Assignment 2

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Number Theory and Applications

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Problem (Question 1). Solve the following system of congruences:

$$x \equiv 2 \pmod 3$$
$$x \equiv 3 \pmod 5$$
$$x \equiv 2 \pmod 7$$

Solution.

$$N = 3 \times 5 \times 7 = 105$$
$$N_1 = 5 \times 7, N_2 = 3 \times 7, N_3 = 3 \times 5$$
$$35x = 1 \pmod 3 \Rightarrow x = 2$$
$$21x = 1 \pmod 5 \Rightarrow x = 1$$
$$15x = 1 \pmod 7 \Rightarrow x = 1$$
$$X = 2 * 35 * 2 + 1 * 21 * 3 + 2 * 15 * 1 = 233$$
$$X = 23 \pmod{105}$$

□

Problem (Question 2). Solve the following system of congruences:

$$x \equiv 11 \pmod{36}$$
$$x \equiv 7 \pmod{40}$$
$$x \equiv 32 \pmod{75}$$

Solution.

$$36 = 2^2 * 3^2 = 4 * 9$$
$$40 = 5 * 2^3 = 5 * 8$$
$$75 = 3 * 5^2 = 3 * 25$$
$$x \equiv 11 \pmod 4 \Rightarrow x \equiv 3 \pmod 4$$
$$x \equiv 11 \pmod 9 \Rightarrow x \equiv 2 \pmod 9$$
$$x \equiv 7 \pmod 5 \Rightarrow x \equiv 2 \pmod 5$$
$$x \equiv 7 \pmod 8 \Rightarrow x \equiv 7 \pmod 8$$
$$x \equiv 32 \pmod{25} \Rightarrow x \equiv 7 \pmod{25}$$
$$x \equiv 32 \pmod 3 \Rightarrow x \equiv 2 \pmod 3$$

Solving these 6 equations is equivalent to solving equation number 2, 4 and 5:
$x \equiv 2 \pmod 9$
$x \equiv 7 \pmod 8$
$x \equiv 7 \pmod{25}$
Solving these we get $X = 407 \pmod{1800}$

□

Problem (Question 3). Solve the following system of congruences:

$$x^2 \equiv 1 \pmod 3$$
$$x \equiv 2 \pmod 4$$

Solution.

$$x^2 \equiv 1 \pmod 3 \Rightarrow x \equiv 1 \pmod 3) or x \equiv 2 \pmod 3$$
$$Case - 1 : x \equiv 1 \pmod 3$$
$$x \equiv 2 \pmod 4$$
$$Solve : 4 \times x \equiv 1 \pmod 3 \Rightarrow x = 1$$
$$3 \times x \equiv 1 \pmod 4 \Rightarrow x = 3$$
$$X = 1 * 4 * 1 + 2 * 3 * 3 = 22$$
$$X \equiv 10 \pmod{12}$$
$$Case - 2 : x \equiv 2 \pmod 3$$
$$x \equiv 2 \pmod 4$$
$$Solve : 4 \times x \equiv 1 \pmod 3 \Rightarrow x = 1$$
$$3 \times x \equiv 1 \pmod 4 \Rightarrow x = 3$$
$$X = 2 * 4 * 1 + 2 * 3 * 3 = 26$$
$$X \equiv 2 \pmod{12}$$

□

Problem (Question 4). For a positive integer p, define the positive integer n to be p-safe if n differs in absolute value by more than 2 from all multiples of p. For example, the set of 10-safe numbers is 3, 4, 5, 6, 7, 13, 14, 15, 16, 17, 23, ... Find the number of positive integers less than or equal to 10,000 which are simultaneously 7-safe, 11-safe, and 13-safe.

Solution. Consider the set of equations:
$x \equiv a \pmod 7$ where a = 3,4
$x \equiv b \pmod{11}$ where b = 3,4....7,8
$x \equiv c \pmod{13}$ where c = 3,4.....9,10
So, the number of possible cases = 2*6*8 = 96 and for each case we will get a unique solution (mod 1001)
So, total number of integers between 1 and 10010 which are simultaneously 7-safe, 11-safe, and 13-safe are $96 \times 10 = 960$.
Now, remove the integers greater than 10000 which fulfil the conditions (i.e. 10006, 10007).
So, the answer is 958. □

Problem (Question 5). There are N permutations $(a_1, a_2, ..., a_{30})$ of 1, 2, ... , 30 such that for $m \in \{2, 3, 5\}$, m divides $a_{n+m} - a_n$ for all integers n with $1 \leq n < n + m \leq 30$. Find the remainder when N is divided by 1000.

Solution. Each position n from 1 to 30 can be represented by triplet (i, j, k), where $i = n \pmod 2$, $j = n \pmod 3$, $k = n \pmod 5$
As L.C.M. of 2,3 and 5 is 30, each n will have a unique index and all the possible indexes will be used
Now, 1 is at position $n = (i_1, j_1, k_1)$, the number of possible positions being $2 \times 3 \times 5 = 30$. Now, 2 cannot be at a place with $i = i_1$. Also, $j \neq j_1$ and $k \neq k_1$. So, number of possible places for $2 = (2-1) \times (3-1) \times (5-1) = 8$
Similarly, for 3 we have, i=1, $j \neq j_1, j_2$ and $k \neq k_1, k_2$. So, number of possible places for 3 are $(1) \times (3 - 2) \times (5 - 2) = 3$.
Likewise, number of possible places for 4= 2, for 5 = 1. This fixes the position of integers from 6 to 30.
So, the number of possible permutations $= 30 \times 8 \times 3 \times 2 = 1440$.
$1440 \pmod{1000} = 440$. □

Problem (Question 6). Implement Chinese Remainder Theorem in code: You are given n numbers which are pairwise co-prime and corresponding remainders when these numbers are divided by some number x. You need to find minimum possible value of x that produces given remainders.

Solution.

```
#Code in Python
k=int(input("Enter the number of eqns:"))
b=[]
n=[]
m=[]
for i in range(k):
    b1,n1=input().split()
    b.append(int(b1))
    n.append(int(n1))
N=1
for i in range(k):
    N = N*n[i]
for i in range(k):
    N1 = N//n[i]
    for j in range(1,n[i]):
        if (j*N1) % n[i]==1:
            m.append(j)
            break
ans=0
for i in range(k):
    ans=ans + N//n[i]*b[i]*m[i]
ans=ans % N
print(ans)
```

∎