PBSR Quiz. September 2013
Answer all questions. Each question carries 10 marks.
Time: 2 hours.

Q1:
There are three tables, each with three drawers. Table 1 has a red ball in two drawers and blue in one drawer. Table 2 has a blue ball in each drawer. Table 3 has a red ball in one drawer and a blue ball in the other two drawers. A table is chosen at random, then a drawer is chosen at random from that table. Find the conditional probability that Table 1 is chosen, given that a blue ball is drawn.

(Chosen at random is to be interpreted as chosen uniformly at random, i.e. all choices are)

Q2:
A fair dice is tossed and if it turns up k, a fair coin is tossed k times. Let X denote the number that turns up on throw of dice and let Y denote the total number of head so observed. Compute
(i) $P(X = 4|Y = 3)$
(ii) $P(X \leq 3|Y = 1)$
(iii) $P(X \geq 2|Y = 0)$

Q3:
Let $X_1, X_2, ..., X_n$ be an i.i.d. sequence of discrete random variables, each with geometric distribution, with parameter p. Let Z be the minimum of these n variables. Find $P(Z \leq t), t \geq 0$ as a function of p.

Q4:
Let X Uniform 1,2,...,n be independent of Y Uniform 1,2,...,n. Let $Z = \max(X, Y)$ and $W = \min(X, Y)$.
(i) Find the joint distribution of (Z,W), i.e. $P(Z \leq s, W \leq t)$ for s, t between 1 and n.
(ii) Find $E[Z|W]$.

$f_x(x)$

PBSR Quiz 2. 21 September 2013
Answer any 4 questions. If you answer all 5, the first four will be counted for your marks.
Each question carries 10 marks.
Time: 2 hours.

~~Q1~~:
Suppose A and B are mutually exclusive events with $P(A) = 0.2$ and $P(B) = 0.3$. Let C be an event such that A and C are independent, B and C are independent and $P(C) = P(A) + P(B)$. Find (i) $P(A \cup C)$, (ii) $P(A \cup B \cup C)$, (iii) $P((B \cup C) \mid (A \cup C))$ and (iv) $P((A \cup C) \mid B)$.

~~Q2~~:
Let X_1, X_2 be independent random variables with GEOMETRIC DISTRIBUTION with parameter p respectively. Let $Y = \min(X_1, X_2)$. Show that Y also has GEOMETRIC DISTRIBUTION with parameter $\tilde{p}$. Compute $\tilde{p}$ when $p = 0.25$

~~Q3~~:
Let X be a random variable with Normal distribution, mean μ variance σ^2. Thus density f of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\tfrac{1}{2\sigma^2}(x-\mu)^2\Big).$$

Let $Y = X^2$. Find the distribution function H and the density h of Y.

~~Q4~~:
Let X, Y be independent random variables with common density given by $f(u) = \lambda \exp(-\lambda u)$. Compute $P(X \le tY)$, where t is a real number, $t > 0$. $\lambda, u > 0$

~~Q5~~:
Let X, Y be random variables with joint density $f(x, y)$ given by

$$f(x,y) = \begin{cases} \frac{1}{8} & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute the marginal densities of X (say g) and Y (say h) and the conditional density $f_{X|Y=0.5}$ of X given $Y = 0.5$.

5. Each integer is colored with exactly one of three possible colors - black, red or white - satisfying the following two rules: the negative of a black number must be colored white, and the sum of two white numbers (not necessarily distinct) must be colored black.

(a) Show that the negative of a white number must be colored black and the sum of two black numbers must be colored white.

(b) Determine all possible colorings of the integers that satisfy these rules.

Sol^n

Mid-Sem PBSR 2023

Answer any 5 questions. **Time: 3 hours.**

Each question carries 20 marks. If you answer more than 5, the first 5 answers will be taken.

1. Let X be a discrete random variable which has $N = \{1, 2, 3, ...\}$ as its range. Suppose that X has the memoryless property: for all positive integers m and n, one has

$$P(X > n + m | X > n) = P(X > m).$$

Prove that X must be a geometric random variable, *i.e.* $P(X = j) = p(1-p)^{j-1}, \quad j \in \{1, 2, 3, \ldots\}$.

2. Let X, Y be independent random variables with Poisson(λ) distribution:

$$P(X = j) = P(Y = j) = \frac{1}{j!}\lambda^j \exp(-\lambda), \quad j \in \{0, 1, 2, 3, \ldots\}.$$

Let $Z = X + Y$.

(i) Obtain $\mathbb{E}(X)$, Var(X), $\mathbb{E}(Z)$, Var(Z), Co-Var(X, Z).

(ii) What is $P[X = k | Z = 10]$, for $k \in \{0, 1, 2, 3, \ldots\}$?

3. A standard light bulb has an average lifetime of four years with a standard deviation of one year. A Super D-Lux lightbulb has an average lifetime of eight years with a standard deviation of three years. A box contains many bulbs – 75% of which are standard bulbs and 25% of which are Super D-Lux bulbs. A bulb is selected at random from the box. What are the average and standard deviation of the lifetime of the selected bulb?.

4. Let X, Y have Uniform$(1, 2, ..., n)$ distribution *i.e.* $P(X = j) = P(Y = j) = \frac{1}{n}, 1 \le j \le n$. Suppose X, Y are independent. Let $Z = \max(X, Y)$ and $W = \min(X, Y)$.

(i) Find the joint distribution of (Z, W).

(ii) Find $\mathbb{E}[Z|W]$.

5. Let X, Y be random variables taking value in $[0, 1]$. Suppose the joint density $f(u, v)$ of X, Y is given by:

$$f(u, v) = cuv^2 \text{ for } 0 \le u \le v \le 1 \text{ of } X, Y \text{ and } f(u, v) = 0 \text{ otherwise.}$$

where $c > 0$ is a suitably chosen constant so that $\int_0^1 \int_0^1 f(u, v)\, dv\, du = 1$.
Find c, $E(X)$, $E(Y)$, Var(X), Var(Y) and Co-Var(X, Y). Also, obtain the Correlation between X and Y.

6. Suppose Y is uniformly distributed on (0, 1), and suppose for $0 < y < 1$ the conditional density of X given $Y = y$ is given by

$$g(x; y) = \frac{2x}{y^2} \quad \text{if } 0 < x < y$$

and $g(x; y) = 0$ otherwise.
(i) Compute the joint p.d.f. of (X, Y) and the marginal density of X.
(ii) Compute the conditional expectation and conditional variance of X given that $Y = y$, for $0 < y < 1$.

7. Suppose X, Y are independent random variables with normal distribution, with $\mathbb{E}(X) = 0$, $\mathbb{E}(Y) = 5$, Var(X)=9, Var(X)=4. Let

$$U = XY, \quad V = X + Y - U.$$

Find mean and variances of U, V. Obtain the correlation coefficients of (i) U, V; (ii) Y, U; (iii) X, V.

Mid-Sem PBSR 2023 (Nov 1 2023)

Answer any 5 questions. **Time: 2 hours 30 minutes.**

Each question carries 20 marks. If you answer more than 5, the first 5 answers will be taken.

1. An urn initially contains 11 white and 9 red balls. Initially a ball is selected at random from the urn. If it is white, it is replaced by 6 red balls (making the total no of balls 25) while if the first ball is red, it is replaced by 5 white balls, taking the total no of balls to 24. A second ball is drawn at random from the urn and its color is noted.
(i) Compute the probability that of the 2 balls selected in the 2 draws, exactly 1 ball is white.
(ii) Compute the conditional probability that both balls are white given that at least one chosen ball is white.

2. Let A, B, C be events such that A, B are independent as well as A, C are independent with $P(A) = 0.4$, $P(B) = 0.25$, $P(C) = 0.3$ and $P(B \cap C) = 0.25$. Find ~~(i)~~ $P(A \cup B)$, $\cancel{P}(B \cup C)$ and $\cancel{P}(A \cup C)$. (ii) Let $D = B \cup C$. What is $\cancel{P(X)}$? Are A and D independent? Explain your answer.
$P(D)$ $P(A \cup B \cup C)$

3. Let X, Y have Geometric distribution *i.e.* $P(X = j) = P(Y = j) = p(1-p)^{j-1}$, $j \in \{1, 2, 3, \ldots\}$. Suppose X, Y are independent. Let $Z = \max(X, Y)$ and $W = \min(X, Y)$.

   (i) Find the joint distribution of (Z, W).

   (ii) What is $\mathbb{P}(W = 2 | Z = 4)$?

4. Let X, Y be random variables taking value in $[0, 1]$. Suppose the joint density $f(u, v)$ of X, Y is given by:
$$f(u, v) = c(u + v) \text{ for } 0 \leq u \leq 1,\ 0 \leq v \leq 1 \text{ and } f(u, v) = 0 \text{ otherwise.}$$
where $c > 0$ is a suitably chosen constant so that $\int_0^1 \int_0^1 f(u, v)\, dv\, du = 1$.
Find ~~(i)~~ c, ~~(ii)~~ the marginal densities of X, Y and (iii) the Correlation coefficient between X and Y.

5. Let $X_1, X_2, \ldots, X_6$ be iid with $P(X_j \leq a) = 0$ if $a \leq 0$; $P(X_j \leq a) = a$ if $0 < a \leq 1$; $P(X_j \leq a) = 1$ if $1 < a$.
Let
$$Y = \min_{\{1 \leq j \leq 6\}} X_j, \quad Z = \max_{\{1 \leq j \leq 6\}} X_j.$$
Find: for $0 < a < b < 1$, (i) $P(Y > a, Z \leq b)$, (ii) $P(Y \leq a, Z \leq b)$, (iii) joint density of (Y, Z).

6. Suppose X, Y are (0, 1) valued random variables with joint density
$$f(x, y) = 2 \quad \text{if } 0 < y < x < 1.$$
and $f(x, y) = 0$ otherwise.
(i) Compute the marginal density of X and the marginal density of Y.
(ii) Compute the conditional density $h(y; x)$ of Y given $X = a$.
(iii) Compute the conditional expectation and conditional variance of Y given that $X = a$, for $0 < a < 1$.

7. Suppose X, Y, Z, W are independent random variables, each with normal distribution, with $\mathbb{E}(X) = -3$, $\mathbb{E}(Y) = 2$, $\mathbb{E}(W) = 4$, $\mathbb{E}(Z) = -5$, Var(X)=9, Var(Y)=4, Var(W)=25, Var(Z)=16. Let
$$U = 2X + 3Y - 4W - 6Z, \quad V = 3X - 4Y - W + 3Z.$$
Find mean and variances of U, V and the correlation coefficient between (i) U, V, (ii) U, W and (iii) Z, V.

End-Sem PBSR 2023

Answer any 5 questions. Each question carries 20 marks. The first 5 answers will be counted, if you answer more.

I suggest that you copy the question and then start answering. Chances of missing part of the question, or misreading will be minimal. If you copy the question wrongly, you will NOT get partial credit.

1. Let X, Y, W be independent random variables, $P(X = 0) = \frac{1}{4}$, $P(X = 1) = \frac{3}{4}$, $P(Y = 0) = \frac{1}{5}$, $P(Y = 1) = \frac{4}{5}$, $P(W = 0) = \frac{1}{3}$, $P(W = 1) = \frac{2}{3}$. Let $Z = WX + (1 - W)Y$. Find (i) $P(Z = 0)$ and $P(Z = 1)$. (ii) Correlation between Y and Z. (iii) Correlation between W and Z. (iv) $P(Z = 1|X = 1)$ and $P(Z = 1|Y = 1)$. (v) $P(W = 1|Z = 0)$ and $P(X = 1|Z = 0)$. (vi) $P(W = 0|Z = 1)$ and $P(Y = 0|Z = 1)$.

2. Let X, Y be random variables taking values in $[0, \infty)$. Suppose the joint density $f(u, v)$ of X, Y is given by $f(u, v) = c(u + v)$ for $0 \le u \le 1$, $0 \le v \le 2$ of X, Y and $f(u, v) = 0$ otherwise. Compute (i) c. (ii) The correlation between X and Y. (iii) For $0 \le a \le 1$, obtain $\mathbb{E}(Y \mid X = a)$. (iv) For $0 \le a \le 1$, $\mathrm{Var}(Y \mid X = a)$.
(part (i) carries 2 points, part (ii), (iii) and (iv) each carry 6 points each).

3. Let X and Y be independent random variables each with Normal distribution with parameters (μ, σ^2). Let U and V be defined by $U = X + Y$ and $V = X - Y$. Find $\mathbb{E}(U)$, $\mathbb{E}(V)$, $\mathrm{Var}(U)$, $\mathrm{Var}(V)$, Co-Var(X, U), Co-Var(X, V), Co-Var(Y, U), Co-Var(Y, U).
(i) Show that (U, V) has bivariate normal distribution. (ii) Are U and V are independent? Explain your answer.

4. Let $X_1, X_2, \ldots, X_n$ be iid with density $f(x; \theta) = cx^2$, if $0 < x < \theta$ and equal to 0 otherwise (where the parameter θ satisfies $0 < \theta < \infty$.) Obtain (i) The constant c; (ii) $\mathbb{E}(X)$; (iii) $\mathrm{Var}(X)$. Find the method of moments estimator as well as MLE of θ based on $X_1, X_2, \ldots, X_n$.
(part (i) carries 2 points, part (ii), (iii) each carry 4 points each. The last part carries 10 points.).

5. A politician believes that his newly announced policy is equally liked by Rural and Urban population in her constituency. A consultant is engaged to undertake a survey - and it is found that out of 1200 people chosen randomly out of the Rural population, 750 respondents support the policy while out of 800 Urban respondents chosen randomly, 450 support the policy.

   As we can see, it appears that the policy is supported more by Rural folks than Urban folks. Does this data give sufficient statistical evidence to dismiss the politicians belief? Use $\alpha = 0.01$ for the appropriate test based on chi-square distribution.
   `(qchisq(0.995,1)= 7.879439, qchisq(0.99,1)=6.634897, qchisq(0.95,1)=3.841459)`

6. A client approaches a consultant with the following problem: There is an experiment whose outcomehas distribution $N(\mu, \sigma^2)$ and it is known that $\sigma^2 = 25$. It is required to test the Null Hypothesis $H_0 : \mu = 4$ versus the alternative $H_1 : \mu = 2$ with size (or Type I error) $\alpha = 0.00135$ and also Type II error $\beta = 0.00135$. Find the smallest n such that their objective can be achieved.
(You may use that `qnorm(0.00135)=-3, qnorm(1-0.00135)=3`)

7. (Each part below carries 10 points)

   (i) Let $X_1, X_2, \ldots, X_n$ be i.i.d. with common distribution being $U(-2a, 3a)$, $a > 0$. Find the method of moments estimator T_1 and the MLE T_2 for the parameter a. Suppose for $n = 9$, the observations on $X_1, X_2, \ldots, X_9$ are 7.74, -6.72, -2.87, 9.72, 2.52, 4.98, -3.92, 3.45, -2.74. Compute T_1, T_2 for this data.

   (ii) Let X denote the (percentage of) alcohol content in the breath and Y denote the (percentage of) alcohol content in the blood. It is believed that the two are related by the model

   $$Y_i = \alpha + \beta X_i + \epsilon_i$$

   where $\{\epsilon_i : i \ge 1\}$ are iid Normal with mean 0 and variance σ^2. It is desired to estimate α, β for the regression model from where data (X_i, Y_i), $1 \le i \le m$ has been observed. A team obtained data on (X_i, Y_i), $1 \le i \le 500$ chosen subjects and it was observed that the estimated means were : $\hat{\mu}_x = 5$, $\hat{\mu}_y = 7$, the estimated variances were $s_x^2 = 2$, $s_y^2 = 3$ and the correlation $r_{x,y} = 0.85$. Find the least square estimates $\hat{\alpha}, \hat{\beta}$ of α, β.