Linear Algebra and its Applications
Mid-semester Examination
26/02/2024

(**Note:** You may use a standalone calculator, not your phone)

1. Find LU factorization of the following matrix: [10 points]

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix}$$

2. Given that the following matrix is symmetric, positive definite find its Cholesky factorization: [10 points]

$$A = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 5 & 6 \\ 4 & 6 & 9 \end{bmatrix}$$

3. Let $w = [0, \dots, 0, w_{k+1}, \dots, w_n]^T$ be a column vector in $\mathbb{R}^n$ such that $0 \leq k \leq n$ and $w_{k+1} \neq 0$. Denote by e_k the k-th standard unit vector with 1 in k-th position and 0 everywhere else. Consider the following $n \times n$ matrix:

$$M_k^w := I_n - w e_k^T.$$

Now answer the following questions: [10 points]

(a) Find the exact operation count needed to multiply two $n \times n$ matrices.

(b) Given an arbitrary $n \times n$ matrix C, find an algorithm to compute the product $M_k^w C$ whose operation count is significantly less (say, $O(n^2)$). Clearly write the algorithm (as a pseudocode) and show the operation count calculations.

4. Let $A = \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix}$ be a 2×2 matrix such that ϵ is a very small real number. Answer the following questions: [15 marks]

(a) Find the condition number $\kappa_\infty(A)$ by explicitly calculating the inverse.

(b) Find the LU decomposition of A using GE *without pivoting.*

(c) What are the ∞-condition numbers of the factors L, U?

(d) Assume $1 - \epsilon^{-1} = -\epsilon^{-1}$ in U and find $\Delta A = LU - A$. What is the condition number of ΔA?

(e) Find the LU decomposition after permuting rows of A.

(f) Assume $1 - \epsilon = 1$ in U and find $\Delta A = LU - A$. What is the condition number of ΔA?

(g) In which of the two scenarios above, the problem of solving $Ax = b$ (for any b and using GE, forward, backward substitutions etc.) is well-conditioned? In which method a small rounding error leads to a large backward error? Explain.

5. A floating-point number representation system is given as follows:

$$\mathcal{F} := \{0\} \cup \{\pm \mathrm{d}_0.\mathrm{d}_1\mathrm{d}_2 \times 10^{e} \mid 1 \le \mathrm{d}_0 \le 9; 0 \le \mathrm{d}_1, \mathrm{d}_2 \le 9; -9 \le e \le 9\}.$$

Now answer the following questions. [15 points]

(a) The number of normalized floating point numbers in $\mathcal{F}$ is:

(b) The smallest positive (nonzero) number in $\mathcal{F}$ is:

(c) The largest positive number in $\mathcal{F}$ is:

(d) The machine epsilon for $\mathcal{F}$ is:

(e) The relative error in representing 0.9995 ~~0.995~~ by an element in $\mathcal{F}$ is

(f) The smallest and largest possible gaps between any two consecutive elements of $\mathcal{F}$ are:

(g) The element of $\mathcal{F}$ that best represents the real number π is:

(h) The most accurate representation of

$$(1.23 \times 10^6) + (4.56 \times 10^4)$$

in $\mathcal{F}$ is:

(i) The most accurate representation of

$$(1.23 \times 10^1) \times (4.56 \times 10^2)$$

in $\mathcal{F}$ is:

(j) Let $x = 1.24 \times 10^1, y = 1.23 \times 10^0, z = 1.00 \times 10^{-3}$. Calculate the following:

$$\mathrm{fl}(\mathrm{fl}(x+y)+z).$$
$$\mathrm{fl}(x+\mathrm{fl}(y+z)).$$

MA41209 (Aug-Nov) 3:1 Linear Algebra
Mid-term Exam 2022 [1]

Duration: 120 minutes (09:00 AM - 11:00 AM).
Total points: 30 (30% of the final grade).

1. Each of the questions below carries one and a half points. Write **T** for a true statement and **F** for a false statement. $1\frac{1}{2} \times 4$

   1.5×4

   6.0

   (a) Let $\mathbf{V}$ be the vector space of n–square matrices over the field $\mathbb{R}$ of real numbers. Let $D : \mathbf{V} \to \mathbb{R}$ be the determinant function, i.e, $D(\mathbf{A}) = \det(\mathbf{A})$. Then D is a linear functional on $\mathbf{V}$.

   (b) There is no nonsingular linear map from $T : \mathbb{R}^5 \to \mathbb{R}^4$.

   (c) Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ square matrices over the field $\mathbb{R}$, and assume that they have the same characteristic polynomial. Then, $\mathbf{A}$ and $\mathbf{B}$ are similar.

   (d) Let T be an invertible linear operator on a vector space $\mathbf{V}$ of finite dimension over field $\mathbb{F}$. Then the constant term of the characteristic polynomial of T equals to zero.

2. Each of the following questions carries two and a half points. $2\frac{1}{2}$.

   (a) Suppose the non-zero vectors $\{v_1, v_2, \cdots, v_m\}$ are linearly dependent. Prove that one of them is a linear combination of the preceding vectors.

   (b) Let T be a linear transformation from a vector space $\mathbf{V}$ into a vector space $\mathbf{W}$. Then T is non-singular if and only if T carries each linearly independent subset of $\mathbf{V}$ onto a linearly independent subset of $\mathbf{W}$.

   (c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation which rotates the vectors counter-clockwise by $90°$. Find the matrix representation of T with respect to the basis $\mathbb{B} = \{(1,3),(2,5)\}$ of $\mathbb{R}^2$.

   (d) Find two linear operators T and U on $\mathbb{R}^2$ such that $T(U(x,y)) \neq U(T(x,y))$.

   (e) Consider the following basis of $\mathbb{R}^2$: $\{v_1 = (2,1), v_2 = (3,1)\}$. Find the dual basis $\{\phi_1, \phi_2\}$.

   (f) Let $\mathbf{A}$ be an $n \times n$ square matrix with entries from the field $\mathbb{R}$. Show that $\mathbf{A}$ is a zero of some non-zero polynomial. (Caution! Do not apply Cayley-Hamilton Theorem here)

   (g) Find the minimal polynomial $m(t)$ of matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}.$$

   (h) Let $T : \mathbf{V} \to \mathbf{V}$ be a linear mapping over field $\mathbb{F}$. Suppose, for $v \in \mathbf{V}$, $T^k(v) = 0$ but $T^{k-1}(v) \neq 0$. Then prove the followings:

[1] Conducted on: September 22, 2022.

Frobenius Pdh lia mene..

Really.. your notes are exceptionally good!

(I) The set $S = \{v, T(v), \cdots, T^{k-1}(v)\}$ is linearly independent.

(II) The subspace $\mathbf{W}$ generated by S is T-invariant.

3. Suppose that $\mathbf{V}$ and $\mathbf{W}$ are finite dimensional vector spaces over field $\mathbb{F}$, and $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation. Then there exists a basis of $\mathbf{V}$ and a basis of $\mathbf{W}$ such that the matrix representation of T has the form

$$\begin{pmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathbf{I}$ is the r−square identity matrix and r is the rank of T. [4 points]

OR

4. Suppose $\mathbf{V}$ and $\mathbf{W}$ are finite dimension vector spaces over field $\mathbb{F}$ and suppose $T : \mathbf{V} \to \mathbf{W}$ is linear transformation. Prove that rank(T) = rank(T^t), where $T^t : \mathbf{W}^* \to \mathbf{V}^*$ is transpose linear mapping of T. [4 points]

Linear Algebra and its Applications
Final Examination
29/04/2024

(**Note**: You may use a calculator, but not your phone. For Questions 1 to 4, no justification is needed.)

1. In each of the following provide an appropriate example of a 2×2 matrix. [5 points]

   (a) A cannot be diagonalized and it is invertible.
   (b) A cannot be diagonalized but it is singular.
   (c) A is diagonalizable and it is non-singular. ✓
   (d) A has orthogonal columns but it is not invertible. ✓
   (e) A has orthogonal columns and it is diagonalizable. ✓

2. Determine the truth value of the following statements. Just write **T** or **F**. [5 points]

   (a) If U is a matrix with orthonormal columns then $\mathsf{U}\mathsf{U}^{\mathsf{T}} = \mathrm{I}$.
   (b) A square matrix with orthonormal columns has real eigenvalues.
   (c) The singular values of an orthonormal matrix are all equal to 1.
   (d) The eigenvalues of an orthogonal matrix need not be all 1.
   (e) For any matrix A, the eigenvalues of $A^{\mathsf{T}}A$ are positive. F

3. Consider the following SVD factorization of a movie-ratings matrix A:

$$\begin{array}{cc} & \begin{array}{cccc} d1 & d2 & d3 & c1 \end{array} \\ \begin{array}{c} U_1 \\ U_2 \\ U_3 \end{array} & \begin{bmatrix} 2 & 2 & 2 & 0 \\ 4 & 4 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \end{array} = \overset{U}{\begin{bmatrix} 0.44 & 0 \\ 0.99 & 0 \\ 0 & 1 \end{bmatrix}} \begin{array}{c} \begin{array}{cc} g1 & g2 \end{array} \\ \begin{bmatrix} 7.74 & 0 \\ 0 & 3 \end{bmatrix} \end{array} \overset{V}{\begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}.$$

$3 \times 2 \quad 2 \times 3 \times 3 \times 1$

Here the rows of A are the users and the columns are the movies labeled *Drama1*, *Drama2*, *Drama3*, *Comedy1*. The matrix U connects users to movie geners (present as latent features of A), the matrix V connects movies to geners and finally, Σ describes the strength of each genere. In this synthetic example it is clear that the number of geners is equal to the rank of the rating matrix. Answer the following questions in order to use the above matrix factorization for recommending movies. [10 points]

   (a) Suppose a new user has watched only *Drama2* movie and rated it 3. Determine the vector $x \in \mathbb{R}^4$ which encodes this information and can be used to determine recommendations.
   (b) Which of the above matrix has to be used to map the ratings vector x to the 2-dimensional "genere space"?
   (c) Determine the representaiton of new user's ratings in the genere space.
   (d) Appropriately map the above representation back into the "movie space" in oder to interprete the genere each movie partakes. Conclude by recommending movies to the new user.
   (e) What is the potential use of the map given by the matrix $\mathsf{U}\mathsf{U}^{\mathsf{T}}$?

4. Let A be an $n \times n$, real matrix and $\mathbf{b} \in \mathbb{R}^n$ be a nonzero vector. Consider the following algorithm and answer the questions based on it: [10 points]

$q_1 = b/||b||_2$
for $k = 1, 2, \ldots$ do
  $v = Aq_k$
  for $j = 1$ to k do
    $h_{jk} = \langle q_j, v\rangle$
    $v = v - h_{jk}q_j$
  end for
  $h_{k+1,k} = ||v||_2$
  $q_{k+1} = v/h_{k+1,k}$
end for

(a) The vectors $\{q_1, \ldots, q_k\}$ form an orthonormal basis of the subspace spanned by which vectors? (Hint: try to answer in terms of A, b).

(b) For each k, denote by Q_k the $n \times k$ matrix whose columnns are $q_1, \ldots, q_k$ and by $H_{k+1,k}$ the $(k+1) \times k$ matrix whose entries are h_{jk}'s. Express the above algorithm in the matrix language using A, Q_k and $H_{k+1,k}$.

(c) What is $Q_k^T A Q_k$ for each $k = 1, \ldots, n$?

(d) Which two matrices have the exact same eigenvalues?

(e) If A is symmetric then each H_k is ... and .... Fill in the blanks.

5. Use QR factorization (Gram-Schmiddt or Householder) method to find x that minimizes $||Ax - b||^2$, where [5 points]

$$A = \begin{bmatrix} 3 & -6 \\ 4 & -8 \\ 0 & 1 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}.$$

6. Find the (approximate) dominant eigenvector of the matrix [10 points]

$$A = \begin{bmatrix} 4 & 5 \\ 6 & 5 \end{bmatrix}$$

using power iteration. Start with the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and perform 4 iterations, with scaling. Use this approximate eigenvector to calculate the dominant eigenvalue.

7. Consider the matrix [15 points]

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

(a) Find the full singular value decomposition of A.

(b) Write down an orthonormal basis for the following four fundamental spaces: row space of A, column space of A, null space of A and null space of A^T.

(c) What is the pseudoinverse of A?

$$A^T = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$