Mathematical Methods - Analysis
Quiz 1
01 September 2023

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1. Determine the truth value of the following statements, write 'T' or 'F' at the end of the sentence.

   (a) Let f be a continuous, real valued function defined on an open interval (a, b). If $f(x) \neq 0$ for any $x \in (a, b)$ then its sign doesn't change. T ✓

   (b) A function f is non-decreasing on an interval if and only if $f'(x) \geq 0$ for all x in that interval. F ✗

   (c) Let f be a twice differentiable function such that f' changes the sign at $x = a$. Then f has a relative extrimum at a. F ✗

   (d) Let f be a twice differentiable function on the domain of its definition. Suppose f is concave down on (a, b) and convex up on (b, c) then $f''(b)$ is necessarily defined. T ✗

2. Give an example of a function f such that $f'(x) < 0$ for all x where f is defined but f is not decreasing. $f(x) = x^2 \cos \frac{1}{x^2}$ on $(0, \varepsilon)$ for very small ε ✗

3. Consider the differentiable function

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin(\frac{\pi}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

   (a) The derivative $f'(0) =$ doesn't exist ✗

   (b) The function is strictly increasing near 0: T/F? F ✓

$f(x) =$

$(x-3)^3$

$3(x-3)^2$

4. Consider the function
$$f(x) = \frac{1}{1 + e^{-x}}.$$

   (a) The function is monotonic increasing on: $(-\infty, 0) \cup (0, \infty)$ ✗

   (b) The function is monotonic decreasing on: ϕ ✓

   (c) The non-stationary point of inflection is: $(0, \frac{1}{2})$ ✓

$ab' + ba'$

$(x-2)^2$

$2(x-2)$

$- \quad +$

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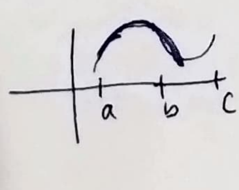

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$f'(0) = \frac{1}{2}$

$f(x) = \left\{ \frac{x}{2} + x^2 \sin\left(\frac{\pi}{x}\right) \right.$

$f'(x) = \frac{1}{2} + 2x \sin\left(\frac{\pi}{x}\right) - \cancel{x^2} \cos \frac{\pi}{\cancel{x}} \cdot \frac{\pi}{\cancel{x^2}}$

Mathematical Methods - Analysis
Mid-semester Exam
05 October 2023

Note: You may use a (non-graphing) calculator.

1. Which of the following functions are differentiable at $x = 0$? [5 points]

   (a) $f(x) = x|x|$.

   (b) $g(x) = x\sqrt{|x|}$.

   (c) $h(x) = x + |x|$.

   (d) $k(x) = x^2 \sin(\frac{\pi}{x})$ for $x \neq 0$ and $k(0) = 0$.

   (e) $l(x) = x \sin(\frac{\pi}{x})$ for $x \neq 0$ and $l(0) = 0$.

2. For which values of a, b is each of the function below differentiable? Draw (a rough sketch of) the graph of the function for the values of a, b you found. [10 points]

   (a)
$$f(x) = \begin{cases} ax + b, & \text{for } x < 0, \\ x - x^2, & \text{for } x \geq 0. \end{cases}$$

   (b)
$$f(x) = \begin{cases} \cos x, & \text{for } x \leq \frac{\pi}{4}, \\ ax + b, & \text{for } x > \frac{\pi}{4}. \end{cases}$$

3. For each of the function below find where f is increasing and where it is decreasing. Also describe the convexity/ cocavity of f using intervals, and determine if there are any inflection points. [15 points]

   (a) $f(x) = x^3 - x$.

   (b) $f(x) = \dfrac{1}{1+x^2}$.

   (c) $f(x) = x + \dfrac{1}{x}$.

4. Solve the following single variable max/ min problems. [15 points]

   (a) It is estimated that the operating rate of a major manufacturer's factories over a certain 365-day period is given by

$$f(t) = 100 + \frac{800t}{t^2 + 90000} \quad 0 \leq t \leq 365$$

   percent. Determine the day on which the operating rate is maximized.

(b) Suppose you are traveling on a straight road parallel to a beach. Moreover, your destination is a point A further on the beach and B is the point on the road closest to it. Distance between A and B is b. Let v be your speed on the road and w be your speed in the sand which is less than v. Right now you are at a point D on the road, which is at a distance a from B. At what point C should you turn off from the road and head across the sand in order to minimize your travel time to A?

5. For each of the multivariate function below find its quadratic approximation near the given point. [15 points]

Recall: The Tayolr polynomial of degree 2 is:

$$f(a+h) = f(a) + \langle \nabla f(a), h \rangle + \frac{1}{2}\langle \mathcal{H}f(a)h, h \rangle.$$

(a) $f(x, y) = x^2 y + y^2$ and $a = (1, 3)$.

(b) $f(x, y) = e^{x^2+y}$ and $a = (0, 0)$.

(c) $f(x, y) = \sqrt{1 + x + 2y}$ and $a = (0, 0)$.

Mathematical Methods - Analysis

End-semester Exam

27 November 2023

Note: You may use a (non-graphing) calculator.

1. Short answer questions [10 points]

   (a) What is the domain of the function $f(x,y) = x\ln(y^2 - x)$?

   (b) What is the domain of the function $g(x,y) = \sqrt{4 - x^2 - y^2}$?

   (c) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if z is defined as an implcit function of x, y by

   $$x^3 + y^3 + z^3 + 6xyz = 1.$$

   (d) Let $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

   (e) Let $f(x,y,z) = \sin(xyz)$, find the directional derivative at $(1,\pi,\pi)$ in the direction of $(1,3,2)$.

   (f) Find the Hessian of $f(x,y,z) = e^{x.y.z}$.

   (g) Is the one-variable, real valued function $f(x) = e^{x-x^2+1}$ convex? Yes or No?

   (h) Evaluate the integral $\displaystyle\int_1^4 (x-1)^{-\frac{2}{3}}dx$.

   (i) Find the one-variable function f given that the slope of the tangent to the graph of f at any point $(x, f(x))$ is (2 points)

   $$f'(x) = xe^{-3x},$$

   and that the graph passes through the point $(0,0)$.

2. Find the local extrema of the following functions [10 points]

   (a) $f(x,y) = (x+y)e^{-xy}$.

   (b) $f(x,y) = x^3y^2(1-x-y)$ for all $x \geq 0, y \geq 0$.

3. Answer the following [15 points]

   (a) For $|x| < 2$, approximate the function $f(x) = \dfrac{1}{x-2}$ by a degree 4 Taylor polynomial and find a good estimate for the error term.

   (b) Find the minima of the function $x^2 + y^2 + z^2$ subject to the constraint $x^2 + 2y^2 - z^2 - 1 = 0$.

   (c) Let $S = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and let $f(x,y) = x^3 + xy$. Find the maximum and the minimum of f on S.

ke^{-kx}

4. Answer the following [15 points]

   (a) During the morning rush hour, local trains on a line every 20 minutes. You arrive at random at the station during rush hour and find no train at the platform. Assuming that the trains run on schedule, use an appropriate uniform density function to find the probability that you will have to wait at least 8 minutes for your train.

   (b) The number of almond pieces in some comercially produced barfi is an exponentially distributed random vairable x with the expected value 10 (pieces).

      i. Find the probability distribution function.

      ii. What is the probability that the barfi has at most 8 almond pieces?

      iii. What is the probability that the barfi has at least 15 almond pieces?

5. Consider the following function [10 points]

$$f(x, y) := (1-x)^2 + 10(y - x^2)^2$$

defined on $\mathbb{R}^2$. Find the global minimum, given that it exists.

Recall the steepest descent method for iteraively locating the global minimum. If p_k is the point after k iterations then

$$p_{k+1} = p_k + \alpha_k d_k,$$

where α_k is the step size and d_k is the descent direction.

Let the initial point $p_0 = (x_0, y_0)$ be $(0, 0)$. Perform 2 iterations of the algorithm with $\alpha_0 = 0.5$ and $\alpha_1 = 0.1$ to find p_1, p_2.