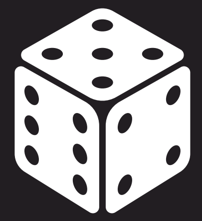
Poisson distribution

Poisson approximation to Binomial

Binomial trials “n” is at least 30
Probability of success “p” is at most 0.05

Binomial $E[X] = np$ $\lambda = np$



How many times can we expect to get a 6 if we throw 600 times?

1 throw $\frac{1}{6}$

600 throws ?

$$(600)\frac{1}{6} = 100$$

How many times can are we expected to throw to get the first 6?

1 throw $\frac{1}{6}$

? 1

$$\frac{1}{1/6} = 6$$

```
binom.pmf(k, n, p)
binom.expect(args=(n, p))
binom.cdf(k, n, p)
```

Suppose we toss a coin once every 10 minutes. The probability of heads is 0.2778

What is the probability of getting one heads in 30 minutes? $n = 3 \quad p = 0.2778$

$P[X=1] =$ `binom.pmf(k=1, n=3, p=0.2778)` $= 0.4346$

What is the expected number of heads in 30 minutes?

$E[X] = n*p = 3*0.2778 = 0.8334$

$E[X] =$ `binom.expect(args=(3, 0.2778))` $= 0.8334$

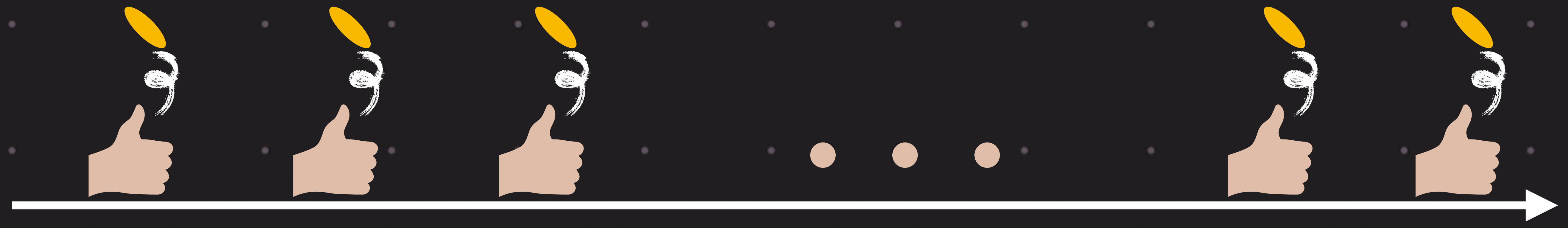
What is the probability of getting one heads in 90 minutes? $n = 9 \quad p = 0.2778$

$P[X=1] =$ `binom.pmf(k=1, n=9, p=0.2778)` $= 0.185$

What is the expected number of heads in 90 minutes?

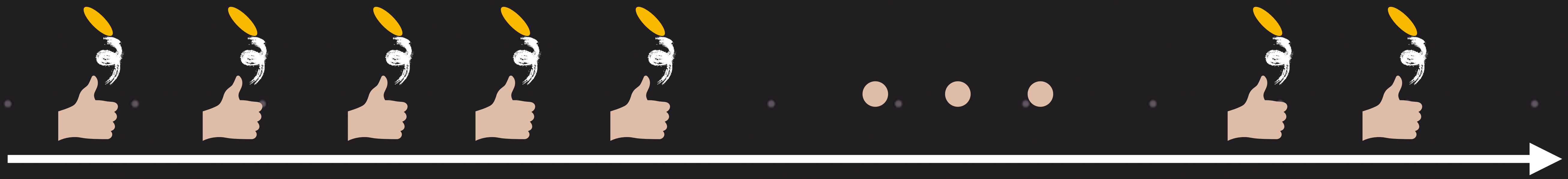
$E[X] = n*p = 9*0.2778 = 2.5$

$E[X] =$ `binom.expect(args=(9, 0.2778))` $= 2.5$



```
binom.pmf(k, n, p)
binom.expect(args=(n, p))
binom.cdf(k, n, p)
```

	Once in 10 mins $p = 0.2778$
30 minutes	$P[X = 1] = 0.4346$ $E[X] = 0.8334$
90 minutes	$P[X = 1] = 0.185$ $E[X] = 2.5$



```
binom.pmf(k, n, p)
binom.expect(args=(n, p))
binom.cdf(k, n, p)
```

Suppose we toss a coin once every minute. The probability of heads is 0.02778

What is the probability of getting one heads in 30 minutes? $n = 30$ $p = 0.02778$

$P[X = 1] =$ `binom.pmf(k=1, n=30, p=0.02778)` $= 0.368$

What is the expected number of heads in 30 minutes?

$E[X] = n * p = 30 * 0.02778 = 0.8334$ $E[X] =$ `binom.expect(args=(30, 0.02778))` $= 0.8334$

What is the probability of getting one heads in 90 minutes? $n = 90$ $p = 0.02778$

$P[X = 1] =$ `binom.pmf(k=1, n=90, p=0.02778)` $= 0.203$

What is the expected number of heads in 90 minutes?

$E[X] = n * p = 90 * 0.02778 = 2.5$ $E[X] =$ `binom.expect(args=(90, 0.02778))` $= 2.5$

```
binom.pmf(k, n, p)
binom.expect(args=(n, p))
binom.cdf(k, n, p)
```

	Once in 10 mins $p = 0.2778$	Once per min $p = 0.02778$
30 minutes	$P[X = 1] = 0.4346$ $E[X] = 0.8334$	$P[X = 1] = 0.368$ $E[X] = 0.8334$
90 minutes	$P[X = 1] = 0.185$ $E[X] = 2.5$	$P[X = 1] = 0.203$ $E[X] = 2.5$

```
binom.pmf(k, n, p)
binom.expect(args=(n, p))
binom.cdf(k, n, p)
```

Suppose we toss a coin 10 times every minute. The probability of heads is 0.002778

What is the probability of getting one heads in 30 minutes? $n = 300$ $p = 0.002778$

$P[X = 1] =$ `binom.pmf(k=1, n=300, p=0.002778)` $= 0.362$

What is the expected number of heads in 30 minutes?

$E[X] = n * p = 300 * 0.002778 = 0.8334$ $E[X] =$ `binom.expect(args=(300, 0.002778))` $= 0.8334$

What is the probability of getting one heads in 90 minutes? $n = 900$ $p = 0.002778$

$P[X = 1] =$ `binom.pmf(k=1, n=900, p=0.002778)` $= 0.205$

What is the expected number of heads in 90 minutes?

$E[X] = n * p = 900 * 0.002778 = 2.5$ $E[X] =$ `binom.expect(args=(90, 0.02778))` $= 2.5$

```
binom.pmf(k, n, p)
binom.expect(args=(n, p))
binom.cdf(k, n, p)
```

	Once in 10 mins $p = 0.2778$	Once per min $p = 0.02778$	Ten times per min $p = 0.002778$
30 minutes	$P[X=1] = 0.4346$ $E[X] = 0.8334$	$P[X=1] = 0.368$ $E[X] = 0.8334$	$P[X=1] = 0.362$ $E[X] = 0.8334$
90 minutes	$P[X=1] = 0.185$ $E[X] = 2.5$	$P[X=1] = 0.203$ $E[X] = 2.5$	$P[X=1] = 0.205$ $E[X] = 2.5$

```
binom.pmf(k, n, p)
binom.expect(args=(n, p))
binom.cdf(k, n, p)
```

Suppose the rate at which "heads" comes is 2.5 in 90 mins

What is the probability of getting one heads in 30 minutes? $\lambda = \frac{2.5}{3} = 0.8334$

$P[X=1] =$ `poisson.pmf(k=1, mu=0.8334)` $= 0.362$

What is the probability of getting one heads in 90 minutes? $\lambda = 2.5$

$P[X=1] =$ `poisson.pmf(k=1, mu=2.5)` $= 0.205$

What is the expected number of heads in 90 minutes?

$E[X] = n * p = 900 * 0.002778 = 2.5$ $E[X] =$ `binom.expect(args=(90, 0.02778))` $= 2.5$

```
binom.pmf(k, n, p)
binom.expect(args=(n, p))
binom.cdf(k, n, p)
```

	Once in 10 mins $p = 0.2778$	Once per min $p = 0.02778$	Ten times per min $p = 0.002778$	Poisson $\lambda = 2.5$ per 90 min
30 minutes	$P[X=1] = 0.4346$ $E[X] = 0.8334$	$P[X=1] = 0.368$ $E[X] = 0.8334$	$P[X=1] = 0.362$ $E[X] = 0.8334$	$P[X=1] = 0.362$ $E[X] = 0.8334$
90 minutes	$P[X=1] = 0.185$ $E[X] = 2.5$	$P[X=1] = 0.203$ $E[X] = 2.5$	$P[X=1] = 0.205$ $E[X] = 2.5$	$P[X=1] = 0.205$ $E[X] = 2.5$

Poisson distribution

```
poisson.pmf(k, mu)
```

$$P[X=k] = \frac{\lambda^k e^{-\lambda}}{k!}$$

Football matches have an average of 2.5 goals in 90 mins

Q1) What is the probability of having one goal in 30 mins?

90 mins	2.5
30 mins	?

$$\lambda = \frac{30 * 2.5}{90} = 0.833$$

$P[X=1] =$ `poisson.pmf(k=1, mu=0.833)` $= 0.362$

Q2) What is the probability of having one goal in 90 mins? $\lambda = 2.5$

$P[X=1] =$ `poisson.pmf(k=1, mu=2.5)` $= 0.205$

Poisson distribution

`poisson.pmf(k, mu)` $P[X=k] = \dfrac{\lambda^k e^{-\lambda}}{k!}$

You receive 240 messages per hour on average - assume Poisson distributed.

Q1) What is the average or expected number of messages in 30 seconds?

1 hour (3600 seconds) — 240 messages

30 seconds — ?

$$\frac{30 * 240}{3600} = 2$$

Q2) What is the probability of one message arriving over a 30 second time interval?

If we consider 30 seconds as one unit interval, then $\lambda = 2$

$P[X=1] =$ `poisson.pmf(k=1, mu=2)` $= 0.27$

$$P[X=1] = \frac{(2)^1 e^{(-2)}}{1!} = 0.27$$

Q3) What is the probability that there are no messages in 15 seconds?

$$\lambda = \frac{15 * 240}{3600} = 1$$

$P[X=0] =$ `poisson.pmf(k=0, mu=1)` $= 0.367$

$$P[X=0] = \frac{(1)^1 e^{(-1)}}{1!} = 0.367$$

Q4) What is the probability that there are 3 messages in 20 seconds?

$$\lambda = \frac{20 * 240}{3600} = 1.33$$

$P[X=3] =$ `poisson.pmf(k=3, mu=1.33)` $= 0.104$

$$P[X=3] = \frac{(1.33)^3 e^{(-1.33)}}{3!} = 0.104$$

Poisson distribution

`poisson.pmf(k, mu)` $P[X = k] = \frac{\lambda^k e^{-\lambda}}{k!}$

You receive 240 messages per hour on average - assume Poisson distributed.

Q1) What is the average time to wait between two messages? $\frac{3600}{240} = 15$ seconds

1 hour (3600 seconds) — 240 messages

? — 1

Q2) What is the average number of messages per second? $\lambda_1 = \frac{1}{15} = 0.067$ per second

Q3) What is the probability of having no messages in 10 seconds? $\lambda_{10} = \frac{10}{15} = 0.67$ per 10-seconds

$$P[X = 0] = \frac{\lambda_{10}^0 e^{-\lambda_{10}}}{0!} = e^{-\lambda_{10}} = e^{-10\lambda_1} = 0.5134$$

$P[X = 0] =$ `poisson.pmf(k=0, mu=10/15)`

$\lambda_{10} = 10 * \lambda_1$

$\lambda_t = t * \lambda_1$

Q4) What is the probability of waiting for more than 10 seconds for the next message?

Let T denote the time to wait for the next message

$$P[T > 10] = e^{-10\lambda_1} = 0.5134$$

Q5) What is the probability of waiting less than or equal to 10 seconds?

$$P[T \leq 10] = 1 - e^{-10\lambda_1} = 0.4865$$

```
from scipy.stats import expon
```

$P[T \leq x] = 1 - e^{-x\lambda}$

$P[T \leq 10] =$ `expon.cdf(x=10, scale=15)` $= 0.4865$

You are working as a data engineer who has to resolve any bugs/ failures of machine learning models in production

$$P[T \leq x] = 1 - e^{-x\lambda}$$

```
expon.cdf(x, scale)
```

The time taken to debug is exponentially distributed with mean of 5 minutes

Q1) Find the probability of debugging in 4 to 5 minutes

$P[4 < T < 5] =$ `expon.cdf(x=5, scale=5) - expon.cdf(x=4, scale=5)` $= 0.0814$

Q2) Find the probability of needing more than 6 minutes to debug

$P[T > 6] =$ `1 - expon.cdf(x=6, scale=5)` $= 0.3012$

Q3) Given that you have already spent 3 minutes, what is the probability of needing more than 9 minutes

$$P[T > 9 \mid T > 3] = \frac{P[(T > 9) \cap (T > 3)]}{P[T > 3]} = \frac{P[T > 9]}{P[T > 3]} = \frac{\texttt{1 - expon.cdf(x=9, scale=5)}}{\texttt{1 - expon.cdf(x=3, scale=5)}} = 0.3012$$

$$\frac{P[T > 9]}{P[T > 3]} = \frac{e^{-9\lambda}}{e^{-3\lambda}} = e^{-6\lambda} = P[T > 6]$$

Memoryless:

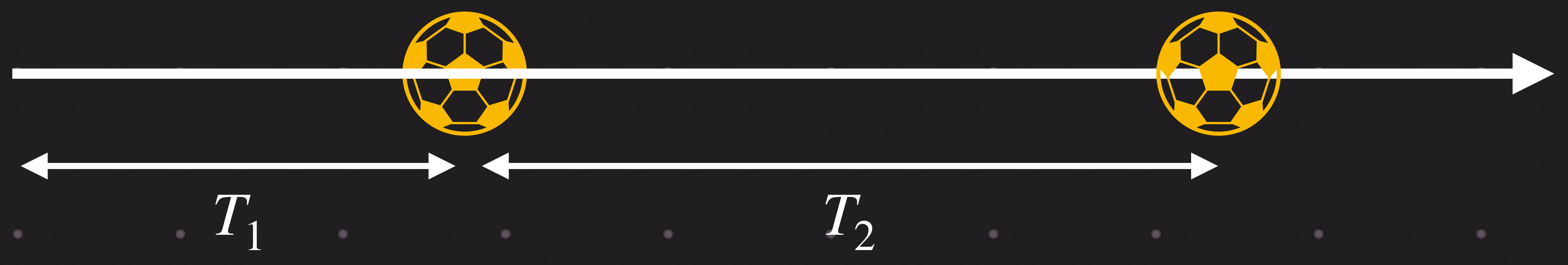
The fact that you took three minutes so far does not affect how much more you might take to debug

Waiting for first goal

Rate = 2.5 / 90 mins Rate = 1.25 / 45 mins Rate = 0.0278 / min

Let T_1 denote the amount of time to wait for first goal

Is T_1 discrete or continuous?



T_1 Time taken for first goal from the start of the match

T_2 Time taken for second goal from point the first goal is scored

Under the assumptions of the poisson process, T_1 and T_2 are independent

These are called “Inter arrival times”, and are exponentially distributed

A call centre gets 3.5 calls per hour

$$P[T \leq x] = 1 - e^{-x\lambda}$$

```
expon.cdf(x, scale)
```

Q) Calculate the probability that the next call will arrive at least 30 minutes after the previous call.

Approach 1: Interval = 1 minute

1 hour (60 min)	3.5
1 minute	?

$$\frac{3.5}{60} = 0.0583$$

$P[T > 30] =$ `1 - expon.cdf(x=30, scale=1/0.0583)` $= 0.1739$

Approach 2: Interval = 30 minute

1 hour (60 min)	3.5
30 minute	?

$$\frac{3.5}{2} = 1.75$$

$P[T > 1] =$ `1 - expon.cdf(x=1, scale=1/1.75)` $= 0.1739$

Log-Normal distribution

Consider an example of number of days of hospitalisation

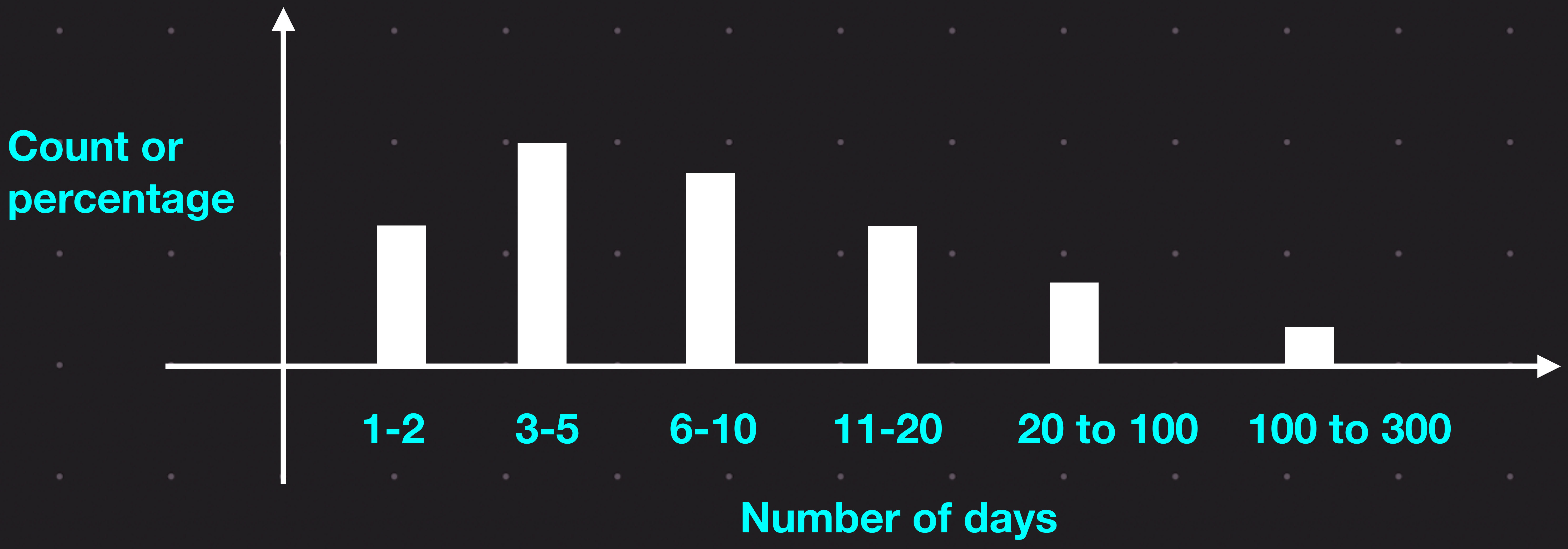
Most people may be hospitalised for 3 to 5 days

Quite often, just 1 or 2 days

But, there can be a few extreme cases in 20 to 30 days

Extremely rare cases of 300 days (coma etc.)

Log-Normal distribution



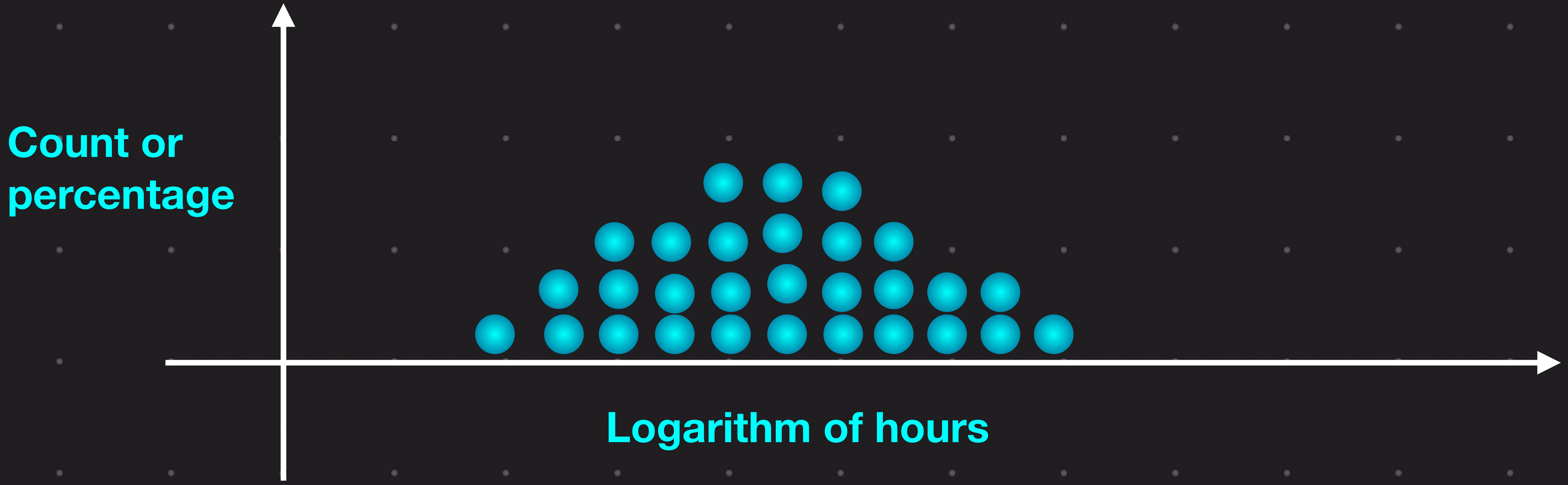
Number of days is discrete

What if we had exact number of hours?

This would be continuous

Log-Normal distribution

Log-Normal distribution



Log-Normal distribution

For the log normal, we use the parameters of the Gaussian: μ and σ

$$E[X] = e^{\mu+\sigma^2/2}$$

$$\text{Var}[X] = (e^{\sigma^2} - 1)\left(e^{2\mu+\sigma^2}\right)$$

Always confirm these equations from some source (eg: wiki)