《微分几何入门与广义相对论》部分习题参考解答

by 薛定谔的大喵[1]

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[1] wyj1234@mail.ustc.edu.cn

目录

第一部分　上册　1

第一章　广义相对论基础　2

第二部分　中册　19

第一部分

上册

第一章　广义相对论基础

习题

1. 试证弯曲时空麦氏方程 $\nabla^a F_{ab} = -4\pi J_b$ 蕴含电荷守恒定律，即 $\nabla_a J^a = 0$ 。注：$\nabla^a F_{ab} = -4\pi J_b$ 等价于式 (7-2-8) 而非 (7-2-9)，故本题表明式 (7-2-8) 而非式 (7-2-9) 可推出电荷守恒。

证明

$$-4\pi \nabla_a J^a = \nabla_a \nabla_b F^{ba} = 0.$$

2. 试证 $\dfrac{\mathrm{D_F}\omega_a}{\mathrm{d}\tau} = \dfrac{\mathrm{D}\omega_a}{\mathrm{d}\tau} + (A_a \wedge Z_b)\,\omega^b \quad \forall \omega_a \in \mathscr{F}_G(0,1).$

证明 $\forall v^a \in \mathscr{F}_G(1,0)$,

$$\begin{aligned}
v^a \frac{\mathrm{D_F}\omega_a}{\mathrm{d}\tau} &= \frac{\mathrm{D_F}\left(v^a \omega_a\right)}{\mathrm{d}\tau} - \omega_a \frac{\mathrm{D_F} v^a}{\mathrm{d}\tau} \\
&= v^a \frac{\mathrm{D}\omega_a}{\mathrm{d}\tau} + \omega_a \frac{\mathrm{D} v^a}{\mathrm{d}\tau} - \omega_a \left(\frac{\mathrm{D} v^a}{\mathrm{d}\tau} + 2A^{[a} Z^{b]} v_b\right) \\
&= v^a \left(\frac{\mathrm{D}\omega_a}{\mathrm{d}\tau} - 2A_{[b} Z_{a]} \omega^b\right) \\
&= v^a \left(\frac{\mathrm{D}\omega_a}{\mathrm{d}\tau} + A_a \wedge Z_b \omega^b\right).
\end{aligned}$$

3. 试证费米导数性质 3.

证明 性质 3 如下：

性质 若 w^a 是 $G(\tau)$ 上的空间矢量场（对线上各点 $w^a Z_a = 0$ ），则

$$\mathrm{D_F} w^a / \mathrm{d}\tau = h^a{}_b \left(\mathrm{D} w^b / \mathrm{d}\tau\right),$$

其中 $h^a{}_b = g_{ab} + Z_a Z_b$ ，$h^a{}_b = g^{ac} h_{cb}$ 是 $G(\tau)$ 上各点的投影映射。

证明 $h^a{}_b = g^{ac}\left(g_{cb} + Z_c Z_b\right) = \delta^a{}_b + Z^a Z_b$,

$$h^a{}_b \frac{\mathrm{D} w^b}{\mathrm{d}\tau} = \left(\delta^a{}_b + Z^a Z_b\right) \frac{\mathrm{D} w^b}{\mathrm{d}\tau}$$

$$
\begin{aligned}
&= \frac{\mathrm{D}w^a}{\mathrm{d}\tau} + Z^a\left(\frac{\mathrm{D}\left(Z_b w^b\right)}{\mathrm{d}\tau} - w^b\frac{\mathrm{D}Z_b}{\mathrm{d}\tau}\right)\\
&= \frac{\mathrm{D}w^a}{\mathrm{d}\tau} - Z^a A^b w_b\\
&= \frac{\mathrm{D}w^a}{\mathrm{d}\tau} + \left(A^a Z^b - Z^a A^b\right) w_b\\
&= \frac{\mathrm{D_F}w^a}{\mathrm{d}\tau}.
\end{aligned}
$$

□

4. 试证类时线 $G(\tau)$ 上长度不变 (且非零) 的矢量场必经受时空转动。提示：令 $u^a \equiv \mathrm{D}v^a/\mathrm{d}\tau$，则 $u_a v^a = 0$。先证：无论 $v_a v^a$ 为零与否，总有 $G(\tau)$ 上矢量场 v'^a 使 $v'_a v^a = 1$ 。再验证 v^a 经受以 $\Omega_{ab} \equiv 2v'_{[a}u_{b]}$ 为角速度 2 形式的时空转动。

证明 1. 记 $u^a = \dfrac{\mathrm{D}v^a}{\mathrm{d}\tau}$ ，则 $\dfrac{\mathrm{D}\left(v_a v^a\right)}{\mathrm{d}\tau} = 2u_a v^a = 0$ 。

2. 若 $v^a v_a \neq 0$，令

$$v'^a = \frac{v^a}{v^b v^b},$$

若 $v^a v_a = 0$，则 $Z^a v_a$ 不为零，因为与类时矢量内积为零则为类空矢量。于是定义

$$v'^a = \frac{Z^a}{Z^b v_b}.$$

3. 定义 $\Omega_{ab} = 2v'_{[a}u_{b]}$，则

$$-\Omega^{ab} v_b = u^a = \frac{\mathrm{D}v^a}{\mathrm{d}\tau},$$

故 v^a 经受以 Ω_{ab} 为角速度 2 形式的时空转动。

5. 设 $\{T,X,Y,Z\}$ 为闵氏时空的洛伦兹坐标系，曲线 $G(\tau)$ 的参数表达式为

$$T = A^{-1}\sinh A\tau,\quad X = A^{-1}\cosh A\tau,\quad Y = Z = 0,\quad (\text{其中 } A \text{ 为常数})$$

(a) 试证 $G(\tau)$ 是类时双曲线（即图 (6-43)[1] 中的 G），τ 是固有时，A 是 $G(\tau)$ 的 4 加速 A^a 的长度。

(b) 试证从 $\{T,X,Y,Z\}$ 坐标系原点 o 出发的与 $G(\tau)$ 有交的任一半直线 $\mu(s)$ 都与 $G(\tau)$ 正交。

(c) 设 (b) 中的 $\mu(s)$ 的参数 s 是 μ 的线长，随着 $\mu(s)$ 取遍所有从 o 出发并与 $G(\tau)$ 有交的半直线，便得 $G(\tau)$ 上的一个空间矢量场 $w^a \equiv (\partial/\partial s)^a$，试证 w^a 沿 $G(\tau)$ 费移。

(d) 令 $Z^a \equiv (\partial/\partial\tau)^a$，选 $\{Z^a, w^a, (\partial/\partial Y)^a, (\partial/\partial Z)^a\}$ 为 $G(\tau)$ 上的正交归一 4 标架场，求出 $G(\tau)$ 的固有坐标系 $\{t,x,y,z\}$ 并指出其坐标域。

答：$T = (A^{-1}+x)\sinh At,\quad X = (A^{-1}+x)\cosh At,\quad Y = y,\quad Z = z$。

[1] 即本文档图 6.13

(e) 写出闵氏时空在上述固有坐标系中的线元表达式。计算闵氏度规在该系的克氏符，验证它满足引理 7-4-3，即式 (7-4-10)[2]。

证明 (a) 由 $\cosh^2 x - \sinh^2 x = 1$ 知 $(AX)^2 - (AT)^2 = 1$，故这是渐近线为 $T = \pm X$ 的双曲线。

以 τ 为参数，

$$\left(\frac{\partial}{\partial \tau}\right)^a = \cosh(A\tau)\left(\frac{\partial}{\partial T}\right)^a + \sinh(A\tau)\left(\frac{\partial}{\partial X}\right)^a,$$

则

$$\eta_{ab}\left(\frac{\partial}{\partial \tau}\right)^a\left(\frac{\partial}{\partial \tau}\right)^b = -\cosh^2(A\tau) + \sinh^2(A\tau) = -1,$$

即切矢归一，τ 为固有时。

将 $\left(\frac{\partial}{\partial \tau}\right)^a$ 延拓为

$$Z^a = AX\left(\frac{\partial}{\partial T}\right)^a + AT\left(\frac{\partial}{\partial X}\right)^a,$$

容易算得观者四加速为

$$\begin{aligned}\hat{A}^a &= Z^b\nabla_b Z^a\big|_{G(\tau)} \\ &= A^2 T\left(\frac{\partial}{\partial T}\right)^a + A^2 X\left(\frac{\partial}{\partial X}\right)^a\bigg|_{G(\tau)} \\ &= A\sinh(A\tau)\left(\frac{\partial}{\partial T}\right)^a + A\cosh(A\tau)\left(\frac{\partial}{\partial X}\right)^a,\end{aligned}$$

则

$$\eta_{ab}\hat{A}^a\hat{A}^b = A^2\left(\cosh^2(A\tau) - \sinh^2(A\tau)\right) = A^2,$$

即四加速的模长为 A。

(b) 与 G 交于 τ 处的 μ 的方程为

$$T = \tanh(A\tau)X,$$

故其在 $G(\tau)$ 处的切矢正比于

$$\left(\frac{\partial}{\partial s}\right)^a = \sinh(A\tau)\left(\frac{\partial}{\partial T}\right)^a + \cosh(A\tau)\left(\frac{\partial}{\partial X}\right)^a,$$

可算得

$$\left(\frac{\partial}{\partial s}\right)^a\left(\frac{\partial}{\partial \tau}\right)_a = -\cosh(A\tau)\sinh(A\tau) + \cosh(A\tau)\sinh(A\tau) = 0.$$

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[2] 正文 (7-4-10) 为

$$\Gamma^0{}_{00} = \Gamma^\sigma{}_{ij} = 0, \quad \Gamma^0{}_{0i} = \Gamma^0{}_{i0} = \Gamma^i{}_{00} = \hat{A}_i,$$

$$\Gamma^i{}_{0j} = \Gamma^i{}_{j0} = -\omega^k\varepsilon_{0kij}, \quad \sigma = 0,1,2,3; \quad i,j,k = 1,2,3.$$

(c) 在 (b) 中给出的 $(\partial/\partial s)^a$ 已经是归一的，因而就是 w^a。由 (b) 和习题 3，知

$$\begin{aligned}
\frac{\mathrm{D_F} w^a}{\mathrm{d}\tau} &= h^a{}_b \frac{\mathrm{D} w^b}{\mathrm{d}\tau} \\
&= h^a{}_b Z^c \nabla_c w^b \\
&= h^a{}_b \left(A\cosh(A\tau) \left(\frac{\partial}{\partial T}\right)^b + A\sinh(A\tau) \left(\frac{\partial}{\partial X}\right)^b \right) \\
&= A h^a{}_b Z^b \\
&= 0,
\end{aligned}$$

其中 $Z^a = (\partial/\partial\tau)^a$，故 w^a 沿 $G(\tau)$ 费移。

(d) 以 $G(0)$ 为坐标原点，$\{t,0,0,0\}$ 对应的点为 $G(t)$，即

$$T = A^{-1}\sinh At, \quad X = A^{-1}\cosh At, \quad Y = Z = 0,$$

而此点处

$$\begin{aligned}
&x w^a + y\left(\frac{\partial}{\partial Y}\right)^a + z\left(\frac{\partial}{\partial Z}\right)^a \\
={}& x\sinh(At)\left(\frac{\partial}{\partial T}\right)^a + x\cosh(At)\left(\frac{\partial}{\partial X}\right)^a + y\left(\frac{\partial}{\partial Y}\right)^a + z\left(\frac{\partial}{\partial Z}\right)^a,
\end{aligned}$$

沿此矢量决定的测地线（直线）走参数为 1 的距离，即

$$\Delta T = x\sinh(At), \quad \Delta X = x\cosh(At), \quad \Delta Y = y, \quad \Delta Z = z\left(\frac{\partial}{\partial Z}\right)^a,$$

故 $\{t,x,y,z\}$ 对应的点为

$$T = \left(A^{-1}+x\right)\sinh At, \quad X = \left(A^{-1}+x\right)\cosh At, \quad Y = y, \quad Z = z. \tag{1.1}$$

(e) 计算得

$$\begin{aligned}
\mathrm{d}s^2 ={}& -\mathrm{d}T^2 + \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2 \\
={}& -\left[(1+Ax)\cosh(At)\,\mathrm{d}t + \sinh(At)\,\mathrm{d}x\right]^2 \\
& + \left[(1+Ax)\sinh(At)\,\mathrm{d}t + \cosh(At)\,\mathrm{d}x\right]^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\
={}& -(1+Ax)^2\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\,,
\end{aligned}$$

容易算得非零克氏符为

$$\Gamma^t{}_{tx} = \Gamma^t{}_{xt} = \frac{A}{1+Ax}, \quad \Gamma^x{}_{tt} = A\,(1+Ax)\,,$$

在线上时

$$\Gamma^t{}_{tx} = \Gamma^t{}_{xt} = \Gamma^x{}_{tt} = A,$$

对 (1.1) 反解得坐标变换

$$t = A^{-1}\tanh^{-1}\left(\frac{T}{X}\right), \quad x = \sqrt{X^2 - T^2} - A^{-1}, \quad y = Y, \quad z = Z,$$

故

$$\begin{aligned}
\left(\frac{\partial}{\partial T}\right)^a &= \frac{\partial t}{\partial T}\left(\frac{\partial}{\partial t}\right)^a + \frac{\partial x}{\partial T}\left(\frac{\partial}{\partial x}\right)^a \\
&= \frac{X}{A\left(X^2 - T^2\right)}\left(\frac{\partial}{\partial t}\right)^a - \frac{T}{\sqrt{X^2 - T^2}}\left(\frac{\partial}{\partial x}\right)^a \\
&= \frac{\cosh At}{1 + Ax}\left(\frac{\partial}{\partial t}\right)^a - \sinh(At)\left(\frac{\partial}{\partial x}\right)^a, \\
\left(\frac{\partial}{\partial X}\right)^a &= \frac{\partial t}{\partial X}\left(\frac{\partial}{\partial t}\right)^a + \frac{\partial x}{\partial X}\left(\frac{\partial}{\partial x}\right)^a \\
&= \frac{T}{A\left(T^2 - X^2\right)}\left(\frac{\partial}{\partial t}\right)^a + \frac{X}{\sqrt{X^2 - T^2}}\left(\frac{\partial}{\partial x}\right)^a \\
&= -\frac{\sinh At}{1 + Ax}\left(\frac{\partial}{\partial t}\right)^a + \cosh(At)\left(\frac{\partial}{\partial x}\right)^a,
\end{aligned}$$

故

$$\begin{aligned}
\hat{A}^a &= A^2 X\left(\frac{\partial}{\partial X}\right)^a + A^2 T\left(\frac{\partial}{\partial T}\right)^a \\
&= A\left(1 + Ax\right)\cosh(At)\left(-\frac{\sinh At}{1 + Ax}\left(\frac{\partial}{\partial t}\right)^a + \cosh(At)\left(\frac{\partial}{\partial x}\right)^a\right) \\
&\quad + A\left(1 + Ax\right)\sinh(At)\left(\frac{\cosh At}{1 + Ax}\left(\frac{\partial}{\partial t}\right)^a - \sinh(At)\left(\frac{\partial}{\partial x}\right)^a\right) \\
&= A\left(1 + Ax\right)\left(\frac{\partial}{\partial x}\right)^a,
\end{aligned}$$

在线上有

$$\hat{A}^a = A\left(\frac{\partial}{\partial x}\right)^a,$$

满足引理，证毕。

6. 设 G 是质点 L 在 $p \in L$ 的瞬时静止自由下落观者（即 G 的 4 速 Z^a 与 L 的 4 速 U^a 在 p 点相切），A^a 是 L 在 p 点的 4 加速，a^a 是 L 在 p 点相对于 G 的 3 加速 [由式 (7-4-3)[3]定义]，试证 $a^a = A^a$。
注：本题可视为命题 6-3-6 在弯曲时空的推广。

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[3]正文式 (7-4-3) 为

$$a^a := \left[\frac{\mathrm{d}^2 x^i(t)}{\mathrm{d}t^2}\right]\left(\frac{\partial}{\partial x^i}\right)^a.$$

证明 记 $G(t)$ 的固有坐标系为 $\{t,x,y,z\}$。在 p 点，有 $U^a=Z^a=\left(\frac{\partial}{\partial t}\right)^a$。对 U^a 做分解，有

$$U^a=\left(\frac{\partial}{\partial\tau_L}\right)^a=\frac{\mathrm{d}t}{\mathrm{d}\tau_L}\left(\frac{\partial}{\partial t}\right)^a+\frac{\mathrm{d}x^i}{\mathrm{d}\tau_L}\left(\frac{\partial}{\partial x^i}\right)^a=\gamma Z^a+\gamma u^a,$$

则

$$\gamma|_p=1,\quad u^a|_p=0,$$

而

$$\begin{aligned}A^a|_p&=\left(Z^b\nabla_bU^a\right)_p\\&=\left(\partial_0U^a+\Gamma^0{}_{ab}U^b\right)_p\\&=\partial_0U^a|_p\\&=\frac{\mathrm{d}\gamma}{\mathrm{d}t}Z^a+\frac{\mathrm{d}\gamma}{\mathrm{d}t}u^a+\gamma\frac{\mathrm{d}u^a}{\mathrm{d}t}\\&=\frac{\mathrm{d}u^a}{\mathrm{d}t},\end{aligned}$$

其中最后一步用到 $\left.\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|_p=0$，这是因为 $\gamma=-U^aZ_a\leqslant1$，故在 p 点 $\gamma|_p=1$ 取到了极值。而 $\frac{\mathrm{d}u^a}{\mathrm{d}t}$ 就是 a^a。

7. 度规 g_{ab} 叫 **里奇平直**的，若 g_{ab} 的里奇张量为零。试证 g_{ab} 是真空爱因斯坦方程的解的充要条件是 g_{ab} 是里奇平直的。

证明 真空爱因斯坦方程为 $R_{ab}-\frac{1}{2}Rg_{ab}=0$。

1. 充分性：若 $R_{ab}=0$，则取迹得 $R=0$，故满足真空场方程。
2. 必要性：设 g_{ab} 满足真空场方程，即 $R_{ab}-\frac{1}{2}Rg_{ab}=0$，取迹得

$$R-2R=-R=0,$$

故

$$R_{ab}-\frac{1}{2}Rg_{ab}=R_{ab}=0,$$

故里奇平直。

8. 设 (M,g_{ab}) 为里奇平直时空（定义见上题），ξ^a 是其中的一个 Killing 矢量场，试证 $F_{ab}:=(\mathrm{d}\xi)_{ab}$ 满足 (M,g_{ab}) 的无源（$J_a=0$）麦氏方程。提示：利用 Killing 场 ξ^a 满足的 $\nabla_a\xi^a=0$（第 4 章习题 11 的结果）。

证明 计算得

$$\begin{aligned}\nabla^aF_{ab}&=\nabla^a\nabla_a\xi_b-\nabla^a\nabla_b\xi_a\\&=-2\nabla^a\nabla_b\xi_a\\&=-2\left(\nabla_b\nabla^a\xi_a+R_b{}^c\xi_c\right)\\&=0,\end{aligned}$$

而第二个方程 $\nabla_{[a}F_{bc]}=0$ 等价于 $(\mathrm{d}F)_{abc}=0$，这是由 $F=\mathrm{d}\xi$ 保证的。

9. 设 $\xi_\mu(\mu=0,1,2,3)$ 为方程 $\partial^b\partial_b\xi_\mu=0$ 在初始条件式 (7-9-10) ~ (7-9-13) [4]下的解，试证由 $\xi_a=\xi_\mu(\mathrm{d}x^\mu)_a$ 及 γ_{ab} 按式 (7-9-8) [5] 构造的 γ'_{ab} 在无源区域既满足洛伦兹规范条件 $\partial^a\bar{\gamma}'_{ab}=0$ 又满足 $\gamma'=0$ 和 $\gamma'_{0i}=0(i=1,2,3)$。提示：(1) 根据解的唯一性定理，只须证明 $\gamma'=0$ 和 $\gamma'_{0i}=0$ 分别是方程 $\partial^c\partial_c\gamma=0$ 和 $\partial^c\partial_c\gamma'_{0i}=0$ 的满足初始条件 $\gamma'|_{\Sigma_0}=0$，$\partial\gamma'/\partial t|_{\Sigma_0}=0$，$\gamma'_{0i}|_{\Sigma_0}=0$ 和 $\partial\gamma'_{0i}/\partial t|_{\Sigma_0}=0$ 的解。(2) 由 $\partial^b\partial_b\xi_\mu=0$ 可得 $\partial^2\xi_\mu/\partial t^2=\nabla^2\xi_\mu$。

证明 由 (7-9-8) 易知，

$$\gamma'=\gamma+2\partial^a\xi_a,$$

则

$$\begin{aligned}\partial^a\bar{\gamma}'_{ab}&=\partial^a\left(\gamma'_{ab}-\frac{1}{2}\eta_{ab}\gamma'\right)\\&=\partial^a\left(\gamma_{ab}+\partial_a\xi_b+\partial_b\xi_a-\frac{1}{2}\eta_{ab}\gamma-\eta_{ab}\partial^c\xi_c\right)\\&=\partial^a\gamma_{ab}+0+\partial^a\partial_b\xi_a-\frac{1}{2}\partial_b\gamma-\partial_b\partial^c\xi_c\\&=0,\end{aligned}$$

其中红色的两项加起来为 $\partial^a\bar{\gamma}_{ab}$，故为零。

在无源区域，$T_{ab}=0$，则线性场方程为

$$\partial^c\partial_c\gamma_{ab}=0,$$

取迹得

$$\partial^c\partial_c\gamma=0,$$

则

$$\partial^c\partial_c\gamma'=\partial^c\partial_c\left(\gamma+2\partial^a\xi_a\right)=0,$$

[4]正文 (7-9-10) ~ (7-9-13) 为

$$2\left(\vec{\nabla}\cdot\vec{\xi}-\partial\xi_0/\partial t\right)\Big|_{\Sigma_0}=-\gamma|_{\Sigma_0}, \tag{7-9-10}$$

$$2\left[-\nabla^2\xi_0+\vec{\nabla}\cdot\left(\partial\vec{\xi}/\partial t\right)\right]_{\Sigma_0}=-\partial\gamma/\partial t|_{\Sigma_0}, \tag{7-9-11}$$

$$\left[(\partial\gamma_i/\partial t)+\left(\partial\xi_0/\partial x^i\right)\right]_{\Sigma_0}=-\gamma_{0i}|_{\Sigma_0},\quad i=1,2,3, \tag{7-9-12}$$

$$\left[\nabla^2\xi_i+\frac{\partial}{\partial x^i}\left(\frac{\partial\xi_0}{\partial t}\right)\right]_{\Sigma_0}=-\left.\frac{\partial\gamma_{0i}}{\partial t}\right|_{\Sigma_0},\quad i=1,2,3. \tag{7-9-13}$$

[5]正文式 (7-9-8) 为

$$\gamma'_{ab}=\gamma_{ab}+\partial_a\xi_b+\partial_b\xi_a, \tag{7-9-8}$$

其中 ξ_a 满足

$$\partial^b\partial_b\xi_a=0. \tag{7-9-9}$$

在边界上又有

$$
\begin{aligned}
\gamma'|_{\Sigma_0} &= (\gamma + 2\partial^a \xi_a)_{\Sigma_0} \\
&= \left(\gamma - 2\frac{\partial \xi_0}{\partial t} + 2\vec{\nabla}\cdot\vec{\xi}\right)_{\Sigma_0} \\
&= 0, \\
\left.\frac{\partial \gamma'}{\partial t}\right|_{\Sigma_0} &= \left(\frac{\partial \gamma}{\partial t} - 2\frac{\partial^2 \xi_0}{\partial t^2} + \vec{\nabla}\cdot\frac{\partial \vec{\xi}}{\partial t}\right)_{\Sigma_0} \\
&= \left(\frac{\partial \gamma}{\partial t} - 2\nabla^2 \xi_0 + \vec{\nabla}\cdot\frac{\partial \vec{\xi}}{\partial t}\right)_{\Sigma_0} \\
&= 0,
\end{aligned}
$$

故知在区域内 γ' 必为零。

再考虑 ${\gamma'}_{0i}$，它也满足拉普拉斯方程

$$
\partial^a \partial_a {\gamma'}_{0i} = 0,
$$

而在边界上

$$
\begin{aligned}
{\gamma'}_{0i}|_{\Sigma_0} &= \left(\gamma_{0i} + \frac{\partial \xi_i}{\partial t} + \frac{\partial \xi_0}{\partial x^i}\right)_{\Sigma_0} \\
&= 0, \\
\left.\frac{\partial {\gamma'}_{0i}}{\partial t}\right|_{\Sigma_0} &= \left(\frac{\partial \gamma_{0i}}{\partial t} + \frac{\partial^2 \xi_i}{\partial t^2} + \frac{\partial}{\partial x^i}\left(\frac{\partial \xi_0}{\partial t}\right)\right)_{\Sigma_0} \\
&= \left(\frac{\partial \gamma_{0i}}{\partial t} + \nabla^2 \xi_i + \frac{\partial}{\partial x^i}\left(\frac{\partial \xi_0}{\partial t}\right)\right)_{\Sigma_0} \\
&= 0,
\end{aligned}
$$

则 ${\gamma'}_{0i}$ 在区域内也为零。

10. 设 γ_{ab} 满足 (a) $\partial^a \bar{\gamma}_{ab} = 0$；(b) $\gamma = 0$；(c) $\gamma_{0i} = 0 (i = 1, 2, 3)$；(d) $\gamma_{00} =$ 常数。试找出一个“无限小”矢量场 ξ^a 使 $\tilde{\gamma}_{ab} \equiv \gamma_{ab} + \partial_a \xi_b + \partial_b \xi_a$ 满足
(a) $\partial^a \bar{\tilde{\gamma}}_{ab} = 0$；(b) $\tilde{\gamma} = 0$；(c) $\tilde{\gamma}_{0i} = 0$；(d) $\tilde{\gamma}_{00} = 0$。

证明 计算得

$$\begin{aligned}
\partial^a \bar{\tilde{\gamma}}_{ab} &= \partial^a \left(\tilde{\gamma}_{ab} - \frac{1}{2}\eta_{ab}\tilde{\gamma}\right) \\
&= \partial^a \left(\gamma_{ab} + \partial_a \xi_b + \partial_b \xi_a - \frac{1}{2}\eta_{ab}\gamma - \eta_{ab}\partial^c \xi_c\right) \\
&= \partial^a \partial_a \xi_b, \\
\tilde{\gamma} &= \gamma + 2\partial^a \xi_a, \\
\tilde{\gamma}_{0i} &= \gamma_{0i} + \frac{\partial \xi_i}{\partial t} + \frac{\partial \xi_0}{\partial x^i} \\
&= \frac{\partial \xi_i}{\partial t} + \frac{\partial \xi_0}{\partial x^i}, \\
\tilde{\gamma}_{00} &= \gamma_{00} + 2\frac{\partial \xi_0}{\partial t},
\end{aligned}$$

故得微分方程组

$$\begin{cases}
\partial^a \partial_a \xi_\mu = 0, \quad \mu = 0,1,2,3, \\
\partial^\mu \xi_\mu = 0, \\
\dfrac{\partial \xi_i}{\partial t} + \dfrac{\partial \xi_0}{\partial x^i} = 0, \quad i = 1,2,3, \\
\dfrac{\partial \xi_0}{\partial t} = -\dfrac{1}{2}\gamma_{00},
\end{cases}$$

先关注 ξ_0，由

$$\begin{cases}
\nabla^2 \xi_0 = 0, \\
\dfrac{\partial \xi_0}{\partial t} = -\dfrac{1}{2}\gamma_{00},
\end{cases}$$

则最简单的解为

$$\xi_0 = -\frac{1}{2}\gamma_{00} t,$$

代回方程组得

$$\begin{cases}
\partial^a \partial_a \xi_i = 0, \quad i = 1,2,3, \\
\partial_i \xi^i = -\dfrac{1}{2}\gamma_{00}, \\
\dfrac{\partial \xi_i}{\partial t} = 0, \quad i = 1,2,3,
\end{cases}$$

则取 $\xi^i = \frac{1}{6}\gamma_{00} x^i$ 即可，即

$$\xi^a = \frac{1}{2}\gamma_{00} t \left(\frac{\partial}{\partial t}\right)^a - \frac{1}{6}\gamma_{00} x^i \left(\frac{\partial}{\partial x^i}\right)^a.$$

11. 试证命题 7-9-2。

证明 命题 7-9-2 为

定理.

$$\begin{aligned}R_{abcd}= & \left[f\left(e^1\right)_a\wedge\left(e^4\right)_b+g\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left(e^4\right)_c\wedge\left(e^1\right)_d\\ & +\left[g\left(e^1\right)_a\wedge\left(e^4\right)_b-f\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left(e^4\right)_c\wedge\left(e^2\right)_d.\end{aligned}$$ ◇

证明 式 (7-9-32) 为

$$\begin{aligned}R_{abc}{}^d &= R_{ab1}{}^3\left(e^1\right)_c(e_3)^d+R_{ab2}{}^3\left(e^2\right)_c(e_3)^d+R_{ab4}{}^1\left(e^4\right)_c\left(e^1\right)^d+R_{ab4}{}^2\left(e^4\right)_c(e_2)^d\\ &=\left[f\left(e^1\right)_a\wedge\left(e^4\right)_b+g\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left[\left(e^1\right)_c(e_3)^d+\left(e^4\right)_c(e_1)^d\right]\\ &\quad+\left[g\left(e^1\right)_a\wedge\left(e^4\right)_b-f\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left[\left(e^2\right)_c(e_3)^d+\left(e^4\right)_c(e_2)^d\right],\end{aligned}$$

由 (7-9-26) 知

$$g_{ab}=\left(e^1\right)_a\left(e^1\right)_b+\left(e^2\right)_a\left(e^2\right)_b-\left(e^3\right)_a\left(e^4\right)_b-\left(e^4\right)_a\left(e^3\right)_b,$$

则

$$\begin{aligned}R_{abcd} &= R_{abc}{}^e g_{ed}\\ &=\left[f\left(e^1\right)_a\wedge\left(e^4\right)_b+g\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left[-\left(e^1\right)_c\left(e^4\right)_d+0\right]\\ &\quad+\left[g\left(e^1\right)_a\wedge\left(e^4\right)_b-f\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left[-\left(e^2\right)_c\left(e^4\right)_d+0\right]\\ &=\left[f\left(e^1\right)_a\wedge\left(e^4\right)_b+g\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left(e^4\right)_c\wedge\left(e^1\right)_d\\ &\quad+\left[g\left(e^1\right)_a\wedge\left(e^4\right)_b-f\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left(e^4\right)_c\wedge\left(e^2\right)_d.\end{aligned}$$ □

12. 验证式 (7-9-41) 后的 (1)∼(3)。

证明 (1)

$$g_{ab}\left(E_1\right)^a\left(E_1\right)^b=g_{ab}\left(\left(\frac{\partial}{\partial x}\right)^a+E^{-1}Z_1K^a\right)\left(\left(\frac{\partial}{\partial x}\right)^b+E^{-1}Z_1K^b\right),$$

易知

$$\begin{aligned}g_{ab}K^a &=\left(\eta_{ab}+2P\left[(\mathrm{d}t)_a-(\mathrm{d}z)_a\right]\left[(\mathrm{d}t)_b-(\mathrm{d}z)_b\right]\right)\left(\left(\frac{\partial}{\partial t}\right)^a+\left(\frac{\partial}{\partial z}\right)^a\right)\\ &=(\mathrm{d}z)_b-(\mathrm{d}t)_b,\\ g_{ab}\left(\frac{\partial}{\partial x}\right)^a &=\left(\eta_{ab}+2P\left[(\mathrm{d}t)_a-(\mathrm{d}z)_a\right]\left[(\mathrm{d}t)_b-(\mathrm{d}z)_b\right]\right)\left(\frac{\partial}{\partial x}\right)^a\\ &=(\mathrm{d}x)_b,\\ g_{ab}\left(\frac{\partial}{\partial y}\right)^a &=\left(\eta_{ab}+2P\left[(\mathrm{d}t)_a-(\mathrm{d}z)_a\right]\left[(\mathrm{d}t)_b-(\mathrm{d}z)_b\right]\right)\left(\frac{\partial}{\partial y}\right)^a\\ &=(\mathrm{d}y)_b,\end{aligned}$$

$$
\begin{aligned}
g_{ab}\left(\frac{\partial}{\partial t}\right)^a &= \left(\eta_{ab}+2P\left[(\mathrm{d}t)_a-(\mathrm{d}z)_a\right]\left[(\mathrm{d}t)_b-(\mathrm{d}z)_b\right]\right)\left(\frac{\partial}{\partial t}\right)^a \\
&= -(\mathrm{d}t)_b+2P\left[(\mathrm{d}t)_b-(\mathrm{d}z)_b\right] \\
&= -(\mathrm{d}t)_b-2PK_b, \\
g_{ab}\left(\frac{\partial}{\partial z}\right)^a &= \left(\eta_{ab}+2P\left[(\mathrm{d}t)_a-(\mathrm{d}z)_a\right]\left[(\mathrm{d}t)_b-(\mathrm{d}z)_b\right]\right)\left(\frac{\partial}{\partial z}\right)^a \\
&= (\mathrm{d}z)_b-2P\left[(\mathrm{d}t)_b-(\mathrm{d}z)_b\right] \\
&= (\mathrm{d}z)_b+2PK_b,
\end{aligned}
$$

则

$$
\begin{aligned}
g_{ab}(E_1)^a &= (\mathrm{d}x)_b+E^{-1}Z_1K_b, \\
g_{ab}(E_2)^a &= (\mathrm{d}y)_b+E^{-1}Z_2K_b, \\
g_{ab}(E_3)^a &= E^{-1}K_b-Z_b,
\end{aligned}
$$

故

$$
\begin{aligned}
g_{ab}(E_1)^a(E_1)^b &= \left((\mathrm{d}x)_b+E^{-1}Z_1K_b\right)\left(\left(\frac{\partial}{\partial x}\right)^b+E^{-1}Z_1K^b\right) \\
&= 1, \\
g_{ab}(E_1)^a(E_2)^b &= \left((\mathrm{d}x)_b+E^{-1}Z_1K_b\right)\left(\left(\frac{\partial}{\partial y}\right)^b+E^{-1}Z_2K^b\right) \\
&= 0, \\
g_{ab}(E_1)^a(E_3)^b &= \left((\mathrm{d}x)_b+E^{-1}Z_1K_b\right)\left(E^{-1}K^b-Z^b\right) \\
&= -Z_1+Z_1 \\
&= 0, \\
g_{ab}(E_2)^a(E_1)^b &= \left((\mathrm{d}y)_b+E^{-1}Z_2K_b\right)\left(\left(\frac{\partial}{\partial x}\right)^b+E^{-1}Z_1K^b\right) \\
&= 0, \\
g_{ab}(E_2)^a(E_2)^b &= \left((\mathrm{d}y)_b+E^{-1}Z_2K_b\right)\left(\left(\frac{\partial}{\partial y}\right)^b+E^{-1}Z_2K^b\right) \\
&= 1, \\
g_{ab}(E_2)^a(E_3)^b &= \left((\mathrm{d}y)_b+E^{-1}Z_2K_b\right)\left(E^{-1}K^b-Z^b\right) \\
&= -Z_2+Z_2 \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
g_{ab}\left(E_3\right)^a\left(E_1\right)^b &= \left(E^{-1}K_b - Z_b\right)\left(\left(\frac{\partial}{\partial x}\right)^b + E^{-1}Z_1K^b\right) \\
&= -Z_1 + Z_1 \\
&= 0, \\
g_{ab}\left(E_3\right)^a\left(E_2\right)^b &= \left(E^{-1}K_b - Z_b\right)\left(\left(\frac{\partial}{\partial y}\right)^b + E^{-1}Z_2K^b\right) \\
&= -Z_2 + Z_2 \\
&= 0, \\
g_{ab}\left(E_3\right)^a\left(E_3\right)^b &= \left(E^{-1}K_b - Z_b\right)\left(E^{-1}K^b - Z^b\right) \\
&= 1 + 1 - 1 \\
&= 1.
\end{aligned}
$$

(2) 先计算 $h^a{}_bK^b = K^a - EZ^a$ 的模方：

$$
\begin{aligned}
\left(K^a - EZ^a\right)\left(K_a - EZ_a\right) &= E^2 + E^2 - E^2 \\
&= E^2,
\end{aligned}
$$

故将其归一化得

$$
\frac{K^a - EZ^a}{E} = E^{-1}K^a - Z^a = \left(E_3\right)^a.
$$

(3) 首先，由于 Z^a 是测地观者，

$$
\begin{aligned}
Z^a\nabla_aE &= -Z^a\nabla_a\left(Z^bK_b\right) \\
&= -\left(Z^a\nabla_aZ^b\right)K_b - Z^aZ_b\nabla_aK_b \\
&= 0,
\end{aligned}
$$

故

$$
\begin{aligned}
Z^b\nabla_b\left(E_3\right)^a &= Z^b\nabla_b\left(E^{-1}K^a - Z^a\right) \\
&= E^{-1}Z^b\nabla_bK^a - Z^b\nabla_bZ^a \\
&= 0,
\end{aligned}
$$

而采用 (7-9-25) 的标架可算得

$$
\begin{aligned}
\nabla_b\left(\frac{\partial}{\partial x}\right)^a &= \nabla_b\left(e_1\right)^a \\
&= -\omega_1{}^{\nu}{}_b\left(e_\nu\right)^a \\
&= -\omega_1{}^{3}{}_b\left(e_3\right)^a \\
&= -\left(fx + gy\right)\left[(\mathrm{d}t)_b - (\mathrm{d}z)_b\right]K^a \\
&= \left(fx + gy\right)K_bK^a,
\end{aligned}
$$

故

$$
\begin{aligned}
Z^b\nabla_b\left(E_1\right)^a &= Z^b\nabla_b\left(\left(\frac{\partial}{\partial x}\right)^a + E^{-1}Z_1K^a\right)\\
&= Z^b\nabla_b\left(\left(\frac{\partial}{\partial x}\right)^a + E^{-1}Z_c\left(\frac{\partial}{\partial x}\right)^c K^a\right)\\
&= Z^b\nabla_b\left(\frac{\partial}{\partial x}\right)^a + E^{-1}Z^bZ_cK^a\nabla_b\left(\frac{\partial}{\partial x}\right)^c\\
&= \left(fz+gy\right)\left(Z^bK_bK^a + E^{-1}Z^bZ_cK^aK_bK^c\right)\\
&= \left(fx+gy\right)\left(-E+E\right)K^a\\
&= 0,
\end{aligned}
$$

同理由于

$$
\begin{aligned}
\nabla_b\left(\frac{\partial}{\partial y}\right)^a &= \nabla_b\left(e_2\right)^a\\
&= -{\omega_2}^{\nu}{}_b\left(e_\nu\right)^a\\
&= -{\omega_2}^{3}{}_b\left(e_3\right)^a\\
&= -\left(gx-fy\right)\left[(\mathrm{d}t)_b-(\mathrm{d}z)_b\right]K^a\\
&= \left(gx-fy\right)K_bK^a,
\end{aligned}
$$

故

$$
\begin{aligned}
Z^b\nabla_b\left(E_2\right)^a &= Z^b\nabla_b\left(\left(\frac{\partial}{\partial y}\right)^a + E^{-1}Z_2K^a\right)\\
&= Z^b\nabla_b\left(\left(\frac{\partial}{\partial y}\right)^a + E^{-1}Z_c\left(\frac{\partial}{\partial y}\right)^c K^a\right)\\
&= Z^b\nabla_b\left(\frac{\partial}{\partial y}\right)^a + E^{-1}Z^bZ_cK^a\nabla_b\left(\frac{\partial}{\partial y}\right)^c\\
&= \left(gx-fy\right)\left(Z^bK_bK^a + E^{-1}Z^bZ_cK^aK_bK^c\right)\\
&= \left(gx-fy\right)\left(-E+E\right)K^a\\
&= 0.
\end{aligned}
$$

13. 试证式 (7-9-43)。[6]

[6]正文 (7-9-43) 为

$$
\left({\psi^i}_j\right) = \begin{bmatrix}\alpha & \beta & 0\\ \beta & -\alpha & 0\\ 0 & 0 & 0\end{bmatrix},\quad \alpha\equiv -E^2f,\quad \beta\equiv -E^2g. \tag{7-9-43}
$$

证明 首先，注意到

$$
\begin{aligned}
(E_1)^a &= (e_1)^a + E^{-1} Z_1 (e_3)^a\,, \\
(E_2)^a &= (e_2)^a + E^{-1} Z_2 (e_3)^a\,, \\
(E_3)^a &= E^{-1} (e_3)^a - Z^a,
\end{aligned}
$$

且 $(e^4)_a = (\mathrm{d}u)_a = -K_a$，易得

$$
\begin{pmatrix}
(e^1)_a (E_1)^a & (e^1)_a (E_2)^a & (e^1)_a (E_3)^a \\
(e^2)_a (E_1)^a & (e^2)_a (E_2)^a & (e^2)_a (E_3)^a \\
(e^4)_a (E_1)^a & (e^4)_a (E_2)^a & (e^4)_a (E_3)^a
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 & -Z_1 \\
0 & 1 & -Z_2 \\
0 & 0 & -E
\end{pmatrix},
$$

并有

$$
\begin{aligned}
(e^1)_a Z^a &= Z_1, \\
(e^2)_a Z^a &= Z_2, \\
(e^4)_a Z^a &= -K_a Z^a \\
&= E,
\end{aligned}
$$

故

$$
\begin{aligned}
\psi^1{}_1 &= -R_{abcd} Z^a (E_1)^b Z^c (E_1)^d \\
&= -\Big(\left[f (e^1)_a \wedge (e^4)_b + g (e^2)_a \wedge (e^4)_b\right] (e^4)_c \wedge (e^1)_d \\
&\qquad + \left[g (e^1)_a \wedge (e^4)_b - f (e^2)_a \wedge (e^4)_b\right] (e^4)_c \wedge (e^2)_d \Big) Z^a (E_1)^b Z^c (E_1)^d \\
&= -\left[f (e^1)_a \wedge (e^4)_b + g (e^2)_a \wedge (e^4)_b\right] E Z^a (E_1)^b + 0 \\
&= E (fE + 0) \\
&= E^2 f \\
&= -\alpha \\
\psi^1{}_2 &= -R_{abcd} Z^a (E_2)^b Z^c (E_1)^d \\
&= -\Big(\left[f (e^1)_a \wedge (e^4)_b + g (e^2)_a \wedge (e^4)_b\right] (e^4)_c \wedge (e^1)_d \\
&\qquad + \left[g (e^1)_a \wedge (e^4)_b - f (e^2)_a \wedge (e^4)_b\right] (e^4)_c \wedge (e^2)_d \Big) Z^a (E_2)^b Z^c (E_1)^d \\
&= -\left[f (e^1)_a \wedge (e^4)_b + g (e^2)_a \wedge (e^4)_b\right] E Z^a (E_2)^b + 0 \\
&= E^2 g \\
&= -\beta,
\end{aligned}
$$

$$
\begin{aligned}
\psi^1{}_3 &= -R_{abcd}Z^a\left(E_3\right)^b Z^c\left(E_1\right)^d \\
&= -\Big(\left[f\left(e^1\right)_a\wedge\left(e^4\right)_b + g\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left(e^4\right)_c\wedge\left(e^1\right)_d \\
&\qquad + \left[g\left(e^1\right)_a\wedge\left(e^4\right)_b - f\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left(e^4\right)_c\wedge\left(e^2\right)_d\Big)Z^a\left(E_3\right)^b Z^c\left(E_1\right)^d \\
&= -\left[f\left(e^1\right)_a\wedge\left(e^4\right)_b + g\left(e^2\right)_a\wedge\left(e^4\right)_b\right]EZ^a\left(E_3\right)^b + 0 \\
&= 0, \\
\psi^2{}_1 &= -R_{abcd}Z^a\left(E_1\right)^b Z^c\left(E_2\right)^d \\
&= -\Big(\left[f\left(e^1\right)_a\wedge\left(e^4\right)_b + g\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left(e^4\right)_c\wedge\left(e^1\right)_d \\
&\qquad + \left[g\left(e^1\right)_a\wedge\left(e^4\right)_b - f\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left(e^4\right)_c\wedge\left(e^2\right)_d\Big)Z^a\left(E_1\right)^b Z^c\left(E_2\right)^d \\
&= -\left[g\left(e^1\right)_a\wedge\left(e^4\right)_b - f\left(e^2\right)_a\wedge\left(e^4\right)_b\right]EZ^a\left(E_1\right)^b \\
&= gE^2 \\
&= -\beta, \\
\psi^2{}_2 &= -R_{abcd}Z^a\left(E_2\right)^b Z^c\left(E_2\right)^d \\
&= -\Big(\left[f\left(e^1\right)_a\wedge\left(e^4\right)_b + g\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left(e^4\right)_c\wedge\left(e^1\right)_d \\
&\qquad + \left[g\left(e^1\right)_a\wedge\left(e^4\right)_b - f\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left(e^4\right)_c\wedge\left(e^2\right)_d\Big)Z^a\left(E_2\right)^b Z^c\left(E_2\right)^d \\
&= -\left[g\left(e^1\right)_a\wedge\left(e^4\right)_b - f\left(e^2\right)_a\wedge\left(e^4\right)_b\right]EZ^a\left(E_2\right)^b \\
&= -fE^2 \\
&= \alpha, \\
\psi^2{}_3 &= -R_{abcd}Z^a\left(E_3\right)^b Z^c\left(E_2\right)^d \\
&= -\Big(\left[f\left(e^1\right)_a\wedge\left(e^4\right)_b + g\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left(e^4\right)_c\wedge\left(e^1\right)_d \\
&\qquad + \left[g\left(e^1\right)_a\wedge\left(e^4\right)_b - f\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left(e^4\right)_c\wedge\left(e^2\right)_d\Big)Z^a\left(E_3\right)^b Z^c\left(E_2\right)^d \\
&= -\left[g\left(e^1\right)_a\wedge\left(e^4\right)_b - f\left(e^2\right)_a\wedge\left(e^4\right)_b\right]EZ^a\left(E_3\right)^b \\
&= 0, \\
\psi^3{}_i &= -R_{abcd}Z^a\left(E_i\right)^b Z^c\left(E_3\right)^d \\
&= -\Big(\left[f\left(e^1\right)_a\wedge\left(e^4\right)_b + g\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left(e^4\right)_c\wedge\left(e^1\right)_d \\
&\qquad + \left[g\left(e^1\right)_a\wedge\left(e^4\right)_b - f\left(e^2\right)_a\wedge\left(e^4\right)_b\right]\left(e^4\right)_c\wedge\left(e^2\right)_d\Big)Z^a\left(E_i\right)^b Z^c\left(E_3\right)^d
\end{aligned}
$$

$$
\begin{aligned}
&= -\Big(\left[f\left(e^1\right)_a \wedge \left(e^4\right)_b + g\left(e^2\right)_a \wedge \left(e^4\right)_b\right]\left(-EZ_1 + Z_1E\right) \\
&\qquad + \left[g\left(e^1\right)_a \wedge \left(e^4\right)_b - f\left(e^2\right)_a \wedge \left(e^4\right)_b\right]\left(-EZ_2 + Z_2E\right)\Big) Z^a \left(E_i\right)^b \\
&= 0,
\end{aligned}
$$

故

$$
\left[\psi^i{}_j\right] = -\begin{pmatrix} \alpha & \beta & 0 \\ \beta & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha \equiv -E^2 f, \quad \beta \equiv -E^2 g.
$$

（好像差了个负号耶……）

14. 试证式 (7-9-36)，即 $\nabla^a\nabla_a P = (\partial^2 P/\partial x^2) + (\partial^2 P/\partial y^2)$。

证明 首先，

$$
\begin{aligned}
\nabla_a P &= (\mathrm{d}P)_a \\
&= \frac{\partial P}{\partial u}(\mathrm{d}u)_a + \frac{\partial P}{\partial x}(\mathrm{d}x)_a + \frac{\partial P}{\partial y}(\mathrm{d}y)_a \\
&= \frac{\partial P}{\partial u}\left(e^4\right)_a + \frac{\partial P}{\partial x}\left(e^1\right)_a + \frac{\partial P}{\partial y}\left(e^2\right)_a,
\end{aligned}
$$

则

$$
\begin{aligned}
\nabla^a\nabla_a P = &\left(e^4\right)_a \nabla^a \frac{\partial P}{\partial u} + \frac{\partial P}{\partial u}\nabla^a\left(e^4\right)_a \\
&+ \left(e^1\right)_a \nabla^a \frac{\partial P}{\partial x} + \frac{\partial P}{\partial x}\nabla^a\left(e^1\right)_a \\
&+ \left(e^2\right)_a \nabla^a \frac{\partial P}{\partial y} + \frac{\partial P}{\partial y}\nabla^a\left(e^2\right)_a,
\end{aligned}
$$

注意到

$$
\nabla^a\left(e^\mu\right)_a = \left(e^\nu\right)^a \omega_\nu{}^\mu{}_a,
$$

且

$$
\begin{aligned}
\left(e^1\right)^a &= g^{1\mu}\left(e_\mu\right)^a \\
&= \left(e_1\right)^a, \\
\left(e^2\right)^a &= g^{2\mu}\left(e_\mu\right)^a \\
&= \left(e_2\right)^a, \\
\left(e^3\right)^a &= g^{3\mu}\left(e_\mu\right)^a \\
&= -\left(e_4\right)^a, \\
\left(e^4\right)^a &= g^{4\mu}\left(e_\mu\right)^a
\end{aligned}
$$

$$= - (e_3)^a ,$$

可知

$$\begin{aligned}
\nabla^a \left(e^1\right)_a &= \left(e^4\right)^a \omega_4{}^1{}_a \\
&= - (e_3)^a \frac{\partial P}{\partial x} \left(e^4\right)_a \\
&= 0, \\
\nabla^a \left(e^2\right)_a &= \left(e^4\right)^a \omega_4{}^2{}_a \\
&= - (e_3)^a \frac{\partial P}{\partial y} \left(e^4\right)_a \\
&= 0, \\
\nabla^a \left(e^3\right)_a &= \left(e^1\right)^a \omega_1{}^3{}_a + \left(e^2\right)^a \omega_2{}^3{}_a \\
&= (e_1)^a \frac{\partial P}{\partial x} \left(e^4\right)_a + (e_2)^a \frac{\partial P}{\partial y} \left(e^4\right)_a \\
&= 0, \\
\nabla^a \left(e^4\right)_a &= 0,
\end{aligned}$$

故

$$\begin{aligned}
\nabla^a \nabla_a P = &= \left(e^4\right)_a \nabla^a \frac{\partial P}{\partial u} + \left(e^1\right)_a \nabla^a \frac{\partial P}{\partial x} + \left(e^2\right)_a \nabla^a \frac{\partial P}{\partial y} \\
&= - (e_3)^a \nabla_a \frac{\partial P}{\partial u} + (e_1)^a \nabla_a \frac{\partial P}{\partial x} + (e_2)^a \nabla_a \frac{\partial P}{\partial y} \\
&= - \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right) \frac{\partial P}{\partial u} + \frac{\partial}{\partial x} \frac{\partial P}{\partial x} + \frac{\partial}{\partial y} \frac{\partial P}{\partial y} \\
&= - \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial z}\right) \frac{\partial^2 P}{\partial u^2} + \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \\
&= \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}.
\end{aligned}$$

第二部分

中册