Research Proposal: An Algebro-geometric Approach to Vertex Operator Algebra Representation Theory

Xu Gao (gausyu@tongji.edu.cn)

Vertex Operator Algebras (VOAs) are the core algebraic structures characterizing the infinite-dimensional symmetries in two-dimensional conformal field theory (CFT). They play a foundational role in Lie theory, modular forms, and mathematical physics. VOAs are crucial in areas such as the Moonshine conjecture[1–3], the Geometric Langlands program[4–6], and the BRST quantization[7].

This project aims to deepen the understanding of representation theory of VOAs through **algebraic geometry**. In particular, it seeks to translate results dependent on complex analytic methods into an algebro-geometric framework, thereby expanding their applications in *arithmetic geometry* and *p-adic physics*.

1 Background and Significance

In *algebraic* perspective, a vertex operator algebra is an algebraic structure using infinitely many operations to characterize operator product expansions in conformal field theory. In *geometric* perspective, these operations correspond to the local expansion and sewing rules of field operators on Riemann surfaces. Revolving around this dual characteristic, VOA theory has evolved along two interconnected paths: namely the **algebraic theory** (e.g., Borcherds' axioms and the proof of the Moonshine conjecture[1,2], the construction of important examples by Frenkel–Lepowsky–Meurman[3], Zhu's modular invariance[8], and the structure theory and representation theory by Dong et al.[9,10]) and the **geometric theory** (e.g., Segal's axioms[11], the conformal block theory of affine Lie algebras by Tsuchiya–Ueno–Yamada[12], and the vertex tensor category theory by Huang–Lepowsky–Zhang[13]). The former is purely formal and self-contained but lacks geometric intuition, making global problems like tensor products and deformation classification difficult to handle. The latter is intuitive but relies on complex analysis, making it difficult to generalize to arithmetic settings.

Recently, the community has sought to unify these approaches. The chiral algebra theory of Beilinson–Drinfeld[4] and the systematic exposition of conformal block theory by Frenkel–Ben-Zvi[5] provided a framework independent of complex analysis, serving as a cornerstone for the geometric Langlands program[6]. Building on this paradigm, this project focuses on the **algebro-geometric approach to conformal blocks**, specifically targeting *orbifold theory*, *tensor structures*, and *arithmetic generalizations*.

2 Research Plan

This project employs the methodology of "**Algebraic Geometrization**," unfolding in two mutually supporting directions: geometric generalization to orbifolds and the algebraic construction of vertex tensor categories. Furthermore, it explores applications in arithmetic geometry.

2.1 Twisted Conformal Blocks and Factorization on Orbifold Curves

Orbifold curves provide a natural geometric model for CFTs with symmetries. Algebraically, *orbifold CFT* corresponds to *twisted representations* of VOAs, for which Dong et al. laid the foundation[14–16]. Geometrically, while Frenkel–Szczesny introduced *twisted conformal blocks*[17] and Hong–Kumar proved factorization for affine Lie algebras[18], a complete algebro-geometric theory for **general VOAs** is lacking. *We propose to extend the theory of Damiolini–Gibney–Tarasca[19] from stable curves to orbifold curves.*

Proposed Methodology:

(1) **Geometric Framework**: Utilizing the fact that stable orbifold curves can be presented as Galois covers of stable curves, we will employ *formal geometry* and *descent theory* to reduce twisted conformal blocks to G-invariant subspaces of conformal blocks on the covering curves. The objective is to construct logarithmic $\mathcal{D}$-modules on the moduli stack of stable orbifold curves $\overline{\mathcal{M}}_{g,G,\xi}$.

(2) **Algebraic Reduction**: We will demonstrate that twisted conformal blocks are controlled by their *restricted conformal blocks*, thereby reducing infinite-dimensional problems to the representation theory of finite-dimensional *twisted Zhu algebras*.

(3) **Proof of Factorization**: Following the strategy of Damiolini–Gibney–Krashen[20], we aim to reduce the factorization theorem to the Wedderburn–Artin decomposition of twisted Zhu algebras via a careful analysis of the formal deformation of nodal orbifold curves.

2.2 Algebro-geometric Approach to Vertex Tensor Categories

The vertex tensor category theory of Huang–Lepowsky–Zhang[13] endows VOA representations with rich tensor structures. However, this construction inherently relies on *complex analytic sewing. This subproject aims to use algebro-geometric conformal block theory to provide an **intrinsic, algebro-geometric approach** to these tensor structures.*

Proposed Methodology:

(1) **The Strong Unital Case**: Boundary degenerations of the moduli stack $\overline{\mathcal{M}}_{g,n}$ encode the associativity and braiding of tensor products[21]. Guaranteed by the *Factorization and Smoothing Theorem* of DGK[20] and the *flatness* of conformal block sheaves, we will construct and verify these structures within a purely algebro-geometric framework.

(2) **Highly Non-semisimple Case**: To address the *"dimension jump"* challenge outside the strong unital case[22], we will introduce *logarithmic algebraic geometry* to analyze the boundary behavior of $\overline{\mathcal{M}}_{g,n}$, establishing a modified factorization theory.

(3) **General Base Rings**: Combining our results with the algebro-geometric construction of little disk operads by Dupont–Panzer–Pym[23], we will generalize these constructions to general base rings (e.g., $\mathbb{Z}$), achieving a true arithmetic generalization.

2.3 Future Direction: Arithmetic Conformal Field Theory

Leveraging the base-change compatibility of our algebraic framework, *we will explore VOA theory in the context of **arithmetic and non-Archimedean geometry**.* The orbifold theory established in Section 2.1 will serve as a geometric analog for arithmetic ramification.

Proposed Methodology:

(1) **Integral Models and Modulo Reduction**: We will construct integral models of conformal block sheaves and study their *flatness* under reduction modulo p. We plan to use twisted logarithmic $\mathcal{D}$-modules to replace Virasoro uniformization, overcoming obstacles in positive characteristic.

(2) **Arithmetic Fundamental Groups**: Grothendieck's *arithmetic fundamental group* relates closely to automorphisms of the tower of moduli stacks[24]. We will investigate the link between these groups and VOA representations via the geometry of arithmetic orbifold curves, aiming to explain number-theoretic phenomena such as the congruence property in CFT[25,26].

(3) p**-adic Conformal Field Theory**: Drawing on the applicant's background in p-adic geometry[27], we will construct p-adic conformal blocks and utilize *Bruhat-Tits buildings* to explore the realization of CFT in non-Archimedean settings.

3 Key Innovations

- **Methodology**: Eliminates reliance on complex analytic sewing. By reformulating conformal blocks using *moduli stacks and formal geometry*, we allow relevant structures to be treated uniformly over general base rings.
- **Subject Matter**: This is the first attempt to establish a factorization theory for twisted conformal blocks on *orbifold curves* within an algebro-geometric framework, and to use *logarithmic geometry* to resolve dimension jumps in non-semisimple VOAs.
- **Vision**: Introduces perspectives from arithmetic and p-adic geometry, offering new geometric models to understand the arithmetic origins of VOA modular properties.

4 Research Foundation

The applicant has received systematic training in algebra, geometry, and number theory. He mastered core VOA theory under Prof. **Chongying Dong** and possesses an interdisciplinary background in p-adic geometry[27] and homological algebra[28] under Prof. **Junecue Suh**.

Preliminary Results: The applicant has published multiple papers in journals such as *J. Algebra*[28,29]. His previous work on totally ramified orbicurves[30] demonstrates his expertise in the geometric aspects of VOA theory. Furthermore, his ongoing collaboration with Prof. **Angela Gibney** and Prof. **Daniel Krashen** provides a solid technical foundation for the proposed algebro-geometric construction of twisted conformal blocks and tensor structures.

5 Timeline and Outlook

This is a medium-to-long-term research plan.

- **Short-term (Year 1)**: Focus on completing the factorization theory of twisted conformal blocks on orbicurves.
- **Mid-term (Year 2)**: Advance the algebro-geometric interpretation of VOA representation categories, particularly aiming for breakthroughs in the non-semisimple case.
- **Long-term**: Further explore the generalization of conformal block theory in arithmetic and non-Archimedean contexts, laying the foundation for Arithmetic CFT.

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