Research Statement

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My research lies at the intersection of **algebraic geometry**, **representation theory**, and **mathematical physics**. My current program centers on **vertex operator algebras (VOAs)**. By replacing traditional complex-analytic methods with algebro-geometric framework, I aim to provide a universal foundation for VOA theory over general base rings.

In addition, I possess a strong background in **arithmetic geometry** and **algebraic topology**. My doctoral work established combinatorial results for **Bruhat-Tits buildings** for p-adic groups, and I actively research tensor triangular geometry in stable homotopy categories. My long-term vision is to bridge these fields by establishing an **arithmetic conformal field theory**.

1 Algebro-geometric Representation Theory of VOAs

The central theme of my recent work is an algebro-geometric approach to VOA representation theory using the Damiolini–Gibney–Tarasca (DGT) framework. This approach allows for the study of conformal blocks over stable curves and general rings, offering insights inaccessible to purely algebraic or analytic methods.

1.1 Twisted Conformal Blocks on Orbicurves

A major difficulty in the algebraic theory of VOAs has been the treatment of twisted modules, which are central to orbifold conformal field theory. To address the complexity inherent in the algebraic study of twisted modules, in collaboration with Jianqi Liu and Yiyi Zhu, we utilize algebraic geometry to treat twisted structures as intrinsic geometric properties of orbicurves. This provides a global DGT-type framework to systematically extend VOA representation theory to the orbifold setting.

- **Key Result (Fusion Rules):** In [1, 2], we established the propagation of vacua and fusion rules for twisted correlation functions on totally ramified orbicurves. We demonstrated that these blocks are controlled by finite-dimensional *twisted Zhu algebras*.
- **Geometric Reduction:** By utilizing *Hurwitz moduli stacks* and formal geometry, we reduced the study of general orbicurves to the totally ramified case. This provides a global pathway toward a comprehensive theory of twisted conformal blocks.

1.2 Algebro-geometric Approach to Vertex Tensor Categories

The construction of vertex tensor categories is mysterious and has long been a difficult conjecture until Huang-Lepowsky-Zhang provided one using complex analytic methods. It is also expected that the conjectured factorization structure of conformal blocks should be the geometric manifestation of the vertex tensor structure. Such a conjecture has been verified in the rational case by DGT, but the non-rational case present significant challenges due to dimension jumps at the boundary of moduli spaces. I am addressing these challenges in collaboration with Angela Gibney and Daniel Krashen.

- **The Strong Identity Condition:** An advancement since DGT is by Damiolini–Gibney–Krashen, who introduced the *strong identity condition* for VOAs, which ensures that the conformal blocks admit well-behaved *smoothings*. In a recent work, we have identified that this condition has pure representation-theoretic interpretations.
- **Logarithmic Geometry & Non-rational VOAs:** When a VOA fails the strong identity condition, conformal blocks can have dimension jump at the boundary of the moduli space. To resolve this, we are utilizing *logarithmic geometry* to modify the factorization structure, ensuring flatness across the boundary divisor. This work aims to provide a tensor category construction that holds for general non-rational VOAs.

1.3 Universal Enveloping Algebras

I am also investigating VOA theory from a topological algebra perspective. In a joint work with Yiyi Zhu [4], we investigated the universal enveloping algebras of permutation orbifolds, proving an isomorphism between the twisted and untwisted universal enveloping algebras $\mathcal{U}_\sigma(V^{\otimes N}) \cong \mathcal{U}(V)$. This approach allows us to generalize VOA results to the broader category of **almost-canonically seminormed rings**, offering a new algebraic foundation for the field.

2 Tensor Triangular Geometry and Topology

I maintain an active research interest in applying tensor triangular geometry to stable homotopy theory and representation theory.

- **Hopf Algebroids:** In joint work with Ang Li [3], we computed the stable Picard group of finite Adams Hopf algebroids. We developed a theory of stable categories for comodules over Hopf algebroids, with applications to the $\mathbb{R}$-motivic Steenrod subalgebra $\mathcal{A}^{\mathbb{R}}(1)$.
- **Stratification:** In [5], I investigated the stratification of Balmer spectra for finite groupoids, reducing the problem to the classification of localizing subcategories for finite groups.

3 Arithmetic Geometry: Bruhat-Tits Buildings

My doctoral thesis focused on the geometry of **Bruhat-Tits buildings** and its application to p-adic representations. This work can serve as a geometric foundation for my future explorations into arithmetic conformal field theory.

- **Simplicial Distance:** The simplicial structure of a building encodes deep information about the lattices in a p-adic representation. In [6], I introduced an explicit characterization of the **simplicial distance** in buildings of split classical types (A_n, B_n, C_n, D_n). I proved that two vertices are at distance r if and only if they admit representative lattices L, L' satisfying specific inclusion properties (e.g., $L \supseteq L' \supseteq \varpi^r L$ for type A_n).
- **Volume Growth:** I analyzed the asymptotic growth of simplicial balls (tangent cones) in these buildings. Using the theory of concave functions on root systems, I established precise volume formulas $SV(r) \sim C \cdot r^{\epsilon(n)} q^{\pi(n) r}$, relating the growth rate to the invariants of the underlying algebraic group.

4 Future Direction: Arithmetic Conformal Field Theory.

My long-term research goal is to synthesize my expertise in VOA geometry and p-adic geometry. I plan to construct an **Arithmetic Conformal Field Theory** by defining p-adic conformal blocks.

In this framework, the **Bruhat-Tits building** will serve as the non-Archimedean analogue of the Teichmüller space, providing the geometric backdrop for p-adic vertex operators. This synthesis promises to open new avenues in both arithmetic geometry and the theory of mathematical physics.

References

[1] X. Gao, J. Liu, Y. Zhu, *Twisted restricted conformal blocks of vertex operator algebras I: g-twisted correlation functions and fusion rules*, **J. Algebra** 675 (2025), 59–132.

[2] X. Gao, J. Liu, Y. Zhu, *Twisted restricted conformal blocks of vertex operator algebras II: twisted restricted conformal blocks on totally ramified orbicurves*, arXiv:2403.00545.

[3] X. Gao, A. Li, *The stable Picard group of finite Adams Hopf algebroids with an application to the $\mathbb{R}$-motivic Steenrod subalgebra $\mathcal{A}^{\mathbb{R}}(1)$*, **J. Pure Appl. Algebra** 228 (2024), no. 11.

[4] X. Gao, Y. Zhu, *Twisted and untwisted universal enveloping algebras for permutation orbifolds*, (Submitted 2025).

[5] X. Gao, *Stratification for finite groupoids*, (Preprint).

[6] X. Gao, *Simplicial distance in Bruhat-Tits buildings of split classical type*, **Ph.D. Dissertation**, UCSC (2023).