EXAMPLE 4.1

In isothermal liquid flows, the fluid density is typically a known constant. What are the dependent field variables in this case? How many equations are needed for a successful mathematical description of such flows? What physical principles supply these equations?

Solution

When the fluid's temperature is constant and its density is a known constant, the thermal energy of fluid elements cannot be changed by heat transfer or work because $dT = dv = 0$, so the thermodynamic characterization of the flow is complete from knowledge of the density. Thus, the dependent field variables are $\mathbf{u}$, the fluid's velocity (momentum per unit mass), and the pressure, p. Here, p is not a thermodynamic variable; instead it is a normal force (per unit area) developed between neighboring fluid particles that either causes or results from fluid-particle acceleration, or arises from body forces. Thus, four equations are needed; one for each component of $\mathbf{u}$, and one for p. These equations are supplied by the principle of mass conservation, and three components of Newton's second law for fluid motion (conservation of momentum).

4.2 CONSERVATION OF MASS

Setting aside nuclear reactions and relativistic effects, mass is neither created nor destroyed. Thus, individual mass elements − molecules, grains, fluid particles, etc. − may be tracked within a flow field because they will not disappear and new elements will not spontaneously appear. The equations representing conservation of mass in a flowing fluid are based on the principle that the mass of a specific collection of neighboring fluid particles is constant. The volume occupied by a specific collection of fluid particles is called a *material volume* $V(t)$. Such a volume moves and deforms within a fluid flow so that it always contains the same mass elements; none enter the volume and none leave it. This implies that a material volume's surface $A(t)$, a material surface, must move at the local fluid velocity $\mathbf{u}$ so that fluid particles inside $V(t)$ remain inside and fluid particles outside $V(t)$ remain outside. Thus, a statement of conservation of mass for a material volume in a flowing fluid is:

$$\frac{d}{dt}\int_{V(t)} \rho(\mathbf{x}, t)dV = 0, \tag{4.1}$$

where ρ is the fluid density. Figure 3.20 depicts a material volume when the control surface velocity $\mathbf{b}$ is equal to $\mathbf{u}$. The primary concept here is equivalent to an infinitely flexible, perfectly sealed thin-walled balloon containing fluid. The balloon's contents play the role of the material volume $V(t)$ with the balloon itself defining the material surface $A(t)$. And, because the balloon is sealed, the total mass of fluid inside the balloon remains constant as the balloon moves, expands, contracts, or deforms.

Based on (4.1), the principle of mass conservation clearly constrains the fluid density. The implications of (4.1) for the fluid velocity field may be better displayed by using Reynolds transport theorem (3.35) with $F = \rho$ and $\mathbf{b} = \mathbf{u}$ to expand the time derivative in (4.1):

$$\int_{V(t)} \frac{\partial \rho(\mathbf{x}, t)}{\partial t} dV + \int_{A(t)} \rho(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t)\cdot\mathbf{n}\, dA = 0. \tag{4.2}$$

This is a mass-balance statement between integrated density changes within $V(t)$ and integrated motion of its surface $A(t)$. Although general and correct, (4.2) may be hard to utilize in practice because the motion and evolution of $V(t)$ and $A(t)$ are determined by the flow, which may be unknown.

To develop the integral equation that represents mass conservation for an *arbitrarily moving* control volume $V^*(t)$ with surface $A^*(t)$, (4.2) must be modified to involve integrations over $V^*(t)$ and $A^*(t)$. This modification is motivated by the frequent need to conserve mass within a volume that is not a material volume, for example a stationary control volume. The first step in this modification is to set $F = \rho$ in (3.35) to obtain:

$$\frac{d}{dt}\int_{V^*(t)} \rho(\mathbf{x}, t) dV - \int_{V^*(t)} \frac{\partial \rho(\mathbf{x}, t)}{\partial t} dV - \int_{A^*(t)} \rho(\mathbf{x}, t)\mathbf{b}\cdot\mathbf{n}\, dA = 0. \tag{4.3}$$

The second step is to choose the arbitrary control volume $V^*(t)$ to be instantaneously coincident with material volume $V(t)$ so that *at the moment of interest* $V(t) = V^*(t)$ and $A(t) = A^*(t)$. At this coincidence moment, the $(d/dt)\int \rho dV$-terms in (4.1) and (4.3) are not equal; however, the volume integration of $\partial\rho/\partial t$ in (4.2) is equal to that in (4.3) and the surface integral of $\rho\mathbf{u}\cdot\mathbf{n}$ over $A(t)$ is equal to that over $A^*(t)$:

$$\int_{V^*(t)} \frac{\partial \rho(\mathbf{x}, t)}{\partial t} dV = \int_{V(t)} \frac{\partial \rho(\mathbf{x}, t)}{\partial t} dV = -\int_{A(t)} \rho(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t)\cdot\mathbf{n}\, dA = -\int_{A^*(t)} \rho(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t)\cdot\mathbf{n}\, dA. \tag{4.4}$$

where the middle equality follows from (4.2). The two ends of (4.4) allow the central volume-integral term in (4.3) to be replaced by a surface integral to find:

$$\frac{d}{dt}\int_{V^*(t)} \rho(\mathbf{x}, t) dV + \int_{A^*(t)} \rho(\mathbf{x}, t)(\mathbf{u}(\mathbf{x}, t) - \mathbf{b})\cdot\mathbf{n}\, dA = 0, \tag{4.5}$$

where $\mathbf{u}$ and $\mathbf{b}$ must both be observed in the same frame of reference; they are not otherwise restricted. This is the general integral statement of conservation of mass for an arbitrarily moving control volume. It can be specialized to stationary, steadily moving, accelerating, or deforming control volumes by appropriate choice of $\mathbf{b}$. In particular, when $\mathbf{b} = \mathbf{u}$, the arbitrary control volume becomes a material volume and (4.5) reduces to (4.1).

The differential equation that represents mass conservation is obtained by applying Gauss' divergence theorem (2.30) to the surface integration in (4.2):

$$\int_{V(t)} \frac{\partial \rho(\mathbf{x}, t)}{\partial t} dV + \int_{A(t)} \rho(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t)\cdot\mathbf{n}\, dA = \int_{V(t)} \left\{ \frac{\partial \rho(\mathbf{x}, t)}{\partial t} + \nabla\cdot(\rho(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t)) \right\} dV = 0. \tag{4.6}$$

The final equality can only be possible if the integrand vanishes at every point in space. If the integrand did not vanish at every point in space, then integrating (4.6) in a small volume around a point where the integrand is nonzero would produce a nonzero integral. Thus, (4.6) requires:

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} + \nabla\cdot(\rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)) = 0 \quad \text{or,} \quad \text{in index notation:}\ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0. \tag{4.7}$$

This relationship is called the *continuity equation*. It expresses the principle of conservation of mass in differential form, but is insufficient for fully determining flow fields because it is a single equation that involves two field quantities, ρ and $\mathbf{u}$, and $\mathbf{u}$ is a vector with three components.

The second term in (4.7) is the divergence of the mass-density flux $\rho\mathbf{u}$. Such *flux divergence* terms frequently arise in conservation statements and can be interpreted as the net loss at a point due to divergence of a flux. For example, the local ρ will decrease with time if $\nabla\cdot(\rho\mathbf{u})$ is positive. Flux divergence terms are also called *transport* terms because they transfer quantities from one region to another without making a net contribution over the entire field. When integrated over the entire domain of interest, their contribution vanishes if there are no sources at the boundaries.

The continuity equation may alternatively be written using the definition of D/Dt (3.5) and $\partial(\rho u_i)/\partial x_i = u_i \partial\rho/\partial x_i + \rho \partial u_i/\partial x_i$ [see (B3.6)]:

$$\frac{1}{\rho(\mathbf{x},t)}\frac{D}{Dt}\rho(\mathbf{x},t) + \nabla\cdot\mathbf{u}(\mathbf{x},t) = 0. \tag{4.8}$$

The derivative $D\rho/Dt$ is the time rate of change of fluid density following a fluid particle. It will be zero for *constant density* flow where $\rho =$ constant throughout the flow field, and for *incompressible* flow where the density of fluid particles does not change but different fluid particles may have different density:

$$\frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla\rho = 0. \tag{4.9}$$

Taken together, (4.8) and (4.9) imply:

$$\nabla\cdot\mathbf{u} = 0 \tag{4.10}$$

for incompressible flows. Constant density flows are a subset of incompressible flows; $\rho =$ constant is a solution of (4.9) but it is not a general solution. A fluid is usually called *incompressible* if its density does not change with *pressure*. Liquids are almost incompressible. Gases are compressible, but for flow speeds less than $\sim$100 m/s (that is, for Mach numbers <0.3) the fractional change of absolute pressure in a room temperature airflow is small. In this and several other situations, density changes in the flow are also small and (4.9) and (4.10) are valid.

The general form of the continuity equation (4.7) is typically required when the derivative $D\rho/Dt$ is nonzero because of changes in the pressure, temperature, or molecular composition of fluid particles.

EXAMPLE 4.2

The density in a horizontal flow $\mathbf{u} = U(y, z)\mathbf{e}_x$ is given by $\rho(\mathbf{x}, t) = f(x - Ut, y, z)$, where $f(x, y, z)$ is the density distribution at $t = 0$. Is this flow incompressible?

Solution

There are two ways to answer this question. First, consider (4.9) and evaluate $D\rho/Dt$, letting $\xi = x - Ut$:

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\nabla\rho = \frac{\partial\rho}{\partial t} + U\frac{\partial\rho}{\partial x} = \frac{\partial\rho}{\partial\xi}\frac{\partial\xi}{\partial t} + U\frac{\partial\rho}{\partial\xi}\frac{\partial\xi}{\partial x} = \frac{\partial\rho}{\partial\xi}(-U) + U\frac{\partial\rho}{\partial\xi}(1) = 0.$$

Second, consider (4.10) and evaluate $\nabla\cdot\mathbf{u}$:

$$\nabla\cdot\mathbf{u} = \frac{\partial U(y,z)}{\partial x} + 0 + 0 = 0.$$

In both cases, the result is zero. This is an incompressible flow, but the density may vary when f is not constant.

4.3 STREAM FUNCTIONS

Consider the steady form of the continuity equation (4.7):

$$\nabla\cdot(\rho\mathbf{u}) = 0. \tag{4.11}$$

The divergence of the curl of any vector field is identically zero (see Exercise 2.21), so $\rho\mathbf{u}$ will satisfy (4.11) when written as the curl of a vector potential $\boldsymbol{\Psi}$:

$$\rho\mathbf{u} = \nabla\times\boldsymbol{\Psi}, \tag{4.12}$$

which can be specified in terms of two scalar functions: $\boldsymbol{\Psi} = \chi\nabla\psi$. Putting this specification for $\boldsymbol{\Psi}$ into (4.12) produces $\rho\mathbf{u} = \nabla\chi\times\nabla\psi$, because the curl of any gradient is identically zero (see Exercise 2.22). Furthermore, $\nabla\chi$ is perpendicular to surfaces of constant χ, and $\nabla\psi$ is perpendicular to surfaces of constant ψ, so the mass flux $\rho\mathbf{u} = \nabla\chi\times\nabla\psi$ will be parallel to surfaces of constant χ and constant ψ. Therefore, three-dimensional streamlines are the intersections of the two stream surfaces, or stream functions in a three-dimensional flow.

The situation is illustrated in Figure 4.1. Consider two members of each of the families of the two stream functions $\chi = a$, $\chi = b$, $\psi = c$, $\psi = d$. The intersections shown as darkened lines in Figure 4.1 are the streamlines. The mass flux $\dot{m}$ through the surface A bounded by the four stream surfaces (shown in gray in Figure 4.1) is calculated with area element dA, normal $\mathbf{n}$ (as shown), and Stokes' theorem.

Defining the mass flux $\dot{m}$ through A, and using Stokes' theorem produces:

$$\dot{m} = \int_A \rho\mathbf{u}\cdot\mathbf{n}\,dA = \int_A (\nabla\times\boldsymbol{\Psi})\cdot\mathbf{n}\,dA = \int_C \boldsymbol{\Psi}\cdot d\mathbf{s} = \int_C \chi\nabla\psi\cdot d\mathbf{s} = \int_C \chi d\psi$$
$$= b(d-c) + a(c-d) = (b-a)(d-c).$$