Picard Groups

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§ I Picard groups of locally ringed spaces

I.1 Definition. Let $(X, \mathscr{O}_X)$ be a locally ringed space. A sheaf $\mathscr{L}$ is **invertible** if there is a sheaf $\mathscr{L}'$ such that $\mathscr{L} \otimes \mathscr{L}' \cong \mathscr{O}_X$ in the category of $\mathscr{O}_X$-modules. The group of isomorphism classes of invertible sheaves is called the **Picard group** of X, denoted by $\mathsf{Pic}(X)$.

I.2 Theorem (St:09NT)**.** *There is a canonical isomorphism of abelian groups*

$$H^1(X, \mathscr{O}_X^\times) \cong \mathsf{Pic}(X).$$

Here $\mathscr{O}_X^\times$ denotes the sheaf of units in $\mathscr{O}_X$.

Hint: Give a bijection between invertible sheaves and $\mathscr{O}_X^\times$-tosors. Show that for any abelian sheaf $\mathscr{F}$, $H^1(X, \mathscr{F})$ classifies $\mathscr{F}$-torsors. □

§ II Picard groups of integral schemes

II.1 Definition. Let X be an integral Noetherian scheme. Let $\mathscr{K}$ be the sheaf of rational functions. Then a **Cartier divisor** is a global section of the quotient sheaf $\mathscr{K}^\times / \mathscr{O}_X^\times$. A Cartier divisor is **principal** if it is in the image of

$$\Gamma(X, \mathscr{K}^\times) \longrightarrow \Gamma(X, \mathscr{K}^\times / \mathscr{O}_X^\times).$$

Two Cartier divisors are **linearly equivalent** if their difference is principal.

Exercise 1. Show that, the group $H^1(X, \mathscr{O}_X^\times)$ classifies Cartier divisors up to linearly equivalence.

II.2 Definition. A **fractional ideal sheaf** is a $\mathscr{O}_X$-submodule of $\mathscr{K}$. A fractional ideal sheaf is **invertible** if it is an invertible $\mathscr{O}_X$-module.

Exercise 2. Show that any Cartier divisor D defines an invertible fractional ideal sheaf $\mathscr{I}(D)$ and vice versa. (*Hint: prove this locally.*)

II.3 Definition. A Cartier divisor D is **effective** if it is in $\Gamma(X, (\mathscr{K}^\times \cap \mathscr{O}_X)/\mathscr{O}_X^\times)$. In this case, we denote $D \geqslant 0$. For two Cartier divisors D and D', we write $D \geqslant D'$ if $D - D' \geqslant 0$.

Exercise 3. Show that: $D \geqslant D'$ if and only if $\mathscr{I}(D) \subseteq \mathscr{I}(D')$

II.4 Definition. For D a Cartier divisor, let $\mathscr{O}_X(D)$ denote the inverse of $\mathscr{I}(X)$.

Exercise 4. Show that, $D \mapsto \mathscr{O}_X(D)$ induces an isomorphism from the class group of Cartier divisors to $\mathsf{Pic}(X)$.

Remark. This gives a proof of Theorem I.2 for integral Noetherian schemes.

II.5 Definition. Let D be a Cartier divisor, then its **support** is

$$\operatorname{supp}(D) := \{x \in X \mid D_x \neq 1\}.$$

Here $D_x \in (\mathscr{K}^\times/\mathscr{O}_X^\times)_x$ is the germ of D at the x.

II.6 Example. If D is an effective Cartier divisor, then $\mathscr{I}(D)$ is an ideal sheaf and $\operatorname{supp}(D)$ is the closed subscheme defined by $\mathscr{I}(D)$. In this case, we also denote $\operatorname{supp}(D)$ by D and we have short exact sequence

$$0 \longrightarrow \mathscr{O}_X(-D) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_D \longrightarrow 0.$$

Exercise 5. Show that the codimension of $\operatorname{supp}(D)$ is at least 1. (*Hint: what if* $\dim(\mathscr{O}_{X,x}) = 0$ *at some* $x \in \operatorname{supp}(D)$?)

§ III Picard groups of normal schemes

III.1 Definition. Let X be a normal integral Noetherian scheme. A **prime** divisor is an irreducible closed subscheme of codimension 1. A **Weil divisor** is an element of the free abelian group $\mathsf{Div}(X)$ generated by prime divisors. A Weil divisor is **effective** if all its coefficients are non-negative.

III.2 Definition. Let C be an irreducible closed subscheme of X with generic point η. Denote the local ring $\mathscr{O}_{X,\eta}$ by $\mathscr{O}_{X,C}$. Then $\operatorname{codim}(C) = \dim(\mathscr{O}_{X,C})$. In the case of C being a prime divisor, $\mathscr{O}_{X,C}$ is a discrete valuation ring. Denote its valuation by ord_C. For any section f of $\mathscr{K}^\times$, define $\operatorname{ord}_C(f)$ as $\operatorname{ord}_C(f_\eta)$, where f_η is the germ of f at η. In particular, any $f \in \Gamma(X, \mathscr{K}^\times)$ defines a Weil divisor

$$\operatorname{div}(f) := \sum_C \operatorname{ord}_C(f)[C].$$

Such kind of Weil divisor is called **principal**. The **(Weil) divisor class group** $\mathsf{Cl}(X)$ is the quotient of $\mathsf{Div}(X)$ by principal Weil divisors.

Exercise 6. Locally, a Cartier divisor D is presented by a section f of $\mathscr{K}^\times$ on some open U, hence $\operatorname{ord}_C(f)$ is defined if $U \cap C \neq \varnothing$. Let $\operatorname{ord}_C(D)$ be $\operatorname{ord}_C(f)$. Show that it does not depend on the choice of (U, f) and furthermore we get a Weil divisor

$$\sum_C \operatorname{ord}_C(D)[C].$$

Exercise 7. Let $\sum_C n_C[C]$ be a Weil divisor, defines a sheaf by letting its sections on U the sections f of $\mathscr{K}^\times$ such that $\mathrm{ord}_C(f) \geqslant -n_C$ for any $C \cap U \neq \varnothing$. Show that, this defines an invertible fractional ideal sheaf.

Exercise 8. Show that, the above constructions induce isomorphisms between the groups $\mathsf{Pic}(X)$ and $\mathsf{Cl}(X)$.

§ IV The exact sequence

Exercise 9. Let X be a onormal integral Noetherian scheme. Let $j\colon U \hookrightarrow X$ be a dense open subscheme and $Z = X \setminus U$.

1. Show that the morphism $\mathscr{O}_X^\times \to j_* j^{-1} \mathscr{O}_X^\times$ is a monomoprhism. (*Hint: look at stalks.*)
2. Show that the cokernel of that morphism is the direct sum of skyscraper sheaves
$$\bigoplus_{C \subseteq Z} \iota_{\eta *}(\mathscr{K}_\eta^\times / \mathscr{O}_{X,\eta}^\times),$$
where C varies through prime divisors contained in Z, η is its generic point and $\iota_\eta\colon \eta \hookrightarrow X$ is the inclusion.
3. Conclude that there is an exact sequence.
$$1 \longrightarrow \mathscr{O}_X(X)^\times \longrightarrow \mathscr{O}_X(U)^\times \longrightarrow \bigoplus_{C \subseteq Z} \mathscr{K}_\eta^\times / \mathscr{O}_{X,\eta}^\times \longrightarrow \mathsf{Pic}(X) \longrightarrow \mathsf{Pic}(U) \longrightarrow 1.$$

Exercise 10. Let X be a one-dimensional integral Noetherian scheme. Let $\pi\colon \widetilde{X} \to X$ be its normalization.

1. Show that the cokernel of the canonical morphism $\pi^\flat\colon \mathscr{O}_X^\times \to \pi_* \mathscr{O}_{\widetilde{X}}^\times$ is the direct sum of skyscraper sheaves
$$\bigoplus_s \iota_{s*}\left(\widetilde{\mathscr{O}}_{X,s}^\times / \mathscr{O}_{X,s}^\times\right),$$
where s varies through singular points of X, $\iota_s\colon s \hookrightarrow X$ is the inclusion and $\widetilde{\mathscr{O}}_{X,s}$ is the integral closure of $\mathscr{O}_{X,s}$ in its fraction field.
2. Conclude that there is an exact sequence.
$$1 \longrightarrow \mathscr{O}_X(X)^\times \longrightarrow \mathscr{O}_{\widetilde{X}}(\widetilde{X})^\times \longrightarrow \bigoplus_s \widetilde{\mathscr{O}}_{X,s}^\times / \mathscr{O}_{X,s}^\times \longrightarrow \mathsf{Pic}(X) \longrightarrow \mathsf{Pic}(U) \longrightarrow 1.$$