Linear Algebra and its Applications
Assignment 1
Due: 30 January 2024

---

1. Clearly write down all the axioms that define a field. Let $p > 1$ be any prime number. Consider the set of all residues modulo p, i.e.,

$$\mathbb{Z}_p := \{\bar{0}, \bar{1}, \ldots, \overline{p-1}\}.$$

Prove that $\mathbb{Z}_p$ is a field (make sure that you clearly demonstrate the operations and verify all the axioms). [5 points]

2. Clearly write down all the axioms that define a vector space over the given field. Let V, W be two vector spaces over $\mathbb{R}$. Show that, $\mathrm{Hom}(V, W)$, the set of all linear maps from V to W is again a vector space. [5 points]

3. For the following problems consider 2×2 matrices with real entries. [10 points]
   (a) Explicitly determine matrices A such that $AB = BA$ for every matrix B (i.e., determine the condition on the entries).
   (b) Give an example of 2 non-zero matrices A, B such that $AB = 0$.
   (c) Give an example of a non-identity and non-zero matrix A such that $A^2 = A$. Can you guess a general form? Justify your answer, briefly.

4. Denote by e_i the n-row vector with 1 in the ith place and 0 everywhere else. Denote by P^{ij} the matrix obtained by exchanging rows i and j of the identity matrix. Denote by $E(c, i)$ the matrix obtained by multiplying the ith row of the indentity matrix by the scalar c. In each of the following case explain (in general terms) what the matrix product will be, assuming all matrices are square of same size. For example, $(e_1^T e_1)A$ is the matrix whose first row is that of A's and all the entries are zero. [10 points]
   (a) $(\mathrm{Id} + e_2^T e_3)A$.
   (b) $E(5, i)AP^{i5}$.
   (c) $(P^{ij}AP^{ij})^T$.
   (d) $(\mathrm{Id} + 2e_1^T e_3)(\mathrm{Id} - 2e_1^T e_3)A$.
   (e) $(\mathrm{Id} + e_i^T e_j)AP^{ji}$.

5. In each of the following situation determine all possible value(s) of the determinant (of the given square matrix): [10 points]
   (a) A is such that $A^k = \mathbf{0}$ for some $k > 1$ (here A^k is the k-fold product of A with itself.
   (b) A is such that $A^T A = \mathrm{Id}$.
   (c) Express $\det(-A)$ in terms of $\det(A)$.
   (d) A is such that $A^T = -A$ then what is $\det(A)$?
   (e) A is such that $A^2 = A$.