Stable lattices in p-adic isometric representations

Xu Gao

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Let G be a group and

$$\rho\colon G \longrightarrow \mathrm{GL}(V)$$

be a representation of G in a finite dimensional vector space V over a non-Archimedean local field K. Let K° be the valuation ring of K. A *lattice* in V is a finitely generated K°-submodule L generating V as a K-vector space. A lattice L is *stable under ρ (or G)* if $\rho(g)L = L$ for all $g \in G$. Such a lattice exists when ρ is *precompact*, that is, the image of ρ has compact closure in $\mathrm{GL}(V)$ with its metric topology. This condition is satisfied if G is finite, or more generally, if G is profinite and ρ is continuous. We keep this assumption from now on.

Then the image of ρ is contained in the subgroup of $\mathrm{GL}(V)$ consisting of automorphisms whose determinant is a unit. Then ρ stabilizes a lattice L if and only if ρ stabilizes its *homothety class*, that is the class of lattices L' different from L by a *homothety*, namely $L' = \lambda L$ for some $\lambda \in K^\times$.

It is then a natural question to count the stable lattices under such a representation. The Jordan-Zassenhaus theorem[1] stated in [Suh21] asserts that there are only finitely many stable lattices up to homotheties if and only if ρ is irreducible. The cardinality $h(\rho)$ of the set $S(\rho)_0$ of homothety classes of stable lattices is then of interesting and is called the *class number* of ρ. In [Suh21], Suh studied the set $S(\rho)_0$ in a geometric way using the Bruhat-Tits building of $\mathsf{SL}(V)$ and give a concrete description of the growth of class number under totally ramified extensions. In that work, the Bruhat-Tits building of $\mathsf{SL}(V)$ plays an important role since $S(\rho)_0$ is naturally the set of vertices of a simplicial subset $S(\rho)$ in the building and the simplicial structure of the building behaves very well under totally ramified extensions.

The purpose of this draft is to extend the story to isometric representations. An *isometric representation* is a (continuous) group homomorphism

$$\rho\colon G \longrightarrow \mathrm{Aut}(V, \mathfrak{b})$$

[1] The classical Jordan-Zassenhaus theorem is about *isomorphism* classes, not *homothety* classes. Suh named it as such because of the similarity in the ideas involved.

where V is a finite dimensional K-vector space and $\mathfrak{b}$ is a non-degenerate bilinear form on it. Such a representation is *split* if the algebraic group $\mathsf{Aut}\,(V, \mathfrak{b})$ is K-split. More precisely, $\mathfrak{b}$ is one of the following:

- $\mathfrak{b}$ is skew-symmetric, then $\mathsf{Aut}\,(V, \mathfrak{b})$ is the *symplectic group* $\mathsf{Sp}\,(V)$ and such a representation is called a *symplectic representation*;

- $\mathfrak{b}$ is symmetric and has largest possible Witt index, then $\mathsf{Aut}\,(V, \mathfrak{b})$ is the *orthogonal group* $\mathsf{O}\,(V)$ and such a representation is called a *orthogonal representation*.

For this purpose, it is necessary to understand the Bruhat-Tits buildings of split classical groups. We will review the theory of Bruhat-Tits buildings in Section 1.

References

[Suh21] J. Suh, *Stable lattices in p-adic representations I. Regular reduction and Schur algebra*, J. Algebra **575** (2021), 192–219.