$A \cdot B \rightsquigarrow$ the direction of relationship

A is invertible iff

A is squar- $\rightsquigarrow$ $Ax = \delta x$ $\quad A_{d\times d}$

$$AX = \Lambda X$$

$X_{n\times d} \rightsquigarrow X^T X \rightsquigarrow \underset{\sim\sim}{d\times d}$ $\rightsquigarrow$ $\dfrac{\bar{X}^T \bar{X}}{n} = \text{Cov}$

$$\bar{X} = \begin{bmatrix} x_{11} - \mu_1 & \cdots \\ x_{21} - \mu_1 & \\ \vdots & \\ x_{n1} - \mu_1 & \end{bmatrix}$$

$\dfrac{x_{11} - \mu_1}{\sigma_1}$ $\rightarrow$ $= \text{Cor}$

Lecture 03
Probability and Statistics

Mahdi Roozbahani
Georgia Tech

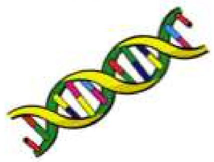
These slides are based on slides from Le Song , Sam Roweis, and Chao Zhang.

Outline

- Probability Distributions
- Joint and Conditional Probability Distributions
- Bayes' Rule
- Mean and Variance
- Properties of Gaussian Distribution
- Maximum Likelihood Estimation

Probability

- A sample space S is the set of all possible outcomes of a conceptual or physical, repeatable experiment. (S can be finite or infinite.)
  - E.g., S may be the set of all possible outcomes of a dice roll: S $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$
  - E.g., S may be the set of all possible nucleotides of a DNA site: S $\begin{pmatrix} A & C & G & T \end{pmatrix}$



  - E.g., S may be the set of all possible time-space positions of an aircraft on a radar screen.
- An Event A is any subset of S
  - Seeing "1" or "6" in a dice roll; observing a "G" at a site; UA007 in space-time interval

Three Key Ingredients in Probability Theory

A **sample space** is a collection of all possible **outcomes**

Random variables X represents **outcomes** in sample space

$X = \{1, 2, \ldots, 6\}$ $\boxed{P(X=1)} = \frac{1}{6}$

Probability of a random variable to happen $p(x) = p(X = x)$

$$p(x) \geq 0$$

$P(X \pm dx)$ $X = 6.000000\ldots$

$= 1$

Continuous variable

Continuous probability distribution

Probability density function

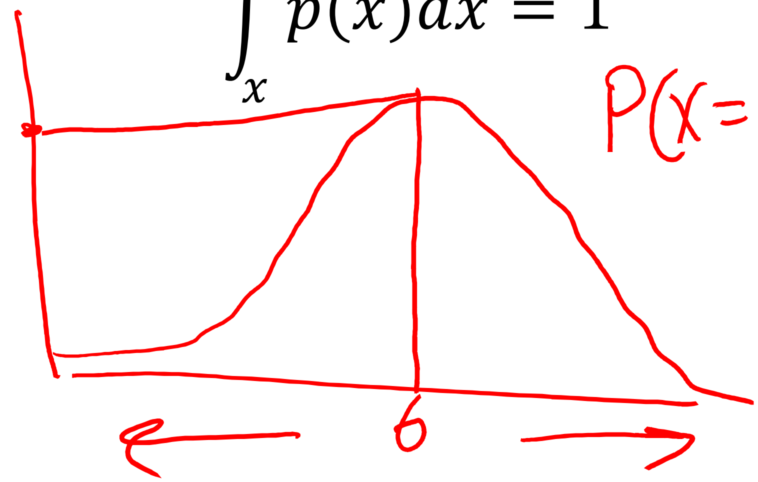
Density or likelihood value

Temperature (real number)

Gaussian Distribution

$$\int_x p(x)dx = 1$$

$P(X=6)$



Discrete variable

Discrete probability distribution

Probability mass function

Probability value

Coin flip (integer)

Bernoulli distribution

$$\sum_{x \epsilon A} p(x) = 1$$

Continuous Probability Functions

$$\frac{1}{b-a} = \frac{1}{7-1} = \frac{1}{6}$$

- Examples:
  - Uniform Density Function:

$$f_x(x) = \begin{cases} \dfrac{1}{b-a} & for\ a \le x \le b \\ 0 & otherwise \end{cases}$$

  - Exponential Density Function:

$$f_x(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}} \qquad for\ x \ge 0$$

$$F_x(x) = 1 - e^{\frac{-x}{\mu}} \qquad for\ x \ge 0$$

$$F_x(X \le x)$$

  - Gaussian(Normal) Density Function

$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Discrete Probability Functions

- Examples:
  - Bernoulli Distribution:
    - $\begin{cases} 1-p & for\ x=0 \\ p & for\ x=1 \end{cases} = \begin{cases} 0.5 \\ 0.5 \end{cases}$

      $\rightarrow P=\frac{1}{2}$

In Bernoulli, just a **single** trial is conducted

  - Binomial Distribution:
    - $P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$

k is number of successes

n-k is number of failures

$\binom{n}{k}$ The total number of ways of selection **k** distinct combinations of **n** trials, **irrespective of order**.

H H H H T

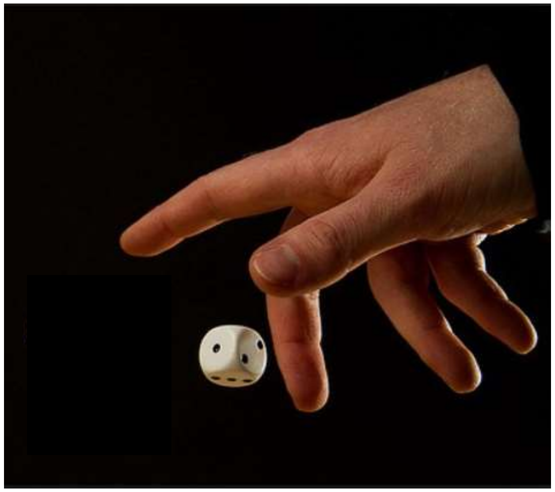
Continuous Probability Functions

- A continuous random variable X is defined on a continuous sample space: an interval on the real line, a region in a high dimensional space, etc.
  - It is meaningless to talk about the probability of the random variable assuming a particular value --- P(x) = 0
  - Instead, we talk about the probability of the random variable assuming a value within a given interval, or half interval, or arbitrary Boolean combination of basic propositions.
  - Cumulative Distribution Function (CDF): $F_x(x) = \mathrm{P}[X \leq x]$
  - Probability Density Function (PDF): $F_x(x) = \int_{-\infty}^{x} f_x(x)\,dx$ or $f_x(x) = \frac{d\,F_x(x)}{dx}$
  - Properties: $f_x(x) \geq 0$ and $\int_{-\infty}^{\infty} f_x(x)\,dx = 1$
  - Interpretation: $f_x(x) = \mathrm{P}[X \in \frac{x,x+\Delta}{\Delta}]$

Outline

- Probability Distributions
- Joint and Conditional Probability Distributions
- Bayes' Rule
- Mean and Variance
- Properties of Gaussian Distribution
- Maximum Likelihood Estimation

Example



X = Throw a dice

Y = Flip a coin

X and **Y** are random variables

N = total number of trials

$\boldsymbol{n_{ij}}$ = Number of occurrence

X

Y

	$x_{i=1} = 1$	$x_{i=2} = 2$	$x_{i=3} = 3$	$x_{i=4} = 4$	$x_{i=5} = 5$	$x_{i=6} = 6$	C_j
$y_{j=2} = tail$	$n_{ij} = 3$	$n_{ij} = 4$	$n_{ij} = 2$	$n_{ij} = 5$	$n_{ij} = 1$	$n_{ij} = 5$	20
$y_{j=1} = head$	$n_{ij} = 2$	$n_{ij} = 2$	$n_{ij} = 4$	$n_{ij} = 2$	$n_{ij} = 4$	$n_{ij} = 1$	15
C_i	5	6	6	7	5	6	N=35

X

Y

	$x_{i=1}=1$	$x_{i=2}=2$	$x_{i=3}=3$	$x_{i=4}=4$	$x_{i=5}=5$	$x_{i=6}=6$	C_j
$y_{j=2}=tail$	$n_{ij}=3$	$n_{ij}=4$	$n_{ij}=2$	$n_{ij}=5$	$n_{ij}=1$	$n_{ij}=5$	20
$y_{j=1}=head$	$n_{ij}=2$	$n_{ij}=2$	$n_{ij}=4$	$n_{ij}=2$	$n_{ij}=4$	$n_{ij}=1$	15
C_i	5	6	6	7	5	6	N=35

$$P(X=1, Y=tail) = \frac{3}{35} = \frac{n_{ij}}{N}$$

$$\rightarrow P(Y=tail \mid X=1) = \frac{3}{5} = \frac{n_{ij}}{C_i}$$

$$P(Y=head) = \frac{15}{35} = \sum_{x} P(Y=head, X=x)$$

X

Y

	$x_{i=1}=1$	$x_{i=2}=2$	$x_{i=3}=3$	$x_{i=4}=4$	$x_{i=5}=5$	$x_{i=6}=6$	C_j
$y_{j=2}=tail$	$n_{ij}=3$	$n_{ij}=4$	$n_{ij}=2$	$n_{ij}=5$	$n_{ij}=1$	$n_{ij}=5$	20
$y_{j=1}=head$	$n_{ij}=2$	$n_{ij}=2$	$n_{ij}=4$	$n_{ij}=2$	$n_{ij}=4$	$n_{ij}=1$	15
C_i	5	6	6	7	5	6	N=35

Probability:

$$p(X = x_i) = \frac{c_i}{N}$$

Joint probability:

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$

Conditional probability:

$$p(Y = y_j | X = x_i) = \frac{n_{ij}}{c_i}$$

Sum rule

$$p(X = x_i) = \sum_{j=1}^{L} p(X = x_i, Y = y_j) \Rightarrow p(X) = \sum_{Y} P(X, Y)$$

Product rule

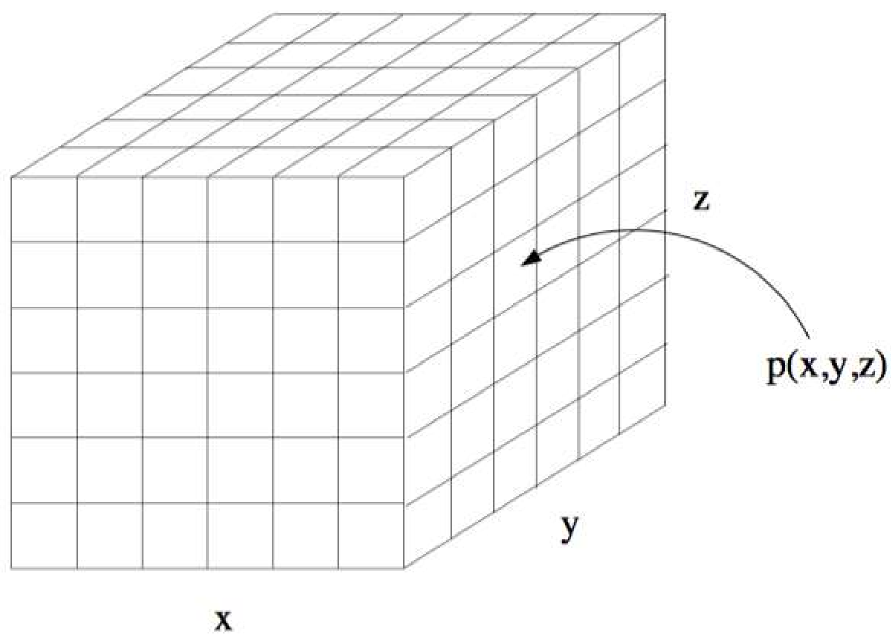
$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N} = \frac{n_{ij}}{c_i}\frac{c_i}{N} = p(Y = y_j | X = x_i) p(X = x_i)$$

$$p(X, Y) = p(Y|X) p(X)$$

P(X, Y) = P(X|Y) P(Y) = P(Y|X) P(X)

Joint Distribution

- Key concept: two or more random variables may interact. Thus, the probability of one taking on a certain value depends on which value(s) the others are taking.
- We call this a joint ensemble and write
$$p(x, y) = \text{prob}(X = x \text{ and } Y = y)$$

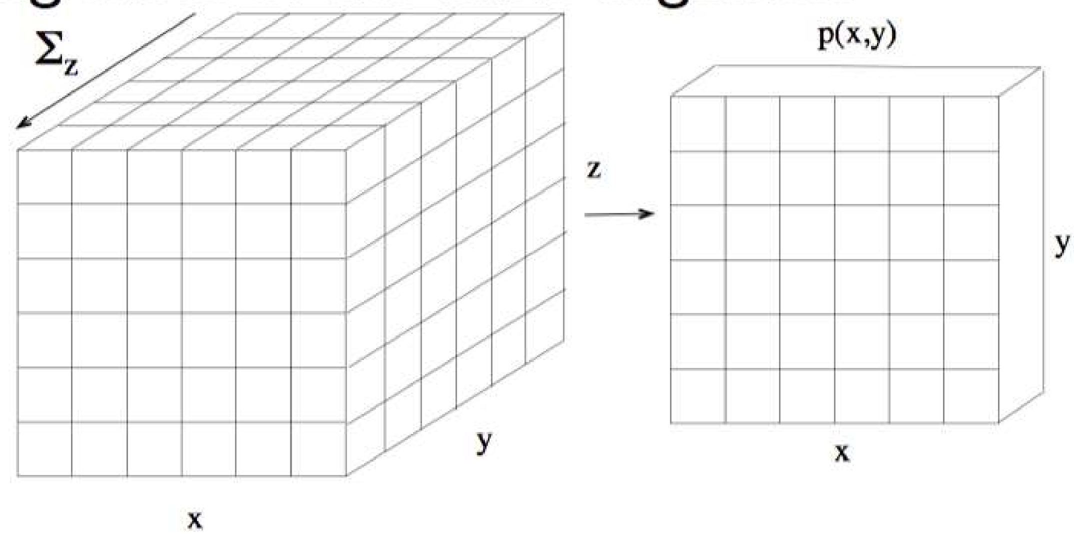


Marginal Distribution

- We can "sum out" part of a joint distribution to get the *marginal distribution* of a subset of variables:

$$p(x) = \sum_y p(x, y)$$

- This is like adding slices of the table together.

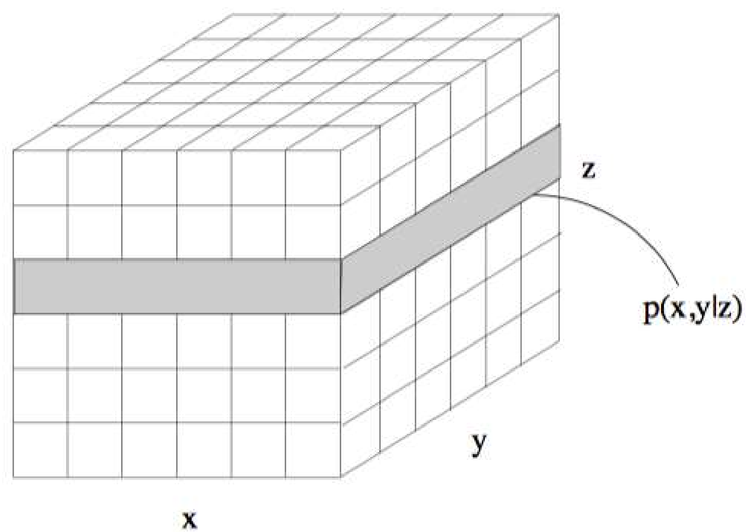


- Another equivalent definition: $p(x) = \sum_y p(x|y)p(y)$.

Conditional Distribution

- If we know that some event has occurred, it changes our belief about the probability of other events.
- This is like taking a "slice" through the joint table.

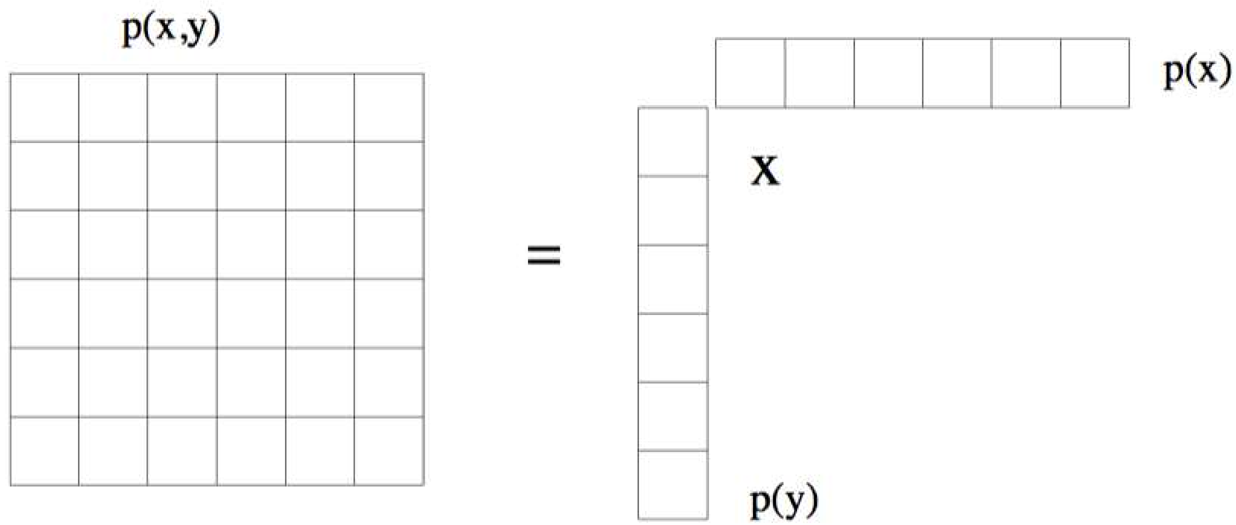
$$\boxed{p(x|y) = p(x,y)/p(y)}$$



Independence & Conditional Independence

- Two variables are independent iff their joint factors:

$$p(x, y) = p(x)p(y)$$



- Two variables are conditionally independent given a third one if for all values of the conditioning variable, the resulting slice factors:

$$p(x, y|z) = p(x|z)p(y|z) \qquad \forall z$$

Conditional Independence

- Examples:

P(Virus | ~~Drink Beer~~) = P(Virus)
iff Virus is independent of Drink Beer

P(Flu | Virus;~~DrinkBeer~~) = P(Flu|Virus)
iff Flu is independent of Drink Beer, given Virus

P(Headache | Flu;~~Virus~~;DrinkBeer) =
P(Headache|Flu;DrinkBeer)
iff Headache is independent of Virus, given Flu and Drink Beer

Assume the above independence, we obtain:
P(Headache;Flu;Virus;DrinkBeer)
=P(Headache | Flu;Virus;DrinkBeer) P(Flu | Virus;DrinkBeer)
P(Virus | Drink Beer) P(DrinkBeer)
=P(Headache|Flu;DrinkBeer) P(Flu|Virus) P(Virus) P(DrinkBeer)

$P(h, F, V, D) =$

$P(h \mid F, \cancel{V}, D)\, P(F, V, D)$

$= P(h \mid F, D)\, P(F, V, D)$

$= P(F \mid V, \cancel{D})\, P(V, D)\, P(h \mid F, D)$

$M = P(F \mid V)\, P(h, F, D)$

$P(V, D)$

$= P(V \mid D)\, P(D)\, M$

$= P(V)\, P(D)\, M$

Outline

- Probability Distributions
- Joint and Conditional Probability Distributions
- Bayes' Rule
- Mean and Variance
- Properties of Gaussian Distribution
- Maximum Likelihood Estimation

Bayes' Rule

- P(X|Y)= Fraction of the worlds in which X is true given that Y is also true.

$P(x|y)P(y) = P(y|x)P(x)$

$P(x,y) = P(y,x)$

- For example:
  - H="Having a headache"
  - F="Coming down with flu"
  - $P(Headche|Flu)$ = fraction of flu-inflicted worlds in which you have a headache. How to calculate?

$P(x|y)P(y)$ or $P(y|x)P(x)$

- Definition:

Likelihood

Prior

$$\underbrace{P(X|Y)}_{\text{Posterior Probability}} = \frac{P(X,Y)}{P(Y)} = \frac{P(Y|X)P(X)}{P(Y)}$$

$P(Y) = \sum_{x} P(y,x)$

Corollary:

$$P(X,Y) = P(Y|X)P(X)$$

This is called Bayes Rule

Bayes' Rule

- $P(Headache|Flu) = \frac{P(Headache,Flu)}{P(Flu)}$

$$= \frac{P(Flu|Headache)P(Headache)}{P(Flu)}$$

Other cases:

- $P(Y|X) = \frac{P(X|Y)P(Y)}{P(X|Y)P(Y)+P(X|\neg Y)P(\neg Y)}$
- $P(Y = y_i|X) = \frac{P(X|Y)P(Y)}{\sum_{i \in S} P(X|Y = y_i)P(Y=y_i)}$
- $P(Y|X,Z) = \frac{P(X|Y,Z)P(Y,Z)}{P(X,Z)} =$

$$\frac{P(X|Y,Z)P(Y,Z)}{P(X|Y,Z)P(Y,Z)+P(X|\neg Y,Z)P(\neg Y,Z)}$$

Outline

- Probability Distributions
- Joint and Conditional Probability Distributions
- Bayes' Rule
- Mean and Variance
- Properties of Gaussian Distribution
- Maximum Likelihood Estimation

Mean and Variance

- Expectation: The mean value, center of mass, first moment:

$$E_X[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x)dx = \mu$$

- N-th moment: $g(x) = x^n$
- N-th central moment: $g(x) = (x - \mu)^n$
- Mean: $E_X[X] = \int_{-\infty}^{\infty} xp_X(x)dx$
  - $E[\alpha X] = \alpha E[X]$
  - $E[\alpha + X] = \alpha + E[X]$
- Variance(Second central moment): $Var(x) = E_X[(X - E_X[X])^2] = E_X[X^2] - E_X[X]^2$
  - $Var(\alpha X) = \alpha^2 Var(X)$
  - $Var(\cancel{\alpha} + X) = Var(X)$

$$\mu = \frac{\sum x_i}{n}$$

$$\sigma^2 = \frac{\sum (x_i - \mu)^2}{n}$$

For Joint Distributions

- Expectation and Covariance:
  - $E[X+Y] = E[X] + E[Y]$
  - $cov(X,Y) = E[(X - E_X[X])(Y - E_Y(Y)] = E[XY] - E[X]E[Y]$
  - $Var(X+Y) = Var(X) + 2cov(X,Y) + Var(Y)$

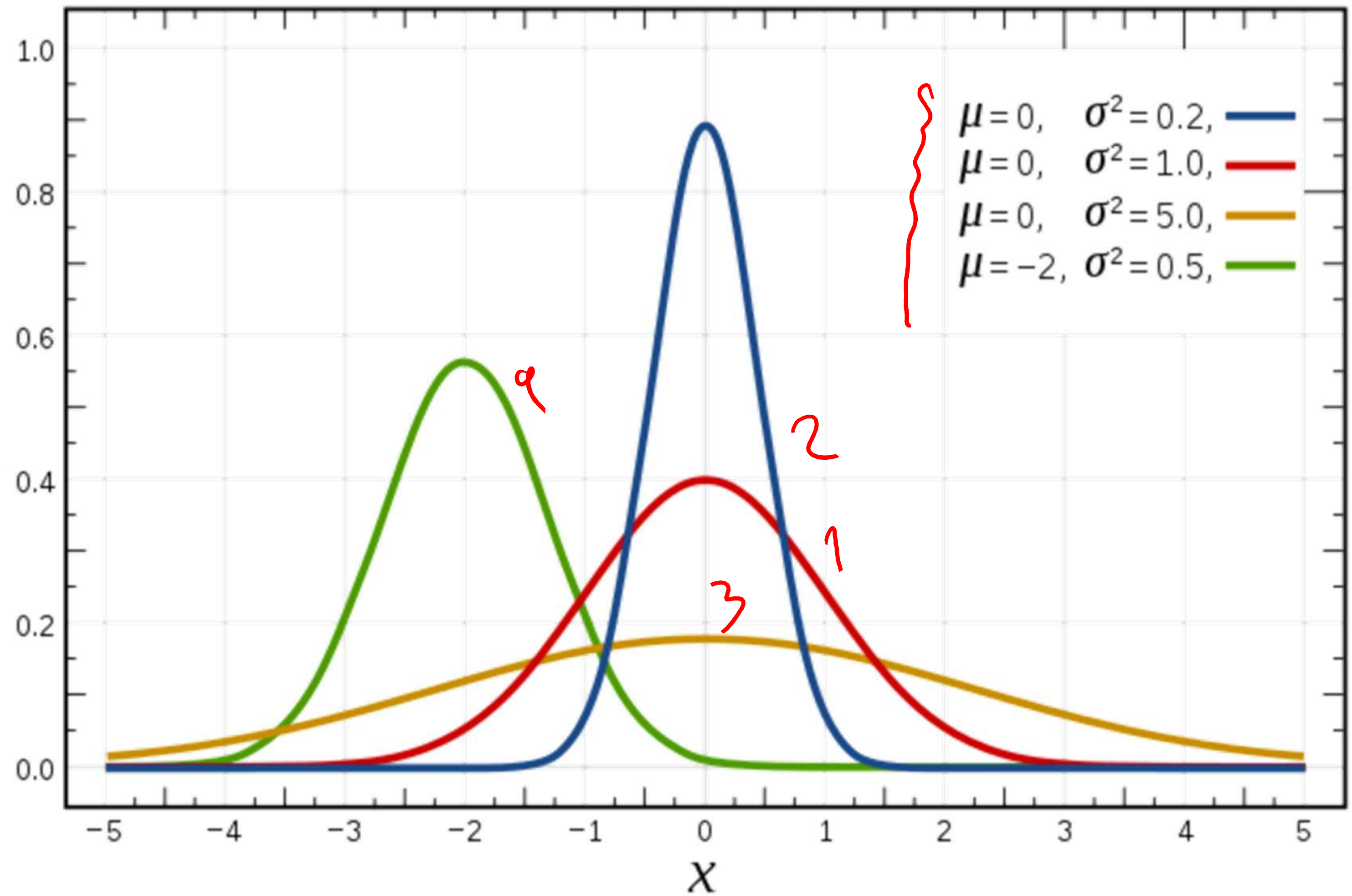
Outline

- Probability Distributions
- Joint and Conditional Probability Distributions
- Bayes' Rule
- Mean and Variance
- Properties of Gaussian Distribution
- Maximum Likelihood Estimation

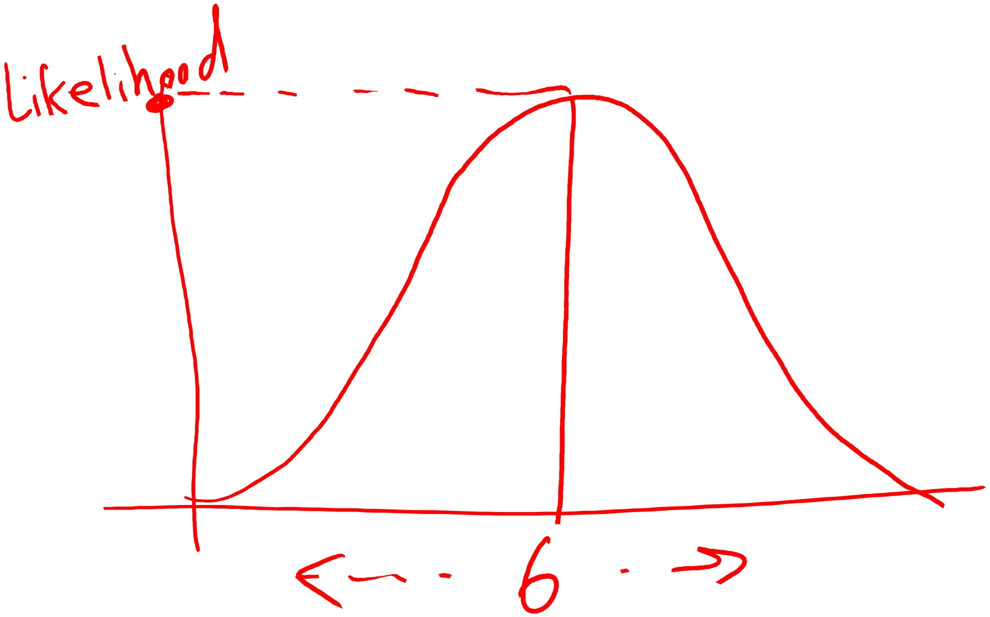
Gaussian Distribution

- **Gaussian Distribution:** $$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Probability density function



[Probability versus likelihood]()



$$P(X+dx \mid \mu, \sigma)$$

$$X=6$$

$$\sigma, \mu \neq \frac{\sum x_i}{n}$$

$$\mu = \frac{\sum x_i}{n} \qquad \sigma = \dots$$

$$\text{Max}\left[P(X=x_1, X=x_2, \dots, X=x_{300}) \right]$$

$$= P(X=x_1)\, P(X=x_2) \dots P(X=x_{300})$$

Multivariate Gaussian Distribution

$$p(x|\mu,\Sigma) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\{-\frac{1}{2}(x-\mu)^{\top}\Sigma^{-1}(x-\mu)\}$$

- Moment Parameterization $\mu = E(X)$

$$\Sigma = Cov(X) = E[(X-\mu)(X-\mu)^{\top}]$$

- Mahalanobis Distance $\Delta^2 = (x-\mu)^{\top}\Sigma^{-1}(x-\mu)$
- Tons of applications (MoG, FA, PPCA, Kalman filter,...)

Multivariate Gaussian Distribution

- Joint Gaussian $P(X_1, X_2)$

$$\mu = \begin{bmatrix}\mu_1 \\ \mu_2\end{bmatrix} \qquad \Sigma = \begin{bmatrix}\Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}$$

- Marginal Gaussian

$$\mu_2^m = \mu_2 \qquad \Sigma_2^m = \Sigma_2$$

- Conditional Gaussian $P(X_1 | X_2 = x_2)$

$$\mu_{1|2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$$

$$\Sigma_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

Properties of Gaussian Distribution

- The linear transform of a Gaussian r.v. is a Gaussian. Remember that no matter how x is distributed

$$E(AX + b) = AE(X) + b$$
$$Cov(AX + b) = ACov(X)A^{\top}$$

this means that for Gaussian distributed quantities:

$$X \sim N(\mu, \Sigma) \quad \rightarrow \quad AX + b \sim N(A\mu + b, A\Sigma A^{\top})$$

- The sum of two independent Gaussian r.v. is a Gaussian

$$\mathrm{Y} = \mathrm{X}_1 + \mathrm{X}_2,\ \mathrm{X}_1 \perp X_2 \ \rightarrow \mu_y = \mu_1 + \mu_2, \Sigma_y = \Sigma_1 + \Sigma_2$$

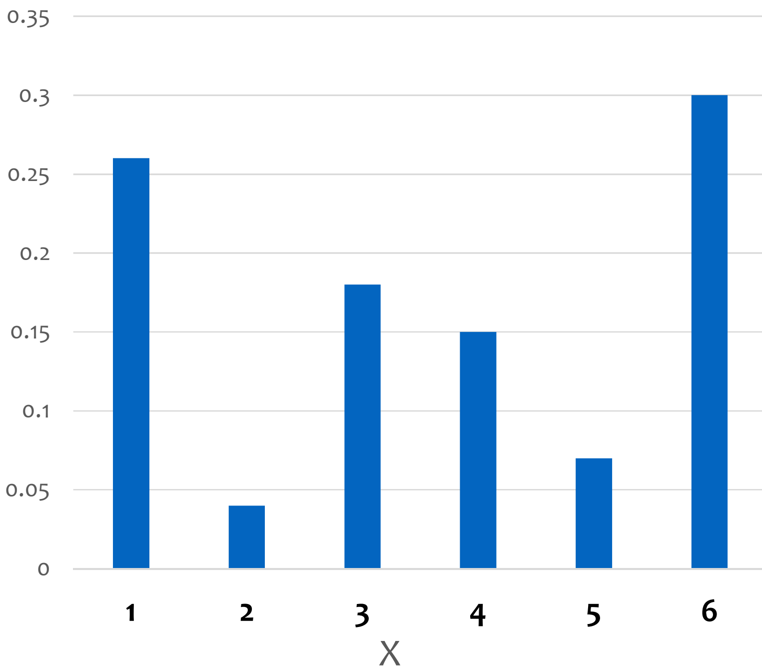
- The multiplication of two Gaussian functions is another Gaussian function (although no longer normalized)

$$N(a, A)N(b, B) \propto N(c, C),$$
$$where\ C = (A^{-1} + B^{-1})^{-1}, c = \mathrm{C}A^{-1}a + CB^{-1}b$$

Central Limit Theorem

$\% 16 = \frac{1}{6}$

Probability mass function of a **biased** dice



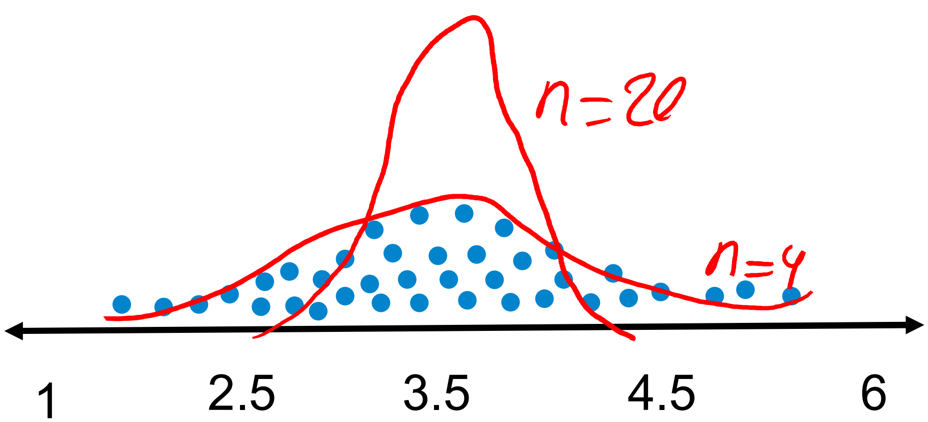
Let's say, I am going to get a sample from this pmf having a size of $n = 4$

$$S_1 = \{1,1,1,6\} \Rightarrow E(S_1) = 2.25$$

$$S_2 = \{1,1,3,6\} \Rightarrow E(S_2) = 2.75$$

$$\vdots$$

$$S_m = \{1,4,6,6\} \Rightarrow E(S_m) = 4.25$$

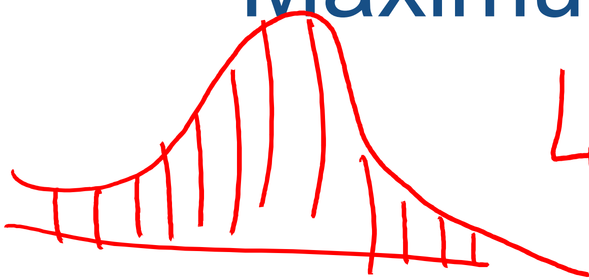


According to CLT, it will follow a bell curve distribution (normal distribution)

Outline

- Probability Distributions
- Joint and Conditional Probability Distributions
- Bayes' Rule
- Mean and Variance
- Properties of Gaussian Distribution
- Maximum Likelihood Estimation

Maximum Likelihood Estimation



$$\underline{L(p) = P(X = x_1, X = x_2, \ldots, X = x_n)}$$

- Probability: inferring probabilistic quantities for data given fixed models (e.g. prob. of events, marginals, conditionals, etc).
- Statistics: inferring a model given fixed data observations (e.g. clustering, classification, regression).

$$\mu = \frac{\sum x_i}{n}$$

Main assumption:

Independent and identically distributed random variables

i.i.d

Maximum Likelihood Estimation

For Bernoulli (i.e. flip a coin):

Objective function: $f(x_i; p) = p^{x_i}(1-p)^{1-x_i}$ $\qquad x_i \in \{0,1\}\ or\ \{head, tail\}$

$$L(p) = p(X = x_1, X = x_2, X = x_3, \dots, X = x_n)$$

i.i.d assumption

$$= p(X = x_1)p(X = x_2) \dots p(X = x_n) = f(x_1; p)f(\cancel{x_2}; \cancel{p}) \dots f(\cancel{x_n}; \cancel{p})$$

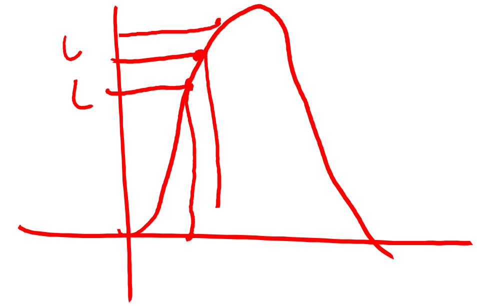
$$L(p) = \prod_{i=1}^{n} f(x_i; p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$

$$L(p) = p^{x_1}(1-p)^{1-x_1} \times p^{x_2}(1-p)^{1-x_2} \dots \times p^{x_n}(1-p)^{1-x_n} =$$

$$= p^{\sum x_i}(1-p)^{\sum(1-x_i)}$$

We don’t like multiplication, let’s convert it into summation

What’s the trick? Take the log



$$L(p) = p^{\sum x_i}(1-p)^{\sum(1-x_i)}$$

$$logL(p) = l(p) = \log(p)\sum_{i=1}^{n} x_i + \log(1-p)\sum_{i=1}^{n}(1-x_i)$$

$\max_{p} l(p)$

How to optimize p?

$$\frac{\partial l(p)}{\partial p} = 0 \qquad \boxed{\frac{\sum_{i=1}^{n} x_i}{p} - \frac{\sum_{i=1}^{n}(1-x_i)}{1-p} = 0}$$

$$p = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\mu = \frac{\sum x_i}{n} \qquad \sigma = \frac{\sum(x_i-\mu)^2}{n}$$