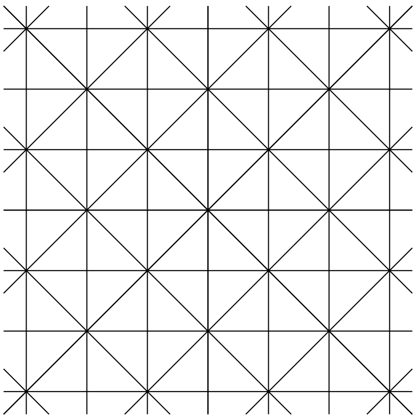
$\widetilde{A_1} \times \mathbf{0}$: a *non-essential* apartment



$\widetilde{A_1} \times \widetilde{A_1}$: a *non-irreducible* apartment



A_2: a *spherical* apartment



$\widetilde{C_2}$: contains *non-special* vertices

1.3.6. Let $\mathscr{A} = (\mathbb{A}, W)$ be an apartment.

The hyperplanes of fixed points of reflections in W are called the *walls* in $\mathscr{A}$. The set $\mathcal{H}$ of walls is stable under W and completely determines it.

A *half-apartment* (also called an *affine root* in [BT72, 1.3.3]) is a closed half-space α of $\mathbb{A}$ bounded by a wall $\partial\alpha$, called its *wall*.

A *facet* in $\mathscr{A}$ is an equivalence class in $\mathbb{A}$ for the relation "x and y are contained in the same half-apartments". A facet F is an open convex subset of in the affine subspace (called the *support* of F) it spans.

The set $\mathcal{F}$ of facets admits an order: a facet F is said to be a *face* of another F', denoted by $F \leqslant F'$, if F is contained in the closure of F'. Such an order gives rise to a polysimplicial complex. To see this, first notice that facets in an apartment are compatible with its decomposition into irreducible components. Hence we may assume our apartment $\mathscr{A}$ is irreducible and essential. Then this can be seen from the fact that any triangulation of a topological space gives rise to a simplicial complex (indeed, this is where the notion comes from). When $\mathscr{A}$ is discrete affine, its facets already triangulate the ambient space. When $\mathscr{A}$ is spherical, its facets triangulate the unit sphere. This is why it is called spherical.

1.3.7. The maximal facets are called *chambers* (or *alcoves*). They are the connected components of the complement of the union of all walls in $\mathbb{A}$. The Weyl group W acts simply transitively on the set $\mathcal{C}$ of chambers [Bou02, chap.V, §3, no.2, th.1].

Let C be a chamber. Then its closure $\overline{C}$ is a fundamental domain of W in $\mathbb{A}$ [Bou02, chap.V, §3, no.3, th.2] and is the intersection of some half-apartments, whose walls are called the *walls* of C. Equivalently, the walls of C are the supports of panels of it, where a *panel* means a maximal proper face of C. Moreover, W is generated by the set S of reflections with respect to the walls of C and the pair (W, S) is a Coxeter system [Bou02, chap.V, §3, no.2, th.1]. The projection of C onto an irreducible component $\mathscr{A}_i$ is again a chamber in it and induces an irreducible Coxeter system (W_i, S_i). Then (W, S) is the product of them. In other words, decomposition of the pair of $(\mathscr{A}, C)$ of an apartment and a chamber is compatible with the decomposition of the Coxeter system (W, S) it defines.

A *type function* on $\mathscr{A}$ is a morphism τ from the complex $\mathcal{F}$ of facets to a Boolean lattice, which maps chambers to the maximal element and is W*-stable* in the sense that for any facet F and any $w \in W$, $\tau(F) = \tau(w.F)$. The image of this function is denoted by $\mathcal{T}$ and its members are called *types*. This notion is essentially the same as a *coloring* as in 1.2.4(c) plus 1.2.4(d). They differs in practice: for a coloring, the target Boolean lattice is viewed as a power set $\mathscr{P}(\mathfrak{I})$ with its usual order $\subseteq$, while for a type function, we use the reverse order $\supseteq$. In other words, a face of type I is of color $\neg I := \mathfrak{I} \setminus I$.

Since any facet is transformed by W to a unique face of C, the type function τ is completely determined by the types of its panels, for which we may viewed as an indexing of S. Indeed, let I be a type, then the set C_I of points $x \in \overline{C}$ such that the reflections $s \in S$ fixing x are indexed by I is a face of C of type I and its stabilizer is the subgroup W_I of W generated by the reflections indexed by I [Bou02, chap.V, §3, no.3, prop.1]. Then $\tau(F) = I$ if and only if F is transformed to C_I.

1.3.8. A reflection group W is said to be *linear* if it fixes a point. This is the case if and only if W is finite [Bou02, chap.V, §3, no.9]. If this is the case, we can identify W with its vectorial part ${}^vW$ by choosing the fixed point to be the origin of $\mathbb{A}$.

Conversely, the vectorial part ${}^vW$ of the Weyl group W can be viewed as a linear reflection group on ${}^v\mathbb{A}$. The spherical apartment ${}^v\mathscr{A} = ({}^v\mathbb{A}, {}^vW)$ obtained in this way is called the *vectorial apartment* of $\mathscr{A}$. The walls (resp. facets, chambers) in ${}^v\mathscr{A}$ are called the *vectorial walls* (resp. *vectorial facets*, *vectorial chambers*) and the set of them is denoted by ${}^v\mathcal{H}$ (resp. ${}^v\mathcal{F}$, ${}^v\mathcal{C}$). Note that the vectorial walls are precisely the directions of walls in $\mathscr{A}$.

1.3.9. Let x be a point in $\mathscr{A}$. The stabilizer W_x of x is a linear reflection group whose vectorial part ${}^vW_x$ is a subgroup of ${}^vW$. The apartment $\mathscr{A}_x = (\mathbb{A}, W_x)$ is called the *spherical apartment at* x. The walls in $\mathscr{A}_x$ are precisely the walls in $\mathscr{A}$ passing through x and the set of them is denoted by $\mathcal{H}_x$. The facets (resp. chambers) in $\mathscr{A}_x$ are called the *vectorial facets with base point* x (resp. *vectorial chambers with base point* x) and the set of them is denoted by $\mathcal{F}_x$ (resp. $\mathcal{C}_x$).

A point $x \in \mathbb{A}$ is said to be *special* if the spherical apartment $\mathscr{A}_x$ is isomorphic to ${}^v\mathscr{A}$, or equivalently, the set $\mathcal{H}_x$ is a complete set of representatives of ${}^v\mathcal{H}$. This can happen only if x belongs to a minimal facet.

1.3.10. The minimal facets are called *vertices*. The set of vertices is denoted by $\mathcal{V}$. When the apartment is essential, they are points. From now on, all apartments are assumed to be essential unless otherwise specified[2].

Under this assumption, every special point is a vertex. Furthermore, any special vertex is an extremal point of the closure of some chamber. Conversely, any chamber admits a special point as an extremal point of its closure [Bou02, chap.V, §3, no.10, prop.11's cor.]. However, not all extremal points, hence not all vertices are special (see $\widetilde{C}_2$ in Example 1.3.5 for an example).

1.4 Root systems

Before moving on to the definition of buildings, let's look at some examples of Euclidean apartments arising from root systems (as well as root data). They are the key examples used in the study of reductive groups.

1.4.1. Let V be a Euclidean vector space and V^* its dual space. For any $a \in V^* \setminus \{0\}$, let r_a be the reflection with respect to the hyperplane $H_a := \mathrm{Ker}(a)$ and $a^\vee$ the vector orthogonal to H_a satisfying $a(a^\vee) = 2$. So for any $x \in V$, we have $r_a(x) = x - a(x)a^\vee$. Note that r_a also induces a reflection on V^*, namely $f \mapsto f - f(a^\vee)a$. A finite spanning subset $\Phi \subseteq V^* \setminus \{0\}$ is called a *root system* on V if

RS1. for any $a \in \Phi$, $r_a(\Phi) = \Phi$;

RS2. for any $a, b \in \Phi$, $a(b^\vee) \in \mathbb{Z}$;

[2] This means we will only focus on *reduced* buildings, rather than *extended* buildings.

and is *reduced* if

RS3. for any $a \in \Phi$, $\mathbb{R}a \cap \Phi = \{\pm a\}$.

From now on, all root systems are assumed to be reduced[3].

Elements of Φ are called *roots* in Φ. For a root $a \in \Phi$, the vector $a^\vee$ is called its *coroot*. A subset $\Psi \subseteq \Phi$ is called a *subroot system* if for any $a \in \Psi$, $r_a(\Psi) = \Psi$, and is said to be *closed* if for any $a, b \in \Psi$ such that $a + b$ is a root, $a + b \in \Psi$.

Any root system Φ admits a *Weyl group* ${}^v W(\Phi)$, that is the reflection group of V generated by r_a for $a \in \Phi$. It is a linear reflection group with walls H_a for $a \in \Phi$. In this way, we get a spherical apartment ${}^v\mathscr{A}(\Phi) := (V, {}^v W(\Phi))$. Note that not all spherical apartments arise in this way (see [Bou02, chap.VI, §2, no.5, prop.9]) and non-isomorphic root systems may have isomorphic Weyl groups (for instance root systems of types B_n and C_n).

1.4.2. A root system Φ is said to be *irreducible* if it cannot be written as the union of two proper subsets such that they are orthogonal to each other. A root system Φ is irreducible if and only if so is its Weyl group ${}^v W(\Phi)$ [Bou02, chap.VI, §1, no.2, prop.5's cor.]. Any root system decomposes into disjoint union of irreducible ones and such a decomposition is compatible with the decomposition of Weyl groups and hence of apartments [Bou02, chap.VI, §1, no.2, prop.6 and 7].

1.4.3. Let Φ be a root system. Then there is a closed subset Φ^+ of Φ such that for any $a \in \Phi$, either $a \in \Phi^+$ or $-a \in \Phi^+$. This set is called the set of *positive roots*. Once such a set is chosen, elements in the set $\Phi^- := -\Phi^+$ are called *negative roots*. A positive root is called a *simple root* if it cannot be written as the sum of two positive roots. The set Δ of simple roots form a *basis* of Φ in the sense that any root is a $\mathbb{Z}$-linear combination of simple roots and its coefficients are either all non-negative or all non-positive [Bou02, chap.VI, §1, no.6, th.3]. The cardinality of the set Δ is called the *rank* of Φ and is independent of the choice of Δ. Indeed, it equals $\dim(V)$.

Let Δ be a basis of Φ. Then the set ${}^v\mathcal{C} = \{x \in V \mid \forall a \in \Delta, a(x) > 0\}$ is a vectorial chamber, called the *Weyl chamber* associated to Δ [Bou02, chap.VI, §1, no.5, th.2]. Conversely, let ${}^v\mathcal{C}$ be a vectorial chamber. Then for any $x \in {}^v\mathcal{C}$, the sets $\Phi^+ = \{a \in \Phi \mid a(x) > 0\}$ and $\Phi^- = \{a \in \Phi \mid a(x) < 0\}$ form a partition of Φ into positive and negative roots and are independent of the choice of x. Then one can obtained a basis Δ by taking the simple roots. But there is a more direct description: they are the roots defining walls of ${}^v\mathcal{C}$ and point inside. As vectorial chambers are Weyl chambers associated to some choice of basis, we call them *Weyl chambers* to specify that they are chambers in the spherical apartment ${}^v\mathscr{A}(\Phi)$.

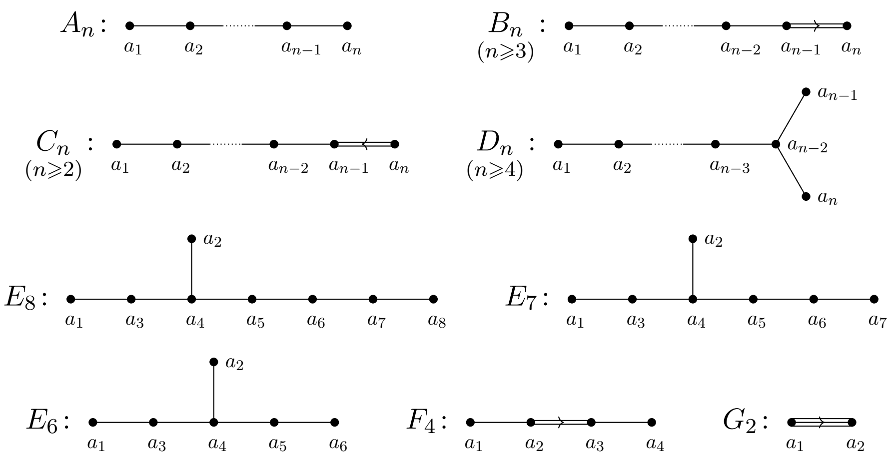
1.4.4. The relation between simple roots and types is the following. First, the Weyl group ${}^v W$ is generated by r_a for $a \in \Delta$ as they are the roots defining walls of ${}^v\mathcal{C}$ and point inside. Therefore a type $I \in \mathcal{T}$ corresponds to a subset of Δ. From now on, we do not distinguish them. Then the face of ${}^v\mathcal{C}$ corresponding to I is the set

[3] This means we will only focus on split reductive groups.

${}^vC_I = \{x \in V \mid \forall a \in I, a(x) = 0, \forall a \in \Delta \setminus I, a(x) > 0\}$. Let Φ_I be the subroot system of Φ generated by I, then the stabilizer ${}^vW_I$ is the Weyl group of it. The set $\Psi = \Phi_I \cup \Phi^+$ has the property that $\Psi \cup (-\Psi) = \Phi$ and is closed. Such kind of subsets of Φ are said to be *parabolic*. Given a parabolic subset Ψ of Φ containing Φ^+, then the simple roots in $\Psi \cap (-\Psi) \cap \Phi^+$ gives the type I. See [Bou02, chap.VI, §1, no.7].

1.4.5. Given a basis Δ of a root system Φ, its *Dynkin diagram* is defined as follows. The vertices are simple roots of Φ and the number of edges between two vertices is $4\cos^2(\theta)$ if the angle between then is θ. Furthermore, these edges are decorated with arrows pointing from longer root to shorter root. It turns out that, up to graph isomorphisms, the Dynkin diagram is independent of the choice of the basis Δ.

From above description, we see that Φ is irreducible if and only if its Dynkin diagram is connected. The Dynkin diagrams of irreducible root systems are classified as follows [Bou02, chap.VI, §4, no.2, th.3], where the subscription n in the notation X_n denote the rank of it.



A spherical apartment is said to be *of type* X_n if it is isomorphic to ${}^v\mathscr{A}(\Phi)$ for an irreducible root system Φ of type X_n.

1.4.6. Let $\mathbb{A}$ be an affine space underlying V with a specified point o. For any $a \in V^*$ and $k \in \mathbb{R}$, denote the affine function $x \mapsto a(x - o) + k$ on V by $a + k$ and denote the closed half-space $\{x \in \mathbb{A} \mid (a + k)(x) \geqslant 0\}$ by α_{a+k}. For each $a \in \Phi$, let Γ_a be a nonempty subset of $\mathbb{R}$. The affine function $a + k$ is called an *affine root* if $a \in \Phi$ and $k \in \Gamma_a$. Let Σ denote the set of closed half-spaces α_{a+k} with $a + k$ an affine root. Then $a + k \mapsto \alpha_{a+k}$ gives rise to a bijection between the set of affine roots and Σ. For this reason, we will not distinguish the affine root $a + k$ and the closed half-space α_{a+k} and will call Σ the *affine root system*[4]. The roots are vectorial part of affine roots. Hence we denote Φ by ${}^v\Sigma$ and call it the *vectorial root system* of Σ.

For $\alpha = \alpha_{a+k}$ an affine root, let ${}^v\alpha$ denote its vectorial part a, let $\partial\alpha$ denote its boundary $\{x \in \mathbb{A} \mid (a + k)(x) = 0\}$, let r_α denote the reflection with respect to $\partial\alpha$, let

[4]Note that, there is a notion called *affine root system*, defined in a similar way as root system, but for affine spaces. In this note, this terminology only refers to those arise from (reduced) root systems.

α^* denote the other affine root sharing the same boundary with α, that is $\overline{\mathbb{A} \setminus \alpha}$, and let α_+ denote the intersection of affine roots containing a neighborhood of α.

1.4.7. Let Σ be an affine root system on a Euclidean affine space $\mathbb{A}$, its *affine Weyl group* $W(\Sigma)$ is the reflection group on $\mathbb{A}$ generated by r_α for all $\alpha \in \Sigma$. In this way, we obtain an apartment $\mathscr{A}(\Sigma) := (\mathbb{A}, W(\Sigma))$ with ${}^v\!\mathscr{A}({}^v\Sigma)$ being its vectorial apartment. Suppose all Γ_a are the same discrete subgroup $\Gamma \neq 0$ of $\mathbb{R}$, then the walls in the apartment $\mathscr{A}(\Sigma)$ are precisely the boundaries $\partial\alpha$ with $\alpha \in \Sigma$ [Bou02, chap.VI, §2, no.1, prop.2]. For x a point in the apartment $\mathscr{A}(\Sigma)$, let Σ_x be the set of affine roots α such that $x \in \partial\alpha$ and let ${}^v\Sigma_x$ be its vectorial part. Then direct computation shows ${}^v\Sigma_x$ is a closed subroot system of ${}^v\Sigma$. Hence the spherical apartment $\mathscr{A}_x$ at x can be identified with ${}^v\!\mathscr{A}({}^v\Sigma_x)$. Also note that Σ_x can be identified with ${}^v\Sigma_x$ by $\alpha \mapsto {}^v\alpha$. In particular, the roots can be identified with the affine roots in Σ_o.

1.4.8. Notations as before. Suppose $\Phi = {}^v\Sigma$ is irreducible and all Γ_a are the same discrete subgroup of $\mathbb{R}$. Let ${}^v C$ be a Weyl chamber of Φ and Δ the simple roots it defines. Then there is a unique root a_0 such that $\|a_0\| \geqslant \|a\|$ for all root a [Bou02, chap.VI, §1, no.8, prop.25]. This a_0 is called the *highest root* with respect to Δ or ${}^v C$. The set $C = (o + {}^v C) \setminus \alpha^*_{-a_0+}$ is a chamber in $\mathscr{A}(\Phi)$ [Bou02, chap.VI, §2, no.2, prop.5] and is called the *fundamental alcove* for Δ.

Let $\widetilde{\Delta}$ denote the set of affine roots α defining walls of C, which means $C \subseteq \alpha$ and $\partial\alpha$ is a wall of C. Then it consists of the simple roots and the affine root $\alpha_0 = \alpha_{-a_0+}$. Such a set $\widetilde{\Delta}$ is a *basis* of Σ in the sense that any affine root is a $\mathbb{Z}$-linear combination of its elements and the coefficients are either all non-negative or all non-positive.

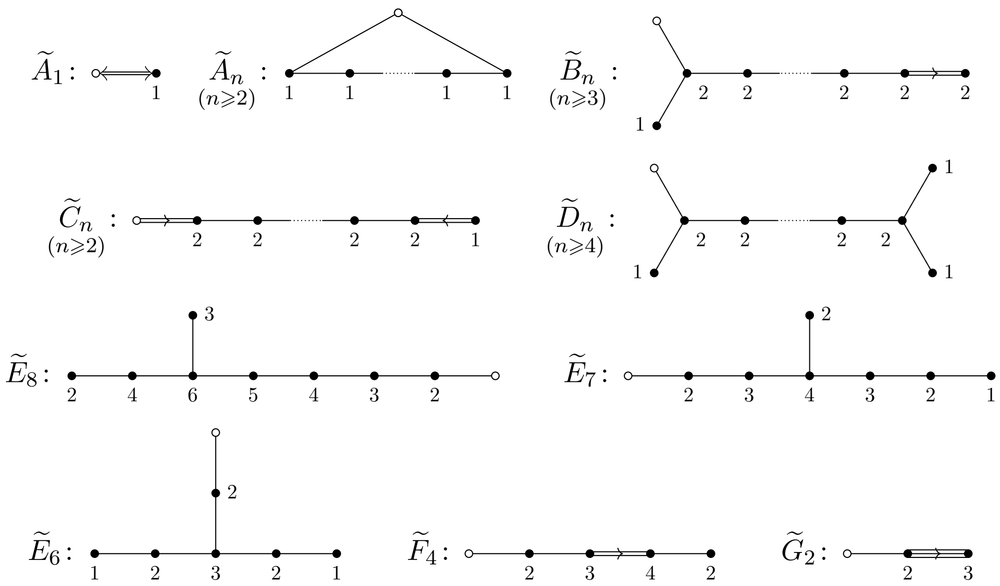
Conversely, let C be a chamber in $\mathscr{A}(\Phi)$ and x a special vertex which is also an extremal point of $\overline{C}$. The affine roots defining walls of C form a basis $\widetilde{\Delta}$ of the affine root system $\widetilde{\Phi}$. Among these affine roots, those vanishing at x give rise to a basis Δ of the root system Φ by taking their vectorial parts and the rest one gives rise to the highest root with respect to Δ by taking the negation of its vectorial part. Since chambers in $\mathscr{A}(\Phi)$ are fundamental alcoves for some basis, we call them *alcoves* to avoid confusion with Weyl chambers.

1.4.9. The types are introduced as follows. The affine Weyl group $W(\Sigma)$ is generated by r_α for $\alpha \in \widetilde{\Delta}$ as they are the affine roots defining walls of C. Therefore a type $I \in \mathcal{T}$ corresponds to a proper subset of $\widetilde{\Delta}$. From now on, we do not distinguish them. Then the face of C corresponding to I is the set

$$C_I = \overline{C} \cap \Big(\bigcap_{\alpha \in I} \partial\alpha\Big) \setminus \Big(\bigcup_{\alpha \in \widetilde{\Delta} \setminus I} \partial\alpha\Big).$$

1.4.10. Let Σ be an irreducible affine root system with $\widetilde{\Delta}$ a basis. Then the *extended Dynkin diagram* of it is defined similarly to Dynkin diagram except in the case of $\widetilde{A}_1$, where there is an left-right double arrow between the two vertices.

The followings are extended Dynkin diagrams of all irreducible affine root systems [Bou02, chap.VI, §4, no.3, prop.4], where the notation $\widetilde{X}_n$ indicates this affine root system arises from the root system of type X_n.



Also note that these Dynkin diagrams are decorated in the following way: the part consists of bold vertices is an ordinary Dynkin diagram and its vertices present the simple roots a_i ($1 \leqslant i \leqslant n$), then the extra hollow vertex presents the (affine root α_0 defined by the) highest root a_0 and each simple root a_i is labelled by its coefficient h_i in the expression

$$a_0 = \sum_{i=1}^{n} h_i a_i$$

presenting the highest root a_0 as $\mathbb{Z}$-linear combination of them.

A discrete affine apartment is said to be *of type* $\widetilde{X}_n$ if it is isomorphic to $\mathscr{A}(\Sigma)$ for an irreducible affine root system Σ of type $\widetilde{X}_n$.

1.5 Root data

Root systems can arise from root data.

1.5.1 Definition. A *(reduced) root datum*[5] $\mathcal{R}$ is a quadruple $(\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ in which

- X and $\mathsf{X}^\vee$ are free $\mathbb{Z}$-modules of finite rank in duality by a pairing
$$\langle \cdot\,, \cdot \rangle \colon \mathsf{X} \times \mathsf{X}^\vee \to \mathbb{Z},$$
- Φ and $\Phi^\vee$ are finite subsets of $\mathsf{X} \setminus \{0\}$ and $\mathsf{X}^\vee \setminus \{0\}$ respectively, in bijection by a correspondence $a \leftrightarrow a^\vee$,

satisfying

[5] in the sense of [SGA3, XXI, 1.1.1].

RD1. for any $a \in \Phi$, $\langle a, a^\vee \rangle = 2$;

RD2. for any $a \in \Phi$, the "reflection" $r_a \colon x \mapsto x - \langle x, a^\vee \rangle a$ preserves Φ and the "reflection" $r_a \colon y \mapsto y - \langle a, y \rangle a^\vee$ preserves $\Phi^\vee$;

RD3. for any $a \in \Phi$, $\mathbb{Z}a \cap \Phi = \{\pm a\}$.

Note that we do not distinguish the two kinds of "reflections" in symbols since they form isomorphic finite groups of automorphisms on X and $\mathsf{X}^\vee$ respectively and therefore it is better to view them as two representations of a same finite group ${}^vW(\mathcal{R})$. This group is called the *Weyl group* of the root datum.

1.5.2. If $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ is a root datum, then its Weyl group acts on the real vector space $\mathsf{X}^\vee_\mathbb{R} := \mathsf{X}^\vee \otimes \mathbb{R}$ and there is a unique inner product on it invariant under the action. Let V be the subspace of $\mathsf{X}^\vee_\mathbb{R}$ spanned by $\Phi^\vee$. Then Φ is a (reduced) root system on the Euclidean vector space V.

In general, V is not the entire $\mathsf{X}^\vee_\mathbb{R}$. When it is, we say $\mathcal{R}$ is *semisimple*. So the apartment associated to root systems can also be viewed as the apartment associated to semisimple root data. As for the non-semisimple ones, they give rise to non-essential apartments and hence are ignored in this note.

The quadruple $\mathcal{R}^\vee = (\mathsf{X}^\vee, \Phi^\vee, \mathsf{X}, \Phi)$ is also a root datum, called the *dual root datum of* $\mathcal{R}$. It is clear that dual root data give rise to coroot systems on the dual spaces.

1.5.3. Let $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ and $\mathcal{R}' = (\mathsf{X}', \Phi', \mathsf{X}'^\vee, \Phi'^\vee)$ be two root data. Then a *morphism* $f \colon \mathcal{R}' \to \mathcal{R}$ between them is a linear map $f \colon \mathsf{X}' \to \mathsf{X}$ inducing a bijection $\Phi \to \Phi'$ and its transpose ${}^tf$ induces a bijection $\Phi'^\vee \to \Phi^\vee$. If f is a morphism of root data, then it also induces bijections between bases, systems of positive roots and Weyl chambers [SGA3, XXI, 6.1.3]. As a consequence, it induces an isomorphism of spherical apartments ${}^v\mathscr{A}(\Phi') \cong {}^v\mathscr{A}(\Phi)$ (and also an isomorphism of affine apartments $\mathscr{A}(\Sigma') \cong \mathscr{A}(\Sigma)$ if $\Phi = {}^v\Sigma$ and $\Phi' = {}^v\Sigma'$ with covariant choice of Γ_a's).

A morphism of root data $f \colon \mathcal{R}' \to \mathcal{R}$ is an *isogeny*[6] if the linear map f is injective and has finite cokernel.

1.5.4. Let $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ be a root datum. Let $\mathsf{X}_0 = \{x \in \mathsf{X} \mid \langle x, \Phi^\vee \rangle = 0\}$ and $\mathsf{X}^\vee_0 = \mathsf{X}^\vee / (V \cap \mathsf{X}^\vee)$. Then X_0 and $\mathsf{X}^\vee_0$ are in duality by the pairing of $\mathcal{R}$ and thus give a trivial root datum $(\mathsf{X}_0, \emptyset, \mathsf{X}^\vee_0, \emptyset)$. It is called the *coradical of* $\mathcal{R}$ and is denoted by $\mathrm{corad}(\mathcal{R})$.

The dual root datum of the coradical of the dual $\mathcal{R}^\vee = (\mathsf{X}^\vee, \Phi^\vee, \mathsf{X}, \Phi)$ is called the *radical of* $\mathcal{R}$ and is denoted by $\mathrm{rad}(\mathcal{R})$. More precisely, let $\mathsf{Y}_0 = \big\{y \in \mathsf{X}^\vee \bigm| \langle \Phi, y \rangle = 0\big\}$ and $\mathsf{Y}^\vee_0 = \mathsf{X} / (V^* \cap \mathsf{X})$, then $\mathrm{rad}(\mathcal{R})$ is the root datum $(\mathsf{Y}^\vee_0, \emptyset, \mathsf{Y}_0, \emptyset)$. It follows that $\mathcal{R}$ is semisimple if and only if $\mathrm{corad}(\mathcal{R}) = 0$ if and only if $\mathrm{rad}(\mathcal{R}) = 0$.

Let $\mathcal{R}^0$ denote the trivial root datum $(\mathsf{X}, \emptyset, \mathsf{X}^\vee, \emptyset)$. Then the inclusion and projection to X induce morphism of root data

$$\mathrm{corad}(\mathcal{R}) \longrightarrow \mathcal{R}^0 \longrightarrow \mathrm{rad}(\mathcal{R}).$$

[6] in the sense of [SGA3, XXI, 6.2.1] and is called a *central isogeny* in [Mil17, 23.2]

and the composition $\mathrm{corad}(\mathcal{R}) \to \mathrm{rad}(\mathcal{R})$ is an isogeny [SGA3, XXI, 6.3.4]. Its cokernel is denoted by $\mathrm{N}(\mathcal{R})$. Note that there is a pairing:

$$\mathrm{N}(\mathcal{R}) \times \mathrm{N}(\mathcal{R}^\vee) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

1.5.5. A *lattice* L in a $\mathbb{R}$-vector space V is a finitely generated $\mathbb{Z}$-submodule of V spanning V. Its *dual lattice* L^* is the lattice in the dual space V^* consisting of those functionals $f \in V^*$ such that $f(L) \subseteq \mathbb{Z}$.

Given a root system Φ on a Euclidean vector space V, there are four lattices:

$\mathcal{Q}$ the *root lattice*, which is the lattice in V generated by the roots;

$\mathcal{Q}^\vee$ the *coroot lattice*, which is the lattice in V^* generated by the coroots;

$\mathcal{P}$ the *weight lattice*, which is the dual lattice of $\mathcal{Q}^\vee$ in V;

$\mathcal{P}^\vee$ the *coweight lattice*, which is the dual lattice of $\mathcal{Q}$ in V^*.

Suppose the root system Φ is given by a root data $\mathcal{R}$. Then X contains $\mathcal{Q}$. If $\mathcal{R}$ is semisimple, then X is a lattice in V between $\mathcal{Q}$ and $\mathcal{P}$. In this case, the quotient $\mathcal{P}/\mathsf{X}$ is a finite group $\pi_1(\mathcal{R})$, called the *fundamental group of* $\mathcal{R}$; the quotient $\mathsf{X}/\mathcal{Q}$ is a finite group $Z(\mathcal{R})$, called the *centre of* $\mathcal{R}$.

1.5.6. Let $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ be a root datum and Y a submodule of X containing Φ. Let $i\colon \mathsf{Y} \to \mathsf{X}$ be the inclusion with the transpose ${}^t i\colon \mathsf{X}^\vee \to \mathsf{Y}^\vee$ and let $\Phi_{\mathsf{Y}} = \Phi$, $\Phi_{\mathsf{Y}}^\vee = {}^t i(\Phi^\vee)$. Then $(\mathsf{Y}, \Phi_{\mathsf{Y}}, \mathsf{Y}^\vee, \Phi_{\mathsf{Y}}^\vee)$ is a root datum and i is a morphism of root data. It is called *the root datum induced by* $\mathcal{R}$ *on* Y and is denoted by $\mathcal{R}_{\mathsf{Y}}$. Let $\mathsf{Y}^\vee$ be a submodule of $\mathsf{X}^\vee$ containing $\Phi^\vee$. Then the dual root datum of the root datum induced by $\mathcal{R}^\vee$ on $\mathsf{Y}^\vee$ is called *the root datum coinduced by* $\mathcal{R}$ *on* $\mathsf{Y}^\vee$ and is denoted by $\mathcal{R}^{\mathsf{Y}^\vee}$.

The followings are some special cases of above.

$\mathrm{ad}(\mathcal{R})$ is the root datum induced by $\mathcal{R}$ on the root lattice $\mathcal{Q}$;

$\mathrm{ss}(\mathcal{R})$ is the root datum induced by $\mathcal{R}$ on $V^* \cap \mathsf{X}$;

$\mathrm{der}(\mathcal{R})$ is the root datum coinduced by $\mathcal{R}$ on $V \cap \mathsf{X}^\vee$;

$\mathrm{sc}(\mathcal{R})$ is the root datum coinduced by $\mathcal{R}$ on the coroot lattice $\mathcal{Q}^\vee$.

Then we have the following commutative diagram of root data [SGA3, XXI, 6.5.5 – 6.5.9]

$$\begin{tikzcd}
\mathrm{ad}(\mathcal{R}) \arrow[r] & \mathrm{ss}(\mathcal{R}) \arrow[rr] \arrow[d] \arrow[dr] & & \mathrm{der}(\mathcal{R}) \arrow[r] \arrow[d] & \mathrm{sc}(\mathcal{R}) \\
& \mathrm{ss}(\mathcal{R}) \times \mathrm{corad}(\mathcal{R}) \arrow[r] & \mathcal{R} \arrow[r] \arrow[ur] & \mathrm{der}(\mathcal{R}) \times \mathrm{rad}(\mathcal{R}) &
\end{tikzcd}$$

where the first line contains three isogenies between four root data and the second line contains two isogenies. Moreover,

(i) $\mathrm{ad}(\mathcal{R})$ is *adjoint*, namely every isogeny to it is an isomorphism;

(ii) $\mathrm{sc}(\mathcal{R})$ is *simply-connected*, namely every isogeny from it is an isomorphism;

(iii) $\mathcal{R}$ is semisimple if and only if the middle triangle consists of isomorphisms;

(iv) if $\mathcal{R}$ is semisimple, its centre $Z(\mathcal{R})$ and fundamental group $\pi_1(\mathcal{R})$ are the cokernels of the isogenies $\mathrm{ad}(\mathcal{R}) \to \mathcal{R}$ and $\mathcal{R} \to \mathrm{sc}(\mathcal{R})$ respectively;

(v) the cokernel of the isogenies $\mathrm{ss}(\mathcal{R}) \times \mathrm{corad}(\mathcal{R}) \to \mathcal{R}$, $\mathrm{der}(\mathcal{R}) \times \mathrm{rad}(\mathcal{R}) \to \mathcal{R}$ and $\mathrm{ss}(\mathcal{R}) \to \mathrm{der}(\mathcal{R})$ are all isomorphic to $\mathrm{N}(\mathcal{R})$;

(vi) $\mathcal{R}$ is the product of a semisimple root datum with a trivial root datum if and only if $\mathrm{N}(\mathcal{R}) = 0$.

(vii) all root data in this diagram have isomorphic root systems and hence isomorphic apartments.

1.6 Euclidean buddings

It's time to give the definition of Euclidean buildings.

1.6.1 Definition. A *(Euclidean) building* is a set $\mathscr{B}$ equipped with a polysimplicial complex $\mathcal{F}$, whose members are subsets of $\mathscr{B}$ and are called *facets*, and a family $\mathcal{A}$ of subsets of $\mathscr{B}$, whose members are called *apartment*, such that the following axioms are satisfied.

EB0. For each apartment $A \in \mathcal{A}$, there is an (abstract) apartment $\mathscr{A}$ together with a bijection between them, exchanging the complex $\mathcal{F}_A$ of facets contained in A and the complex of facets in $\mathscr{A}$.

Note that, this allows us to view each apartments in $\mathscr{B}$ as Euclidean affine spaces and hence it makes sense to talk about isometries between them.

EB1. For any two facets F, F', there is an apartment A containing them.

EB2. If A, A' are two apartments containing both F and F', then there is an isomorphism between A and A' fixing F and F' pointwise.

Here an isomorphism between A and A' is an isometry between them exchanging the posets $\mathcal{F}_A$ and $\mathcal{F}_{A'}$.

Note that, from the definition, all apartments $A \in \mathcal{A}$ are isomorphic to an abstract one $\mathscr{A}$. Then $\mathscr{B}$ is said to be *of type* $\mathscr{A}$ and is said to be *spherical* (resp. *discrete affine*, etc.) if so is $\mathscr{A}$. The Weyl group W of $\mathscr{A}$ is also called the *Weyl group* of $\mathscr{B}$.

Remark. One can see that the combinatorial information of $\mathscr{B}$ is encoded in the polysimplicial complex $\mathcal{F}$ and hence is completely determined by it up to a choice of the family $\mathcal{A}$. One can compare the axioms **EB0.–EB2.** with **B0.–B2.**.

Remark. The notions of *walls*, *chambers*, *vertices* and *types* in a building is defined similarly as in an apartment and we will use the same notations as there. Furthermore, there is a *type function* $\tau\colon \mathcal{F} \to \mathcal{T}$ extending the type function on an apartment to the entire building uniquely.

Remark. We have assumed that apartments are essential. In particular, the buildings in Bruhat-Tits theory used in this note are the *reduced buildings*, rather than *extended buildings*. However, this is harmless as we focus more on the polysimplicial structure and we do want the vertices being points.

1.6.2. An *morphism* between buildings $\mathscr{B}$ and $\mathscr{B}'$ is a continuous map inducing a *chamber map* between $\mathcal{F}$ and $\mathcal{F}'$ and maps apartments in apartments. Then an *automorphism* of a building is an isometry transforming a facet (resp. apartment) in a facet (resp. apartment). Any building can be decomposed into a product of a trivial building with irreducible essential buildings, similarly as in 1.3.3. However, there is no guarantee that such a decomposition gives a good corresponding on the family $\mathcal{A}$.

1.6.3. An automorphism is said to be *type-preserving* if it leaves the type function τ invariant. For instance, any $w \in W$ is such an automorphism. A group G of automorphisms is said to be *strongly transitive* if it acts transitively on the pairs (C, A) where C is a chamber in the apartment A. This is the case if and only if G acts transitively on apartments and in any apartment A, the following conditions for a pair of chambers C, C' in A are equivalent:

(i) C and C' are conjugated by the Weyl group W;

(ii) C and C' are conjugated by the stabilizer $N_G(A)$ of A in G;

(iii) C and C' are conjugated by G.

When a group G of automorphisms is strongly transitive and type-preserving, we have

$$W \cong N_G(A)/C_G(A),$$

where $C_G(A)$ is the fixator of an apartment A in G.

1.6.4. Let G be a strongly transitive and type-preserving group of automorphisms and F be a facet in an apartment A. The stabilizer (which is also the fixator)

$$G_F := N_G(F) = C_G(F)$$

of F is called a *parabolic subgroup* of G. The parabolic group G_F acts transitively on the apartments containing F. Indeed, one can deduce this from the fact that G_F acts transitively on chambers containing F since G is strongly transitive. Here the former is due to that G acts transitively on chambers and is type-preserving.

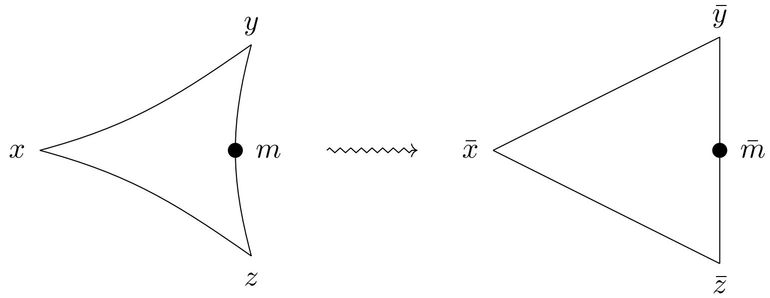
Moreover, we have the *Bruhat decomposition* [Rou09, 6.9]

$$G = G_F.N_G(A).G_F.$$

In particular, if $F = C$ is a chamber, then

$$G = \bigsqcup_{w \in W} G_C w G_C.$$

1.6.5. The apartments are Euclidean affine spaces, hence have metrics. Those metrics are compatible in the sense that the agree on any overlap, hence are glued into a metric $d(-,-)$ on the entire building $\mathscr{B}$ in a consistent way. Then $\mathscr{B}$ equipped with this metric is a complete metric space having the *CAT(0)-property* [Rou09, 6.5], which means that geodesic triangles in $\mathscr{B}$ are at least as thin as in a Euclidean plane: saying x, y, z are three points in $\mathscr{B}$ forming a geodesic triangle and $\bar{x}, \bar{y}, \bar{z}$ are three points in a Euclidean plane having the same pointwise distance as x, y, z, then for any point m in the geodesic segment $[x, y]$ in the triangle and $\bar{m}$ the corresponding point in the segment $[\bar{x}, \bar{y}]$ (namely, $d(\bar{x}, \bar{m}) = d(x, m)$), then $d(z, m) \leqslant d(\bar{z}, \bar{m})$.



Consequences of the CAT(0)-property include: the geodesic segments between points are unique [Rou09, 6.6]; any group of isometries stabilizing a nonempty bounded subset has a fixed point [Rou09, 7.1]; the distance from a point to a nonempty closed convex subset is achieved by a unique point [Rou09, 7.3]. For more details, see [Rou09, §6 and 7].

1.6.6. A *bornology* on a set X is a collection $\mathcal{B}$ of subsets of X such that it covers X and is stable under inclusion and finite unions. Once such a bornology is chosen, its members are called *bounded subsets* of X. For instance, any metric space has a canonical bornology induced by its metric. Another example is any topological space, where the bornology consists of all relatively compact subsets. A *morphism* between bornological sets is a map preserving the bornologies.

A *bornological group* is a group G equipped with a bornology on it stable under multiplication. For instance, let G be an isometry group on a metric space X, then there is a canonical bornology whose members are subsets M such that the set $M.x$ is bounded in X for some $x \in X$.

Let $\varphi \colon G' \to G$ be a group homomorphism and G a bornological group. Then we can canonically pullback the bornology on G to G': a subset of G' is bounded when its image is bounded in G.

So we can talk about *bounded subgroups* of a group G acting on the building $\mathscr{B}$ regardless its own topology or bornology. But if G is topological or bornological, it makes sense to ask if its bornology is the same as the pullback one. It worth to point out that this is the case when G acts continuously on $\mathscr{B}$.

Let G be a strongly transitive and type-preserving group of automorphisms of $\mathscr{B}$, then for any subgroup H of G, the following conditions are equivalent [Rou09, 7.2].

(i) H is bounded;

(ii) H fixes a point in $\mathscr{B}$;

(iii) H is contained in a parabolic subgroup of G.

In particular, the maximal bounded subgroups of G are the maximal parabolic subgroups and hence the stabilizers of vertices.

In general, if G is not type-preserving, then maximal bounded subgroups of G are still stabilizers of points, but: *1,* not all such stabilizers are maximal; *2,* not all such stabilizers are stabilizers of vertices.

§ 2 Reductive groups and Tits buildings

Tits' building theory was applied to study the structure of reductive groups over arbitrary field, a family of linear algebraic groups play important roles in mathematics. Throughout this section, we fix a ground field K and an algebraic closure K^{a} (resp. separable closure K^{s}) of it.

2.1 Algebraic groups

We first recall some basic notions on algebraic groups.

2.1.1 Definition. By an *algebraic group* (defined over K), we mean a group object in the category $\mathbf{Sch}_K$ of schemes of finite type over K.

The definition implies that algebraic groups are in particular group-valued functors from the category $\mathbf{Alg}_K$ of finitely generated K-algebras.

An algebraic group is *affine* (resp. *smooth*, *connected*, etc.) if so is its underlying scheme. In particular, affine algebraic groups are precisely representable group-valued functors.

We will use bold letters like G to denote algebraic groups defined over K. For any K-algebra R, the group scheme obtained by base change $\mathsf{G} \otimes_K R$ is denoted by G_R and the group of R-points is denoted by $\mathsf{G}(R)$ (but if we used notations with parenthesis, e.g. $\mathsf{GL}(V)$, to denote an algebraic group, then its group of R-points is denoted by padding R into the parenthesis as the last parameter, e.g. $\mathsf{GL}(V, R)$). Moreover, $\mathsf{G}(K)$ is simply denoted by G and $\mathsf{G}_R(R) \cong \mathsf{G}(R)$ is simply denoted by G_R. We also use $g \in \mathsf{G}$ to mean that g is a R-point of G for some K-algebra R.

Many group-theoretical constructions apply to algebraic groups. For G an algebraic group and H a subgroup, we use $\mathsf{N}_{\mathsf{G}}(\mathsf{H})$ (resp. $\mathsf{Z}_{\mathsf{G}}(\mathsf{H})$) to denote *normalizer* (resp. *centralizer*) of H in G. In particular, $\mathsf{Z}(\mathsf{G})$ denote the centre of G.

Remark (Cohomology of algebraic groups). Definition 2.1.1 is equivalent to say that an algebraic group is a locally presentable group-valued sheaf on the site K_{fppf} whose underlying category is $\mathbf{Alg}_K^{\mathrm{opp}}$ and is equipped with the *fppf topology*[7]. Let R be an object in this site, its *fppf covering* is a family of K-algebra homomorphisms $R \to R_i$ of finite presentation such that $R \to \prod_i R_i$ is faithfully flat. Therefore a functor F from $\mathbf{Alg}_K$ is a sheaf on K_{fppf} if and only if it satisfies the followings [Mil17, 5.65].

(a) (*Local*) For any K-algebras $R_1, \cdots, R_m$,

$$\mathsf{F}(R_1 \times \cdots \times R_m) \cong \mathsf{F}(R_1) \times \cdots \times \mathsf{F}(R_m).$$

(b) (*Decent*) For any faithfully flat K-algebra homomorphism $R' \to R$, the sequence

$$\mathsf{F}(R) \longrightarrow \mathsf{F}(R') \rightrightarrows \mathsf{F}(R' \otimes_R R')$$

[7]The name *fppf* is short of "fidèlement plate de présentation finie", that is, "faithfully flat and of finite presentation". Note that any finitely generated k-algebra R is noetherian, hence all morphisms of finite type in the category $\mathbf{Alg}_K$ are actually of finite presentation.

is exact, where the homomorphisms $R' \to R' \otimes_R R'$ are $r \mapsto r \otimes 1$ and $r \mapsto 1 \otimes r$ respectively.

So a sequence of algebraic groups

$$1 \longrightarrow \mathsf{N} \longrightarrow \mathsf{G} \longrightarrow \mathsf{Q} \longrightarrow 1.$$

is *exact* means N is isomorphic to the kernel of $\mathsf{G} \to \mathsf{Q}$ and $\mathsf{G} \to \mathsf{Q}$ is surjective as a sheaf homomorphism. The later turns out to say that $\mathsf{G} \to \mathsf{Q}$ is faithfully flat [Mil17, 5.43]. When consider homomorphisms between smooth algebraic groups, this is equivalent to say that $\mathsf{G} \to \mathsf{Q}$ is surjective on closed points [Mil17, 1.71]. Hence to verify a homomorphism between smooth algebraic groups is surjective, it is sufficient to verify on K^{a}-points.

Hence in general we do not have a short exact sequence of the groups of K-points. Instead, there is a long exact sequence:

$$1 \longrightarrow \mathsf{N}(K) \longrightarrow \mathsf{G}(K) \longrightarrow \mathsf{Q}(K) \longrightarrow \mathrm{H}^1(K, \mathsf{N}) \longrightarrow \mathrm{H}^1(K, \mathsf{G}) \longrightarrow \mathrm{H}^1(K, \mathsf{Q}).$$

It turns out that [Mil17, 3.50], if G is a smooth algebraic group, then $\mathrm{H}^1(K, \mathsf{G})$ is canonically isomorphic to the *Galois cohomology* $\mathrm{H}^1(\Gamma, \mathsf{G}(K^{\mathrm{s}}))$ with $\Gamma = \mathsf{Gal}(K^{\mathrm{s}}/K)$. The followings are some useful results in Galois cohomology.

(i) (*Hilbert's theorem 90*) If L/K is a Galois extension, then $\mathrm{H}^1(\mathsf{Gal}(L/K), L^\times) = 0$.

(ii) (*Lang's theorem* [Mil17, 17.98]) If G is a smooth connected algebraic group over a finite field K, then $\mathrm{H}^1(K, \mathsf{G}) = 0$.

2.1.2. Let G be an algebraic group. Its *neutral component* G° is the largest connected subgroup of G. Its *component group* $\pi_0(\mathsf{G})$ is the universal étale scheme under G. Then there is an exact sequence [Mil17, 2.37]:

$$1 \longrightarrow \mathsf{G}^\circ \longrightarrow \mathsf{G} \longrightarrow \pi_0(\mathsf{G}) \longrightarrow 1.$$

The above formations are compatible with field extensions and products.

The following conditions on an algebraic group G are equivalent [Mil17, 1.36]:

(i) G is irreducible;

(ii) G is connected;

(iii) G is geometrically connected;

(iv) $\pi_0(\mathsf{G})$ equals the trivial group 1.

2.1.3 Example. Here we give some algebraic groups presented as functors.

(a) The functor $R \rightsquigarrow (R, +)$ mapping a K-algebra to its underlying abelian group defines an algebraic group $\mathbb{G}_{\mathrm{a}}$, called the *additive group*.

(b) The functor $R \rightsquigarrow (R^\times, \times)$ mapping a K-algebra to its unit group defines an algebraic group $\mathbb{G}_\mathrm{m}$, called the *multiplicative group.*

(c) The functor $R \rightsquigarrow \{r \in R \mid r^n = 1\}$ mapping a K-algebra to its set of n-th roots of unity defines an algebraic group μ_n, called the *group of n-th roots of unity.*

(d) Let G be a finite group. The constant functor $R \rightsquigarrow G$ is not a scheme, but its sheafification $R \rightsquigarrow \mathrm{Map}(\pi_0(R), G)$, where $\pi_0(R)$ is the set of connected components of $\mathrm{Spec}(R)$, defines an algebraic group $\underline{G}$. Such an algebraic group is called a *constant algebraic group.*

(e) Let V be a finite-dimensional vector space over K, then the functor $R \rightsquigarrow V_R := V \otimes_K R$ defines an algebraic group $\mathbb{W}(V)$, called the *additive group of V*. Any choice of basis of V gives rise to an isomorphism from this group to a product of copies of $\mathbb{G}_\mathrm{a}$.

(f) The functor mapping a K-algebra R to the additive group of $m \times n$ matrices with entries in R defines an algebraic group $\mathsf{M}_{m\times n}$.

(g) Let V be a finite-dimensional vector space over K, then the functor $R \rightsquigarrow \mathrm{End}(V_R)$ defines an algebraic group $\mathsf{End}(V)$. When V is of dimension n, any choice of basis of V gives an isomorphism from this group to $\mathsf{M}_{n\times n}$.

(h) The functor mapping a K-algebra R to the group of invertible $n \times n$ matrices with entries in R defines an algebraic group GL_n, called the *general linear group.*

(i) Let V be a finite-dimensional vector space over K, then the functor $R \rightsquigarrow \mathrm{Aut}(V_R)$ defines an algebraic group $\mathsf{GL}(V)$, called the *general linear group of V*. When V is of dimension n, any choice of basis of V gives rise to an isomorphism from this group to GL_n.

All above functors are representable. Hence above algebraic groups are affine.

2.1.4. A *representation* of an algebraic group G is a homomorphism of group-valued functors $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$, where V is a vector space over K and $\mathsf{GL}(V)$ is the functor $R \rightsquigarrow \mathrm{Aut}(V_R)$. When V is finite-dimensional, this is a homomorphism of algebraic groups. Such a representation is *faithful* if ρ is injective.

An algebraic group is *linear* if it admits a finite-dimensional faithful representation. Equivalently, an algebraic group is linear if it is isomorphic to an algebraic subgroup of some GL_n. It turns out that [Mil17, 1.43 and 4.10]:

$$\text{affine algebraic group} = \text{linear algebraic group}.$$

2.1.5 Example. Here we give some linear algebraic groups.

(a) The functor $R \rightsquigarrow \{g \in \mathsf{GL}_n(R) \mid \det(g) = 1\}$ mapping a K-algebra R to the group of invertible $n\times n$ matrices with entries in R and determinant 1 defines an algebraic subgroup SL_n of GL_n, called the *special linear group.*

(b) The functor $R \rightsquigarrow \{(g_{ij}) \in \mathsf{GL}_n(R) \mid g_{ij} = 0 \text{ for } i > j\}$ mapping a K-algebra R to the group of upper triangular invertible $n \times n$ matrices with entries in R defines an algebraic subgroup T_n of GL_n.

(c) The functor $R \rightsquigarrow \{(g_{ij}) \in \mathsf{GL}_n(R) \mid g_{ij} = 0 \text{ for } i > j \text{ and } g_{ij} = 1 \text{ if } i = j\}$ mapping a K-algebra R to the group of upper triangular invertible $n \times n$ matrices with entries in R and diagonal entries 1 defines an algebraic subgroup U_n of T_n.

(d) The functor $R \rightsquigarrow \{\mathrm{diag}(t_1, \cdots, t_n) \in \mathsf{GL}_n \mid t_1, \cdots, t_n \in R\}$ mapping a K-algebra R to the group of invertible diagonal $n \times n$ matrices with entries in R defines an algebraic subgroup D_n of T_n. Note that $\mathsf{D}_n \cong \mathbb{G}_{\mathrm{m}}^n$.

(e) The functor $R \rightsquigarrow \{g \in \mathsf{GL}(V, R) \mid \det(g) = 1\}$ mapping a K-algebra R to the group of R-automorphisms of V_R having determinant 1 defines an algebraic subgroup $\mathsf{SL}(V)$ of $\mathsf{GL}(V)$, called the *special linear group of* V.

(f) The quotient of GL_n (resp. $\mathsf{GL}(V)$) by the norm subgroup of scalars is a linear algebraic group. It is denoted by PGL_n (resp. $\mathsf{PGL}(V)$) and is called the *projective linear group*.

2.1.6. An algebraic group G is *unipotent* if its every finite-dimensional representation $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$ is *unipotent*, namely there exists a G-stable flag

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_m = V,$$

such that G acts trivially on each factor V_i/V_{i-1}. Equivalently, an algebraic group is unipotent if it is isomorphic to an algebraic subgroup of some U_n.

For any $g \in \mathsf{G}(K^{\mathrm{a}})$, we have *Jordan–Chevalley decomposition* [Mil17, 9.18]: there exist unique elements $g_s, g_u \in \mathsf{G}(K^{\mathrm{a}})$ such that

$$g = g_s g_u = g_u g_s,$$

and for any representation $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$, the linear operator $\rho(g_s)$ is semisimple and $\rho(g_u)$ is unipotent. An element $g \in \mathsf{G}(K^{\mathrm{a}})$ is said to be *semisimple* (reps. *unipotent*) if $g = g_s$ (resp. $g = g_u$). A smooth algebraic group G is unipotent if and only if all elements of $\mathsf{G}(K^{\mathrm{a}})$ are unipotent [Mil17, 14.12].

2.1.7. An algebraic group is a *vector group* if it is isomorphic to a product of copies of $\mathbb{G}_{\mathrm{a}}$. Let V be a finite-dimensional vector space over K, then the algebraic group $\mathbb{W}(V)$ is a vector group. A vector group is in particular a vector bundle on K_{fppf}.

For V a vector bundle on K_{fppf}, let $\mathsf{V}^\times$ denote the open subscheme of V obtained by deleting the zero section. Then the action of $\mathbb{G}_{\mathrm{a}}$ on V induces an action of $\mathbb{G}_{\mathrm{m}}$ on $\mathsf{V}^\times$. In particular, if L is a one-dimensional vector space over K, then $\mathbb{W}(L)$ is a line bundle and $\mathbb{W}(L)^\times$ is a homogeneous principal $\mathbb{G}_{\mathrm{m}}$-bundle [SGA3, XIX, 4.3-4.4].

2.1.8. For R a K-algebra, its *algebra of dual numbers* is the algebra $R[\epsilon]/(\epsilon^2)$. Let $\mathscr{D}$ denote the functor sending each R to its algebra of dual numbers. For X a K-scheme, the composition $\mathsf{X} \circ \mathscr{D}$ is also a K-scheme, called the *tangent bundle of* X and is denoted by $\mathsf{T}(\mathsf{X})$. For any point x of X, the pullback of $\mathsf{T}(\mathsf{X})$ along $x \hookrightarrow \mathsf{X}$ is called the *tangent space of* X *at* x and is dented by $\mathsf{T}_x(\mathsf{X})$.

Let G be an algebraic group and e be its identity. Then both $\mathsf{T}(\mathsf{G})$ and $\mathsf{T}_e(\mathsf{G})$ are algebraic groups and we have a split short exact sequence [SGA3, II, 3.9.0.2]

$$\begin{tikzcd} 1 \arrow[r] & \mathsf{T}_e(\mathsf{G}) \arrow[r, "i"] & \mathsf{T}(\mathsf{G}) \arrow[r, "\mathrm{pr}"] & \mathsf{G} \arrow[r] \arrow[l, bend left, dotted, "s"] & 1. \end{tikzcd}$$

Let $\varphi\colon \mathsf{G} \to \mathsf{G}'$ be a homomorphism of algebraic groups, then there is a unique morphism of K_{fppf}-vector bundles $\mathrm{d}\varphi\colon \mathsf{T}_e(\mathsf{G}) \to \mathsf{T}_e\big(\mathsf{G}'\big)$ making the following diagram commute

$$\begin{tikzcd} 1 \arrow[r] & \mathsf{T}_e(\mathsf{G}) \arrow[r] \arrow[d, dotted, "\mathrm{d}\varphi"] & \mathsf{T}(\mathsf{G}) \arrow[r] \arrow[d, "\mathsf{T}(\varphi)"] & \mathsf{G} \arrow[r] \arrow[d, "\varphi"] & 1. \\ 1 \arrow[r] & \mathsf{T}_e\big(\mathsf{G}'\big) \arrow[r] & \mathsf{T}\big(\mathsf{G}'\big) \arrow[r] & \mathsf{G}' \arrow[r] & 1. \end{tikzcd}$$

The morphism $\mathrm{d}\varphi$ is called the *differential of* φ. We will not distinguish it with the K-linear map on K-points $\mathrm{d}\varphi(K)\colon \mathsf{T}_e(\mathsf{G}, K) \to \mathsf{T}_e\big(\mathsf{G}', K\big)$.

Let $\mathfrak{g}$ denote the vector space $\mathsf{T}_e(\mathsf{G}, K)$, hence $\mathsf{T}_e(\mathsf{G}) = \mathbb{W}(\mathfrak{g})$. Then the action of G on itself by conjugations induces a representation $\mathrm{Ad}\colon \mathsf{G} \to \mathsf{GL}(\mathfrak{g})$ of G on $\mathfrak{g}$: for any $g \in \mathsf{G}$, the endomorphism $\mathrm{Ad}(g)$ is the differential of $\mathrm{inn}(g)$ (conjugated by g). This representation is called the *adjoint representation* of G [SGA3, II, 4.1]. Let $\mathrm{ad}\colon \mathsf{T}_e(\mathsf{G}) \to \mathsf{End}(\mathfrak{g})$ denote the differential of the adjoint representation and for any $X, Y \in \mathfrak{g}$, define $[X, Y]$ as $\mathrm{ad}(X).Y$. Then this gives rise to a Lie bracket

$$\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}\colon \qquad X, Y \longmapsto [X, Y].$$

This Lie algebra is called the *Lie algebra of G* and is denoted by $\mathrm{Lie}(\mathsf{G})$.

2.1.9 Example. The Lie algebras of $\mathbb{G}_{\mathrm{m}}$ and $\mathbb{G}_{\mathrm{a}}$ are the trivial Lie algebra K. The Lie algebras of GL_n, SL_n, T_n, U_n and D_n are the Lie algebras $\mathfrak{gl}_n$ of all matrices, $\mathfrak{sl}_n$ of trace zero matrices, $\mathfrak{t}_n$ of all upper triangular matrices, $\mathfrak{u}_n$ of strict upper triangular matrices and $\mathfrak{d}_n$ of all diagonal matrices respectively.

2.1.10. The above constructions give rise to an equivalence of categories between vector groups and finite-dimensional vector spaces over K [Mil17, 10.9]. Moreover, when K is of characteristic zero and G is a unipotent group over it, there is an isomorphism of schemes (and of algebraic groups if G is further commutative)

$$\exp\colon \mathsf{T}_e(\mathsf{G}) = \mathbb{W}(\mathrm{Lie}(\mathsf{G})) \longrightarrow \mathsf{G},$$

called the *exponential map* [Mil17, 14.32].

2.1.11. An algebraic group is a *torus* if it becomes isomorphic to a product of copies of $\mathbb{G}_{\mathrm{m}}$ over some finite containing K. A torus over K is *split* if it is already isomorphic to a product of copies of $\mathbb{G}_{\mathrm{m}}$ over K.

An algebraic group G is *diagonalizable* if its every representation is *diagonalizable*, namely it is a sum of one-dimensional representations. Equivalently, an algebraic group is unipotent if it is isomorphic to an algebraic subgroup of some D_n.

An algebraic group G is *of multiplicative type* if it becomes diagonalizable over some finite containing K. All tori are of multiplicative type. A smooth commutative algebraic group G is of multiplicative type if and only if all elements of $\mathsf{G}(K^{\mathrm{a}})$ are semisimple [Mil17, 12.21].

A *character* of an algebraic group G is a homomorphism $\chi\colon \mathsf{G} \to \mathbb{G}_{\mathrm{m}}$. Let χ and χ' be two characters of G, then the sum $\chi + \chi'$ is defined as

$$(\chi + \chi')(g) = \chi(g) \cdot \chi'(g), \qquad \forall g \in \mathsf{G}.$$

This is again a character and the set of characters is an abelian group, denoted by $\mathsf{X}(\mathsf{G})$. The *character group* of G is the abelian group $\mathsf{X}^*(\mathsf{G}) := \mathrm{Hom}(\mathsf{G}_{K^{\mathrm{s}}}, \mathbb{G}_{\mathrm{m},K^{\mathrm{s}}})$.

A *cocharacter* of an algebraic group G is a homomorphism $\lambda\colon \mathbb{G}_{\mathrm{m}} \to \mathsf{G}$. Suppose G is commutative. Then the sum $\lambda + \lambda'$ of two cocharacters of G is defined as:

$$(\lambda + \lambda')(z) = \lambda(z) \cdot \lambda'(z), \qquad \forall z \in \mathbb{G}_{\mathrm{m}}.$$

This is again a cocharacter. The *cocharacter group* of G is then the abelian group $\mathsf{X}_*(\mathsf{G}) := \mathrm{Hom}(\mathbb{G}_{\mathrm{m},K^{\mathrm{s}}}, \mathsf{G}_{K^{\mathrm{s}}})$.

2.1.12 Example. Let $\mathsf{G} = \mathsf{D}_n$. For each $1 \leqslant i \leqslant n$, define $\chi_i\colon \mathsf{D}_n \to \mathbb{G}_{\mathrm{m}}$ as the character

$$\mathrm{diag}(t_1, \cdots, t_n) \mapsto t_i$$

and $\lambda_i\colon \mathbb{G}_{\mathrm{m}} \to \mathsf{D}_n$ as the cocharacter

$$z \mapsto \mathrm{diag}(1, \cdots, z, \cdots, 1)$$

with z at the i-th position. Then

(i) characters of D_n are $\mathbb{Z}$-linear combinations of $\chi_1, \cdots, \chi_n$;

(ii) cocharacters of D_n are $\mathbb{Z}$-linear combinations of $\lambda_1, \cdots, \lambda_n$.

Therefore if T is a torus of dimension n, then its character group $\mathsf{X}^*(\mathsf{T})$ (resp. cocharacter group $\mathsf{X}_*(\mathsf{T})$) is isomorphic to $\mathbb{Z}^n$ and consists of all characters (resp. cocharacters) if T is split.

Let χ be a character and λ be a cocharacter of T. Then the composition $\chi \circ \lambda$ is an endomorphism $z \mapsto z^{\langle \chi, \lambda \rangle}$ of $\mathbb{G}_{\mathrm{m}}$, which can be identified with the integer $\langle \chi, \lambda \rangle \in \mathbb{Z}$. In this way, we get a perfect pairing of $\mathbb{Z}$-modules

$$\langle \cdot\,, \cdot \rangle\colon \mathsf{X}^*(\mathsf{T}) \times \mathsf{X}_*(\mathsf{T}) \to \mathbb{Z}.$$

making $\mathsf{X}^*(\mathsf{T})$ and $\mathsf{X}_*(\mathsf{T})$ in duality.

2.1.13 Example. Let G be a diagonalizable algebraic group. Then $\mathsf{X}(\mathsf{G})$ is a finitely generated abelian group and (here p is the characteristic of K) [Mil17, 12.5]

(i) G is smooth if and only if $\mathsf{X}(\mathsf{G})$ has no p-torsion;

(ii) G is connected if and only if $\mathsf{X}(\mathsf{G})$ has no torsion other than p-torsions;

(iii) G is smooth and connected if and only if $\mathsf{X}(\mathsf{G})$ is free.

Moreover, the functor $\mathsf{G} \rightsquigarrow \mathsf{X}(\mathsf{G})$ gives a contravariant equivalence from the category of diagonalizable algebraic groups to the category of finitely generated abelian groups [Mil17, 12.9].

More general, the functor $\mathsf{G} \rightsquigarrow \mathsf{X}^*(\mathsf{G})$ gives a contravariant equivalence from the category of algebraic groups of multiplicative type over K to the category of finitely generated $\mathbb{Z}$-modules equipped with a continuous action of the absolute Galois group of K [Mil17, 12.23].

In particular, an algebraic group of multiplicative type is a torus if and only if it is smooth and connected.

2.1.14. An algebraic group G is *trigonalizable* if its every finite-dimensional representation $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$ is *trigonalizable*, namely there exists a G-stable maximal flag

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_m = V, \qquad \dim(V_i) = i.$$

Equivalently, an algebraic group is trigonalizable if it is isomorphic to an algebraic subgroup of some T_n [Mil17, 16.2]. All unipotent algebraic groups are trigonalizable.

An algebraic group G is *solvable* if it has a subnormal series

$$\mathsf{G} \supsetneq \mathsf{G}_0 \supsetneq \mathsf{G}_1 \supsetneq \cdots \supsetneq \mathsf{G}_m = 1$$

such that each factor $\mathsf{G}_i \,/\, \mathsf{G}_{i+1}$ is commutative. A solvable algebraic group G is *split* if it has a subnormal series (G_i) in which each factor $\mathsf{G}_i \,/\, \mathsf{G}_{i+1}$ is isomorphic to either $\mathbb{G}_{\mathrm{a}}$ or $\mathbb{G}_{\mathrm{m}}$. Hence split solvable algebraic groups are trigonalizable [Mil17, 16.52].

Any trigonalizable algebraic group G has a subnormal series (G_i) in which G_0 is unipotent, $\mathsf{G} \,/\, \mathsf{G}_0$ is diagonalizable and each factor $\mathsf{G}_i \,/\, \mathsf{G}_{i+1}$ is $(\mathsf{G} \,/\, \mathsf{G}_0)$-equivariantly embedded into $\mathbb{G}_{\mathrm{a}}$ [Mil17, 16.21]. Therefore trigonalizable algebraic groups are solvable. Conversely, every smooth connected solvable algebraic group becomes trigonalizable after some finite filed extension [Mil17, 16.30].

2.1.15 Example. T_n is trigonalizable and hence solvable. It has a normal series

$$\mathsf{T}_n \supsetneq \mathsf{U}_n = \mathsf{U}_n^{(0)} \supsetneq \mathsf{U}_n^{(1)} \supsetneq \cdots \supsetneq \mathsf{U}_n^{(m)} = 1,$$

where $m = \binom{n}{2}$ and for each $0 \leqslant r \leqslant m$,

$$\mathsf{U}_n^{(r)}\colon R \rightsquigarrow \left\{(u_{ij}) \in \mathsf{U}_n(R) \;\middle|\; u_{ij} = 0 \text{ for } \tfrac{1}{2}(j-i-1)(2n-j+i)+i \leqslant r\right\}.$$

In which, U_n is the largest solvable normal subgroup of T_n (and is in fact smooth and connected), the quotient $\mathsf{U}_n \,/\, \mathsf{T}_n$ is isomorphic to D_n and each factor $\mathsf{U}_n^{(r)} \,/\, \mathsf{U}_n^{(r+1)}$ is isomorphic to $\mathbb{G}_{\mathrm{a}}$.

2.2 Reductive groups

Let's introduce the notion of reductive groups.

2.2.1. Let G be a smooth connected linear algebraic group.

(i) [Mil17, 6.44] There is a largest smooth connected solvable norm subgroup $\mathscr{R}(\mathsf{G})$ of G. It is called the *radical* of G.

(ii) [Mil17, 6.46] There is a largest smooth connected unipotent norm subgroup $\mathscr{R}_u(\mathsf{G})$ of G. It is called the *unipotent radical* of G.

Since unipotent groups are solvable, $\mathscr{R}_u(\mathsf{G})$ is a subgroup of $\mathscr{R}(\mathsf{G})$.

2.2.2 Definition. An algebraic group G is *reductive* (resp. *semisimple*) if its *geometric unipotent radical* $\mathscr{R}_u(\mathsf{G}_{K^{\mathrm{a}}})$ (resp. *geometric radical* $\mathscr{R}(\mathsf{G}_{K^{\mathrm{a}}})$) is trivial.

The formation of $\mathscr{R}_u(\mathsf{G})$ and $\mathscr{R}(\mathsf{G})$ commute with separable field extensions [Mil17, 19.1 and 19.9]. Hence when K is perfect, G is reductive (resp. semisimple) if and only if $\mathscr{R}_u(\mathsf{G})$ (resp. $\mathscr{R}(\mathsf{G})$) is trivial.

2.2.3 Example. For any finite-dimensional vector space V, $\mathsf{SL}(V)$ is semisimple, while $\mathsf{GL}(V)$ is reductive but not semisimple.

Since any torus becomes a product of copies of $\mathbb{G}_{\mathrm{m}} = \mathsf{GL}_1$ over a finite field extension, it is reductive. Conversely, if G is a solvable reductive group, then since $\mathscr{R}_u(\mathsf{G}_{K^{\mathrm{a}}})$ is trivial, it is a torus by [Mil17, 16.33].

2.2.4. Let G be a reductive group. There are various semisimple groups related to it.

(a) The radical $\mathscr{R}(\mathsf{G})$ is a *central* torus, namely it is contained in the centre $\mathsf{Z}(\mathsf{G})$. Therefore the quotient $\mathsf{G}\,/\,\mathsf{Z}(\mathsf{G})$ is semisimple. It is furthermore *adjoint*, namely it is semisimple with trivial centre, and is called the *adjoint group of* G with notation G^{ad}.

(b) The radical $\mathscr{R}(\mathsf{G})$ turns out to be the largest subtorus of $\mathsf{Z}(\mathsf{G})$ and hence the formation of $\mathscr{R}(\mathsf{G})$ commute with field extensions [Mil17, 19.21]. Therefore the quotient $\mathsf{G}^{\mathrm{ss}} := \mathsf{G}\,/\,\mathscr{R}(\mathsf{G})$ is semisimple.

(c) The derived group $\mathsf{G}^{\mathrm{der}}$ is semisimple [Mil17, 19.21]: its geometric radical $\mathscr{R}\left(\mathsf{G}^{\mathrm{der}}_{K^{\mathrm{a}}}\right)$ is normal in $\mathsf{G}_{K^{\mathrm{a}}}$ hence $\mathscr{R}\left(\mathsf{G}^{\mathrm{der}}_{K^{\mathrm{a}}}\right) \subseteq \mathscr{R}(\mathsf{G}_{K^{\mathrm{a}}})$ and is central. But $\mathsf{Z}(\mathsf{G}) \cap \mathsf{G}^{\mathrm{der}}$ is finite hence $\mathscr{R}\left(\mathsf{G}^{\mathrm{der}}_{K^{\mathrm{a}}}\right)$ is trivial.

2.2.5 Example. The above semisimple groups associated to $\mathsf{G} = \mathsf{GL}_n$ are the following:

(a) $\mathsf{Z}(\mathsf{GL}_n) \cong \mathbb{G}_{\mathrm{m}}$, hence $\mathsf{G}^{\mathrm{ad}} = \mathsf{PGL}_n$;

(b) $\mathscr{R}(\mathsf{GL}_n) = \mathsf{Z}(\mathsf{GL}_n)$, hence we obtain PGL_n again;

(c) the derived group of GL_n is SL_n.

2.2.6. Let G be a reductive group with $\mathsf{Z}(\mathsf{G})$ its centre, G^{ad} its adjoint group, $\mathsf{G}^{\mathrm{der}}$ its derived group, G^{Ab} its abelianization and let $\mathsf{Z}\big(\mathsf{G}^{\mathrm{der}}\big)$ be the centre of $\mathsf{G}^{\mathrm{der}}$. We have the following *deconstruction of* G [Mil17, 19.25]:

$$\begin{tikzcd}
\mathsf{Z}\big(\mathsf{G}^{\mathrm{der}}\big) \arrow[r, hook] \arrow[d, hook] & \mathsf{G}^{\mathrm{der}} \arrow[d, hook] \arrow[dr, bend left, two heads] & \\
\mathsf{Z}(\mathsf{G}) \arrow[r, hook] \arrow[dr, bend right, two heads] & \mathsf{G} \arrow[r, two heads] \arrow[d, two heads] & \mathsf{G}^{\mathrm{ad}} \\
 & \mathsf{G}^{\mathrm{Ab}} &
\end{tikzcd}$$

where the square is bicartesian, namely $\mathsf{Z}\big(\mathsf{G}^{\mathrm{der}}\big) = \mathsf{Z}(\mathsf{G}) \cap \mathsf{G}^{\mathrm{der}}$ and $\mathsf{G} = \mathsf{Z}(\mathsf{G}) \cdot \mathsf{G}^{\mathrm{der}}$, and all rows and columns are exact sequences.

Conversely, if we have a triple $(\mathsf{H}, \mathsf{D}, \varphi)$ with H a semisimple algebraic group, D an algebraic group of multiplicative type, and $\varphi\colon \mathsf{Z}(\mathsf{H}) \to \mathsf{D}$ a monomorphism whose cokernel is a torus T. Then the homomorphism

$$\mathsf{Z}(\mathsf{H}) \longrightarrow \mathsf{H} \times \mathsf{D}\colon z \longmapsto (z, \varphi(z)^{-1})$$

is normal and its cokernel, denoted by G, is reductive and with the following deconstruction [Mil17, 19.27]

$$\begin{tikzcd}
\mathsf{Z}(\mathsf{H}) \arrow[r, hook] \arrow[d, hook] & \mathsf{H} \arrow[d, hook] \arrow[dr, bend left, two heads] & \\
\mathsf{D} \arrow[r, hook] \arrow[dr, bend right, two heads] & \mathsf{G} \arrow[r, two heads] \arrow[d, two heads] & \mathsf{H}^{\mathrm{ad}} \\
 & \mathsf{T} &
\end{tikzcd}$$

Namely, $\mathsf{Z}(\mathsf{G}) \cong \mathsf{D}, \mathsf{G}^{\mathrm{ad}} \cong \mathsf{H}^{\mathrm{ad}}, \mathsf{G}^{\mathrm{der}} \cong \mathsf{H}$ and $\mathsf{G}^{\mathrm{Ab}} \cong \mathsf{T}$.

More general, one can start from a triple $(\mathsf{H}, \mathsf{D}, \varphi)$ with φ not necessarily injective. Then we can replace H by the $\mathsf{H} \,/\, \mathrm{Ker}(\varphi)$ and everything follows.

2.2.7. Let G be a reductive group with radical $\mathscr{R}(\mathsf{G})$, semisimple quotient G^{ss}, derived group $\mathsf{G}^{\mathrm{der}}$ and abelianization G^{Ab}. Then by [Mil17, 12.46], $\mathsf{G} = \mathscr{R}(\mathsf{G}) \cdot \mathsf{G}^{\mathrm{der}}$ and hence we have another deconstruction of G:

$$\begin{tikzcd}
\mathscr{R}(\mathsf{G}) \cap \mathsf{G}^{\mathrm{der}} \arrow[r, hook] \arrow[d, hook] & \mathsf{G}^{\mathrm{der}} \arrow[d, hook] \arrow[dr, bend left, two heads] & \\
\mathscr{R}(\mathsf{G}) \arrow[r, hook] \arrow[dr, bend right, two heads] & \mathsf{G} \arrow[r, two heads] \arrow[d, two heads] & \mathsf{G}^{\mathrm{ss}} \\
 & \mathsf{G}^{\mathrm{Ab}} &
\end{tikzcd}$$

In particular, a reductive group G is a product of a semisimple group and a torus if and only if $\mathscr{R}(\mathsf{G}) \cap \mathsf{G}^{\mathrm{der}} = 1$.

2.2.8 Example. Let $\mathsf{G} = \mathsf{GL}_n$. Then we have the following deconstruction

$$\begin{tikzcd}
\mu_n \arrow[r, hook] \arrow[d, hook] & \mathsf{SL}_n \arrow[d, hook] \arrow[dr, bend left, two heads] & \\
\mathbb{G}_{\mathrm{m}} \arrow[r, hook, "\lambda\mapsto\lambda I_n"] \arrow[dr, bend right, two heads, "\lambda\mapsto\lambda^n"'] & \mathsf{GL}_n \arrow[r, two heads] \arrow[d, two heads, "\det"] & \mathsf{PGL}_n \\
& \mathbb{G}_{\mathrm{m}} &
\end{tikzcd}$$

Conversely, GL_n can be recovered from the triple $(\mathsf{SL}_n, \mathbb{G}_{\mathrm{m}}, \mu_n \hookrightarrow \mathbb{G}_{\mathrm{m}})$.

Similar conclusion applies to $\mathsf{GL}(V)$.

2.2.9. Let G be a reductive group. It is *splittable* if it has a split maximal torus. A *split reductive group* is a pair (G, T) of a reductive group and a split maximal torus in it. A *homomorphism* between split reductive groups is a homomorphism of algebraic group preserving the split maximal torus. It turns out that, any two maximal split tori (hence split maximal tori if G is splittable) in G are conjugate by an element of G [Mil17, 25.10], while two (not necessarily split) maximal tori are only conjugate over a finite separable extension [Mil17, 17.87].

Let G be a splittable reductive group. Then its *rank* is the dimension of one (hence any) split maximal torus in it and its *semisimple rank* is the rank of $\mathsf{G}/\mathscr{R}(\mathsf{G})$. Since the centre $\mathsf{Z}(\mathsf{G})$ is contained in every maximal torus [Mil17, 17.61], the semisimple rank of G equals $\operatorname{rank}(\mathsf{G}) - \dim(\mathsf{Z}(\mathsf{G}))$.

2.2.10 Example. D_n is a split maximal torus in GL_n and it induces a split maximal torus in PGL_n by modulo the scalars $\mathbb{G}_{\mathrm{m}}$ and a split maximal torus in SL_n by intersecting with it. Hence GL_n is splittable with rank n and semisimple rank $n-1$.

2.2.11 Example. Let V be a vector space over K of dimension n. Then the conjugacy classes of maximal tori in $\mathsf{GL}(V)$ are one-one corresponding to the isomorphism classes of étale K-algebras of degree n: a maximal torus T gives a decomposition $V = \bigoplus_i V_i$ into simple T-modules and thus finite separable extensions $K_i = \operatorname{End}_{\mathsf{T}}(V_i)$ and étale K-algebra $A = \prod_i K_i$ of degree n; conversely, as V is a free A-module of rank 1, it decomposes into vector spaces V_i, one-dimensional over K_i, and the A-equivariant automorphisms preserving this decomposition form a maximal torus T such that $\mathsf{T}(K) = A^\times$.

In particular, the only conjugacy class of split maximal tori in $\mathsf{GL}(V)$ corresponds to the étale algebra K^n.

2.2.12. A homomorphism between smooth connected algebraic groups is said to be an *isogeny* if it is surjective and has finite kernel. An *isogeny of split reductive groups* $(\mathsf{G}', \mathsf{T}') \to (\mathsf{G}, \mathsf{T})$ is a homomorphism of split reductive groups such that $\varphi\colon \mathsf{G}' \to \mathsf{G}$ is an isogeny.

An isogeny is *central* if its kernel is central, namely contained in the centre, and is *multiplicative* if its kernel is of multiplicative type. A multiplicative isogeny is central (since every normal multiplicative subgroup of a connected algebraic group is central [Mil17, 12.38]) and the converse is true if its domain is reductive (since the centre of a reductive group is of multiplicative type [Mil17, 17.62]).

A smooth connected algebraic group is *simply connected* if every multiplicative isogeny to it is an isomorphism. Let G be a smooth connected algebraic group. A *universal covering* on it is an multiplicative isogeny $\widetilde{\mathsf{G}} \to \mathsf{G}$ with $\widetilde{\mathsf{G}}$ a simply connected one. When the universal covering exists, its kernel is called the *fundamental group* $\pi_1(\mathsf{G})$ of G.

2.2.13 Example. In 2.2.6 and 2.2.7, the homomorphisms $\mathsf{G}^{\mathrm{der}} \to \mathsf{G}^{\mathrm{ad}}$, $\mathsf{G}^{\mathrm{der}} \to \mathsf{G}^{\mathrm{ss}}$, $\mathsf{Z}(\mathsf{G}) \to \mathsf{G}^{\mathrm{Ab}}$ and $\mathscr{R}(\mathsf{G}) \to \mathsf{G}^{\mathrm{Ab}}$ are isogenies. In particular, the homomorphism $\mathsf{SL}_n \to \mathsf{PGL}_n$ in Example 2.2.8 is a universal covering (hence $\pi_1(\mathsf{PGL}_n) \cong \mu_n$) and it induces a central isogeny of split reductive groups $(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) \to (\mathsf{PGL}_n, \mathsf{D}_n / \mathbb{G}_{\mathrm{m}})$.

2.3 Root systems and root groups

Given a split reductive group, there is a root system associated to it.

2.3.1. Let (G, T) be a split reductive group. Since T is diagonalizable, it acts (via the adjoint representation) on $\mathfrak{g} := \mathrm{Lie}(\mathsf{G})$ diagonalizably and we hav decomposition:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{a \in \mathsf{X}^*(\mathsf{T})} \mathfrak{g}_a,$$

where $\mathfrak{t} = \mathfrak{g}^{\mathsf{T}} = \mathrm{Lie}(\mathsf{T})$ [Mil17, 10.34] and $\mathfrak{g}_a$ is the subspace on which T acts through a nontrivial character a. A character a is a *root* if $\mathfrak{g}_a$ is nontrivial. The set of roots is denoted by $\Phi(\mathsf{G}, \mathsf{T})$, called the *root system* of the split reductive group (G, T).

2.3.2 Example. The pair $(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$ is a split reductive group with Lie algebra the one dimensional vector space K. The adjoint action of $\mathbb{G}_{\mathrm{m}}$ on K is trivial, hence the root system of $(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$ is empty.

2.3.3 Example. Consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$. The action of D_n on $\mathfrak{gl}_n := \mathrm{Lie}(\mathsf{GL}_n)$ is

$$(\mathrm{diag}(t_1, \cdots, t_n), (g_{ij})_{i,j}) \longmapsto (t_i g_{ij} t_j^{-1})_{i,j}.$$

By Example 2.1.12, the characters of D_n are of the form $c_1\chi_1 + \cdots + c_n\chi_n$. If $(g_{ij})_{i,j}$ is an eigenvector of $c_1\chi_1 + \cdots + c_n\chi_n$, then for any $t_1, \cdots, t_n \in R$, we have

$$\forall i, j : t_i g_{ij} t_j^{-1} = (t_1^{c_1} \cdots t_n^{c_n}) g_{ij}.$$

Therefore: *1,* the Lie algebra $\mathfrak{d}_n$ of D_n consists of all diagonal matrices; *2,* the root system $\Phi(\mathsf{GL}_n, \mathsf{D}_n) = \{\chi_i - \chi_j \mid 1 \leqslant i \neq j \leqslant n\}$; *3,* for each $a = \chi_i - \chi_j$, the Lie algebra $\mathfrak{g}_a$ is generated by E_{ij}, the matrix with 1 in the ij position and 0 elsewhere.

2.3.4 Example. Consider the split reductive group $(\mathsf{PGL}_n, \mathsf{D}_n / \mathbb{G}_{\mathrm{m}})$. A character $c_1\chi_1 + \cdots + c_n\chi_n$ of D_n factors through $\mathsf{D}_n / \mathbb{G}_{\mathrm{m}}$ if and only if $c_1 + \cdots + c_n = 0$. Hence

$$\mathsf{X}^*(\mathsf{D}_n / \mathbb{G}_{\mathrm{m}}) = \{c_1\chi_1 + \cdots + c_n\chi_n \mid c_1, \cdots, c_n \in \mathbb{Z}, c_1 + \cdots + c_n = 0\}$$

and we see that $\Phi(\mathsf{PGL}_n, \mathsf{D}_n / \mathbb{G}_\mathrm{m}) = \{\chi_i - \chi_j \mid 1 \leqslant i \neq j \leqslant n\}$.

The Lie algebra of PGL_n and $\mathsf{D}_n / \mathbb{G}_\mathrm{m}$ are $\mathfrak{pgl}_n := \mathfrak{gl}_n / KI_n$ and $\mathfrak{d}_n / KI_n$. For each $a = \chi_i - \chi_j$, the Lie algebra $\mathfrak{g}_a$ is generated by E_{ij}.

2.3.5 Example. Consider the split reductive group $(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$. Then two characters $c_1\chi_1 + \cdots + c_n\chi_n$ and $c_1'\chi_1 + \cdots + c_n'\chi_n$ of D_n may give rise to the same character of $\mathsf{D}_n \cap \mathsf{SL}_n$. This is the case precisely when $c_i - c_i'$ is a constant. Hence

$$\mathsf{X}^*(\mathsf{D}_n \cap \mathsf{SL}_n) = (\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n)/\mathbb{Z}(\chi_1 + \cdots + \chi_n)$$

and we see that $\Phi(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) = \{\overline{\chi_i} - \overline{\chi_j} \mid 1 \leqslant i \neq j \leqslant n\}$.

The Lie algebra of SL_n and $\mathsf{D}_n \cap \mathsf{SL}_n$ are $\mathfrak{sl}_n$, consisting of matrices with trace 0, and $\mathfrak{d}_n \cap \mathfrak{sl}_n$. For each $a = \overline{\chi_i} - \overline{\chi_j}$, the Lie algebra $\mathfrak{g}_a$ is generated by E_{ij}.

2.3.6. Let (G, T) be a split reductive group. Then $\mathsf{N} = \mathsf{N}_\mathsf{G}(\mathsf{T})$ acts on T, hence on $\mathsf{X}^*(\mathsf{T})$ by conjugations. The centralizer $\mathsf{Z}_\mathsf{G}(\mathsf{T})$ (which equals T itself [Mil17, 17.84]) acts trivially, hence we have an action of the quotient N / T on $\mathsf{X}^*(\mathsf{T})$. It turns out that, this quotient is precisely $\pi_0(\mathsf{N})$ [Mil17, 17.39], and is furthermore constant [Mil17, 21.1]. The finite group ${}^vW(\mathsf{G}, \mathsf{T}) = N/T$ is called the *Weyl group* of (G, T).

2.3.7 Example. Consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$. Then N consists of invertible monomial matrices and the regular representation of $\mathfrak{S}_n$ gives a semi-direct product $\mathsf{N} = \mathsf{D}_n \rtimes \underline{\mathfrak{S}_n}$. Hence the Weyl group ${}^vW(\mathsf{GL}_n, \mathsf{D}_n)$ is isomorphic to $\mathfrak{S}_n$.

Similar arguments apply to $(\mathsf{PGL}_n, \mathsf{D}_n / \mathbb{G}_\mathrm{m})$ and $(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$ and their Weyl groups are ${}^vW(\mathsf{PGL}_n, \mathsf{D}_n / \mathbb{G}_\mathrm{m}) \cong \mathfrak{S}_n$ and ${}^vW(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) \cong \mathfrak{S}_n$.

2.3.8. Let (G, T) be a split reductive group and $a \in \Phi(\mathsf{G}, \mathsf{T})$ a root of it. Then there is a unique homomorphism $u_a \colon \mathbb{W}(\mathfrak{g}_a) \to \mathsf{G}$ such that its differential $\mathrm{d}u_a$ is the inclusion $\mathfrak{g}_a \hookrightarrow \mathfrak{g}$. Let U_a denote the image of u_a. It is called the *root group* of G and satisfies the following properties [Mil17, 21.11 and 21.19; SGA3, XX, 1.5, XXII, 1.1].

(i) U_a has Lie algebra $\mathfrak{g}_a$ and a smooth subgroup of G contains U_a if and only if its Lie algebra contains $\mathfrak{g}_a$.

(ii) U_a is normalized by T and T acts on U_a through the character a:

$$\mathrm{inn}(t).u_a(X) = u_a(a(t)X),$$

for all $t \in \mathsf{T}$ and $X \in \mathbb{W}(\mathfrak{g}_a)$.

(iii) the morphism

$$\mathbb{W}(\mathfrak{g}_{-a}) \times \mathsf{T} \times \mathbb{W}(\mathfrak{g}_a) \longrightarrow \mathsf{G}$$

defined by $(Y, t, X) \mapsto u_{-a}(Y) \cdot t \cdot u_a(X)$ is an open immersion.

Moreover, if $n \in \mathsf{N}_\mathsf{G}(\mathsf{T})$. Then $b = a \circ \mathrm{inn}(n)$ is a root and we have the following commutative diagram [SGA3, XXII, 1.4].

$$\begin{tikzcd}
\mathbb{W}(\mathfrak{g}_a) \arrow[r, "u_a"] \arrow[d, "\mathrm{Ad}(n)"'] & \mathsf{G} \arrow[d, "\mathrm{inn}(n)"] \\
\mathbb{W}(\mathfrak{g}_b) \arrow[r, "u_b"] & \mathsf{G}
\end{tikzcd}$$

Indeed, both u_a and $\mathrm{inn}(n)^{-1} \circ u_b \circ \mathrm{Ad}(n)$ have the differential $\mathfrak{g}_a \hookrightarrow \mathfrak{g}$.

2.3.9 Example. Consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$ and its root $a = \chi_i - \chi_j$. Then U_a is the algebraic group

$$R \rightsquigarrow I_n + RE_{ij}.$$

The homomorphism $u_a \colon \mathbb{W}(\mathfrak{g}_a) \to \mathsf{GL}_n$ is

$$xE_{ij} \mapsto I_n + xE_{ij}.$$

For any $t = \mathrm{diag}(t_1, \cdots, t_n) \in \mathsf{T}$, we have

$$\mathrm{inn}(t).u_a(xE_{ij}) = I_n + t_i x t_j^{-1} E_{ij} = u_a(t_i t_j^{-1} x E_{ij}) = u_a(a(t) x E_{ij}).$$

2.3.10. Notations as above. Then there is a natural duality on the one-dimensional vector groups

$$\mathbb{W}(\mathfrak{g}_a) \times \mathbb{W}(\mathfrak{g}_{-a}) \longrightarrow \mathbb{G}_{\mathrm{a}} \colon \qquad (X, Y) \longmapsto \langle X, Y \rangle.$$

and a unique cocharacter $a^\vee \colon \mathbb{G}_{\mathrm{m}} \to \mathsf{T}$ such that [SGA3, XX, 2.1]:

(i) for any $X \in \mathbb{W}(\mathfrak{g}_a)$ and $Y \in \mathbb{W}(\mathfrak{g}_{-a})$, the product $X \cdot Y$ lies in the image of the open immersion in 2.3.8(iii) if and only if $1 + \langle X, Y \rangle \in \mathbb{G}_{\mathrm{m}}$;

(ii) under these conditions we have the formula

$$u_a(X) \cdot u_{-a}(Y) = u_{-a}((1 + \langle X, Y \rangle)^{-1} Y) \cdot a^\vee(1 + \langle X, Y \rangle) \cdot u_a((1 + \langle X, Y \rangle)^{-1} X);$$

(iii) $\langle a, a^\vee \rangle = 2$.

The above duality induces a pairing of $\mathbb{G}_{\mathrm{m}}$-bundles [SGA3, XX, 2.6]:

$$\mathbb{W}(\mathfrak{g}_a)^\times \times \mathbb{W}(\mathfrak{g}_{-a})^\times \longrightarrow \mathbb{G}_{\mathrm{m}} \colon \qquad (X, Y) \longmapsto XY.$$

Then for any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, there is a unique $X^{-1} \in \mathbb{W}(\mathfrak{g}_{-a})^\times$ such that $XX^{-1} = 1$. This gives rise to an isomorphism $(-)^{-1}$ compatible with the action of $\mathbb{G}_{\mathrm{m}}$. Then for any $x \in \mathbb{G}_{\mathrm{m}}$ and $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, we have [SGA3, XX, 2.7]:

$$a^\vee(x) = u_{-a}((x^{-1} - 1)X^{-1}) u_a(X) u_{-a}((x - 1)X^{-1}) u_a(-x^{-1}X).$$

This cocharacter is called the *coroot associated to the root* a.

The root a and its coroot $a^\vee$ induces the following Lie algebra homomorphisms

$$K \xrightarrow{\mathrm{d}a^\vee} \mathfrak{t} \xrightarrow{\mathrm{d}a} K.$$

The vector $H_a := \mathrm{d}a^\vee(1)$ is called the *infinitesimal coroot vector*. Then $H_{-a} = -H_a$ and for any $X \in \mathbb{W}(\mathfrak{g}_a)$, $Y \in \mathbb{W}(\mathfrak{g}_{-a})$ and $H \in \mathbb{W}(\mathfrak{t})$, we have [SGA3, XX, 2.10]:

$$[H, X] = \mathrm{d}a(H)X, \qquad [H, Y] = -\mathrm{d}a(H)Y, \qquad [X, Y] = \langle X, Y \rangle H_a.$$

Hence if $H_a \neq 0$, then for any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, the followings define an embedding from the Lie algebra $\mathfrak{sl}_2$:

$$E_{12} \mapsto X, \qquad E_{21} \mapsto X^{-1}, \qquad E_{11} - E_{22} \mapsto H_a.$$

Remark. One can fix such a embedding and hence fix a choice of basis of $\mathfrak{g}_a$ (as well as $\mathfrak{g}_{-a}$). Then one can identify them with K and denote the composition $\mathbb{G}_\mathrm{a} \cong \mathbb{W}(\mathfrak{g}_a) \to \mathsf{G}$ (resp. $\mathbb{G}_\mathrm{a} \cong \mathbb{W}(\mathfrak{g}_a) \to \mathsf{G}$) by u_a (resp. u_{-a}).

2.3.11 Example. Consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$ and its root $a = \chi_i - \chi_j$. To take the advantage of calculations on $\mathfrak{gl}_2$, we can define a homomorphism ξ_{ij} mapping a 2×2-matrix $M = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathfrak{gl}_2(R)$ to the $n \times n$-matrix $\xi(M)$ satisfying

$$\xi(M).e_k = \begin{cases} xe_i + ze_j & k = i, \\ ye_i + we_j & k = j, \\ e_k & \text{otherwise}, \end{cases}$$

where $e_1, e_2, \cdots, e_n$ is the standard basis of R^n. Then we have

$$xE_{ij} = \xi_{ij}\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) \qquad \text{and} \qquad u_a(x) = \xi_{ij}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right).$$

Also note that $\xi_{ji} = \xi_{ij} \circ \text{transpose}$.

Then the duality is

$$\langle xE_{ij}, yE_{ji} \rangle = xy.$$

The coroot $a^\vee$ associated to a is $\lambda_i - \lambda_j$ and one can verify that

$$\begin{aligned} & u_a(x) \cdot u_{-a}(y) \\ =& \xi_{ij}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\right) \\ =& \xi_{ij}\left(\begin{pmatrix} 1+xy & x \\ y & 1 \end{pmatrix}\right) \\ =& \xi_{ij}\left(\begin{pmatrix} 1 & 0 \\ (1+xy)^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1+xy & 0 \\ 0 & (1+xy)^{-1} \end{pmatrix}\begin{pmatrix} 1 & (1+xy)^{-1} \\ 0 & 1 \end{pmatrix}\right) \\ =& u_{-a}((1+xy)^{-1}) \cdot a^\vee(1+xy) \cdot u_a((1+xy)^{-1}). \end{aligned}$$

The differentials of a and $a^\vee$ are

$$\begin{aligned} \mathrm{d}a &= \mathrm{d}\chi_i - \mathrm{d}\chi_j \colon \operatorname{diag}(t_1, \cdots, t_n) \mapsto t_i - t_j, \\ \mathrm{d}a^\vee &= \mathrm{d}\lambda_i - \mathrm{d}\lambda_j \colon z \mapsto \xi_{ij}\left(\begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix}\right). \end{aligned}$$

In particular, the infinitesimal coroot vector associated to a is

$$H_a = \xi_{ij}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right).$$

For any $H = \mathrm{diag}(t_1, \cdots, t_n) \in \mathfrak{t}$, we have

$$[H, xE_{ij}] = \xi_{ij}\left(\begin{pmatrix} 0 & (t_i x - t_j x) \\ 0 & 0 \end{pmatrix}\right) = \mathrm{d}a(H)xE_{ij},$$
$$[H, yE_{ji}] = \xi_{ij}\left(\begin{pmatrix} 0 & 0 \\ (t_j y - t_i y) & 0 \end{pmatrix}\right) = -\mathrm{d}a(H)yE_{ji},$$
$$[xE_{ij}, yE_{ji}] = \xi_{ij}\left(\begin{pmatrix} xy & 0 \\ 0 & -xy \end{pmatrix}\right) = xyH_a.$$

2.3.12. Notations as aboves. Let L_a be the algebraic subgroup of G generated by U_a, U_{-a} and T, called the *Levi subgroup associated to* a. Then L_a is the centralizer of the largest subtorus of $\mathrm{Ker}(a)$ and the pair $(\mathsf{L}_a, \mathsf{T})$ is a split reductive group of semisimple rank 1 [SGA3, XIX, 1.12 and XXII, 1.1; Mil17, 21.11 and 21.23]. The Lie algebra of it admits a decomposition

$$\mathrm{Lie}(\mathsf{L}_a) = \mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$$

and the Weyl group ${}^vW(\mathsf{L}_a, \mathsf{T})$ contains exactly one nontrivial element r_a given by the formula

$$r_a \colon \chi \longmapsto \chi - \langle \chi, a^\vee \rangle a.$$

Moreover, for any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, let

$$m_a(X) = u_a(X) \cdot u_{-a}(-X^{-1}) \cdot u_a(X).$$

Then we have [SGA3, XX, 3.1; Mil17, 20.39]:

(i) $m_a(X) \in \mathsf{N}_{\mathsf{L}_a}(\mathsf{T})$;

(ii) let M_a° be the image of $m_a \colon \mathbb{W}(\mathfrak{g}_a)^\times \to \mathsf{N}_{\mathsf{L}_a}(\mathsf{T})$, then $\mathsf{M}_a = \mathsf{T} \cdot \mathsf{M}_a^\circ$ is a right congruence class modulo T: indeed, for any $z \in \mathbb{G}_{\mathrm{m}}$ and $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, we have

$$m_a(zX) = a^\vee(z) m_a(X);$$

(iii) this right congruence class is precisely r_a: indeed, for any $t \in \mathsf{T}$ and $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, we have

$$\mathrm{inn}(m_a(X)).t = t \cdot a^\vee(a(t))^{-1};$$

(iv) for any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$ and $Y \in \mathbb{W}(\mathfrak{g}_{-a})^\times$, we have

$$m_a(X) m_{-a}(Y) = a^\vee(XY).$$

2.3.13 Example. Consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$ and its root $a = \chi_i - \chi_j$. Then the algebraic subgroup L_a is

$$R \rightsquigarrow \mathsf{D}_n(R) + RE_{ij} + RE_{ji}.$$

Its Lie algebra is $\mathfrak{d}_n + KE_{ij} + KE_{ji}$, which is precisely $\mathfrak{d}_n \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$.

The normalizer of D_n in L_a is precisely the monomial matrices belonging to L_a. Hence the Weyl group ${}^vW(\mathsf{L}_a, \mathsf{T})$ contains exactly one nontrivial element $r_a = (i, j)$, the permutation of i-th and j-th coordinates.

The map m_a is

$$xE_{ij} \longmapsto \xi_{ij}\left(\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\-x^{-1} & 1\end{pmatrix}\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}\right) = \xi_{ij}\left(\begin{pmatrix}0 & x\\-x^{-1} & 0\end{pmatrix}\right).$$

We have

$$\begin{aligned}m_a(x) &= \xi_{ij}\left(\begin{pmatrix}0 & x\\-x^{-1} & 0\end{pmatrix}\right)\\ &= \xi_{ij}\left(\begin{pmatrix}x & 0\\0 & -x^{-1}\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\right)\\ &= a^\vee(x)m_a(1).\end{aligned}$$

The action of $m_a(x)$ on $t = \mathrm{diag}(t_1, \cdots, t_n) \in \mathsf{T}$ is

$$\begin{aligned}\mathrm{inn}(m_a(x)).t &= \mathrm{inn}\left(\xi_{ij}\left(\begin{pmatrix}0 & x\\-x^{-1} & 0\end{pmatrix}\right)\right).\,\mathrm{diag}(t_1, \cdots, t_n)\\ &= \mathrm{inn}\big(a^\vee(x)\big).\,\mathrm{inn}\left(\xi_{ij}\left(\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\right)\right).\,\mathrm{diag}(t_1, \cdots, t_n)\\ &= \mathrm{inn}\big(a^\vee(x)\big).\,\mathrm{diag}\big(t_{\sigma(1)}, \cdots, t_{\sigma(1)}\big) \qquad \text{with } \sigma = (i, j)\\ &= \mathrm{diag}\big(t_{\sigma(1)}, \cdots, t_{\sigma(1)}\big) \qquad \text{with } \sigma = (i, j)\\ &= \mathrm{diag}(t_1, \cdots, t_n)\xi_{ij}\left(\begin{pmatrix}t_i^{-1}t_j & 0\\0 & t_it_j^{-1}\end{pmatrix}\right)\\ &= \mathrm{diag}(t_1, \cdots, t_n)a^\vee(a(t))^{-1}.\end{aligned}$$

We also have

$$\begin{aligned}m_a(x)\cdot m_{-a}(y) &= \xi_{ij}\left(\begin{pmatrix}0 & x\\-x^{-1} & 0\end{pmatrix}\begin{pmatrix}0 & -y^{-1}\\y & 0\end{pmatrix}\right)\\ &= \xi_{ij}\left(\begin{pmatrix}xy & 0\\0 & (xy)^{-1}\end{pmatrix}\right)\\ &= a^\vee(xy).\end{aligned}$$

2.4 Root data

2.4.1 Definition. Let (G, T) be a split reductive group. Then there is a *root datum* $\mathcal{R}(\mathsf{G}, \mathsf{T}) = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ associated to it [SGA3, XXII, 1.14; Mil17, 21.c], where

- the $\mathbb{Z}$-module X is the character group $\mathsf{X}^*(\mathsf{T})$;
- the root system Φ is the root system $\Phi(\mathsf{G}, \mathsf{T})$;

- the dual $\mathbb{Z}$-module $\mathsf{X}^\vee$ is the cocharacter group $\mathsf{X}_*(\mathsf{T})$;
- the coroot system $\Phi^\vee$ is the set $\Phi^\vee(\mathsf{G},\mathsf{T})$ of coroots $a^\vee$ associated to the roots $a \in \Phi(\mathsf{G},\mathsf{T})$.

Let V denote the subspace of $\mathsf{X}_*(\mathsf{T}) \otimes \mathbb{R}$ spanned by $\Phi^\vee(\mathsf{G},\mathsf{T})$ equipped with a ${}^vW(\mathsf{G},\mathsf{T})$-invariant inner product, called the *coroot space*. Then we get a spherical apartment ${}^v\mathscr{A}(\mathsf{G},\mathsf{T})$ with underlying Euclidean vector space V on which the Weyl group ${}^vW(\mathsf{G},\mathsf{T})$ acts as its reflection group.

2.4.2. The *rank* of a root datum $\mathcal{R} = (\mathsf{X},\Phi,\mathsf{X}^\vee,\Phi^\vee)$ is the rank of the $\mathbb{Z}$-module X and its *semisimple rank* is the dimension of the coroot space V. Let (G,T) be a split reductive group. Then the rank (resp. semisimple rank) of the root datum $\mathcal{R}(\mathsf{G},\mathsf{T})$ is the rank (resp. semisimple rank) of G.

2.4.3 Example. Consider the split reductive group $(\mathbb{G}_{\mathrm{m}},\mathbb{G}_{\mathrm{m}})$. Then by Example 2.3.2, the root datum $\mathcal{R}(\mathbb{G}_{\mathrm{m}},\mathbb{G}_{\mathrm{m}})$ is $(\mathbb{Z},\emptyset,\mathbb{Z},\emptyset)$.

2.4.4 Example. Consider the split reductive group $(\mathsf{GL}_n,\mathsf{D}_n)$. Then by Examples 2.3.3, 2.3.9 and 2.3.11, the root datum $\mathcal{R}(\mathsf{GL}_n,\mathsf{D}_n)$ is

- $\mathsf{X} = \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n$;
- $\Phi = \{\chi_i - \chi_j \mid 1 \leqslant i \neq j \leqslant n\}$;
- $\mathsf{X}^\vee = \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_n$;
- $\Phi^\vee = \{\lambda_i - \lambda_j \mid 1 \leqslant i \neq j \leqslant n\}$.

The coroot space is

$$V = \{c_1\lambda_1 + \cdots + c_n\lambda_n \mid c_1,\cdots,c_n \in \mathbb{R}, c_1 + \cdots + c_n = 0\}.$$

2.4.5 Example. Consider the split reductive group $(\mathsf{PGL}_n,\mathsf{D}_n/\mathbb{G}_{\mathrm{m}})$. Two cocharacters $c_1\lambda_1+\cdots+c_n\lambda_n$ and $c'_1\lambda_1+\cdots+c'_n\lambda_n$ of D_n give rise to the same cocharacter of $\mathsf{D}_n/\mathbb{G}_{\mathrm{m}}$ precisely when $c_i - c'_i$ is a constant. Hence

$$\mathsf{X}_*(\mathsf{D}_n/\mathbb{G}_{\mathrm{m}}) = (\mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_n)/\mathbb{Z}(\lambda_1 + \cdots + \lambda_n)$$

and we see that $\Phi^\vee(\mathsf{PGL}_n,\mathsf{D}_n/\mathbb{G}_{\mathrm{m}}) = \left\{\overline{\lambda_i} - \overline{\lambda_j} \mid 1 \leqslant i \neq j \leqslant n\right\}$ with coroot space

$$V = \left\{c_1\overline{\lambda_1} + \cdots + c_n\overline{\lambda_n} \mid c_1,\cdots,c_n \in \mathbb{R}, c_1 + \cdots + c_n = 0\right\}.$$

Then, by Example 2.3.4, the root datum $\mathcal{R}(\mathsf{PGL}_n,\mathsf{D}_n/\mathbb{G}_{\mathrm{m}})$ is

- $\mathsf{X} = \{c_1\chi_1 + \cdots + c_n\chi_n \mid c_1,\cdots,c_n \in \mathbb{Z}, c_1 + \cdots + c_n = 0\}$;
- $\Phi = \{\chi_i - \chi_j \mid 1 \leqslant i \neq j \leqslant n\}$;
- $\mathsf{X}^\vee = (\mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_n)/\mathbb{Z}(\lambda_1 + \cdots + \lambda_n)$;

- $\Phi^\vee = \{\overline{\lambda_i} - \overline{\lambda_j} \mid 1 \leqslant i \neq j \leqslant n\}$.

2.4.6 Example. Consider the split reductive group $(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$. A cocharacter $c_1\lambda_1 + \cdots + c_n\lambda_n$ of factors through $\mathsf{D}_n \cap \mathsf{SL}_n$ if and only if $c_1 + \cdots + c_n = 0$. Hence

$$\mathsf{X}_*(\mathsf{D}_n \cap \mathsf{SL}_n) = \{c_1\lambda_1 + \cdots + c_n\lambda_n \mid c_1, \cdots, c_n \in \mathbb{Z}, c_1 + \cdots + c_n = 0\}$$

and we see that $\Phi^\vee(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) = \{\lambda_i - \lambda_j \mid 1 \leqslant i \neq j \leqslant n\}$ with coroot space

$$V = \{c_1\lambda_1 + \cdots + c_n\lambda_n \mid c_1, \cdots, c_n \in \mathbb{R}, c_1 + \cdots + c_n = 0\}.$$

Then, by Example 2.3.5, the root datum $\mathcal{R}(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$ is

- $\mathsf{X} = (\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n)/\mathbb{Z}(\chi_1 + \cdots + \chi_n)$;
- $\Phi = \{\overline{\chi_i} - \overline{\chi_j} \mid 1 \leqslant i \neq j \leqslant n\}$;
- $\mathsf{X}^\vee = \{c_1\lambda_1 + \cdots + c_n\lambda_n \mid c_1, \cdots, c_n \in \mathbb{Z}, c_1 + \cdots + c_n = 0\}$;
- $\Phi^\vee = \{\lambda_i - \lambda_j \mid 1 \leqslant i \neq j \leqslant n\}$.

2.4.7. Let $\varphi\colon (\mathsf{G}, \mathsf{T}) \to (\mathsf{G}', \mathsf{T}')$ be a homomorphism between split reductive groups. Then it induces a linear map $f = \varphi^*\colon \mathsf{X}^*(\mathsf{T}') \to \mathsf{X}^*(\mathsf{T})$. Then f is a morphism of root data if and only if there is a bijection $u\colon \Phi(\mathsf{G}, \mathsf{T}) \to \Phi(\mathsf{G}', \mathsf{T}')$ such that

$$f(u(a)) = a, \qquad {}^t f(a^\vee) = u(a)^\vee.$$

A homomorphism of split reductive groups $\varphi\colon (\mathsf{G}, \mathsf{T}) \to (\mathsf{G}', \mathsf{T}')$ induces an isogeny of root data $\varphi^*\colon \mathcal{R}(\mathsf{G}', \mathsf{T}') \to \mathcal{R}(\mathsf{G}, \mathsf{T})$ if and only if it is a central isogeny. Moreover, all isogenies of root data arise in this way [SGA3, XXII, 4.2.11; Mil17, 23.25].

2.4.8 Example. Consider the inclusion $\varphi\colon (\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \to (\mathsf{GL}_n, \mathsf{D}_n)$. Then the linear map $f = \varphi^*\colon \mathsf{X}^*(\mathsf{GL}_n, \mathsf{D}_n) \to \mathsf{X}^*(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$ is the linear map

$$\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n \longrightarrow \mathbb{Z}$$

mapping χ_i to 1. Then this is not a morphism of root data since it does not induce a bijection on roots.

2.4.9 Example. Consider the determinant homomorphism $\varphi\colon (\mathsf{GL}_n, \mathsf{D}_n) \to (\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$. Then the linear map $f = \varphi^*\colon \mathsf{X}^*(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \to \mathsf{X}^*(\mathsf{GL}_n, \mathsf{D}_n)$ is the linear map

$$\mathbb{Z} \longrightarrow \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n$$

mapping 1 to $\chi_1 + \cdots + \chi_n$. Then this is not a morphism of root data since it does not induce a bijection on roots.

2.4.10 Example. Consider the isogeny $\varphi\colon (\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \to (\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$ sending t to t^n. Then the linear map $f = \varphi^*\colon \mathsf{X}^*(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \to \mathsf{X}^*(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$ is the linear map

$$\mathbb{Z} \longrightarrow \mathbb{Z}\colon 1 \longmapsto n.$$

It is an isogeny of root data with finite cokernel μ_n.

2.4.11 Example. Consider the inclusion $\varphi\colon (\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) \to (\mathsf{GL}_n, \mathsf{D}_n)$. Then the linear map $f = \varphi^*\colon\ \mathsf{X}^*(\mathsf{GL}_n, \mathsf{D}_n) \to \mathsf{X}^*(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$ is the projection

$$\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n \longrightarrow (\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n)/\mathbb{Z}(\chi_1 + \cdots + \chi_n).$$

This is a morphism of root data but not an isogeny since f is not injective.

2.4.12 Example. Consider the quotient map $\varphi\colon (\mathsf{GL}_n, \mathsf{D}_n) \to (\mathsf{PGL}_n, \mathsf{D}_n / \mathbb{G}_{\mathrm{m}})$. Then the linear map $f = \varphi^*\colon\ \mathsf{X}^*(\mathsf{PGL}_n, \mathsf{D}_n / \mathbb{G}_{\mathrm{m}}) \to \mathsf{X}^*(\mathsf{GL}_n, \mathsf{D}_n)$ is the inclusion

$$\{c_1 + \cdots + c_n = 0\} \cap \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n \hookrightarrow \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n.$$

This is a morphism of root data but not an isogeny since f has infinite cokernel.

2.4.13 Example. Consider the composition $\varphi\colon (\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) \to (\mathsf{PGL}_n, \mathsf{D}_n / \mathbb{G}_{\mathrm{m}})$ of previous two homomorphisms. Then the linear map $f = \varphi^*\colon\ \mathsf{X}^*(\mathsf{PGL}_n, \mathsf{D}_n / \mathbb{G}_{\mathrm{m}}) \to \mathsf{X}^*(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$ is

$$\{c_1 + \cdots + c_n = 0\} \cap \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n \longrightarrow (\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n)/\mathbb{Z}(\chi_1 + \cdots + \chi_n)$$

This is an isogeny of root data with finite cokernel $\mu_n\overline{\chi_1}$.

2.4.14. Let (G, T) be a split reductive group. Then by 2.2.6 and 2.2.7, we have the following commutative diagram of split reductive groups:

$$\begin{tikzcd}
\mathrm{sc}(\mathsf{G},\mathsf{T}) \arrow[r] & \mathrm{der}(\mathsf{G},\mathsf{T}) \arrow[rr] \arrow[dr] & & \mathrm{ss}(\mathsf{G},\mathsf{T}) \arrow[r] & \mathrm{ad}(\mathsf{G},\mathsf{T}) \\
& & (\mathsf{G},\mathsf{T}) \arrow[ur] \arrow[dr] & & \\
& \mathrm{rad}(\mathsf{G},\mathsf{T}) \arrow[rr] \arrow[ur] & & \mathrm{corad}(\mathsf{G},\mathsf{T}) &
\end{tikzcd}$$

where the horizontal arrows are isogenies, the diagonals are short exact sequences and

- $\mathrm{ad}(\mathsf{G}, \mathsf{T})$ is the pair of the adjoint group G^{ad} and the image of T in it, namely $\mathsf{T} / \mathsf{Z}(\mathsf{G})$;
- $\mathrm{ss}(\mathsf{G}, \mathsf{T})$ is the pair of the semisimple quotient G^{ss} and the image of T in it, namely $\mathsf{T} / \mathscr{R}(\mathsf{G})$;
- $\mathrm{der}(\mathsf{G}, \mathsf{T})$ is the pair of the derived group $\mathsf{G}^{\mathrm{der}}$ and the preimage of T in it, namely $\mathsf{T} \cap \mathsf{G}^{\mathrm{der}}$;
- $\mathrm{sc}(\mathsf{G}, \mathsf{T})$ is the universal covering of all aboves;
- $\mathrm{rad}(\mathsf{G}, \mathsf{T})$ is the pair of the radical $\mathscr{R}(\mathsf{G})$ and the trivial torus 1;
- $\mathrm{corad}(\mathsf{G}, \mathsf{T})$ is the pair of the abelianization G^{Ab} and the trivial torus 1.

Moreover, we have [SGA3, XXII, 4.3.7, 6.2.1 and 6.2.3; Mil17, 23.a]

(i) $\mathcal{R}(\mathrm{ad}(\mathsf{G}, \mathsf{T})) = \mathrm{ad}(\mathcal{R}(\mathsf{G}, \mathsf{T}))$;

(ii) $\mathcal{R}(\mathrm{ss}(\mathsf{G}, \mathsf{T})) = \mathrm{ss}(\mathcal{R}(\mathsf{G}, \mathsf{T}))$;

(iii) $\mathcal{R}(\mathrm{der}(\mathsf{G}, \mathsf{T})) = \mathrm{der}(\mathcal{R}(\mathsf{G}, \mathsf{T}))$;

(iv) $\mathcal{R}(\mathrm{sc}(\mathsf{G}, \mathsf{T})) = \mathrm{sc}(\mathcal{R}(\mathsf{G}, \mathsf{T}))$;

(v) $\mathcal{R}(\mathrm{rad}(\mathsf{G}, \mathsf{T})) = \mathrm{rad}(\mathcal{R}(\mathsf{G}, \mathsf{T}))$;

(vi) $\mathcal{R}(\mathrm{corad}(\mathsf{G}, \mathsf{T})) = \mathrm{corad}(\mathcal{R}(\mathsf{G}, \mathsf{T}))$;

(vii) the morphisms between above root data come from the homomorphisms between corresponding split reductive groups.

2.4.15 Example. Consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$. Then the deconstruction in Example 2.2.8 gives the following isogenies of root data.

$$\mathcal{R}(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) \xleftarrow[2.4.11]{} \mathcal{R}(\mathsf{GL}_n, \mathsf{D}_n) \xleftarrow[2.4.12]{} \mathcal{R}(\mathsf{PGL}_n, \mathsf{D}_n / \mathbb{G}_{\mathrm{m}})$$

$$\mathcal{R}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \xleftarrow[2.4.10]{} \mathcal{R}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$$

where $\mathcal{R}(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$ is described in Example 2.4.4, $\mathcal{R}(\mathsf{GL}_n, \mathsf{D}_n)$ in Example 2.4.5, $\mathcal{R}(\mathsf{PGL}_n, \mathsf{D}_n / \mathbb{G}_{\mathrm{m}})$ in Example 2.4.6, and $\mathcal{R}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$ in Example 2.4.3 respectively.

2.5 Tits buildings

Let G be a reductive group. Associated to it, there is a spherical building ${}^{v}\mathscr{B}(\mathsf{G})$ equipped with a natural G-action, called its *Tits building*. In this subsection, Tits buildings will be introduced for splittable reductive groups and we will see that the underlying building only depends on the root system and the ground field.

2.5.1. Let G be a reductive group. A *parabolic subgroup* of it is a smooth subgroup P such that G / P is a complete variety. A subgroup T of G is *Borel* if it is smooth, connected, solvable, and parabolic. It turns out that a smooth subgroup P is parabolic if and only if $\mathsf{P}_{K^{\mathrm{a}}}$ contains a Borel subgroup in $\mathsf{G}_{K^{\mathrm{a}}}$ [Mil17, 17.16] and every parabolic subgroup is connected and equal to its own normalizer since this is so over K^{a} [Mil17, 17.49]. When G has a Borel subgroup, it is said to be *quasi-split*. In this case, Borel subgroups are exactly the minimal parabolic subgroups and maximal connected solvable subgroups [Mil17, 17.19] and any two of them are conjugated by an element of G [Mil17, 25.8]. If the Borel subgroup is furthermore split (as a solvable algebraic group, namely it admits a normal series whose factors are isomorphic to either $\mathbb{G}_{\mathrm{a}}$ or $\mathbb{G}_{\mathrm{m}}$), then G is said to be *split*. It turns out that, G is split if and only if it is splittable [Mil17, 21.64].

Let $\pi \colon \mathsf{G} \to \mathsf{Q}$ be a quotient map and H a smooth subgroup of G. Then if H is parabolic (resp. Borel), so is $\pi(\mathsf{H})$. Moreover, every such subgroup of Q arises in this way [Mil17, 17.20]. This allows us to reduce the study of (the poset of) parabolic subgroups from reductive groups to simply-connected semisimple groups. The *Tits building* of a reductive group is essentially this poset [Tit74, 5.2].

2.5.2. Let (G, T) be a split reductive group. Then the followings sets are equipollent and the Weyl group ${}^vW(\mathsf{G}, \mathsf{T})$ acts simply transitively on them.

(i) The set of Borel subgroups B of G containing T.

(ii) The set of Weyl chambers ${}^v\mathscr{C}$ in the spherical apartment ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T})$.

(iii) The set of systems of positive roots Φ^+ in the root system $\Phi(\mathsf{G}, \mathsf{T})$.

(iv) The set of bases Δ of $\Phi(\mathsf{G}, \mathsf{T})$.

Indeed, if a system of positive roots Φ^+ is given, then B is generated by T and U_a for all $a \in \Phi^+$ and if a Borel subgroup B containing T is given, then the set of roots a whose Lie algebra $\mathfrak{g}_a$ is contained in the Lie algebra of T forms a system of positive roots Φ^+ [Mil17, 21.d].

More general, after choosing one element of the above equipollent sets. We have the following equipollent sets.

(i) The set of parabolic subgroups P of G containing B.

(ii) The set of faces ${}^vF$ of the Weyl chambers ${}^v\mathscr{C}$.

(iii) The set of parabolic subset Ψ of $\Phi(\mathsf{G}, \mathsf{T})$ containing Φ^+.

(iv) The set of types I on ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T})$.

Indeed, if a parabolic subset Ψ is given, then P is generated by T and U_a for all $a \in \Psi$ and if a parabolic subgroup P containing B is given, then the set of roots a whose Lie algebra $\mathfrak{g}_a$ is contained in the Lie algebra of P forms a parabolic subset Ψ [Mil17, 21.i].

Fix a Borel subgroup B of G containing T. Let I be a type and P_I the parabolic subgroup corresponding to it. Then the unipotent radical of P_I is generated by U_a for all $a \in \Phi^+ \setminus \Psi$ and the reductive quotient of P_I is isomorphic to the centralizer L_I of the largest subtorus contained in $\mathrm{Ker}(a)$ for all $a \in I$ [Mil17, 21.91]. This reductive group is called the *Levi subgroup associated to I* and $(\mathsf{L}_I, \mathsf{T})$ is a split reductive group with root datum $(\mathsf{X}^*, \Phi_I, \mathsf{X}_*, \Phi_I^\vee)$ and Weyl group ${}^vW_I$ [Mil17, 21.90].

2.5.3 Example. Consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$. Then the subgroup T_n of upper triangular invertible matrices is a Borel subgroup containing D_n. It corresponds to the system of positive roots $\Phi^+ = \{\chi_i - \chi_j \mid 1 \leqslant i < j \leqslant n\}$ with basis $\Delta = \{a_1 = \chi_1 - \chi_2, \cdots, a_{n-1} = \chi_{n-1} - \chi_n\}$. The Weyl chamber corresponding to it is

$$ {}^v\mathscr{C} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1 > x_2 > \cdots > x_n\}. $$

Let $I = \Delta \setminus \{l_1 = k_1, l_2 = k_1 + k_2, \cdots, l_{t-1} = k_1 + k_2 + \cdots + k_{t-1}\}$ be a type on the apartment, identified with a subset of Δ. Then the parabolic subgroup P_I, its unipotent radical $\mathscr{R}_u(\mathsf{P}_I)$ and the Levi subgroup L_I consist of the matrices of the following forms respectively

$$\begin{pmatrix} A_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & A_t \end{pmatrix}, \qquad \begin{pmatrix} I_{k_1} & * & * \\ 0 & \ddots & * \\ 0 & 0 & I_{k_t} \end{pmatrix}, \qquad \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_t \end{pmatrix},$$

where A_i is a $k_i \times k_i$ matrix. The facet corresponding to them is

$${}^vF = \big\{(x_1, \cdots, x_n) \in \mathbb{R}^n \,\big|\, x_1 = \cdots = x_{l_1} > x_{l_1+1} \cdots x_{l_{t-1}} > x_{l_{t-1}+1} = \cdots = x_n\big\}.$$

2.5.4 Theorem ([Rou09, §10; Tit74, §5])**.** *Let (G, T) be a split reductive group with Weyl group ${}^vW$, coroot space V, normalizer N of T and the root groups U_a. Then there is a unique (up to unique isomorphism) G-set ${}^v\mathscr{B}(\mathsf{G})$ containing V and satisfying the followings.*

(i) *${}^v\mathscr{B}(\mathsf{G}) = \bigcup_{g \in G} g.V$;*

(ii) *N stabilizes V and acts on it through ${}^vW$;*

(iii) *the fixator of $\alpha_{a+0} := \{x \in V \mid a(x) \geqslant 0\}$ is $T \cdot U_a$.*

Then ${}^v\mathscr{B}(\mathsf{G})$ is a building of type ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T})$. In fact, since N stabilizes V and preserves its apartment structure, each $g.V$ is endowed with such a structure and moreover they agree on intersections.

2.5.5 Definition. The building ${}^v\mathscr{B}(\mathsf{G})$ is called the *Tits building* of G.

Remark. Apartments in ${}^v\mathscr{B}(\mathsf{G})$ are one-one corresponding to split maximal tori. In fact, each $g.V$ endowed with its apartment structure is precisely the spherical apartment ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T}^g)$.

The action of G on ${}^v\mathscr{B}(\mathsf{G})$ is strongly transitive and type-preserving. It is also worth to mention that ${}^v\mathscr{B}(\mathsf{G})$ is further a $\mathrm{Aut}(\mathsf{G})$-set. Indeed, if φ is an automorphism of G, then $\varphi(\mathsf{T})$ is also a split maximal torus and the pushforward along φ defines a homomorphism from ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T})$ to ${}^v\mathscr{A}(\mathsf{G}, \varphi(\mathsf{T}))$.

2.5.6 Example. The simplicial complex structure on the Tits building of GL_n can be described as in 1.1.2.

2.5.7. Let G be a splittable reductive group. Let φ be a homomorphism in the following sequence.

$$\mathsf{G}^{\mathrm{sc}} \longrightarrow \mathsf{G}^{\mathrm{der}} \longrightarrow \mathsf{G} \longrightarrow \mathsf{G}^{\mathrm{ss}} \longrightarrow \mathsf{G}^{\mathrm{ad}}$$

Then for any split maximal torus T in G, its image or preimage under φ is again a split maximal torus T' and such a corresponding $\mathsf{T} \mapsto \mathsf{T}'$ gives rise to a bijection between the set of maximal tori. Therefore by 1.5.6 and 2.4.14, we see that all above reductive groups have isomorphic Tits buildings.

Conversely, any root datum arises from a splittable reductive group [Mil17, 23.55; SGA3, XXV, 1.2]. Hence we see that the Tits building ${}^v\mathscr{B}(\mathsf{G})$ depends only on the root system Φ and the ground field K and any root system gives rise to such a building. So we can denote this building by ${}^v\mathscr{B}(\Phi, K)$.

§ 3 Bruhat-Tits buildings

The datum of a split reductive group give rise to a root group datum. If the ground field K is valued, then there are valuations on such a datum. Bruhat and Tits [BT72, BT84a] introduced an affine building based on such data.

Throughout this section, the the ground field K is assumed to be equipped with a discrete valuation $\mathrm{val}\colon K \to \mathbb{R} \cup \{\infty\}$. Its valuation group $\mathrm{val}(K^\times)$ is denoted by Γ and we fix the following associated notations.

$$\begin{aligned} K^\circ &:= \{x \in K \mid \mathrm{val}(x) \geqslant 0\}, \\ (K^\circ)^\times &:= \{x \in K \mid \mathrm{val}(x) = 0\}, \\ K^{\circ\circ} &:= \{x \in K \mid \mathrm{val}(x) > 0\}, \\ \kappa &:= K^\circ / K^{\circ\circ}. \end{aligned}$$

3.1 Valuations on root group data

3.1.1 Definition. Let Φ be a root system and G be a group. A *root group datum*[8] *of type* Φ *in* G is a system $(T, (U_a, M_a)_{a\in\Phi})$, where T is a subgroup of G and for each $a \in \Phi$, U_a is a non-trivial subgroup of G and M_a is a right congruence class modulo T, satisfying the following axioms.

RGD1. For any $a, b \in \Phi$, the commutator group $[U_a, U_b]$ is contained in the group generated by the U_c for $c = ia + jb \in \Phi$ with $i, j > 0$.

RGD2. For each $a \in \Phi$, the class M_a satisfies $U_{-a}^* := U_{-a} \setminus \{1\} \subseteq U_a M_a U_a$.

RGD3. For any $a, b \in \Phi$ and each $m \in M_a$, we have $\mathrm{inn}(m).U_b \subseteq U_{r_a(b)}$.

RGD4. Let Φ^+ be a system of positive roots in Φ and if U^+ (resp. U^-) is the subgroup of G generated by the U_a for $a \in \Phi^+$ (resp. $a \in \Phi^-$), then $TU^+ \cap U^- = \{1\}$.

This root group datum is said to be *generating* when G is generated by the subgroups T and U_a for $a \in \Phi$.

3.1.2. Let $(T, (U_a, M_a)_{a\in\Phi})$ be a root group datum. We have the following consequences of above axioms [BT72, 6.1.2].

(i) $U_a \neq U_{-a}$ and $U_a M_a U_a \cap N_G(U_a) = \emptyset$.

(ii) For any $u \in U_{-a}^*$, there is a unique triple $(u', m, u'') \in U_a \times G \times U_a$ such that $u = u'mu''$, $\mathrm{inn}(m).U_a = U_{-a}$ and $\mathrm{inn}(m).U_{-a} = U_a$. Moreover, $m \in M_a$ and $u' \neq 1$.

Let $m(-)\colon U_{-a}^* \to M_a$ denote the map $u \mapsto m$ in above and put M_a° being its image.

(iii) T normalizes U_a and M_a.

[8]It is called a *(reduced) root datum* in [BT72, 6.1.1].

(iv) $M_a = M_a^{-1} = M_{-a}$ and $T \cup M_a$ is a subgroup of G.

(v) Let L_a be the subgroup of G generated by U_a, U_{-a} and T. Then

$$L_a = U_a M_a U_a \cup TU_a.$$

(vi) $N_G(U_a) \cap L_a = TU_a$ and

$$M_a = \{g \in L_a \mid \mathrm{inn}(g).U_a = U_{-a} \text{ and } \mathrm{inn}(g).U_{-a} = U_a\}.$$

So M_a is completely determined by U_a, U_{-a} and T. Hence we can say $(T, (U_a)_{a\in\Phi})$ is a root group datum without mention M_a.

(vii) Let N be the subgroup of G generated by T and M_a for all $a \in \Phi$. Then, if Φ is nonempty, N is already generated by M_a's and normalizes T. Moreover, there is an epimorphism ${}^v\nu \colon N \to {}^vW(\Phi)$ such that for each $a \in \Phi$ and $m \in N$, we have $\mathrm{inn}(m).U_a = U_b$ with $b = {}^v\nu(m).a$. In particular, we have ${}^v\nu(M_a) = \{r_a\}$. Also note that $\mathrm{Ker}({}^v\nu) = T$ [BT72, 6.1.11].

(viii) Suppose Φ is nonempty. Let N° be the subgroup of G generated by M_a° for all $a \in \Phi$ and $T^\circ = N^\circ \cap T$. Then $(T^\circ, (U_a, M_a^\circ)_{a\in\Phi})$ is a generating root group datum on the subgroup G° of G generated by U_a for all $a \in \Phi$.

3.1.3 Example ([BT72, 6.1.3.b; BT65]). Let (G, T) be a split reductive group over K, $(\mathsf{U}_a)_{a\in\Phi}$ be the root groups associated to the root system Φ of (G, T) and $(\mathsf{M}_a)_{a\in\Phi}$ be the right congruence classes in 2.3.12. Then $(T, (U_a, M_a)_{a\in\Phi})$ forms a generating root group datum in G:

RGD1. Let a and b be two roots in Φ. Then for any $i, j > 0$ such that $ia + jb \in \Phi$, there is a linear function [SGA3, XXII, 5.5.4]

$$f_{a,b;i,j} \colon \mathfrak{g}_a^{\otimes i} \otimes_K \mathfrak{g}_b^{\otimes j} \longrightarrow \mathfrak{g}_{ia+jb}$$

such that for any $X \in \mathbb{W}(\mathfrak{g}_a)$ and $Y \in \mathbb{W}(\mathfrak{g}_b)$, we have

$$[u_a(X), u_b(Y)] = \prod_{ia+jb\in\Phi} u_{ia+jb}(f_{a,b;i,j}(X^i \otimes Y^j))$$

where the product is taking in any order.

RGD2. Let $a \in \Phi$. Let $\mathsf{U}_{-a}^* = u_{-a}(\mathbb{W}(\mathfrak{g}_{-a})^\times)$, then $\mathsf{U}_{-a}^*(K) = U_{-a}^*$. Taking any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, then we have $X^{-1} \in \mathbb{W}(\mathfrak{g}_a)^\times$ and

$$\begin{aligned} u_{-a}(X^{-1}) &= u_a(X)u_a(-X)u_{-a}(-(-X)^{-1})u_a(-X)u_a(X) \\ &= u_a(X)m_a(-X)u_a(X). \end{aligned}$$

RGD3. This follows from 2.3.8 and 2.3.12(iii).

RGD4. There are closed immersions [SGA3, XXII, 5.5.1 and 5.6.5; Mil17, 21.68]

$$\mathsf{T}\times\prod_{a\in\Phi^+}\mathsf{U}_a\longrightarrow\mathsf{G}\qquad\text{and}\qquad\prod_{a\in\Phi^+}\mathsf{U}_a\longrightarrow\mathsf{G},$$

with images $\mathsf{T}\cdot\mathsf{U}_+$ and U_+. Where $\mathsf{T}\cdot\mathsf{U}_+$ is a Borel subgroup of G corresponding to the system of positive roots Φ^+, while U_+ is its unipotent radical and is generated by the root groups U_a for all $a\in\Phi^+$. Similarly we have Borel subgroup $\mathsf{T}\cdot\mathsf{U}_-$ and its unipotent radical U_-. Then the Borel subgroups $\mathsf{T}\cdot\mathsf{U}_+$ and $\mathsf{T}\cdot\mathsf{U}_-$ are *opposite*, namely their intersection is T [SGA3, 5.9.2; Mil17, 21.84]. Therefore $\mathsf{T}\cdot\mathsf{U}_+\cap\mathsf{U}_-$ is trivial.

Moreover, G is generated by T and the root groups U_a for all $a\in\Phi$ [Mil17, 21.11].

We also verify the corollaries in 3.1.2:

(i) This is clear.

(ii) We have already seen in above discussion that if $u=u_{-a}(X^{-1})$, then the triple $(u_a(X),m_a(-X),u_a(X))$ satisfies the requirements. Suppose (u',m,u'') is another triple, then $m_a(-X)=u_a(-X)u'mu''u_a(-X)\in M_a$ and hence it maps U_a to U_{-a} by conjugate. Then $u_a(-X)u'$ normalizes U_{-a} and $u''u_a(-X)$ normalizes U_a, hence $u'=u_a(X)$ and $u''=u_a(X)$ by 3.1.2(i).

(iii) The first follows from 2.3.8(ii) and as for the second: let $t\in\mathsf{T}$ and $X\in\mathbb{W}(\mathfrak{g}_a)^\times$, then we have

$$\begin{aligned}\operatorname{inn}(t).m_a(X)&=\operatorname{inn}(t).(u_a(X)u_{-a}(-X^{-1})u_a(X))\\&=(\operatorname{inn}(t).u_a(X))(\operatorname{inn}(t).u_{-a}(-X^{-1}))(\operatorname{inn}(t).u_a(X))\\&=u_a(a(t)X)\cdot u_{-a}((-a)(t)(-X^{-1}))\cdot u_a(a(t)X)\\&=u_a(a(t)X)\cdot u_{-a}(-(a(t)X)^{-1})\cdot u_a(a(t)X)\\&=m_a(a(t)X).\end{aligned}$$

(iv) For any $X\in\mathbb{W}(\mathfrak{g}_a)^\times$, we have

$$\begin{aligned}m_a(X)^{-1}&=(u_a(X)u_{-a}(-X^{-1})u_a(X))^{-1}\\&=u_a(-X)u_{-a}(X^{-1})u_a(-X)\\&=m_a(-X).\end{aligned}$$

By 2.3.12(iv), we also have

$$m_a(X)^{-1}=m_{-a}(X^{-1}).$$

These prove the first part. As for the second: let $X,Y\in\mathbb{W}(\mathfrak{g}_a)^\times$, then we have

$$m_a(X)m_a(Y)=m_a(X)m_{-a}(-Y^{-1})=a^\vee(\langle X,-Y^{-1}\rangle)\in\mathsf{T}.$$

(v) This follows from the Bruhat decomposition [SGA3, 5.7.4; Mil17, 21.73] for L_a:

$$\mathsf{L}_a = \mathsf{B} \sqcup \mathscr{R}_u(\mathsf{B}) w\, \mathsf{B},$$

where B is the Borel subgroup $\mathsf{T} \cdot \mathsf{U}_a$ of L_a with unipotent radical $\mathscr{R}_u(\mathsf{B}) = \mathsf{U}_a$ and w is the only nontrivial element of ${}^vW(\mathsf{L}_a, \mathsf{T})$, hence M_a.

(vi) The first follows from the normalizer theorem [Mil17, 17.50]. As for the second: suppose $g \in \mathsf{L}_a$ has the property that $\mathrm{inn}(g).\, \mathsf{U}_a = \mathsf{U}_{-a}$ and $\mathrm{inn}(g).\, \mathsf{U}_{-a} = \mathsf{U}_a$. Then $g \notin \mathsf{T} \cdot \mathsf{U}_a$ and hence $g \in \mathsf{U}_a \cdot \mathsf{M}_a \cdot \mathsf{U}_a$. But $\mathsf{U}_a \cap \mathsf{N}_\mathsf{G}(\mathsf{U}_{-a})$ is trivial. Hence $g \in \mathsf{M}_a$.

(vii) $\mathsf{N} = \mathsf{N}_\mathsf{G}(\mathsf{T})$ is generated by M_a for all $a \in \Phi$, the epimorphism ${}^v\nu\colon N \to {}^vW(\Phi)$ is the quotient map $N \to {}^vW$ and the statement follows from 2.3.12.

(viii) By [Mil17, 21.49], $\mathsf{G}^{\mathrm{der}}$ is generated by U_a for all $a \in \Phi$. Then it is clear that $G^\circ = \mathsf{G}^{\mathrm{der}}(K)$, $T^\circ = T \cap G^\circ$ and $N^\circ = N_{G^\circ}(T^\circ)$.

Note that the above facts already will imply Theorem 2.5.4 using either *Tits system* or similar construction in 3.2.8.

3.1.4 Definition. A *valuation on the root group datum* $(G, T, (U_a)_{a\in\Phi})$ is a family $\varphi = (\varphi_a)_{a\in\Phi}$ of functions $\varphi_a\colon U_a \to \mathbb{R} \cup \{\infty\}$ satisfying the following axioms.

V0. For each $a \in \Phi$, the image of φ_a contains at least three elements.

V1. For each $a \in \Phi$ and any $\lambda \in \mathbb{R} \cup \{\infty\}$, the set $U_{a,\lambda} := \varphi_a^{-1}([\lambda, \infty])$ is a subgroup of U_a and $U_{a,\infty} = \{1\}$.

V2. For each $a \in \Phi$ and any $m \in M_a$, the function $u \mapsto \varphi_{-a}(u) - \varphi_a(mum^{-1})$ is constant on U^*_{-a}.

V3. For any pair $a, b \in \Phi$ not proportional and any $\lambda, \mu \in \mathbb{R} \cup \{\infty\}$, the commutator group $[U_{a,\lambda}, U_{b,\mu}]$ is contained in the subgroup generated by $U_{ia+jb, i\lambda+j\mu}$ for all $i, j > 0$ such that $ia + jb \in \Phi$.

V4. For each $a \in \Phi$ and any $u \in U_a$, $u', u'' \in U_{-a}$, if $u'uu'' \in M_a$, then $\varphi_{-a}(u') = \varphi_{-a}(u'') = -\varphi_a(u)$.

For each $a \in \Phi$, let Γ_a denote the set $\varphi_a(U_a^*)$ and for any $k \in \Gamma_a$, let $M_{a,k}$ be the intersection of M_a and $U_{-a}\varphi_a^{-1}(k)U_{-a}$.

3.1.5 Example. Consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$ over K. Denote $a_{ij} = \chi_i - \chi_j \in \Phi$ and define $\varphi = (\varphi_{a_{ij}})_{a_{ij}\in\Phi}$ as

$$\varphi_{a_{ij}}(u_{a_{ij}}(z)) = \mathrm{val}(z).$$

Note that we have

$$\varphi_{a_{ij}}\left(\xi_{ij}\left(\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\right)\right) = \mathrm{val}(z), \qquad \varphi_{-a_{ij}}\left(\xi_{ij}\left(\begin{pmatrix}1 & 0\\ z & 1\end{pmatrix}\right)\right) = \mathrm{val}(z).$$

Then φ is a valuation on the root group datum $(D_n, (U_{a_{ij}})_{a_{ij}\in\Phi})$ with $\Gamma_{a_{ij}} = \Gamma$:

V0. This clear as val is nontrivial.

V1. For any $\lambda \in \mathbb{R} \cup \{\infty\}$, we have

$$U_{a_{ij},\lambda} = \left\{ \xi_{ij}\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) \in U_{a_{ij}} \;\middle|\; \operatorname{val}(z) \geqslant \lambda \right\}.$$

Then $U_{a_{ij},\infty} = \{I_n\}$ and for any $x, y \in K$ with $\operatorname{val}(x), \operatorname{val}(y) \geqslant \lambda$, we have

$$u_{a_{ij}}(x) \cdot u_{a_{ij}}(y)^{-1} = u_{a_{ij}}(x - y),$$

and its valuation is $\operatorname{val}(x-y) \geqslant \min\{\operatorname{val}(x), \operatorname{val}(y)\} \geqslant \lambda$.

V2. For any $x, y, z \in K^\times$ and

$$m = \xi_{ij}\left(\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\right) \in M_{a_{ij}}, \qquad u = \xi_{ij}\left(\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\right) \in U^*_{-a_{ij}},$$

we have

$$mum^{-1} = \xi_{ij}\left(\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} 0 & y^{-1} \\ x^{-1} & 0 \end{pmatrix}\right) = \xi_{ij}\left(\begin{pmatrix} 1 & xzy^{-1} \\ 0 & 1 \end{pmatrix}\right).$$

Therefore

$$\begin{aligned} \varphi_{-a_{ij}}(u) - \varphi_{a_{ij}}(mum^{-1}) &= \varphi_{-a_{ij}}\left(u_{-a_{ij}}(z)\right) - \varphi_{a_{ij}}\left(u_{a_{ij}}(xzy^{-1})\right) \\ &= \operatorname{val}(z) - \operatorname{val}\left(xzy^{-1}\right) \\ &= -\operatorname{val}(x) + \operatorname{val}(y), \end{aligned}$$

independently on u.

V3. We need the following *commutator formula*:

$$[u_{a_{ij}}(x), u_{a_{kl}}(y)] = \begin{cases} u_{a_{il}}(xy) & i \neq l, k = j, \\ u_{a_{kj}}(-xy) & i = l, k \neq j, \\ I_n & i \neq l, k \neq j. \end{cases}$$

From which, we see that if $\varphi_{a_{ij}}(u) \geqslant \lambda$ and $\varphi_{a_{kl}}(v) \geqslant \mu$, then either $[u, v] = I_n$ or $a_{ij} + a_{kl} \in \Phi$ and $\varphi_{a_{ij}+a_{kl}}([u, v]) \geqslant \lambda + \mu$.

V4. For any $x, y, z \in K$ and $u = u_{a_{ij}}(x)$, $u' = u_{-a_{ij}}(y)$, $u'' = u_{-a_{ij}}(z)$, we have

$$u'uu'' = \xi_{ij}\left(\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\right) = \xi_{ij}\left(\begin{pmatrix} 1 + xz & x \\ y + z + xyz & 1 + xy \end{pmatrix}\right).$$

If $u'uu'' \in M_{a_{ij}}$, then we must have

$$1 + xz = 1 + xy = 0, \qquad \text{and} \qquad -x^{-1} = y + z + xyz.$$

Hence we have $y = z = -x^{-1}$ have thus $\varphi_{-a_{ij}}(u') = \varphi_{-a_{ij}}(u'') = -\varphi_{a_{ij}}(u)$.

Now, let $\operatorname{val}(x) = k$. Then we see that

$$M_{a_{ij},k} = \left\{ \xi_{ij}\left(\begin{pmatrix} 0 & z \\ -z^{-1} & 0 \end{pmatrix}\right) \;\middle|\; z \in K, \operatorname{val}(z) = k \right\}.$$

3.1.6. Given a root group datum $(T, (U_a)_{a\in\Phi})$ in G and let φ be a valuation on it. Then for any vector v in the ambient space V of Φ, the family $\psi = (\psi_a)_{a\in\Phi}$ given by $\psi_a\colon u \mapsto \varphi_a(u) + a(v)$ is a valuation [BT72, 6.2.5] and is denoted by $\varphi + v$. The valuations φ and $\psi = \varphi + v$ are said to be *equipollent*. The mapping $(\varphi, v) \mapsto \varphi + v$ defines an action of V on the set of valuations and each equipollent class is an orbit.

Let $\mathbb{A}$ denote the set of valuations equipollent to φ. Then $\mathbb{A}$ is an affine space underlying V and 1.4.6 applies. For $\alpha = \alpha_{a+k}$ with $a \in \Phi$, $k \in \Gamma_a$, let $U_\alpha = U_{a,k}$ and $U_{\alpha+} = \bigcup_{h>k} U_{a,h}$ (note that $U_{\alpha+} = U_{\alpha_+}$ if Γ_a is discrete). It is clear that the affine root system Σ and the mapping $\alpha \mapsto U_\alpha$ depends only on the equipollent class of φ.

3.1.7 Example. Continue Example 3.1.5. The coroot space V is (by Example 2.4.4)

$$V = \{c_1\lambda_1 + \cdots + c_n\lambda_n \mid c_1, \cdots, c_n \in \mathbb{R}, c_1 + \cdots + c_n = 0\}.$$

The for any $v = c_1\lambda_1 + \cdots + c_n\lambda_n$, the valuation $\varphi + v$ is given by

$$u_{a_{ij}}(x) \longmapsto \mathrm{val}(x) + a_{ij}(v) = \mathrm{val}(x) + c_i - c_j.$$

Let $k \in \Gamma_a$, then $\varphi + v \in \alpha_{a+k}$ if and only if $c_i - c_j + k \geqslant 0$.

3.1.8. Let $m \in N$ and $w = {}^v\nu(m) \in {}^vW$. Then the family $\psi = (\psi_a)_{a\in\Phi}$ given by $\psi_a\colon u \mapsto \varphi_{w^{-1}.a}(m^{-1}um)$ is a valuation [BT72, 6.2.5] and is denoted by $m.\varphi$. We thus obtain an action of N on the set of valuations such that for any $m \in N$ and $v \in V$, we have $m.(\varphi + v) = m.\varphi + {}^v\nu(m).v$.

The action of N stabilizes $\mathbb{A}$ and for any $m \in N$, the map $\nu(m)\colon \varphi \mapsto m.\varphi$ itself is an automorphism of the Euclidean affine space $\mathbb{A}$ whose vectorial part is ${}^v\nu(m)$ [BT72, 6.2.10]. For each $a \in \Phi$ and $k \in \Gamma_a$, the image of $M_{a,k}$ under ν is the reflection r_{a+k}. So the automorphism $\nu(m)$ maps affine roots to affine roots and we have $mU_\alpha m^{-1} = U_{\nu(m).\alpha}$. In particular, for $u \in U_a^*$, $\nu(m(u)) = r_{a+\varphi_a(u)}$ [BT72, 6.2.12]. Therefore, the valuation φ is completely determined by the homomorphism $\nu\colon N \to \mathrm{Aut}(\mathbb{A})$.

3.1.9 Example. Continue Examples 3.1.5 and 3.1.7. Then N is the group of monomial matrices in $\mathsf{GL}_n(K)$. Let $m \in N$ and we can write it as

$$m = \sum_{k=1}^{n} x_k E_{\sigma(k)k},$$

where $\sigma \in \mathfrak{S}_n$ is a permutation such that $w = {}^v\nu(m)$ is identified with σ through ${}^vW \cong \mathfrak{S}_n$. Then, for any $u = u_{a_{ij}}(y) \in U_{a_{ij}}$, we have

$$\begin{aligned} m^{-1}um &= \left(\sum_{k=1}^{n} x^{-1}_{\sigma^{-1}(k)} E_{\sigma^{-1}(k)k}\right)(I_n + yE_{ij})\left(\sum_{k=1}^{n} x_k E_{\sigma(k)k}\right) \\ &= I_n + x^{-1}_{\sigma^{-1}(i)} y x_{\sigma^{-1}(j)} E_{\sigma^{-1}(i)\sigma^{-1}(j)} \\ &= u_{a_{\sigma^{-1}(i)\sigma^{-1}(j)}}\left(x^{-1}_{\sigma^{-1}(i)} y x_{\sigma^{-1}(j)}\right) \in U_{a_{\sigma^{-1}(i)\sigma^{-1}(j)}} = U_{w^{-1}.a_{ij}}. \end{aligned}$$

Hence the valuation $m.\varphi$ is given by

$$(m.\varphi)_{a_{ij}} : u = u_{a_{ij}}(y) \longmapsto \varphi_{w^{-1}.a_{ij}}(m^{-1}um) = \mathrm{val}\Big(x_{\sigma^{-1}(i)}^{-1} y x_{\sigma^{-1}(j)}\Big).$$

From above computations, it is also clear that $mU_\alpha m^{-1} = U_{\nu(m).\alpha}$ holds for any affine root α.

For any $v \in V$, one can verify that

$$\begin{aligned}(m.(\varphi + v))_{a_{ij}}(u) &= (\varphi + v)_{w^{-1}.a_{ij}}(m^{-1}um) \\ &= \varphi_{w^{-1}.a_{ij}}(m^{-1}um) + (w^{-1}.a_{ij})(v) \\ &= \mathrm{val}\Big(x_{\sigma^{-1}(i)}^{-1} y x_{\sigma^{-1}(j)}\Big) + a_{ij}(w.v) \\ &= (m.\varphi + w.v)_{a_{ij}}(u).\end{aligned}$$

Also note that

$$\begin{aligned}(m.\varphi)_{a_{ij}}(u) - \varphi_{a_{ij}}(u) &= \mathrm{val}\Big(x_{\sigma^{-1}(i)}^{-1} y x_{\sigma^{-1}(j)}\Big) - \mathrm{val}(y) \\ &= \mathrm{val}\Big(x_{\sigma^{-1}(i)}^{-1}\Big) - \mathrm{val}\Big(x_{\sigma^{-1}(j)}^{-1}\Big) \\ &= \left\langle a_{ij}, \sum_{k=1}^{n} \mathrm{val}\Big(x_{\sigma^{-1}(k)}^{-1}\Big)\lambda_k \right\rangle.\end{aligned}$$

Hence we see that $m.(\varphi + v) = m.\varphi + w.v = \varphi + w.v + v_m$, where

$$v_m = \sum_{k=1}^{n} \left(\mathrm{val}\Big(x_{\sigma^{-1}(k)}^{-1}\Big) + \frac{1}{n}\,\mathrm{val}(\det(m))\right)\lambda_k \in V.$$

Namely, the affine transformation $\nu(m)$ has vectorial part w and translation part v_m.

If $u = u_{a_{ij}}(x) \in U_{a_{ij}}^*$, then

$$m(u) = m_{a_{ij}}(x) = \xi_{ij}\left(\begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}\right).$$

Hence ${}^v\nu(m(u)) = (i, j)$ and the translation part is

$$v_{m(u)} = \mathrm{val}\big(x^{-1}\big)\lambda_i + \mathrm{val}(-x)\lambda_j = -\,\mathrm{val}(x)a_{ij}^\vee.$$

Then we have

$$\begin{aligned}\nu(m(u)).(\varphi + v) &= \varphi + (i, j).v - \mathrm{val}(x)a_{ij}^\vee \\ &= \varphi + r_{a_{ij}}(v) - \varphi_{a_{ij}}(u)a_{ij}^\vee \\ &= \varphi + v - a_{ij}(v)a_{ij}^\vee - \varphi_{a_{ij}}(u)a_{ij}^\vee \\ &= r_{a_{ij}+\varphi_{a_{ij}}(u)}(\varphi + v).\end{aligned}$$

3.1.10. Let $H = \mathrm{Ker}(\nu)$ and $\widehat{W} = \nu(N)$. Let W denote the subgroup of $\widehat{W}$ generated by r_{a+k} with $a \in \Phi$ and $k \in \Gamma_a$. It is a normal subgroup because N permutes $M_{a,k}$. Let $N' = \nu^{-1}(W)$. It is usually not the entire N. We say the root group datum (together with the valuation φ) is *simply-connected* when $N' = N$. Let $T' = T \cap N'$ and let G' be the subgroup of G generated by N' and the U_a for $a \in \Phi$. Since $M_a \cap N' \neq \emptyset$ for all $a \in \Phi$, we see that $(T', (U_a)_{a\in\Phi})$ is a simply-connected generating root group datum in G' [BT72, 6.2.11]. Recall that (3.1.2(viii)) N° is generated by M_a° for all $a \in \Phi$, hence $N^\circ \subseteq N'$ and therefore the generating root group datum $(T^\circ, (U_a)_{a\in\Phi})$ on G° is also simply-connected.

The valuation φ is said to be *special* if $0 \in \Gamma_a$ for all $a \in \Phi$. If this is the case, then the group W (resp. $\widehat{W}$) can be decomposed as $W = W_\varphi \ltimes \mathrm{Ker}(W \to {}^vW)$ (resp. $\widehat{W} = W_\varphi \ltimes \nu(T)$) [BT72, 6.2.19], where W_φ is the stabilizer of φ.

The valuation φ is said to be *discrete* if Γ_a is a discrete subset of $\mathbb{R}$ for all $a \in \Phi$. If this is the case, then W is the affine Weyl group $W(\Sigma)$ for the affine root system Σ [BT72, 6.2.22].

Suppose Φ is irreducible and φ is discrete and special. Then all Γ_a are the same discrete subgroup Γ of $\mathbb{R}$ [BT72, 6.2.23]. So 1.4.7 applies and we get an apartment $\mathscr{A}(\Sigma)$. Then $\mathrm{Ker}(W \to {}^vW) = \mathcal{Q}^\vee \otimes \Gamma$ and $\nu(T)$ is between $\mathcal{Q}^\vee \otimes \Gamma$ and $\mathcal{P}^\vee \otimes \Gamma$ [BT72, 6.2.20].

3.1.11 Example. Continue Examples 3.1.5, 3.1.7 and 3.1.9. Let $m \in N$ with related notations as before. Then for $m \in \mathrm{Ker}(\nu)$, one must have both ${}^v\nu(m) = \mathrm{id}$ and $v_m = 0$. Hence m is diagonal and for all $1 \leqslant k \leqslant n$, $\mathrm{val}(x_k) = \frac{1}{n}\mathrm{val}(\det(m))$. Therefore

$$H = \{\mathrm{diag}(x_1, \cdots, x_n) \in \mathsf{D}_n(K) \mid \mathrm{val}(x_1) = \cdots = \mathrm{val}(x_n)\}.$$

It is clear from the computations above that for any $m \in N$, its translation part v_m is in $\mathcal{P}^\vee \otimes \Gamma$. But we emphasize that the translation group $\nu(T)$ should be $\mathsf{X}_{ss}^\vee \otimes \Gamma$. Here $\mathsf{X}_{ss}^\vee$ is the dual lattice of $V^* \cap \mathsf{X}$ and is the cocharacter group of the semisimple quotient of $(\mathsf{GL}_n, \mathsf{D}_n)$. On the other hand, the translation group of W is clearly $\mathcal{Q}^\vee \otimes \Gamma$.

Let $m \in N$ with related notations as before. Then for $m \in N'$, one must have $v_m \in \mathcal{Q}^\vee \otimes \Gamma$, which is equivalent to say that for all $1 \leqslant k \leqslant n$, $\mathrm{val}\big(x_k^{-1}\big) + \frac{1}{n}\mathrm{val}(\det(m)) \in \Gamma$, which is the case if and only if $\frac{1}{n}\mathrm{val}(\det(m)) \in \Gamma$. Therefore

$$N' = \{m \in N \mid \mathrm{val}(\det(m)) \in n\Gamma\}.$$

Therefore we have

$$G' = \{g \in \mathsf{GL}_n(K) \mid \mathrm{val}(\det(g)) \in n\Gamma\}.$$

It is worth to mention that there is a group

$$\mathsf{GL}_n(K)^1 := \{g \in \mathsf{GL}_n(K) \mid \mathrm{val}(\det(g)) = 0\},$$

between G' and $G^\circ = \mathsf{SL}_n(K)$. Hence for this group, the generating root group datum $(D_n \cap \mathsf{GL}_n(K)^1, (U_{a_{ij}})_{a_{ij}\in\Phi})$ is simply-connected.

3.2 Bruhat-Tits building

Given a root group datum $(T, (U_a, M_a)_{a\in\Phi})$ in G with a valuation $\varphi = (\varphi_a)_{a\in\Phi}$ on it, Bruhat and Tits [BT72] associate an affine building equipped with natural G-action to these data.

3.2.1. Let Ω be a nonempty subset of $\mathbb{A}$ and let U_Ω denote the subgroup generated by U_α for all affine roots $\alpha \supseteq \Omega$. Then the image of $N \cap U_\Omega$ under $\nu\colon N \to \widehat{W}$ is generated by the reflections r_α for affine roots α such that $\Omega \subseteq \partial\alpha$ and is identified with the Weyl group of $\Phi_\Omega := \{a \in \Phi \mid \exists\alpha, {}^v\alpha = a, \partial\alpha \supseteq \Omega\}$ [BT72, 7.1.3]. Let N_Ω denote its preimage and let $P_\Omega = H \cdot U_\Omega$. Then

$$N_\Omega = N \cap P_\Omega.$$

Let $\widehat{N}_\Omega$ denote the fixator of Ω in N:

$$\widehat{N}_\Omega := \{n \in N \mid \nu(n).x = x \text{ for all } x \in \Omega\}.$$

Then $\widehat{N}_\Omega$ contains N_Ω and normalizes P_Ω. Hence

$$\widehat{P}_\Omega := \widehat{N}_\Omega \cdot P_\Omega = \widehat{N}_\Omega \cdot U_\Omega$$

is a group having P_Ω and U_Ω as its normal subgroups. By 3.1.8, for any $n \in N$, we have

$$\mathrm{inn}(n).P_\Omega = P_{\nu(n).\Omega} \qquad \text{and} \qquad \mathrm{inn}(n).\widehat{P}_\Omega = \widehat{P}_{\nu(n).\Omega}.$$

Note that the map $\Omega \mapsto U_\Omega$ (resp. Φ_Ω, N_Ω, P_Ω, $\widehat{N}_\Omega$, $\widehat{P}_\Omega$) reverses the order of inclusions.

Remark. For $x \in \mathbb{A}$ a point, $\widehat{N}_x = \nu^{-1}(\widehat{W}_x)$. Hence if the generating root group datum is simply-connected, we have $\widehat{N}_x = N_x$ and hence $\widehat{P}_x = P_x$.

3.2.2 Example. Continue Examples 3.1.5, 3.1.7, 3.1.9 and 3.1.11.

First consider $\Omega = \alpha_{a_{ij}+k} \in \Sigma$. Then $U_\Omega = U_{a_{ij},k}$, $\Phi_\Omega = \emptyset$ and

$$P_\Omega = H \cdot U_{a_{ij},k} = \left\{ \mathrm{diag}(x_1, \cdots, x_n) + yE_{ij} \in \mathsf{GL}_n(K) \;\middle|\; \begin{array}{l} \mathrm{val}(x_1) = \cdots = \mathrm{val}(x_n), \\ \mathrm{val}(y) - \mathrm{val}(x_i) \geqslant k \end{array} \right\}.$$

In particular, $N_\Omega = H$. Note that we also have $\widehat{N}_\Omega = H$ since Ω contains an open in $\mathbb{A}$. Therefore $\widehat{P}_\Omega = P_\Omega$.

Next, consider $x = \varphi + v \in \mathbb{A}$. Then U_x is generated by $U_{a_{ij},-a_{ij}(v)}$ for all $a_{ij} \in \Phi$ and $\Phi_x = \{a_{ij} \in \Phi \mid a_{ij}(v) \in \Gamma\}$. Then W_x is generated by $r_{a,-a(v)}$ for all $a \in \Phi_x$ but $\widehat{W}_x$ may be larger in general: it contains $r_{a,-a(v)}$ even when $a(v) \in \frac{1}{n}\Gamma \setminus \Gamma$. Now, suppose x is special. Then $\widehat{W}_x = W_x \cong {}^vW$ and $\widehat{P}_x = P_x$ is generated by $H \cdot U_{a_{ij},-a_{ij}(v)}$ for all $a_{ij} \in \Phi$.

To see what are $\widehat{P}_\Omega$ and P_Ω in general, we need more knowledge on such subgroups.

3.2.3 Proposition ([BT72, 7.1.11])**.** *Let Ω be a nonempty subset of $\mathbb{A}$. Then*

$$\widehat{P}_\Omega = \bigcap_{x\in\Omega} \widehat{P}_x.$$

So in particular, $\widehat{P}_\Omega \cap \widehat{P}_{\Omega'} = \widehat{P}_{\Omega\cup\Omega'}$.

3.2.4. Let Ω be a nonempty subset of $\mathbb{A}$. The *enclosure* $\mathrm{cl}(\Omega)$ of Ω is the intersection of all affine roots α containing Ω. It turns out that [BT72, 7.1.2 and 7.1.9]

$$U_{\mathrm{cl}(\Omega)} = U_\Omega, \qquad \widehat{N}_{\mathrm{cl}(\Omega)} = \widehat{N}_\Omega \qquad \text{and} \qquad \widehat{P}_{\mathrm{cl}(\Omega)} = \widehat{P}_\Omega.$$

From its definition, we see that $\mathrm{cl}(\Omega)$ must be a disjoint union of some facets in $\mathbb{A}$. With Proposition 3.2.3, we conclude that the groups $\widehat{P}_\Omega$ are all of the form

$$\bigcap_F \widehat{P}_F,$$

where F ranges over all facets in $\mathbb{A}$ covered by $\mathrm{cl}(\Omega)$.

Hence to understand the subgroups $\widehat{P}_\Omega$, it suffices to understand those $\widehat{P}_F$.

3.2.5. Let ${}^v\!C$ be a Weyl chamber in $\mathbb{A}$ and $\Phi^+_{{}^v\!C}$ (resp. $\Phi^-_{{}^v\!C}$) the system of positive (resp. negative) roots in Φ defined by ${}^v\!C$. Let $U^+_{{}^v\!C}$ (resp. $U^-_{{}^v\!C}$) the subgroup of G generated by the U_a for $a \in \Phi^+_{{}^v\!C}$ (resp. $a \in \Phi^-_{{}^v\!C}$). Then for any $x \in \mathbb{A}$, we have $U_{x\pm{}^v\!C} \subseteq U^\pm_{{}^v\!C}$ and $\widehat{N}_{x\pm{}^v\!C} = N_{x\pm{}^v\!C} = H$. As a consequence, $\widehat{P}_{x+{}^v\!C} = P_{x+{}^v\!C}$. Denote it by $B_{x,{}^v\!C}$.

In general, let Ω be a nonempty subset of $\mathbb{A}$. Then we have [BT72, 7.1.4]

$$P_\Omega \cap U^\pm_{{}^v\!C} = U_{\Omega\pm{}^v\!C} \qquad \text{and} \qquad P_\Omega = N_\Omega \cdot U_{\Omega+{}^v\!C} \cdot U_{\Omega-{}^v\!C}.$$

As a consequence, we have [BT72, 7.1.8]

$$\widehat{P}_\Omega \cap U^\pm_{{}^v\!C} = U_{\Omega\pm{}^v\!C}, \qquad \widehat{P}_\Omega \cap N = \widehat{N}_\Omega \qquad \text{and} \qquad \widehat{P}_\Omega = \widehat{N}_\Omega \cdot U_{\Omega+{}^v\!C} \cdot U_{\Omega-{}^v\!C}.$$

3.2.6 Theorem (Bruhat decomposition [BT72, 7.3.4])**.** *Let ${}^v\!C$ and ${}^v\!C'$ be two Weyl chambers and x, x' be two points in $\mathbb{A}$.*

(i) *We have*

$$G = B_{x,{}^v\!C} \cdot N \cdot B_{x',{}^v\!C'}.$$

(ii) *More precisely, the canonical map from N to the set of double cosets induces a bijection from $\widehat{W} \cong N/H$ to $B_{x,{}^v\!C}\backslash G/B_{x',{}^v\!C'}$.*

3.2.7 Example. Continue Example 3.2.2. Let Ω be a nonempty subset of $\mathbb{A}$. We claim that[9]

$$\widehat{P}_\Omega = \left\{ g = (g_{ij})_{i,j} \in \mathsf{GL}_n(K) \;\middle|\; \forall i,j : \mathrm{val}(g_{ij}) - \frac{1}{n}\mathrm{val}(\det(g)) \geqslant -\inf_{x\in\Omega} a_{ij}(x) \right\}.$$

[9] slightly different from [BT72, 10.2.9] due to different conventions on the root group datum

Denote the right hand side by L_Ω. Then it is clear that

$$L_\Omega = \bigcap_{x\in\Omega} L_x.$$

Therefore it suffices to show $\widehat{P}_x = L_x$.

First, we have $H \subseteq L_x$ and for any $a_{ij} \in \Phi$,

$$L_x \cap U_{a_{ij}} = U_{a_{ij},-a_{ij}(x)} = P_x \cap U_{a_{ij}}.$$

Therefore, $P_x \subseteq L_x$. Let ${}^vC$ be any Weyl chamber, then we have $B_{x,{}^vC} \subseteq P_x \subseteq L_x$. Hence by Theorem 3.2.6, we have

$$L_x = B_{x,{}^vC} \cdot (L_x \cap N) \cdot B_{x,{}^vC}.$$

Therefore it suffices to show $L_x \cap N = \widehat{N}_x$.

If $m = \sum_{k=1}^n x_k E_{\sigma(k)k} \in L_x \cap N$ with $\sigma = {}^v\nu(m)$, then by Example 3.1.9, we have

$$\mathrm{val}\big(x_{\sigma^{-1}(k)}\big) - \frac{1}{n}\,\mathrm{val}(\det(m)) \geqslant -a_{k\sigma^{-1}(k)}(x).$$

Hence

$$\sum_{k=1}^n \mathrm{val}(x_k) - \mathrm{val}(\det(m)) \geqslant 0,$$

which should be an equality. Therefore for all $1 \leqslant k \leqslant n$, we have

$$\mathrm{val}\Big(x_{\sigma^{-1}(k)}^{-1}\Big) + \frac{1}{n}\,\mathrm{val}(\det(m)) = a_{k\sigma^{-1}(k)}(x).$$

Again by Example 3.1.9, we have (where $x = \varphi + v$)

$$m.x - x = \sigma.v - v + \sum_{k=1}^n a_{k\sigma^{-1}(k)}(x)\lambda_k.$$

Note that

$$\sigma.v - v = \sum_{k=1}^n \langle a_{\sigma^{-1}(k)k}, v\rangle \lambda_k$$

Therefore for any $x \in \Omega$, we have

$$m.x - x = \sum_{k=1}^n \left(a_{\sigma^{-1}(k)k}(x) + a_{k\sigma^{-1}(k)}(x)\right)\lambda_k = 0.$$

This shows $L_x \cap N \subseteq \widehat{N}_x$.

Conversely, if $m = \sum_{k=1}^n x_k E_{\sigma(k)k} \in \widehat{N}_x$ with $\sigma = {}^v\nu(m)$, then $m.x = x$. Which, by similar argument as above, implies

$$\mathrm{val}\Big(x_{\sigma^{-1}(k)}^{-1}\Big) + \frac{1}{n}\,\mathrm{val}(\det(m)) = a_{k\sigma^{-1}(k)}(x).$$

Since other entries of m are 0, the inequality holds trivially. Therefore $m \in L_x$.

3.2.8 Definition. The *Bruhat-Tits building of* G (with the root group datum and the valuation φ being given) is the quotient set $\mathscr{B}^{\varphi}$ of $G \times \mathbb{A}$ under the following equivalent relation [BT72, 7.4.1]:

$$(g, x) \sim (h, y) \iff \exists n \in N : y = \nu(n).x,\ g^{-1}hn \in \widehat{P}_x.$$

We will simply denote this set by $\mathscr{B}$ if there is no ambiguity.

Remark. Let $\lambda\colon \Phi \to \mathbb{R}_{>0}$ be a function, constant on each irreducible component, and let $v \in V$. Then the family $\psi_a\colon u \mapsto \lambda(a)\varphi_a(u) + a(v)$ defines a valuation. Let ψ be another valuation. It is said to be *equivalent to* φ if there exists a function $\lambda\colon \Phi \to \mathbb{R}_{>0}$, constant on each irreducible component, and a vector $v \in V$, such that $\psi = \lambda\varphi + v$.

Remark. The left multiplication of G on $G \times \mathbb{A}$ is compatible with above equivalent relation, hence $\mathscr{B}$ inherits a G-action. Identifying $\mathbb{A}$ as the subset $\{1\} \times \mathbb{A}$ of $\mathscr{B}$, we have:

1. $\mathscr{B} = \bigcup_{g \in G} g.\mathbb{A}$;
2. each U_α fixes $\alpha \in \Sigma$ pointwise [BT72, 7.4.5];
3. for each nonempty $\Omega \subseteq \mathbb{A}$, its fixator is $\widehat{P}_\Omega$ and it acts transitively on apartments containing Ω [BT72, 7.4.4, 7.4.9];
4. the stabilizer (resp. fixator) of $\mathbb{A}$ is N (resp. H) [BT72, 7.4.10].

Then one can carry apartment structure on $\mathbb{A}$ to each $g.\mathbb{A}$ and see that they agree on the intersections [BT72, 7.4.18]. Hence $\mathscr{B}$ is a building of type $\mathscr{A}(\Sigma)$. The action of G on it is strongly transitively by the construction but is not necessarily type-preserving since the affine Weyl group W of $\mathscr{A}(\Sigma)$ is usually not the entire $\widehat{W}$. The subgroup of type-preserving automorphisms is then the group $G' = \nu^{-1}(W)$ introduced in 3.1.10.

3.2.9. The *bornology defined by* φ is

3.2.10 Example. Continue Example 3.2.7.

3.3 Concave functions

One important ingredient in Bruhat-Tits theory is the subgroups associated to concave functions. They are refinements of parabolic subgroups.

3.3.1 Definition ([BT72, 6.4.3])**.** Let Φ be a root system and denote $\widetilde{\Phi} = \Phi \cup \{0\}$. A *concave function on* $\widetilde{\Phi}$ is a function $f\colon \widetilde{\Phi} \to \widetilde{\mathbb{R}}$ such that

C. for any finite family (a_i) in $\widetilde{\Phi}$ such that $\sum_i a_i \in \widetilde{\Phi}$, we have

$$\sum_i f(a_i) \geqslant f(\sum_i a_i).$$

Here $\widetilde{\mathbb{R}}$ is *the ordered monoid of extended real numbers*. Formally, $\widetilde{\mathbb{R}}$ is the union of

$$\mathbb{R}, \qquad \mathbb{R}+ := \{k+ \mid k \in \mathbb{R}\} \qquad \text{and} \qquad \{\infty\}$$

The commutative addition on $\mathbb{R}$ is extended to $\widetilde{\mathbb{R}}$ as follows:

- for all $k, l \in \mathbb{R}$, $k + (l+) = (k+) + (l+) = (k + l)+$;
- for all $\lambda \in \widetilde{\mathbb{R}}$, $\lambda + \infty = \infty$.

The total order on $\mathbb{R}$ is extended to $\widetilde{\mathbb{R}}$ as follows:

- for all $k, l \in \mathbb{R}$ such that $k < l$, $k < k+ < l$;
- for all $\lambda \in \widetilde{\mathbb{R}}$ such that $\lambda \neq \infty$, $\lambda < \infty$.

Note that the axiom is equivalent to the followings:

C1. for any roots $a, b \in \Phi$ such that $a + b \in \Phi$, we have $f(a) + f(b) \geqslant f(a + b)$;

C2. for any root $a \in \Phi$, we have $f(a) + f(-a) \geqslant f(0)$;

C3. $f(0) \geqslant 0$.

A concave function f on $\widetilde{\Phi}$ is said to be a *concave function on* Φ if $f(0) = 0$.

3.3.2. Let f be a concave function on Φ. Let U_f denote the subgroup generated by $U_{a,f(a)}$ for all $a \in \Phi$.

3.3.3. Let Ω be a nonempty subset of $\mathbb{A}$ and let U_Ω denote the subgroup generated by U_α for all affine roots $\alpha \supseteq \Omega$. Define $f_\Omega \colon \Phi \to \mathbb{R} \cup \{\infty\}$ by

$$f_\Omega(a) = \inf\{k \in \mathbb{R} \mid \Omega \subseteq \alpha_{a+k}\}.$$

Then f_Ω defines a concave function $f_\Omega \colon \widetilde{\Phi} \to \widetilde{\mathbb{R}}$ by putting $f_\Omega(0) = 0$.

For f a concave function on Φ, denote U_f the subgroup generated by $U_{a,f(a)}$ for all $a \in \Phi$ and let $P_f = T^\circ.U_f$, $N_f = N \cap U_f$ and $\Phi_f = \{a \in \Phi \mid f(a) + f(-a) = 0\}$. In particular, for $f = f_\Omega$, U_f coincides with U_Ω and we will denote P_f and Φ_f by P_Ω and Φ_Ω.

Then denote

$$U_f^+ = U^+ \cap U_f, \qquad U_f^- = U^- \cap U_f, \qquad U_{f,a} = U_a \cap U_f,$$
$$T_f^\circ = T^\circ \cap U_f, \qquad N_f = N \cap U_f.$$

Then we have the following facts [BT72, 6.4.9].

1. For each $a \in \Phi$, $U_{f,a} = U_{a,f(a)}$.
2. The homomorphism $\prod_{a \in \Phi^\pm} U_{f,a} \to U_f^\pm$ is bijective regardless of the order of factors.
3. We have $U_f = U_f^-.N_f.U_f^+$.

3.4 Bruhat-Tits buildings

3.4.1. In the rest, K will be a field equipped with a (trivial or discrete) valuation $\mathrm{val}\colon K \to \Gamma \cup \{\infty\}$. We fix the following associated notations.

$$\begin{aligned} K^\circ &:= \{x \in K \mid \mathrm{val}(x) \geqslant 0\}, \\ (K^\circ)^\times &:= \{x \in K \mid \mathrm{val}(x) = 0\}, \\ K^{\circ\circ} &:= \{x \in K \mid \mathrm{val}(x) > 0\}, \\ \kappa &:= K^\circ / K^{\circ\circ}. \end{aligned}$$

We further assume K is complete with respect to $\mathrm{val}(\,\cdot\,)$ and κ is a finite field with cardinality q and characteristic p.

3.5 Moy-Prasad filtrations

3.5.1. Let f be a concave function. Define $f^*\colon \Phi \to \widetilde{\mathbb{R}}$ as follows:

$$f^*(a) = \begin{cases} f(a)+ & \text{if } a \in \Phi_f, \\ f(a) & \text{if } a \notin \Phi_f. \end{cases}$$

Here $\widetilde{\mathbb{R}}$ is the ordered monoid of extended real numbers[10] and for each $k \in \mathbb{R}$, $k+$ is the smallest extended real number larger than k. Then f^* is a concave function in the sense of 3.3.1 with $\mathbb{R} \cup \{\infty\}$ replaced by $\widetilde{\mathbb{R}}$ [BT72, 6.4.23]. For each $a \in \Phi$ and any $u \in U_{-a,f(-a)}, v \in U_{a,f^*(a)}$ Let T_{f,f^*} denote the subgroup of T° generated by

§ 4 Buildings for classical groups

4.1

4.2 Norms and buildings

The Bruhat-Tits building associated to a classical group has a concrete interpretation.

4.2.1. Let V be a K-vector space. A *norm* (defined over K) on V is a map $\alpha\colon V \to \mathbb{R} \cup \{\infty\}$ such that for any $x, y \in V$ and $t \in K$,

1. $\alpha(tx) = \mathrm{val}(t) + \alpha(x)$;
2. $\alpha(x + y) \geqslant \inf\{\alpha(x), \alpha(y)\}$;
3. $\alpha(x) = \infty$ if and only if $x = 0$.

[10] Formally, $\widetilde{\mathbb{R}}$ is the union of $\mathbb{R}$, $\mathbb{R}+ := \{k+ \mid k \in \mathbb{R}\}$ and $\{\infty\}$. The commutative addition on $\mathbb{R}$ is extended as follows: $k + (l+) = (k+) + (l+) = (k + l)+$ for all $k, l \in \mathbb{R}$ and $\lambda + \infty = \infty$ for all $\lambda \in \widetilde{\mathbb{R}}$. The total order on $\mathbb{R}$ is extended as follows: $k < k+ < l$ for all $k, l \in \mathbb{R}$ such that $k < l$ and $\lambda < \infty$ for all $\lambda \neq \infty$.

The set of norms on V is denoted by $\mathscr{N}(K,V)$.

If α is a norm, then so is $\alpha + c$ for any $c \in \mathbb{R}$. Such a norm is said to be *homothetic* to α. The set of homothetic classes of norms on V is denoted by $\mathscr{X}(K,V)$.

Let α be a norm and g be an automorphism of V. Then $\alpha \circ g^{-1}$ is also a norm, denoted by $g.\alpha$. In such a way, $\mathrm{GL}(V)$ acts on $\mathscr{N}(K,V)$. Moreover, this action respects homotheties, making $\mathscr{X}(K,V)$ a $\mathrm{GL}(V)$-set.

A family $\mathscr{W}$ of subspaces of V is said to be *splitting* for α if V admits a decomposition $V = \bigoplus_{W\in\mathscr{W}} W$ respecting α in the sense that for any tuple $(x_W)_{W\in\mathscr{W}}$ with $x_W \in W$, we have

$$\alpha(\sum_{W\in\mathscr{W}} x_w) = \inf_{W\in\mathscr{W}} \alpha(x_W).$$

A *frame* in V is a family $\mathscr{L}$ of lines (i.e. one-dimensional subspaces) such that V admits a decomposition $V = \bigoplus_{L\in\mathscr{L}} L$. Given a frame $\mathscr{L}$, the set $\{\alpha \in \mathscr{N}(K,V) \mid \mathscr{L} \text{ is splitting for } \alpha\}$ is invariant under homotheties and its homothetic quotient is denoted by $\mathscr{A}(\mathscr{L})$.

4.2.2 Theorem. *Let V be a K-vector space. Then the action of $\mathrm{GL}(V)$ on $\mathscr{X}(K,V)$ gives rise to a Bruhat-Tits building strcuture with each apartment being $\mathscr{A}(\mathscr{L})$ for some frame $\mathscr{L}$. Moreover, $\mathscr{X}(K,V) \cong \mathscr{B}(\mathsf{GL}(V))$.*

1. Let $\mathsf{N}(\mathscr{L})$ be the stabilizer of $\mathscr{L}$, that is the algebraic subgroup of $\mathsf{GL}(V)$ representing the functor

   $$R \rightsquigarrow \big\{g \in \mathrm{GL}(V_R) \,\big|\, \forall L \in \mathscr{L}, \exists L' \in \mathscr{L} : g.L \subseteq L'\big\}.$$

   Then its K-points $N(\mathscr{L})$ is the stabilizer of $\mathscr{A}(\mathscr{L})$.

2. Any element of $N(\mathscr{L})$ permutes $\mathscr{L}$. We thus get a surjection $N(\mathscr{L}) \twoheadrightarrow \mathfrak{S}_n$. Let D_n denote its kernel. Then D_n is precisely those automorphisms diagonalized by $\mathscr{L}$. Hence $D_n \cong \mathbb{R}^n$.

3. Therefore $\mathscr{A}(\mathscr{L})$ is the apartment $\mathscr{A}(D_n)$.

§ 5 Classical groups

5.0.1 Example. (a) The functor $R \rightsquigarrow \left\{A \in \mathsf{GL}_{2n}(R) \,\middle|\, {}^tAJ_{2n}A = J_{2n}\right\}$, where $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and I_n is the identity matrix, defines an algebraic subgroup Sp_{2n} of GL_{2n}, called the *symplectic group.*

(b) Let V be a finite-dimensional vector space over K and $\mathfrak{b}$ be a *symplectic form* on it, then the functor $R \rightsquigarrow \mathrm{Aut}(V_R, \mathfrak{b})$ mapping a K-algebra R to the group of R-automorphisms of V_R leaving the form $\mathfrak{b}$ invariant defines an algebraic subgroup $\mathsf{Sp}(V)$ of $\mathsf{GL}(V)$, called the *symplectic group of* V.

(c) The functor $R \rightsquigarrow \left\{A \in \mathsf{GL}_{2n}(R) \,\middle|\, {}^tAJ_{2n}A = \lambda J_{2n} \text{ for some } \lambda \in R^\times\right\}$ defines an algebraic subgroup GSp_{2n} of GL_{2n}, called the *symplectic similitude group.*

(d) Let V be a finite-dimensional vector space over K and $\mathfrak{b}$ be a *symplectic form* on it, then the functor $R \rightsquigarrow \{\varphi \in \mathsf{GL}(V, R) \mid \mathfrak{b} \circ \varphi = \lambda\mathfrak{b} \text{ for some } \lambda \in R^\times\}$ defines an algebraic subgroup $\mathsf{GSp}(V)$ of $\mathsf{GL}(V)$, called the *symplectic similitude group of* V.

(e) Let V be a finite-dimensional vector space over K and $\mathfrak{q}$ be a *quadratic form* on it, then the functor $R \rightsquigarrow \mathrm{Aut}(V_R, \mathfrak{q})$ mapping a K-algebra R to the group of R-automorphisms of V_R leaving the form $\mathfrak{q}$ invariant defines an algebraic subgroup $\mathsf{O}(V, \mathfrak{q})$ of $\mathsf{GL}(V)$, called the *orthogonal group of* $(V, \mathfrak{q})$.

(f) The neutral component of $\mathsf{O}(V, \mathfrak{q})$ is denoted by $\mathsf{SO}(V, \mathfrak{q})$ and called the *special orthogonal group of* $(V, \mathfrak{q})$. When the characteristic of K is not 2, it equals the intersection of $\mathsf{O}(V, \mathfrak{q})$ and $\mathsf{SL}(V)$.

(g) Let $\mathfrak{q}$ be the quadratic form

$$(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) \longmapsto x_1y_1 + x_2y_2 + \cdots + x_ny_n,$$

then the algebraic group $\mathsf{O}(K^{2n}, \mathfrak{q})$ (resp. $\mathsf{SO}(K^{2n}, \mathfrak{q})$) is simply denoted by O_{2n} (resp. SO_{2n}) and called the *orthogonal group* (resp. *special orthogonal group*).

(h) Let $\mathfrak{q}$ be the quadratic form

$$(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n, z) \longmapsto x_1y_1 + x_2y_2 + \cdots + x_ny_n + z^2,$$

then the algebraic group $\mathsf{O}(K^{2n+1}, \mathfrak{q})$ (resp. $\mathsf{SO}(K^{2n+1}, \mathfrak{q})$) is simply denoted by O_{2n+1} (resp. SO_{2n+1}) and called the *orthogonal group* (resp. *special orthogonal group*).

(i) Let V be a finite-dimensional vector space over K and $\mathfrak{q}$ be a *quadratic form* on it, then the functor $R \rightsquigarrow \{\varphi \in \mathsf{GL}(V) \mid \mathfrak{q} \circ \varphi = \lambda\mathfrak{q} \text{ for some } \lambda \in R^\times\}$ defines an algebraic subgroup $\mathsf{GO}(V, \mathfrak{q})$ of $\mathsf{GL}(V)$, called the *orthogonal similitude group of* $(V, \mathfrak{q})$.

(j) Let $\mathfrak{q}$ be the quadratic form

$$(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) \longmapsto x_1y_1 + x_2y_2 + \cdots + x_ny_n,$$

then the algebraic group $\mathsf{GO}\big(K^{2n}, \mathfrak{q}\big)$ is simply denoted by GO_{2n} and called the *orthogonal similitude group*.

(k) Let $\mathfrak{q}$ be the quadratic form

$$(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n, z) \longmapsto x_1y_1 + x_2y_2 + \cdots + x_ny_n + z^2,$$

then the algebraic group $\mathsf{GO}\big(K^{2n+1}, \mathfrak{q}\big)$ is simply denoted by GO_{2n+1} and called the *orthogonal similitude group*.

5.0.2 Example. In Example 2.1.3,

(a) SL_n, $\mathsf{SL}(V)$, Sp_{2n}, $\mathsf{Sp}(V)$, SO_{2n}, SO_{2n+1} and $\mathsf{SO}(V, \mathfrak{q})$ are semisimple;

(b) GL_n, $\mathsf{GL}(V)$, GSp_{2n}, $\mathsf{GSp}(V)$, GO°_{2n}, GO°_{2n+1} and $\mathsf{GO}^\circ(V, \mathfrak{q})$ are reductive but not semisimple.

(c) Any torus is reductive. Conversely, if G is a solvable reductive group, then since $\mathscr{R}_u(\mathsf{G}_{K^{\mathrm{a}}})$ is trivial, it is a torus by [Mil17, 16.33].

The algebraic groups in (a) and (b) are called *classical groups*. We refer [Gro02] for a treatment of them.

5.0.3 Example. Let $\mathsf{G} = \mathsf{GSp}_{2n}$. Then its deconstruction is

$$\begin{tikzcd}
\mu_2 \arrow[r, hook] \arrow[d, hook] & \mathsf{Sp}_{2n} \arrow[d, hook] \arrow[dr, bend left, two heads] & \\
\mathbb{G}_{\mathrm{m}} \arrow[r, "\lambda\mapsto\lambda I_n"] \arrow[dr, bend right, two heads, "\lambda\mapsto\lambda^2"'] & \mathsf{GSp}_{2n} \arrow[r, two heads] \arrow[d, two heads, "\text{ratio}"] & \mathsf{PSp}_{2n} \\
& \mathbb{G}_{\mathrm{m}} &
\end{tikzcd}$$

where ration maps a matrix $A \in \mathsf{GSp}_{2n}$ to the number $\lambda \in \mathbb{G}_{\mathrm{m}}$ such that ${}^tAJ_{2n}A = \lambda J_{2n}$ and PSp_{2n} is called the *projective symplectic group*.

Conversely, GSp_{2n} can be recovered from the triple $(\mathsf{Sp}_{2n}, \mathbb{G}_{\mathrm{m}}, \mu_2 \hookrightarrow \mathbb{G}_{\mathrm{m}})$.

Similar conclusion applies to $\mathsf{GSp}(V)$.

5.0.4 Example. Let $\mathsf{G} = \mathsf{GSp}_{2n}$. Then its deconstruction is

$$\begin{tikzcd}
\mu_2 \arrow[r, hook] \arrow[d, hook] & \mathsf{Sp}_{2n} \arrow[d, hook] \arrow[dr, bend left, two heads] & \\
\mathbb{G}_{\mathrm{m}} \arrow[r, "\lambda\mapsto\lambda I_n"] \arrow[dr, bend right, two heads, "\lambda\mapsto\lambda^2"'] & \mathsf{GSp}_{2n} \arrow[r, two heads] \arrow[d, two heads, "\text{ratio}"] & \mathsf{PSp}_{2n} \\
& \mathbb{G}_{\mathrm{m}} &
\end{tikzcd}$$

where ration maps a matrix $A \in \mathsf{GSp}_{2n}$ to the number $\lambda \in \mathbb{G}_\mathrm{m}$ such that ${}^tAJ_{2n}A = \lambda J_{2n}$ and PSp_{2n} is called the *projective symplectic group*.

Conversely, GSp_{2n} can be recovered from the triple $(\mathsf{Sp}_{2n}, \mathbb{G}_\mathrm{m}, \mu_2 \hookrightarrow \mathbb{G}_\mathrm{m})$.

Similar conclusion applies to $\mathsf{GSp}(V)$.

5.0.5 Example. In Example 5.0.2(b),

(a) GL_n (and hence $\mathsf{GL}(V)$ and their derived groups SL_n and $\mathsf{SL}(V)$) is splittable and D_n is a split maximal torus in it. Its rank is n and its semisimple rank is $n-1$.

(b) GSp_{2n} (and hence $\mathsf{GSp}(V)$ and their derived groups Sp_{2n} and $\mathsf{Sp}(V)$) is splittable and $\mathsf{D}_{2n} \cap \mathsf{GSp}_{2n}$ is a split maximal torus in it. Its rank is $n+1$ and its semisimple rank is n.

5.0.6 Example. In Example 5.0.2(a),

(a) SL_n, $\mathsf{SL}(V)$, Sp_{2n} and $\mathsf{Sp}(V)$ are simply-connected;

(b) SO_{2n}, SO_{2n+1} and $\mathsf{SO}(V, \mathfrak{q})$ are not simply-connected and their fundamental groups are isomorphic to $\mathbb{Z}/2$.

(i) (*Lang's theorem* [Mil17, 17.98]) If G is a smooth connected algebraic group over a finite field K, then $\mathrm{H}^1(K, \mathsf{G})$ vanishes.

(ii) [Mil17, 3.46] $\mathrm{H}^1(K, \mathsf{G})$ vanishes for all $\mathsf{G} = \mathsf{GL}_n, \mathsf{SL}_n, \mathsf{Sp}_{2n}$.

(iii) [Mil17, 25.61; BT87b, 4.3] If G is a simply-connected semisimple group over a local field K, then $\mathrm{H}^1(K, \mathsf{G})$ vanishes.

Index of Notations

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