ASTRONOMY & ASTROPHYSICS

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PROBLEM SET-01

1. Data:

Important Conversion factors:
$M_\odot \approx 2 \times 10^{30}$ Kg
pc $\approx 3 \times 10^{16}$ M
yr $\approx 3 \times 10^{7}$ S

Given:

d (distance b/w sun & galactic center) = 8 Kpc

$\vec{V}$ (velocity) = 220 Km/sec

"Circular path"

To find: Mass of the Galaxy (rough estimation)

Solution:

Assuming that most of the Milky way Galaxy mass will lie at the center, and by neglecting the other masses that are further away in the galaxy.

We can write the Gravitational force and equate it with centripetal force:

$$\frac{G \times M_\odot M_{mw}}{d^2} = M_\odot \frac{V^2}{d}$$

$$= \frac{6.6713\times10^{-11}\times\cancel{2\times10^{30}}\times M_{MW}}{(21\times10^{19})^2} = \cancel{M_\odot}\times\frac{(220)^2\times\cancel{2\times10^{30}}}{21\times10^{19}}$$

$$\rightarrow \quad M_{MW} \approx 1.1\times10^{11} M_\odot$$

Haven't taken approx values.

The experimental result says $M_{MW} = 10^{12} M_\odot$, in the galaxies with ~~ae~~ well observed rotation curves, the discrepancy in the mass occurs ~~due to~~ when the acceleration is below a critical acceleration.

Modified Newtonian dynamics says deviation from $1/r$ occurs beyond a fundamental length scale. The Milogrim suggestion says force law becomes $1/r$ below a critical acceleration.

2.) Data:

Given: $d = 4\,pc$

$M_1 = 2$

$L_2 = 3.9 \times 10^{26}\,W$

$M_2 = 5$

To find: $M_1 = ?$

(Absolute magnitude)

$L_1 = ?$

Solution:

We have the relation of absolute and apparent magnitudes as:

$$m_1 - M = 5 \log_{10}\left(\frac{d}{10\,[pc]}\right)$$

$$\Rightarrow 2 - M_1 = 5 \log_{10}\left(\frac{4}{10}\right)$$

$$M = 2 - 5 \log \frac{4}{10}$$

$$\approx 2 - 5 \times (-0.4)$$

$$= 2 + 2$$

$$= 4$$

So we have $M_1 = 4$; $M_2 = 5$

$$M_1 - M_2 = -2.5 \log \frac{L_1}{L_2}$$

$$\Rightarrow 4-5 = -2.5 \log_{10} \frac{L_1}{3.9\times10^{26}}$$

$$\Rightarrow 10^{0.4} = \frac{L_1}{3.9\times10^{26}\,W}$$

$$\Rightarrow L_1 \approx 2.5 \times 3.9\times10^{26}\,W$$

$$= 9.8\times10^{26}\,W$$

3.

(a)

$$\theta = \frac{1.22\lambda}{D}$$

$D \rightarrow$ Diameter of telescope/antenna.

Given: $D = 10$ Km.

$\lambda = 0.2 - 6$ m

$$\theta = \frac{1.22\times3.5}{10000} \text{ radians.}$$

$$= 4.27\times10^{-4} \times 3600 \times \frac{180}{\pi} \text{ arc sec.}$$

$$= 88.0437''$$

(b) Taking $\lambda = 550 \times 10^{-9} m$ for optical light.

$$D = \frac{1.22 \times 550 \times 10^{-9}}{4.27 \times 10^{-4}}$$

$$= 157.14285 \times 10^{-5}$$

$$= 1.6 \times 10^{-3} \, m$$

$$= 1.6 \, mm$$

(c) This is quite difficult to achieve because the air through which light rays will pass are in turbulent motion even before reaching the telescope. One solution might be to place is above the Earth's atmosphere.

4.) (a) To show: Energy density of black body at temp. T is given by:

$$U = a_B T^4$$

where; $$a_B = \frac{8\pi K_B^4}{c^3 h^3} \int_0^\infty \frac{x^3 \, dx}{e^x - 1}$$

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Proof: Specific energy emitted by a black body is given by:

$$U_\nu = \frac{8\pi h \nu^3}{c^3 e^{h\nu/K_BT} - 1}$$

then the total energy density of a black body at temp T is:

$$U = \int_0^\infty U_\nu \, d\nu$$

$$= \int_0^\infty \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/K_BT} - 1} \, d\nu$$

Now; let $x = \frac{h\nu}{K_BT}$ $\Rightarrow dx = \frac{h}{K_BT} d\nu$ & $\nu = \frac{K_BT}{h} x$.

$$U = \frac{8\pi h}{c^3} \int \frac{\left(\frac{K_BT}{h}\right)^3 x^3}{(e^x - 1)} \times \left(\frac{K_BT}{h}\right) dz$$

$$U = \frac{8\pi K_B^4}{h^3 c^3} T^4 \int_0^\infty \frac{x^3}{e^x - 1} dx$$

$U = a_B T^4$; where $a_B = \frac{8\pi K_B^4}{c^3 h^3} \int_0^\infty \frac{x^3 dx}{e^x - 1}$

(b) To show: Energy radiated in unit time from unit area on the surface of black body is given by σT^4, where:

$$\sigma = \frac{c a_B}{4}$$

Proof: $F = \int_\nu \int I_\nu \cos\theta \, d\Omega \, d\nu$

Since black body is isotropic; The intensity is given as:

$$I_\nu = \frac{c}{4\pi} U_\nu$$

From the (a) part we got: $U(T) = a_B T^4$

thus: $$F = \frac{c a_B}{4\pi} T^4 \int \cos\theta \underbrace{\sin\theta \, d\phi \, d\theta}_{d\Omega}.$$

We know that emission from half the sphere is visible;

$$F = \frac{c a_B}{4\pi} T^4 \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{c a_B}{4\pi} T^4 \cdot \frac{1}{2} \cdot 2\pi$$

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(b) ~~[illegible]~~ $= \frac{c\, a_B}{4\pi} T^4$

Thus; $F = \sigma T^4$; where $\sigma = \frac{c\, a_B}{4}$

(c)

Specific energy $U_\nu = \frac{8\pi h \nu^3}{c^3\, e^{h\nu/k_B T} - 1}$

Taking it's derivative w.r.t ν & equating to 0.

$$\frac{dU_\nu}{d\nu} = 0$$

$$\Rightarrow \frac{d}{d\nu}\left(\frac{\nu^3}{e^{h\nu/k_B T} - 1}\right) = 0 \quad , \text{ with } x = \frac{\nu_{max}}{T}$$

$$\Rightarrow 3 e^{\frac{h x_0}{k_B}} - 3 - \frac{h x_0}{k_B} e^{\frac{h x_0}{k_B}} = 0$$

Numerical solution was done using mathematica:

$$\frac{\nu_{max}}{T} \approx 5.88 \times 10^{10}\ Hz\, K^{-1}.$$

(d) $$U_\lambda = \frac{8\pi h}{\lambda^5} \frac{1}{\left(e^{hc/K_BT\lambda} - 1\right)}$$

$$\frac{dU_\lambda}{d\lambda} = 8\pi h \left[-5\lambda^{-6} \frac{1}{\left(e^{hc/K_BT\lambda} - 1\right)} + \frac{1}{\lambda^5} \frac{1}{\left(e^{hc/K_BT\lambda} - 1\right)^2} \cdot \frac{hc}{K_BT\lambda^2} \cdot e^{hc/K_BT\lambda} \right]$$

For $U_\lambda =$ Maximum $\Rightarrow \frac{dU_\lambda}{d\lambda} = 0$

$$\frac{5}{\lambda^6} \frac{1}{\cancel{\left(e^{hc/K_BT\lambda} - 1\right)}} = \frac{1}{\lambda^5} \frac{1}{\left(e^{hc/K_BT\lambda} - 1\right)^{\cancel{2}}} \frac{hc}{K_BT\lambda^2} e^{hc/K_BT\lambda}$$

$$5e^{hc/K_BT\lambda} - 5 - \frac{hc}{K_BT\lambda} e^{hc/K_BT\lambda} = 0$$

Let, $\frac{c}{\lambda T} = x = \frac{c}{\lambda_{max} T}$ $[U_\lambda = max]$

$$\Rightarrow 5e^{hx/K_B} - 5 - \frac{hx}{K_B} e^{\frac{hx}{K_B}} = 0$$

As we can see the above equ is not same as equ in part(c)

$$\therefore \frac{\nu_{max}}{T} \approx 5.88 \times 10^{10}\ Hz\,K^{-1}.$$

5.) (a) To find: $B_\nu(T)$ in the limit $h\nu << KT$. (Rayleigh-Jeans law)

Solution: From the definition; we can write $B_\nu(T)$ as:

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/K_BT} - 1}$$

Taking the limit $h\nu << KT$.

Taking the term $e^{\frac{h\nu}{KT}}$

$$e^{h\nu/KT} = 1 + \frac{h\nu}{K_BT} + \frac{1}{2!}\left(\frac{h\nu}{K_BT}\right)^2 + \frac{1}{3!}\left(\frac{h\nu}{K_BT}\right)^3 + \dots$$

$$= 1 + \frac{h\nu}{K_BT} \quad \left\{ \text{as } \frac{h\nu}{K_BT} << 1 \right\}$$

On substituting we get:

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{1 + \frac{h\nu}{K_BT} - 1}$$

$$= \frac{2h\nu^3}{c^2} \frac{K_BT}{h\nu}$$

$$B_\nu(T) = \frac{2\nu^2}{c^2} K_BT$$

(b) For deriving the Wein's law;

we know that Plank's lawsays:

$$h\nu >> KT$$

$$\Rightarrow \exp\left(\frac{h\nu}{K_B T}\right) >> 1.$$

$$B_\nu(T) \approx \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{K_B T}\right)}$$

$$= \frac{2h\nu^3}{c^2} \exp\left(-\frac{h\nu}{K_B T}\right)$$

(d) The plot has been attached as .py file..

* The spectral flux density at surface is given as: $\pi B_\nu(T)$

Therefore we need to multiply all plots by ~~a val~~ π.

(e) $$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/K_B T} - 1}$$

$$\frac{dB_\nu(T)}{dT} = \frac{2h\nu^3}{c^2} - \frac{1}{\left(e^{\frac{h\nu}{K_B T}} - 1\right)^2} e^{h\nu/K_B T} \cdot \left(-\frac{1}{T^2}\right)$$

$$= \frac{2h\nu^3}{c^2} \frac{1}{\left(e^{h\nu/K_B T} - 1\right)^2} \frac{e^{h\nu/K_B T}}{T^2}$$

We know that $\left(e^{h\nu/k_BT}-1\right)^2 > 0$ & also $\nu, T > 0$

$$\Rightarrow \frac{dB_\nu}{dT} > 0 \Rightarrow \text{Monotonically increasing.}$$

6.) For deriving the equation.

$$I_\nu(t_\nu, \mu) = B_\nu + \mu \frac{dP_\nu}{dT_\nu}$$

For an optical depth t_ν which is close to T_ν, we can write source function as:

$$S_0(t_\nu) = B_\nu(T_\nu) + (t_\nu - T_\nu)\frac{dB_\nu}{dT_\nu} \quad \text{—(1)}$$

The intensity can be given as:

for $0 \leq \mu \leq 1$: $I_\nu(T_\nu, \mu) = \int_{T_\nu}^{\infty} S_\nu e^{-(t_\nu - T_\nu)/\mu} \frac{dt_\nu}{\mu}$

for $0 \geq \mu \geq -1$: $I_\nu(t_\nu, \mu) = \int_{0}^{T_\nu} S_\nu e^{-(t_\nu - T_\nu)/-\mu} \frac{dt_\nu}{(-\mu)}$

Substituting the ① in the above equs:

$$I_\nu(t_\nu, \mu) = \int_{T_\nu}^{0} S_\nu \, e^{-(t_\nu - T_\nu)/\mu} \frac{dt_\nu}{\mu}$$

$$= \int_{T_\nu}^{\infty} \left[B_\nu(T_\nu) \, e^{-(t_\nu - T_\nu)/\mu} \frac{dt_\nu}{\mu} + (t_\nu - T_\nu) \frac{dB_\nu}{dT_\nu} e^{-(t_\nu - T_\nu)/\mu} \frac{dt_\nu}{\mu} \right]$$

$$= B_\nu \int_{T_\nu}^{\infty} e^{-(t_\nu - T_\nu)/\mu} \frac{dt_\nu}{\mu} + \frac{dB_\nu}{dT_\nu} \left[\int_{T_\nu}^{\infty} (t_\nu - T_\nu) \, e^{-(t_\nu - T_\nu)/\mu} \frac{dt_\nu}{\mu} \right]$$

Let $\frac{(t_\nu - T_\nu)}{\mu} = x \qquad \frac{dt_\nu}{\mu} = dx \qquad t_\nu \to T_\nu \quad x \to 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t_\nu \to \infty \quad x \to \infty$

Then we will get,

$$I_\nu(T_\nu, \mu) = B_\nu \int_0^{\infty} e^{-x} \, dx + \frac{dB_\nu}{dT_\nu} \int_0^{\infty} \mu x e^{-x} \, dx$$

$$= B_\nu \left[\frac{e^{-x}}{-1} \right]_0^{\infty} + \mu \frac{dB_\nu}{dT_\nu} \left[\left| \frac{x e^{-x}}{(-1)} \right|_0^{\infty} + \left| \frac{e^{-x}}{-1} \right|_0^{\infty} \right]$$

$$= B_\nu + \mu \frac{dB_\nu}{dT_\nu}$$

If $-1 \le \mu < 0$; then above eqn can be written as:

$$I_\nu(T_\nu,\mu) = \int_0^{T_\nu} S_\nu\, e^{-(T_\nu - t_\nu)/-\mu} \frac{dt_\nu}{(-\mu)}$$

$$= \int_0^{T_\nu} \left(B_\nu - (T_\nu - t_\nu)\frac{dB_\nu}{dT_\nu}\right) e^{-(T_\nu - t_\nu)/-\mu} \frac{dt_\nu}{(-\mu)}$$

$$= \int_0^{T_\nu} B_\nu\, e^{(T_\nu - t_\nu)/\mu} \frac{dt_\nu}{-\mu} - \int_0^{T_\nu} (T_\nu - t_\nu)\frac{dB_\nu}{dT_\nu} e^{(T_\nu - t_\nu)} \frac{dt_\nu}{(-\mu)}$$

If we take $\frac{(T_\nu - t_\nu)}{\mu} = x \qquad \frac{dt_\nu}{-\mu} = dx.$ As $t_\nu \to 0 \Rightarrow x \to \frac{T_\nu}{\mu}$

$t_\nu \to T_\nu \Rightarrow x \to 0$

$$I_\nu(t_\nu,\mu) = \int_{T_\nu/\mu}^{0} B_\nu e^x dx - \mu \frac{dB_\nu}{dT_\nu}\int_{\frac{T_\nu}{\mu}}^{0} x e^x dx$$

$$= B_\nu e^x \Big|_{\frac{T_\nu}{\mu}}^{0} - \mu \frac{dB_\nu}{dT_\nu} \left| x e^x - e^x \right|_{\frac{T_\nu}{\mu}}^{0}$$

$$= B_\nu\left[1 - e^{\frac{T_\nu}{\mu}}\right] - \mu \frac{dB_\nu}{dT_\nu}\left[-1 - \left(\frac{T_\nu}{\mu} \cdot e^{\frac{T_\nu}{\mu}} - e^{\frac{T_\nu}{\mu}}\right)\right]$$

$$= B_\nu\left[1 - e^{\frac{T_\nu}{\mu}}\right] - \mu \frac{dB_\nu}{dT_\nu}\left(e^{\frac{T_\nu}{\mu}} - \frac{T_\nu}{\mu} e^{\frac{T_\nu}{\mu}} - 1\right)$$

For $T_\nu >>> 1$ & $\mu < 0 \Rightarrow e^{T_\nu/\mu} << 1.$

$$\therefore I_\nu(t_\nu;\mu) = B_\nu + \mu \frac{dB_\nu}{T_\nu}$$ // Proved.

7.) From Eddington approximation:

$$S = \frac{3F}{4\pi}(\tau + q) \quad — ①$$

Now;

$$I_\nu(\tau, \mu) = \int_{\tau_\nu}^{\infty} S_\nu e^{-(t_\nu - T_\nu/\mu)} \frac{dt_\nu}{\mu}$$

Assuming constant flux; $\frac{dF}{d\tau} = 0$

$$\Rightarrow J = S$$

Now; specific intensity is given as:

$$I(\tau, \mu \geq 0) = \int_\tau^\infty S e^{-(t-\tau)/\mu} \frac{dt}{\mu} \quad — ②$$

Substituting equ① in ②;

$$I(\tau, \mu) = \frac{3F}{\mu} \int_\tau^\infty e^{-(t-\tau)/\mu} \left(\frac{dt}{\mu}\right)(t+q)$$

$$= \frac{3F}{4\pi} \int_\tau^\infty t e^{-(t-\tau)\mu} dt/\mu + q\int_\tau^\infty e^{-(t-\tau)/\mu} dt/\mu$$

if $\left(\frac{t-\tau}{\mu}\right) = x$

$$dx = \frac{dt}{\mu}$$

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$$\Rightarrow I = \frac{3F}{4\pi}\int_0^\infty (\mu x + \tau)\, e^{-x}\, dx + q\int_0^\infty e^{-x}\, dx$$

$$\Rightarrow I(\tau,\mu) = \frac{3F}{4\pi}\left[\mu + \tau + q\right]$$

(b) when $\tau = 0$

$$I(0,\mu) = \frac{3F}{4\pi}\left[\mu + q\right]$$

(c)

$$T^4 = \frac{3}{4} T_{eff}^4 \left(\tau + \frac{2}{3}\right)$$

For source as Blackbody, $T = T_{eff}$

Therefore $T_{eff}^4 = \frac{3}{4} T_{eff}^4 \left(\tau + \frac{2}{3}\right)$

$$\Rightarrow \tau = 2/3$$

∴ The avg. optical depth of light reaches us : $\tau = 2/3$

(d) Optical depth ; $\tau = 2/3$

Probability of escaping a photon $= e^{-\tau}$

$$= e^{-2/3}$$

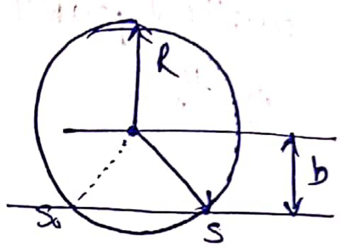
$$= 0.513.$$

8) (a) Eqn of radiative transfer can be given as:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

If our gas cloud is thin, we assume absorbtion is minimum, hence α_ν can be neglected. i.e. $\alpha_\nu \approx 0$.

$$\frac{dI_\nu}{ds} = j_\nu \Rightarrow I_\nu$$



$$\frac{dI_\nu}{ds} = j_\nu$$

$$I_\nu = I_\nu(S_0) + \int_{S_0}^{S} j(s')\,ds$$

Let's consider $I_\nu(S_0) = 0$, j_ν to be constant.

$$I_\nu(s) = \int_{S_0}^{S} j_\nu(s')\,ds'$$

$$s = b\tan\theta \Rightarrow ds = b\,d(\tan\theta)$$

$$I_\nu(s) = 2j_\nu b \int_0^{\cos^{-1}(b/R)} d\tan\theta$$

$$= 2j_\nu b \tan\theta \Big|_0^{\cos^{-1}(b/R)}$$

$$= 2 j_\nu b \left(\sqrt{1 - b^2/R^2} \Big/ b/R \right) = 2 j_\nu \sqrt{R^2 - b^2}$$

(b) We can write emission coefficient j_ν as

$$\frac{j_\nu}{4} = \frac{P_\nu}{4\pi}$$ => P_ν is the radiative power per unit vol.

The total emitted power is then $= \int P_\nu \times \text{volume} \times d\nu$

$$= \frac{4}{3}\pi R^3 P \quad -①$$

From Stephan-Boltzman equ:

$$\text{flux} = \sigma T_{eff}^4$$

And the total power can be given as: = flux × areas

$$= 4\pi R^2 \times \sigma T_{eff}^4 \quad -②$$

Equating ① & ②, we can write:

$$\frac{4}{3}\pi R^3 P = 4\pi R^2 \sigma T_{eff}^4$$

$$T_{eff}^4 = \frac{RP}{3\sigma} \Rightarrow T_{eff} = \left(\frac{RP}{3\sigma}\right)^{1/3}$$

(c) If the cloud is optically thick, then $\tau_\nu >> 1$.

then $\frac{dI_\nu}{ds} = -I_\nu + S_\nu$ as I_ν tries to approach S_ν.

Then we get $S_\nu = B_\nu(T) = I_\nu(T)$ (From Kirchoff law).

So, the intensity at a distance B from the center is independent of ray path.

For the same case the object is a blackbody. $T \equiv T_{eff}$.