DSML : Python Intermediate. Dhruv. Jawali.

Basic Calculus:

Topics for today:

(a) Functions: mapping from input to output.

(b) Limits: Left-hand limit, Right-hand limit, 2-sided limit.

(c) Continuity: How to identify continuous functions.

(d) Derivatives: calculating derivatives, interpreting it as slope.

How comfortable are you with these topics: functions, limits, and derivatives?

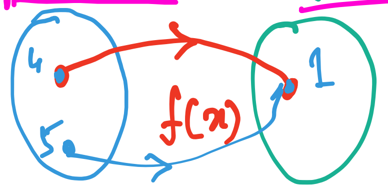
23 responses from 23 users

A	Very comfortable	22%
B	Somewhat comfortable	61%
C	Not comfortable	9%
D	Don't remember anything	9%

(a) Functions.

Set from where we take inputs. ← Domain

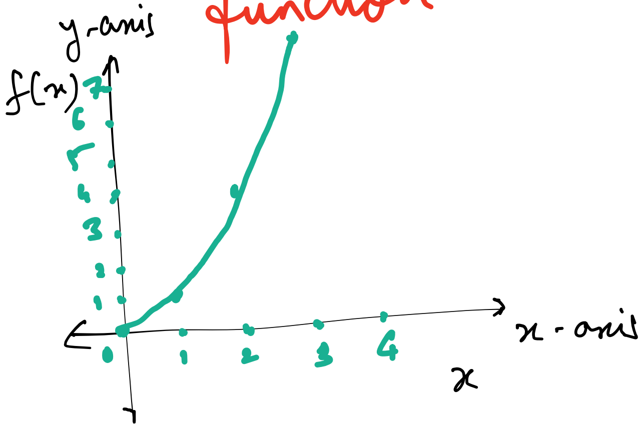
Range. → set where we keep the outputs.



$$\boxed{f(x) = x^2.}$$

f(x) → value returned by the function.

↑ Input to the function.



How do we visualize functions?

(a) Use Desmos.

(b) Code it up and use python to visualize.

(c) Plot by hand.

1] Quadratic function : x^2

2] Line : $2x + 1$

① $\boxed{ax + by + c = 0.}$

② $\boxed{y = mx + c.}$

m → slope

c → y-intercept.

3] Exponential : e^x $e \rightarrow 2.718\ldots$

2^x , 3^x , a^x

4] Logarithm : $\ln(x) \rightarrow \log_e(x)$

$\log_2(x)$

log -> inverse of the exponential.

$e^2 = y$

$2^7 = y$

$\log_2(y) = 7$.

$\ln y = 2$.

inverse of exponential.

$\frac{1}{\infty} = 0.$

Domain log → Only real positive numbers

$\underbrace{\log(-ve)}$ ↓ not defined.

$\log(0) := -\infty.$

Which of the following type of functions is not continuous?

52 users have participated

A	Quadratic functions	6%
B	Exponential functions	4%
C	Logarithm functions	13%
D	All of the above are continuous.	77%

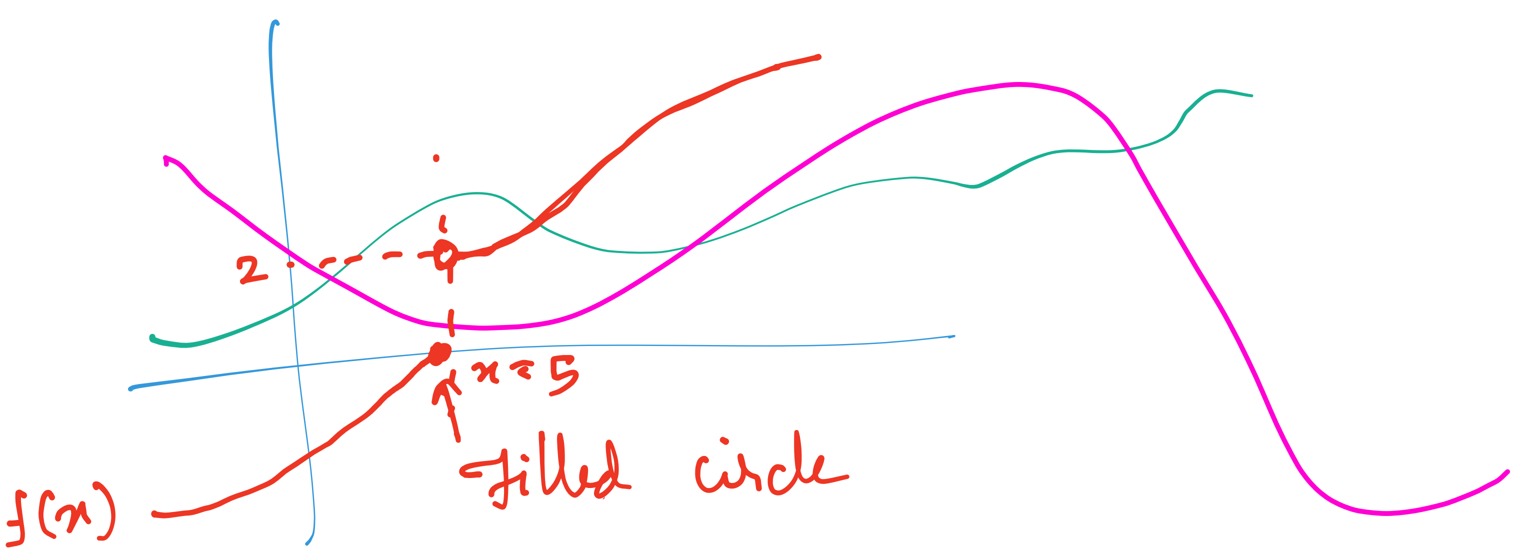
2^{-10} 2^{-100} 2^{-1000} $2^{-10,000}$

$2^{\boxed{-\infty}} = \underset{y}{0}$

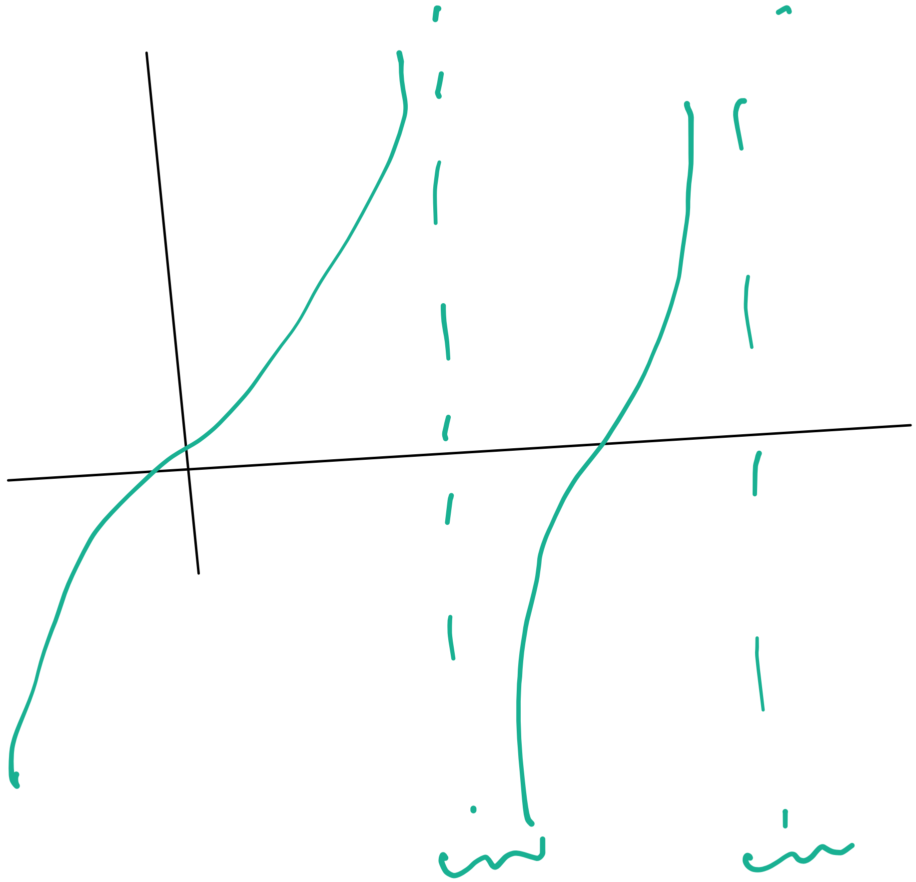
$\log_2(0) = -\infty.$

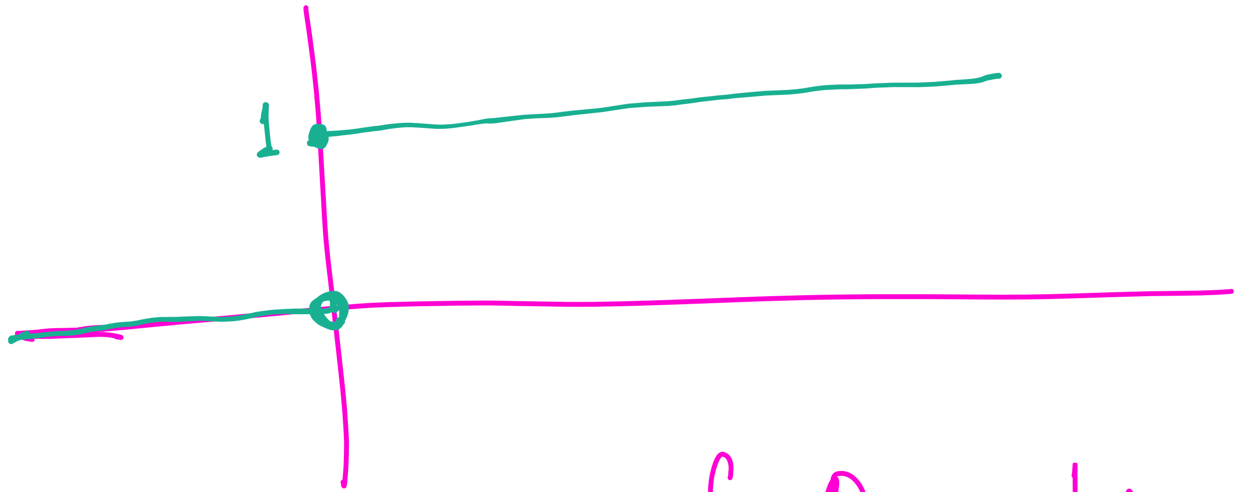
What is a continuous function?

Intuition: If while drawing the function (by hand), I don't need to lift the pen up, the function is continuous.



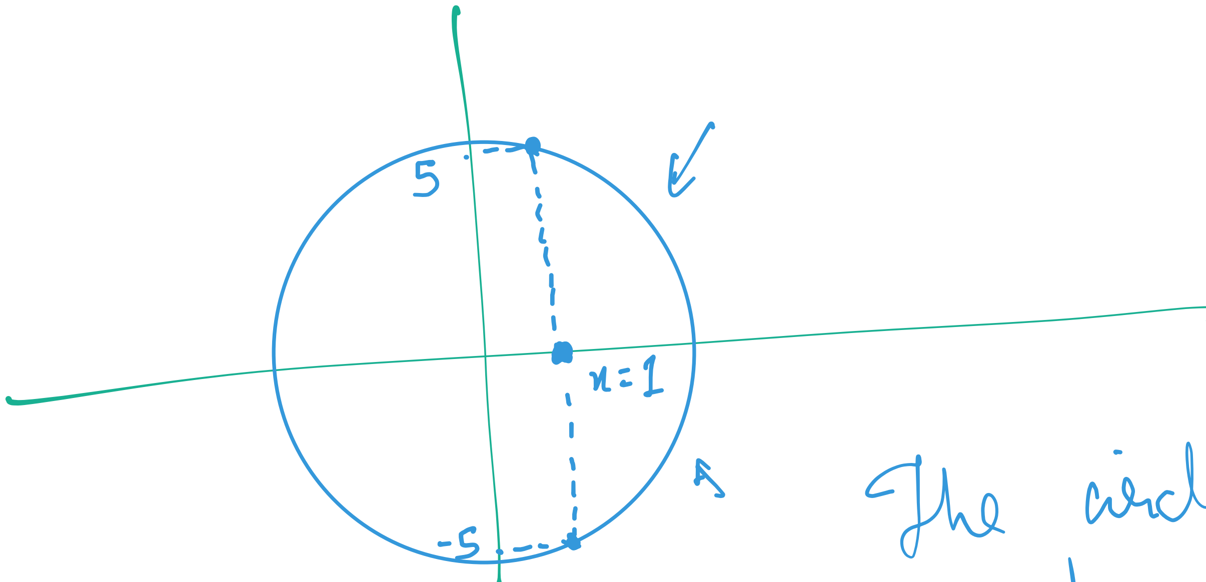
at x = 5, f(x) = 0.
At x = 5, we have a discontinuity.



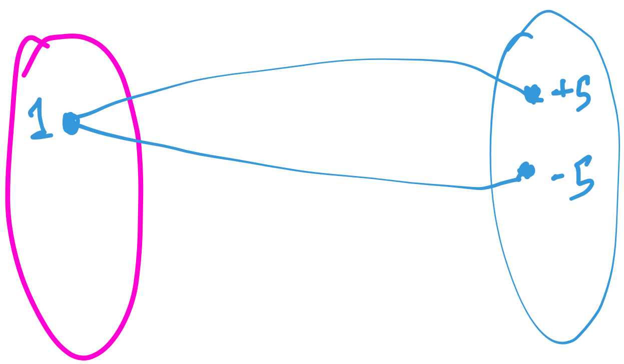


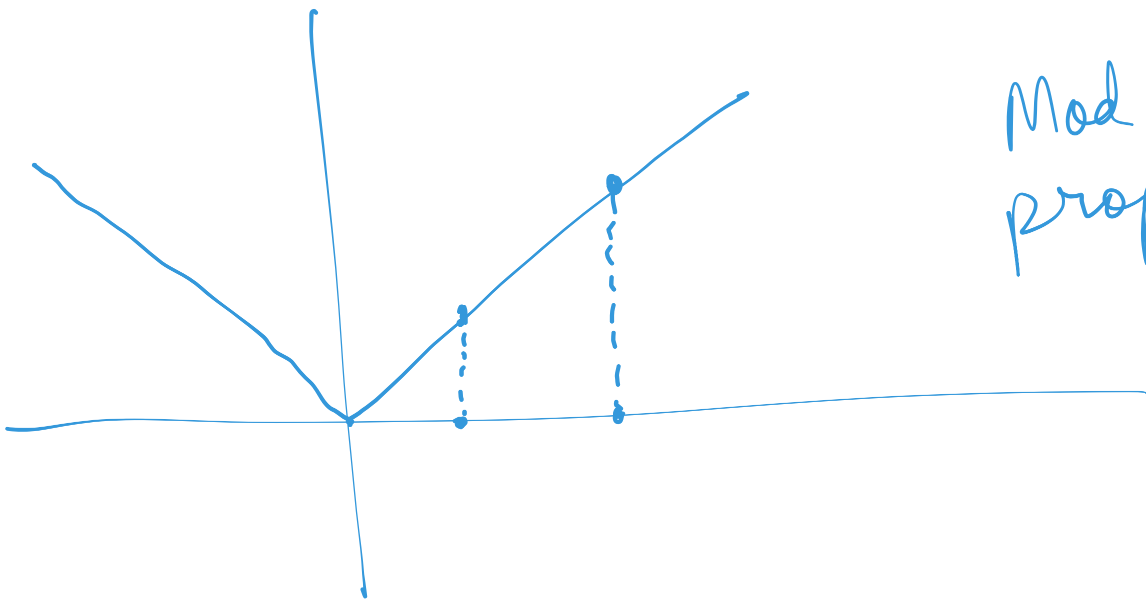
$$u(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0. \end{cases}$$

$u(0) =$



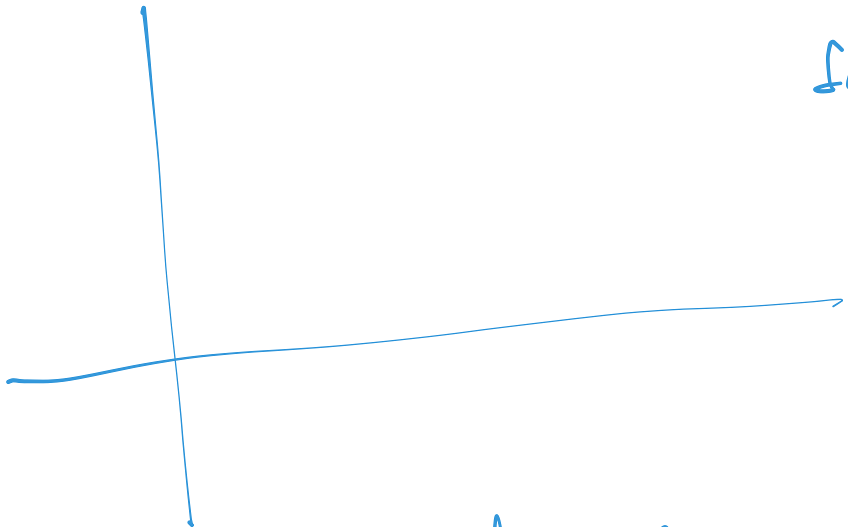
The circle is not a function.





Mod is a proper function.

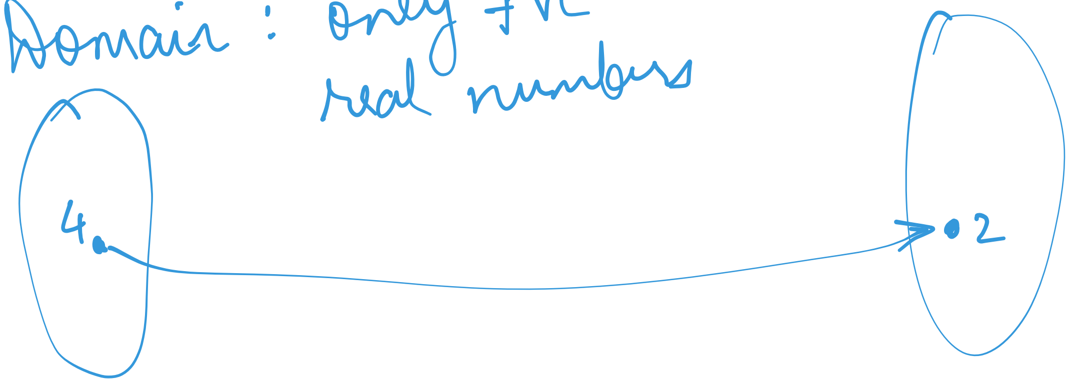
$$|x| = \begin{cases} x & \text{for } x > 0 \\ -x & \text{for } x \leq 0. \end{cases}$$



sqrt. is also a proper function.

Domain : only +ve real numbers

Range : +ve real numbers



$\boxed{\sqrt{4} = +2}$

$\sqrt{a} \rightarrow$ always +ve.

$$f(x) = \begin{cases} x^2 & , \quad x \geq a \\ x & , \quad x < a. \end{cases}$$

What is the value of a so that $f(x)$ is continuous.

For which value of `a` is the following function continuous?
`f(x) = x^2 if x >= a`, and `f(x) = x if x < a`.

50 users have participated

A	1	78%
B	-1	14%
C	2	6%
D	-2	2%

(b) Limits.

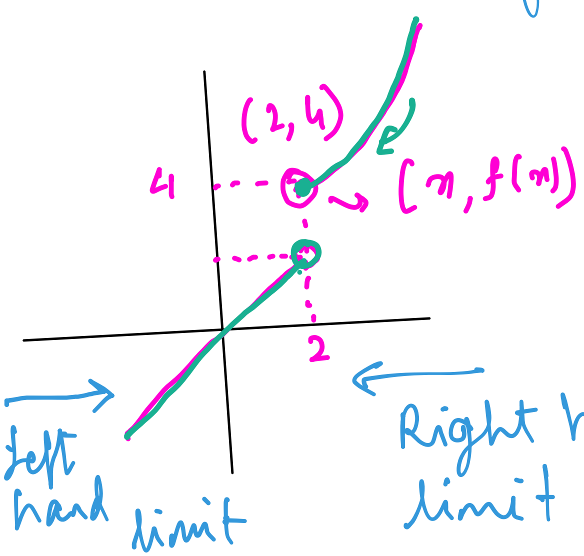
(i) Left hand limit : $f(x)$.

$$\text{LHL} = 2 \qquad \left\{ \lim_{x \to a^-} f(x) \rightarrow \text{Left-hand limit of } f(x). \right.$$

$a = 2$

(ii) Right hand limit : $f(x)$

RHL = 4



$$\lim_{x \to a^+} f(x) \rightarrow \text{Right-hand limit.}$$

$$f(x) = \begin{cases} x^2, & x \geq 2 \\ x, & x < 2 \end{cases}$$

x^2 at $2 = 2^2$
$= 4$

Summary: If the left-hand limit at a is equal to the right-hand limit at a, the limit is said to exist at a.

Test for continuity:

If the limit exists for all inputs x in the domain of $f(x)$, then $f(x)$ is said to be continuous.

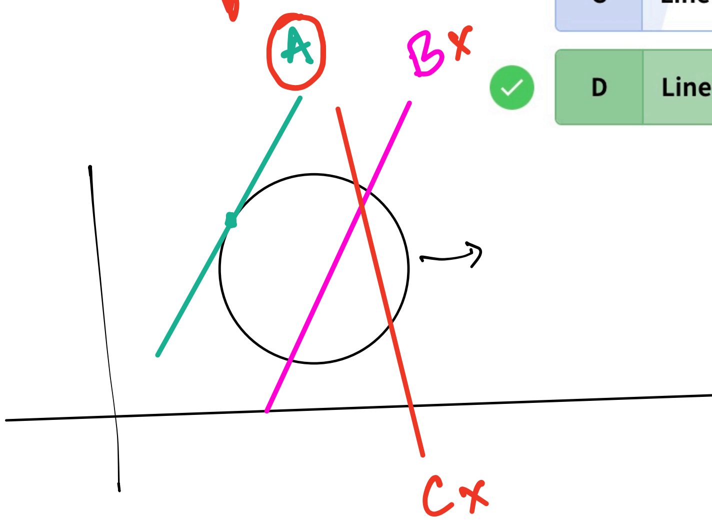
(c) Tangents :

[redacted]

In the figure shown before, which lines were tangents?

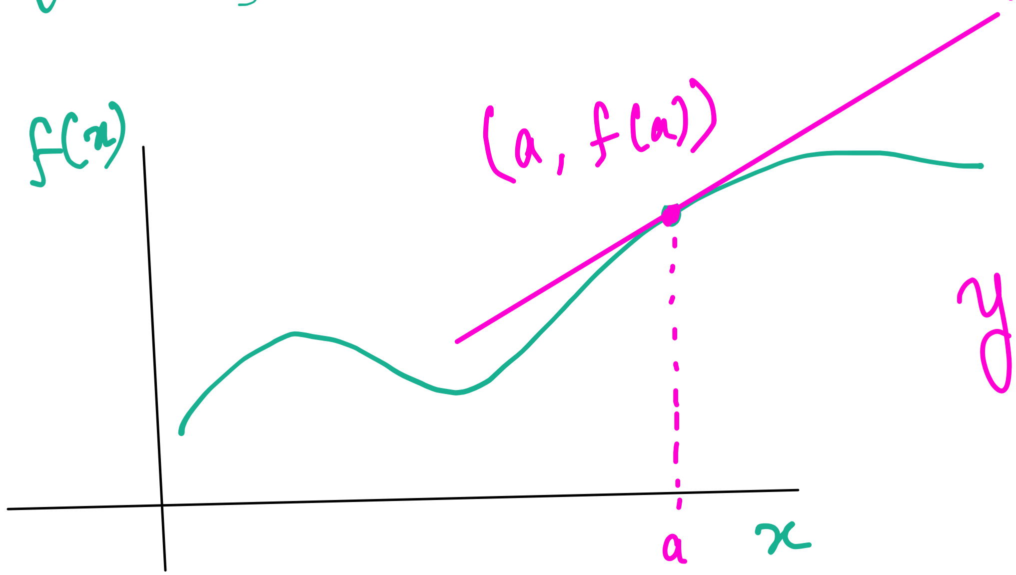
33 users have participated

A	Line A and Line C	12%
B	Line C	6%
C	Line B	3%
D	Line A	79%



B, C → secants.
↓
a line which touches the circle at 2 different points.

$f(x)$



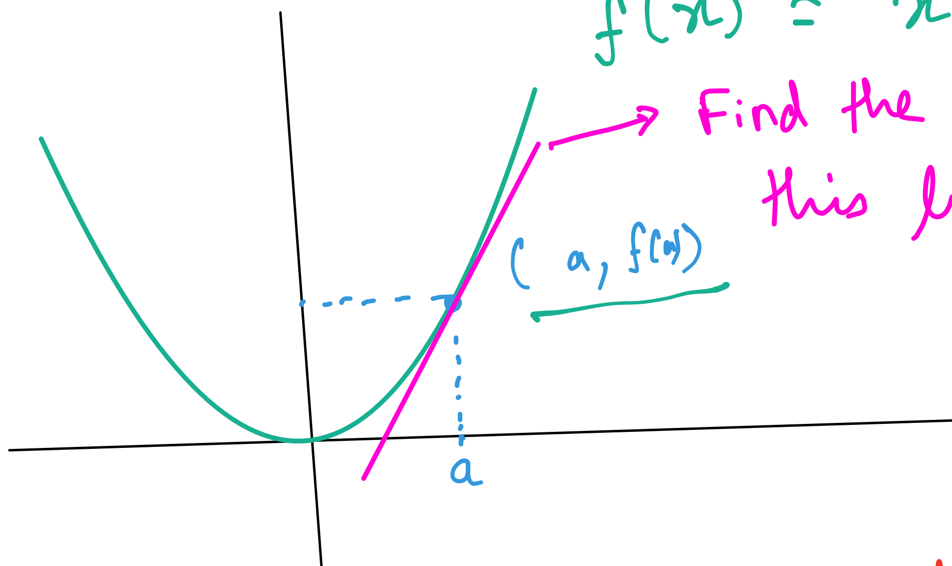
Tangent: Some line which touches $f(x)$ at $x = a$.

$$y = \underline{m}x + \underline{c}.$$

Tangent with respect to $f(x)$ at $x = a$?

What is the equation of a line which is tangent with respect to $f(x)$ at $x = a$?

$f(x) = x^2$.



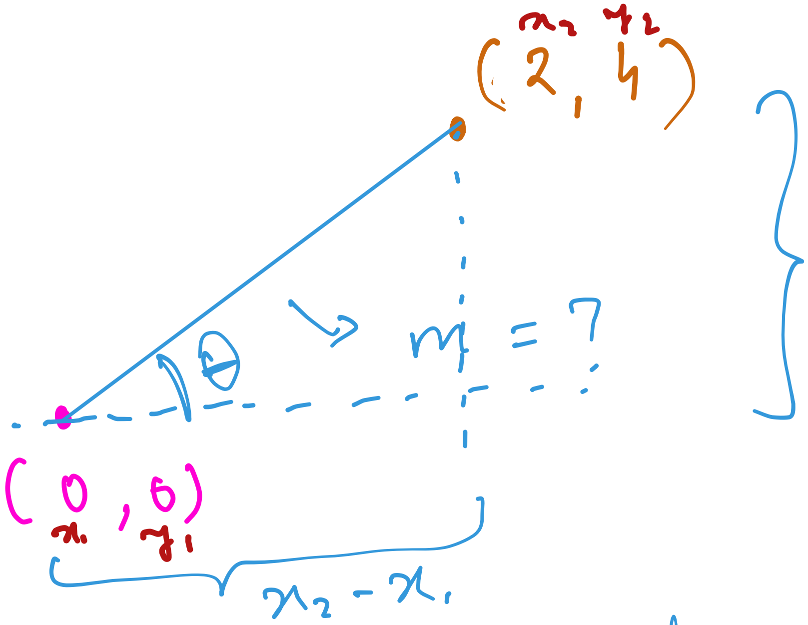
$(a, f(a))$

→ Find the equation of this line.

$a = 2$

$(a, f(a)) = (2, 4)$

→ "To find the slope of the tangent, we use the derivative!!"



$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side.}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

$$(y - \underset{0}{y_1}) = m(x - \underset{0}{x_1})$$

$$y - 0 = \left(\frac{4-0}{2-0}\right)(x - 0)$$

$$y = 2x$$

Summary: To find the equation of a line, we need 2 things: (a) slope.
(b) one point passing through the line

X Continue →

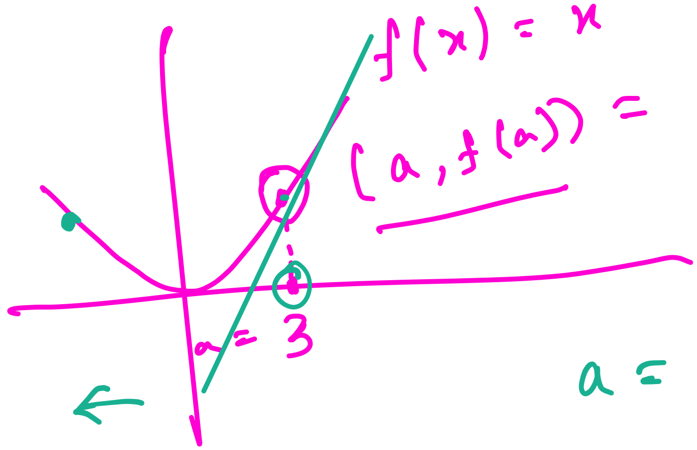
We want to find the equation of the tangent line to the function f(x) = x^2 at a point x = 3. What is one point that the tangent line passes through?

39 users have participated

A	(3, 3)	0%
B	(3, 6)	10%
C	(3, 9)	85%
D	(3, 12)	5%

$f(x) = x^2$

$(a, f(a)) = (3, 9)$



$a = 3$

d) Derivatives:

The derivative of a function $f(x)$ is also a function.

$$\underbrace{\frac{d}{dx}} f(x) = f'(x)$$

$$\frac{d}{dx} g(x) = g'(x)$$

How to obtain the derivative of $f(x)$?

(a) Ab initio → Discuss this last.

(b) Use some well known rules.

			Derivatives
(a)	Quadratic	: $f(x) = x^2$	$f'(x) = 2x$
(b)	Line	: $f(x) = mx + c$	$f'(x) = m$
(c)	Exponential	: $f(x) = e^x$	$f'(x) = e^x$
(d)	Logarithm.	: $f(x) = \ln x$	$f'(x) = \frac{1}{x}$

Rule 1 : Derivative of a monomial.

$$f(x) = \underline{x^n}$$

Monomial.

$$\boxed{f'(x) = n \cdot x^{n-1}}$$

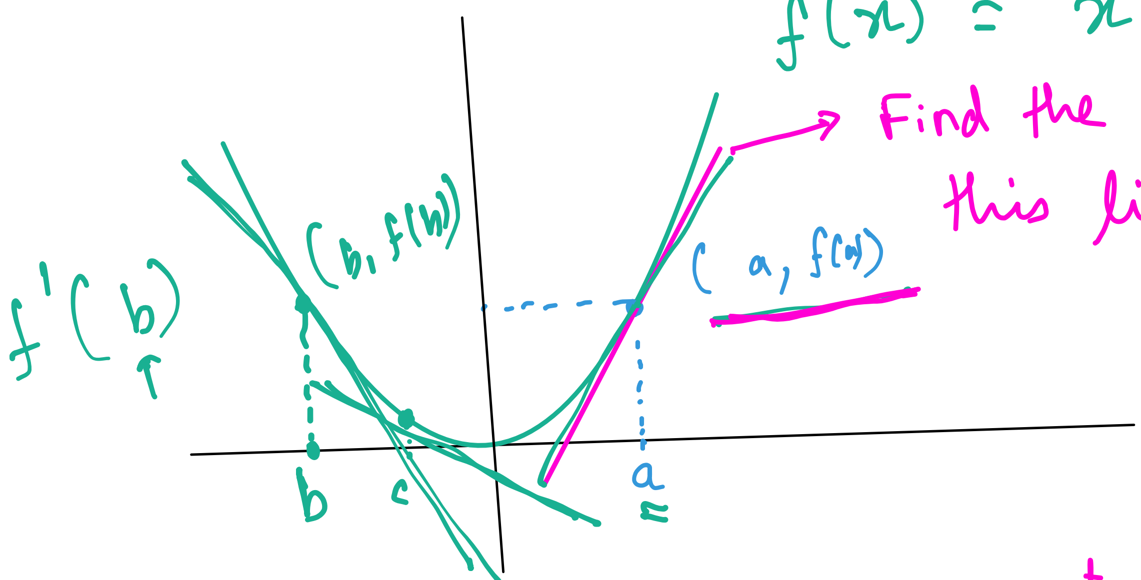
$$x^{\boxed{2}} = 2 \cdot x^{2-1} = 2 \cdot x$$

$$x^{-2} = -2 \cdot x^{-2-1} = \frac{-2}{x^3}$$

We can apply Rule no. 1 to solve our earlier problem!

What is the equation of a line which is tangent with respect to $f(x)$ at $x = a$?

$f(x) = x^2$.



→ Find the equation of this line.

$a = 2$

$(a, f(a)) = \boxed{(2, 4)}$

Using rule 1, we can get slope!!

$f'(x) = 2x$ $\qquad$ $m = f'(a) = 2 \cdot 2$

$= 4$

$$y - y_1 = \underline{m}(x - x_1)$$

$m = 4$

$(x_1, y_1) = (\underline{2}, \underline{4})$

$$y - 4 = 4(x - 2)$$

$$y = 4x - 8 + 4$$

$$\boxed{y = 4x - 4}$$ → Equation of the tangent!!

$\boxed{a}$

1 $(a, f(a)) \rightarrow (x_1, y_1)$

2 $f'(a) \rightarrow \underline{m}$

$$y - y_1 = m(x - x_1)$$

Rule 2: Linearity Rule.

$$h(x) = \underline{a\,f(x)} + \underline{b\,g(x)}$$

$$\frac{d}{dx} h(x) = a \cdot \frac{d}{dx} f(x) + b \cdot \frac{d}{dx} g(x)$$

Eg: $h(x) = 3 \cdot x^2 + 2 \cdot e^x$

$$h'(x) = 3 \cdot \underbrace{\frac{d}{dx} x^2} + 2 \cdot \frac{d\,e^x}{dx}$$

$$= 3 \cdot (2x) + 2e^x$$

$$= 6x + 2e^x.$$

Rule 3: Product Rule:

$$h(x) = f(x) \cdot g(x).$$

$$\frac{d}{dx} h(x) = \left[\frac{d}{dx} f(x)\right] \cdot g(x) + \left[\frac{d}{dx} g(x)\right] \cdot f(x)$$

Eg:

$$h(x) = x^2 \cdot \ln x$$

$$\frac{d}{dx} h(x) = \underbrace{\frac{d}{dx}(x^2)} \cdot \ln(x) + \frac{d}{dx} \ln(x) \cdot x^2$$

$$= 2x \cdot \ln(x) + \frac{1}{\cancel{x}} \cdot x^{\cancel{2}}$$

$$= 2x \cdot \ln(x) + x.$$

Rule 4: Chain Rule:

$$h(x) = f(g(x))$$

$$\frac{d}{dx} h(x) = \underbrace{\frac{d}{dx} f(x) \Big|_{x = g(x)}} \cdot g'(x)$$

$$\Rightarrow h'(x) = f'(g(x)) \cdot g'(x)$$

Eg: $h(x) = e^{x^2} \rightarrow \underline{f(x) = e^{x}}$ $\quad \underline{g(x) = x^2}$

$f'(x) = e^x \quad g'(x) = 2x$

$$h'(x) = f'(g(x)) \cdot g'(x)$$

$$= e^{x^2} \cdot 2x = 2x \cdot e^{x^2}$$

How do we get derivatives for more complicated functions?

→ Approximately get the derivative using the ab initio definition!!

$$\underline{f'(a)} = \underbrace{\lim_{h \to 0}}_{\text{ignore}} \boxed{\frac{f(a+h) - f(a)}{h}}$$

$h \to$ a very small number.

$$f(x) = x^2.$$

$$f'(x) = 2x.$$

$$\text{inputs} = [1, 2, 3, 4, 5]$$

$$\text{outputs} = [1^2, 2^2, 3^2, 4^2, 5^2]$$

$$= [1, 4, 9, 16, 25]$$

$$\text{derivatives} = [2 \cdot 1, 2 \cdot 2, 2 \cdot 3, 2 \cdot 4, 2 \cdot 5]$$

$$= [2, 4, 6, 8, 10]$$

$f(x) = \underline{x^3}$

$f'(x) = \underline{3x^2}$

ip. = [1, 2, 3, 4, 5]

o.p = [1, 8, 27, 64, 125]

derivatives = $[3 \cdot 1^2, 3 \cdot 2^2, 3 \cdot 3^2, 3 \cdot 4^2, 3 \cdot 5^2]$

$= [\underline{3}, \underline{12}, \underline{27}, \underline{48}, \underline{75}]$

Application: Find maxima & minima.

Q] Given $f(x)$, find the maxima & minima.

Ans.

1. Find $f'(x)$.
2. Find all x where $f'(x) = 0$.

$x \rightarrow$ candidates for maxima and minima.

3. $f''(x) \rightarrow$ positive $\rightarrow$ minima
   negative $\rightarrow$ maxima.

$f(x) = 3x^2 - 2x + 3.$

Slack: @Dhruv Jawali

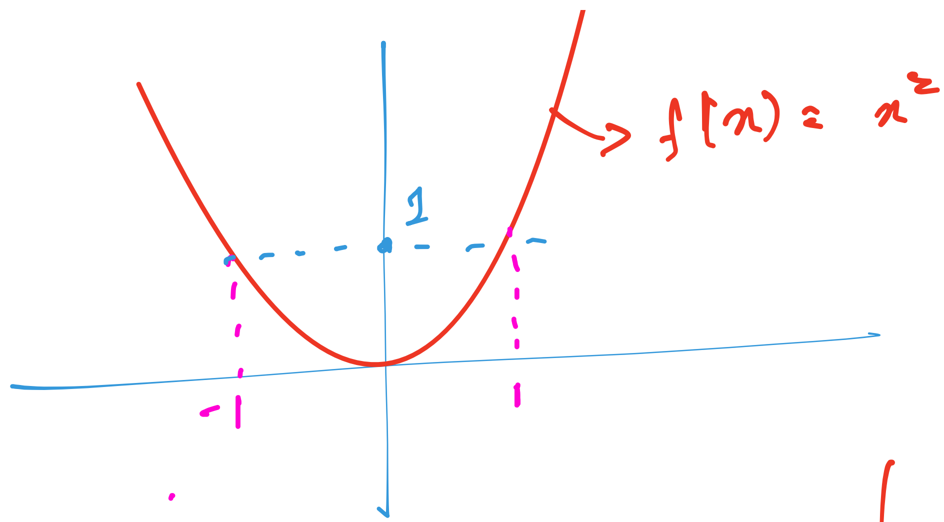
We can improve performance by choosing h as small as possible.

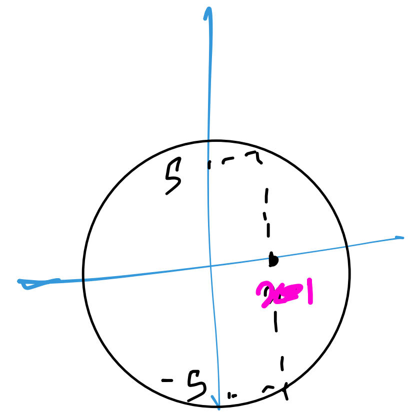
$f(x) = x^2.$

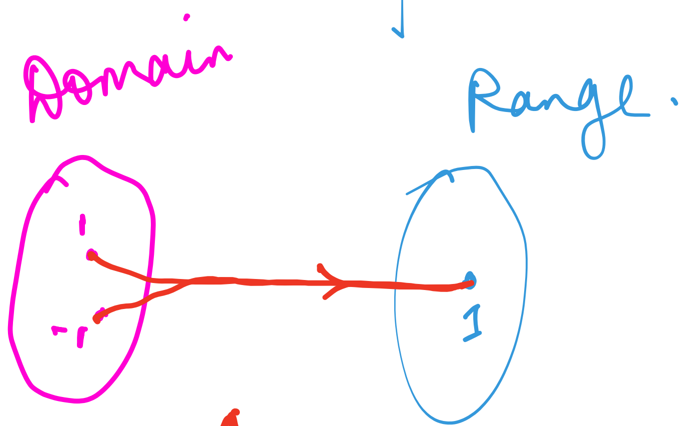
$f'(x) = 2x.$

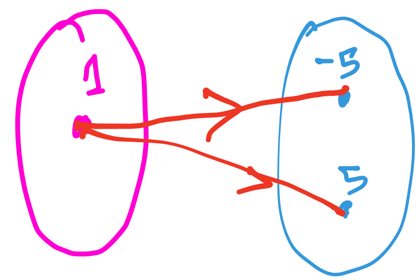
$f''(x) = 2.$ → positive!

→ minima.







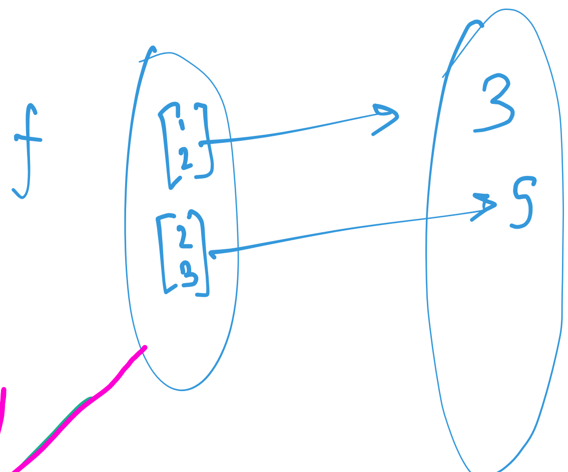


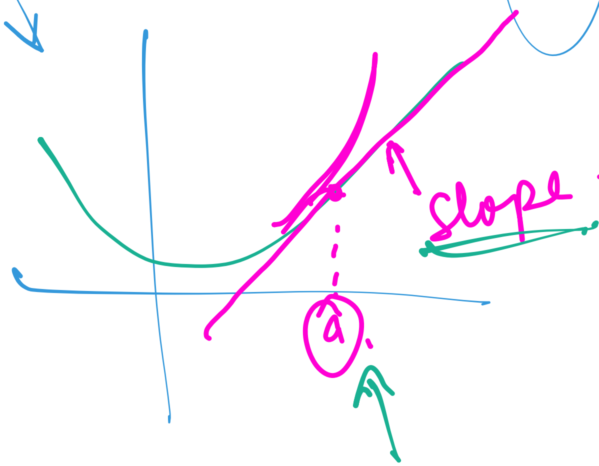
This is many-to-one
And, this is allowed.

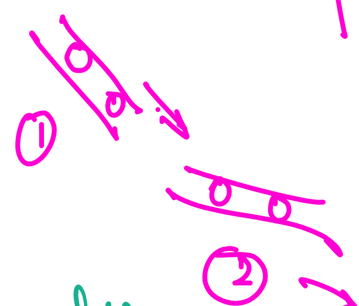
This is one-to-many and not allowed!!

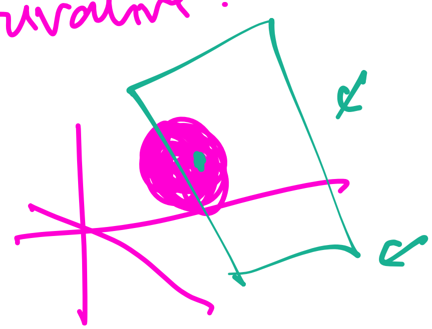
1 Derivatives : f : numbers → number.

2 Gradients : f : vectors → numbers









Angles of orientation.

Probability $= \dfrac{N \rightarrow \text{counts}}{D \rightarrow \text{counts.}}$

Permutations &

Combinations.

Rule 4: Chain Rule → find the derivative of functions like

$$h(x) = f(g(x))$$

eg: $f(x) = \ln(x) \longrightarrow f'(x) = 1/x$.

$g(x) = x^2 + 3x \longrightarrow g'(x) = 2x + 3$

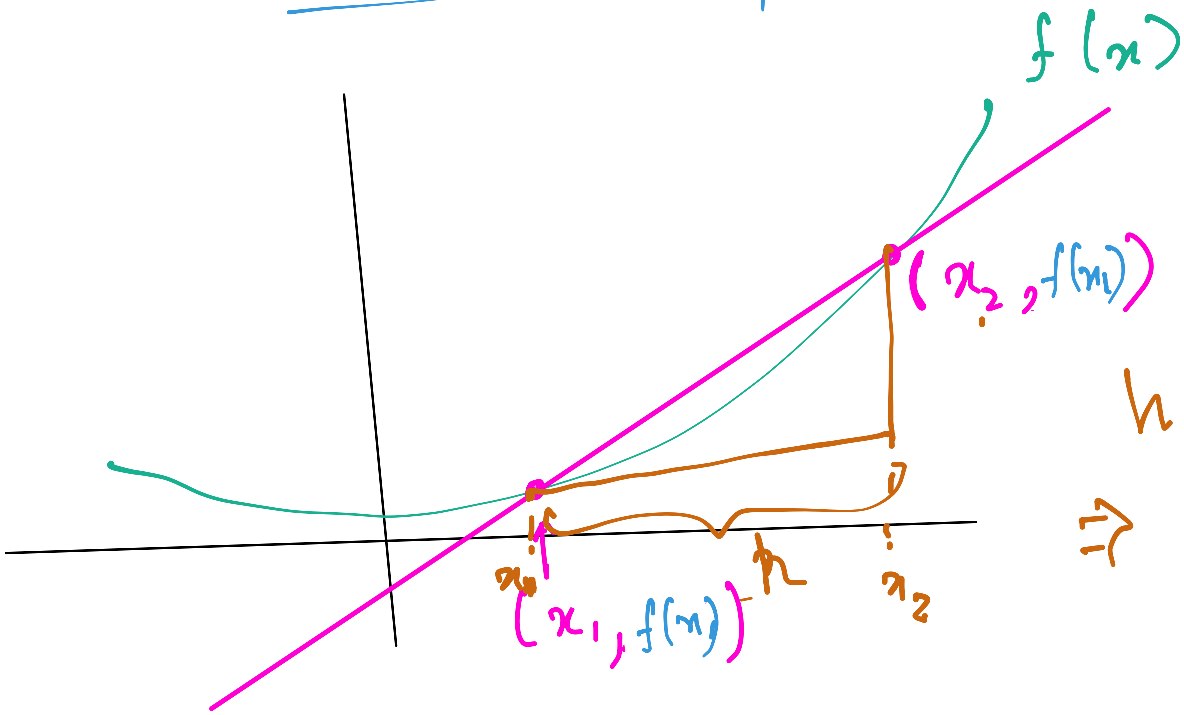
$$f(g(x)) = \ln(x^2 + 3x)$$

Rule: $h'(x) = \underline{f'(g(x))} \cdot \underline{g'(x)}$.

$$= \frac{1}{(x^2+3x)} (2x+3)$$

$$= \frac{2x+3}{x(x+3)}$$

Derivation of the ab-initio rule:



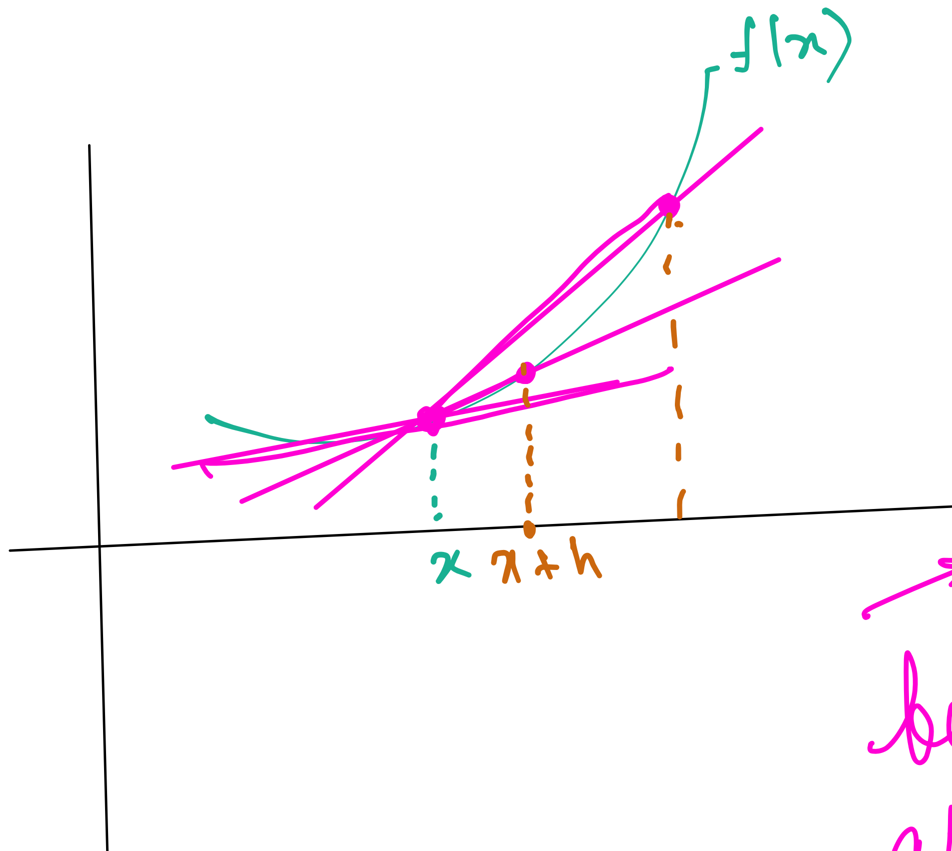
$$h = \underline{x_2 - x_1}$$

$$\Rightarrow \underline{x_2} = h + x_1$$

slope: $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

slope of a secant! $\rightarrow \left\{ \dfrac{f(x_1 + h) - f(x_1)}{h} \right.$

Slope of the secant : $\dfrac{f(x+h) - f(x)}{h}$



as $h \to 0$,
the secant which passes through $(x, f(x))$ becomes a tangent at $(x, f(x))$.

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$