Assignment 2: Solutions to the Questions

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Question 1

Given the function:

$$f(x,y,z) = \begin{cases} k\,x\,y\,z^2, & \text{if } 0 < x, y < 1 \text{ and } 0 < z < 3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Normalization

To ensure that $f(x,y,z)$ is a valid probability density function, we require:

$$\int_0^3 \int_0^1 \int_0^1 f(x,y,z)\,dx\,dy\,dz = 1.$$

Thus,

$$\begin{aligned} \int_0^3 \int_0^1 \int_0^1 k\,x\,y\,z^2\,dx\,dy\,dz &= k \int_0^3 z^2\,dz \int_0^1 x\,dx \int_0^1 y\,dy \\ &= k\left(\frac{3^3}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ &= k \cdot 9 \cdot \frac{1}{4} = \frac{9k}{4} = 1. \end{aligned}$$

Hence,

$$k = \frac{4}{9}.$$

(b) Expectation and Marginal Density of X

The marginal density function of X is found by integrating out y and z:

$$f_X(x) = \int_0^3 \int_0^1 \frac{4}{9}\,x\,y\,z^2\,dy\,dz.$$

First, integrate with respect to y:

$$\int_0^1 y\,dy = \frac{1}{2},$$

so that

$$f_X(x) = \int_0^3 \frac{4x\,z^2}{9} \cdot \frac{1}{2}\,dz = \frac{4x}{18} \int_0^3 z^2\,dz.$$

Next, since

$$\int_0^3 z^2\,dz = \frac{3^3}{3} = 9,$$

we have

$$f_X(x) = \frac{4x}{18} \cdot 9 = 2x.$$

Then, the expectation is

$$E(x) = \int_0^1 x\,(2x)\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}.$$

(c) Marginal Densities of Y and Z

Similarly, we obtain:

$$f_Y(y) = \int_0^3 \int_0^1 \frac{4}{9}\,x\,y\,z^2\,dx\,dz = 2y,$$

and

$$f_Z(z) = \int_0^1 \int_0^1 \frac{4}{9}\,x\,y\,z^2\,dx\,dy = \frac{z^2}{9}.$$

(d) Conditional Density

Since

$$f(x, y, z) = f_X(x)\,f_Y(y)\,f_Z(z),$$

the variables are independent. Thus, the conditional density is

$$f_{X|Y=1/2}(x) = f_X(x) = 2x.$$

(e) Covariance of X and Z

Because x and z are independent, their covariance is

$$\text{cov}(x, z) = 0.$$

Question 2

The variables x_1, x_2, and x_3 are mutually independent with unit variance. Define

$$x^\top = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

so that

$$\mathrm{Cov}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, define

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 - x_2 \\ x_1 - x_3 \end{bmatrix} = A\,x, \quad \text{where} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

The covariance matrix of y is then

$$\mathrm{Cov}(y) = A\ \mathrm{Cov}(x)\, A^\top = A\, A^\top.$$

A direct calculation gives

$$\mathrm{Cov}(y) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

The corresponding correlation matrix is

$$\mathrm{Corr}(y) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{bmatrix},$$

where each correlation coefficient is defined as

$$\mathrm{corr}_{ij} = \frac{\mathrm{cov}_{ij}}{\sigma_i \sigma_j}.$$

Question 3

Let $U \sim N_3(0, I)$ and

$$\mu^\top = \begin{bmatrix} 10 & 4 & 7 \end{bmatrix}.$$

(i)

Consider the matrix

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 2 & -1 & 3 \end{bmatrix}.$$

Define the random vector

$$X = \mu + B\,U.$$

Since $U \sim N_3(0, I)$, it follows that

$$X \sim N_3\Big(\mu,\ B\,B^\top\Big).$$

A calculation shows that

$$B\,B^\top = \begin{bmatrix} 5 & 3 & 3 \\ 3 & 3 & -2 \\ 3 & -2 & 14 \end{bmatrix}.$$

Its Row Echelon form is given by

$$\begin{bmatrix} 5 & 3 & 3 \\ 0 & 6/5 & -19/5 \\ 0 & 0 & 1/6 \end{bmatrix}$$

which confirms that this covariance matrix has full rank:

(ii)

Now, consider

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & -1 \end{bmatrix},$$

and let $U' \sim N_2(0, I)$ (using the last two components of U). Then,

$$X = \mu + B\,U'$$

implies

$$X \sim N_3\Big(\mu,\, B\,B^\top\Big),$$

with

$$B\,B^\top = \begin{bmatrix} 5 & 3 & 3 \\ 3 & 2 & 1 \\ 3 & 1 & 5 \end{bmatrix}.$$

Its Row Echelon form is given by

$$\begin{bmatrix} 5 & 3 & 3 \\ 0 & 1/5 & -4/5 \\ 0 & 0 & 0 \end{bmatrix}$$

which confirms that this covariance matrix has rank 2.

Question 4

Using the same $\mu^\top = \begin{bmatrix} 10 & 4 & 7 \end{bmatrix}$ as in Question 3:

(a)

(i) Suppose

$$X \sim N_3\left(\begin{bmatrix} 10 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 9 & -3 & -3 \\ -3 & 5 & 1 \\ -3 & 1 & 5 \end{bmatrix}\right).$$

Writing $X = \mu + B\,U$ with $U \sim N_3(0, I)$, we show that

$$B = \Sigma^{1/2},$$

obtained by the Mahalanobis transformation:

$$U = \Sigma^{-1/2}(X - \mu) \quad \Longrightarrow \quad X = \mu + \Sigma^{1/2} U.$$

Using eigenvector analysis,

$$\Sigma^{1/2} = \sum_{i=1}^{3} \sqrt{\lambda_i}\, e_i\, e_i^\top,$$

with eigenvalues and eigenvectors

$$\lambda_1 = 3, \quad e_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda_2 = 4, \quad e_2 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \lambda_3 = 12, \quad e_3 = \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}.$$

Thus, we have

$$\begin{aligned}
\Sigma^{1/2} &= \sqrt{3}\, e_1 e_1^\top + \sqrt{4}\, e_2 e_2^\top + \sqrt{12}\, e_3 e_3^\top \\
&= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} + 2\sqrt{3} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{bmatrix} \\
&= \begin{bmatrix} \frac{5}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{2\sqrt{3}+3}{3} & \frac{2\sqrt{3}-3}{3} \\ -\frac{1}{\sqrt{3}} & \frac{2\sqrt{3}-3}{3} & \frac{2\sqrt{3}+3}{3} \end{bmatrix} = B.
\end{aligned}$$

(ii) Similarly, consider

$$X \simeq N_3\left(\begin{bmatrix} 10 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 8 & -4 & -4 \\ -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix} \right).$$

Again, writing $X = \mu + \Sigma^{1/2}U$, we determine $\Sigma^{1/2}$ via eigen-decomposition:

$$\Sigma^{1/2} = \sum_{i=1}^{3} \sqrt{\lambda_i}\, e_i\, e_i^\top = B,$$

with the eigenvalues and eigenvectors (for this case) given by

$$\lambda_1 = 0, \quad e_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda_2 = 4, \quad e_2 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \lambda_3 = 2\sqrt{3}, \quad e_3 = \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}.$$

Thus, in this instance,

$$\begin{aligned}
&\Sigma^{1/2} \\
&= \sum_{i=1}^{3} \sqrt{\lambda_i} e_i e_i^\top \\
&= 0 + 2 \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + 2\sqrt{3} \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{4}{\sqrt{3}} & -\frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} & \frac{3+\sqrt{3}}{3} & \frac{\sqrt{3}-3}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} & \frac{\sqrt{3}-3}{\sqrt{3}} & \frac{\sqrt{3}+3}{\sqrt{3}} \end{bmatrix} = B.
\end{aligned}$$

(b) Structural Relationships among the Components

(i) For

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \sim N_3\left(\begin{bmatrix} 10 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 9 & -3 & -3 \\ -3 & 5 & 1 \\ -3 & 1 & 5 \end{bmatrix} \right),$$

we note that x_1 is negatively correlated with both x_2 and x_3, while x_2 and x_3 are positively correlated.

(ii) For

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \sim N_3\left(\begin{bmatrix} 10 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 8 & -4 & -4 \\ -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix}\right),$$

x_1 remains negatively correlated with x_2 and x_3, but here x_2 and x_3 are independent (the correlation coefficient is 0).

Question 5

(i) Distance from $3a$

Assume that

$$x \sim N_3\left(\begin{bmatrix}10\\4\\7\end{bmatrix}, \begin{bmatrix}5 & 3 & 3\\3 & 3 & -2\\3 & -2 & 14\end{bmatrix}\right).$$

Marginal Distribution of $\begin{bmatrix}x_1\\x_2\end{bmatrix}$: To obtain the marginal distribution for $\begin{bmatrix}x_1\\x_2\end{bmatrix}$, we extract the corresponding sub-covariance matrix:

$$\Sigma' = \begin{bmatrix}5 & 3\\3 & 3\end{bmatrix},$$

and the corresponding mean vector is

$$\mu' = \begin{bmatrix}10\\4\end{bmatrix}.$$

Thus, we have

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} \sim N_2(\mu', \Sigma'),$$

which is non-degenerate.

Conditional Distribution of x_1: The conditional distribution of x_1 given the other variables (denoted by $\bar{x}_1$) is specified as

$$\bar{x}_1 \sim N_1\left(\frac{48\,x_2 + 18\,x_3 + 83}{38}, \frac{1}{38}\right).$$

Again, note that this sub-distribution is non-degenerate.

(b) Structural Relationships among the Components x_i

(i) Consider the vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \sim N_3\left(\begin{bmatrix} 10 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 9 & -3 & -3 \\ -3 & 5 & 1 \\ -3 & 1 & 5 \end{bmatrix}\right).$$

In this case, the variable x_1 is negatively correlated with both x_2 and x_3, while x_2 is positively correlated with x_3.

(ii) Consider now

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \sim N_3\left(\begin{bmatrix} 10 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 8 & -4 & -4 \\ -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix}\right).$$

Here, x_1 still shares a negative correlation with both x_2 and x_3, but additionally, x_2 and x_3 are independent (their correlation coefficient is 0).

(iii) Distribution from Q4 (iii)

Let

$$X \sim N_3\left(\begin{bmatrix} 10 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 9 & -3 & -3 \\ -3 & 5 & 1 \\ -3 & 1 & 5 \end{bmatrix}\right).$$

Marginal Distribution: The marginal distribution of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N_2\left(\begin{bmatrix} 10 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 & -3 \\ -3 & 5 \end{bmatrix}\right),$$

which is non-degenerate.

Conditional Distributions: The conditional distribution of x_1 given $\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$ is

$$\bar{x}_1 \sim N_1\left(\frac{-x_2 - x_3 + 31}{2}, 6\right).$$

Furthermore, the conditional distribution of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ given x_3 is approximately

$$\bar{x}_{12} \approx N_2\left(\begin{bmatrix} 10 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} \frac{1}{5}(x_3 - 7),\ \begin{bmatrix} 9 & -3 \\ -3 & 5 \end{bmatrix} - \begin{bmatrix} -3 \\ 1 \end{bmatrix} \frac{1}{5} \begin{bmatrix} -3 & 1 \end{bmatrix}\right).$$

After simplification, we obtain

$$\bar{x}_{12} \sim N_2\left(\begin{bmatrix} -\frac{3x_3+71}{5} \\ \frac{x_3+13}{5} \end{bmatrix}, \begin{bmatrix} \frac{36}{5} & -\frac{12}{5} \\ -\frac{12}{5} & \frac{24}{5} \end{bmatrix}\right),$$

which is non-degenerate.

(iv) Distribution from Q4 (iv)

Let

$$X \sim N_3\left(\begin{bmatrix} 10 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 8 & -4 & -4 \\ -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix}\right).$$

Marginal Distribution: The marginal distribution of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N_2\left(\begin{bmatrix} 10 \\ 4 \end{bmatrix}, \begin{bmatrix} 8 & -4 \\ -4 & 4 \end{bmatrix}\right),$$

which is non-degenerate.

Conditional Distributions: The conditional distribution of x_1 given $\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$ is computed as

$$\bar{x}_1 \sim N_1\left(10 + \begin{bmatrix} -4 & -4 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}^{-1} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix},\ 8 - \begin{bmatrix} -4 & -4 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ -4 \end{bmatrix}\right).$$

This expression simplifies to

$$\bar{x}_1 \sim N_1\big(-x - y + 21,\ 0\big),$$

which indicates that the variance is zero.

Moreover, the conditional distribution of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ given x_3 is approximated by

$$\bar{x}_{12} \approx N_2\left(\begin{bmatrix} 10 \\ 4 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \frac{1}{4}(x_3 - 7), \begin{bmatrix} 8 & -4 \\ -4 & 4 \end{bmatrix} - \begin{bmatrix} -4 \\ 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -4 & 0 \end{bmatrix}\right).$$

After further simplification, we obtain

$$\bar{x}_{12} \sim N_2\left(\begin{bmatrix} -x_3 + 17 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}\right),$$

which is non-degenerate.

The above results summarize the marginal and conditional distributions for various cases and illustrate the structural relationships among the components x_i in the given multivariate normal settings.