中图分类号: O154 学校代码: 10055
UDC: 512 密级: 公开

南開大學

硕 士 学 位 论 文

Pre-Lie 代数的扩张与非阿贝尔上同调

Extensions and non-abelian cohomology of pre-Lie algebras

论文作者 高煦 指导教师 白承铭 教授

申请学位 理学硕士 培养单位 陈省身数学研究所

学科专业 基础数学 研究方向 李理论与数学物理

答辩委员会主席 苏育才教授 评 阅 人 冯惠涛、王红

南开大学研究生院

二〇一五年四月

南开大学学位论文使用授权书

本人完全了解《南开大学关于研究生学位论文收藏和利用管理办法》关于南开大学 (简称“学校”) 研究生学位论文收藏和利用的管理规定，同意向南开大学提交本人的学位论文电子版及相应的纸质本，并委托印刷存档论文。

本人了解南开大学拥有在《中华人民共和国著作权法》规定范围内的学位论文使用权，同意在以下几方面向学校授权。即：

1. 学校将学位论文编入《南开大学博硕士学位论文全文数据库》，并作为资料在学校图书馆等场所提供阅览，在校园网上提供论文目录检索、文摘以及论文全文浏览、下载等信息服务；

2. 学校可以采用影印、缩印或其他复制手段保存学位论文；学校根据规定向教育部指定的收藏和存档单位提交学位论文；

3. 非公开学位论文在解密后的使用权同公开论文。

4. 同意学校将本人向有关电子出版单位授权的学位论文 (含电子版和授权书) 转交相关授权单位。

本人承诺：本人的学位论文是在南开大学学习期间创作完成的作品，并已通过论文答辩；提交的学位论文电子版与纸质本论文的内容一致，如因不同造成不良后果由本人自负。

本人签署本授权书一份，交图书馆留存。

学位论文作者暨授权人 (亲笔) 签字：高煦

2015 年 6 月 8 日

南开大学研究生学位论文作者信息

论文题目	Pre-Lie 代数的扩张与非阿贝尔上同调				
姓名	高煦	学号	2120120012	答辩日期	2015 年 5 月 29 日
论文类别	博士 □	学历硕士 ☑	硕士专业学位 □	同等学力硕士 □	划☑选择
学院 (单位)	陈省身数学研究所	学科/专业 (专业学位) 名称	基础数学		
联系电话	15620798461	电子邮箱	GauSyu@Gmail.com		
通讯地址 (邮编)：	天津市南开区卫津路 94 号南开大学省身楼 (300071)				
非公开论文编号		备注			

注：本授权书适用我校授予的所有博士、硕士的学位论文。由作者填写一份并签字后交校图书馆，如已批准为非公开学位论文，须附批准通过的《南开大学研究生申请非公开学位论文审批表》和“非公开学位论文标注说明”页。

南开大学学位论文原创性声明

本人郑重声明：所呈交的学位论文，是本人在导师指导下进行研究工作所取得的研究成果。除文中已经注明引用的内容外，本学位论文的研究成果不包含任何他人创作的、已公开发表或者没有公开发表的作品的内容。对本论文所涉及的研究工作做出贡献的其他个人和集体，均已在文中以明确方式标明。本学位论文原创性声明的法律责任由本人承担。

学位论文作者签名：高聪　　　　2015 年　6 月　8 日

非公开学位论文标注说明

根据南开大学有关规定，非公开学位论文须经指导教师同意、作者本人申请和相关部门批准方能标注。未经批准的均为公开学位论文，公开学位论文本说明为空白。

论文题目			
申请密级	□ 限制 (≤2 年)　□ 秘密 (≤10 年)　□ 机密 (≤20 年)		
保密期限	20　年　月　日至 20　年　月　日		
审批表编号		**批准日期**	20　年　月　日

南开大学学位办公室盖章 (有效)

限制 □ 2 年（最长 2 年，可少于 2 年）

秘密 □10 年（最长 5 年，可少于 5 年）

机密 □20 年（最长10年，可少于10年）

中文摘要

本文研究了 pre-Lie 代数的扩张，定义了 pre-Lie 代数的二阶非阿贝尔上同调，并证明 pre-Lie 代数的扩张为其所分类. 考虑阿贝尔扩张的分类，又得到其与二阶阿贝尔上同调的关系. 然后，本文通过在 pre-Lie 代数的 Chevalley-Eilenberg 复形上定义一种 pre-Lie 代数结构，得到一个分次微分李代数 $\mathfrak{L}^{\bullet}$. Pre-Lie 代数扩张的范畴等价于 $\mathfrak{L}^{\bullet}$ 上的 Deligne 群胚，从而为其连通分支之集合所分类. 同时，二阶阿贝尔上同调可以通过考察 $\mathfrak{L}^{\bullet}$ 的切线复形而得到. 最后，本文指出虽然 pre-Lie 代数的二阶非阿贝尔上同调并不天然是 $\mathcal{PL}_{\infty}$-代数的内在上同调，但 pre-Lie 代数的中心扩张能被一种二项 $\mathcal{PL}_{\infty}$-代数的内在上同调所刻画.

关键词： 非阿贝尔上同调；扩张；Deligne 群胚；Pre-Lie 代数；代数形变理论；内在上同调

Abstract

In this paper, the second non-abelian pre-Lie algebra cohomology is introduced by considering the classification of extensions of pre-Lie algebras. This notion is related to the second abelian cohomology, which classifies abelian extensions, and thus gets its name. Then the author introduces a pre-Lie structure on the Chevalley-Eilenberg complex of pre-Lie algebras and constructs a differential graded Lie algebra $\mathfrak{L}^\bullet$. The author then proves that the category of extensions of pre-Lie algebras is equivalent to the Deligne groupoid $\mathrm{Del}(\mathfrak{L}^\bullet)$. Then the second non-abelian pre-Lie algebra cohomology comes naturally as the set of connected components of the Deligne groupoid $\mathrm{Del}(\mathfrak{L}^\bullet)$. Further, the second abelian cohomology arises from the tangent complex of $\mathfrak{L}^\bullet$. Finally, the author points out that although the second non-abelian pre-Lie algebra cohomology is not naturally an intrinsic cohomology, the central extensions of pre-Lie algebras can be classified by an intrinsic cohomology of 2-term $\mathcal{PL}_\infty$-algebras.

Key Words: Non-abelian cohomology; Extensions; Deligne groupoid; Pre-Lie algebras; Algebraic deformation theory; Intrinsic cohomology

CONTENTS

Chapter 1 Introduction

Extensions of groups are classified by non-abelian group cohomology. Specially, abelian extensions are classified by abelian group cohomology. This fact has been found by Eilenberg and Maclane in the 1940s ([11]). Then there are a lot of analogous results for Lie algebras ([1, 16, 17]), Lie superalgebras ([2, 12]) and Lie algebroids ([5]).

Pre-Lie algebras, also called left-symmetric algebras, quasi-associative algebras, Koszul-Vinberg algebras etc., are nonassociative algebras whose commutators form Lie algebras and whose left multiplications form representations of the commutator Lie algebras. They have appeared in Cayley's work on rooted tree algebras in 1896 ([9]). Then they were forgotten for a long time before arising from the study of several topics in geometry and algebra in 1960s, such as convex homogenous cones ([28]), affine manifolds and affine structures on Lie groups ([20, 24]), deformation of associative algebras ([14]) etc. A survey of this history and the algebraic theory of pre-Lie algebras have been given by Burde in [7]. In 1980s, Kim studied the extensions of pre-Lie algebras and defined the non-abelian cohomology for pre-Lie algebras in [18, 19].

However, the collection of all extensions of an algebra by another is naturally a groupoid, rather than a set, so the second non-abelian cohomology should be viewed as the set of connected components of a groupoid. Moreover, if this groupoid arises as an action groupoid, it is naturally connected to the deformation theory.

In 2012, Frégier found that the second non-abelian cohomology classifying Lie algebra extensions can also be described in terms of the Deligne groupoid, seeing [13]. The notion of Deligne groupoid comes from the ideas of Deligne on deformation theory, which were transmitted via [15]. I refer [23] for more information about this approach of deformation theory.

The Deligne groupoid used in Frégier's work arises from a differential graded Lie algebra (DGLA for short) related to the Chevalley-Eilenberg complex of Lie algebras. However, this DGLA is obtained by taking the graded commutator of a graded pre-Lie algebra, thus it is natural to ask if the analogous results hold for pre-Lie algebras.

There is a general perspective on cohomology, that is the intrinsic cohomology for a higher category. It is nothing but the set of connected components of the hom-space with possible extra structures induced from the coefficents. This idea was essentially established in [6] and developed recently by Lurie ([21]). A comprehensive account can be found in [26].

In [30], the author pointed out that the extensions of Lie algebras can be described by a hom-space of 2-term L_∞-algebras and the second non-abelian Lie algebra cohomology therefore can be viewed as a special case of the intrinsic cohomology of L_∞-algebras in a natural way. However, this approach relies on the antisymmetry of Lie bracket. So it is doubtful whether this can be done for other kind of algebras.

In this paper, I study the extensions of pre-Lie algebras and the second non-abelian pre-Lie algebra cohomology. By constructing a suitable DGLA, I get the analogous results of Frégier for pre-Lie algebras. Lacking of enough symmetry, the approach encoding the second non-abelian cohomology into an intrinsic cohomology fails for pre-Lie algebras. However, I show that the central extensions of pre-Lie algebras can be classified by an intrinsic cohomology.

The whole article is organized as follows. First, in chapter 2, I summarize some well-known results on Lie algebra extensions with a little different style of presentation, and make some conventions. For self-contained, I also explain the notion of Deligne groupoid and homotopy algebras. Some notions from higher category theory are used. So, I explain them in a strict 2-categorical context and put this part as an appendix.

Then, in chapter 3, I study the extensions of pre-Lie algebras and define the second non-abelian pre-Lie algebra cohomology to classify them. I also discuss the relationship between the non-abelian cohomology with the abelian one, which classifies abelian extensions. Although those results have been given by Kim ([18, 19]), for self-contained, I explain them in details.

Then, in chapter 4, I introduce a pre-Lie structure on the Chevalley-Eilenberg complex of pre-Lie algebras and construct the DGLA $\mathfrak{L}^\bullet$. I prove that the category of extensions of pre-Lie algebras is equivalent to the Deligne groupoid $\mathrm{Del}(\mathfrak{L}^\bullet)$ of the DGLA $\mathfrak{L}^\bullet$. Then, the second non-abelian pre-Lie algebra cohomology arises naturally as the set of connected components of $\mathrm{Del}(\mathfrak{L}^\bullet)$ and the groups of non-abelian 1-cocycles arises naturally as the automorphism groups in $\mathrm{Del}(\mathfrak{L}^\bullet)$. Further, the abelian cohomology arises from the tangent complex of $\mathfrak{L}^\bullet$.

Finally, in chapter 5, I recall the notion of 2-term $\mathcal{PL}_\infty$-algebras. Then I explain why the approach for Lie algebras fails for pre-Lie algebras. After all, I show that the central extensions can be classified by an intrinsic cohomology of 2-term $\mathcal{PL}_\infty$-algebras.

Chapter 2 Preliminaries

Throughout this paper, all vector spaces are over a given field k. To simplify notations, I also make the following conventions:

1. If there is no ambiguity, the composite symbol $\circ$ will be omitted.
2. If ρ is a linear map from some vector space A to the generalized linear Lie algebra $\mathfrak{gl}(V)$ for some vector space V, then for all $x \in A$, $\rho(x)$ will be also written as ρ_x when it is viewed as a linear map on V.
3. The subscript $\mathfrak{g}$ of a Lie bracket $[,]_{\mathfrak{g}}$ emphasizes which algebra this bracket is working, and will be omitted if there is no ambiguity.
4. The cyclic sum notation $\sum_{x,y,z}^{\circlearrowleft}$ is used frequently, where the summation is taken over all cyclic permutation of x, y and z.

Some terminologies and basic facts from category theory and homology algebra, such as 5-lemma, complex, groupoid etc., are frequently used. There are many textbooks on those topics, like [8, 22, 29].

2.1 Extensions of Lie algebras and the non-abelian cohomology

2.1.1 Let $\mathfrak{g}, \widehat{\mathfrak{g}}, \mathfrak{h}$ be Lie algebras. $\widehat{\mathfrak{g}}$ is said to be an ***extension*** of $\mathfrak{g}$ by $\mathfrak{h}$ if there exist a short exact sequence

$$0 \longrightarrow \mathfrak{h} \xrightarrow{\mathfrak{i}} \widehat{\mathfrak{g}} \xrightarrow{\mathfrak{p}} \mathfrak{g} \longrightarrow 0.$$

A ***splitting*** of $\widehat{\mathfrak{g}}$ is a linear map $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ such that $\mathfrak{p} \circ \sigma = \mathrm{id}$.

A *morphism* $\theta\colon \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}'$ of two extensions is a Lie algebra morphism θ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{h} \arrow[r, "\mathfrak{i}"] \arrow[d, equal] & \widehat{\mathfrak{g}} \arrow[r, "\mathfrak{p}'"] \arrow[d, "\theta"] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{h} \arrow[r, "\mathfrak{i}'"] & \widehat{\mathfrak{g}}' \arrow[r, "\mathfrak{p}'"] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

By 5-lemma, this θ, if exists, must be an isomorphism. If this is the case, the two extensions $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}'$ are said to be *isomorphic*.

Now all extensions of $\mathfrak{g}$ by $\mathfrak{h}$, together with the morphisms between them, form a category, denoted by $\mathcal{E}xt_{Lie}(\mathfrak{g},\mathfrak{h})$. Moreover, this category is a ***groupoid***, that is a category whose every morphism is invertible. One can then consider the set $\mathrm{Ext}_{Lie}(\mathfrak{g},\mathfrak{h})$ of isomorphism classes of $\mathcal{E}xt_{Lie}(\mathfrak{g},\mathfrak{h})$.

Remark $\mathcal{E}xt_{Lie}(\mathfrak{g},\mathfrak{h})$ is naturally a *pointed groupoid* as it contains a special object, which is the Lie algebra direct sum $\mathfrak{g}\oplus\mathfrak{h}$. Thus $\mathcal{E}xt_{Lie}(\mathfrak{g},\mathfrak{h})$ is naturally a *pointed set*.

2.1.2 A ***non-abelian*** 2-***cocycle*** on $\mathfrak{g}$ with values in $\mathfrak{h}$ is a couple (ω,ρ) of an alternating bilinear map $\omega\colon \wedge^2\mathfrak{g}\to\mathfrak{h}$ and a linear map $\rho\colon \mathfrak{g}\to\mathrm{Der}(\mathfrak{h})$ satisfying the following equalities for all $x,y,z\in\mathfrak{g}$:

$$[\rho(x),\rho(y)]-\rho([x,y])=\mathrm{ad}(\omega(x,y)),$$
$$\sum_{x,y,z}^{\circlearrowleft}\rho_x\omega(y,z)-\omega([x,y],z)=0.$$

The set of those 2-cocycles is denoted by $Z^2_{Lie}(\mathfrak{g},\mathfrak{h})$.

Two 2-cocycles (ω,ρ) and (ω',ρ') are *equivalent*, if there exists a linear map $\varphi\colon\mathfrak{g}\to\mathfrak{h}$ such that $\rho'-\rho=\mathrm{ad}\circ\varphi$, and for all $x,y\in\mathfrak{g}$,

$$\omega'(x,y)-\omega(x,y)=\rho_x\varphi(y)-\rho_y\varphi(x)+[\varphi(x),\varphi(y)]-\varphi([x,y]).$$

The set $H^2_{Lie}(\mathfrak{g},\mathfrak{h})$ of equivalence classes of non-abelian 2-cocycle is called the ***second non-abelian cohomology*** of the Lie algebra $\mathfrak{g}$ with values in $\mathfrak{h}$.

2.1.3 *Extensions of* $\mathfrak{g}$ *by* $\mathfrak{h}$ *are classified by* $H^2_{Lie}(\mathfrak{g},\mathfrak{h})$.

More precisely, any 2-cocycle (ω,ρ) defines a Lie bracket on $\mathfrak{g}\oplus\mathfrak{h}$ via

$$[x+u,y+v]_{\omega,\rho}:=[x,y]_{\mathfrak{g}}+\omega(x,y)+\rho_x(v)-\rho_y(u)+[u,v]_{\mathfrak{h}},\quad \forall x,y\in\mathfrak{g},u,v\in\mathfrak{h}.$$

This gives a Lie algebra structure on $\mathfrak{g}\oplus\mathfrak{h}$, called the ***semidirect product*** $\mathfrak{g}\ltimes_{\omega,\rho}\mathfrak{h}$, and one can see it is an extension of $\mathfrak{g}$ by $\mathfrak{h}$.

Conversely, given an extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathfrak{h}$, by choosing a splitting σ and identify $\mathfrak{h}$ with its image in $\widehat{\mathfrak{g}}$, one can define an alternating bilinear map $\omega\colon\wedge^2\mathfrak{g}\to\mathfrak{h}$ and a linear map $\rho\colon\mathfrak{g}\to\mathrm{Der}(\mathfrak{h})$ as follows:

$$\omega(x,y):=[\sigma(x),\sigma(y)]_{\widehat{\mathfrak{g}}}-\sigma([x,y]_{\mathfrak{g}}),\quad \forall x,y\in\mathfrak{g},$$
$$\rho_x(u):=[\sigma(x),u]_{\widehat{\mathfrak{g}}},\quad \forall x\in\mathfrak{g},y\in\mathfrak{h}.$$

One can check this couple (ω,ρ) is a 2-cocycle and the cohomological class of it is independent of the choice of σ.

Finally, the equivalence of 2-cocycles corresponding to the isomorphism of extensions. The details can be found in [1, 16, 17].

2.2 The notion of Deligne groupoid

In this section, I recall the definition of Deligne groupoid. For self-contained, I explain the approach in details.

2.2.1 A ***$\mathbb{Z}$-graded vector space*** is a direct sum $V^\bullet = \bigoplus_{n\in\mathbb{Z}} V^n$ of countable many vector spaces. An element in a term V^n is called a *homogeneous element* of *degree* $|x| = n$. A *graded linear map* $\varphi\colon V^\bullet \to W^\bullet$ of degree k is a linear map satisfying $\varphi(V^n) \subset W^{n+k}$. The *graded tensor product* of two graded vector spaces $V^\bullet$ and $W^\bullet$ is a graded vector space $V^\bullet \otimes W^\bullet$ with grading:

$$(V^\bullet \otimes W^\bullet)^n := \bigoplus_{i+j=n} V^i \otimes W^j.$$

A *graded bilinear map* is precisely a graded linear map from graded tensor product.

A ***graded vector space*** is a $\mathbb{Z}$-graded vector space whose every term in negative degree vanishes. A $\mathbb{Z}$-graded vector space is said to be *lower-bounded* if it can be identified with a graded vector space by a degree shifting.

A ***graded Lie algebra*** is a graded vector space $\mathfrak{g} = \bigoplus_{n\geqslant 0} \mathfrak{g}^n$ equipped with a graded bilinear map $[,]\colon \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ of degree 0 satisfies

1. the *graded antisymmetry*, i.e. $[x,y] = -(-1)^{|x||y|}[y,x]$,
2. the *graded Jacobin identity*, i.e. $\sum^{\circlearrowleft}{}_{x,y,z}(-1)^{|x||z|}[x,[y,z]] = 0$.

Here x,y,z are all homogeneous elements in $\mathfrak{g}$. For any $x \in \mathfrak{g}^n$, the *graded adjoint transformation* $\mathfrak{ad}_x$ is defined as $\mathfrak{ad}_x := [x,-]$. This is a graded linear map of degree n.

A ***differential graded Lie algebra*** (***DGLA*** for short), is a graded Lie algebra $\mathfrak{g}$ equipped with a *cohomological derivation* $d\colon \mathfrak{g} \to \mathfrak{g}$ of degree 1, that means a graded linear map of degree 1 satisfies $d^2 = 0$ and

$$d[x,y] = [dx,y] + (-1)^{|x|}[x,dy],$$

for all homogeneous elements x,y of $\mathfrak{g}$.

2.2.2 Let $\mathfrak{g}$ be a nilpotent Lie algebra. One can use the *Baker-Campbell-Hausdorff formula*

$$x*y := x+y+\frac{1}{2}[x,y]+\frac{1}{12}([x,[x,y]]-[y,[x,y]])-\frac{1}{24}[y,[x,[x,y]]]+\cdots$$

to give a group structure on $\mathfrak{g}$. This group will be denoted by $\exp(\mathfrak{g})$. The full formula can be found in some textbooks like [3, ch.II]. Alternatively, one can also view $\mathfrak{g}$ as the tangent Lie algebra of a Lie group G, then the exponential map exp is well defined. Anyhow, one can get a connected Lie group $\exp(\mathfrak{g})$ such that its tangent Lie algebra is $\mathfrak{g}$ and that the exponential map is bijective.

Note that, when $\mathfrak{g}$ is abelian, the group $\exp(\mathfrak{g})$ will coincide with the underlying abelian group of $\mathfrak{g}$.

2.2.3 Let $(\mathfrak{g}, d)$ be an $\mathfrak{ad}_0$-*nilpotent* DGLA, that means for all $x \in \mathfrak{g}^0$, the graded adjoint transformation $\mathfrak{ad}_x$ is nilpotent. Then the set of ***Maurer-Cartan elements*** is defined as

$$MC(\mathfrak{g}) := \left\{ \alpha \in \mathfrak{g}^1 \middle| d\alpha + \frac{1}{2}[\alpha, \alpha] = 0 \right\}.$$

Any $\alpha \in \mathfrak{g}^1$ defines a graded derivation of degree 1 by the formula $d_\alpha = d + \mathrm{ad}_\alpha$, called $\mathfrak{g}$-***connection***. Specially, Maurer-Cartan elements define ***flat connections*** in the sense of $d_\alpha^2 = 0$. If this is the case, one can see $(\mathfrak{g}, d_\alpha)$ becomes another DGLA, called the ***tangent complex*** $\mathfrak{g}_\alpha$ of $\mathfrak{g}$ at α.

The subspace $\mathfrak{g}^0$ of $\mathfrak{g}$ itself is then a nilpotent Lie algebra. The corresponding Lie group $\exp(\mathfrak{g}^0)$ acts on $\mathfrak{g}$ as

$$\begin{aligned} \exp(\mathfrak{g}^0) &\longrightarrow \mathrm{GL}(\mathfrak{g}) \\ \exp(x) &\longmapsto e^{\mathfrak{ad}_x} := \sum_{n\geqslant 0} \frac{\mathfrak{ad}_x^n}{n!}. \end{aligned}$$

Then this group acts on $\mathfrak{gl}(\mathfrak{g})$ by conjugation. Such an action is the same as the adjoint action of $\mathrm{GL}(\mathfrak{g})$ on $\mathfrak{gl}(\mathfrak{g})$, thus one has

$$\exp(x)\varphi\exp(-x) = e^{\mathfrak{ad}_x} \circ \varphi \circ e^{-\mathfrak{ad}_x} = e^{\mathrm{ad}_{\mathfrak{ad}_x}}(\varphi), \quad \forall \varphi \in \mathfrak{gl}(\mathfrak{g}).$$

Here the notation ad_φ denotes the adjoint transformation of $\mathfrak{gl}(\mathfrak{g})$.

Remark 1 In a DGLA $(\mathfrak{g}, d)$, for any $x \in \mathfrak{g}^0, \alpha \in \mathfrak{g}^1$, one has

$$\mathrm{ad}_{\mathfrak{ad}_x}(d + \mathfrak{ad}_\alpha) = \mathrm{ad}_{\mathfrak{ad}_x(\alpha) - dx}.$$

Indeed, for any $y \in \mathfrak{g}$, one has

$$\begin{aligned}[\mathfrak{ad}_x, d + \mathfrak{ad}_\alpha](y) &= [\mathfrak{ad}_x, d](y) + [\mathfrak{ad}_x, \mathfrak{ad}_\alpha](y) \\ &= [x, dy] - d[x, y] + \mathfrak{ad}_{[x,\alpha]}(y) \\ &= (\mathfrak{ad}_{[x,\alpha] - dx})(y),\end{aligned}$$

which shows the required equality.

Remark 2 Let $\mathfrak{g}$ be a Lie algebra and $f(X)$ be a polynomial, then a linear map ϕ on $\mathfrak{g}$ is a homomorphism if and only if the following equality holds:

$$f(\mathrm{ad}_{\phi(x)})(\phi(y)) = \phi(f(\mathfrak{ad}_x)(y)), \quad \forall x, y \in \mathfrak{g}.$$

2.2.4 By the above remarks, one can write down the action of $\exp(\mathfrak{g}^0)$ on the set of $\mathfrak{g}$-connections explicitly as

$$\begin{aligned}\exp(x)(d + \mathfrak{ad}_\alpha)\exp(-x) &= e^{\mathrm{ad}_{\mathfrak{ad}_x}}(d + \mathfrak{ad}_\alpha) \\ &= d + \mathfrak{ad}_\alpha + \frac{e^{\mathrm{ad}_{\mathfrak{ad}_x}} - \mathrm{id}}{\mathrm{ad}_{\mathfrak{ad}_x}}(\mathrm{ad}_{\mathfrak{ad}_x(\alpha) - dx}) \\ &= d + \mathfrak{ad}_\alpha + \mathfrak{ad}\left(\frac{e^{\mathfrak{ad}_x} - \mathrm{id}}{\mathfrak{ad}_x}(\mathfrak{ad}_x(\alpha) - dx)\right) \\ &= d + \mathfrak{ad}\left(\alpha + \frac{e^{\mathfrak{ad}_x} - \mathrm{id}}{\mathfrak{ad}_x}(\mathfrak{ad}_x(\alpha) - dx)\right).\end{aligned}$$

Thus, the action of $\exp(\mathfrak{g}^0)$ transforms $\mathfrak{g}$-connections to $\mathfrak{g}$-connections, and defines so-called ***gauge transformations*** on the set of $\mathfrak{g}$-connections. Note that

$$(\exp(x)(d + \mathfrak{ad}_\alpha)\exp(-x))^2 = \exp(x)(d + \mathfrak{ad}_\alpha)^2\exp(-x).$$

Thus the gauge transformations preserve flat connections.

The gauge transformations on the set of $\mathfrak{g}$-connections then induce the ***gauge action*** of $\exp(\mathfrak{g}^0)$ on $\mathfrak{g}^1$ by

$$\exp(x).\alpha = \alpha + \frac{e^{\mathfrak{ad}_x} - \mathrm{id}}{\mathfrak{ad}_x}(\mathfrak{ad}_x(\alpha) - dx),$$

and this action preserves Maurer-Cartan elements.

2.2.5 Given an action of a group G on the set S, the ***action groupoid*** $S//G$ is the groupoid consists of the following data:

- *objects* are the elements of S,
- *morphisms* are triples (s,s',g), where $s \in S$ is the "source", $s' \in S$ is the "target" and g is an element of G such that $g.s = s'$,
- the *composite* is induced by the multiplication of G.

For an $\mathfrak{ad}_0$-nilpotent DGLA $(\mathfrak{g},d)$, its ***Deligne groupoid*** $\mathrm{Del}(\mathfrak{g})$ is defined to be the action groupoid under the gauge action:

$$\mathrm{Del}(\mathfrak{g}) := MC(\mathfrak{g})//\exp(\mathfrak{g}^0).$$

2.3 Describe H^2_{Lie} in terms of Deligne groupoids

Before going forward, I make some convention on tensor and exterior notations.

First, the 0-th tensor power $\otimes^0 V$ always mean the ground field k.

I also identify the exterior product $\wedge^n V$ as a subspace of the tensor product $\otimes^n V$ by setting

$$x_1 \wedge \cdots \wedge x_n := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\sigma(x_1 \otimes \cdots \otimes x_n), \quad \forall x_1, \cdots, x_n \in V,$$

where S_n denotes the group of n-permutataions and each $\sigma \in S_n$ acts on the tensors by permuting its components, i.e.

$$\sigma(x_1 \otimes \cdots \otimes x_n) := x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}.$$

For this reason, I will also denote the resulted tensor by $x_{\sigma^{-1}}$

In this way, any n-linear map f induces an alternating n-linear map $f_\wedge$, called its ***antisymmetrization***, as follows: note that an n-linear map f is actually a linear map from $\otimes^n V$, thus one can define $f_\wedge$ as

$$\begin{aligned} f_\wedge(x_1, \cdots, x_n) :&= f(x_1 \wedge \cdots \wedge x_n) \\ &= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) f(x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}) \\ &= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) f(x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(n)}). \end{aligned}$$

For instance, $\wedge^2 V$ is identified as a subspace of $\otimes^2 V$ via $x \wedge y = x \otimes y - y \otimes x$, and any bilinear map ω on V induces its antisymmetrization $\omega_\wedge$ via

$$\omega_\wedge(x,y) = \omega(x,y) - \omega(y,x).$$

Finally, any linear endomorphism f on V can be extended to a multilinear map via

$$f(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_1 \otimes \cdots \otimes f(x_k) \otimes \cdots \otimes x_n.$$

2.3.1 Let $\mathfrak{g}$ be a Lie algebra and M a $\mathfrak{g}$-module. The ***Chevalley-Eilenberg complex*** is the graded vector space $C^\bullet(\mathfrak{g}, M)$ of alternating multilinear maps from $\mathfrak{g}$ to M:

$$C^n(\mathfrak{g}, M) := \mathrm{Hom}(\wedge^n \mathfrak{g}, M),$$

with the *differential* δ which maps any $f \in C^{n-1}(\mathfrak{g}, M)$ to

$$\begin{aligned}(\delta f)(x_1, \cdots, x_n) := &\sum_{k=1}^{n} (-1)^k x_k f(x_1, \cdots, \widehat{x_k}, \cdots, x_n) \\ &- \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \cdots, \widehat{x_i}, \cdots, \widehat{x_j}, \cdots, x_n),\end{aligned} \tag{2.1}$$

where $\widehat{\ }$ indicates the omission of the underneath term.

Consider now the complex $C^{\bullet+1}(\mathfrak{g}, \mathfrak{g})$, which is obtained from $C^\bullet(\mathfrak{g}, \mathfrak{g})$ by degree shifting $n \mapsto n-1$. On this complex, one has the *graded composite* of $f \in C^{n+1}(\mathfrak{g}, \mathfrak{g})$ and $g \in C^{m+1}(\mathfrak{g}, \mathfrak{g})$ defined as

$$(f \diamond g)(x_0, \cdots, x_{n+m}) := \sum_{\sigma} \mathrm{sgn}(\sigma) g(f(x_{\sigma^{-1}(0)}, \cdots, x_{\sigma^{-1}(n)}), x_{\sigma^{-1}(n+1)}, \cdots, x_{\sigma^{-1}(n+m)})$$

where σ is taken over the set of all $(n+1,m)$***-shuffles***, i.e., permutations such that $\sigma^{-1}(0) < \cdots < \sigma^{-1}(n)$ and $\sigma^{-1}(n+1) < \cdots < \sigma^{-1}(n+m)$.

The ***Nijenhuis-Richardson bracket*** ([25]) on $C^{\bullet+1}(\mathfrak{g}, \mathfrak{g})$ is defined to be the graded commutator of the composite:

$$[f, g] := f \diamond g - (-1)^{|f||g|} g \diamond f.$$

Remark 1 One can see that the Chevalley-Eilenberg complex $(C^{\bullet+1}(\mathfrak{g}, \mathfrak{g}), \delta)$ together with the Nijenhuis-Richardson bracket, forms a DGLA, and that in this DGLA, one has $\delta = \mathfrak{ad}_\omega$, where $\omega \in C^{1+1}(\mathfrak{g}, \mathfrak{g})$ is the Lie bracket of $\mathfrak{g}$.

Remark 2 In such a DGLA, one has $\frac{1}{2}[\alpha,\alpha] = \alpha \diamond \alpha$ for all $\alpha \in C^{1+1}(\mathfrak{g},\mathfrak{g})$. So to define Maurer-Cartan elements, what one really needs is the graded composite, instead of the graded bracket.

2.3.2 Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras and $\mathfrak{g}\oplus\mathfrak{h}$ be the Lie algebra direct sum of them. Then $\mathfrak{h}$ is a $\mathfrak{g}\oplus\mathfrak{h}$-module via the adjoint action of $\mathfrak{h}$ on itself. The complex $C^\bullet_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$ is defined as the subcomplex of $C^\bullet(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$ by

$$C^\bullet(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h}) = C^\bullet_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h}) \oplus C^\bullet_>(\mathfrak{h},\mathfrak{h}).$$

One can verify that $C^{\bullet+1}_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$ is a sub-DGLA of $C^{\bullet+1}(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{g}\oplus\mathfrak{h})$ endowed with the Nijenhuis-Richardson graded Lie bracket.

Denote this DGLA by $\mathfrak{L}$. One can prove further that $\mathfrak{L}^0$ is abelian and that $\mathfrak{L}$ is $\mathfrak{ad}_0$-nilpotent, see [13] for more details. Then one can define the Maurer-Cartan elements of $\mathfrak{L}$ and further the Deligne groupoid $\mathrm{Del}(\mathfrak{L})$ following 2.2.3–2.2.5.

2.3.3 *Extensions of* $\mathfrak{g}$ *by* $\mathfrak{h}$ *are classified by* $\pi_0\,\mathrm{Del}(\mathfrak{L})$.

More precisely, one can prove that $Z^2_{Lie}(\mathfrak{g},\mathfrak{h}) \cong MC(\mathfrak{L})$ and that two 2-cycles are equivalent if and only if their corresponding Maurer-Cartan elements are connected by a morphism in $\mathrm{Del}(\mathfrak{L})$. Thus $H^2_{Lie}(\mathfrak{g},\mathfrak{h})$ is isomorphic to the set of connected components of $\mathrm{Del}(\mathfrak{L})$, i.e. $\pi_0\,\mathrm{Del}(\mathfrak{L})$. See [13] for more details on the proof.

2.3.4 Let $\mathfrak{g}$ be a Lie algebra and M be a $\mathfrak{g}$-module. Denote the antisymmetrization of its $\mathfrak{g}$-module structure by α, then $\alpha \in MC(\mathfrak{L})$ and one has the tangent complex $\mathfrak{L}_\alpha$. One can further prove that the Chevalley-Eilenberg complex $C^\bullet(\mathfrak{g},M)$ is a sub-DGLA of it.

2.4 The notion of homotopy algebras

From now on, I will use some notions from higher category theory. As I only use them in an elementary way, one can refer the appendix for necessary information instead of the monograph [21].

2.4.1 A ***chain complex*** $C_\bullet$ is a graded vector space equipped with a graded linear map d of degree -1 satisfying $d^2 = 0$ called the ***differential***. A chain complex is said to be

concentrated in first n ***terms*** if $C_k = 0$ for all $k > n$. Any chain complex $C_\bullet$ admits a chain complex concentrated in first n terms

$$\cdots \to 0 \to Z_n \to \cdots \to C_0,$$

where Z_n is the kernel of the differential d in C_n. This chain complex is called ***n-truncation*** of $C_\bullet$ and is denoted by $\tau_{\leqslant n} C_\bullet$.

A ***chain map*** $F : C_\bullet \to D_\bullet$ between two chain complexes is a graded linear map of degree 0 compatible with the differentials, that means the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r] \arrow[d, "F"] & C_{n-1} \arrow[r] \arrow[d, "F"] & \cdots \\
\cdots \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Let $F, G \colon C_\bullet \to D_\bullet$ be two parallel chain maps. A ***chain homotopy*** $\Phi \colon F \Rightarrow G$ between them is a graded linear map $\Phi \colon C_\bullet \to D_\bullet$ of degree -1 such that

$$F - G = \Phi \circ d + d \circ \Phi.$$

The *vertical composite* of two chain homotopies is then the sum of them and the *horizontal composite* of two chain homotopiesn is the composite of them as graded linear maps.

One can verify that the chain complexes together with chain maps between them as 1-*morphisms* and chain homotopies between chain maps as 2-*morphisms* form a 2-*category*, denoted by **Ch**. This is further a $(2,1)$***-category***, that is a 2-category whose every 2-morphism is invertible.

Dually, one has cochain complexes, which are graded vector spaces equipped with a graded linear map d of degree 1 satisfying $d^2 = 0$. One also has cochain maps and cochain homotopies.

2.4.2 The ***tensor product*** of two cochain complexes $C_\bullet$ and $D_\bullet$ is the cochain complex $(C_\bullet \otimes D_\bullet, \delta)$, where $C_\bullet \otimes D_\bullet$ is the tensor product of graded vector spaces and

$$\delta(x,y) = (d(x),y) + (-1)^{|x|}(x, d(y)), \quad \forall x \in C_\bullet, y \in D_\bullet.$$

This makes **Ch** become a monoidal category and then many algebra structures on vector spaces can be generalized on chain complexes. For instance, a DGLA is a Lie aglebra on a cochain complex, but one can also define it on a chain complex.

Remark 1 Note that the graded transposition operation $s_{12}\colon C_\bullet\otimes C_\bullet\to C_\bullet\otimes C_\bullet$ maps $x\otimes y$ to $(-1)^{|x||y|}y\otimes x$ instead of $y\otimes x$ and general permutation operations s_σ are generated by the garded transposition operations via the formula $s_{\sigma\tau}=s_\sigma\circ s_\tau$.

Remark 2 Recall that the *tensor product* of two linear maps $f\colon V\to W$ and $g\colon V'\to W'$ is a linear map $f\otimes g\colon V\otimes V'\to W\otimes W'$ which maps $u\otimes v$ to $f(u)\otimes g(v)$.

2.4.3 The *interval chain complex* $I_\bullet$ is the chain complex

$$\cdots\longrightarrow 0\longrightarrow k\xrightarrow{(\mathrm{id},-\mathrm{id})}k\oplus k,$$

where the term $k\oplus k$ is in degree 0.

Then the tensor product of $I_\bullet$ and $C_\bullet$ is the chain complex

$$\cdots\longrightarrow C_2\oplus C_2\oplus C_1\xrightarrow{\delta}C_1\oplus C_1\oplus C_0\xrightarrow{\delta}C_0\oplus C_0\oplus 0,$$

where the term $C_0\oplus C_0\oplus 0$ is in degree 0 and the differential δ is

$$\delta(x_1,x_2,y)=(d(x_1)+y,d(x_2)-y,-d(y)),\quad \forall x_1,x_2\in C_{n+1},y\in C_n,\forall n\in\mathbb{N}.$$

2.4.4 Let $F,G\colon C_\bullet\to D_\bullet$ be two parallel chain maps. A chain homotopy $\Phi\colon F\Rightarrow G$ between them can also be encoded into a chain map as follows:

For any F,G two graded linear maps of degree 0 and Φ a graded linear map of degree 1, one has the graded linear map $(F,G,\Phi)\colon C_\bullet\oplus C_\bullet\oplus C_{\bullet-1}\to D_\bullet$ of degree 0. Note that $C_\bullet\otimes I_\bullet=C_\bullet\oplus C_\bullet\oplus C_{\bullet-1}$ as graded vector spaces. Then (F,G,Φ) is a chain map if and only if the following diagram commutes for all $n\in\mathbb{N}$.

$$\begin{tikzcd}
C_{n+1}\oplus C_{n+1}\oplus C_n \arrow[r, "{(F,G,\Phi)}"] \arrow[d, "\delta"] & D_{n+1} \arrow[d, "d"] \\
C_n\oplus C_n\oplus C_{n-1} \arrow[r, "{(F,G,\Phi)}"] & D_n
\end{tikzcd}$$

Restricted on $C_\bullet\oplus 0\oplus 0$ and $0\oplus C_\bullet\oplus 0$, this condition says that F,G are chain maps. Restricted on $0\oplus 0\oplus C_\bullet$, the condition becomes

$$d\circ\Phi=F-G-\Phi\circ d,$$

which says Φ is a chain homotopy between F and G.

2.4.5 As chain homotopies can be encoded into chain maps, it makes sense to define chain homotopies between them. In this way, one has chain maps as 1-morphisms, chain homotopies between chain maps as 2-morphisms, chain homotopies between the chain homotopies between chain maps as 3-morphisms, etc. Therefore, **Ch** is actually an $(\infty,1)$-category.

A graded algebra (for instance a Lie algebra on a chain complex) can be viewed as a chain complex $C_\bullet$ equipped a graded binary operation, which is a chain map $C_\bullet \otimes C_\bullet \to C_\bullet$, satisfying a graded equality, which is a commutative diagram consisting of chain maps induced by the graded binary operation.

Then the idea of homotopy algebras arise: a ***homotopy algebra*** is a chain complex $C_\bullet$ equipped a family of graded linear maps $l_n\colon \otimes^n C_\bullet \to C_\bullet$ of degree $n-2$ for all $n \geqslant 2$. Here $l_2\colon C_\bullet \otimes C_\bullet \to C_\bullet$ is the graded binary operation, $l_3\colon \otimes^3 C_\bullet \to C_\bullet$ is a chain homotopy between the two chain maps coming from the commutative diagram where they should be equal. Then l_2, l_3 form a large diagram which should be automatically commutative when l_3 is trivial, and now be encoded into a homotopy l_4. Continue this progress, one gets all l_n.

If the underlying chain complex $C_\bullet$ is concentrated in first n terms, this homotopy algebra will be called a ***n*-term** one and in this case l_k will be trivial when $k \geqslant n+2$. Note that when $n=1$, all homotopies become trivial, thus the 1-term homotopy algebras are precisely the original algebras.

2.4.6 As an example, a ***2-term L_∞-algebra*** is a chain complex $C_\bullet$ concentrated in first 2 terms and equipped a graded antisymmetric bracket $\mu\colon C_\bullet \otimes C_\bullet \to C_\bullet$ and a chain homotopy

$$l_3\colon \mu \circ (\mathrm{id} \otimes \mu) \circ (\mathrm{id} + s_{123} + s_{123}^2) \Rightarrow 0,$$

which is total graded antisymmetric in the sense that

$$l_3 \circ s_\sigma = (\operatorname{sgn}\sigma) l_3.$$

and satisfies

$$\delta l_3 = 0,$$

where δ is defined by Eq. (2.1) with $\mathfrak{g} = C_0$, $M = C_1$ and the action is given by adjoint.

A *morphism* $\mathfrak{F}\colon C_\bullet \to D_\bullet$ between 2-term L_∞-algebras consists of a chain map $F\colon C_\bullet \to D_\bullet$ and a chain homotopy $\mathfrak{F}_2\colon F\circ\mu \Rightarrow \mu\circ(F\otimes F)$ which is graded antisymmetric and satisfying

$$\begin{aligned} &F(l_3(x,y,z)) - l_3(F(x),F(y),F(z)) \\ =&\mathscr{F}_2(x,[y,z]) + \mathscr{F}_2(y,[z,x]) + \mathscr{F}_2(z,[x,y]) \\ &+[F(x),\mathscr{F}_2(y,z)] + [F(y),\mathscr{F}_2(z,x)] + [F(z),\mathscr{F}_2(x,y)] \end{aligned}$$

for all $x,y,z\in C_\bullet$.

A *homotopy* $\Phi\colon \mathscr{F}\Rightarrow\mathscr{F}'$ between two morphisms $\mathscr{F},\mathscr{F}'\colon C_\bullet\to D_\bullet$ is a chain homotopy $\Phi\colon F\Rightarrow F'$satisfying

$$\mathscr{F}_2'(x,y)-\mathscr{F}_2(x,y) = [F(x),\Phi(y)]+[\Phi(x),F(y)]+[\Phi(x),d\Phi(y)]-\Phi([x,y]), \forall x,y\in C_0.$$

One can also define the composites of morphisms and homotopies and verify that 2-term L_∞-algebras form a $(2,1)$-category.

Remark Note that the DGLAs are defined on cochain complexes, while L_∞-algebras are defined on chain complexes. One can also define DGLAs on chain complexes, then DGLAs are special kind of L_∞-algebras. But by taking the n-truncation of a DGLA $\mathfrak{g}^\bullet$ and changing the degree via $k\mapsto n-k$, one also obtains a n-term L_∞-algebra. Likewise, morphisms between concentrated DGLAs can be identified with morphisms between concentrated L_∞-algebras.

For instance, the 2-truncation of the DGLA $(C^{\bullet+1}(\mathfrak{g},\mathfrak{g}),[,],\delta)$

$$C^0(\mathfrak{g},\mathfrak{g}) \xrightarrow{\delta} Z^1(\mathfrak{g},\mathfrak{g}),$$

where $Z^1(\mathfrak{g},\mathfrak{g}) = \ker\delta\cap C^1(\mathfrak{g},\mathfrak{g})$, can be viewed as a concentrated chain complex by putting $C^0(\mathfrak{g},\mathfrak{g})$ on degree 1 and $Z^1(\mathfrak{g},\mathfrak{g})$ on degeree 0. This chain complex, together with $\mu=[,]$ and $l_3=0$, form a 2-term L_∞-algebra. Note that $Z^1(\mathfrak{g},\mathfrak{g})$ is the set $\mathrm{der}(\mathfrak{g})$ of derivations on $\mathfrak{g}$, so this 2-term L_∞-algebra is denoted by $\mathrm{Der}(\mathfrak{g})$.

2.5 Describe H^2_{Lie} in terms of 2-term L_∞-algebras

2.5.1 Recall that, in a $(2,1)$-category, the hom-set $\mathrm{Hom}(A,B)$ is naturally a groupoid since there are equivalences between morphisms. Denote this groupoid by $\mathrm{Hom}(A,B)$ again and call it ***hom-space*** to emphasize that it is more then a set.

As a groupoid, it is natural to consider the set $\pi_0 \operatorname{Hom}(A,B)$ of connected components of $\operatorname{Hom}(A,B)$. In the case that B is a pointed object, $\operatorname{Hom}(A,B)$ is naturally a pointed groupoid and then $\pi_0 \operatorname{Hom}(A,B)$ is naturally a pointed set.

This set $\pi_0 \operatorname{Hom}(A,B)$ is called the ***intrinsic cohomology*** of A with values in B and denoted by $H(A,B)$.

2.5.2 Note that $H^2_{Lie}(\mathfrak{g},\mathfrak{h})$ can be described in terms of Deligne groupoids $\operatorname{Del}(\mathfrak{L})$. By the canonical isomorphism

$$\wedge^\bullet(\mathfrak{g}\oplus\mathfrak{h}) \cong \wedge^\bullet\mathfrak{g}\otimes\wedge^\bullet\mathfrak{h}$$

One can identify each term $C^{n+1}_{>}(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$ as

$$\prod_{i=1}^{n+1}\operatorname{Hom}(\wedge^i\mathfrak{g}\otimes\wedge^{n+1-i}\mathfrak{h},\mathfrak{h}) \cong \prod_{i=1}^{n+1}\operatorname{Hom}(\wedge^i\mathfrak{g},C^{n-i+1}(\mathfrak{h},\mathfrak{h})).$$

Specially, one has

$$\mathfrak{L}^1 \cong \operatorname{Hom}(\mathfrak{g},C^{0+1}(\mathfrak{h},\mathfrak{h}))\oplus\operatorname{Hom}(\wedge^2\mathfrak{g},\mathfrak{h}).$$

View $\mathfrak{g}$ as a complex concentrated in degree 0, then the right side is precisely the set of pairs of a chain map and a graded linear map of degree 1 from $\mathfrak{g}$ to $C^{\bullet+1}(\mathfrak{h},\mathfrak{h})$. Moreover, the Maurer-Cartan elements are precisely the data form morphisms between 2-term L_∞-algebras $0\to\mathfrak{g}$ and $\operatorname{Der}(\mathfrak{h})$ and the gauge action on $MC(\mathfrak{L})$ coincides with the homotopies between those morphisms.

Consequently, one has the following equivalences of groupoids

$$\mathcal{E}xt_{Lie}(\mathfrak{g},\mathfrak{h}) \cong \operatorname{Del}(\mathfrak{L}) \cong \operatorname{Hom}(\mathfrak{g},\operatorname{Der}(\mathfrak{h})),$$

and isomorphisms of pointed sets

$$H^2_{Lie}(\mathfrak{g},\mathfrak{h}) \cong \pi_0\operatorname{Del}(\mathfrak{L}) \cong H(\mathfrak{g},\operatorname{Der}(\mathfrak{h})).$$

Remark One can see that $H^2_{Lie}(\mathfrak{g},\mathfrak{h})$ is natural in $\mathfrak{g}$ in the sense that $\mathfrak{g}\mapsto H^2_{Lie}(\mathfrak{g},\mathfrak{h})$ induces a contravariant functor from Lie algebras to pointed sets. However, $H^2_{Lie}(\mathfrak{g},\mathfrak{h})$ is not natural in $\mathfrak{h}$ since Der is not a functor.

Chapter 3 From pre-Lie algebra extensions to the second non-abelian pre-Lie algebra cohomology

Most of the results in this chapter have been given by Kim in [18, 19]. For self-contained, I explain them in details.

3.1 The notion of pre-Lie algebra extensions

3.1.1 A ***pre-Lie algebra*** is a vector space A equipped with a bilinear product $\cdot$ such that the *associator* $(x,y,z) := (x \cdot y) \cdot z - x \cdot (y \cdot z)$ is symmetric in x, y, i.e.

$$(x,y,z) = (y,x,z), \quad \forall x,y,z \in A. \tag{3.1}$$

The *commutator* $[x,y] := x \cdot y - y \cdot x$ defines a Lie algebra structure on A, which is called the ***sub-adjacent Lie algebra*** of A and denoted by $\mathfrak{g}(A)$.

There are two natural linear maps $L, R \colon A \to \mathfrak{gl}(A)$ on A provided by the pre-Lie algebra structure, that is the *left* and *right multiplications*:

$$L_x(y) := x \cdot y, \quad R_x(y) := y \cdot x, \quad \forall x,y \in A.$$

One can see Eq. (3.1) equals to the follows:

$$[L_x, L_y] = L_{[x,y]}, \quad \forall x,y \in A. \tag{3.2}$$

Example Every vector space M admits a *trivial pre-Lie algebra* structure by

$$u \cdot v = 0, \quad \forall u,v \in M.$$

Example If a subspace B of a pre-Lie algebra A is closed under the multiplication $\cdot$ of A, then B has a multiplication given by restricting $\cdot$ on B and is a pre-Lie algebra, called a ***subalgebra*** of A.

Remark If there is no ambiguity, the symbols $\cdot$, L and R always denote the multiplication, left and right multiplications of a given pre-Lie algebra.

3.1.2 A ***representation*** of a pre-Lie algebra A, or an $A-$***module***, is a vector space M equipped with a Lie algebra action l of $\mathfrak{g}(A)$ on M and a linear map $r\colon A \to \mathfrak{gl}(M)$ such that the following equality holds:

$$[l_x, r_y] = r_{x\cdot y} - r_y r_x, \quad \forall x, y \in A.$$

In this case, (l, r) is also called an ***action*** of A on M.

Example The pre-Lie algebra A together with L and R is a representation of A itself.

3.1.3 An ***ideal*** I of a pre-Lie algebra A is a subspace of A such that for all $x \in A$ and $y \in I$, one has $x \cdot y \in I$ and $y \cdot x \in I$. This condition is equivalent to say (I, L, R) is an $A-$module.

Let I be an ideal of A, then the multiplication on A induces a multiplication on the quotient A/I and makes it being a pre-Lie algebra.

Example The set of *zero-divisors* of A is defined to be

$$\mathrm{Ann}(A) := \ker L \cap \ker R = \{x \in A | L_x = R_x = 0\}.$$

One can verify that it is an ideal of A.

3.1.4 Let $A, \widehat{A}, B$ be pre-Lie algebras. $\widehat{A}$ is said to be an ***extension*** of A by B if there exist a short exact sequence

$$0 \longrightarrow B \xrightarrow{\mathfrak{i}} \widehat{A} \xrightarrow{\mathfrak{p}} A \longrightarrow 0.$$

A ***splitting*** of $\widehat{A}$ is a linear map $\sigma\colon A \to \widehat{A}$ such that $\mathfrak{p} \circ \sigma = \mathrm{id}$.

A *morphism* $\theta\colon \widehat{A} \to \widehat{A}'$ of two extensions is a pre-Lie algebra morphism θ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "\mathfrak{i}"] \arrow[d, equal] & \widehat{A} \arrow[r, "\mathfrak{p}"] \arrow[d, "\theta"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "\mathfrak{i}'"] & \widehat{A}' \arrow[r, "\mathfrak{p}'"] & A \arrow[r] & 0
\end{tikzcd}$$

By 5-lemma, this θ, if exists, must be an isomorphism. If this is the case, the two extensions $\widehat{A}$ and $\widehat{A}'$ are said to be *isomorphic*.

Now all extensions of A by B, together with the morphisms between them, form a groupoid $\mathcal{E}xt_{preLie}(A,B)$. One can then consider the set $\mathrm{Ext}_{preLie}(A,B)$ of connected components, i.e. isomorphism classes, of this groupoid.

Consider the commutators, one can see that if $\widehat{A}$ is a pre-Lie algebra extension of A by B, then $\mathfrak{g}(\widehat{A})$ is a Lie algebra extension of $\mathfrak{g}(A)$ by $\mathfrak{g}(B)$, and that if the morphism $\theta\colon \widehat{A} \to \widehat{A}'$ gives rise to a morphisms between pre-Lie algebra extensions of A by B, then it also gives rise to a morphisms between Lie algebra extensions of $\mathfrak{g}(A)$ by $\mathfrak{g}(B)$. Thus one has a functor $\mathfrak{g}(-)\colon \mathcal{E}xt_{preLie}(A,B) \to \mathcal{E}xt_{Lie}(\mathfrak{g}(A),\mathfrak{g}(B))$. This functor is in general not *surjective* or *essentially surjective*. Indeed, even a Lie algebra extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}(A)$ by $\mathfrak{g}(B)$ is isomorphic to a sub-adjacent Lie algebra of some pre-Lie algebra on the same space, this pre-Lie algebra may not be a pre-Lie algebra extension of A by B. So I only consider the image of this functor and denote it by $\mathcal{E}xt_{\mathfrak{g}}(A,B)$.

3.1.5 When the pre-Lie algebra structure on B is trivial, the extension is said to be ***abelian***.

3.2 Toward non-abelian 2-cocycles

3.2.1 Let $(\widehat{A},*)$ be an extension of A by B, and $\sigma\colon A \to \widehat{A}$ a splitting of $\widehat{A}$. After identifying B with its image in $\widehat{A}$, one can define the bilinear map $\omega\colon \otimes^2 A \to B$ and linear maps $l,r\colon A \to \mathfrak{gl}(B)$ as follows:

$$\omega(x,y) := \sigma(x)*\sigma(y) - \sigma(x\cdot y), \quad \forall x,y \in A, \tag{3.3}$$

$$l_x(v) := \sigma(x)*v, \quad \forall x \in A, v \in B, \tag{3.4}$$

$$r_y(u) := u*\sigma(y), \quad \forall y \in A, u \in B. \tag{3.5}$$

Consider the actions of them on commutators, the antisymmetrization of ω defines an alternating bilinear map $\omega_\wedge\colon \wedge^2\mathfrak{g}(A) \to \mathfrak{g}(B)$, and the difference $l-r$ defines a linear map $\rho\colon \mathfrak{g}(A) \to \mathfrak{gl}(\mathfrak{g}(B))$. This couple $(\omega_\wedge,\rho)$ is precisely the Lie algebra 2-cocycle defined by the Lie algebra extension $\mathfrak{g}(\widehat{A})$ of $\mathfrak{g}(A)$ by $\mathfrak{g}(B)$ together with the chosen of the splitting σ.

Given a splitting, one has $\widehat{A} \cong A \oplus B$ and the pre-Lie algebra structure on $\widehat{A}$ can be transferred to $A \oplus B$ by define $\widehat{L}$ to be

$$\widehat{L}_{x+u}(y+v) = L_x(y) + \omega(x,y) + l_x(v) + r_y(u) + L_u(v), \quad \forall x,y \in A, \forall u,v \in B. \tag{3.6}$$

It is a representation of the Lie algebra structure on $A \oplus B$ given by the Lie algebra 2-cocycle $(\omega_\wedge, \rho)$:

$$[x+u, y+v]_{\omega_\wedge,\rho} := [x,y]_A + \omega_\wedge(x,y) + \rho_x(v) - \rho_y(u) + [u,v]_B, \quad \forall x,y \in A, u,v \in B. \tag{3.7}$$

Conversely, one has

Proposition 3.2.2 *Let A, B be two pre-Lie algebras, $\omega\colon \otimes^2 A \to B$ be a bilinear map and $l, r\colon A \to \mathfrak{gl}(B)$ be two linear maps, then Eq.* (3.6) *defines a pre-Lie algebra structure on $A \oplus B$ if and only if ω, l, r satisfy the following equalities for all $x,y,z \in A, u,v \in B$:*

$$r_x([u,v]) = u \cdot r_x(v) - v \cdot r_x(u) \tag{3.8}$$

$$l_x(u \cdot v) = \rho_x(u) \cdot v + u \cdot l_x(v) \tag{3.9}$$

$$[l_x, l_y] - l_{[x,y]} = L_{\omega_\wedge(x,y)} \tag{3.10}$$

$$[l_x, r_y] - r_{x \cdot y} + r_y r_x = R_{\omega(x,y)} \tag{3.11}$$

$$\omega([x,y],z) - \omega(x, y\cdot z) + \omega(y, x\cdot z) = l_x\omega(y,z) - l_y\omega(x,z) - r_z(\omega_\wedge(x,y)). \tag{3.12}$$

If this is the case, the couple $(\omega_\wedge, \rho)$, where $\omega_\wedge$ is the antisymmetrization of ω and $\rho = l - r$, will be a Lie algebra 2-cocyle and Eq. (3.7) *defines a Lie algebra structure on $A \oplus B$ making $\widehat{L}$ being its representation.*

Proof. Note that the bracket defined by Eq. (3.7) is actually the commutator of the multiplication defined by Eq. (3.6), thus the last statements follow immediately from the previous one.

By Eq. (3.2), one can see Eq. (3.6) defines a pre-Lie algebra structure on $A \oplus B$ if and only if the following equalities hold:

$$[\widehat{L}_x, \widehat{L}_y] = \widehat{L}_{[x,y]_{\omega_\wedge,\rho}}, \quad \forall x,y \in A, \tag{3.13}$$

$$[\widehat{L}_x, \widehat{L}_u] = \widehat{L}_{[x,u]_{\omega_\wedge,\rho}}, \quad \forall x \in A, \forall u \in B, \tag{3.14}$$

$$[\widehat{L}_u, \widehat{L}_v] = \widehat{L}_{[u,v]_{\omega_\wedge,\rho}}, \quad \forall u,v \in B. \tag{3.15}$$

Apply two sides of Eq. (3.13) on any $z \in A$, one has

$$\begin{aligned}
[\widehat{L}_x, \widehat{L}_y](z) &= \widehat{L}_x(L_y(z) + \omega(y,z)) - \widehat{L}_y(L_x(z) + \omega(x,z)) \\
&= L_x L_y(z) + \omega(x, L_y(z)) + l_x \omega(y,z) \\
&\quad - L_y L_x(z) - \omega(y, L_x(z)) - l_y \omega(x,z), \\
\widehat{L}_{[x,y]_{\omega_\wedge,\rho}}(z) &= \widehat{L}_{[x,y]+\omega_\wedge(x,y)}(z) \\
&= L_{[x,y]}(z) + \omega([x,y],z) + r_z(\omega_\wedge(x,y)).
\end{aligned}$$

Thus

$$\omega([x,y],z) - \omega(x, y\cdot z) + \omega(y, x\cdot z) = l_x \omega(y,z) - l_y \omega(x,z) - r_z(\omega_\wedge(x,y)),$$

which shows Eq. (3.12).

Apply two sides of Eq. (3.13) on any $u \in B$, one has

$$\begin{aligned}
[\widehat{L}_x, \widehat{L}_y](u) &= l_x l_y(u) - l_y l_x(u) = [l_x, l_y](u), \\
\widehat{L}_{[x,y]_{\omega_\wedge,\rho}}(u) &= \widehat{L}_{[x,y]+\omega_\wedge(x,y)}(u) = l_{[x,y]} u + \omega_\wedge(x,y) \cdot u.
\end{aligned}$$

Thus

$$[l_x, l_y] - l_{[x,y]} = L_{\omega_\wedge(x,y)},$$

which shows Eq. (3.10).

Apply two sides of Eq. (3.14) on any $y \in A$, one has

$$\begin{aligned}
[\widehat{L}_x, \widehat{L}_u](y) &= \widehat{L}_x(r_y(u)) - \widehat{L}_u(L_x(y) + \omega(x,y)) \\
&= l_x r_y(u) - r_{x\cdot y}(u) - u \cdot \omega(x,y), \\
\widehat{L}_{[x,u]_{\omega_\wedge,\rho}}(y) &= \widehat{L}_{\rho_x(u)}(y) = r_y \rho_x(u).
\end{aligned}$$

Thus

$$[l_x, r_y] - r_{x\cdot y} + r_y r_x = R_{\omega(x,y)},$$

which shows Eq. (3.11).

Apply two sides of Eq. (3.14) on any $v \in B$, one has

$$\begin{aligned}
[\widehat{L}_x, \widehat{L}_u](v) &= l_x(u\cdot v) - u \cdot l_x(v), \\
\widehat{L}_{[x,u]_{\omega_\wedge,\rho}}(v) &= \widehat{L}_{\rho_x(u)}(v) = \rho_x(u) \cdot v.
\end{aligned}$$

Thus

$$l_x(u \cdot v) = \rho_x(u) \cdot v + u \cdot l_x(v),$$

which shows Eq. (3.9).

Apply two sides of Eq. (3.15) on any $x \in A$, one has

$$[\widehat{L}_u, \widehat{L}_v](x) = u \cdot r_x(v) - v \cdot r_x(u),$$
$$\widehat{L}_{[u,v]_{\omega_\wedge,\rho}}(x) = r_x([u,v]).$$

Thus

$$r_x([u,v]) = u \cdot r_x(v) - v \cdot r_x(u),$$

which shows Eq. (3.8).

Finally, since the restriction of $\widehat{L}$ on B coincides with its original left multiplication L, apply two sides of Eq. (3.15) on any $w \in B$ will give a trivial equality. □

Remark The pre-Lie algebra defined by Eq. (3.6) is called the ***semidirect product*** and is denoted by $A \ltimes_{\omega,l,r} B$. One can see $\mathfrak{g}(A \ltimes_{\omega,l,r} B) = \mathfrak{g}(A) \ltimes_{\omega_\wedge,l-r} \mathfrak{g}(B)$.

Example There is a natural pre-Lie algebra structure on $A \oplus B$, it is given by

$$(x+u) * (y+v) := x \cdot y + u \cdot v, \quad \forall x,y \in A, u,v \in B.$$

In other words, it is $A \ltimes_{0,0,0} B$. This pre-Lie algebra is called the ***direct sum*** of A and B and is denoted by $A \oplus B$ again. Direct sums give rise to the ***split extensions***.

3.2.3 A ***non-abelian*** 2***-cocycle*** on A with values in B is a triple (ω, l, r) of a bilinear map $\omega\colon \otimes^2 A \to B$ and two linear maps $l, r\colon A \to \mathfrak{gl}(B)$, satisfies Eq. (3.8)–(3.12). The set of them is denoted by $Z^2_{preLie}(A,B)$.

Two 2-cocycle (ω, l, r) and (ω', l', r') are *equivalent*, if there exists a linear map $\varphi\colon A \to B$ such that the following equalities hold for all $x, y \in A$:

$$l' - l = L \circ \varphi, \tag{3.16}$$

$$r' - r = R \circ \varphi, \tag{3.17}$$

$$\omega'(x,y) - \omega(x,y) = l_x\varphi(y) + r_y\varphi(x) + \varphi(x) \cdot \varphi(y) - \varphi(x \cdot y). \tag{3.18}$$

The ***second non-abelian cohomology*** $H^2_{preLie}(A,B)$ is then defined to be the quotient of $Z^2_{preLie}(A,B)$ by the above equivalence relation.

Any pre-Lie algebra 2-cocyle (ω,l,r) on A with values in B provides a Lie algebra 2-cocyle $(\omega_\wedge, l-r)$ on $\mathfrak{g}(A)$ with values in $\mathfrak{g}(B)$, two pre-Lie algebra 2-cocyles are said to be *pre-equal* if they provide the same Lie algebra 2-cocyle, and *pre-equivalent* if they provide equivalent Lie algebra 2-cocyles. The quotient of $Z^2_{preLie}(A,B)$ by the pre-equal relation is denoted by $Z^2_{\mathfrak{g}}(A,B)$, and the quotient of $H^2_{preLie}(A,B)$, and thus of $Z^2_{preLie}(A,B)$, by the pre-equivalent relation is denoted by $H^2_{\mathfrak{g}}(A,B)$.

Combine proposition 3.2.2 and 2.1.3, one has

Corrollay 3.2.4 *Let A,B be two pre-Lie algebras, then $Z^2_{\mathfrak{g}}(A,B)$ can be identified with a subset of $Z^2_{Lie}(\mathfrak{g}(A),\mathfrak{g}(B))$ via $(\omega,l,r)\mapsto(\omega_\wedge,l-r)$, whose elements correspond to the objects of $\mathcal{E}xt_{\mathfrak{g}}(A,B)$. Consequently, $H^2_{\mathfrak{g}}(A,B)$ can be identified with a subset of $H^2_{Lie}(\mathfrak{g}(A),\mathfrak{g}(B))$, which classifies $\mathcal{E}xt_{\mathfrak{g}}(A,B)$.*

3.3 The classification theorem

Like Lie algebra case, one has

Theorem 3.3.1 *Let A,B be two pre-Lie algebras, the extensions of A by B are classified by the second non-abelian cohomology $H^2_{preLie}(A,B)$.*

Proof. Let $\widehat{A}$ be an extension of A by B. By choosing a splitting $\sigma\colon A\to\widehat{A}$, one obtain a 2-cocycle (ω,l,r) via Eq. (3.3)–(3.5). First of all, the cohomological class of this 2-cocyle is independent of the choice of splittings. Indeed, let σ,σ' be two different splittings and (ω,l,r) and (ω',l',r') be corresponding 2-cocycles. Set $\varphi=\sigma'-\sigma$, then its image lies in B, and for all $x,y\in A, u\in B$, one has

$$\begin{aligned}
l'_x(u)-l_x(u) &= \sigma'(x)*u-\sigma(x)*u=\varphi(x)*u,\\
r'_x(u)-r_x(u) &= u*\sigma'(x)-u*\sigma(x)=u*\varphi(x),\\
\omega'(x,y)-\omega(x,y) &= \sigma'(x)*\sigma'(y)-\sigma'(x\cdot y)-\sigma(x)*\sigma(y)+\sigma(x\cdot y)\\
&= l_x\varphi(y)+r_y\varphi(x)+\varphi(x)\cdot\varphi(y)-\varphi(x\cdot y),
\end{aligned}$$

which shows Eq. (3.16)–(3.18).

Secondly, morphisms of extensions give rise to equivalence of 2-cocycles. Let $\theta\colon \widehat{A} \to \widehat{A}'$ be a morphism of pre-Lie algebra extensions of A by B and σ, σ' be two splittings such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & \widehat{A} \arrow[r] \arrow[d, "\theta"] & A \arrow[r] \arrow[d, equal] \arrow[l, dashed, bend right, "\sigma"'] & 0 \\
0 \arrow[r] & B \arrow[r] & \widehat{A}' \arrow[r] & A \arrow[r] \arrow[l, dashed, bend left, "\sigma'"] & 0
\end{tikzcd}$$

Let (ω, l, r) and (ω', l', r') be the corresponding 2-cocycles of $(\widehat{A}, \sigma)$ and $(\widehat{A}', \sigma')$.

Set $\varphi = \theta^{-1}\sigma' - \sigma$ and note that $\theta|_B = \mathrm{id}$. one has

$$\begin{aligned}
l'_x(u) &= \sigma'(x) * u = \theta^{-1}(\sigma'(x) * u) = \theta^{-1}\sigma'(x) * u \\
&= \sigma(x) * u + \varphi(x) * u = l_x(u) + L_{\varphi(x)}(u).
\end{aligned}$$

Therefore, $l' - l = L \circ \varphi$. Similarly, $r' - r = R \circ \varphi$. As for Eq. (3.18), consider that for all $x, y \in A$, one has

$$\begin{aligned}
\omega'(x,y) &= \theta^{-1}\omega'(x,y) \\
&= \theta^{-1}(\sigma'(x) * \sigma'(y) - \sigma'(x \cdot y)) \\
&= \theta^{-1}\sigma'(x) * \theta^{-1}\sigma'(y) - \theta^{-1}\sigma'(x \cdot y) \\
&= (\sigma + \varphi)(x) * (\sigma + \varphi)(y) - (\sigma + \varphi)(x \cdot y) \\
&= \omega(x,y) + l_x\varphi(y) + r_y\varphi(x) + \varphi(x) \cdot \varphi(y) - \varphi(x \cdot y),
\end{aligned}$$

which shows Eq. (3.18).

Conversely, one has seen that non-abelian 2-cocycles defines pre-Lie algebra structures on $A \oplus B$ and thus provide extensions. Let (ω, l, r) and (ω', l', r') be two equivalent 2-cocycles, and $\varphi\colon A \to B$ be the map satisfying Eq. (3.16)–(3.18). Define $\theta\colon A \oplus B \to A \oplus B$ as follows:

$$\theta(x + u) = x - \varphi(x) + u, \quad \forall x \in A, u \in B.$$

One can verify that this map gives rise to a morphism of the extensions from $A \ltimes_{\omega,l,r} B$ to $A \ltimes_{\omega',l',r'} B$. This finishes the proof. $\square$

3.3.2 From the above proof, one can see the morphisms from $A \ltimes_{\omega,l,r} B$ to $A \ltimes_{\omega',l',r'} B$ are one-one corresponding to the linear maps $\varphi\colon A \to B$ satisfying Eq. (3.16)–(3.18). Specially, automorphisms of $A \ltimes_{\omega,l,r} B$ are one-one corresponding to the linear maps $\varphi\colon A \to \mathrm{Ann}(B)$ satisfying

$$l_x\varphi(y) + r_y\varphi(x) - \varphi(x \cdot y) = 0. \tag{3.19}$$

A ***non-abelian* 1*-cocycle*** on A with values in B respect to the 2-cocycle (l, r, ω) is a linear map $\varphi\colon A \to \mathrm{Ann}(B)$ satisfying Eq. (3.19). The set of them is denoted by $Z^1_{preLie}(A, B, (\omega, l, r))$. Note that the non-abelian 1-cocycles are independent on ω.

3.3.3 Now one can consider the abelian extensions. In this case, Eq. (3.8) and (3.9) trivially hold. Then Eq. (3.10) and (3.11) show that the couple (l, r) gives an A-module structure on B. Finally, Eq. (3.12) is equivalent to say ω is a 2-cocycle of A with values in the A-module (B, l, r) in the sense of usual abelian cohomology of pre-Lie algebras. I refer [7] and [4] for the definition and details. Note that the two definitions are different although they agree at low degree terms.

If (B, l, r) is an A-module, one may be interesting only on the extensions which induce the same A-module structure on B with the original one (l, r). This subcategory is denoted by $\mathcal{E}xt_{preLie}(A, (B, l, r))$ and is classified by the second abelian cohomology $H^2_{preLie}(A, (B, l, r))$.

Note that when the pre-Lie algebra structure on B is trivial, Eq. (3.16) and (3.17) force equivalent non-abelian 2-cocyles give the same A-module structure on B, thus one has

$$\mathcal{E}xt_{preLie}(A, B) = \bigsqcup_{(l,r)} \mathcal{E}xt_{preLie}(A, (B, l, r)),$$
$$H^2_{preLie}(A, B) = \bigsqcup_{(l,r)} H^2_{preLie}(A, (B, l, r)),$$

where the union is disjoint and taken over all possible A-module structures (l, r) on B.

3.3.4 Now one can consider the automorphisms of abelian extensions. Let (B, l, r) be an A-module and $A \ltimes_{\omega,l,r} B$ an extension of A by B belonging to $\mathcal{E}xt_{preLie}(A, (B, l, r))$. Then the automorphisms of this extension is one-one corresponding to the non-abelian

1-cocycles on A with values in B respect to the 2-cocycle (l, r, ω), which are precisely the 1-cocycles of A with values in the A-module (B, l, r) in the sense of usual abelian cohomology of pre-Lie algebras.

Chapter 4 Describe H^2_{preLie} in terms of Deligne groupoids

4.1 The DGLA structure on $C^{\bullet+1}(A,A)$

To construct the DGLA $\mathfrak{L}^\bullet$, one need the Chevalley-Eilenberg complex for pre-Lie algebras. However, there are two possible complexes, I use the one defined in [4].

4.1.1 Let A be a pre-Lie algebra and $l, r\colon A \to \mathfrak{gl}(M)$ a representation of A. Then one can define the action $(\mathfrak{l}, \mathfrak{r})$ of A on $\mathrm{Hom}(\otimes^n A, M)$ as follows:

$$(\mathfrak{l}_a(f))(x) = l_a(f(x)) - f(L_a(x)), \quad (\mathfrak{r}_a(f))(x) = r_a(f(x)), \quad \forall x \in \otimes^n A. \tag{4.1}$$

Note that when $n = 0$, one has $\otimes^0 A = k$, so $\mathrm{Hom}(\otimes^0 A, M) \cong M$, and that the action $(\mathfrak{l}, \mathfrak{r})$ coincides with (l, r).

For any $f \in \mathrm{Hom}(\otimes^n A, M)$, one can define $\delta f \in \mathrm{Hom}(\otimes^{n+1} A, M)$ by the formula

$$\begin{aligned}(\delta f)(x_1, \cdots, x_{n+1}) := &\sum_{k=1}^{n} (-1)^k (\mathfrak{l}_{x_k} f)(x_1, \cdots, \widehat{x_k}, \cdots, x_{n+1}) \\ &+ \sum_{k=1}^{n} (-1)^k (\mathfrak{r}_{x_{n+1}} f)(x_1, \cdots, \widehat{x_k}, \cdots, x_n, x_k),\end{aligned} \tag{4.2}$$

where $\widehat{\ }$ indicates the omission of the underneath term.

Note that this definition only works for $n \geqslant 1$, as for $u \in M \cong \mathrm{Hom}(\otimes^0 A, M)$, set $(\delta u)(x)$ to be $-l_x(u) + r_x(u)$. One can see that $(\delta^2 u)(x,y) = (l_x l_y - l_{x\cdot y})(u)$, thus one can define the set of ***Jacobin elements*** as

$$J(M) := \left\{ u \in M \middle| (l_x l_y - l_{x\cdot y})(u) = 0, \forall x, y \in A \right\}.$$

Now, let $C^0(A,M) = J(M)$ and $C^n(A,M) = \mathrm{Hom}(\otimes^n A, M)$ for $n \geqslant 1$. Then one can verify that δ^2 vanishes on $C^\bullet(A,M)$ and thus $(C^\bullet(A,M), \delta)$ becomes a non-negative cochain complex.

4.1.2 Denote the differential of $C^{\bullet+1}(A,A)$ by d. Following 2.3.1, one should define a graded Lie bracket $[,]$ on $C^{\bullet+1}(A,A)$ such that $d = \mathfrak{ad}_\mu$, where $\mu \in C^{1+1}(A,A)$ is the pre-Lie multiplication of A.

Recall that in 2.3.1, the Nijenhuis-Richardson graded Lie bracket is given as the graded commutator of the graded composite operation. One can see this graded composite operation gives a ***graded pre-Lie algebra*** structure on the Chevalley-Eilenberg complex in the sense that the associator (f,g,h) has the *graded left symmetry property*:

$$(f,g,h) = (-1)^{|f||g|}(g,f,h).$$

Therefore, it suffices to define a graded pre-Lie multiplication on $C^{\bullet+1}(A,A)$.

4.1.3 To do this, I first identify the complex $C^{\bullet+1}(A,A)$ with $C^{\bullet}(A,\mathfrak{gl}(A))$ via

$$\begin{aligned} C^{n+1}(A,A) &\longrightarrow C^{n}(A,\mathfrak{gl}(A)) \\ f &\longmapsto \widetilde{f} \end{aligned}$$

where $\widetilde{f}$ is defined via

$$\widetilde{f}(x)(y) = f(x,y), \quad \forall x \in \otimes^n A, y \in A.$$

Now, I define a binary operation $\diamond$ on $C^{\bullet}(A,\mathfrak{gl}(A))$ by

$$(f \diamond g)(x,y) := g(f(x)(y)) + g(y) \circ f(x),$$

where $f,g \in C^{\bullet}(A,\mathfrak{gl}(A))$ and $x \in \otimes^{|f|}A, y \in \otimes^{|g|}A$.

For any $f,g,h \in C^{\bullet}(A,\mathfrak{gl}(A))$ and $x \in \otimes^{|f|}A, y \in \otimes^{|g|}A, z \in \otimes^{|h|}A$, one has

$$\begin{aligned} ((f \diamond g) \diamond h)(x,y,z) &= h((f \diamond g)(x,y)(z)) + h(z) \circ ((f \diamond g)(x,y)) \\ &= h(g(f(x)(y))(z)) + h(g(y)(f(x)(z))) \\ &\quad + h(z) \circ g(f(x)(y)) + h(z) \circ g(y) \circ f(x), \\ (f \diamond (g \diamond h))(x,y,z) &= (g \diamond h)(f(x)(y,z)) + ((g \diamond h)(y,z)) \circ f(x) \\ &= (g \diamond h)(f(x)(y),z) + (g \diamond h)(y, f(x)(z)) \\ &\quad + ((g \diamond h)(y,z)) \circ f(x) \\ &= h(g(f(x)(y))(z)) + h(z) \circ g(f(x)(y)) \\ &\quad + h(g(y)(f(x)(z))) + h(f(x)(z)) \circ g(y) \\ &\quad + h(g(y)(z)) \circ f(x) + h(z) \circ g(y) \circ f(x) \end{aligned}$$

Therefore the associator is

$$(f,g,h)(x,y,z) = -h(g(y)(z)) \circ f(x) - h(f(x)(z)) \circ g(y).$$

Thus

$$(f,g,h)(x,y,z) = (g,f,h)(y,x,z), \tag{4.3}$$

which shows that $\diamond$ is not a graded pre-Lie multiplication as desired. However, after antisymmetrizing, one has

$$(f,g,h)_\wedge(x,y,z) = (g,f,h)_\wedge(y,x,z) = (-1)^{|f||g|}(g,f,h)_\wedge(x,y,z).$$

This suggests one to define the correct graded composite of f and g as the antisymmetrization $(f \diamond g)_\wedge$, which is

$$(f \diamond g)_\wedge(x) = \sum_{\sigma \in S_{|f|+|g|}} \mathrm{sgn}(\sigma)(f \diamond g)(x_{\sigma^{-1}}), \quad \forall x \in \otimes^{|f|+|g|}A.$$

However, this brings too many terms.

Note that to force $\diamond$ becomes graded left symmetric, one only need the variable (x,y,z) has graded left symmetry property on x and y as a whole, so I define the binary operation $\star$ by

$$(f \star g)(x) = \sum_{\sigma \in Sh(|f|,|g|)} \mathrm{sgn}(\sigma)(f \diamond g)(x_{\sigma_1^{-1}}, x_{\sigma_2^{-1}}),$$

where σ is taken over the set of all $(|f|,|g|)$-shuffles and thus the resulted tensor $x_{\sigma^{-1}}$ can be written into two parts $(x_{\sigma_1^{-1}}, x_{\sigma_2^{-1}})$, where $x_{\sigma_1^{-1}} \in \otimes^{|f|}A, x_{\sigma_2^{-1}} \in \otimes^{|g|}A$ and in each of them, the order of components follows the original one.

Example Let $f \in C^1(A, \mathfrak{gl}(A))$ and $g \in C^3(A, \mathfrak{gl}(A))$, taking $x = x_1 \otimes \cdots \otimes x_4 \in \otimes^4 A$ then one has

$$\begin{aligned}(f \star g)(x) =& (f \diamond g)(x_1,x_2,x_3,x_4) - (f \diamond g)(x_2,x_1,x_3,x_4) \\ &+ (f \diamond g)(x_3,x_1,x_2,x_4) - (f \diamond g)(x_4,x_1,x_2,x_3) \\ =& g(f(x_1)(x_2,x_3,x_4)) + g(x_2,x_3,x_4) \circ f(x_1) \\ &- g(f(x_2)(x_1,x_3,x_4)) - g(x_1,x_3,x_4) \circ f(x_2) \\ &+ g(f(x_3)(x_1,x_2,x_4)) + g(x_1,x_2,x_4) \circ f(x_3) \\ &- g(f(x_4)(x_1,x_2,x_3)) - g(x_1,x_2,x_3) \circ f(x_4).\end{aligned}$$

Before going forward, I need a lemma on shuffles:

Lemma 4.1.4 *There exist canonical isomorphisms:*

$$Sh(n,m,l) \cong Sh(n,m) \times Sh(n+m,l),$$
$$Sh(n,m,l) \cong Sh(m,l) \times Sh(n,m+l).$$

*Here $Sh(m,n,l)$ denote the set of all (m,n,l)-**shuffles**, i.e. permutations σ satisfying*

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(n),$$
$$\sigma^{-1}(n+1) < \cdots < \sigma^{-1}(n+m),$$
$$\sigma^{-1}(n+m+1) < \cdots < \sigma^{-1}(n+m+l).$$

Proof. An (n,m)-shuffle σ can be totally determined by the total ordered set $\{\sigma^{-1}(1) < \cdots < \sigma^{-1}(n)\}$, which can be viewed as a monotone map $\sigma_1^{-1}\colon [n] \to [n+m]$, where $[n]$ denotes the total ordered set $\{1 < \cdots < n\}$. Similarly, σ can also be determined by a monotone map $\sigma_2^{-1}\colon [n] \to [n+m]$, the complement of σ_1^{-1}. Thus one has the canonical isomorphism:

$$Sh(m,n) \cong \mathrm{Hom}([n],[n+m]) \cong \mathrm{Hom}([m],[n+m]). \tag{4.4}$$

An (n,m,l)-shuffle σ can be totally determined by the pair of total ordered sets $\{\sigma^{-1}(1) < \cdots < \sigma^{-1}(n)\}$ and $\{\sigma^{-1}(n+m+1) < \cdots < \sigma^{-1}(n+m+l)\}$, which gives rise to a pair of monotone maps $\sigma_1^{-1}\colon [n] \to [n+m+l]$ and $\sigma_3^{-1}\colon [l] \to [n+m+l]$, but also requires the images of them are disjoint. To defuse the extra requirement, one can replace σ_1^{-1} by its complement $\sigma_{23}^{-1}\colon [m+l] \to [n+m+l]$, then the requirement is equivalent to say there exists a monotone map $\tau^{-1}\colon [l] \to [m+l]$ such that $\sigma_3^{-1} = \sigma_{23}^{-1}\tau^{-1}$. Thus one has

$$Sh(n,m,l) \cong \mathrm{Hom}([n],[n+m+l]) \times \mathrm{Hom}([l],[m+l]). \tag{4.5}$$

Similarly, by replacing σ_3^{-1} with its complement, one gets

$$Sh(n,m,l) \cong \mathrm{Hom}([l],[n+m+l]) \times \mathrm{Hom}([n],[n+m]). \tag{4.6}$$

Comparing Eq. (4.4)–(4.6), the desired isomorphisms are obvious. □

Remark By the lemma, a pair (σ,τ) of a $(n,m+l)$-shuffle σ and a (m,l)-shuffle τ can be identified with a (n,m,l)-shuffle determined by the pair of monotone maps $(\sigma_1^{-1},\tau_2^{-1})$, which is obviously the composite $\tau\sigma$.

Proposition 4.1.5 $(C^\bullet(A,\mathfrak{gl}(A)),\star)$ *is a graded pre-Lie algebra.*

Proof. For all $f,g,h\in C^\bullet(A,\mathfrak{gl}(A))$ and $x\in\otimes^{|f|+|g|+|h|}A$. By lemma 4.1.4 and the remark above, one has

$$\begin{aligned}((f\star g)\star h)(x) &= \sum_{\substack{\sigma\in Sh(|f|,|g|)\\ \tau\in Sh(|f|+|g|,|h|)}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)((f\diamond g)\diamond h)(x_{\tau_1^{-1}\sigma_1^{-1}},x_{\tau_1^{-1}\sigma_2^{-1}},x_{\tau_2^{-1}})\\ &= \sum_{\pi\in Sh(|f|,|g|,|h|)} \operatorname{sgn}(\pi)((f\diamond g)\diamond h)(x_{\pi_1^{-1}},x_{\pi_2^{-1}},x_{\pi_3^{-1}}),\end{aligned}$$

here the resulted tensor $x_{\tau^{-1}\sigma^{-1}}$ are written into three parts: the shuffle τ gives rise to two monotone maps τ_1^{-1} and τ_2^{-1}, which induce two subsequences $x_{\tau_1^{-1}}$ and $x_{\tau_2^{-1}}$ of the components of $x_{\tau^{-1}}$, then the shuffle σ gives rise to two monotone maps σ_1^{-1} and σ_2^{-1}, which induce two subsequences $x_{\tau_1^{-1}\sigma_1^{-1}}$ and $x_{\tau_1^{-1}\sigma_2^{-1}}$ of the components of $x_{\tau_1^{-1}}$.

Likewise, one has

$$\begin{aligned}(f\star(g\star h))(x,y,z) &= \sum_{\substack{\sigma\in Sh(|f|,|g|+|h|)\\ \tau\in Sh(|g|,|h|)}} \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma)(f\diamond(g\diamond h))(x_{\sigma_1^{-1}},y_{\sigma_2^{-1}\tau_1^{-1}},z_{\sigma_2^{-1}\tau_2^{-1}})\\ &= \sum_{\pi\in Sh(|f|,|g|,|h|)} \operatorname{sgn}(\pi)(f\diamond(g\diamond h))(x_{\pi_1^{-1}},y_{\pi_2^{-1}},z_{\pi_3^{-1}}),\end{aligned}$$

Therefore the associator $(f,g,h)_\star$ of $\star$ is

$$\begin{aligned}(f,g,h)_\star(x) &= \sum_{\sigma\in Sh(|f|,|g|,|h|)} \operatorname{sgn}(\sigma)(f,g,h)(x_{\sigma_1^{-1}},x_{\pi_2^{-1}},x_{\sigma_3^{-1}})\\ &= \sum_{\sigma\in Sh(|f|,|g|,|h|)} \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)(f,g,h)(x_{\sigma_1^{-1}\pi_1^{-1}},x_{\sigma_2^{-1}\pi_2^{-1}},x_{\sigma_3^{-1}})\\ &= \sum_{\sigma\in Sh(|f|,|g|,|h|)} \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)(g,f,h)(x_{\sigma_1^{-1}},x_{\sigma_2^{-1}},x_{\sigma_3^{-1}})\\ &= \operatorname{sgn}(\pi)(g,f,h)_\star(x),\end{aligned}$$

where π is the $(|f|,|g|)$-shuffle interchanging the total ordered sets $\{1<\cdots<|f|\}$ and $\{|f|+1<\cdots<|f|+|g|\}$, thus $\operatorname{sgn}(\pi)=(-1)^{|f||g|}$. This finishes the proof. $\square$

4.1.6 Now the graded pre-Lie multiplication $\star$ on $C^\bullet(A,\mathfrak{gl}(A))$ is translated to the graded pre-Lie multiplication $\star$ on $C^{\bullet+1}(A,A)$ via

$$\widetilde{(f\star g)}=\widetilde{f}\star\widetilde{g}.$$

Then, the graded Lie bracket can be given by

$$[f,g]:=f\star g-(-1)^{|f||g|}g\star f.$$

Corrollay 4.1.7 *$(C^{\bullet+1}(A,A),[,])$ is a graded Lie algebra.*

Proof. The graded antisymmetry is obvious. The graded Jacobin identity follows immediately from the graded left symmetry property. □

4.1.8 Consider now the differential d, one should show that $(C^{\bullet+1}(A,A),[,],d)$ is a DGLA. This can be done by showing $d=\mathfrak{ad}_\mu$, where $\mu\in C^{1+1}(A,A)$ denote the pre-Lie multiplication of A and $\mathfrak{ad}$ denote the graded adjoint action induced by $[,]$.

Proposition 4.1.9 *In $(C^{\bullet+1}(A,A),[,])$, one has $d=\mathfrak{ad}_\mu$.*

Proof. For any $f\in C^{n-1+1}(A,A)$ and $x_1,\cdots,x_{n+1}\in A$, one has

$$\begin{aligned}(\mu\star f)(x_1,\cdots,x_{n+1})&=\sum_{k=1}^n(-1)^{k-1}(\widetilde{\mu}\diamond\widetilde{f})(x_k,x_1,\cdots,\widetilde{x_k},\cdots,x_n)(x_{n+1})\\&=\sum_{k=1}^n(-1)^{k-1}\Big(\widetilde{f}(\widetilde{\mu}(x_k)(x_1,\cdots,\widetilde{x_k},\cdots,x_n))\\&\qquad+\widetilde{f}(x_1,\cdots,\widetilde{x_k},\cdots,x_n)\circ\widetilde{\mu}(x_k)\Big)(x_{n+1})\\&=\sum_{k=1}^n(-1)^{k-1}f(l_{x_k}(x_1,\cdots,\widetilde{x_k},\cdots,x_n,x_{n+1})),\\(f\star\mu)(x_1,\cdots,x_{n+1})&=\sum_{k=1}^n(-1)^{n-k}(\widetilde{f}\diamond\widetilde{\mu})(x_1,\cdots,\widetilde{x_k},\cdots,x_n,x_k)(x_{n+1})\\&=\sum_{k=1}^n(-1)^{n-k}\Big(\widetilde{\mu}(\widetilde{f}(x_1,\cdots,\widetilde{x_k},\cdots,x_n)(x_k))\\&\qquad+\widetilde{\mu}(x_k)\circ\widetilde{f}(x_1,\cdots,\widetilde{x_k},\cdots,x_n)\Big)(x_{n+1})\\&=\sum_{k=1}^n(-1)^{n-k}\left(r_{x_{n+1}}f(x_1,\cdots,\widetilde{x_k},\cdots,x_n,x_k)\right.\\&\qquad\left.+l_{x_k}f(x_1,\cdots,\widetilde{x_k},\cdots,x_n,x_{n+1})\right),\end{aligned}$$

where l, r denote the left and right multiplication of A. Let $(\mathfrak{l}, \mathfrak{r})$ denote the action of A on $C^{\bullet+1}(A,A)$ induced by (l,r), which is defined by Eq. (4.1). Then one has

$$\begin{aligned}
[\mu, f](x_1, \cdots, x_{n+1}) &= (\mu \star f - (-1)^{n-1} f \star \mu)(x_1, \cdots, x_n, x_{n+1}) \\
&= \sum_{k=1}^{n} (-1)^{k-1} f(l_{x_k}(x_1, \cdots, \widetilde{x_k}, \cdots, x_n, x_{n+1})) \\
&\quad - \sum_{k=1}^{n} (-1)^{k-1} \big(r_{x_{n+1}} f(x_1, \cdots, \widetilde{x_k}, \cdots, x_n, x_k) \\
&\qquad\qquad + l_{x_k} f(x_1, \cdots, \widetilde{x_k}, \cdots, x_n, x_{n+1}) \big) \\
&= \sum_{k=1}^{n} (-1)^{k} \big((\mathfrak{l}_{x_k} f)(x_1, \cdots, \widehat{x_k}, \cdots, x_{n+1}) \\
&\qquad\qquad + (\mathfrak{r}_{x_{n+1}} f)(x_1, \cdots, \widehat{x_k}, \cdots, x_n, x_k) \big) \\
&= (df)(x_1, \cdots, x_{n+1}).
\end{aligned}$$

This shows $d = \mathfrak{ad}_\mu$. □

Remark The above proposition also provides a simple proof of $d^2 = 0$: as $d = \mathfrak{ad}_\mu$, it suffices to show $[\mu, \mu] = 0$, but one has

$$\frac{1}{2}[\mu, \mu](x,y,z) = (\mu \star \mu)(x,y,z) = (x \cdot y) \cdot z + y \cdot (x \cdot z) - (y \cdot x) \cdot z - x \cdot (y \cdot z) = 0.$$

Corrollay 4.1.10 *$(C^{\bullet+1}(A,A), [,], d)$ is a DGLA.*

Proof. Note that any graded adjoint transformation $\mathfrak{ad}_x$ on a graded Lie algebra is a graded derivation of degree $|x|$, and that $|\mu| = 1$ in $C^{\bullet+1}(A,A)$. Then the statement is obvious. □

4.2 Construction of the DGLA $\mathfrak{L}^\bullet$

4.2.1 Let A and B be two pre-Lie algebras and $A \oplus B$ be the pre-Lie algebra direct sum of them. Then B is an $A \oplus B$-module via the left and right multiplication of B on itself. The complex $C^\bullet_>(A \oplus B, B)$ is defined as the subcomplex of $C^\bullet(A \oplus B, B)$ by

$$C^\bullet(A \oplus B, B) = C^\bullet_>(A \oplus B, B) \oplus C^\bullet(B, B).$$

Proposition 4.2.2 *$C^{\bullet+1}_>(A \oplus B, B)$ is a sub-DGLA of $C^{\bullet+1}(A \oplus B, A \oplus B)$ endowed with the bracket $[,]$ defined in 4.1.6.*

Proof. An element $f \in C^n(A \oplus B, A \oplus B)$ lies in $C^n_>(A \oplus B, B)$ if and only if it vanishes on $\otimes^n B$ and its image lies in B. For such an element f, it is obvious that df vanishes on $\otimes^{n+1} B$ and its image lies in B, i.e. $df \in C^{n+1}_>(A \oplus B, B)$. Similarly, $C^{\bullet+1}_>(A \oplus B, B)$ is closed under $\star$, a fortiori $[,]$. □

Denote this DGLA by $\mathfrak{L}^\bullet$, one further has

Proposition 4.2.3 *$\mathfrak{L}^\bullet$ is $\mathfrak{ad}_0$-nilpotent, further, $\mathfrak{L}^0$ is abelian.*

Proof. Note that $\mathfrak{L}^0 = \mathrm{Hom}(A, B)$, thus $\star$ gives a trivial pre-Lie algebra structure on $\mathfrak{L}^0$. As for the $\mathfrak{ad}_0$-nilpotent property, consider

$$C^{n,m} := \left\{ f \in C^{n+m}(A \oplus B, B) \middle| f \text{ vanishes outside } A \otimes^{n,m} B \right\},$$

where $A \otimes^{n,m} B$ is the subspace of $\otimes^{n+m}(A \oplus B)$ obtained by

$$A \otimes^{n,m} B := \bigoplus_{\sigma \in Sh(n,m)} X_{\sigma^{-1}(1)} \otimes \cdots \otimes X_{\sigma^{-1}(n+m)},$$

where $X_1 = \cdots = X_n = A$ and $X_{n+1} = \cdots = X_{n+m} = B$.

One can see

$$\mathfrak{L}^\bullet = \bigoplus_{n \in \mathbb{N}^*, m \in \mathbb{N}} C^{n,m}.$$

For all $f \in \mathfrak{L}^0$, $g \in C^{n,m}$, $x \in \otimes^{n+m-1}(A \oplus B)$ and $y \in A \oplus B$, one has

$$\begin{aligned} \mathfrak{ad}_f(g)(x,y) &= \sum_{\sigma \in S_{|g|}} \mathrm{sgn}(\sigma) \left(g(f(x_{\sigma^{-1}}, y)) - f(g(x_{\sigma^{-1}}, y)) \right) \\ &= \sum_{\sigma \in S_{|g|}} \mathrm{sgn}(\sigma) g(f(x_{\sigma^{-1}}, y)). \end{aligned}$$

Note that f maps A to B and vanishes on B, thus $\mathfrak{ad}_f$ induces linear maps:

$$\cdots \xrightarrow{\mathfrak{ad}_f} C^{n,m} \xrightarrow{\mathfrak{ad}_f} C^{n+1,m-1} \xrightarrow{\cdots}$$

Further, one has $\mathfrak{ad}_f(C^{n,0}) = 0$. This shows that $\mathfrak{ad}_f$ is nilpotent. □

Remark From the above proof, one can see $\mathfrak{ad}_f^{n+1}(\mathfrak{L}^n) = 0$ for all $f \in \mathfrak{L}^0$.

4.3 $H^2_{preLie} \cong \pi_0 \mathrm{Del}(\mathfrak{L}^\bullet)$

4.3.1 As the DGLA $\mathfrak{L}^\bullet$ is $\mathfrak{ad}_0$-nilpotent, one can define the Maurer-Cartan elements of it and further the Deligne groupoid $\mathrm{Del}(\mathfrak{L}^\bullet)$ following 2.2.3–2.2.5.

Note that $\mathfrak{L}^1 = \mathrm{Hom}(\otimes^2 A, B) \oplus \mathrm{Hom}(A \otimes B, B) \oplus \mathrm{Hom}(B \otimes A, B)$. One can thus view any triple (ω, l, r) of a bilinear map $\omega\colon \otimes^2 A \to B$ together with two linear maps $l, r\colon A \to \mathfrak{gl}(B)$ as an element of $\mathfrak{L}^1$ via:

$$l(x,u) := l_x(u), \qquad r(u,x) := r_x(u), \qquad \forall x \in A, u \in B.$$

Proposition 4.3.2 *Let A, B be two pre-Lie algebras. A triple (ω, l, r) of a bilinear map $\omega\colon \otimes^2 A \to B$ and two linear maps $l, r\colon A \to \mathfrak{gl}(B)$ is a non-abelian 2-cocycle, thus Eq.* (3.6) *defines a pre-Lie algebra structure on $A \oplus B$, if and only if $\omega + l + r \in MC(\mathfrak{L}^\bullet)$.*

Proof. Let $c = \omega + l + r$, then $c \in MC(\mathfrak{L}^\bullet)$ if and only if $Q = dc + \frac{1}{2}[c,c]$ vanishes if and only if the following equalities hold for all $x, y, z \in A$ and $u, v, w \in B$:

$$Q(x,y,z) = 0, \qquad Q(x,y,u) = 0, \qquad Q(x,u,y) = 0, \tag{4.7}$$

$$Q(x,u,v) = 0, \qquad Q(u,v,x) = 0, \qquad Q(u,v,w) = 0. \tag{4.8}$$

To calculate them, one should first write down the explicit formula of Q. For all $e_1, e_2, e_3 \in A \oplus B$, one has

$$\begin{aligned} Q(e_1,e_2,e_3) = &(dc + c \star c)(e_1,e_2,e_3) \\ = &- e_1 \cdot c(e_2,e_3) + c(e_1 \cdot e_2, e_3) + c(e_2, e_1 \cdot e_3) + e_2 \cdot c(e_1,e_3) \\ &- c(e_2 \cdot e_1, e_3) - c(e_1, e_2 \cdot e_3) - c(e_2,e_1) \cdot e_3 + c(e_1,e_2) \cdot e_3 \\ &+ c(c(e_1,e_2),e_3) + c(e_2, c(e_1,e_3)) - c(c(e_2,e_1),e_3) - c(e_1, c(e_2,e_3)). \end{aligned}$$

By the above formulas, one can expand the left sides of Eq. (4.7)–(4.8) as follows:

$$\begin{aligned} Q(x,y,z) &= \omega([x,y],z) - \omega(x, y \cdot z) + \omega(y, x \cdot z) - l_x \omega(y,z) + l_y \omega(x,z) + r_z \omega_\wedge(x,y) \\ Q(x,y,u) &= l_{[x,y]}(u) + \omega_\wedge(x,y) \cdot u + l_y l_x(u) - l_x l_y(u) \\ Q(x,u,y) &= r_{x \cdot y}(u) + u \cdot \omega(x,y) + r_y l_x(u) - r_y r_x(u) - l_x r_y(u), \\ Q(x,u,v) &= u \cdot l_x(v) - l_x(u \cdot v) - r_x(u) \cdot v + l_x(u) \cdot v, \\ Q(u,v,x) &= -u \cdot r_x(v) + r_x([u,v]) + v \cdot r_x(u), \\ Q(u,v,w) &= 0. \end{aligned}$$

One can then immediately see that $c \in MC(\mathfrak{L}^\bullet)$ if and only if Eq. (3.8)–(3.12) holds. This finishes the proof. □

Remark By proposition 4.3.2, one can identify $Z^2_{preLie}(A,B)$ with $MC(\mathfrak{L}^\bullet)$. However, there is an equivalence relation on $Z^2_{preLie}(A,B)$ and there is the gauge action on $MC(\mathfrak{L}^\bullet)$. Thus it is natural to consider the relationship between them.

On the other hand, combine proposition 4.3.2, proposition 3.2.2 and 3.2.1, one gets the following surjective map:

$$\begin{aligned} MC(\mathfrak{L}^\bullet) &\xrightarrow{\mathscr{F}} \mathcal{E}xt_{preLie}(A,B) \\ \omega + l + r &\longmapsto A \ltimes_{\omega,l,r} B. \end{aligned}$$

Since $MC(\mathfrak{L}^\bullet)$ is the set of objects of $\mathrm{Del}(\mathfrak{L}^\bullet)$, one may ask if this map can be further a functor.

4.3.3 Note that $\mathfrak{L}^0$ is abelian, thus $\exp(\mathfrak{L}^0) \cong (\mathfrak{L}^0,+) = \mathrm{Hom}(A,B)$. For all $\varphi \in \mathfrak{L}^0$, since $\mathfrak{ad}^2_\varphi(\mathfrak{L}^1) = 0$, the gauge action on $\mathfrak{L}^1$ thus read:

$$\varphi.\alpha = \alpha + \mathfrak{ad}_\varphi(\alpha) - d\varphi - \frac{1}{2}\mathfrak{ad}_\varphi(d\varphi), \quad \forall \alpha \in \mathfrak{L}^1.$$

For any $\alpha = \omega + l + r \in MC(\mathfrak{L}^\bullet)$ with $\omega \in \mathrm{Hom}(\otimes^2 A, B)$, $l \in \mathrm{Hom}(A \otimes B, B)$ and $r \in \mathrm{Hom}(B \otimes A, B)$, the gauge action of φ translate it to $\varphi.\alpha$. To write down it, taking any $x, y \in A$ and $u \in B$, one has

$$\begin{aligned} (\varphi.\alpha)(x,y) &= \omega(x,y) + l(x,\varphi(y)) + r(\varphi(x),y) - d\varphi(x,y) - \frac{1}{2}(\varphi \star d\varphi)(x,y) \\ &= \omega(x,y) + l_x\varphi(y) + r_y\varphi(x) - \varphi(x\cdot y) + \varphi(x)\cdot\varphi(y), \end{aligned} \tag{4.9}$$

$$(\varphi.\alpha)(x,u) = l(x,u) - d\varphi(x,u) - \frac{1}{2}(\varphi \star d\varphi)(x,u) = l_x(u) + \varphi(x)\cdot u, \tag{4.10}$$

$$(\varphi.\alpha)(u,x) = r(u,x) - d\varphi(u,x) - \frac{1}{2}(\varphi \star d\varphi)(u,x) = r_x(u) + u\cdot\varphi(x). \tag{4.11}$$

Compare them with Eq. (3.16)–(3.15), one can see that $\varphi.\alpha$ lies in the same cohomology class of α. Further, by the proof of theorem 3.3.1, the above action of φ induces a morphism θ of extensions of A by B via

$$\theta(x+u) = x - \varphi(x) + u, \quad \forall x \in A, u \in B. \tag{4.12}$$

Thus, by setting $\mathscr{F}(\alpha \to \varphi.\alpha) = \theta$, the map $\mathscr{F}$ becomes a functor

$$\mathscr{F}:\ \mathrm{Del}(\mathfrak{L}^\bullet) \to \mathcal{E}xt_{preLie}(A,B).$$

Theorem 4.3.4 *$\mathscr{F}: \mathrm{Del}(\mathfrak{L}^\bullet) \to \mathcal{E}xt_{preLie}(A,B)$ is an equivalence of categories.*

Proof. one has seen $\mathscr{F}$ is surjective, a fortiori essentially surjective, thus it suffices to show $\mathscr{F}$ is fully faithful.

Let $\mathscr{F}(\varphi) = \theta$, then Eq. (4.12) shows that $\varphi(x) = x - \theta(x)$ for all $x \in A$. The faithfulness thus follows.

By theorem 3.3.1, any morphism of extensions of A by B gives to equivalence of corresponding 2-cocycles. Thus, to show $\mathscr{F}$ is full, it suffices to show equivalent 2-cocycles are connected by some gauge transformation.

Indeed, let (ω,l,r) and (ω',l',r') be two equivalent 2-cocycles. Then there exists some $\varphi \in \mathrm{Hom}(A,B) = \mathfrak{L}^0$ such that Eq. (3.16)–(3.15) hold. Then Eq. (4.9)–(4.11) show that $\varphi.(\omega+l+r) = \omega'+l'+r'$ as desired. □

Taking the set π_0 of connected components of the two categories, one has

Corrollay 4.3.5 *Extensions of A by B are classified by $\pi_0\,\mathrm{Del}(\mathfrak{L}^\bullet)$.*

Remark For any $\alpha \in MC(\mathfrak{L}^\bullet)$, consider the *stabilizer* $S_\alpha := \{\varphi \in \mathfrak{L}^0 | \varphi.\alpha = \alpha\}$ of α, which is a subgroup of $\mathfrak{L}^0$ and one has $S_\alpha = \mathrm{Aut}_{\mathrm{Del}(\mathfrak{L}^\bullet)}(\alpha)$ by definition. But the later one is isomorphic to $\mathrm{Aut}_{\mathcal{E}xt_{preLie}(A,B)}(A \ltimes_\alpha B)$ by theorem 4.3.4.

If α' is another element in $MC(\mathfrak{L}^\bullet)$ and there exists $\varphi \in \mathfrak{L}^0$ such that $\varphi.\alpha' = \alpha$, then one has $S_{\alpha'} = \varphi^{-1} S_\alpha \varphi$ and so that $\mathrm{Aut}_{\mathcal{E}xt_{preLie}(A,B)}(A \ltimes_{\alpha'} B) \cong \mathrm{Aut}_{\mathcal{E}xt_{preLie}(A,B)}(A \ltimes_\alpha B)$.

Written α as $\omega+l+r$ so that (ω,l,r) is a 2-cocycle. Then Eq. (4.9)–(4.11) show that $S_\alpha = Z^1_{preLie}(A,B,(\omega,l,r))$.

4.4 Abelian cohomology as tangent complex

4.4.1 Now I consider the abelian extensions. Let A be a pre-Lie algebra and (B,l,r) an A-module. As the semiproduct $A \ltimes_{0,l,r} B$ is already an extension of A by B, one has $l+r \in MC(\mathfrak{L}^\bullet)$. Denote $l+r$ by α, then one gets the tangent complex $\mathfrak{L}^\bullet_\alpha$ at α. Let d denote the differential of $\mathfrak{L}^\bullet$, thus the differential of $\mathfrak{L}^\bullet_\alpha$ is $d_\alpha = d + \mathfrak{ad}_\alpha$.

Proposition 4.4.2 *$(C^{\bullet+1}(A,B),\delta)$ is a sub-DGLA of $(\mathfrak{L}^\bullet_\alpha, d_\alpha)$.*

Proof. One can see the bracket defined in 4.1.6 vanishes on $C^{\bullet+1}(A,B)$, thus it is an abelian subalgebra of $\mathfrak{L}^\bullet_\alpha$. It remains to show that the two differentials d and d_α coincide.

For any $f \in C^{n-1+1}(A,B)$ and $x \in \otimes^{n+1}A$, note that f vanishes outside $\otimes^n A$, one has

$$df(x) = \sum_{k=1}^{n}(-1)^{k-1} f(L_{x_k}(x_1,\cdots,\widehat{x_k},\cdots,x_{n+1})), \qquad (\alpha \star f)(x) = 0,$$

and

$$(f \star \alpha)(x) = \sum_{k=1}^{n}(-1)^{n-k}\left(r_{x_{n+1}} f(x_1,\cdots,\widehat{x_k},\cdots,x_n,x_k) + l_{x_k} f(x_1,\cdots,\widehat{x_k},\cdots,x_{n+1})\right).$$

Thus

$$d_\alpha f = df + [\alpha, f] = df - (-1)^{n-1} f \star \alpha = \delta f.$$

as desired. □

4.4.3 By theorem 4.3.4, an element $\omega \in C^{1+1}(A,B)$ gives rise to an abelian extension which induces the same A-module structure on B with the original one if and only if $\omega + \alpha \in MC(\mathfrak{L}^\bullet)$. Let $MC(\mathfrak{L}^\bullet)_\alpha$ denote the set

$$\left\{\omega \in C^{1+1}(A,B) \middle| \omega + \alpha \in MC(\mathfrak{L}^\bullet)\right\}.$$

By Eq. (4.9)–(4.11), the gauge action of $\varphi \in \mathrm{Hom}(A,B)$ maps $\omega + \alpha$ to

$$\varphi.(\omega+\alpha) = \omega - d\varphi - [\alpha,\varphi] + \alpha = \omega - d_\alpha(\varphi) + \alpha. \tag{4.13}$$

Therefore $MC(\mathfrak{L}^\bullet)_\alpha$ is preserved under the gauge action and thus induces a subgroupoid $\mathrm{Del}(\mathfrak{L}^\bullet)_\alpha$ of $\mathrm{Del}(\mathfrak{L}^\bullet)$.

By theorem 4.3.4, the functor $\mathscr{F}$ induces an equivalence of categories:

$$\mathrm{Del}(\mathfrak{L}^\bullet)_\alpha \xrightarrow{\mathscr{F}} \mathcal{E}xt_{preLie}(A,(B,l,r)).$$

On the other hand, note that $\omega \star \omega = 0$, one has

$$d(\omega+\alpha) + \frac{1}{2}[\omega+\alpha,\omega+\alpha] = d\omega + [\alpha,\omega] + d\alpha + \frac{1}{2}[\alpha,\alpha] = d\omega + [\alpha,\omega].$$

Thus $\omega + \alpha \in MC(\mathfrak{L}^\bullet)$ if and only if $\omega \in \ker d_\alpha = \ker\delta$, in other words, ω is an abelian 2-cocycle of A with values in the A-module (B,l,r).

Moreover, Eq. (4.13) shows that the gauge action on $MC(\mathfrak{L}^\bullet)_\alpha$ gives rise to the abelian 2-coboundaries, i.e. elements of $\delta(C^1(A,B))$. Therefore,

$$\pi_0 \mathrm{Del}(\mathfrak{L}^\bullet)_\alpha \cong \frac{\ker(C^2(A,B) \xrightarrow{\delta} C^3(A,B))}{\mathrm{im}(C^1(A,B) \xrightarrow{\delta} C^2(A,B))} =: H^2_{preLie}(A,(B,l,r)).$$

Thus, $\mathcal{E}xt_{preLie}(A,(B,l,r))$ is classified by the abelian cohomology $H^2_{preLie}(A,(B,l,r))$ as mentioned before.

4.4.4 Consider $\mathrm{Aut}_{\mathrm{Del}(\mathfrak{L}^\bullet)_\alpha}(\omega)$ for all $\omega \in MC(\mathfrak{L}^\bullet)_\alpha$. By Eq. (4.13), $\varphi \in \mathrm{Hom}(A,B)$ belongs to $\mathrm{Aut}_{\mathrm{Del}(\mathfrak{L}^\bullet)_\alpha}(\omega)$ if and only if $d_\alpha(\varphi) = 0$, in other words, φ is an abelian 1-cocycle of A with values in the A-module (B,l,r). Therefore

$$\mathrm{Aut}_{\mathrm{Del}(\mathfrak{L}^\bullet)_\alpha}(\omega) \cong Z^1_{preLie}(A,(B,l,r)).$$

Thus, automorphisms of an abelian extension of a pre-Lie algebra A by an A-module (B,l,r) is one-one corresponding to the abelian 1-cocycles as mentioned before.

Chapter 5 Toward the intrinsic cohomology

5.1 The notion of 2-term $\mathcal{PL}_\infty$-algebras

The terminology of a $\mathcal{PL}_\infty$-algebra has been introduced in [10]. It is a right-symmetric algebra up to homotopy. By a slight modification, one could obtain a left-symmetric one up to homotopy, which is the homotopic version of the notion of pre-Lie algebras which used in this paper.

Since to follow 2.5.2, one only needs 2-term $\mathcal{PL}_\infty$-algebras, so one can restrict in the subcategory of 2-term $\mathcal{PL}_\infty$-algebras, which are obtained by truncation of $\mathcal{PL}_\infty$-algebras. I refer [27] for more information on the 2-term $\mathcal{PL}_\infty$-algebras and their properties.

5.1.1 A ***2-term $\mathcal{PL}_\infty$-algebra*** consists of

- the *underlying complex* $\mathfrak{A}\colon \cdots \to 0 \to A_1 \xrightarrow{d} A_0$, which is a chain complex concentrated in first 2 terms,
- the *multiplication* $\mu\colon \mathfrak{A}\otimes\mathfrak{A}\to\mathfrak{A}$, which is chain map,
- the *left-symmetry law* $\gamma\colon \alpha \Rightarrow \alpha_{12}$, which is a chain homotopy between the two parallel chain maps $\alpha, \alpha_{12}\colon \mathfrak{A}\otimes\mathfrak{A}\otimes\mathfrak{A}\to\mathfrak{A}$, where α denotes the associator of μ and α_{12} is defined by

$$\alpha_{12}(x,y,z) = \alpha(y,x,z), \quad \forall x,y,z\in\mathfrak{A},$$

satisfying the following equalities:

$$\gamma = \gamma_{12}, \quad \delta(\gamma) = 0,$$

where δ is defined by Eq. (4.2) with $A = A_0$, $M = A_1$ and l, r the left and right multiplication induced by μ. Note that although in general A_0 is not a pre-Lie algebra and $l, r\colon A_0 \to \mathfrak{gl}(A_1)$ is not a representation of A_0, the formula Eq. (4.2) still makes sense.

A 2-term $\mathcal{PL}_\infty$-algebra $(\mathfrak{A}, d, \mu, l_3)$ is said to be ***strict*** if $l_3 = 0$.

From now on, if there is no ambiguity, a 2-term $\mathcal{PL}_\infty$-algebra $(\mathfrak{A},d,\mu,l_3)$ will be simply denoted as $\mathfrak{A}$ and the multiplication $\mu(x,y)$ will be written as $x\cdot y$.

5.1.2 A *morphism* $\mathscr{F}\colon(\mathfrak{A},d,\mu,l_3)\to(\mathfrak{A}',d',\mu',l_3')$ between 2-term $\mathcal{PL}_\infty$-algebras consists of a chain morphism F between the 2-term chain complexes $(\mathfrak{A},d)$ and $(\mathfrak{A}',d')$ and a chain homotopy $\mathscr{F}_2\colon\mu'\circ(F\otimes F)\Rightarrow F\circ\mu$ satisfying

$$\begin{aligned}
&F\circ l_3(x,y,z)-l_3'(F(x),F(y),F(z))\\
=&\mathscr{F}_2(\mu(x,y)-\mu(y,x),z)-\mathscr{F}_2(x,\mu(y,z))+\mathscr{F}_2(y,\mu(x,z))\\
&-\mu'(F(x),\mathscr{F}_2(y,z))+\mu'(F(y),\mathscr{F}_2(x,z))+\mu'(\mathscr{F}_2(x,y)-\mathscr{F}_2(y,x),F(z)).
\end{aligned}$$

for all $x,y,z\in A_0$.

Let $\mathscr{F}:\mathfrak{A}\to\mathfrak{A}'$ and $\mathscr{G}:\mathfrak{A}'\to\mathfrak{A}''$ be two morphisms between 2-term $\mathcal{PL}_\infty$-algebras, the composite of them is the morphism $\mathscr{G}\circ\mathscr{F}:\mathfrak{A}\to\mathfrak{A}''$ consisting of the chain map $G\circ F$ and the chain homotopy $(\mathscr{G}\circ\mathscr{F})_2$ defined by

$$(\mathscr{G}\circ\mathscr{F})_2:=\mathscr{G}_2\circ(F\otimes F)+G\circ\mathscr{F}_2$$

5.1.3 A *homotopy* $\Phi\colon\mathscr{F}\Rightarrow\mathscr{F}'$ between two morphisms $\mathscr{F},\mathscr{F}'\colon\mathfrak{A}\to\mathfrak{A}'$ is a chain homotopy $\Phi\colon F\Rightarrow F'$ satisfying an extra condition that it induces a homotopy $\overline{\Phi}$ between the chain homotopies $\mathscr{F}$ and $\mathscr{F}'$. To write down this extra condition, note that the chain homotopy $\mathscr{F}_2$, regarded as a chain map, is

$$\begin{tikzcd}
(\otimes^2\mathfrak{A})_2 \arrow[r] \arrow[d, "\delta"] & 0 \arrow[d] \\
(\otimes^2\mathfrak{A})_2\oplus(\otimes^2\mathfrak{A})_2\oplus(\otimes^2\mathfrak{A})_1 \arrow[r] \arrow[d, "\delta"] & 0 \arrow[d] \\
(\otimes^2\mathfrak{A})_1\oplus(\otimes^2\mathfrak{A})_1\oplus\otimes^2A_0 \arrow[r, "\mathscr{F}_2"] \arrow[d, "\delta"] & A_1' \arrow[d, "d"] \\
\otimes^2A_0\oplus\otimes^2A_0 \arrow[r, "(\mu'\circ(F\otimes F),F\circ\mu)"] & A_0'
\end{tikzcd}$$

where $(\otimes^2\mathfrak{A})_1=A_1\otimes\otimes A_0\oplus A_0\otimes A_1$ and $(\otimes^2\mathfrak{A})_2=A_1\otimes A_1$. Note that this $\overline{\Phi}$ is a graded linear map from $(\otimes^2\mathfrak{A})\otimes I_\bullet$ to $\mathfrak{A}'$ of degree 1 and satisfying

$$d\circ\overline{\Phi}=(\mu'\circ(F\otimes F),F\circ\mu)-(\mu'\circ(F'\otimes F'),F'\circ\mu),\quad\overline{\Phi}\circ\delta=\mathscr{F}_2-\mathscr{F}_2'.$$

The first equality suggests one to define this $\overline{\Phi}$ as

$$\overline{\Phi}=\left(\mu'\circ(F'\otimes\Phi+\Phi\otimes F'+\Phi\otimes d\Phi),\Phi\circ\mu\right),$$

and then the second one provides the explicit formula for the extra condition

$$\begin{aligned}&\mu'(F'(x),\Phi(y))+\mu'(\Phi(x),F'(y))+\mu'(\Phi(x),d\Phi(y))-\Phi(\mu(x,y))\\&=\mathscr{F}_2(x,y)-\mathscr{F}_2'(x,y),\quad \forall x,y\in A_0.\end{aligned}\tag{5.1}$$

The *vertical* and horizontal composite of homotopies are the same as chain homotopies.

5.1.4 The 2-term $\mathcal{PL}_\infty$-algebras, together with morphisms between them and homotopies between morphisms, form a 2-category, denoted by $2PreL_\infty$. Moreover, this is a $(2,1)-$category since every homotopy Φ has an inverse $-\Phi$. $2PreL_\infty$ is then a subcategory of the $(\infty,1)-$category $PreL_\infty$ of all $\mathcal{PL}_\infty$-algebras.

Then as a $(2,1)$-categories, the *intrinsic cohomology* of 2-term $\mathcal{PL}_\infty$-algebras is the set of connected component of their hom-space

$$H(\mathfrak{A},\mathfrak{A}'):=\pi_0\,\mathrm{Hom}(\mathfrak{A},\mathfrak{A}').$$

5.2 Central extensions of pre-Lie algebras

Although it is natural to question if one can follow 2.5.2 to encode the second non-abelian pre-Lie algebra cohomology into an intrinsic cohomology. But this can not be down since for pre-Lie algebras, $\mathfrak{L}^1$ is not canonically isomorphic to $\mathrm{Hom}(\otimes^2 A,B)\oplus\mathrm{Hom}(A,C^1(B,B))$. One way to drop off the extra term $\mathrm{Hom}(B\otimes A,B)$ is to consider only the central extensions.

5.2.1 Recall the ***center*** $Z(A)$ of a pre-Lie algebra A is defined to be the set of elements commute with every elements of A, in other words, $Z(A)=\ker(L-R)$.

Let A be a pre-Lie algebra, an ***central extension*** is an pre-Lie algebra extension $\widehat{A}$ of A by B such that the image of B lies in the center of $\widehat{A}$. The full subcategory of central extensions of A by B is denoted by $\mathcal{E}xt_{cen}(A,B)$. Note that, the injectivity of $B\to\widehat{A}$ forces B to be a commutative algebra. If this is not the case, set $\mathcal{E}xt_{cen}(A,B)=\varnothing$.

Following 3.2.1, by choosing a splitting and identify B with its image, one can identify $\widehat{A}$ with some semiproduct $A\ltimes_{\omega,l,r}B$. Moreover, since B lies in the center of $\widehat{A}$, one has $l=r$. Then proposition 3.2.2 becomes

Proposition 5.2.2 *Let A be a pre-Lie algebras and B a commutative algebra, a pair (ω, l) of a bilinear map $\omega\colon \otimes^2 A \to B$ and a linear map $l\colon A \to \mathfrak{gl}(B)$ produces a central extension $A \ltimes_{\omega,l,l} B$ via Eq.* (3.6) *if and only if it satisfies the following equalities for all $x, y, z \in A, u, v \in B$:*

$$l_x(u \cdot v) = u \cdot l_x(v) \tag{5.2}$$

$$l_x l_y - l_{x \cdot y} = L_{\omega(x,y)} \tag{5.3}$$

$$\omega([x,y],z) - \omega(x, y \cdot z) + \omega(y, x \cdot z) = l_x \omega(y,z) - l_y \omega(x,z) - l_z(\omega_\wedge(x,y)). \tag{5.4}$$

Such kind of triples (ω, l, l) are called ***central* 2*-cocycles***, obviously they are 2-cocycles and one thus get a subset $Z^2_{cen}(A,B)$ of $Z^2_{preLie}(A,B)$. Moreover, it is closed under the equivalence of 2-cocycles and thus provides a quotient $H^2_{cen}(A,B)$, which is a subset of $H^2_{preLie}(A,B)$ and classifying $\mathcal{E}xt_{cen}(A,B)$.

Aside, by 3.3.2, the automorphisms of a central extension corresponding to the central 2-cocycle (ω, l, l) are described by $Z^1_{preLie}(A, B, (\omega, l, l))$.

5.2.3 The Eq. (5.2) means l belongs to the ***multiplier*** $M(B)$ of the commutative algebra B, that is the subspace of $\mathfrak{gl}(B)$ defined as

$$M(B) := \{l \in \mathfrak{gl}(B) | l(u \cdot v) = l(u) \cdot v, \forall u, v \in B\}.$$

One can see it contains the left multiplication L of B and is closed under the composite operation of linear maps. Therefore, $M(B)$ is a sub-associative algebra of $\mathfrak{gl}(B)$.

Moreover, $M(B)$ can be regarded as a 2-term $\mathcal{PL}_\infty$-algebra $\mathfrak{M}(B)$ as follows:

- the underlying complex is $\cdots \to 0 \to B \xrightarrow{L} M(B)$
- the multiplication is given by

$$f \cdot g := f \circ g, \quad f \cdot u = u \cdot f := f(u), \quad \forall f, g \in M(B), u \in B.$$

- the left symmetry law vanishes.

Now Eq. (5.2)–(5.4) is obviously equivalent to say (l, ω) is a morphism between the 2-term $\mathcal{PL}_\infty$-algebras $0 \to A$ and $\mathfrak{M}(B)$.

Given two morphisms (l', ω') and (l, ω), a homotopy between them is a chain homotopy $\Phi\colon l' \Rightarrow l$ satisfying Eq. (5.1). A chain homotopy Φ between l' and l is a graded linear map from $0 \to A$ to $\mathfrak{M}(B)$ of degree 1, which is equivalent to a linear map $\varphi\colon A \to B$ satifying

$$l' = l = L \circ \varphi,$$

which is Eq. (3.16), and also Eq. (3.17) by the commutativity of B. Then the condition Eq. (5.1) is equivalent to

$$\omega'(x,y) - \omega(x,y) = l_x\varphi(y) + r_y\varphi(x) + \varphi(x)\cdot\varphi(y) - \varphi(x\cdot y), \quad \forall x,y \in A.$$

Therefore, a homotopy between two morphisms (l', ω') and (l, ω) is precisely an equivalence of 2-cocycles.

Consequently, one has

Theorem 5.2.4 *There following groupoids are equivalent:*

$$\mathcal{E}xt_{cen}(A,B) \cong \mathrm{Hom}(A, \mathfrak{M}(B))$$

Consequently, central extensions of A by B are classified by the intrinsic cohomology $H(A, \mathfrak{M}(B))$.

5.2.5 Now one consider the split central extension $B \to A \oplus B \to A$. By 3.3.2, the automorphisms of it are corresponding to the 1-cocycles respect to the 2-cocycle $(0,0,0)$, which are linear maps $\varphi\colon A \to \mathrm{Ann}(B)$ satisfying $\varphi(x\cdot y) = 0$ for all $x, y \in A$. One can see those maps are precisely the homomorphisms between A and $\mathrm{Ann}(B)$.

Remark One can see $H(A, \mathfrak{M}(B))$ is natural in A, but not in B since $\mathfrak{M}$ is not a functor from commutative algebras to associative algebras. However, $H(A, \mathfrak{M}(B))$ is natural in $\mathfrak{M}(B)$. Then, by the knowledge from intrinsic cohomology, whenever there is a *fibration sequence* of 2-term $\mathcal{PL}_\infty$-algebras

$$\mathfrak{M}(B_1) \to \mathfrak{M}(B_2) \to \mathfrak{M}(B_3),$$

Then there is a long exact sequence of pointed sets

$$\begin{aligned} 0 \to H(A, \Omega\mathfrak{M}(B_1)) \to H(A, \Omega\mathfrak{M}(B_2)) \to H(A, \Omega\mathfrak{M}(B_3)) \\ \to H(A, \mathfrak{M}(B_1)) \to H(A, \mathfrak{M}(B_2)) \to H(A, \mathfrak{M}(B_3)). \end{aligned}$$

One can verify that for any commutative algebra B, the *looping* $\Omega\mathfrak{M}(B)$ of the 2-term $\mathcal{PL}_\infty$-algebra $\mathfrak{M}(B)$ is precisely the subalgebra $\mathrm{Ann}(B)$ of B and $H(A, \mathrm{Ann}(B))$ is a abelian group under the addition operation.

Appendix A glance of higher category theory

A.1 The notion of higher category theory and intrinsic cohomology

A.1.1 The original notion of category can be extended to involve higher morphisms, such as 2-*morphisms* between the original morphisms and 3-*morphisms* between 2-morphisms. A ***n*-category** is then an extended category involving *k-morphisms* for all $1 \leqslant k \leqslant n$, and an **∞-*category*** is an extended category involving k-morphisms for all $k \geqslant 1$. Those higher morphisms should satisfy suitable associative and unital composition laws. In the ***strict*** case, the composition laws for higher morphisms are the same for the 1-morphisms with the extra requirement that different ways of composites are compatible.

For instance, a (*strict*) 2-***category*** $\mathcal{C}$ consists of

- 0-morphisms, i.e. objects,
- 1-morphisms, i.e. the original morphisms, between objects, and their composites

$$\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \qquad \rightsquigarrow \qquad \cdot \xrightarrow{g\circ f} \cdot$$

- 2-morphisms between 1-morphisms, and the *vertical* and *horizontal composites*

$$\begin{tikzcd} \cdot \arrow[r, bend left=70] \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta"] & \cdot \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} \cdot \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & \cdot \end{tikzcd}$$

$$\begin{tikzcd} \cdot \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow\beta"] & \cdot \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow\alpha"] & \cdot \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} \cdot \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & \cdot \end{tikzcd}$$

and satisfies the following axioms:

1. the composites satisfy the associative law,
2. every object A admits an 1-morphism id_A called the identity of A, which is the identity under the composite operation of 1-morphisms,

3. every 1-morphism f admits a 2-morphism id_f called the identity of f, which is the identity under the vertical composite operation of 2-morphisms,
4. for every object A, the 2-morphism $\mathrm{id}_{\mathrm{id}_A}$ is the identity under the horizontal composite operation of 2-morphisms,
5. the vertical and horizontal composites satisfy the *interchange law*: for all quadruples $(\alpha, \alpha', \beta, \beta')$ of 2-morphisms of the form

$$\begin{tikzcd} \cdot \arrow[r, bend left=70] \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta"] & \cdot \arrow[r, bend left=70] \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha'"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta'"] & \cdot \end{tikzcd}$$

the following equality holds.

$$(\beta \circ \alpha) * (\beta' \circ \alpha') = (\beta * \beta') \circ (\alpha * \alpha').$$

However, the composites as well as composition laws can be replaced by some weaker ones, then one gets various of weak higher categories. The technical definitions can be found in [21]. I omit it since it is not necessary in this paper.

A.1.2 In original category theory, there are functors between categories and natural transformations between functors. Under suitable size hypothesis, all small categories together with functors between them and natural transformations between those functors form a large 2-category **Cat**.

In higher category theory, functors should also work on all kind of higher morphisms. Therefore there are not only functors between categories and natural transformations between functors but also 2-transformations between natural transformations, 3-transformations between 2-transformations etc. Under suitable size hypothesis, those data for small n-categories form a large $(n+1)$-category $n\,\mathbf{Cat}$. Here n can be taken as ∞ by setting $\infty + 1 = \infty$.

A.1.3 In original category theory, an *isomorphism* f is an *invertible* morphism, that means it has an *inverse* g, which is a morphism satisfying $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$. In higher category theory, the same notion can be defined for all higher morphisms. So it makes sense to say two morphisms are isomorphic.

But then a weak version of invertibility arises since the equalities $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$ can be weakened to be isomorphisms. Then one can define various of weak versions of invertibility by similar consideration. Morphisms satisfying any kind of invertibility are called *equivalences*.

A (n,r)***-category*** is a n-category whose every k-morphism is invertible for all $k > r$. All small (n,r)-categories form a sub$(n+1)$-category of $n\,\mathbf{Cat}$, denoted by $(n,r)\,\mathbf{Cat}$. Specially, $(n,0)$-categories can be viewed as the generalization of groupoids and are called ***n-groupoids***. The $(n+1)$-category of all small n-groupoids is denoted by $n\,\mathbf{Grpd}$.

Remark The notion of (n,r)-categories can be weakened by replacing invertibility with weak version, which is not necessary throughout this paper. Under this convention, the only weak version of invertibility in a $(n,1)$-category is the one mentioned above.

A.1.4 For any pair of objects (A,B) in a higher category $\mathcal{C}$, there exists a natural higher category structure on the set $\mathrm{Hom}_{\mathcal{C}}(A,B)$ of 1-morphisms between them: the objects are 1-morphisms of $\mathcal{C}$ between A and B, the 1-morphisms are 2-morphisms of $\mathcal{C}$ between those 1-morphisms etc. This higher category $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is called the ***hom-space*** of A and B.

In a $(n,1)$-category $\mathcal{C}$, a hom-space $\mathrm{Hom}(A,B)$ is a n-groupoid. Like groupoids, one can taking the set of connected components of $\mathrm{Hom}(A,B)$. This set is called the ***intrinsic cohomology*** of A with values in B, and denoted by $H(A,B)$. In other words, the intrinsic cohomology is the composite of the following functors:

$$\mathcal{C} \times \mathcal{C} \xrightarrow{\mathrm{Hom}(-,-)} n\,\mathbf{Grpd} \xrightarrow{\pi_0} \mathbf{Set}\,.$$

If $\mathcal{C}$ satisfies good properties, the intrinsic cohomology will behave well.

A.2 Intrinsic cohomology in $(2,1)$-categorical context

In this section, I recall some facts about intrinsic cohomology and prove them in $(2,1)$-categorical context. Those facts hold in general higher categories, but it is not necessary for this paper.

A.2.1 Recall that in original category theory, a square diagram

$$\begin{tikzcd} \cdot \arrow[r, "f'"] \arrow[d, "g'"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "f"] & \cdot \end{tikzcd}$$

is said to *commute* if $f \circ g' = g \circ f'$. In higher category theory, this diagram is said to ***commute up to homotopy*** if $f \circ g'$ and $g \circ f'$ are different by an equivalence. Note that, in a $(2,1)$-category, this means $f \circ g'$ and $g \circ f'$ are isomorphic.

A.2.2 Recall that in an original category, a *cartesian diagram* is a commutative square diagram

$$\begin{tikzcd} A \times_C B \arrow[r, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

such that for any commutative square diagram

$$\begin{tikzcd} D \arrow[r, "f''"] \arrow[d, "g''"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

there exists a unique morphism u such that $f'' = f' \circ u$ and $g'' = g' \circ u$. If this is the case, f' (resp. g') is said to be the *pullback* of f (resp. g) along g (resp. f) and $A \times_C B$ is called the *fibred product* of f and g. A category is said to be *having pullbacks* if for every pair of morphisms $\cdot \to \cdot \leftarrow \cdot$, the pullbacks exist.

Note that a commutative square diagram of the form

$$\begin{tikzcd} D \arrow[r, "f''"] \arrow[d, "g''"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

is nothing but an element of the fibred product $\mathrm{Hom}(D,A) \times_{\mathrm{Hom}(D,C)} \mathrm{Hom}(D,B)$ of hom-sets. Then the above universal property can be encoded into the following natural isomorphism in **Set**

$$\mathrm{Hom}(D, A \times_C B) \cong \mathrm{Hom}(D,A) \times_{\mathrm{Hom}(D,C)} \mathrm{Hom}(D,B).$$

Those notions can be generalized into higher categories and can be weakened by replacing commutative diagrams with diagrams commuting up to homotopy.

A.2.3 Recall that, in the original category theory, the *comma category* $(\mathscr{F} \downarrow \mathscr{G})$ of two functors $\mathscr{F}\colon \mathcal{A} \to \mathcal{C}$ and $\mathscr{G}\colon \mathcal{B} \to \mathcal{C}$ is the category in which

- objects are triples (A, f, B) of an object A in $\mathcal{A}$, an object B in $\mathcal{B}$ and a morphism $f\colon \mathscr{F}(A) \longrightarrow \mathscr{G}(B)$ in $\mathcal{C}$,
- morphisms from (A, f, B) to (A', f', B') are pairs (g, h) of a morphism $g\colon A \longrightarrow A'$ in $\mathcal{A}$ and a morphism $h\colon B \longrightarrow B'$ in $\mathcal{B}$ making the following diagram commute

$$\begin{tikzcd} \mathscr{F}(A) \arrow[r, "f"] \arrow[d, "\mathscr{F}(g)"'] & \mathscr{G}(B) \arrow[d, "\mathscr{G}(h)"] \\ \mathscr{F}(A') \arrow[r, "f'"] & \mathscr{G}(B') \end{tikzcd}$$

- composite of (g, h) and (g', h') is $(g \circ g', h \circ h')$,
- identity of (A, f, B) is $(\mathrm{id}_A, \mathrm{id}_B)$.

Consider a pair of morphisms $A \xrightarrow{f} C \xleftarrow{g} B$ in a $(2,1)$-category, let f_* and g_* denote the induced functors $\mathrm{Hom}(D, A) \xrightarrow{f_*} \mathrm{Hom}(D, C) \xleftarrow{g_*} \mathrm{Hom}(D, B)$. One can see that $(f_* \downarrow g_*)$ is precisely the category of homotoy commutative square diagrams of the form

$$\begin{tikzcd} D \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ A \arrow[r, "f"] \arrow[ur, phantom, "\nearrow\!\!\!\nearrow"] & C \end{tikzcd}$$

Thus the ***homotopy fibred product*** should be defined as the object $A \times_C^h B$ such that it induces a natural equivalence of groupoids

$$\mathrm{Hom}(D, A \times_C^h B) \cong (f_* \downarrow g_*).$$

As a special case, the ***comma fibred product*** is defined to be the object $A \times_C^h B$ inducing a natural isomorphism instead of merely equivalence. To simplify the discussion, I only consider the comma fibred products and call them homotopy fibred products.

A.2.4 Now, one can write down the explicit definition of *homotopy fibred products*. A ***homotopy cartesian diagram*** is a square diagram which commutes up to homotopy

$$\begin{tikzcd} A \times_C^h B \arrow[r, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] \arrow[ur, phantom, "\alpha\nearrow\!\!\!\nearrow"] & C \end{tikzcd}$$

together with the equivalence $\alpha\colon f \circ g' \Rightarrow g \circ f'$ satisfying

- 1-*universal property*: for any square diagram which commutes up to homotopy

$$\begin{tikzcd} D \arrow[r, "f''"] \arrow[d, "g''"'] \arrow[dr, phantom, "\alpha' \Nearrow"] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

  together with the equivalence $\alpha'\colon f\circ g'' \Rightarrow g\circ f''$, there exists a unique morphism u such that $f'' = f'\circ u$, $g'' = g'\circ u$ and $\alpha' = \alpha * u$. Here $\alpha * u$ denotes the horizontal composite of α and id_u.
- 2-*universal property*: for any two morphisms $u, v\colon D \to A\times^h_C B$ and 2-morphisms $\phi\colon f'\circ u \Rightarrow f'\circ v$ and $\psi\colon g'\circ u \Rightarrow g'\circ v$ satisfying

$$(g*\phi)\circ(\alpha*u) = (\alpha*v)\circ(f*\psi),$$

  then there exists a unique 2-morphism $\theta\colon u \Rightarrow v$ such that $\phi = f' * \theta, \psi = g' * \theta$.

If this is the case, f' (resp. g') is said to be the ***homotopy pullback*** of f (resp. g) along g (resp. f) and $A\times^h_C B$ is called the ***homotopy fibred product*** of f and g. A higher category is said to be *having homotopy pullbacks* if for every pair of morphisms $\cdot \to \cdot \leftarrow \cdot$, the homotopy pullbacks exist.

Lemma A.2.5 (Pasting lemma) *In a higher category having homotopy pullbacks, consider the following diagram of morphisms.*

$$\begin{tikzcd} \cdot \arrow[r, "g'"] \arrow[d, "h''"'] & \cdot \arrow[r, "f'"] \arrow[d, "h'"'] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "g"] & \cdot \arrow[r, "f"] & \cdot \end{tikzcd}$$

If h' is the homotopy pullback of h along f and h'' is the homotopy pullback of h' along g, then h'' is the homotopy pullback of h along $f\circ g$.

Proof. If there exist equivalences $\alpha\colon f\circ h' \Rightarrow h\circ f'$ and $\beta\colon g\circ h'' \Rightarrow h'\circ g'$ making the above two squares become homotopy cartesian diagrams, then one can verify that $(\alpha * g')\circ(f*\beta)\colon f\circ g\circ h'' \Rightarrow h\circ f'\circ g'$ is an equivalence making the outer rectangle become a homotopy cartesian diagram. $\square$

A.2.6 Recall that in an original category, the *initial object* (resp. *terminal object*) is the object A such that $\mathrm{Hom}(A,X)$ (resp. $\mathrm{Hom}(X,A)$) is a singleton for all objects X. If an initial object is also a terminal object, then it is called the ***zero object*** and is usually denoted by 0. A category with zero object is called a ***pointed category***. In a pointed category having pullbacks, the *kernel* $\ker f \to A$ of a morphism $f\colon A \to B$ is the pullback of $0 \to B$ along f. The kernel of a kernel $\ker f \to A$ is then a pullback of $0 \to B$ along $0 \to B$, which is $0 \to 0$.

Now consider a pointed higher category having homotopy pullbacks. The ***homotopy kernel*** $\ker f \to A$ of a morphism $f\colon A \to B$ is defined to be the homotopy pullback of $0 \to B$ along f. The homotopy kernel of a homotopy kernel $\ker f \to A$ is then a pullback of $0 \to B$ along $0 \to B$. However, it is in general not $0 \to 0$.

Remark 1 For convenience, one can use the term *kernel* to infer either the object $\ker f$ or the morphism $\ker f \to A$ if there is no ambiguity.

Remark 2 Any category $\mathcal{C}$ having terminal object 0 admits a pointed category $\mathcal{C}^{0/}$, the *category of points of objects* of $\mathcal{C}$, whose objects are morphisms $0 \to X$ in $\mathcal{C}$ and a morphism from $0 \to A$ to $0 \to B$ is a morphism $f\colon A \to B$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ A \arrow[rr, "f"] & & B \end{tikzcd}$$

An object in this category is called a ***pointed object*** in $\mathcal{C}$. Let $\cdot \xrightarrow{f} \cdot \xleftarrow{g} \cdot$ be morphisms between pointed objects, then there already exists a commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "f"] & \cdot \end{tikzcd}$$

Therefore if X is the homotoy fibred product of f and g in $\mathcal{C}$, then there will be a unique morphism $0 \to X$ making it become the homotoy fibred product in $\mathcal{C}^{0/}$.

Example The category **Set** has no nontrivial 2-morphisms, thus homotopy pullbacks in **Set** are precisely the usual pullbacks. As the terminal object in **Set** is the singleton, a ***pointed set*** is just a set equipped with a specific element. The kernel of a map f between pointed sets (A,a) and (B,b) is the inverse image $f^{-1}(b)$ of that specific element b. When $\ker f = A$, this map is said to be a ***zero map***.

Example The 2-category **Grpd** of groupoids has a terminal object 0, which is the category having only one object and one morphism. A pointed object in **Grpd** is a groupoid $\mathcal{G}$ together with a functor $x\colon 0 \to \mathcal{G}$, which can be viewed as a specific object in $\mathcal{G}$. Then the homotopy fibred product of x and x is the comma category $(x \downarrow x)$, which can be identified with the group of automorphisms of x in $\mathcal{G}$.

A.2.7 In a higher category having pullbacks and terminal object, the ***looping*** ΩX of a pointed object $0 \to X$ is the homotopy fibred product of $0 \to X$ and itself. The previous example shows that the looping of a groupoid is a group, which is not an accident.

Indeed, let $0 \to X$ be a pointed object in a $(2,1)$-category $\mathcal{C}$, then for any object Y in $\mathcal{C}$, the universal property of homotopy fibred product provides a natural bijection from the set $\mathrm{Hom}(Y, \Omega X)$ of 1-morphisms to the set of 2-automorphisms of $Y \to 0 \to X$, which is naturally a group. Thus the bijection gives a group structure on $\mathrm{Hom}(Y, \Omega X)$. Moreover, this group structure is natural in the sense that any morphism $Y \to Y'$ in $\mathcal{C}$ (resp. $X \to X'$ in $0/\mathcal{C}$) induces a group homomorphism $\mathrm{Hom}(Y', \Omega X) \to \mathrm{Hom}(Y, \Omega X)$ (resp. $\mathrm{Hom}(Y, \Omega X) \to \mathrm{Hom}(Y, \Omega X')$) rather than a map.

Recall that an object G in a category $\mathcal{C}$ is called a ***group object*** if for any objects X in $\mathcal{C}$, there exists a group structure on $\mathrm{Hom}(X, G)$ and for any morphism $f\colon X \to Y$, the map $\mathrm{Hom}(Y, G) \to \mathrm{Hom}(X, G)$ is a group homomorphism. In other words, $\mathrm{Hom}(-, G)$ induces a functor from $\mathcal{C}$ to the category **Grp** of groups. A morphism between group objects is a morphism $f\colon G \to G'$ in $\mathcal{C}$ such that it induces a natural transformation $\mathrm{Hom}(-, G) \Rightarrow \mathrm{Hom}(-, G')$ of functors from $\mathcal{C}$ to **Grp**. Then one obtains a functor

$$\Omega\colon \mathcal{C}^{0/} \to \mathbf{Grp}(\mathcal{C}),$$

where $\mathbf{Grp}(\mathcal{C})$ denotes the full subcategory of $\mathcal{C}$ consisting of all group objects. Any group object admits a special point corresponding to the unity of the group $\mathrm{Hom}(0, G)$, therefore, one can composite Ω and gets functors $\Omega^2, \Omega^3, \cdots$. In higher category theory, the similar result can be generalized by using the notion of $\infty-$*group objects*.

A.2.8 A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be a ***fibration sequence*** if f is the homotopy kernel of g. If this is the case, by lemma A.2.5, the kernel of f will be the looping of C. Thus there exists a long fibration sequence

$$\cdots \to \Omega^2 C \to \Omega A \xrightarrow{\Omega f} \Omega B \xrightarrow{\Omega g} \Omega C \to A \xrightarrow{f} B \xrightarrow{g} C.$$

Since the homotopy pullbacks are preserved by the covariant hom-functors, one obtains the long fibration sequence of pointed groupoids for every object X:

$$\cdots \to \mathrm{Hom}(X,\Omega B) \to \mathrm{Hom}(X,\Omega C) \to \mathrm{Hom}(X,A) \to \mathrm{Hom}(X,B) \to \mathrm{Hom}(X,C).$$

Then one can use the following lemma A.2.9 to obtain a long exact sequence

$$\cdots \to H(X,\Omega B) \to H(X,\Omega C) \to H(X,A) \to H(X,B) \to H(X,C).$$

Note that here this is a long exact sequence of pointed sets in the sense that for any adjacent two maps $\cdot \xrightarrow{u} \cdot \xrightarrow{v} \cdot$, one has $\mathrm{im}\, u = \ker v$. Aside, in the above sequence, only the first three terms $H(X,A)$, $H(X,B)$ and $H(X,C)$ are not groups, the remaining sequence is a long exact sequence of groups.

For this reason, one can define the ***graded intrinsic cohomology*** by

$$H^{-n}(X,A) := H(X,\Omega^n A).$$

Lemma A.2.9 *The functor π_0 maps fibration sequences of pointed groupoids to exact sequence of pointed sets.*

Proof. Let $\mathcal{A} \xrightarrow{\mathscr{F}} \mathcal{B} \xrightarrow{\mathscr{G}} \mathcal{C}$ be a fibration sequences of pointed groupoids. Then one can identify $\mathcal{A}$ as the comma category $(\mathscr{G} \downarrow 0)$. Then $\mathscr{F}(\mathcal{A})$ is precisely the subcategory of $\mathcal{B}$ consisting of all those objects whose images in $\mathcal{C}$ are isomorphic to the specific object of $\mathcal{C}$. Thus $\mathrm{im}(\pi_0\mathscr{F}) = \ker(\pi_0\mathscr{G})$. □

References

[1] Alekseevsky D., Michor P. W. and Ruppert W. Extensions of Lie algebras. ArXiv e-prints, May 2000. arXiv: `math/0005042 [math.DG]`.

[2] Alekseevsky D., Michor P. W. and Ruppert W. A. F. Extensions of super Lie algebras. J. Lie Theory, 2005, 15(1): 125–134.

[3] Bourbaki N. Lie groups and Lie algebras. Chapters 1–3. Springer-Verlag, Berlin, 1998: 155–164.

[4] Boyom M. N. The cohomology of Koszul-Vinberg algebras. Pacific J. Math. 2006, 225(1): 119–153.

[5] Brahic O. Extensions of Lie brackets. J. Geom. Phys. 2010, 60(2): 352–374.

[6] Brown K. S. Abstract homotopy theory and generalized sheaf cohomology. Trans. Amer. Math. Soc. 1973, 186: 419–458.

[7] Burde D. Left-symmetric algebras, or pre-Lie algebras in geometry and physics. Cent. Eur. J. Math. 2006, 4(3): 323–357.

[8] Cartan H., Eilenberg S. Homological algebra. Princeton University Press, Princeton, NJ, 1999: xvi+390.

[9] Cayley A. On the Theory of the Analytical Forms called Trees. In: The collected mathematical papers of Arthur Cayley. Volume 3. Cambridge University Press, Cambridge, 2009: 242–246.

[10] Chapoton F., Livernet M. Pre-Lie algebras and the rooted trees operad. Internat. Math. Res. Notices, 2001, (8): 395–408.

[11] Eilenberg S., MacLane S. Cohomology theory in abstract groups. II. Group extensions with a non-Abelian kernel. Ann. of Math. (2), 1947, 48: 326–341.

[12] Fialowski A., Penkava M. Extensions of (super) Lie algebras. Commun. Contemp. Math. 2009, 11(5): 709–737.

[13] Frégier Y. Non-abelian cohomology of extensions of Lie algebras as Deligne groupoid. J. Algebra, 2014, 398: 243–257.

[14] Gerstenhaber M. The cohomology structure of an associative ring. Ann. of Math. (2), 1963, 78: 267–288.

[15] Goldman W. M., Millson J. J. The deformation theory of representations of fundamental groups of compact Kähler manifolds. Inst. Hautes Études Sci. Publ. Math. 1988, (67): 43–96.

[16] Hochschild G. Cohomology classes of finite type and finite dimensional kernels for Lie algebras. Amer. J. Math. 1954, 76: 763–778.

[17] Inassaridze N., Khmaladze E. and Ladra M. Non-abelian cohomology and extensions of Lie algebras. J. Lie Theory, 2008, 18(2): 413–432.

[18] Kim H. Complete left-invariant affine structures on nilpotent Lie groups. J. Differential Geom. 1986, 24(3): 373–394.

[19] Kim H. Extensions of left-symmetric algebras. Algebras Groups Geom. 1987, 4(1): 73–117.

[20] Koszul J.-L. Domaines bornés homogènes et orbites de groupes de transformations affines. Bull. Soc. Math. France, 1961, 89: 515–533.

[21] Lurie J. Higher topos theory. Princeton University Press, Princeton, NJ, 2009: xviii+925.

[22] Mac Lane S. Categories for the working mathematician. Springer-Verlag, New York, 1998: xii+314.

[23] Manetti M. Lectures on deformations of complex manifolds (deformations from differential graded viewpoint). Rend. Mat. Appl. (7), 2004, 24(1): 1–183.

[24] Matsushima Y. Affine structures on complex manifolds. Osaka J. Math. 1968, 5: 215–222.

[25] Nijenhuis A., Richardson Jr. R. W. Cohomology and deformations of algebraic structures. Bull. Amer. Math. Soc. 1964, 70: 406–411.

[26] Nikolaus T., Schreiber U. and Stevenson D. Principal infinity-bundles - General theory. ArXiv e-prints, July 2012. arXiv: `1207.0248 [math.AT]`.

[27] Sheng Y. Categorification of Pre-Lie Algebras and Solutions of 2-graded Classical Yang-Baxter Equations. ArXiv e-prints, Mar. 2014. arXiv: `1403.2144 [math-ph]`.

[28] Vinberg E. The theory of convex homogeneous cones. Trans. Mosc. Math. Soc. 1963, 12: 340–403.

[29] Weibel C. A. An introduction to homological algebra. Cambridge University Press, Cambridge, 1994: xiv+450.

[30] *n*Lab. nonabelian Lie algebra cohomology. `http://ncatlab.org/nlab/show/nonabelian+Lie+algebra+cohomology`, retrieved on 02/19/2015.

Index

致谢

衷心感谢我的导师白承铭教授。感谢他不辞辛劳不厌其烦地为我修改论文。感谢他三年以来对我在学习和研究上的悉心指导和耐心帮助。同时也感谢蒋剑剑学长、张汉斌同学和朱亦艺同学的建议和帮助，感谢一直以来关心和帮助过我的人！

个人简历

高煦，男，1989 年 8 月出生于四川省雅安市，2012 年 7 月毕业于南开大学数学科学学院，获理学学士学位。2012 年 9 月进入南开大学陈省身数学研究所攻读基础数学专业研究生，师从白承铭教授，主要从事李理论与数学物理方面的研究。