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ASTROPHYSICS
PROBLEM SET-03

1.) We know;

$$P = k\rho^{(1+1/n)} = k\rho^{4/3}$$

$$= \left[\frac{3}{a}\left(\frac{k_B}{\mu m_H}\right)^4 \frac{1-\beta}{\beta^4}\right]^{1/3} \rho^{4/3}$$

Also known is $\theta = \left(\frac{\rho}{\rho_c}\right)^{1/n}$ $\xi = \frac{r}{\alpha}$; where:

$$\alpha = \left[\frac{(n+1)k\rho_c^{\frac{1-n}{n}}}{4\pi G}\right]^{1/2}$$

$$\Rightarrow \alpha^2 = \frac{k\rho^{-2/3}}{\pi G}$$

$$\frac{d}{dr} = \frac{d\xi}{dr}\frac{d}{d\xi} = \frac{1}{\alpha}\frac{d}{d\xi}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{2/3}}\frac{d\rho}{dr}\right) = \frac{-3\pi G\rho}{k}$$

$$\frac{1}{r^2\alpha^2}\frac{1}{\alpha}\frac{d}{d\xi}\left(\frac{r^2}{\alpha^2}\frac{\alpha^2}{\rho^{2/3}}\frac{1}{\alpha}\frac{d\rho}{d\xi}\right) = -\frac{3\pi G\rho}{k}$$

After substituting $\rho = \rho_c\theta^n$, we get the form as:

$$\frac{1}{\xi^2}\frac{1}{\alpha^3}\frac{d}{d\xi}\left(\xi^2\frac{\alpha}{\rho_c^{2/3}\theta^{2n/3}}\frac{d\rho_c\theta^n}{d\xi}\right) = -\frac{3\pi G\rho_c\theta^n}{k}$$

$$\frac{\rho_c^{1/3}}{\alpha^2\xi^2}\frac{d}{d\xi}\left(\frac{\xi^2}{\theta^{2n/3}}\frac{d\theta^n}{d\xi}\right) = \frac{-3\pi G}{k}\rho_c\theta^n$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \rightarrow \text{Lane Emden equation.}$$

For $n=3$, we have.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^3 \quad (1)$$

To obtain as a pair of coupled 1st order differential equation:

Let's take: $$\xi^2\frac{d\theta}{d\xi} = -\psi \rightarrow (2)$$

Then we can write:

$$\frac{1}{\xi^2}\frac{d\psi}{d\xi} = \theta^3 \Rightarrow \frac{d\psi}{d\xi} = \theta^3\xi^2$$

Generalizing it for any 'n'.

$$\frac{d\psi}{d\xi} = \theta^n\xi^2$$

We know that; $$M = 4\pi a^3 \rho_c \int_0^{\xi} \xi^2\theta^n d\xi$$

Substituting for $\xi^2\theta^n$ we can write:

$$M(r) = 4\pi a^3 \rho_c \psi(\xi)$$

$\therefore$ The coupled 1st order DE's are:

$$\xi^2\frac{d\theta}{d\xi} = -\psi$$

$$\frac{d\psi}{d\xi} = \theta^n\xi^2 \quad \text{Where } \psi(\xi) = \frac{M(r)}{4\pi a^3\rho_c}$$

(c) Boundary conditions at $\xi = 0$.

(i) At $\xi = 0$, $\theta(\xi = 0) = 1$ $\longrightarrow$ 1st B.C.

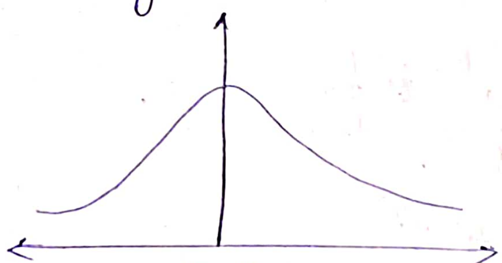
As $\xi = \frac{r}{a}$ $\Rightarrow$ at $r = 0$, $\xi = 0$.

$\Rightarrow \rho = \rho_c$ at $r = 0$.

as $\rho = \rho_c \theta^n$ as $\theta(\xi = 0) = 1$.

(ii) $\left(\frac{d\theta}{d\xi}\right)_{\xi = 0} = 0$ $\longrightarrow$ 2nd B.C.

As there should be one sharp edge or no cusp at the center of star.



(d) Given: $\xi_3 = 6.90$ and $-\xi_3^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_3} = 2.02$ i.e. $\varphi(\xi) = 2.02$.

$$\xi_3 = \frac{R}{\alpha} \text{ and } \alpha = \left[\frac{K\rho_c^{-2/3}}{\pi G}\right]^{1/2}$$

$$\Rightarrow R = 6.90 \times \sqrt{\frac{K}{\rho_c^{2/3}\pi G}}$$

$$R = \frac{6.90}{\sqrt{\pi G \rho_c^{2/3}}} \sqrt{\left[\frac{3}{a}\left(\frac{k_B}{\mu m_H}\right)^4 + \frac{1-\beta}{\beta^4}\right]^{1/3}}$$

$$R = \frac{6.90}{\sqrt{\pi G \rho_c^{2/3}}} \left[\frac{3}{a}\left(\frac{k_B}{m_H}\right)^4\right]^{1/6} \frac{(1-\beta)^{1/6}}{(\mu\beta)^{2/3}\rho_c^{1/3}}$$ //

e) To derive the expression for mass M:

$$M = \frac{18.1\, M_\odot}{\mu^2}\left(\frac{1-\beta}{\beta^4}\right)^{1/2}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho \Rightarrow dm = 4\pi r^2 \rho\, dr$$

$$M = \int 4\pi r^2 \rho\, dr = \int 4\pi \alpha^2 \xi^2 \rho_c \theta^3 \alpha\, d\xi$$

$$M = 4\pi \alpha^3 \rho_c \int_0^{\xi_3} \rho_c \theta^3 d\xi$$

$$= 4\pi\alpha^3 \rho_c \left| -\xi_3^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_3}\right)$$

$$= 4\pi \rho_c \times 2.02 \times \left(\frac{4k}{4\pi G \rho_c^{2/3}}\right)^{3/2}$$

$$= 4\pi\rho_c \times 2.02 \times \left[\frac{4\times\left[\frac{3}{a}\left(\frac{k_B}{\mu m_H}\right)^4\left(\frac{1-\beta}{\beta^4}\right)\right]^{1/3}}{4\pi G \rho_c^{2/3}}\right]^{3/2}$$

$$= 4\pi \times 2.02 \times \left[\frac{\rho_c}{\rho}\times\left[\frac{3}{a}\left(\frac{k_B}{\mu m_H}\right)^4\left(\frac{1-\beta}{\beta^4}\right)\right]^{1/2}\right]\times\left(\frac{1}{\pi G}\right)^{3/2}$$

$$= \frac{4\pi \times 2.02 \times \left(\frac{3}{a}\right)^{1/2}\left(\frac{k_B}{m_H}\right)^2 \frac{M_\odot}{\mu^2}\left[\frac{1-\beta}{\beta^4}\right]^{1/2}}{(\pi G)^{3/2} M_\odot}$$

$$= \frac{18.32}{\mu^2} M_\odot \left(\frac{1-\beta}{\beta^4}\right)^{1/2}$$

f) Given: $X = 0.73$, $Z = 0.02$.

Let's assume that the star is fully ionized.

$$Y = 1 - (0.73 + 0.02) = 0.25$$

Then $\mu = \left(2X + \frac{3X}{4} + \frac{Z}{2}\right)^{-1}$

$$= \left(2 \times 0.73 + \frac{3 \times 0.25}{4} + \frac{0.02}{2}\right)^{-1}$$

$$= (1.6575)^{-1} = 0.603$$

(g) As we have proved above:

$$M = \frac{M_\odot \times 18.1}{\mu^2} \left(\frac{1-\beta}{\beta^4}\right)^{1/2} \quad \left[\text{for Sun } M = M_\odot\right]$$

$$\frac{\mu^2}{18.1} = \left(\frac{1-\beta}{\beta^4}\right)^{1/2} \Rightarrow \frac{1-\beta}{\beta^4} = \left(\frac{\mu^2}{18.1}\right)^2$$

$$= \left[\frac{(0.0603)^2}{18.1}\right]^2$$

Solving for β, we get : $\beta = 0.99$

$$= 0.9996$$

This implies $\frac{P_{gas}}{P} = 0.99 \Rightarrow P_{rad} = 4.0 \times 10^{-4} P$

$$P_{rad} << P_{gas}.$$

(k) The ~~a~~ radiation pressure of black body radiation is given as: $\boxed{P_{rad} = \frac{1}{3} a T^4}$

$$P_{gas} = \left(\frac{\rho}{\mu m_H}\right) K_B T$$

At center of the sun $\rightarrow$ $P_{rad} = \frac{a}{3} \times (1.6 \times 10^7)^4$

$$= 2.528 \times 10^{-16} \times (1.6 \times 10^7)^4$$

$$= 1.657 \times 10^{13} \text{ Pa}$$

$$P_{gas} = \left(\frac{1.6 \times 10^5}{0.0603 \times 1.67 \times 10^{-27}}\right) \times 1.38 \times 10^{-23} \times 1.6 \times 10^7 \text{ Pa}$$

$$\boxed{P_{gas} = 3.507 \times 10^{16} \text{ Pa}}$$

$$P = P_{gas} + P_{rad} = 3.509 \times 10^{16}$$

$$\beta = \frac{P_{gas}}{P} = \frac{3.507}{3.509} = 9.999 \times 10^{-1}$$

i)

The answers from (g) & (h) are quite close.

2.) (a) Using radiative temperature gradient eqn, we know that:

$$\frac{dT}{dr} = \frac{-3}{4ac} \frac{\kappa \rho}{T^3} \frac{L}{4\pi r^2}$$

$$\frac{T_{wd} - T_c}{R_{wd} - 0} = \frac{3}{4ac} \frac{\kappa \rho}{T_c^3} \frac{L_{wD}}{4\pi R_{wD}^2}$$

$$T_c = \left[\frac{3}{4ac} \frac{\kappa \rho_c}{4\pi R_{wD}} L_{wD}\right]^{1/4}$$ ~ Sirius B ($L_{wD} \approx 0.03 L_\odot$)

$M_{WD} = 1 M_\odot$ & $R_{WD} = 0.01 R_\odot$, taking the central avg. density $\rho_c \sim \dfrac{M_\odot}{\frac{4}{3}\pi R_\odot^3}$

We can find K then →

$$K\rho = n_e \sigma_{TH} \quad \text{\& also } n_e = \frac{\rho(1+X)}{2m_H}$$

$$K = \frac{1+X \, \sigma_{TH}}{2m_H} = 0.013\,(1+X)\ m^2 kg^{-1}$$

$$\approx 0.02\,(1+X)$$

For $e^{\ominus}$ scattering: $X = 0$.

$$\Rightarrow K = 0.02.$$

Using this we get: $T_c = \left[\frac{3^2}{16\sigma} \times \frac{0.02 \times M}{(4\pi)^2 \cdot R^4_{WD}} \times 0.03\, L_\odot\right]^{1/4}$

We know $L_\odot = 3.846 \times 10^{26}\, W$
$M_\odot = 1.989 \times 10^{30}\, kg$
$R_\odot = 6.96 \times 10^8\, m$

$$\Rightarrow T_c = \left[\frac{9 \times 0.02 \times 1.989 \times 10^{30} \times 3.846 \times 10^{26} \times 0.03}{16 \times 5.6703 \times (4\pi)^2 \times 10^{-8} \times (6.96 \times 10^{8})^4}\right]^{1/4}$$

$$= 5.92 \times 10^7\, K$$

So the central temperature is of the order $10^7 K$.

) For the to burn we need central temp: in the range 10^8-10^9K. But the core temp is of the order of 10^7K.

So fusion of H occurs & He doesn't undergo fusion, with temp only pp-1, chain occurs.

$\therefore$ CNO cycle cannot occur in this temp & hence ~~beta~~ heavier element will be present only in trace amounts.

(c)

The fermi Energy E_F is given by:

$$E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$

For an $e^\ominus$; $n = n_e = \frac{\rho}{\mu M_H}$

$$\text{Then, } E_F = \frac{h^2}{2m_e}\left(\frac{3\pi^2 \rho}{\mu M_H}\right)^{2/3}$$

Avg. thermal energy of $e^\ominus$ is $\frac{3}{2}k_BT$. If $\frac{3}{2}k_BT < E_F$ then the probability of an $e^\ominus$ to make a transition to an unoccupied state ~~is~~ is less and the $e^\ominus$ gas will not be generated. For degenerate gas:

$$\frac{3}{2}k_BT < E_F$$

$$E_F > k_BT \Rightarrow \frac{E_F}{k_B} \gg T$$

(d) $P_{gas} = \frac{\rho}{\mu m_H} k_B T$

For electrons, $\mu_e \approx \frac{2}{1+X} \approx 2$

$P_{gas} = \frac{\rho}{2 m_H} k_B T$

For non-relativistic case: $P_e = \frac{8\pi P_F^5}{15 h^3 m_e}$ where $P_F = \left(\frac{3\rho}{8\pi \mu_e m_H}\right)^{1/3} h$

$= K_{NR} \rho^{5/3} = \frac{1.00 \times 10^7 \rho^{5/3}}{\mu_e}$.

Then $\frac{P_e}{P_{gas}} = \frac{1.00 \times 10^7 \rho^{5/3} / \mu_e^{5/3}}{\frac{\rho k_B T}{\mu_e m_H}} = \frac{1.0 \times 10^7 m_H}{k_B} \left(\frac{\rho}{\mu}\right)^{2/3} T^{-1}$

Upon taking $\rho = \bar{\rho} = \frac{3M}{4\pi R^3}$, we get $\frac{P_e}{P} = m_e \frac{1.0 \times 10^{+7}}{1.38 \times 10^{-23}} \times \left(\frac{3 M_\odot}{4\pi (r_{wp})^2 \times 2}\right)^{2/3} T^{-1}$

$= 1.9924 \times 10^7 \, T^{-1}$

$M_\odot = 1.989 \times 10^{30}$ kg

$R_\odot = 6.96 \times 10^8$ m.

For ultra relativistic case, $P_e = \frac{2\pi c P_f^4}{3h^3}$

$= K_{UR} \rho^{4/3} = \frac{1.24 \times 10^{10} \rho^{4/3}}{\mu_e^{4/3}}$

Then $\frac{P_e}{P_{gas}} = \frac{1.24 \times 10^{10}}{\frac{\rho k_B T}{\mu m_H}} \left(\frac{\rho}{\mu_e}\right)^{4/3}$

$$= \frac{1.24 \times 10^{10} \times m_H}{k_B} \left(\frac{\rho}{\mu_e}\right)^{1/3}$$

$$= 1.335 \times 10^{7}\ T^{-1}$$

We can see that the dominant distribution of pressure in WDs is due to degeneracy pressure than gas ~~pres~~ pressure.

3.) Cosmic rays are highly energetic charged particles ($e^{\ominus}$s, protons and heavy nuclei) continously bombarding Earth from all directions.

Yet, the origin of cosmic rays are still unknown as they gett scatter around the universe too much before reaching Earth. Though it is believed that the cosmic rays get accelerated primarily due to supernovae explosions.

Now if we go for the energy density of cosmic rays. The relation b/w the energy spectrum and energy density is as follows from the relation b/w flux & no. density of cosmic rays:

$$\text{Flux}\left(\frac{\text{Particles}}{cm^2\ s\ sr}\right) = \frac{\rho_{CR}\,\beta c}{4\pi}$$

The energy density ρ_E is therefore:

$$\rho_E = 4\pi \int E\,\frac{dN}{dE}\,\frac{dE}{\beta c} = \int \frac{4\pi E^2}{\beta c}\,\frac{dN}{dE}\, \ln E$$

where E is the energy of cosmic rays.
c is the speed of light.

The above eqn is useful so as to understand the spectrum of cosmic rays and through that it can be concluded that, these particles aren't arriving from the Sun.

The above eqn helps me to understand the mechanism for cosmic ray confinement, i.e. the coupling between the charged particles and the tangled magnetic field lines that thread the interstellar medium.
And this can be plausibly seen by comparing the energy densities of cosmic rays to the energy in magnetic fields.

On further note to understand the cosmic ray acceleration.

Friedmann equations are used:

$$\frac{\dot{a}^2 + kc^2}{a^2} = \frac{8G\rho + \Lambda c^2}{3}$$

where 'a' is the scale factor,

G, Λ, c are universal constant.