Assignment 1

Gaurang Dangayach
Number Theory and Applications

May 8, 2022

Problem (Question 1). Prove that for positive integer n we have $169|3^{3n+3} - 26n - 27$.

Solution.

$$\begin{aligned}
3^{3n+3} - 26n - 27 &= 27^{n+1} - 26n - 27 \\
&= (26+1)^{n+1} - 26n - 26 - 1 \\
&= (C_0^{n+1}.26^0 + C_1^{n+1}.26^1 + C_2^{n+1}.26^2 + ... + C_{n+1}^{n+1}.26^{n+1}) - 26(n+1) - 1 \\
&= 1 + (n+1).26 + 26^2(C_2^{n+1} + ... + C_{n+1}^{n+1}.26^{n-1}) - 26(n+1) - 1 \\
&= 169.4(C_2^{n+1} + ... + C_{n+1}^{n+1}.26^{n-1}) \\
&= 169.k
\end{aligned}$$

□

Problem (Question 2). Prove that for positive integer n we have $n^2|(n+1)^n - 1$.

Solution. Using binomial expansion, we can write:

$$\begin{aligned}
(1+n)^n &= C_0^n.n^0 + C_1^n.n^1 + C_2^n.n^2 + ... + C_n^n.n^n \\
(1+n)^n - 1 &= n.n + C_2^n.n^2 + ... + C_n^n.n^n \\
(1+n)^n - 1 &= n^2(1 + +C_2^n + ... + C_n^n.n^{n-2})
\end{aligned}$$

So, we can write $(1+n)^n - 1$ as $n^2.k$ and hence it is divisible by n^2 □

Problem (Question 3). Prove that if for integers a and b we have $7|a^2 + b^2$, then $7|a$ and $7|b$.

Solution.

$$a^2 + b^2 (\mod 7) = a(\mod 7).a(\mod 7) + b(\mod 7).b(\mod 7) = 0$$

Now, consider the cases, $a(\mod 7) = 0, 1, 2, 3, 4, 5, 6$. Corresponding to these cases, $a(\mod 7).a(\mod 7) = 0, 1, 4, 2, 2, 4, 1$. Similar is true for b. We can see that only possible solution that can give us $a(\mod 7).a(\mod 7) + b(\mod 7).b(\mod 7) = 0$ is $a(\mod 7) = b(\mod 7) = 0$. □

Problem (Question 4). For numbers $2k - 1$ and $9k + 4$, find their greatest common divisor as a function of k.

Solution.

$$\begin{aligned}
\gcd(2k-1, 9k+4) &= \gcd(2k-1, k+8) \\
&= \gcd(k-9, k+8) \\
&= \gcd(k-9, 17)
\end{aligned}$$

Now, if $17|k+9$ then $\gcd(2k-1, 9k+4) = 17$ else $\gcd(2k-1, 9k+4) = 1$. □

Problem (Question 5). Find the remainder when 2^81 is divided by 17.

Solution.

$$\begin{aligned} 2^{81}(\mod 17) &= 2^{4.20+1}(\mod 17) \\ &= 16^{20}(\mod 17) + 2(\mod 17) \\ &= (-1)^{20}(\mod 17) + 2 \\ &= 1 + 2(\mod 17) \\ &= 3(\mod 17) \end{aligned}$$

□

Problem (Question 6). Prove that $2^n + 6.9^n$ is always divisible by 7 for any positive integer n.

Solution.

$$\begin{aligned} 2^n + 6.9^n(\mod 7) &= 2^n(\mod 7) + 6.9^n(\mod 7) \\ &= 2^n(\mod 7) + 6(\mod 7).9^n(\mod 7) \\ &= 2^n(\mod 7) + (-1).2^n(\mod 7) \\ &= 2^n - 2^n(\mod 7) \\ &= 0 \end{aligned}$$

□

Problem (Question 7). The two-digit integers form 19 to 92 are written consecutively to form a large integer

$$N = 192021......909192$$

Suppose that $3k$ is the highest power of 3 that is a factor of N. What is k?

Solution. Consider the summation of the digits of N:

$$\begin{aligned} 1 + 9 + 10.\sum_{i=2}^{8} i + 7.\sum_{i=1}^{9} i + 9 + 0 + 9 + 1 + 9 + 2 &= 3.9 + 4 + 17.\sum_{i=2}^{8} i + 70 \\ &= 3.9 + 4 + 17.35 + 70 \end{aligned}$$

Consider $3.9 + 4 + 17.35 + 70(\mod 3) = 0 + 1 + (-1).(-1) + 1 = 0$ Consider $3.9 + 4 + 17.35 + 70(\mod 9) = 0 + 4 + (-1).(-1) + (-2) = 3$ So, N is divisible by 3 but not by 9. So, $k = 1$

□

Problem (Question 8). Show that there are no integer solutions to $x^2 + y^2 = 10^z1$ for $z > 1$.

Solution. Let us evaluate the LHS and RHS in $\mod 4$. LHS: Consider $x = 0, 1, 2, 3(\mod 4)$ then $x^2(\mod 4) = 0, 1, 0, 1$ respectively. Similar is for y, so $x^2 + y^2(\mod 4) = 0/1/2$.
RHS:

$$\begin{aligned} 10^z - 1(\mod 4) &= 10^z(\mod 4) - 1(\mod 4) \\ &= 2^z(\mod 4) - 1(\mod 4) \end{aligned}$$

Now, as $z > 1$, 2^z is divisible by 4

$$\begin{aligned} &= 0 - 1(\mod 4) \\ &= -1(\mod 4) \end{aligned}$$

□