Catalan Numbers and Dyck Paths

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1. Introduction

1.1. Legal strings

Let us call a 'legal string' of parentheses the one with the property that, as we move along the string from left to right, we never will have seen more right parentheses than left. Let us count now the number of ways one can arrange n pairs of parentheses into a legal string. Note that a pair of parenthesis consists of a left and a right one. For $n = 3$, i.e. for 3 pairs of parentheses, it is easy to check that there are exactly 5 legal strings:

$$((()));\quad (()());\quad (())();\quad ()()();\quad ()(()) \tag{1.1}$$

Let $f(n)$ be the total number of legal strings of n pairs of parentheses for $n \geq 0$. $f(0) = 1$. It is apparent from equation (1.1) that **'('** & **')'** are right and left parentheses respectively.

For a legal string of n parentheses, we associate a unique non negative integer k as follows: as we move along the string scanning it from left to right, certainly after we have seen all the n pairs, the number of left parentheses will equal the number of right parentheses; however these two numbers might get equal before we have scanned all the n pairs! For instance, consider the last string in the above example. After just 1 pair (we say $k = 1$) have been scanned, we find that all the parentheses that were opened are now closed. Similarly, in the 3rd string above, just after 2 pairs (here, we say, $k = 2$) we find that all the parentheses that were opened have now closed. Let us name k as the Legal

Number (LN), The value of LN ($= k$) associated with each strings in the above are 3, 3, 2, 1, 1 respectively.

In fact, for any legal string, the integer, k is the smallest positive integer such that the first $2k$ characters of the legal string form a legal string! We say that a legal string of $2n$ parentheses is *primitive* if $k = n$. For instance, the first two strings of the above example are *primitive.* Both have $k = 3$.

Let us ask a question: how many legal strings of $2n$ parentheses will have a given legal number, say k ? If ω be such a string, then the first 2k characters of ω will not only be a legal string but will be a *primitive* string. But a question arises: in how many ways can we choose the first $2k$ characters of ω or in other words, how many *primitive* strings of length $2k$ are there? The last $2n - 2k$ characters of ω, on the other hand will form an arbitrary legal string. Since it is arbitrary & legal, there are exactly $f(n - k)$ ways to choose the last $2n - 2k$ characters.

Lemma 1.1.1 If $k \geq 1$ & $g(k)$ is the number of primitive legal strings of length $2k$, and $f(k)$, the number of all legal strings of $2k$ parentheses, then

$$g(k) = f(k - 1) \tag{1.2}$$

Proof. Let ω be any legal string of $k-1$ pairs of parentheses. If we put a left parenthesis in the beginning and a right parenthesis in the end of ω then lets say we get a new legal string $\omega^{'}$. Then in $\omega^{'}$, we wont find that the parentheses that have been opened getting closed until the last parenthesis. Hence $\omega^{'}$ is a primitive legal string of length $2k$. Conversely, considering a primitive legal string, $\omega^{'}$ of length $2k$, then deleting the initial left and the terminal right parenthesis would result in an arbitrary legal string, ω of length $2k - 2$. Hence there are as many primitive strings of length $2k$ as there are all legal strings of length $2k - 2$, i.e., there are $f(k - 1)$ of them. $\square$

Thus the total number of legal strings of $2n$ characters that have a given Legal Number, k is $g(k)f(n - k) = f(k - 1)f(n - k)$, using (1.2). Since every legal string has an unique Legal Number, k, we have,

$$f(n) = \sum_k f(k - 1)f(n - k) \qquad (n \neq 0; f(0) = 1). \tag{1.3}$$

1.2. The Generating Function

We wish to find the generating function for the numbers counted by $f(n)$. The reason is that it would be found out that these numbers represent a very important class of numbers in combinatorics can appear quite often. The sum on the right of (1.3) is related to the product of 2 *ordinary power series generating function* (OPSGF or opsgf). So, let $F(x) = \sum_{k\geq 0} f(k)x^k$ be the opsgf of $\{f(n); n \geq 0\}$.In order to go further, one needs to get accustomed to the **Cauchy product of two infinite power series:** Consider the following two power series $\displaystyle\sum_{i=0}^{\infty} a_i x^i$ *and* $\displaystyle\sum_{j=0}^{\infty} b_j x^j$. The Cauchy product of these two power series is (defined) as follows: $\displaystyle\left(\sum_{i=0}^{\infty} a_i x^i\right)\cdot\left(\sum_{j=0}^{\infty} b_j x^j\right) = \sum_{k=0}^{\infty} c_k x^k$ where $\displaystyle c_k = \sum_{l=0}^{k} a_l b_{k-l}$. It is now easy to observe that the RHS of (1.3) is the coefficient of x^n in the Cauchy product of the power series $F(x)$ and the power series $\sum_{k\geq 0} f(k-1)x^k$. The later series is nothing but $xF(x)$. Therefore, if we multiply both the sides of (1.3) by x^n and sum over all the terms and conditions that $n \neq 0$; $f(0) = 1$, the LHS becomes $F(x) - 1$ and the RHS becomes $xF(x)^2$. Hence we have,

$$F(x) - 1 = xF(x)^2. \tag{1.4}$$

Solving for quadratic equation in $F(x)$, we get,

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2x}. \tag{1.5}$$

Now, if we choose the '+' signed solution, then $\lim_{x\to 0} F(x) = \infty \neq 1$. But if we choose the '-' sign, then using L'Hospital's rule one can easily show that $\lim_{x\to 0} F(x) = \lim_{x\to 0} \dfrac{1-\sqrt{1-4x}}{2x} = 1 = f(0)$.

1.3. The explicit formula for the numbers $f(n)$

The numbers, $f(n)$ are celebrated as **The Catalan Numbers** and one can find an explicit formula for these numbers $f(n)$. Lets try to find it!

1.3.1. Newton's Generalised Binomial Theorem Let us define for an arbitrary number r, $\displaystyle\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}$. If x and y are real numbers with $|x| > |y|$, and r is any complex number, then we have,

$$(x+y)^r = \sum_{k=0}^{\infty}\binom{r}{k} x^{r-k}y^k = x^r + rx^{r-1}y + \frac{r(r-1)}{2!}x^{r-2}y^2 + \frac{r(r-1)(r-2)}{3!}x^{r-3}y^3 + \cdots.$$

which is the **Newton's generalised binomial theorem.**

Now, $\binom{\frac{1}{2}}{k} = \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!} = \frac{(-1)^{k-1}}{2^k} \frac{1\times2\times3\times4\times\cdots\times(2k-3)\times(2k-2)}{2\times4\times6\times\cdots\times(2k-2)\times(k!)} = \frac{(-1)^{k-1}}{k\times2^{2k-1}} \frac{(2k-2)!}{(k-1)!^2}$ $\implies \binom{\frac{1}{2}}{k} = \frac{(-1)^{k-1}}{k\times2^{2k-1}} \binom{2k-2}{k-1}$. Using Newton's Generalised Binomial Theorem for $x = 1$, replacing y with $-4x$ and putting $r = \frac{1}{2}$, we get,

$$\sqrt{1-4x} = (1-4x)^{\frac{1}{2}} = 1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} (-4x)^k.$$

Now, $\frac{(-1)^{k-1}}{k\times2^{2k-1}} \binom{2k-2}{k-1} (-4x)^k = \frac{(-1)^{2k-1}2^{2k}}{k\times2^{2k-1}} \binom{2k-2}{k-1} x^k =$ - $\frac{2}{k} \binom{2k-2}{k-1} x^k$. Therefore,

$$\frac{1-\sqrt{1-4x}}{2x} = \sum_{k=1}^{\infty} \frac{1}{k}\binom{2k-2}{k-1} x^{k-1} = \sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k} x^k = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n.$$

This implies

$$F(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 + \cdots \quad (1.6)$$

Thus $f(n) = \frac{1}{n+1}\binom{2n}{n}$ and these numbers are the Catalan numbers. It can be verified in (1.1) that for $n = 3$ pairs of parentheses, there were exactly 5 legal strings. 5 is the coefficient of x^3 in $F(x)$. These numbers will be investigated more thoroughly in the later sections.

2. A brief history of the Catalan Numbers

2.1. Euler, Goldbach & Segner

In the world of combinatorial mathematics, Catalan numbers cover an extensive range and is quite popular. The knowledge about these numbers were once obscure. Catalan numbers were rediscovered many times in the past until recently. Here is a quick exploration of its 200 years of history.

One of the first instances of the use of Catalan numbers can be found in, *Quick Methods for Accurate Values of Circle Segments* written by the Chinese Mathematician, Ming Antu (1692-1763). On September 4, 1751, Euler wrote a letter to his mentor and friend Goldbach (Both of them shared 196 letters between them) about the discovery of a closed formula for the Catalan numbers. The name Catalan Numbers were not used back then so these numbers were rather represented as: C_n, the number of triangulations of $(n+2)$-gon, and using which he gives the values of C_n upto $n = 8$ (which he computed by hand). All these values, including $C_8 = 1430$ were correct. Euler then observes symmetry and in some ratios that arose and concluded that they had a pattern and guessed the following formula for the Catalan numbers:

$$C_{n-2} = \frac{2 \times 6 \times 10 \cdots \times (4n-10)}{2 \times 3 \times 4 \cdots \times (n-1)} \tag{2.1}$$

Both of them arrived at the generating function satisfied by the Catalan numbers and found out that it satisfied the equation (but could not prove it):

$$1 + xA(x) = A(x)^{\frac{1}{2}} \tag{2.2}$$

During that time, Euler also shared a friendship with Johann Andreas von Segner (1704-1777), who was older to him by 3 years. In a letter Euler suggested Segner to count the number of triangulations of an n-gon and check for patterns instead of revealing his discoveries. Segner accepted his challenge and in 1758 wrote a paper back to him, whose main result is a recurrence relation which he states and proves combinatorially:

$$C_{n+1} = C_0C_n + C_1C_{n-1} + C_2C_{n-2} + \cdots + C_nC_0 \tag{2.3}$$

Using (2.3) he calculates the values of C_n upto $n = 18$ but he makes a calculation mistake for n=13 because of which the the larger values obtained for C_n were incorrect. Using equation (2.3) Euler proved equation (2.2). To summarize, a combination of results of Euler, Segner and Goldbach were needed to prove equation (2.1). Unfortunately, it took about 80 years until the first complete proof was published.

Nicolas Fuss (1755-1826), Swiss born mathematician who married Euler's granddaughter and became a well known mathematician in his own right. Fuss introduced what is now known as the Fuss-Catalan numbers.

2.2. The French school during 1838-1843

In 1838, a Jewish French mathematician and mathematical historian, Olry Terquem (1782-1862) asked Liouville if he knows a way to derive Euler's formula (2.1) from Segner's recurrence relation (2.3). Liouville in turn passed on this problem to various mathematicians who specialized in geometry. This resulted in a rigorous study of these numbers. Gabriel Lamé (1795-1870) wrote a letter to Liouville with a solution, using a double counting argument. Inspired by this solution of Lamé, in 1838, French and Belgium mathematician Eugéne Charles Catalan (who was a student of Liouville) became interested in the problem. He was the first one to obtain what now is the standard formula of these numbers:

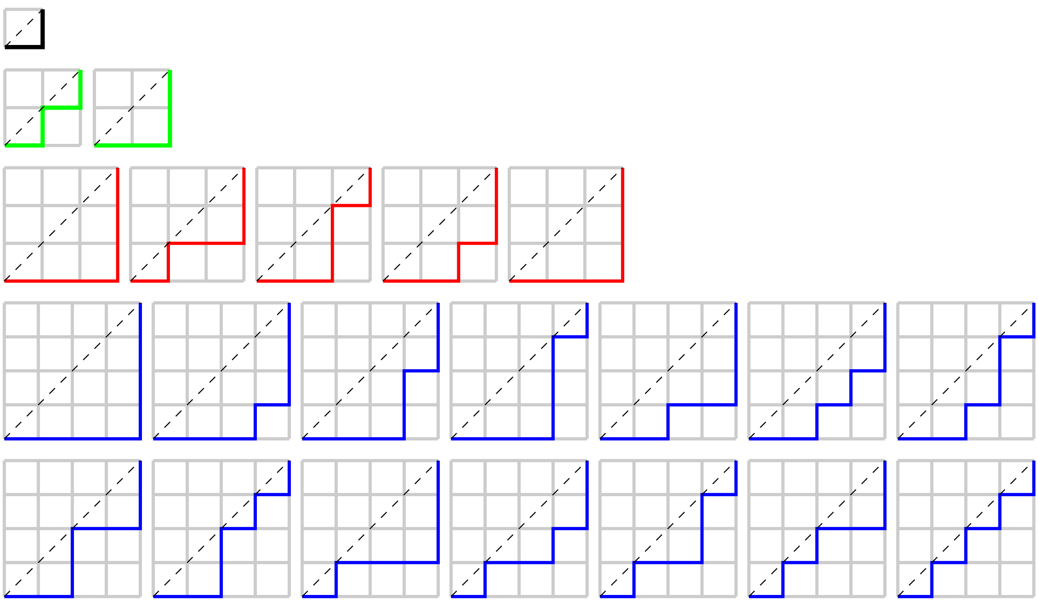
$$C_n = \frac{2n!}{n!(n+1)!} = \binom{2n}{n} - \binom{2n}{n-1} \tag{2.4}$$

Catalan revisited this problem throughout his life. For instance, even after around 50 years, in 1878, he published *Sur les nombres de Segner* which was a notable work on the divisibility of the Catalan numbers.

3. Dyck Paths

3.1. What is a dyck path?

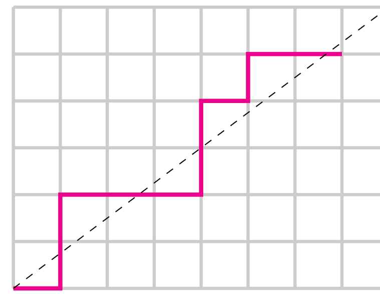
A Dyck Path is a series of equal length steps that form a staircase walk from $(0,0)$ to (n,n) that will lie strictly below, or touching, the diagonal of the $n \times n$ square. We say it then, that it is a "a dyck path of order n". Here are the sets of dyck paths of orders ranging from 1 to 4.



The first 1×1 square can sustain only 1 dyck path (coloured black). In the second row of 2×2 squares, there can be only 2 dyck paths (coloured green). Similarly, there are 5 dyck paths of order 3 and 14 dyck paths of order 4. Do you see a pattern in the numbers of dyck paths of order n? Indeed! if you have guessed it right, then they are the Catalan numbers: 1, 2, 5, 14, 42, 132, $\cdots$. The dyck paths are one of the *more than hundred* representations of the Catalan numbers. Without going much detail into these representations, lets have a look at *Why and how do the dyck paths count the Catalan numbers?*

3.2. Calculating the number of dyck paths of order n

3.2.1. Lattice paths We define a **lattice** as an arrangement of points in a regular periodic pattern in 2 or 3 dimensions. We say a 2 dimensional lattice is *integral* if the points in the co-ordinate plane are integers. Consider two points in this plane (p, q) and (r, s) with $p \geq q$ & $r \geq s$. We define a **a rectangular lattice path** from (r, s) to (p, q) is a path from (r, s) to (p, q) that is made up of horizontal steps $H = (1, 0)$ and vertical steps $V = (0, 1)$ For example:



The above figure shows a rectangular lattice from $(0, 0)$ to $(7, 5)$. It is uniquely determined by the sequence

$$H, V, V, H, H, H, V, V, H, V, H, H$$

of 7 H's & 5 V'S. Note that some of the lattice paths cross the diagonal of the rectangle. If there were to exist a sequence of lattice paths such that it never crossed the diagonal, we would call it **subdiagonal rectangular lattice paths**, similar to what we have seen in the case of dyck paths.

3.2.2. Counting rectangular lattice paths In order to count dyck paths, we need to count the number of subdiagonal rectangular lattice paths between two points and to count that we need to know the number of rectangular lattice paths from one point to the other. Hence we state and prove the following Theorem:

Theorem: The number of rectangular lattice paths from (r, s) to (p, q) equals the binomial coefficient

$$\binom{p-r+q-s}{p-r}. \tag{3.1}$$

Proof. Observe that a rectangular lattice path from (r, s) to (p, q) is uniquely determined by a sequence consisting of $p - r$ horizontal steps (H's) and $q - s$ vertical steps(v's). Hence from a total of $p - r + q - s$ steps, $p - r$ of them have to be H's and the rest of them are V's. This can thus be done in

$$\binom{p-r+q-s}{p-r} \times \binom{q-s}{q-s} = \binom{p-r+q-s}{p-r}$$

ways. $\square$

Without loss of generality we may assume (r, s) as the origin. Then the number of rectangular lattice paths from $(0,0)$ to (p, q) equals

$$\binom{p+q}{p} = \binom{p+q}{q}. \tag{3.2}$$

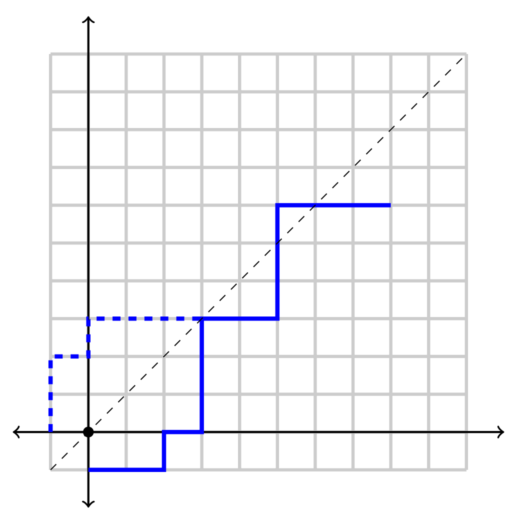
3.2.3. Counting subdiagonal rectangular lattice paths In this section, we wish to calculate the number of subdiagonal rectangular lattice paths from $(0,0)$ to (p, q). Observe that $p \geq q$, otherwise if $q > p$, then the path will have to cross the diagonal at some point. We lay the following theorem to count such paths and prove it.

Theorem: Let p & q be positive integers with $p \geq q$. Then the number of subdiagonal rectangular lattice paths from $(0,0)$ to (p, q) equals

$$\frac{p-q+1}{p+1}\binom{p+q}{q}. \tag{3.3}$$

Proof. Let $l(p, q)$ be the number of rectangular lattice paths γ from $(0,0)$ to (p, q) that cross the diagonal. Then the number of subdiagonal rectangular lattice paths from $(0,0)$ to (p, q) equals $\binom{p+q}{q}$ - $l(p, q)$. So, we try to find the value of $l(p, q)$. For every γ let γ' be the path that is constructed by shifting the paths of γ one step down. This establishes a 1 to 1 correspondence between all the paths γ from $(0,0)$ to (p, q) and the paths γ' from $(0,-1)$ to $(p, q-1)$. Note that every path γ' from $(0,-1)$ to $(p, q-1)$ will either touch or cross the diagonal of the lattice.

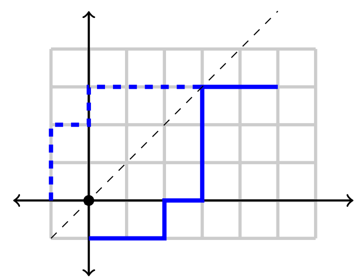
Consider a path γ' from $(0,-1)$ to $(p, q-1)$ that touches the diagonal of the lattice. Let γ_1' be the subpath of γ' from $(0,-1)$ to the point that touches diagonal first say at (d, d). Let γ_2' be the remaining subpath of γ'. Now we wish to make a construction. We reflect γ_1' about the diagonal and obtain a new path γ_1^* from $(-1,0)$ to (d, d). Let us name the path formed by the concatenation of γ_1^* & γ_2' by γ^*. The whole construction has been illustrated below with an example for reference.



In the above diagram, the co-ordinate axes has been drawn and their intersection, the origin has been marked with a black dot. Here,
$\gamma = H,H,V,H,V,V,V,H,H,V,V,V,H,H,H$ starting from $(0,0)$ and thus
$\gamma' = H,H,V,H,V,V,V,H,H,V,V,V,H,H,H$ starting from $(0,-1)$ where, $\gamma'_1 = H,H,V,H,V,V,V$;
$\gamma_1^* = V,V,H,V,H,H,H$ starting from $(-1,0)$ and
$\gamma^* = V,V,H,V,H,H,H,H,H,V,V,V,H,H,H$ starting from $(0,-1)$
Since $q \leq p$, we have $q-1 < p$ then the point $(p, q-1)$ must lie below the diagonal. Thus every rectangular lattice path θ from $(-1,0)$ to $(p,q-1)$ must cross the diagonal since $(-1,0)$ is above it and $(p,q-1)$ is below it. Now, if we reflect the part of θ from $(-1,0)$ to the first crossing point, we get a path from $(0,-1)$ to $(p,q-1)$ that touches the diagonal and not cross it (since $(p,q-1)$ is below it). This has been illustrated below. Note that the diagonal is the line $y = x$.

Thus there is a 1 to 1 correspondence between γ^* & γ'. Therefore, the number of lattice paths from $(-1,0)$ to $(p,q-1)$ equals $l(p,q)$ and by theorem (13), we have

$$l(p,q) = \binom{p+1+q-1}{q-1} = \binom{p+q}{q-1}. \tag{3.4}$$



Then the number of subdiagonal rectangular lattice paths from $(0,0)$ to (p,q) equals

$$\binom{p+q}{q} - l(p,q) = \binom{p+q}{q} - \binom{p+q}{q-1} = \frac{(p+q)!}{p!q!} - \frac{(p+q)!}{(q-1)!(p+1)!} = \frac{p-q+1}{p+1}\binom{p+q}{q},$$

which completes our proof. □

3.2.4. Number of dyck paths of order n Observe that a dyck path of order n is a subdiagonal rectangular lattice path from $(0,0)$ to (n,n). Plugging it in theorem (15), we get the number of dyck paths of order n to be $\frac{1}{n+1}\binom{2n}{n}$ (since $p = q = n$), which are the catalan numbers.

More information on the things discussed here can be found in the wonderful books by Brualdi [Bru99], Stanley [Sta15] and Wilf [Wil06].

References

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