Tensor products

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Abstract

This note summarize some concepts and properties related to tensor products.

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§1 Tensor products of abelian groups

I Bilinear maps and tensor products

A **binary morphism** in a category is an arrow whose source is a pair of objects and the target is an object. This concept is very general but vague. In particular, one may define it more specific, making it related to the morphisms.

In the category **Ab** of abelian groups, a binary morphism is defined to be a **bilinear map**. That is a binary function f from the underlying sets of two abelian groups A and B and target to an abelian group C such that by fixing any of the two inputs, for example $b \in B$, the binary function becomes a group homomorphism $f(-, b)\colon A \to C$.

Remark. Since a binary function from two sets can be represented by a function from their cartesian product, it is common to write a bilinear map from A and B to C as $A \times B \to C$. However, this notation brings some confusion since it is NOT a homomorphism from the product $A \times B$ to C. For this reason, we denote the bilinear map by $A, B \to C$ instead.

Among all bilinear maps from two abelian groups A and B, there exists a universal one, called the **tensor product** of A and B and denoted by $A \otimes B$. In other words, the tensor product $A \otimes B$, together with the bilinear map $\otimes\colon A, B \to A \otimes B$, has the following universal property:

> if $A, B \to C$ is a bilinear map of abelian groups, then there exists a unique homomorphism $A \otimes B \to C$ such that this bilinear map can be factored as $A, B \xrightarrow{\otimes} A \otimes B \to C$.

Or, we can represent this universal property by Hom,

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Bil}(A, B; C)$$

where $\mathrm{Bil}(A, B; C)$ denote the set of bilinear maps from A and B to C.

Thus the through the tensor products, bilinear maps can be represented as homomorphisms in **Ab**. That may be the best reason to introduce this notion.

Note that, the tensor products are NOT the products or coproducts in **Ab** since if they are, the bilinear maps must also be homomorphisms which is not true.

Constructions of tensor products

To see more explicit, we construct the tensor product $A \otimes B$ as follow.

We start from the fact that by forgetting abelian group structures, the bilinear map $A, B \xrightarrow{\otimes} A \otimes B$ is just a function of the underlying sets[1]

$$|A| \times |B| \xrightarrow{\otimes} |A \otimes B|.$$

Therefore, this function must has a factorization

$$|A| \times |B| \xrightarrow{i} |F(|A| \times |B|)| \xrightarrow{p} |A \otimes B|,$$

where $F(|A| \times |B|)$ is the free abelian group generated by $|A| \times |B|$ while p is the unique group homomorphism admitted by the universal property of $F(|A| \times |B|)$.

Since we expect $A \otimes B$ to be universal among all bilinear maps, thus p must be surjective. Otherwise, p can factor through the image and one can verify that

Proposition 1.1. *If $A, B \to C$ is a bilinear map which can factor as a function $|A| \times |B| \to D$ followed by a monomorphism $D \to C$. Then $A, B \to D$ is also a bilinear map.*

Therefore, $A \otimes B \cong F(|A| \times |B|)/\ker p$

Note that the composite $p \circ i$ is a bilinear map if and only if

$$p \circ i(a + a', b) = p \circ i(a, b) + p \circ i(a', b), \forall a, a' \in A, b \in B,$$
$$p \circ i(a, b + b') = p \circ i(a, b) + p \circ i(a, b'), \forall a \in A, b, b' \in B.$$

Let N be the subgroup of $F(|A| \times |B|)$ generated by the union of

$$\left\{ i(a + a', b) - i(a, b) - i(a', b) \,\middle|\, a, a' \in A, b \in B \right\}$$

[1]Here we denote the underlying set of an abelian group A by $|A|$ for distinguish.

and

$$\left\{ i(a, b+b') - i(a, b) - i(a, b') \,\middle|\, a \in A, b, b' \in B \right\}.$$

Then the composite $p \circ i$ is a bilinear map if and only if $N \subset \ker p$. Since we except $A \otimes B$ to be universal, this containing should be also universal. That means $\ker p$ must be the smallest one among all such kernels constructed from bilinear maps. Indeed, it is N.

Thus we conclude: $A \otimes B \cong F(|A| \times |B|)/N$.

The elements in $A \otimes B$ are usually called **tensors**. Among them, the image of $(a, b) \in A, B$ in $A \otimes B$ is called an **elementary tensor** or **pure tensor** and denoted by $a \otimes b$.

Note that, by the universal property, $A \otimes B$ is naturally isomorphic to $B \otimes A$. This natural isomorphism is called the **symmetric braiding** and denoted by $\gamma_{A,B}$.

Naturality of tensor products

We now go further, to show that the tensor products actually provide a binary functor:

$$\otimes \colon \mathbf{Ab} \times \mathbf{Ab} \longrightarrow \mathbf{Ab}.$$

Given an abelian group A, it is not difficult to see that $X \mapsto A \otimes X$ and $X \mapsto X \otimes A$ are functors. We denote the corresponding homomorphisms of $f \colon B \to C$ under those functors by $A \otimes f$ and $f \otimes A$ respectively. In the case of not confusing, we will omit the identity and simply identify them with f.

Moreover, we have

Proposition 1.2. *Let $f \colon A \to B$ and $g \colon C \to D$ be two homomorphisms, then the following diagram commutes.*

$$\begin{tikzcd}
A \otimes C \arrow[r] \arrow[d] & B \otimes C \arrow[d] \\
A \otimes D \arrow[r] & B \otimes D
\end{tikzcd}$$

The result homomorphism $A \otimes C \to B \otimes D$ is usually denoted by $f \otimes g$.

Proof. Consider an arbitrary bilinear map $h \colon B, D \to T$, then one can see $h(f(-), g(-)) \colon A, C \to T$ is a bilinear map. Thus the unique homomorphism

$A \otimes C \to T$ factors through a unique homomorphism $A \otimes C \to B \otimes D$. However, one can verify that both of the composites in the above square satisfy this property. Thus they must be the same homomorphism. $\square$

Remark. By the naturality of tensor products, we can define the tensor product of natural transformations pointwise.

In **Ab**, the group of integers $\mathbb{Z}$ play a special role in the theory of tensor products. Indeed, it serves like a unit of multiplication.

Theorem 1.3. *For any abelian group A, we have two natural isomorphisms:*

$$\mathbb{Z} \otimes A \cong A, \qquad A \otimes \mathbb{Z} \cong A.$$

Proof. Since we already have a natural isomorphism $\mathbb{Z} \otimes A \cong A \otimes \mathbb{Z}$, we only need to prove the first one.

Let T be an arbitrary abelian group, by Yoneda lemma, we only need to show there exists a natural (on T) bijection:

$$\mathrm{Bil}(\mathbb{Z}, A; T) \cong \mathrm{Hom}(A, T).$$

Let $f\colon \mathbb{Z}, A \to T$ be a bilinear map, then $f(1, -)\colon A \to T$ is a homomorphism. Conversely, If $g\colon A \to T$ is a homomorphism, then we can define a bilinear map $\tilde{g}\colon \mathbb{Z}, A \to T$ by $\tilde{g}(n, a) := g(na)$, where na denote the sum of n terms of $a \in A$.

It is not difficult to verify that $f \mapsto f(1, -)$ and $g \mapsto \tilde{g}$ are inverse for each, thus provide bijections as desired. Moreover, the naturality of those bijections are obvious. $\square$

Remark. The natural isomorphisms $\lambda_A\colon \mathbb{Z} \otimes A \to A$ and $\rho_A\colon A \otimes \mathbb{Z} \to A$ are called the **left unitor** and **right unitor**.

II Basic properties of tensor products

A distribution law

We prove a distribution law.

Theorem 1.4. *Let A be an abelian group and $\{B_i\}_{i\in I}$ a set of abelian groups. Then we have the following natural isomorphisms*

$$A \bigotimes \left(\bigoplus_{i\in I} B_i\right) \cong \bigoplus_{i\in I}\left(A \bigotimes B_i\right).$$

Proof. Let T be an arbitrary abelian group, by Yoneda Lemma, we only need to show there exists a natural (on T) bijection

$$\mathrm{Bil}(A, \bigoplus_{i\in I} B_i; T) \cong \prod_{i\in I} \mathrm{Bil}(A, B_i; T).$$

where the right is naturally isomorphic to $\mathrm{Hom}(\bigoplus_{i\in I}(A\otimes B_i), T)$.

Indeed, for any bilinear maps $f\colon A, \bigoplus_{i\in I} B_i \to T$, by fixing $a \in A$, we get a homomorphism $f(a,-)\colon \bigoplus_{i\in I} B_i \to T$, thus a set of homomorphisms $\{f(a, \imath_i(-))\}_{i\in I}$, where $\imath_i$ denote the canonical map $B_i \to \bigoplus_{i\in I} B_i$. One can verify that $f(-, \imath_i(-))\colon A, B_i \to T$ is a bilinear map. Thus we get

$$\begin{aligned} \mathrm{Bil}(A, \bigoplus_{i\in I} B_i; T) &\longrightarrow \prod_{i\in I} \mathrm{Bil}(A, B_i; T) \\ f &\longmapsto \{f(-, \imath_i(-))\}_{i\in I}. \end{aligned}$$

Conversely, given a set of bilinear maps $\{g_i\colon A, B_i \to T\}_{i\in I}$, we can get a set of homomorphisms $\{g_i(a,-)\colon B_i \to T\}_{i\in I}$ by fixing $a \in A$. Then they factor through a unique homomorphism $g^{(a)}\colon \bigoplus_{i\in I} B_i \to T$. For any $a \in A$ and $b \in \bigoplus_{i\in I} B_i$, let $g(a,b) := g^{(a)}(b)$. One can verify that g is a bilinear map. Thus we get

$$\begin{aligned} \prod_{i\in I} \mathrm{Bil}(A, B_i; T) &\longrightarrow \mathrm{Bil}(A, \bigoplus_{i\in I} B_i; T) \\ \{g_i\}_{i\in I} &\longmapsto g. \end{aligned}$$

Moreover, by combining those two maps, one can see they are inverse for each other. Thus they are bijections as desired. The naturality of those bijections on T is obvious. $\square$

Remark. From the bijections above, one can see that for an elementary tensor $a \otimes (b_i)_{i\in I} \in A \otimes (\bigoplus_{i\in I} B_i)$, its image in $\bigoplus_{i\in I}(A\otimes B_i)$ is $(a\otimes b_i)_{i\in I}$. Conversely, for $(a_i \otimes b_i)_{i\in I} \in \bigoplus_{i\in I}(A\otimes B_i)$, its image in $A \otimes (\bigoplus_{i\in I} B_i)$ is $\sum_{i\in I} a_i \otimes \imath_i(b_i)$. One can also verify from this corresponding directly.

However, $A \otimes (\prod_{i\in I} B_i)$ is NOT isomorphic to $\prod_{i\in I}(A \otimes B_i)$ in general. For example, $\mathbb{Q} \otimes \mathbb{Z}/p^i\mathbb{Z} = 0$ for any $i \geqslant 1$, but $\mathbb{Q} \otimes (\prod_{i\geqslant 1} \mathbb{Z}/p^i\mathbb{Z})$ is not 0. See the follow subsections for details.

However, we have

Proposition 1.5. *Let A be an abelian group and $\{B_i\}_{i\in I}$ a set of abelian groups. Then we have the following natural homomorphism*

$$A \bigotimes \left(\prod_{i\in I} B_i\right) \to \prod_{i\in I} \left(A \bigotimes B_i\right).$$

Proof. First, there exists a set of natural homomorphisms $p_i\colon \prod_{i\in I} B_i \to B_i$, i.e. the canonical projrctions. Then we get a set of natural homomorphisms $A \otimes p_i\colon A \otimes (\prod_{i\in I} B_i) \to A \otimes B_i$, thus a natural homomorphism from the universal property of product as desired. $\square$

Tensor products and hom-functors

Recall that the set $\mathrm{Hom}(A, B)$ of group homomorphisms between two abelian groups has a natural abelian group structure. For distinguish, we denote this abelian group by $\mathrm{hom}(A, B)$. Moreover, one can verify that $\mathrm{hom}(-, -)$, like $\mathrm{Hom}(-, -)$, is a bifunctor, but its codomain is **Ab** instead of **Set**.

We have

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Bil}(A, B; C).$$

But every bilinear map $f\colon A, B \to C$ can be view as a group homomorphism from A to $\mathrm{hom}(B, C)$ by

$$\begin{aligned} A &\longrightarrow \mathrm{hom}(B, C) \\ a &\longmapsto f(a, -) \end{aligned}$$

Thus $- \otimes B$ is the left adjoint of the covariant functor $\mathrm{hom}(B, -)$.

Exactness

Recall that a functor between *finitely complete* categories is said to be **left exact** if it preserves finite limits. Dually, a functor between finitely cocomplete categories is said to be **right exact** if it preserves finite colimits.

One general criteria is that: a left adjoint functor is right exact and a right adjoint functor is left exact. Indeed, we have

Lemma 1.6. *Let $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ be a pair of adjoint functors. Then L preserves colimits in $\mathcal{C}$, R preserves limits in $\mathcal{D}$.*

Proof. Let $D\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{D}$ be a diagram whose limit $\varprojlim D$ exists. Then we have a sequence of natural isomorphisms, natural in $X \in \operatorname{ob}\mathcal{C}$:

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{C}}(X, R(\varprojlim D)) &\cong \operatorname{Hom}_{\mathcal{D}}(L(X), \varprojlim D) \\
&\cong \varprojlim \operatorname{Hom}_{\mathcal{D}}(L(X), D) \\
&\cong \varprojlim \operatorname{Hom}_{\mathcal{C}}(X, R(D)) \\
&\cong \operatorname{Hom}_{\mathcal{C}}(X, \varprojlim R(D))
\end{aligned}$$

where we used the adjunction isomorphism and the fact that any hom-functor preserves limits. Because this is natural in X the Yoneda lemma implies that we have an isomorphism

$$R \varprojlim D \cong \varprojlim R(D)$$

The argument that shows the preservation of colimits by L is analogous. $\square$

From this lemma we can see that the tensor product functor $-\otimes B$ is right exact.

III Multilinear maps and $n-$fold tensor products

The notion of binary morphisms, and also bilinear maps, can be easily generalized to multimorphisms and multilinear maps.

A $0-$**linear map** is just an element, thus the **nullary tensor product** is just the initial object 0 in **Ab**.

A $n-$**linear map** is a $n-$ary function from the underlying sets of n abelian groups $A_1, A_2, \cdots, A_n$ and target to an abelian group B such that by fixing any $n-1$ components of the n inputs, the $n-$ary function becomes a group homomorphism. Then the $n-$**ary tensor product** is the universal $n-$linear map.

The $n-$ary tensor product can easily be constructed from the binary ones. To see this, we prove the following result as an example.

Theorem 1.7. *Let A, B, C be three abelian groups, then*

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

Proof. We check that both $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ satisfy the universal property of ternary tensor products.

Let $f\colon A, B, C \to T$ be a trilinear map. Then, by fixing $c \in C$, we get a bilinear map $f(-,-,c)\colon A, B \to T$, thus it factors through a homomorphism $f_c\colon A \otimes B \to T$. Conversely, by fixing an elementary tensor $a \otimes b$, which is equivalent to fixing two inputs $a \in A$ and $b \in B$, we get a homomorphism $f(a, b, -)\colon C \to T$. Extending this to a general tensor $x = \sum_{i \in I} a_i \otimes b_i$, we get a homomorphism $f_x := \sum_{i \in I} f(a_i, b_i, -)$. From those f_c $(c \in C)$ and f_x $(x \in A \otimes B)$, we get a bilinear map $(A \otimes B), C \to T$, thus it factors through a homomorphism $\bar{f}\colon (A \otimes B) \otimes C \to T$. This $\bar{f}$ is unique since every homomorphism in our construction is unique.

For $A \otimes (B \otimes C)$, the proof is similar. □

Remark. By the universal property, there exists a unique isomorphism

$$\alpha_{A,B,C}\colon (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C).$$

Moreover, this isomorphism is natural. This natural isomorphism α is called the **associator** of **Ab**.

This theorem tells us that a ternary tensor product can be constructed by $3-$*fold tensor product*, that means a parenthesizing of the expression $A \otimes B \otimes C$, and the two possible constructions can be uniquely identified through the associator.

In general, we should prove that the $n-$**fold tensor product**, i.e. a parenthesizing of the expression $A_1 \otimes A_2 \otimes \cdots \otimes A_n$, produces a $n-$**ary tensor product** of n abelian groups $A_1, A_2, \cdots, A_n$, and all such constructions can be uniquely identified through chains of associators.

Indeed, it is not difficult to prove that $(\cdots(A_1 \otimes A_2) \otimes \cdots) \otimes A_n$ is a $n-$ary tensor product by induction on n. Then, an easy combinatorial argument shows that one can identify any two parenthesizings of the expression $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ through a chain of associators.

For now, it seems that everything is done and therefore the parentheses in the expression $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ can be ignored and we can just denote the $n-$fold tensor product by it. However, some ambiguities still exist: there may be two or more different chains connecting two different parenthesizings thus they may provide different identifications, which is not expected.

For example, the following diagram shows two possible chains of associators connecting $((A \otimes B) \otimes C) \otimes D$ and $A \otimes (B \otimes (C \otimes D))$. (One can verify they provide the same identification, this is the **pentagon identity**.)

$$\begin{tikzcd}
& (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C\otimes D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A\otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C}\otimes D"'] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "A\otimes\alpha_{B,C,D}"']
\end{tikzcd}$$

Moreover, $n-$fold tensor product is not the only way to produce $n-$ary tensor products. For example, we can identify a object with the tensor product of it with $\mathbb{Z}$ through the unitors, we can also identify a tensor product by a permutation of its expression through a chain of symmetric braidings, we can even identify different expressions by a composite of all the stuff. The followings are some typical examples.

- (the **triangle identity**)

$$\begin{tikzcd}
(A \otimes \mathbb{Z}) \otimes B \arrow[rr, "\alpha_{A,\mathbb{Z},B}"] \arrow[dr, "\rho_A\otimes B"'] & & A \otimes (\mathbb{Z} \otimes B) \arrow[dl, "A\otimes\lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

- (the **hexagon identity**)

$$
\begin{tikzcd}
 & A \otimes (B \otimes C) \arrow[dr, "\gamma_{A,B\otimes C}"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha_{A,B,C}"] \arrow[d, "\gamma_{A,B}\otimes C"'] & & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[dr, "\alpha_{B,A,C}"'] & & B \otimes (C \otimes A) \\
 & B \otimes (A \otimes C) \arrow[ur, "B\otimes\gamma_{A,C}"'] &
\end{tikzcd}
$$

- (the **involution law**)

$$
\begin{tikzcd}
A \otimes B \arrow[rr, "1_{A\otimes B}"] \arrow[dr, "\gamma_{A,B}"'] & & A \otimes B \\
 & B \otimes A \arrow[ur, "\gamma_{B,A}"'] &
\end{tikzcd}
$$

All those problem come down to a general statement about the commutativities of certain diagrams. This is the "*coherence theorem*". It holds in any *monoidal category* or *symmetric monoidal category*, which we will define in next subsection.

To prove the coherence theorem in general is not easy, but for **Ab**, it is not a big problem since we our tensor products come from the universal multilinear maps. The trick is, every expression in our consideration comes from a universal multilinear map from the factors, and every associator, unitor or symmetric braiding commutes with the universal multilinear maps. Therefore, a composite of them must commutes with the universal multilinear maps of the start and end. Thus, by the universal property, it must equals to the unique one.

$$
\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \cdots\cdots \arrow[r] & \bullet \arrow[r] & \bullet \\
 & & A_1, A_2, \cdots, A_n \arrow[ull] \arrow[ul] \arrow[ur] \arrow[urr] & &
\end{tikzcd}
$$

§2 Monoidal categories

I Monoidal categories

We have seen that the tensor product on **Ab** is *commutative*, up to natural isomorphism s, *associative*, up to natural isomorphism α, and has an identity $\mathbb{Z}$ satisfying the left and right *unity law*, up to natural isomorphisms λ and ρ respectively.

Therefore, the tensor product on **Ab** is very similar to the multiplication on an abelian monoid, except the operation laws hold only up to isomorphisms. This leads to the notion of *symmetric monoidal categories*. They are categories equipped with extra structures and behave like abelian monoids.

More precisely, a **monoidal category** is a category $\mathcal{C}$ equipped with a bifunctor

$$\otimes \colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C},$$

called the **tensor product**, an object

$$\mathbf{1} \in \operatorname{ob}\mathcal{C},$$

called the **unit object** or **tensor unit**, a natural isomorphism

$$\alpha_{A,B,C} \colon (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C),$$

called the **associator**, a natural isomorphism

$$\lambda_A \colon \mathbf{1} \otimes A \xrightarrow{\cong} A,$$

called the **left unitor**, and a natural isomorphism

$$\rho_A \colon A \otimes \mathbf{1} \xrightarrow{\cong} A,$$

called the **right unitor**, such that the *pentagon identity* hold for every $A, B, C, D \in \operatorname{ob}\mathcal{C}$,

$$\begin{tikzcd}
& (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C\otimes D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A\otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C}\otimes D"'] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "A\otimes\alpha_{B,C,D}"']
\end{tikzcd}$$

and the *triangle identity* holds for every $A, B \in \operatorname{ob}\mathcal{C}$.

$$\begin{tikzcd}
(A \otimes \mathbf{1}) \otimes B \arrow[rr, "\alpha_{A,\mathbf{1},B}"] \arrow[dr, "\rho_A \otimes B"'] & & A \otimes (\mathbf{1} \otimes B) \arrow[dl, "A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

Moreover, a **symmetric monoidal category** is a monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ equipped a natural isomorphism

$$\gamma_{A,B} \colon A \otimes B \xrightarrow{\cong} B \otimes A,$$

called the **symmetric braiding**, such that the *hexagon identity* holds for every $A, B, C \in \operatorname{ob}\mathcal{C}$,

$$\begin{tikzcd}
& A \otimes (B \otimes C) \arrow[dr, "\gamma_{A,B\otimes C}"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha_{A,B,C}"] \arrow[d, "\gamma_{A,B} \otimes C"'] & & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[dr, "\alpha_{B,A,C}"'] & & B \otimes (C \otimes A) \\
& B \otimes (A \otimes C) \arrow[ur, "B \otimes \gamma_{A,C}"'] &
\end{tikzcd}$$

and the *involution law* holds for every $A, B \in \operatorname{ob}\mathcal{C}$.

$$\begin{tikzcd}
A \otimes B \arrow[rr, "1_{A\otimes B}"] \arrow[dr, "\gamma_{A,B}"'] & & A \otimes B \\
& B \otimes A \arrow[ur, "\gamma_{B,A}"'] &
\end{tikzcd}$$

Remark. The monoidal categories are also called **tensor categories**. The symmetric monoidal categories are specail cases of *braiding monoidal categories*, which are defined like symmetric monoidal categories except the involution law fails and another hexagon identity is required.

Therefore, the discussion of naturality of tensor products of abelian groups can be summarized as follow:

Theorem 2.1. $(\mathbf{Ab}, \otimes, \mathbb{Z}, \alpha, \lambda, \rho, \gamma)$ *is a symmetric monoidal category.*

There are other familiar examples of monoidal categories: The cartesian products provide a symmetric monoidal category structure on **Set**, whose tansor unit is the singleton. The cartesian products also provide another symmetric monoidal category structure on **Ab**, whose tensor unit is the trivial group. Such kind of monoidal categories are so-called **cartesian** ones.

Monoidal functors and monoidal natural transformations

Since monoidal categories are categories equipped with extra structures, so the correct morphisms between them should be functors preserving those extra structures. They are the monoidal functors.

More precisely, a **monoidal functor** is a functor $F\colon \mathcal{C} \to \mathcal{C}'$ between monoidal categories, equipped with a natural isomorphism

$$\Phi_{A,B}\colon F(A) \otimes F(B) \xrightarrow{\cong} F(A \otimes B),$$

and an isomorphism

$$\phi\colon \mathbf{1} \xrightarrow{\cong} F(\mathbf{1}),$$

such that the following diagram commutes for every $A, B, C \in \operatorname{ob}\mathcal{C}$,

$$\begin{tikzcd}[column sep=small]
& F(A \otimes B) \otimes F(C) \arrow[dr, "\Phi_{A\otimes B,C}"] & \\
(F(A) \otimes F(B)) \otimes F(C) \arrow[ur, "\Phi_{A,B}\otimes F(C)"] \arrow[d, "\alpha_{F(A),F(B),F(C)}"] & & F((A \otimes B) \otimes C) \arrow[d, "F(\alpha_{A,B,C})"] \\
F(A) \otimes (F(B) \otimes F(C)) \arrow[dr, "F(A)\otimes\Phi_{B,C}"'] & & F(A \otimes (B \otimes C)) \\
& F(A) \otimes F(B \otimes C) \arrow[ur, "\Phi_{A,B\otimes C}"'] &
\end{tikzcd}$$

and the following diagrams commute for every $A \in \operatorname{ob}\mathcal{C}$.

$$\begin{tikzcd}
\mathbf{1} \otimes F(A) \arrow[r, "\lambda_{F(A)}"] \arrow[d, "\phi\otimes F(A)"'] & F(A) \\
F(\mathbf{1}) \otimes F(A) \arrow[r, "\Phi_{\mathbf{1},A}"] & F(\mathbf{1} \otimes A) \arrow[u, "F(\lambda_A)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A) \otimes \mathbf{1} \arrow[r, "\rho_{F(A)}"] \arrow[d, "F(A)\otimes\phi"'] & F(A) \\
F(A) \otimes F(\mathbf{1}) \arrow[r, "\Phi_{A,\mathbf{1}}"] & F(A \otimes \mathbf{1}) \arrow[u, "F(\rho_A)"']
\end{tikzcd}$$

Remark. Note that the above diagrams show that the isomorphism ϕ is uniquely determined, thus when define the notion of monoidal functor, we can omit this isomorphism and just require $\mathbf{1}$ and $F(\mathbf{1})$ are isomorphic.

A **symmetric monoidal functor** is simply a monoidal functor $F\colon \mathcal{C} \to \mathcal{C}'$ making the following diagram commute for every $A, B \in \operatorname{ob}\mathcal{C}$.

$$\begin{tikzcd}
F(A) \otimes F(B) \arrow[r, "\gamma_{F(A),F(B)}"] \arrow[d, "\Phi_{A,B}"'] & F(B) \otimes F(A) \arrow[d, "\Phi_{B,A}"] \\
F(A \otimes B) \arrow[r, "F(\gamma_{A,B})"] & F(B \otimes A)
\end{tikzcd}$$

When ϕ and Φ are identities, we say that F is a **strict monoidal functor**. For example, the identity functor of a monoidal category has a trivial obvious strict monoidal functor structure.

Like functors, monoidal functors can be composed. More precisely, if F and G are two functors between monoidal categories and F has a monoidal functor structure. Then if G has a monoidal functor structure, then $G \circ F$ has a natural monoidal functor structure given by

$$G(F(A)) \otimes G(F(B)) \cong G(F(A) \otimes F(B)) \cong G(F(A \otimes B)).$$

A monoidal functor is said to be an **equivalence** of monoidal categories if it is an equivalence of categories. This definition seems not correct since the right concept of "equivalence" of monoidal categories should be stronger than the concept of equivalence of categories.

To explain this in detail, we should define the *monoidal natural transformations.* They are the natural transformations respect the monoidal structure in an obvious way.

More precisely, a **monoidal natural transformation** is a natural transformation $\eta\colon F \Rightarrow G$ between two monoidal functors (or, symmetric monoidal functors) (F, Φ, ϕ) and (G, Ψ, ψ) making the following diagrams commute for every $A, B \in \operatorname{ob}\mathcal{C}$.

$$\begin{tikzcd}
F(A) \otimes F(B) \arrow[r, "\eta_A \otimes \eta_B"] \arrow[d, "\Phi_{A,B}"'] & G(A) \otimes G(B) \arrow[d, "\Psi_{A,B}"] \\
F(A \otimes B) \arrow[r, "\eta_{A \otimes B}"] & G(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbf{1} \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\
F(\mathbf{1}) \arrow[rr, "\eta_{\mathbf{1}}"] & & G(\mathbf{1})
\end{tikzcd}$$

A monoidal functor $F\colon \mathcal{C} \to \mathcal{C}'$ between two monoidal categories is said to be an **monoidal equivalence** and thus $\mathcal{C}$ and $\mathcal{C}'$ are said to be **monoidally equivalent** if it has a **monoidal weak inverse**, that is a monoidal functor $G\colon \mathcal{C}' \to \mathcal{C}$ such that there exist monoidal natural isomorphisms $F \circ G \cong \mathbf{I}_{\mathcal{C}'}$ and $\mathbf{I}_{\mathcal{C}} \cong G \circ F$. The **symmetric monoidal equivalence** and **symmetric monoidal weak inverse** are defined likewise.

However, we have

Proposition 2.2. *If a monoidal functor is an equivalence of monoidal categories, then it is a monoidal equivalence.*

Proof. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be an equivalence of monoidal categories and G be its weak inverse. Then we have a natural isomorphism:

$$G(A \otimes B) \cong G(F(G(A)) \otimes F(G(B))) \cong G(F(G(A) \otimes G(B))) \cong G(A) \otimes G(B).$$

It is straightforward to see that this gives G a monoidal functor structure. Moreover, it is not difficult to verify that the natural isomorphisms $F \circ G \cong \mathbf{I}_{\mathcal{C}'}$ and $\mathbf{I}_{\mathcal{C}} \cong G \circ F$ are monoidal. $\square$

II Coherence theorems

Now we should prove the coherence theorems for monoidal categories and symmetric monoidal categories. But before going further, we should clarify that the theorems should talk about the commutativities of "formal" diagrams instead of individual diagrams because in a particular monoidal category, a same object can accidentally become tensor products defined from different sorts, thus provides accidental identities and brings some diagrams beyond our expectation and may be unfortunately not commutative. A suitable approach is to consider the diagrams of functors and natural transformations instead of objects and morphisms.

So, the coherence theorem states

Theorem 2.3 (Coherence theorem for monoidal categories)**.** (Mac Lane, 1963) *In a monoidal category, any two chains of associators and unit isomorphisms connecting two functors built from tensor products and identities will provide the same identification.*

Remark. The coherence theorem is first proved by Mac Lane and a well-known version can be found in his famous textbook *Categories for the Working Mathematician*, where the statement is a little different. Our statement basically follows a course note of Etingof (2000).

Other coherence conditions

To prove the coherence theorem, we need some lemmas, they are actually the extra axioms given by Mac Lane (1963) and showed by Kelly (1964) later that they could be implied from the *pentagon* and *triangle* identities.

Lemma 2.4. *In a monoidal category, the following diagrams commute.*

$$
\begin{tikzcd}
(\mathbf{1}\otimes A)\otimes B \arrow[rr, "\alpha_{\mathbf{1},A,B}"] \arrow[dr, "\lambda_A\otimes B"'] & & \mathbf{1}\otimes(A\otimes B) \arrow[dl, "\lambda_{A\otimes B}"] \\
& A\otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes(B\otimes\mathbf{1}) \arrow[rr, "\alpha_{A,B,\mathbf{1}}"] \arrow[dr, "A\otimes\rho_B"'] & & (A\otimes B)\otimes\mathbf{1} \arrow[dl, "\rho_{A\otimes B}"] \\
& A\otimes B &
\end{tikzcd}
$$

Proof. We only prove the first. Note that the functor $\mathbf{1}\otimes -$ is isomorphic to the identity, thus an equivalence. Therefore it suffices to check the commutativity of the triangle below.

$$
\begin{tikzcd}
\mathbf{1}\otimes((\mathbf{1}\otimes A)\otimes B) \arrow[rr, "\mathbf{1}\otimes\alpha_{\mathbf{1},A,B}"] \arrow[dr, "\mathbf{1}\otimes(\lambda_A\otimes B)"'] & & \mathbf{1}\otimes(\mathbf{1}\otimes(A\otimes B)) \arrow[dl, "\mathbf{1}\otimes\lambda_{A\otimes B}"] \\
& \mathbf{1}\otimes(A\otimes B) &
\end{tikzcd}
$$

To do this, we consider the following diagram,

$$
\begin{tikzcd}
\mathbf{1}\otimes((\mathbf{1}\otimes A)\otimes B) \arrow[rr, "\mathbf{1}\otimes\alpha_{\mathbf{1},A,B}"] \arrow[dr, "\mathbf{1}\otimes(\lambda_A\otimes B)"'] & \mathrm{I} & \mathbf{1}\otimes(\mathbf{1}\otimes(A\otimes B)) \arrow[dl, "\mathbf{1}\otimes\lambda_{A\otimes B}"] \\
& \mathbf{1}\otimes(A\otimes B) \arrow[r, phantom, "\mathrm{II}"] & \\
\mathrm{V} & & (\mathbf{1}\otimes\mathbf{1})\otimes(A\otimes B) \arrow[ul, "\rho_{\mathbf{1}}\otimes(A\otimes B)"'] \arrow[uu, "\alpha_{\mathbf{1},\mathbf{1},A\otimes B}"'] \\
& (\mathbf{1}\otimes A)\otimes B \arrow[uu, "\alpha_{\mathbf{1},A,B}"] \arrow[ur, phantom, "\mathrm{III}"] & \\
(\mathbf{1}\otimes(\mathbf{1}\otimes A))\otimes B \arrow[ur, "(\mathbf{1}\otimes\lambda_A)\otimes B"] \arrow[uuuu, "\alpha_{\mathbf{1},\mathbf{1}\otimes A,B}"] & \mathrm{IV} & ((\mathbf{1}\otimes\mathbf{1})\otimes A)\otimes B \arrow[ll, "\alpha_{\mathbf{1},\mathbf{1},A}\otimes B"'] \arrow[ul, "(\rho_{\mathbf{1}}\otimes A)\otimes B"'] \arrow[uu, "\alpha_{\mathbf{1}\otimes\mathbf{1},A,B}"']
\end{tikzcd}
$$

Here, the outside commutes by the *pentagon identity*, regions II and IV by the *triangle identity*, and regions III and V by the naturality of α. It follows that region I commutes as desired. $\square$

Lemma 2.5. *In a monoidal category, $\lambda_{\mathbf{1}} = \rho_{\mathbf{1}}$.*

Proof. From the *triangle identity*, we have $(\mathbf{1} \otimes \lambda_{\mathbf{1}}) \circ \alpha_{\mathbf{1},\mathbf{1},\mathbf{1}} = \rho_{\mathbf{1}} \otimes \mathbf{1}$. On the other hand, by Lemma 2.4, we have $(\mathbf{1} \otimes \lambda_{\mathbf{1}}) \circ \alpha_{\mathbf{1},\mathbf{1},\mathbf{1}} = \lambda_{\mathbf{1}} \otimes \mathbf{1}$. Therefore $\rho_{\mathbf{1}} \otimes \mathbf{1} = \lambda_{\mathbf{1}} \otimes \mathbf{1}$. Then $\lambda_{\mathbf{1}} = \rho_{\mathbf{1}}$ since $- \otimes \mathbf{1}$ is an equivalence. $\square$

The strictness theorem

It is obvious that if the associator α and the two unitors λ and ρ are all identities, then the coherence theorem for monoidal categories trivially holds. Such kind of monoidal categories are said to be **strict**.

Then, the main step of proving the coherence theorem is to prove the following strictness theorem.

Theorem 2.6 (Strictness theorem for monoidal categories)**.** (Joyal and Street, 1993) *Every monoidal category is monoidally equivalent to a strict monoidal category.*

Proof. Let $\mathcal{C}$ be a monoidal category, we now construct a strict monoidal category $\mathcal{C}_s$ as follows. The objects of $\mathcal{C}_s$ are pairs (F, ζ), where $F \colon \mathcal{C} \to \mathcal{C}$ is a functor and

$$\zeta_{A,B} \colon F(A) \otimes B \xrightarrow{\cong} F(A \otimes B)$$

is a natural isomorphism, such that the following diagram commutes for any $A, B, C \in \operatorname{ob}\mathcal{C}$.

$$\begin{tikzcd}
 & F(A) \otimes (B \otimes C) \arrow[dr, "\zeta_{A,B\otimes C}"] & \\
(F(A) \otimes B) \otimes C \arrow[ur, "\alpha_{F(A),B,C}"] \arrow[d, "\zeta_{A,B}\otimes C"'] & & F(A \otimes (B \otimes C)) \\
F(A \otimes B) \otimes C \arrow[rr, "\zeta_{A\otimes B,C}"] & & F((A \otimes B) \otimes C) \arrow[u, "F(\alpha_{A,B,C})"']
\end{tikzcd}$$

A morphism $\theta\colon (F,\zeta) \to (F',\zeta')$ in $\mathcal{C}_s$ is a natural transformation $\theta\colon F \Rightarrow F'$ such that the following square commutes for any $A, B \in \operatorname{ob}\mathcal{C}$.

$$\begin{tikzcd}
F(A) \otimes B \arrow[r, "\theta_A \otimes B"] \arrow[d, "\zeta_{A,B}"'] & F'(A) \otimes B \arrow[d, "\zeta'_{A,B}"] \\
F(A \otimes B) \arrow[r, "\theta_{A\otimes B}"] & F'(A \otimes B)
\end{tikzcd}$$

Composition of morphisms is the *vertical composition* of natural transformations. The tensor product of objects is given by

$$(F,\zeta) \otimes (F',\zeta') := (F \circ F', \tilde{\zeta}),$$

where $\tilde{\zeta}$ is given by the composition

$$F(F'(A)) \otimes B \overset{\zeta_{F'(A),B}}{\longrightarrow} F(F'(A) \otimes B) \overset{F(\zeta'_{A,B})}{\longrightarrow} F(F'(A \otimes B)),$$

and the tensor product of morphisms is the *horizontal composition* of natural transformations. Then one can verify that $\mathcal{C}_s$ is a strict monoidal category with identity functor $\mathbf{I}$ as the tensor unit.

Next, we define a functor $L\colon \mathcal{C} \to \mathcal{C}_s$ as follows:

$$L(A) = (A \otimes -, \alpha_{A,-,-}), \qquad L(f) = f \otimes -.$$

Now, we claim that this functor L is a monoidal equivalence.

First, it is easy to verify that for any $(F,\zeta) \in \operatorname{ob}\mathcal{C}_s$, (F,ζ) is isomorphic to $L(F(\mathbf{1}))$ by check the commutativity of the following square. Here the natural transformation θ is defined by $\theta_A := F(\lambda_A) \circ \zeta_{\mathbf{1},A}$.

$$\begin{tikzcd}
(F(\mathbf{1}) \otimes A) \otimes B \arrow[r, "\theta_A \otimes B"] \arrow[d, "\alpha_{\mathbf{1},A,B}"'] & F(A) \otimes B \arrow[d, "\zeta_{A,B}"] \\
F(\mathbf{1}) \otimes (A \otimes B) \arrow[r, "\theta_{A\otimes B}"] & F(A \otimes B)
\end{tikzcd}$$

Thus L is essentially surjective.

Then us should show that L is full. Let $\theta\colon L(A) \to L(B)$ be a morphism in $\mathcal{C}_s$, we define $f\colon A \to B$ to be the composite

$$A \xrightarrow{\rho_A^{-1}} A \otimes \mathbf{1} \xrightarrow{\theta_{\mathbf{1}}} B \otimes \mathbf{1} \xrightarrow{\rho_B} B.$$

To show $\theta = L(f)$, it suffices to show that for any $C \in \operatorname{ob}\mathcal{C}$, $\theta_C = f \otimes C$. Indeed, this follows from the commutativity of the diagram,

$$\begin{tikzcd}
A \otimes C \arrow[r, "\rho_A^{-1}\otimes C"] \arrow[d, "f\otimes C"'] & (A \otimes \mathbf{1}) \otimes C \arrow[r, "\alpha_{A,\mathbf{1},C}"] \arrow[d, "\theta_{\mathbf{1}}\otimes C"'] & A \otimes (\mathbf{1} \otimes C) \arrow[r, "A\otimes\lambda_C"] \arrow[d, "\theta_{\mathbf{1}\otimes C}"'] & A \otimes C \arrow[d, "\theta_C"'] \\
B \otimes C \arrow[r, "\rho_B^{-1}\otimes C"] & (B \otimes \mathbf{1}) \otimes C \arrow[r, "\alpha_{B,\mathbf{1},C}"] & B \otimes (\mathbf{1} \otimes C) \arrow[r, "B\otimes\lambda_C"] & B \otimes C
\end{tikzcd}$$

where the rows are the identities by the *triangle identity*, the left square commutes by the definition of f, the right square commutes by naturality of θ, and the central square commutes since θ is a morphism in $\mathcal{C}_s$.

Next, if $L(f) = L(g)$ for some morphisms f, g in $\mathcal{C}$ then, in particular $f \otimes \mathbf{1} = g \otimes \mathbf{1}$ so that $f = g$. Thus L is faithful.

Finally, we define a monoidal functor structure

$$\Psi_{A,B} \colon L(A) \otimes L(B) \xrightarrow{\cong} L(A \otimes B), \qquad \psi \colon \mathbf{I} \xrightarrow{\cong} L(\mathbf{1})$$

on L as follows. First,

$$\begin{aligned} L(A) \otimes L(B) &= (A \otimes (B \otimes -), \zeta)\,, \\ L(A \otimes B) &= \big((A \otimes B) \otimes -, \zeta'\big)\,. \end{aligned}$$

(Here we omit the explicit formulas for ζ and ζ').

Thus we can define $\Psi_{A,B}$ to be $\alpha^{-1}_{A,B,-}$. To show it is a morphism in $\mathcal{C}_s$, we should check the commutativity of the square for any $C, D \in \operatorname{ob}\mathcal{C}$.

$$\begin{tikzcd}
L(A) \circ L(B)(C) \otimes D \arrow[r, "\alpha^{-1}_{A,B,C}\otimes D"] \arrow[d, "\zeta_{C,D}"'] & L(A \otimes B)(C) \otimes D \arrow[d, "\zeta'_{C,D}"] \\
L(A) \circ L(B)(C \otimes D) \arrow[r, "\alpha^{-1}_{A,B,C\otimes D}"] & L(A \otimes B)(C \otimes D)
\end{tikzcd}$$

But $\zeta_{C,D}$ is the composite

$$(A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha_{A,B\otimes C,D}} A \otimes ((B \otimes C) \otimes D) \xrightarrow{A\otimes\alpha_{B,C,D}} A \otimes (B \otimes (C \otimes D)),$$

and $\zeta'_{C,D}$ is $\alpha_{A\otimes B,C,D}$. Thus the square becomes the *pentagon identity*.

The tensor unit of $\mathcal{C}_s$ is the identity functor $\mathbf{I}$, so the isomorphism ϕ can be defined as $\phi_A = \lambda_A^{-1} \colon A \xrightarrow{\cong} \mathbf{1} \otimes A$. To show it is a morphism in $\mathcal{C}_s$, we need to verify the commutativity of the square for any $A, B \in \mathrm{ob}\,\mathcal{C}$.

$$\begin{tikzcd}
A \otimes B \arrow[r, "\lambda_A^{-1} \otimes B"] \arrow[d, equal] & (\mathbf{1} \otimes A) \otimes B \arrow[d, "\alpha_{\mathbf{1},A,B}"] \\
A \otimes B \arrow[r, "\lambda_{A \otimes B}^{-1}"] & \mathbf{1} \otimes (A \otimes B)
\end{tikzcd}$$

It follows directly from Lemma 2.4.

Now we check that (Φ, ϕ) give a monoidal functor structure on L. To check the hexagon condition of (L, Φ, ϕ), i.e.

$$\begin{tikzcd}
& L(A \otimes B) \otimes L(C) \arrow[dr, "\Phi_{A \otimes B, C}"] & \\
(L(A) \otimes L(B)) \otimes L(C) \arrow[ur, "\Phi_{A,B} \otimes L(C)"] \arrow[d, equal] & & L((A \otimes B) \otimes C) \arrow[d, "L(\alpha_{A,B,C})"] \\
L(A) \otimes (L(B) \otimes L(C)) \arrow[dr, "L(A) \otimes \Phi_{B,C}"'] & & L(A \otimes (B \otimes C)) \\
& L(A) \otimes L(B \otimes C) \arrow[ur, "\Phi_{A, B \otimes C}"'] &
\end{tikzcd}$$

we apply it on an object D. Then it becomes the *pentagon identity*, thus holds. The rest two conditions are

$$\begin{tikzcd}
\mathbf{1} \otimes L(A) \arrow[r, equal] \arrow[d, "\phi \otimes L(A)"'] & L(A) \\
L(\mathbf{1}) \otimes L(A) \arrow[r, "\Phi_{\mathbf{1},A}"] & L(\mathbf{1} \otimes A) \arrow[u, "L(\lambda_A)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
L(A) \otimes \mathbf{1} \arrow[r, equal] \arrow[d, "L(A) \otimes \phi"'] & L(A) \\
L(A) \otimes L(\mathbf{1}) \arrow[r, "\Phi_{A,\mathbf{1}}"] & L(A \otimes \mathbf{1}) \arrow[u, "L(\rho_A)"']
\end{tikzcd}$$

we apply them on an object B, then they become

$$\begin{tikzcd}
& A \otimes B \arrow[dl, "\lambda_{A \otimes B}^{-1}"'] & \\
\mathbf{1} \otimes (A \otimes B) \arrow[rr, "\alpha_{\mathbf{1},A,B}"] & & (\mathbf{1} \otimes A) \otimes B \arrow[ul, "\lambda_A \otimes B"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \otimes B \arrow[dl, "A \otimes \lambda_B^{-1}"'] & \\
A \otimes (\mathbf{1} \otimes B) \arrow[rr, "\alpha_{A,\mathbf{1},B}"] & & (A \otimes \mathbf{1}) \otimes B \arrow[ul, "\rho_A \otimes B"']
\end{tikzcd}$$

where the left triangle commute by Lemma 2.4 and the right by the *triangle identity*. □

The coherence theorems

Once we have the strictness theorem, the coherence theorem follows almost directly.

Proof for Theorem 2.3. Let $F\colon \mathcal{C} \to \mathcal{C}_s$ be a monoidal equivalence from a monoidal category to a strict monoidal category. Then, for two chains θ and θ' connected two expressions f and g, we have the following commutative diagram by the definition of monoidal functors.

$$\begin{tikzcd}
f(F(A_1), F(A_2), \cdots, F(A_n)) \arrow[r, "\theta_s", "\theta'_s"'] \arrow[d, "\Phi_f"'] & g(F(A_1), F(A_2), \cdots, F(A_n)) \arrow[d, "\Phi_g"] \\
F(f(A_1, A_2, \cdots, A_n)) \arrow[r, "F(\theta)", "F(\theta')"'] & F(g(A_1, A_2, \cdots, A_n))
\end{tikzcd}$$

Here, the columns are natural isomorphisms and rows are chains of associators and unit isomorphisms. Since $\mathcal{C}_s$ is strict, both θ_s and θ'_s are just chains of identities, thus equal. Then we have

$$F(\theta) \circ \Phi_f = \Phi_g = F(\theta') \circ \Phi_f.$$

Since Φ_f is an isomorphism and F is an equivalence, $\theta = \theta'$ as desired. $\square$

Now we give the coherence theorem for symmetric monoidal categories and prove it.

Theorem 2.7 (Coherence theorem for symmetric monoidal categories)**.** *In a symmetric monoidal category, any two chains of associators, unit isomorphisms and braiding isomorphisms connecting two functors built from tensor products, identities and permutations will provide the same identification.*

Proof. Let $\mathcal{C}$ be a symmetric monoidal category. Then, by Theorem 2.3, we can safely assume that it is strict, thus all $n-$ary functors built from tensor products and identities are equal. Let's call it the $n-$**ary tensor product functor**. Then we only need to show that any two chains of braiding isomorphisms connecting two permutations of the $n-$ary tensor product functor will provide the same identification. Moreover, we only need to check this for the *closed chains*, that is a chain whose start equals to the end.

Note that, the *hexagon identity* in the strict case shows that we can replace any braiding isomorphism by a chain of *adjacent transposition braiding isomorphisms*, that is a braiding isomorphism whose domain and codomain differ by an adjacent transposition. For example, the braiding isomorphism $\gamma_{A,B\otimes C}\colon A\otimes B\otimes C\to B\otimes C\otimes A$ is equal to the composite

$$A\otimes B\otimes C \xrightarrow{\gamma_{A,B}\otimes C} B\otimes A\otimes C \xrightarrow{B\otimes\gamma_{A,C}} B\otimes C\otimes A.$$

Then, we can label every adjacent transposition braiding isomorphism by the corresponding adjacent transposition. We define the label of a chain to be the *formal* product of the labels of its components. That means we do not take products in the nth symmetric group $\mathfrak{S}_n$, instead, we should take products in the free group $\mathfrak{F}_n$ generated by the adjacent transpositions. Thus a chain is closed if and only if its label belongs to the kernel of the canonical map $\mathfrak{F}_n\to\mathfrak{S}_n$.

Recall that this kernel is generated by the following elements

$$\begin{aligned} {\sigma_i}^2, &\quad 1\leqslant i\leqslant n-1;\\ (\sigma_i\sigma_j)^2, &\quad 1\leqslant i<j-1\leqslant n-2;\\ (\sigma_i\sigma_{i+1})^3, &\quad 1\leqslant i\leqslant n-2. \end{aligned}$$

Then, we can decompose a closed chain as a composite of some *elementary* ones, each of them is labelled by one of the above elements.

Thus, it suffices to verify that every elementary closed chain provides the trivial identification. For chains labelled by ${\sigma_i}^2$, this follows by the *involution law*; for chains labelled by $(\sigma_i\sigma_j)^2$, this follows from the naturality of γ; for chains labelled by $(\sigma_i\sigma_{i+1})^3$, this follows from the commutativity of the diagram below.

$$\begin{tikzcd}
 & A\otimes B\otimes C \arrow[dl, "\gamma_{A,B}\otimes C"'] \arrow[dr, "A\otimes\gamma_{B,C}"] & \\
B\otimes A\otimes C \arrow[d, "B\otimes\gamma_{A,C}"'] & & A\otimes C\otimes B \arrow[d, "\gamma_{A,C}\otimes B"] \\
B\otimes C\otimes A \arrow[dr, "\gamma_{B,C}\otimes A"'] & & C\otimes A\otimes B \arrow[dl, "C\otimes\gamma_{A,B}"] \\
 & C\otimes B\otimes A &
\end{tikzcd}$$

This is the **Yang-Baxter identity**, one can verify it directly from the *hexagon identity.* $\square$

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III Enriched categories

The structure of monoidal category allows us to describe the axioms of hom-sets by notions

More precisely, an $\mathcal{K}-$**category** $\mathcal{C}$ (or $\mathcal{K}-$**enriched category** or **category enriched over** $\mathcal{K}$) consists of the following data:

- a collection $\operatorname{ob}\mathcal{C}$ of objects.
- for every two objects A and B in $\mathcal{C}$, an object $\mathcal{C}(A,B) \in \operatorname{ob}\mathcal{K}$ called the **hom-object**.
- for every three objects A, B and C in $\mathcal{C}$, a morphism

  $$\bullet_{A,B,C} \colon \mathcal{C}(B,C) \otimes \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A,C)$$

  called the **composition**.
- for every object A in $\mathcal{C}$, a morphism $1_A \colon \mathbf{1} \to \mathcal{C}(A,A)$, called the **identity**.

subject to the following axioms:

1. the composition is *associative*, that means for every $A, B, C, D \in \mathrm{ob}\,\mathcal{C}$, the following diagram commutes;

$$\begin{tikzcd}
(\mathcal{C}(C,D) \otimes \mathcal{C}(B,C)) \otimes \mathcal{C}(A,B) \arrow[r, "\bullet_{B,C,D}"] \arrow[dd, "\alpha"', "\cong"] & \mathcal{C}(B,D) \otimes \mathcal{C}(A,B) \arrow[d, "\bullet_{A,B,D}"'] \\
& \mathcal{C}(A,D) \\
\mathcal{C}(C,D) \otimes (\mathcal{C}(B,C) \otimes \mathcal{C}(A,B)) \arrow[r, "\bullet_{A,B,C}"] & \mathcal{C}(C,D) \otimes \mathcal{C}(A,C) \arrow[u, "\bullet_{A,C,D}"]
\end{tikzcd}$$

2. composition is *unital*, that means for every $A, B \in \mathrm{ob}\,\mathcal{C}$, the following diagram commutes.

$$\begin{tikzcd}
\mathbf{1} \otimes \mathcal{C}(A,B) \arrow[rr, "1_B"] \arrow[dr, "\lambda"', "\cong"] & & \mathcal{C}(B,B) \otimes \mathcal{C}(A,B) \arrow[dl, "\bullet_{A,B,B}"] \\
& \mathcal{C}(A,B) & \\
\mathcal{C}(A,B) \otimes \mathbf{1} \arrow[rr, "1_A"] \arrow[ur, "\rho", "\cong"'] & & \mathcal{C}(A,B) \otimes \mathcal{C}(A,A) \arrow[ul, "\bullet_{A,A,B}"']
\end{tikzcd}$$

Given two $\mathcal{K}-$categories $\mathcal{C}$ and $\mathcal{D}$, a $\mathcal{K}-$**enriched functor** $F\colon \mathcal{C} \to \mathcal{D}$ consists of following data:

- A mapping between the collection of objects;
- A collection of morphisms $\mathcal{C}(A,B) \to \mathcal{D}(FA,FB)$ natural in $A, B \in \mathrm{ob}\,\mathcal{C}$.

subject to the following axioms:

1. the functor is compatible with enriched compositions, that means for every $A, B, C \in \mathrm{ob}\,\mathcal{C}$, the following diagram commutes;

$$\begin{tikzcd}
\mathcal{C}(B,C) \otimes \mathcal{C}(A,B) \arrow[r, "F"] \arrow[d, "\bullet_{A,B,C}"'] & \mathcal{D}(FB,FC) \otimes \mathcal{D}(FA,FB) \arrow[d, "\bullet_{FA,FB,FC}"'] \\
\mathcal{C}(A,C) \arrow[r, "F"] & \mathcal{D}(FA,FC)
\end{tikzcd}$$

2. the functor is compatible with enriched identity, that means for every $A \in \mathrm{ob}\,\mathcal{C}$, the following diagram commutes.

$$\begin{tikzcd} & \mathbf{1} \arrow[dl, "1_A"'] \arrow[dr, "1_{FA}"] & \\ \mathcal{C}(A,A) \arrow[rr, "F"] & & \mathcal{D}(FA,FA) \end{tikzcd}$$

For two $\mathcal{K}-$enriched functors $F, G\colon \mathcal{C} \to \mathcal{D}$, a $\mathcal{K}-$**enriched natural transformation** $\alpha\colon F \Rightarrow G$ consists of a collection of morphisms $\eta_A\colon \mathbf{1} \to \mathcal{D}(FA, GA)$ natural in $A \in \mathrm{ob}\,\mathcal{C}$ such that the following diagram commutes for every $A, B \in \mathrm{ob}\,\mathcal{C}$.

$$\begin{tikzcd} \mathbf{1} \otimes \mathcal{C}(A,B) \arrow[r, "\eta_B \otimes F"] \arrow[d, "\lambda"'] & \mathcal{D}(FB,GB) \otimes \mathcal{D}(FA,FB) \arrow[d, "\bullet_{FA,FB,GB}"'] \\ \mathcal{C}(A,B) & \mathcal{D}(FA,GB) \\ \mathcal{C}(A,B) \otimes \mathbf{1} \arrow[r, "G \otimes \eta_A"] \arrow[u, "\rho"] & \mathcal{D}(GA,GB) \otimes \mathcal{D}(FA,GA) \arrow[u, "\bullet_{FA,GA,GB}"] \end{tikzcd}$$

The **vertical composite** $\varepsilon \circ \eta$ of two $\mathcal{K}-$enriched natural transformations $\eta\colon F \Rightarrow G$ and $\varepsilon\colon G \Rightarrow H$ is obtained by the obvious composition of morphisms:

$$\mathbf{1} \cong \mathbf{1} \otimes \mathbf{1} \xrightarrow{\varepsilon \otimes \eta} \mathcal{D}(GA, HA) \otimes \mathcal{D}(FA, GA) \xrightarrow{\bullet} \mathcal{D}(FA, HA).$$

As for the **horizontal composite** $\varepsilon * \eta$ of two $\mathcal{K}-$enriched natural transformations:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \mathcal{E} \end{tikzcd}$$

there are two possible composites:

$$\begin{tikzcd} & \mathbf{1} \otimes \mathcal{D}(FA,GA) \arrow[r, "\varepsilon_{GA} \otimes F'"] & \mathcal{E}(F'GA,G'GA) \otimes \mathcal{E}(F'FA,F'GA) \arrow[d, "\bullet_{F'FA,F'GA,G'GA}"'] \\ \mathbf{1} \arrow[r, "\cong"] & \mathbf{1} \otimes \mathbf{1} \arrow[u, "\eta"] \arrow[d, "\eta"'] & \mathcal{E}(F'FA,G'GA) \\ & \mathcal{D}(FA,GA) \otimes \mathbf{1} \arrow[r, "G' \otimes \varepsilon_{FA}"] & \mathcal{E}(G'FA,G'GA) \otimes \mathcal{E}(F'FA,G'FA) \arrow[u, "\bullet_{F'FA,G'FA,G'GA}"] \end{tikzcd}$$

Proposition 2.8. *The two composites in the above diagram provide the same morphism.*

Proof. Replacing $\mathbf{1} \otimes \mathbf{1}$ by $\mathcal{D}(FA, GA)$, the commutativity of the diagram is just the naturality of ε, so we only need to show the following lemma. $\square$

Lemma 2.9. *For any morphism $f\colon \mathbf{1} \to A$ in a monoidal category, the following diagram commutes.*

$$\begin{tikzcd}
\mathbf{1} \otimes \mathbf{1} \arrow[r, "f"] \arrow[d, "f"'] & \mathbf{1} \otimes A \arrow[d, "\lambda_A"'] \\
A \otimes \mathbf{1} \arrow[r, "\rho_A"] & A
\end{tikzcd}$$

Proof. It suffices to show the following diagrams commute,

$$\begin{tikzcd}
\mathbf{1} \otimes \mathbf{1} \arrow[r, "f"] \arrow[d, "\lambda_{\mathbf{1}}"'] & \mathbf{1} \otimes A \arrow[d, "\lambda_A"'] \\
\mathbf{1} \arrow[r, "f"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \otimes \mathbf{1} \arrow[r, "f"] \arrow[d, "\rho_{\mathbf{1}}"'] & A \otimes \mathbf{1} \arrow[d, "\rho_A"'] \\
\mathbf{1} \arrow[r, "f"] & A
\end{tikzcd}$$

which follow from the naturality of λ and ρ. $\square$

Proposition 2.10 (Interchange law)**.** *Consider this situation*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow\eta"] \arrow[r, phantom, shift right=3ex, "\Downarrow\varepsilon"] & \mathcal{D} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow\eta'"] \arrow[r, phantom, shift right=3ex, "\Downarrow\varepsilon'"] & \mathcal{E}
\end{tikzcd}$$

Where $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are $\mathcal{K}-$enriched categories and $\eta, \varepsilon, \eta', \varepsilon'$ are $\mathcal{K}-$enriched natural transformations. Then the following equality holds:

$$(\varepsilon' \circ \eta') * (\varepsilon \circ \eta) = (\varepsilon' * \varepsilon) \circ (\eta' * \eta).$$

Proof. Omitting the natural isomorphisms come from the monoidal category structure of $\mathcal{K}$ and the composition $\bullet$, the above two composites are given by

$$(\varepsilon' \circ \eta') * (\varepsilon \circ \eta) = \varepsilon' \otimes \eta' \otimes (F' \circ \varepsilon) \otimes (F' \circ \eta),$$
$$(\varepsilon' * \varepsilon) \circ (\eta' * \eta) = \varepsilon' \otimes (G' \circ \varepsilon) \otimes \eta' \otimes (F' \circ \eta).$$

Therefore, it suffices to show $\eta' \otimes (F' \circ \varepsilon) = (G' \circ \varepsilon) \otimes \eta'$, which is 2.8 $\square$

Base change

Every $\mathcal{K}-$category $\mathcal{C}$ has a **underlying category**, usually denoted by $\mathcal{C}_0$, whose objects are precisely the objects of $\mathcal{C}$, while the hom-set $\mathrm{Hom}(A,B)$ is defined as $\mathrm{Hom}_{\mathcal{K}}(\mathbf{1},\mathcal{C}(A,B))$.

A special case is that, when $\mathrm{Hom}_{\mathcal{K}}(\mathbf{1},-)\colon \mathcal{K} \to \mathbf{Set}$ has a left adjoint $-\cdot \mathbf{1}\colon \mathbf{Set} \to \mathcal{K}$ (one can verify that, $S \cdot \mathbf{1}$ must be the coproduct of an $S-$indexed set of copies of $\mathbf{1}$), then any ordinary category $\mathcal{C}$ can be regarded as enriched in $\mathcal{K}$ by forming the composite

$$\mathrm{ob}\,\mathcal{C} \times \mathrm{ob}\,\mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}} \mathbf{Set} \xrightarrow{-\cdot\mathbf{1}} \mathcal{K}.$$

More generally, any monoidal functor $F\colon \mathcal{K} \to \mathcal{L}$ between monoidal categories provides a **base change** of enriched categories over them. Indeed, by applying F on the hom-objects, any category enriched over $\mathcal{K}$ gives rise to one enriched over $\mathcal{L}$. One can also verify that this base change is a $2-$functor from $\mathcal{K}-\mathbf{Cat}$ to $\mathcal{L}-\mathbf{Cat}$.

Internal hom

We have seen that the category of abelian groups has a structure of symmetric monoidal category, and in this monoidal category, there exists a right adjoint of the tensor product, which is the *internal hom*.

More precisely, an **internal hom** of a monoidal category $(\mathcal{C},\otimes)$ is a bifunctor

$$\mathrm{hom}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

such that for every $A \in \mathrm{ob}\,\mathcal{C}$, there exists a pair of adjoint functors

$$-\otimes A \vdash \mathrm{hom}(A,-).$$

If a monoidal category has an internal hom, then it will be called a **closed monoidal category**.

Let $(\mathcal{C},\otimes)$ be a closed monoidal category, then by the adjunction, we have a natural isomorphism:

$$\mathrm{Hom}(C \otimes A, B) \cong \mathrm{Hom}(C, \mathrm{hom}(A,B)).$$

Particularly, we have

$$\mathrm{Hom}(\hom(A,B)\otimes A, B) \cong \mathrm{Hom}(\hom(A,B), \hom(A,B)).$$

So, corresponding to the identity $1_{\hom(A,B)}\colon \hom(A,B) \to \hom(A,B)$, there exist a unique morphism from $\hom(A,B)\otimes A$ to B, which is called the **adjunct** of $1_{\hom(A,B)}$. Since the isomorphism is natural in both $A \in \mathrm{ob}\,\mathcal{C}$ and $B \in \mathrm{ob}\,\mathcal{C}$, the adjuncts form a natural transformation:

$$\mathrm{eval}_{A,B} : \hom(A,B)\otimes A \longrightarrow B.$$

This is called the **evaluation map**.

The evaluation map makes the internal hom working like the usual hom-set, even there is actually no elements in the objects. One can see that in a concrete category, the internal hom is just the hom-set equipped extra structures so that it becomes an object.

Now we consider the composite of morphisms. To present this concept using internal hom, we need a natural transformation called the **composition**:

$$\bullet_{A,B,C}\colon \hom(B,C)\otimes\hom(A,B) \longrightarrow \hom(A,C).$$

Recall how two functions $f\colon A \to B$ and $g\colon B \to C$ be composed into one function $g\circ f\colon A \to C$: first, we evaluate f with a value in A, say x, then we evaluate g with the value $f(x)$. This suggests we to define the composition $\bullet_{A,B,C}$ as the *adjunct* of the composite of evaluation maps

$$\hom(B,C)\otimes\hom(A,B)\otimes A \overset{\mathrm{eval}_{A,B}}{\longrightarrow} \hom(B,C)\otimes B \overset{\mathrm{eval}_{B,C}}{\longrightarrow} C.$$

§3 Tensor products of rings

When A and B are both rings, there exists a ring structure on $A \otimes B$ as follow.

Consider two elementary tensors $a_1 \otimes b_1$ and $a_2 \otimes b_2$, we define their product in $A \otimes B$ to be $a_1 a_2 \otimes b_1 b_2$. This definition can be uniquely linearly extended to all tensors. Then one can verify that $A \otimes B$ is a ring with identity $1 \otimes 1$ under this multiplication.

We should show that this ring is the *tensor product* in **Ring**. However, one should notice that the ideal that using bilinear maps to define binary morphisms in **Ring** is not so suitable as in **Ab**.

The problem is that when fix one of the two inputs, say $b \in B$, the result function $\otimes(-, b)\colon A \to A \otimes B$ is NOT a ring homomorphism. Even if we abandon $A \otimes B$ and consider an arbitrary ring C and expect $f\colon A, B \to C$ to be a bilinear map of rings. Then by fixing an arbitrary element $b \in B$ and the requirement that $f(-, b)\colon A \to C$ is a ring homomorphism, $f(1, b)$ must be the identity of C. On the other hand, $f(1, -)\colon B \to C$ is also required to be a ring homomorphism, thus for any $b, b' \in B$, we have

$$f(1, b) + f(1, b') = f(1, b + b').$$

Combine the two requirements, we have to require $1 = f(1, b) = 0$ in C. Thus C must be the zero ring.

Therefore, we should conclude that the linear map is not a suitable concept for rings.

Note that when we try to fix one of the two inputs to get a ring homomorphism. The above discussion shows that we can only fix the identities to make sure $\otimes(-, 1)\colon A \to A \otimes B$ and $\otimes(1, -)\colon B \to A \otimes B$ become ring homomorphisms. But this is just a *sink* of two ring homomorphisms $A \to A \otimes B$ and $B \to A \otimes B$ instead of anything could be called bilinear.

Therefore, a **tensor product** in **Ring** is given by a sink of two ring homomorphisms satisfying some universal properties.

Theorem 3.1. *The tensor product $A \otimes B$, together the two ring homomorphisms $j_1\colon A \to A \otimes B$ and $j_2\colon B \to A \otimes B$, has the following universal property:*

If $f\colon A \to T$ and $g\colon B \to T$ are two ring homomorphisms such that their images commute in T, then there exists a unique homomorphism $u\colon A \otimes B \to T$ such that $u \circ j_1 = f$ and $u \circ j_2 = g$.

Or, we can represent this universal property by Hom,[2]

$$\mathrm{Hom}(A \otimes B, T) \cong \{(f,g) \in \mathrm{Hom}(A,T) \times \mathrm{Hom}(B,T) | \, [\mathrm{im}\, f, \mathrm{im}\, g] = 0\}$$

Proof. Let $f\colon A \to T$ and $g\colon B \to T$ be two ring homomorphisms such that their images commute in T. We construct a bilinear map h as follow:

$$\begin{aligned} h\colon A, B &\longrightarrow T \\ (a,b) &\longmapsto f(a)g(b). \end{aligned}$$

One can verify that h is a bilinear map of abelian groups. Thus there exists a unique group homomorphism $u\colon A \otimes B \to T$ such that $u \circ j_1 = f$ and $u \circ j_2 = g$.

We now prove that u is also a ring homomorphism. Indeed, we have $h(1 \otimes 1) = f(1)g(1) = 1$. Moreover, for any two elementary tensors $a_1 \otimes b_1$ and $a_2 \otimes b_2$, we have

$$\begin{aligned} h(a_1 \otimes b_1)h(a_2 \otimes b_2) &= f(a_1)g(b_1)f(a_2)g(b_2) \\ &= f(a_1)f(a_2)g(b_1)g(b_2) \text{ (since } [\mathrm{im}\, f, \mathrm{im}\, g] = 0) \\ &= f(a_1a_2)g(b_1b_2) \\ &= h(a_1a_2 \otimes b_1b_2) \\ &= h((a_1 \otimes b_1)(a_2 \otimes b_2)). \end{aligned}$$

Then u is also a ring homomorphism and thus is the desired unique one. □

Remark. Let $f\colon A \to T$ and $g\colon B \to T$ be two ring homomorphisms such that their images commute in T. Then, there exists a unique homomorphism $v\colon A \amalg B \to T$ such that $v \circ i_1 = f$ and $v \circ i_2 = g$, where $i_1\colon A \to A \amalg B$ and $i_2\colon B \to A \amalg B$ are the canonical maps to the coproduct. Thus, we have $[\mathrm{im}\, f, \mathrm{im}\, g] = 0$, which is equivalent to $[\mathrm{im}\, i_1, \mathrm{im}\, i_2] \in \ker v$.

[2] Here we use the Lie bracket to simply notations. For two elements a, b in a ring, their Lie bracket is defined as $[a,b] := ab - ba$. For two subsets A, B of a ring, their Lie bracket is the additive subgroup generated by $\{[a,b] | \, a \in A, b \in B\}$.

Let I be the ideal generated by $[\operatorname{im} i_1, \operatorname{im} i_2]$, and $q\colon A \amalg B \to A \amalg B/I$ be the quotient map. Then the universal property of $A \otimes B$ is equivalent to say $A \otimes B \cong A \amalg B/I$.

When A and B are both commutative rings, it is easy to see that $A \otimes B$ is also a commutative ring. Thus we can determine it in **CRing** from the similar universal property as in **Ring**. However, since we are considering commutative rings, all elements commute. Thus by the remark after Theorem 3.1, we conclude that

Corrollay 3.2. *In* **CRing***, tensor products are precisely the coproducts.*

§4 Tensor products of modules

Let A be a commutative ring, we now consider the tensor products in the category $A\,\mathbf{Mod}$ of $A-$modules. $\mathrm{Hom}_A(-,-)$ denote the hom-set in this category. One may also note that it is an *concrete abelian category.*

I Bilinear maps and tensor products

The binary morphisms in $A\,\mathbf{Mod}$ are just $A-$**bilinear maps**. That is a bilinear map f from the underlying abelian groups of two $A-$modules M and N and target to an $A-$module T such that by fixing any of the two inputs, for example $y \in N$, the bilinear map becomes an $A-$homomorphism $f(-,b)\colon M \to T$.

Since a bilinear map from two abelian groups can be represented by a group homomorphism from their tensor product, we can write an $A-$bilinear map from M and N to T as $M \otimes N \to T$. But this notation may cause some confusions, so we still use the notation $M, N \to T$.

Among all $A-$bilinear maps from two $A-$modules M and N, there exists a universal one, called the **tensor product** of M and N (over A) and denoted by $M \otimes_A N$. In other words, the tensor product $M \otimes_A N$, together with the $A-$bilinear map $M, N \to M \otimes_A N$, has the following universal property:

> if $M \otimes N \to T$ is an $A-$bilinear map of $A-$modules, then there exists a unique $A-$homomorphism $M \otimes_A N \to T$ such that this $A-$bilinear map can be factored as $M, N \to M \otimes_A N \to T$.

Or, we can represent this universal property by Hom_A,

$$\mathrm{Hom}_A(M \otimes_A N, T) \cong \mathrm{Bil}_A(M, N; T)$$

where $\mathrm{Bil}_A(M, N; T)$ denote the set of $A-$bilinear maps from M and N to T.

Thus through the tensor products, $A-$bilinear maps can be represented as $A-$homomorphisms in $A\,\mathbf{Mod}$.

Note that, the tensor products are NOT the products or coproducts in $A\,\mathbf{Mod}$ since if they are, then the $A-$bilinear maps must always be $A-$homomorphisms which is not true.

Constructions of tensor products

To see more explicit, we construct the tensor product $M \otimes_A N$ as follow.

First note that, by forgetting $A-$module structures, the $A-$bilinear map $M, N \to M \otimes_A N$ is just a group homomorphism of the underlying abelian groups $q\colon M \otimes N \to M \otimes_A N$. Since we expect $M \otimes_A N$ to be universal among all $A-$bilinear maps, thus q must be surjective. Otherwise, q can factor through the image and one can verify that

Proposition 4.1. *If $M, N \to T$ is an $A-$bilinear map which can factor as a group homomorphism $M \otimes N \to S$ followed by a monomorphism $S \to T$. Then $M, N \to S$ is also an $A-$bilinear map.*

Therefore, $M \otimes_A N \cong M \otimes N / \ker q$

Note that, a group homomorphism $f\colon M \otimes N \to T$ is an $A-$bilinear map if and only if it coequalizes the $A-$module action on N and M as the composite of $A-$module action on T with f itself.

More precisely, recall that a left $A-$module action on N is just a group homomorphism $n\colon A \otimes N \to N$ satisfying certain conditions, thus "by fixing input of M, f becomes an $A-$homomorphism" means nothing but a commutative diagram

$$\begin{tikzcd}
M \otimes A \otimes N \arrow[r, "M\otimes n"] \arrow[d, "\tilde{f}"'] & M \otimes N \arrow[d, "f"] \\
A \otimes T \arrow[r, "t"] & T
\end{tikzcd}$$

Where t is the $A-$module action on T and $\tilde{f}$ denote the composite

$$M \otimes A \otimes N \xrightarrow{\cong} A \otimes M \otimes N \xrightarrow{A \otimes f} T.$$

By similar reasoning on the right $A-$module action $m\colon M \otimes A \to M$ on M, we conclude that f is a $A-$bilinear map if and only if

$$f \circ (M \otimes n) = t \circ \tilde{f} = f \circ (m \otimes N).$$

In other words, f is an $A-$bilinear map if and only if

$$f(ax, y) = af(x, y) = f(x, ay), \forall a \in A, x \in M, y \in N.$$

Thus $M \otimes_A N$ must be a quotient of the coequalizer of the two group homomorphisms

$$M \otimes A \otimes N \rightrightarrows M \otimes N.$$

But the coequalizer has a natural $A-$module action comes from the universal property of coequalizer, which is compatible with the quotient map, thus it is $M \otimes_A N$.

More precisely: $M \otimes_A N \cong M \otimes N / ax \otimes y \sim x \otimes ay$.

The elements in $M \otimes_A N$ are also called **tensors**. We will also denote the image of an elementary tensor $x \otimes y \in M \otimes N$ by $x \otimes y$ again.

Note that, by the universal property, $M \otimes_A N$ is naturally isomorphic to $N \otimes_A M$. This natural isomorphism is called the **symmetric braiding** and denoted by $\gamma_{M,N}$.

Naturality of tensor products

We now go further, to show that the tensor products actually provide a binary functor:

$$\otimes_A \colon A\,\mathbf{Mod} \times A\,\mathbf{Mod} \longrightarrow A\,\mathbf{Mod}.$$

Given an $A-$module M, it is not difficult to see that $X \mapsto M \otimes_A X$ and $X \mapsto X \otimes_A M$ are functors. We denote the corresponding homomorphisms of $f \colon N_1 \to N_2$ under those functors by $M \otimes_A f$ and $f \otimes_A M$ respectively. In the case of not confusing, we will omit the identity and simply identify them with f.

Moreover, we have

Proposition 4.2. *Let $f \colon M_1 \to N_1$ and $g \colon M_2 \to N_2$ be two $A-$homomorphisms, then the following diagram commutes.*

$$\begin{tikzcd} M_1 \otimes_A N_1 \arrow[r] \arrow[d] & M_2 \otimes_A N_1 \arrow[d] \\ M_1 \otimes_A N_2 \arrow[r] & M_2 \otimes_A N_2 \end{tikzcd}$$

The result homomorphism $M_1 \otimes_A N_1 \to M_2 \otimes_A N_2$ is usually denoted by $f \otimes_A g$.

Proof. Consider an arbitrary $A-$bilinear map $h\colon M_2, N_2 \to T$, then one can see $h(f(-), g(-))\colon M_1, N_1 \to T$ is an $A-$bilinear map. Thus the unique homomorphism $M_1 \otimes_A N_1 \to T$ factors through a unique homomorphism $M_1 \otimes_A N_1 \to M_2 \otimes_A N_2$. However, one can verify that both of the composites in the above square satisfy this property. Thus they must be the same homomorphism. □

Remark. With the naturality of tensor products, we can define the tensor product of natural transformations pointwise.

In $A\,\mathbf{Mod}$, the A itself plays a special role in the theory of tensor products. Indeed, it serves like a unit of multiplication.

Theorem 4.3. *For any $A-$module M, we have two natural isomorphisms:*

$$A \otimes_A M \cong M, \qquad M \otimes_A A \cong M.$$

Proof. Since we already have a natural isomorphism $A \otimes_A M \cong M \otimes_A A$, we only need to prove the first one.

Let T be an arbitrary $A-$module, by Yoneda lemma, we only need to show there exists a natural (on T) bijection:

$$\mathrm{Bil}_A(A, M; T) \cong \mathrm{Hom}_A(M, T).$$

Let $f\colon A, M \to T$ be a bilinear map, then $f(1, -)\colon M \to T$ is a homomorphism. Conversely, If $g\colon A \to T$ is a homomorphism, then we can define an $A-$bilinear map $\tilde{g}\colon A, M \to T$ by $\tilde{g}(a, x) := g(ax)$.

It is not difficult to verify that $f \mapsto f(1, -)$ and $g \mapsto \tilde{g}$ are inverse for each, thus provide bijections as desired. Moreover, the naturality of those bijections are obvious. □

Remark. The natural isomorphisms $\lambda_M\colon A \otimes_A M \to M$ and $\rho_M\colon M \otimes_A A \to M$ are called the **left unitor** and **right unitor**.

II Basic properties of tensor products

A distribution law

We prove a distribution law.

Theorem 4.4. *Let M be an $A-$module and $\{N_i\}_{i\in I}$ a set of $A-$modules. Then we have the following natural isomorphisms*

$$M \bigotimes_A \left(\bigoplus_{i\in I} N_i \right) \cong \bigoplus_{i\in I} \left(M \bigotimes_A N_i \right).$$

Proof. Let T be an arbitrary $A-$module, by Yoneda Lemma, we only need to show there exists a natural (on T) bijection

$$\mathrm{Bil}_A(M, \bigoplus_{i\in I} N_i; T) \cong \prod_{i\in I} \mathrm{Bil}_A(M, N_i; T).$$

where the right is naturally isomorphic to $\mathrm{Hom}(\bigoplus_{i\in I}(M \otimes_A N_i), T)$.

Indeed, for any $A-$bilinear maps $f\colon M, \bigoplus_{i\in I} N_i \to T$, by fixing $x \in M$, we get a homomorphism $f(x,-)\colon \bigoplus_{i\in I} M_i \to T$, thus a set of homomorphisms $\{f(x, \imath_i(-))\}_{i\in I}$, where $\imath_i$ denote the canonical map $N_i \to \bigoplus_{i\in I} N_i$. One can verify that $f(-, \imath_i(-))\colon M, N_i \to T$ is an $A-$bilinear map. Thus we get

$$\begin{aligned} \mathrm{Bil}_A(M, \bigoplus_{i\in I} N_i; T) &\longrightarrow \prod_{i\in I} \mathrm{Bil}(M, N_i; T) \\ f &\longmapsto \{f(-, \imath_i(-))\}_{i\in I}. \end{aligned}$$

Conversely, given a set of $A-$bilinear maps $\{g_i\colon M, N_i \to T\}_{i\in I}$, we can get a set of homomorphisms $\{g_i(x,-)\colon N_i \to T\}_{i\in I}$ by fixing $x \in M$. Then they factor through a unique homomorphism $g^{(x)}\colon \bigoplus_{i\in I} N_i \to T$. For any $x \in M$ and $y \in \bigoplus_{i\in I} N_i$, let $g(x,y) := g^{(x)}(y)$. One can verify that g is an $A-$bilinear map. Thus we get

$$\begin{aligned} \prod_{i\in I} \mathrm{Bil}(M, N_i; T) &\longrightarrow \mathrm{Bil}(M, \bigoplus_{i\in I} N_i; T) \\ \{g_i\}_{i\in I} &\longmapsto g. \end{aligned}$$

Moreover, by combining those two maps, one can see they are inverse for each other. Thus they are bijections as desired. The naturality of those bijections on T is obvious. $\square$

Remark. From the bijections above, one can see that for an elementary tensor $x \otimes (y_i)_{i\in I} \in M \otimes_A (\bigoplus_{i\in I} N_i)$, its image in $\bigoplus_{i\in I}(M \otimes_A N_i)$ is $(x \otimes$

$y_i)_{i\in I}$. Conversely, for $(x_i \otimes y_i)_{i\in I} \in \bigoplus_{i\in I}(M \otimes_A N_i)$, its image in $M \otimes_A (\bigoplus_{i\in I} N_i)$ is $\sum_{i\in I} x_i \otimes \imath_i(y_i)$. One can also verify from this corresponding directly.

However, $M \otimes_A (\prod_{i\in I} N_i)$ is NOT isomorphic to $\prod_{i\in I}(M \otimes_A N_i)$ in general.

However, we have

Proposition 4.5. *Let M be an $A-$module and $\{N_i\}_{i\in I}$ a set of $A-$modules. Then we have the following natural homomorphism*

$$M \bigotimes_A \left(\prod_{i\in I} N_i\right) \to \prod_{i\in I}\left(M \bigotimes_A N_i\right).$$

Proof. First, there exists a set of natural homomorphisms $p_i \colon \prod_{i\in I} N_i \to N_i$, i.e. the canonical projrctions. Then we get a set of natural homomorphisms $M\otimes_A p_i \colon M\otimes_A(\prod_{i\in I} N_i) \to M\otimes_A N_i$, thus a natural homomorphism from the universal property of product as desired. □

Tensor products and hom-functors

Recall that the set $\mathrm{Hom}_A(M, N)$ of $A-$homomorphisms between two $A-$modules has a natural $A-$module structure. For distinguish, we denote this $A-$module by $\mathrm{hom}_A(M, N)$. Moreover, one can verify that $\mathrm{hom}_A(-, -)$, like $\mathrm{Hom}_A(-, -)$, is a bifunctor, but its codomain is $A\,\mathbf{Mod}$ instead of $\mathbf{Set}$.

We have

$$\mathrm{Hom}_A(M \otimes N, T) \cong \mathrm{Bil}_A(M, N; T).$$

But every $A-$bilinear map $f \colon M, N \to T$ can also be view as an $A-$homomorphism from M to $\mathrm{hom}_A(N, T)$ by

$$\begin{aligned} M &\longrightarrow \mathrm{hom}_A(N, T) \\ x &\longmapsto f(x, -) \end{aligned}$$

Thus $- \otimes_A N$ is the left adjoint of the covariant functor $\mathrm{hom}_A(N, -)$. Then by Lemma 1.6, it preverses colimits, in particular, it is right exact.

III Multilinear maps and $n-$fold tensor products

Like in **Ab**, multilinear maps can also be define for $A-$modules.

A $0-$**linear map** is just an element, thus the **nullary tensor product** is just the initial object 0 in $A\,$**Mod**.

A $n-$**linear map** is a $n-$ary function from the underlying sets of $A-$modules $M_1, M_2, \cdots, M_n$ targeting to an $A-$module N such that by fixing any $n-1$ components of the n inputs, the $n-$ary function becomes an $A-$homomorphism. Then the $n-$**ary tensor product** is the universal $n-$linear map.

Thus the $n-$**ary tensor product** is the coequalizer of the $A-$module actions on components of the tensor product $M_1 \otimes M_2 \otimes \cdots \otimes M_n$ of the underlying abelian groups of $A-$modules $M_1, M_2, \cdots, M_n$, equipped with natural $A-$modlue action on it.

Since the $n-$ary coequalizer can be constructed from the binary ones, thus so does the $n-$ary tensor product.

To see this, we prove the following result as an example.

Theorem 4.6. *Let M_1, M_2, M_3 be three $A-$modules, then*

$$(M_1 \otimes_A M_2) \otimes_A M_3 \cong M_1 \otimes_A (M_2 \otimes_A M_3)$$

Proof. We check that both $(M_1 \otimes_A M_2) \otimes_A M_3$ and $M_1 \otimes_A (M_2 \otimes_A M_3)$ satisfy the universal property of ternary tensor products.

First of all, note that $(M_1 \otimes_A M_2) \otimes_A M_3$ is the coequalizer of the $A-$module actions of $M_1 \otimes_A M_2$ and M_3 (denote by the solid arrow and the dashed arrow at the bottom in the diagram below), and that the $A-$module action of $M_1 \otimes_A M_2$ comes from being the coequalizer of the $A-$module actions of M_1 and M_2 (denoted by the solid arrows in the diagram below).

$$\begin{tikzcd}
A \otimes M_1 \otimes M_2 \otimes M_3 \arrow[r, shift left=2] \arrow[r] \arrow[r, dashed, shift right=2] \arrow[d, "A\otimes p"'] & M_1 \otimes M_2 \otimes M_3 \arrow[d, "p"] & \\
A \otimes (M_1 \otimes_A M_2) \otimes M_3 \arrow[r, shift left=1] \arrow[r, dashed, shift right=1] & (M_1 \otimes_A M_2) \otimes M_3 \arrow[r, "q"] & (M_1 \otimes_A M_2) \otimes_A M_3
\end{tikzcd}$$

Then one can immediately see that $(M_1 \otimes_A M_2) \otimes_A M_3$ coequalizes the $A-$module actions of M_1, M_2 and M_3.

Let $f\colon M_1, M_2, M_3\colon T$ be a trilinear map of $A-$modules, then since f coequalize the $A-$module actions of M_1 and M_2, thus there exists a unique group homomorphism $g\colon (M_1 \otimes_A M_2) \otimes M_3 \to T$ compatible with the $A-$module actions of $M_1 \otimes_A M_2$ and T such that $f = g \circ p$.

Then the previous diagram now becomes the following one. Here, the horizontal arrows are $A-$module actions except q, and the solid arrows form commutative squares.

$$\begin{tikzcd}
A \otimes M_1 \otimes M_2 \otimes M_3 \arrow[r, shift left] \arrow[r, dashed, shift right] \arrow[d, "A\otimes p"'] & M_1 \otimes M_2 \otimes M_3 \arrow[d, "p"] & \\
A \otimes (M_1 \otimes_A M_2) \otimes M_3 \arrow[r, shift left] \arrow[r, dashed, shift right] \arrow[d, "A\otimes g"'] & (M_1 \otimes_A M_2) \otimes M_3 \arrow[r, "q"] \arrow[d, "g"] & (M_1 \otimes_A M_2) \otimes_A M_3 \\
A \otimes T \arrow[r] & T &
\end{tikzcd}$$

Then since p is an epimorphism, so is $A \otimes p$ and thus the commutativity of the outer square and the upper squares induce the commutativity of the lower ones. Thus g coequalizes the $A-$module actions of $M_1 \otimes_A M_2$ and M_3. Since q is the coequalizer, there exists a unique $A-$homomorphism $u\colon (M_1 \otimes_A M_2) \otimes_A M_3 \to T$ such that $g = u \circ q$. Then $f = u \circ (q \circ p)$ as desired and the uniqueness of u comes obviously.

For $M_1 \otimes_A (M_2 \otimes_A M_3)$, the proof is similar. □

Remark. By the universal property, there exists a unique isomorphism

$$\alpha_{M_1,M_2,M_3}\colon (M_1 \otimes_A M_2) \otimes_A M_3 \xrightarrow{\cong} M_1 \otimes_A (M_2 \otimes_A M_3).$$

Moreover, this isomorphism is natural. This natural isomorphism α is called the **associator** of $A\,\mathbf{Mod}$.

Furthermore, one can verify the *pentagon identity*, *triangle identity*, *hexagon identity* and *involution law*. Then we conclude:

Theorem 4.7. $(A\,\mathbf{Mod}, \otimes_A, A, \alpha, \lambda, \rho, \gamma)$ *is a symmetric monoidal category.*

Therefore, the *$n-$ary tensor products* can be constructed as *$n-$fold tensor products.*

§5 Tensor products of algebras

§6 Tensor products of chain complexes

§7 Tensor products of others

Index

Annotated Bibliography

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Further readings on monoidal categories