Analysis of continuous Random Variables

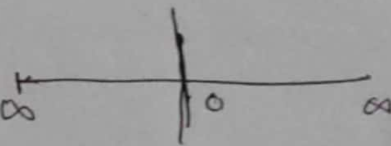
A continuous random variable is a random variable with an interval of real number called its range.

Probability density function:

For a continuous random variable X, a probability density function (p d f) is a function such that

(i) $f(x) \geq 0$

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$



③ $\int_{a}^{b} f(x)\,dx = P(a \leq x \leq b)$ = area under $f(x)$ from a to b. for any a & b.

If X is a continuous random variable for any x_1 & x_2

$$P(x_1 \leq X \leq x_2) = P(x_1 < x \leq x_2) = P(x_1 \leq x < x_2)$$
$$= P(x_1 < X < x_2)$$

CHAPTER-2

ANALYSIS OF CONTINUOUS RANDOM VARIABLES

SECTION -1

1. Suppose that $f(x)=e^{-x}$ for $0<x$. Determine the following probabilities:
(a) $P(1<X)$ (b) $P(1<X<2.5)$ (c) $P(X=3)$ (d) $P(X<4)$ (e) $P(3\le X)$
(f) Determine x such that $P(x<X)=0.10$
(g) Determine x such that $P(X\le x)=0.10$.

2. Suppose that $f(x)=x/8$ for 3<x<5.Determine the following probabilities:
(a) $P(X<4)$ (b) $P(X>3.5)$ (c) $P(4<X<5)$ (d) $P(X<4.5)$ (e) $P(X<3.5 \text{ or } X>4.5)$

3. Suppose that $f(x)=e^{-(x-4)}$ for 4< x. Determine the following probabilities:
(a) $P(1<X)$ (b) $P(2\le X<5)$ (c) $P(5<X)$ (d) $P(8<X<12)$
(e)Determine x such that $P(X<x)=0.90$.

4. Suppose that $f(x)=1.5x^2$ for -1< x< 1. Determine the following probabilities:
(a) $P(0<X)$ (b) $P(0.5<X)$ (c) $P(-0.5\le X\le 0.5)$ (d) $P(X<-2)$
(e) $P(X<0 \text{ or } X>-0.5)$(f) Determine x such that $P(X<x)=0.05$.

5. The probability density function of the time to failure of an electronic component in a copier (in hours) is

$f(x)=\dfrac{e^{\frac{-x}{1000}}}{1000}$ *for* $x>0$.Determine the probability that
(a) A component lasts more than 3000 hours before failure.
(b) A component fails in the interval from 1000 to 2000 hours.
(c) A component fails before 1000 hours.
(d) Determine the number of hours at which 10% of all components have failed.

6. The probability density function of the net weight in pounds of a packaged chemical herbicide is f(x)=2.0 for $49.75<x<50.25$ pounds.
(a) Determine the probability that a package weighs more than 50 pounds.
(b) How much chemical is contained in 90% of all packages?

7. The probability density function of the length of a hinge for fastening a door is f(x)=1.25 $74.6.<x<75.4$ millimeters. Determine the following:
(a) $P(X<74.8)$ (b) $P(X<74.8 \text{ or } X>75.2)$
(c) If the specifications for this process are from 74.7 to 75.3 millimeters, what proportion of hinges meets specifications?

8. The probability density function of the length of a metal rod is f(x)=2 for 2.3 < x< 2.8 meters.a) If the specifications for this process are from 2.25 to 2.75 meters, what proportion of the bars fail to meet the specifications?
(b) Assume that the probability density function is f(x)=2 for an interval of length 0.5 meters. Over what value the density should be centered to achieve the greatest proportion of bars within specifications?

Chapter 2
Section-1

1 Ans: $f(x) = e^{-x}$ for $0 < x$ $(x > 0)$

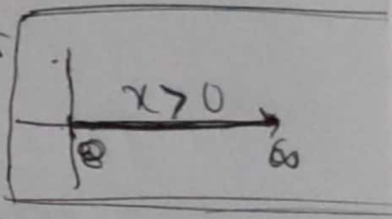
$e^{-\infty} = \frac{1}{e^{\infty}} = 0$

$\int e^{ax} dx = \frac{e^{ax}}{a}$

a) $P(1 < X) = P(X > 1) = \int_1^{\infty} e^{-x} dx$

$P(1 < X) = \left[\frac{e^{-x}}{-1}\right]_1^{\infty} = -e^{-\infty} + e^{-1} = e^{-1} = 0.3679$

b) $P(1 < X < 2.5) = \int_1^{2.5} e^{-x} dx = \left[-e^{-x}\right]_1^{2.5}$



$= -e^{-2.5} + e^{-1} = 0.285$

c) $P(X = 3) = \int_3^3 e^{-x} dx = \left[-e^{-x}\right]_3^3 = 0$

d) $P(X < 4) = P(X \le 4) = \int_0^4 e^{-x} dx = \left[-e^{-x}\right]_0^4$

$= -e^{-4} + e^{0} = 0.9817$

e) $P(3 \le X) = P(X \geq 3) = \int_3^{\infty} e^{-x} dx = \left[-e^{-x}\right]_3^{\infty}$

$= -e^{-\infty} + e^{-3} = 0 + \frac{1}{e^3} = 0.0798$

f) Determine x such that $P(x < X) = 0.10$

$P(x < X) = P(X > x) = \int_x^{\infty} e^{-x} dx = \left[-e^{-x}\right]_x^{\infty}$

$P(x < X) = -e^{-\infty} + e^{-x}$

$P(x < X) = e^{-x} = 0.1$

$\therefore \; e^{-x} = 0.1 \Rightarrow \log e^{-x} = \log(0.1)$

$-x = \log(0.1) \Rightarrow$

$\Rightarrow x = -\log(0.1)$ | ln

$\boxed{x = 2.30}$

1 g) Determine x such that $P(X \leq x) = 0.10$

$\therefore \; 0.1 = P(X \leq x) = \int_0^x e^{-x} dx = \left[-e^{-x}\right]_0^x = -e^{-x} + 1$

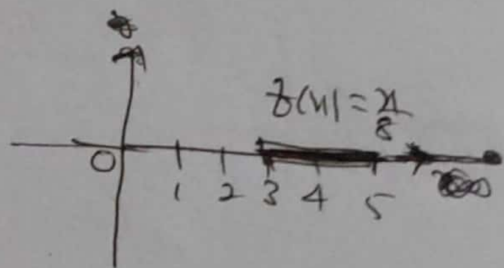
$0.1 = 1 - e^{-x} \Rightarrow e^{-x} = 1 - 0.1$

$e^{-x} = 0.9 \Rightarrow \ln(e^{-x}) = \ln(0.9)$

$-x = \ln(0.9) \Rightarrow x = -\ln(0.9)$

$\boxed{x = 0.105}$

2) $f(x) = \frac{x}{8}$ for $3 < x < 5$



a) $P(X < 4) = \int_3^4 f(x)\,dx$

$= \int_3^4 \frac{x}{8}\,dx = \frac{1}{8}\left[\frac{x^2}{2}\right]_3^4$

$= \frac{1}{16}\left[4^2 - 3^2\right] = \frac{7}{16} = 0.4375$

b) $P(X > 3.5) = \int_{3.5}^5 f(x)\,dx = \int_{3.5}^5 \frac{x}{8}\,dx$

$= \left[\frac{x^2}{16}\right]_{3.5}^5 = \frac{1}{16}\left[5^2 - (3.5)^2\right]$

$= 0.7069$

(2) (c) $P(4<X<5) = \int_4^5 f(x)\,dx = \int_4^5 \frac{x}{8}\,dx$

$$= \frac{1}{8}\left[\frac{x^2}{2}\right]_4^5 = \frac{1}{16}[25-16] = 0.5625$$

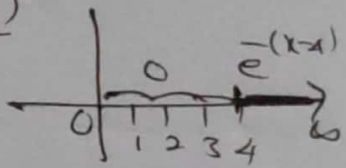
d) $P(X<4.5) = \int_3^{4.5} \frac{x}{8}\,dx = \left[\frac{x^2}{16}\right]_3^{4.5} = 0.7031$

e) $P(X>4.5) + P(X<3.5) = \int_{4.5}^{5} \frac{x}{8}\,dx + \int_3^{3.5} \frac{x}{8}\,dx$

$$= \left[\frac{x^2}{16}\right]_{4.5}^{5} + \left[\frac{x^2}{16}\right]_3^{3.5}$$

$$= \frac{1}{16}\left[5^2-(4.5)^2\right] + \frac{1}{16}\left[(3.5)^2-(3)^2\right] = 0.5$$

(3) $f(x) = e^{-(x-4)}$ for $4<x$. (or $x>4$)



a) $P(1<X) = P(X>1)$

$$= \int_1^4 f(x)\,dx + \int_4^\infty e^{-(x-4)}\,dx$$

$$= 0 + \left[-e^{-(x-4)}\right]_4^\infty = -e^{-\infty} + e^0 = 0+1 = 1$$

b) $P(2 \le X \le 5) = \int_4^5 e^4 e^{-x}\,dx = e^4 \int_4^5 e^{-x}\,dx$

$$= e^4\left[-e^{-x}\right]_4^5$$

$f(x) = e^{-(x-4)}$
$= e^{-x+4}$
$= e^{-x} \cdot e^4$

$$= e^4\left[-e^{-5} + e^{-4}\right] = -e^4 e^{-5} + e^4 e^{-4}$$

$$= -e^{-1} + e^0 = 1 - \frac{1}{e} = 0.632$$

(3) (c) $P(5 < X) = P(X > 5) = \int_5^\infty f(x)\,dx$

$= \int_5^\infty e^4 e^{-x}\,dx = e^4\left[-e^{-x}\right]_5^\infty = e^4\left[-e^{-\infty} + e^{-5}\right]$

$= e^4\left[0 + e^{-5}\right] = e^{-1} = \frac{1}{e} = 0.368$

(d) $P(8 < X < 12) = \int_8^{12} f(x)\,dx = \int_8^{12} e^4 e^{-x}\,dx$

$= e^4\left[-e^{-x}\right]_8^{12} = e^4\left[-e^{-12} + e^{-8}\right] = 0.018$

(e) <u>Determine x such that $P(X < x) = 0.90$</u>

$P(X < x) = 0.90$

$\int_4^x f(x)\,dx = 0.9 \Rightarrow \int_4^x e^4 e^{-x}\,dx = 0.9$

$e^4\left[-e^{-x}\right]_4^x = 0.9$

$e^4\left[-e^{-x} + e^{-4}\right] = 0.9 \Rightarrow -e^{-x} + e^{-4} = \frac{0.9}{e^4}$

$-e^{-x} + e^{-4} = 0.013$

$e^{-x} = e^{-4} - 0.013$

$e^{-x} = 5.317 \times 10^{-3}$

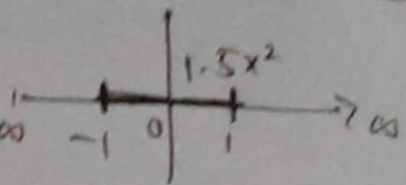
$\ln(e^{-x}) = \ln(5.317 \times 10^{-3})$

$-x = \ln(5.317 \times 10^{-3})$

$x = -\ln(5.317 \times 10^{-3})$

$\boxed{x = 5.23}$

(1) $f(x) = 1.5x^2$ for $-1 < x < 1$ (6)



a) $P(0 < X) = P(X > 0)$

$= \int_0^1 f(x)\,dx = \int_0^1 1.5x^2\,dx = \frac{3}{2}\left[\frac{x^3}{3}\right]_0^1$

$= \frac{1}{2}[1-0] = \frac{1}{2} = 0.5$

b) $P(0.5 < X) = P(X > 0.5) = \int_{0.5}^1 f(x)\,dx$

$= \int_{0.5}^1 1.5x^2\,dx = \frac{\cancel{3}}{2}\left[\frac{x^3}{\cancel{3}}\right]_{0.5}^1$

$= \frac{1}{2}\left[1-\frac{1}{8}\right] = \frac{1}{2}\times\frac{7}{8} = \frac{7}{16} = 0.4375$

c) $P(-0.5 \le X \le 0.5) = \int_{-0.5}^{0.5} 1.5x^2\,dx$

$= \frac{3}{2}\left[\frac{x^3}{3}\right]_{-0.5}^{0.5} = \frac{1}{2}\left[(0.5)^3 - (-0.5)^3\right] = 0.125$

d) $P(X < -2) = 0$ $\because f(x) = 0$ for $x \notin (-1, 1)$

e) $P(X < 0 \text{ or } X > -0.5) = \int_{-1}^{0} 1.5x^2\,dx + \int_{-0.5}^{1} 1.5x^2\,dx$

$= \frac{3}{2}\left[\frac{x^3}{3}\right]_{-1}^{0} + \frac{3}{2}\left[\frac{x^3}{3}\right]_{-0.5}^{1}$

$= 1.06$

f) Determine x such that $P(X < x) = 0.05$

$\int_x^1 \frac{3}{2}x^2\,dx = 0.05 \Rightarrow \frac{\cancel{3}}{2}\left[\frac{x^3}{\cancel{3}}\right]_x^1 = 0.05$

$0.5 - 0.5x^3 = 0.05 \Rightarrow 0.5x^3 = 0.45$

$\boxed{x = 0.9655}$

(5) $f(x) = \frac{e^{-\frac{x}{1000}}}{1000}$ for $x > 0$ $f(x) = 10^{-3} e^{-\frac{x}{1000}}$

a) $P(X > 2000) = \int_{2000}^{\infty} 10^{-3} e^{-\frac{x}{1000}} dx$

$= 10^{-3} \left[\frac{-e^{-\frac{x}{1000}}}{\frac{1}{1000}} \right]_{2000}^{\infty} = \left[\cancel{-e^{-\infty}}^{\to 0} + e^{-3} \right]$

$= e^{-2} = 0.04979$

b) $P(1000 < X < 2000) = \frac{1}{1000} \int_{1000}^{2000} e^{-\frac{x}{1000}} dx$

$= \frac{1}{\cancel{1000}} \left[\frac{-e^{-\frac{x}{1000}}}{\frac{1}{\cancel{1000}}} \right]_{1000}^{2000} = -e^{-2} + e^{-1} = 0.2325$

c) $P(X < 1000) = \int_{0}^{1000} \frac{1}{1000} e^{-\frac{x}{1000}} dx$

$= \frac{1}{1000} \left[\frac{-e^{-\frac{x}{1000}}}{\frac{1}{1000}} \right]_{0}^{1000} dx$

$= -e^{-1} + e^{0} = 1 - \frac{1}{e} = 0.6321$

d) $P(X < 1) = 10\%$

$\int_{0}^{x} \frac{1}{1000} e^{-\frac{x}{1000}} dx = 0.1 \Rightarrow \frac{1}{1000} \left[\frac{-e^{-\frac{x}{1000}}}{\frac{1}{1000}} \right]_{0}^{x} = 0.1$

$-e^{-\frac{x}{1000}} + e^{-0} = 0.1 \Rightarrow e^{-\frac{x}{1000}} = 1 - 0.1$

$e^{-\frac{x}{1000}} = 0.9 \Rightarrow \frac{-x}{1000} = \ln(0.9)$

$x = 105.3648$

(6) $f(x) = 2.0$ for $49.75 < x < 50.25$ (8)

a) $P(X > 50) = \int_{50}^{50.25} f(x)\,dx = \int_{50}^{50.25} 2\,dx = 2[x]_{50}^{50.25}$

$= 2[50.25 - 50] = 0.5$

b) $P(X > x) = 90\%$

$\int_{x}^{50.25} f(x)\,dx = 0.9 \Rightarrow \int_{x}^{50.25} 2\,dx = 0.9$

$2[x]_{x}^{50.25} = 0.9 \Rightarrow 2(50.25) - 2(x) = 0.9$

$2x = 100.5 - 0.9 \Rightarrow 2x = 99.6$

$\therefore \boxed{x = 49.8}$

(7) $f(x) = 1.25$ for $74.6 < x < 75.4$

a) $P(X < 74.8) = \int_{74.6}^{74.8} f(x)\,dx = \int_{74.8}^{75.4} 1.25\,dx$

$= 1.25[x]_{74.6}^{74.8} = 1.25[75.8 - 74.6] = 0.25$

b) $P(X < 74.8 \text{ or } X > 75.2) = \int_{74.6}^{74.8} 1.25\,dx + \int_{75.2}^{75.4} 1.25\,dx$

$= 1.25[x]_{74.6}^{74.8} + 1.25[x]_{75.2}^{75.4}$

$= 1.25[74.8 - 74.6] + 1.25(75.4 - 75.2] = 0.5$

⑥ ⓒ $P(74.7 < X < 75.3) = \int_{74.7}^{75.3} 1.25\,dx = 1.25[x]_{74.7}^{75.3}$ ⑨

$= 1.25[75.3 - 74.7] = 0.75$

Hence 0.75 proportion of hinges meets the specification.

⑧ $f(x) = 2$ for $2.3 < x < 2.8$

a) $P(2.25 < x < 2.75) = \int_{2.25}^{2.75} f(x)\,dx = \int_{2.30}^{2.75} 2\,dx = 2[x]_{2.30}^{2.75}$

$= 2[2.75 - 2.30] = 0.9$ for success

$\therefore$ $100 - 90 = 10\%$ of the rods are not with in expectation (for failure)

b) If we suppose that X is centered. occurs $x = 2.5$, the probability density formula is transferred into $f(x) = 2$, $2.25 < x < 2.75$

$\therefore P(2.25 < x < 2.75) = 2 \times 0.5 = 1$

$\therefore$ The answer is 2.5

SECTION -2.

1. Suppose the cumulative distribution function of the random variable X is

$$F(x)=\begin{cases}0 & x<0\\ 0.2x & 0\le x<5\\ 1 & 5\le x\end{cases}$$

Determine the following:
(a) $P(X<2.8)$ (b) $P(X>1.5)$ (c) $P(X<-2)$ (d) $P(X>6)$

2. Suppose the cumulative distribution function of the random variable X is

$$F(x)=\begin{cases}0 & x<-2\\ 0.25x+0.5 & -2\le x<2\\ 1 & 2\le x\end{cases}$$

Determine the following:
(a) $P(X\le 1.8)$ (b) $P(X>-1.5)$ (c) $P(X<-2)$ (d) $P(-1<X<1)$

3. The gap width is an important property of a magnetic recording head. In coded units, if the width is a continuous random variable over the range from $0<x<2$ with $f(x)=0.5x$, determine the cumulative distribution function of the gap width.

4. The probability density function of the time customers arrive at a terminal (in minutes after 8:00 A.M) is $f(x)=\dfrac{e^{\frac{-x}{10}}}{10}$ for $0<x$. Determine the probability that
(a) The first customer arrives by 9:00 A.M
(b) The first customer arrives between 8:15 A.M and 8:30 A.M.
(c) Two or more customers arrive before 8:40 A.M among five that arrive at the terminal. Assume customer arrivals are independent.
(d) Determine the cumulative distribution function and use the cumuluative distribution function to determine the probability that the first customer arrives between 8:15 A.M and 8:30 A.M.

5. Determine the cumulative distribution function for the distribution in $f(x)=e^{-x}$ for $0<x$.
6. Determine the cumulative distribution function for the distribution in $f(x)=x/8$ for $3<x<5$.
7. Determine the cumulative distribution function for the distribution in $f(x)=e^{-(x-4)}$ for $4<x$.

8. Determine the cumulative distribution function for the distribution in $f(x)=\dfrac{e^{\frac{-x}{1000}}}{1000}$ *for* $x>0$. Use the cumulative distribution function to determine the probability that a component lasts more than 3000 hours before failure.

9. Determine the cumulative distribution function for the distribution in f(x)=2 for $2.3<x<2.8$ meters.. Use the cumulative distribution function to determine the probability that a length exceeds 2.7meters.

Determine the probability density function for each of the following cumulative distribution functions.

10 $F(x)=1-e^{-2x}$ x>0

11 $F(x)=\begin{cases}0 & x<0\\ 0.2x & 0\le x<4\\ 0.04x+0.64 & 4\le x<9\\ 1 & 9\le x\end{cases}$

12 $F(x)=\begin{cases}0 & x<-2\\ 0.25x+0.5 & -2\le x<1\\ 0.5x+0.25 & 1\le x<1.5\\ 1 & 1.5\le x\end{cases}$

Cumulative Distribution function of a continuous random variable

$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\,du$ for $-\infty < x < \infty$

Note: $\frac{d}{dx}[F(x)] = f(x)$

Problems on section-2

(1) $F(x) = \begin{cases} 0 & x < 0 \\ 0.2x & 0 \le x < 5 \\ 1 & 5 \le x \end{cases}$

a) $P(X < 2.8) = 0.2x = 0.2(2.8) = 0.56 = F(2.8)$

b) $P(X > 1.5) = 1 - P(X \le 1.5) = 1 - F(1.5)$
$= 1 - 0.2(1.5) = 0.7$

c) $P(X < -2) = 0 \quad \because F(-2) = 0$

d) $P(X > 6) = 1 - P(X \le 6) = 1 - 1 = 0$, $\boxed{1 - F(6) = 1 - 1 = 0}$

(2) $F(x) = \begin{cases} 0 & x < -2 \\ 0.25x + 0.5 & -2 \le x \le 2 \\ 1 & 2 \le x \end{cases}$

a) $P(X < 1.8) = 0.25x + 0.5 = 0.25(1.8) + 0.5 = 0.95$

b) $P(X > -1.5) = 1 - P(X \le -1.5)$
$= 1 - [0.25(-1.5) + 0.5]$
$= 0.725$

c) $P(X < -2) = 0$ (or) $[0.25(-2) + 0.5 = 0]$

~~d) $P(-1 < X < 1) =$~~

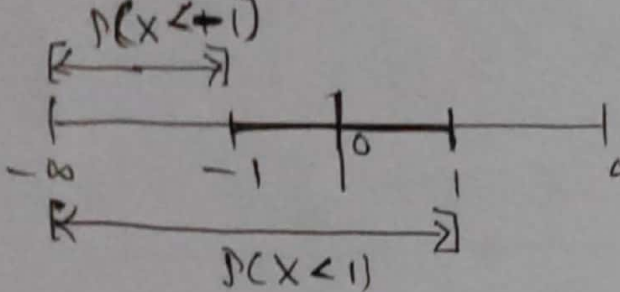
(2) a) $P(-1 < X < 1) = P(X < 1) - P(X < -1)$

$= [0.25x + 0.5] - [0.25x + 0.5]$

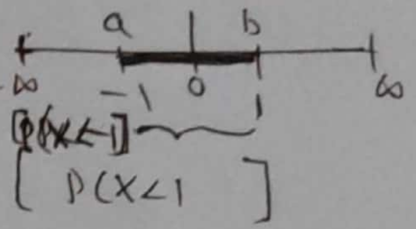
$= [0.25(1) + 0.5] - [0.25(-1) + 0.5]$

$= 0.75 - 0.25$

$= 0.5$



Note:
$P(a < x < b) = P(x < b) - P(x < a)$



(3) a) Given $f(x) = 0.5x \quad 0 < x < 2$

$F(x) = 0$ for $x < 0$

$F(x) = \int_0^x 0.5x\,dx = 0.5\left[\frac{x^2}{2}\right]_0^x = 0.25[x^2 - 0]$

$F(x) = 0.25x^2$ & $F(x) = 1 \quad 2 \le x$

$$\therefore F(x) = \begin{cases} 0 & x < 0 \\ 0.25x^2 & 0 \le x < 2 \\ 1 & 2 \le x. \end{cases}$$

(4) Note $P(a < x < b) = P(x < b) - P(x < a)$

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(5) $f(x) = e^{-x}$ for $0 < x$ $(x > 0)$

$F(x) = 0 \quad x < 0 \quad (0 \geq x)$

$F(x) = \int_0^x e^{-x}\,dx = \left[-e^{-x}\right]_0^x = -e^{-x} + e^0 = 1 - e^{-x} \quad 0 < x$

(or) $x > 0$

$\therefore F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases}$

(6) $f(x) = \frac{x}{8}$ for $3 < x < 5$

$\therefore F(x) = 0 \quad x < 3$

$F(x) = \int_3^x \frac{x}{8}\,dx = \frac{1}{8}\left[\frac{x^2}{2}\right]_3^x = \frac{1}{16}\left[x^2 - 9\right]$

$F(x) = 1 \quad x > 5$

$F(x) = \begin{cases} 0 & x < 3 \\ \frac{x^2 - 9}{16} & 3 \leq x < 5 \\ 1 & x \geq 5 \end{cases}$

(7) $f(x) = e^{-(x-4)} \quad 4 < x \ (x > 4)$

$F(x) = 0 \quad x < 4$

$F(x) = \int_4^x e^{-(x-4)}\,dx = \left[-e^{-(x-4)}\right]_4^x = -e^{-(x-4)} + e^0$

$(x \geq 4)$

$F(x) = 1 - e^{-(x-4)} \quad (x \geq 4)$

$F(x) = \begin{cases} 0 & x < 4 \\ 1 - e^{-(x-4)} & 4 \leq x \ (x \geq 4) \end{cases}$

(6) $f(x) = \frac{e^{-\frac{x}{1000}}}{1000}$ $x > 0$ (13)

$F(x) = 0$ $x < 0$

$F(x) = \int_0^x \frac{e^{-\frac{x}{1000}}}{1000} dx = \frac{1}{1000}\left[\frac{-e^{-\frac{x}{1000}}}{\frac{1}{1000}}\right]_0^x$

$F(x) = -e^{-\frac{x}{1000}} + 1 = 1 - e^{-\frac{x}{1000}}$ $x \geq 0$

$\therefore F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\frac{x}{1000}} & x \geq 0 \end{cases}$

$P(X > 3000) = 1 - F(x \leq 3000)$

$= 1 - \int_0^{3000} \frac{e^{-\frac{x}{1000}}}{1000} dx = 1 - \left[\frac{1}{1000}\left(\frac{-e^{-\frac{x}{1000}}}{\frac{1}{1000}}\right)\right]_0^{3000}$

$= 1 - \left[-e^{-3} + e^{0}\right] = 1 + e^{-3} - 1$

$= e^{-3} = 0.0498$

(7) $f(x) = 2$ for $2.3 < x < 2.8$

$F(x) = 0$ $x < 2.3$

$F(x) = \int_{2.3}^{x} 2\, dx = 2[x]_{2.3}^{x} = 2[x - 2.3] = 2x - 4.6$

$F(x) = 1$ $x \geq 2.8$

$F(x) = \begin{cases} 0 & x < 2.3 \\ 2x - 4.6 & 2.3 \leq x < 2.8 \\ 1 & x \geq 2.8 \end{cases}$

$F(X > 2.7) = 1 - F(x \leq 2.7) = 1 - [2x - 4.6]$

$= 1 - [2(2.7) - 4.6] = 0.2$

(14)

(10) $F(x) = 1 - e^{-2x}, x > 0$

$f(x) = \frac{d}{dx}[F(x)] = \frac{d}{dx}(1 - e^{-2x})$

$f(x > 0) = 2e^{-2x}, \quad f(x < 0) = 0$

$$f(x) = \begin{cases} 0 & x \le 0 \\ 2e^{-2x} & x > 0 \end{cases}$$

P.d.f

$$f(x) = \begin{cases} \frac{d}{dx}(0) = 0 & x \le 0 \\ \frac{d}{dx}(1 - e^{-2x}) = 2e^{-2x} & x > 0 \end{cases}$$

(11) $$F(x) = \begin{cases} 0 & x < 0 \\ 0.2x & 0 \le x < 4 \\ 0.04x + 0.64 & 4 \le x < 9 \\ 1 & 9 \le x \end{cases}$$

$f(x < 0) = 0$

$f(0 \le x < 4) = 0.2$

$f(4 \le x < 9) = 0.04$

$P(9 \le x) = 0$

$$f(x) = \begin{cases} 0 & x < 0 \\ 0.2 & 0 \le x < 4 \\ 0.04 & 4 \le x < 9 \\ 0 & x \ge 9 \end{cases}$$

$$f(x) = \begin{cases} \frac{d}{dx}(0) & x < 0 \\ \frac{d}{dx}(0.2x) & 0 \le x < 4 \\ \frac{d}{dx}(0.04x + 0.64) & 4 \le x < 9 \\ \frac{d}{dx}(1) & x \ge 9 \end{cases}$$

(12) $$F(x) = \begin{cases} 0 & x < -2 \\ 0.25x + 0.5 & -2 \le x < 1 \\ 0.5x + 0.25 & 1 \le x < 1.5 \\ 1 & 1.5 \le x \end{cases}$$

$f(x < -2) = 0$

$f(-2 \le x < 1) = 0.25$

$f(1 \le x < 1.5) = 0.5$

$f(x \ge 1.5) = 0$

$$\therefore f(x) = \begin{cases} 0 & x < -2 \\ 0.25 & -2 \le x < 1 \\ 0.5 & 1 \le x < 1.5 \\ 0 & x > 1.5 \text{ (or) } (1.5 \le x) \end{cases}$$

SECTION -3

1. Suppose $f(x) = 0.25$ for $0 < x < 4$. Determine the mean and variance of X.

2. Suppose $f(x) = 1.5x^2$ for $-1 < x < 1$. Determine the mean and variance of X.

3. Suppose that $f(x) = x/8$ for $3 < x < 5$. Determine the mean and variance of X.

4. Suppose that contamination particle size (in micrometers) can be modeled as $f(x) = 2x^{-3}$ for $1 < x$. Determine the mean of X.

5. Suppose the probability density function of the length of computer cables is $f(x) = 0.1$ from 1200 to 1210 millimeters.
(a) Determine the mean and standard deviation of the cable length.
(b) If the length specifications are $1195 < x < 1205$ millimeters, what proportion of cables are within specifications?

6. The thickness of a conductive coating in micrometers has a density function of $600x^{-2}$ for $100\,\mu m < x < 120\,\mu m$
(a) Determine the mean and variance of the coating thickness.
(b) If the coating costs \$0.50 per micrometer of thickness on each part, what is the average cost of the coating per part?

7. The probability density function of the weight of packages delivered by a post office is
$f(x) = \dfrac{70}{(69x^2)}$ *for* $1 < x < 70$ Pounds.
(a) Determine the mean and variance of the weight.
(b) If the shipping cost \$2.50 per pound, what is the average shipping cost of a package?
(c) Determine the probability that the weight of a package exceeds 50 pounds.

8. Integration by parts is required. The probability density function for the diameter of a drilled hole in millimeters is $10e^{-10(x-5)}$ for x > 5 mm. Although the target diameter is 5millimeters, vibrations, tool wear, and other nuisances produce diameters larger than 5 millimeters.
(a) Determine the mean and variance of the diameter of the holes.
(b) Determine the probability that a diameter exceeds 5.1 millimeters.

Mean and Variance of continuous Random variable

Suppose X is a continuous Random variable with probability density function (pdf) $f(x)$ the mean (or) Expected value $E(x)$

$$\boxed{E(X) = \mu = \text{mean} = \int_{-\infty}^{\infty} x\, f(x)\, dx}$$

Standard deviation $= \sqrt{\text{variance}}$

$$\text{Variance } \sigma^2 = E(X^2) - [E(X)]^2$$

$$\text{where } E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx.$$

Problems Section-3

(1) $f(x) = 0.25$ for $0 < x < 4$

$$\text{Mean} = E(X) = \int_0^4 x\, f(x)\, dx = \int_0^4 x(0.25)\, dx$$

$$= 0.25\left[\frac{x^2}{2}\right]_0^4 = 0.25[8-0] = 2$$

$$E(X^2) = \int_0^4 x^2 f(x)\, dx = \int_0^4 0.25x^2\, dx = 0.25\left[\frac{x^3}{3}\right]_0^4$$

$$E(X^2) = 0.25\left[\frac{4^3}{3} - 0\right] = 5.33$$

$$\text{Variance } (\sigma^2) = E(X^2) - [E(X)]^2$$

$$= 5.33 - (2)^2$$

$$\boxed{\text{Variance} = 1.33}$$

(2) $f(x) = 1.5x^2$ for $-1 < x < 1$

$$\text{Mean} = E(X) = \int_{-1}^{1} x f(x)\,dx = \int_{-1}^{1} 1.5x^3\,dx$$

$$= 1.5\left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{1.5}{4}\left[(1)^2 - (-1)^2\right] = 0$$

$$E(X^2) = \int_{-1}^{1} x^2 f(x)\,dx = \int_{-1}^{1} x^2 (1.5) x^2\,dx$$

$$= \int_{-1}^{1} 1.5x^4\,dx = 1.5\left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{1.5}{5}\left[(1)^5 - (-1)^5\right]$$

$$\boxed{E(X^2) = 0.6}$$

$$\text{Variance} = \sigma^2 = E(X^2) - [E(X)]^2 = 0.6 - (0)^2$$

$$\boxed{\text{variance} = 0.6}$$

(3) $f(x) = \frac{x}{8}$ for $3 < x < 5$

$$\text{Mean} = E(X) = \int_{+3}^{5} x f(x)\,dx = \int_{+3}^{5} x \frac{(x)}{8}\,dx$$

$$= \int_{+3}^{5} \frac{x^2}{8}\,dx = \frac{1}{8}\left[\frac{x^3}{3}\right]_3^5 = \frac{1}{24}\left[5^3 - 3^3\right]$$

$$\boxed{\text{Mean} = 4.08}$$

$$E(X^2) = \int_{3}^{5} x^2 f(x)\,dx = \int_{3}^{5} \frac{x^3}{8}\,dx = \left[\frac{x^4}{32}\right]_3^5 = 17$$

$$\text{variance} = \sigma^2 = E(X^2) - [E(X)]^2 = 17 - (4.08)^2$$

$$= 0.3536$$

4) $f(x) = 2x^{-3}$ for $1 < x$ $(x > 1)$

$$\text{Mean} = E(X) = \int_1^\infty x f(x)\,dx = \int_1^\infty x \frac{2}{x^3}\,dx = \int_1^\infty \frac{2}{x^2}\,dx$$

$$E(X) = \left[-\frac{2}{x}\right]_1^\infty = -\frac{2}{\infty} + \frac{2}{1} = 0 + 2$$

$$\boxed{E(X) = 2}$$

$$\int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = \frac{-1}{x}$$

5) a) $f(x) = 0.1$ for $1200 < x < 1210$

$$\text{Mean} = \int_{1200}^{1210} 0.1x\,dx = 0.1\left[\frac{x^2}{2}\right]_{1200}^{1210} = 1205$$

$$E(X^2) = \int_{1200}^{1210} 0.1\,x^2\,dx = (0.1)\left[\frac{x^3}{3}\right]_{1200}^{1210}$$

$$E(X^2) = 1452033.33$$

$$\text{variance} = \sigma^2 = E(X^2) - [E(X)]^2$$

$$\boxed{\text{variance} = 8.33}$$

$$S.D = \sqrt{\text{variance}} = \sqrt{8.33} = 2.887$$

b) clearly centering the process at centre of specification results in greater proportion of specification.

$\therefore$ $1200 < X < 1205$

$$= \int_{1195}^{1200} 0\,dx + \int_{1200}^{1205} 0.1\,dx = 0.5$$

$$P(1200 < X < 1205) = \int_{1200}^{1205} 0.1\,dx = 0.1\,[x]_{1200}^{1205} = 0.1 \times 5 = 0.5$$

(6) $f(x) = 600x^{-2}$ $\quad 100\text{ cm} < x < 120\text{ cm}$ (18)

a) Mean $= E(x) = \int_{100}^{120} 600x^{-2} \times x\, dx = \int_{100}^{120} \frac{600}{x} dx$

$= 600[\log x]_{100}^{120} = 109.39$

$E(x^2) = \int_{100}^{120} 600x^{-2} \cdot x^2 dx = \int_{100}^{120} 600\, dx = 600[20]$

$E(x^2) = 12000$

variance $= \sigma^2 = E(x^2) - [E(x)]^2$

$= 12000 - (109.39)^2 = 33.83$

b) Average cost $= 0.5 \times 109.39 = 54.7$

(7) a) $f(x) = \frac{70}{69x^2}$ for $1 < x < 70$

Mean $= E(x) = \int_1^{70} \frac{70}{69x^2} x\, dx = \frac{70}{69} \int_1^{70} \frac{1}{x} dx$

Mean $= \frac{70}{69} [\log x]_1^{70} = 4.31$

$E(x^2) = \int_1^{70} \frac{70}{69x^2} x^2 dx = \frac{70}{69} [x]_1^{70} = \frac{70}{69} [69] = 70$

$\sigma^2 =$ variance $= E(x^2) - (E(x))^2$

$= 70 - (4.31)^2 = 51.42$

b) Average cost $= 4.31 \times 2.5 = 10.775$

(c) $P(X > 50) = \int_{50}^{70} \frac{70}{69x^2} dx = \left[-\frac{70}{69} \frac{1}{x}\right]_{50}^{70}$

$= 5.79 \times 10^{-3}$

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8) $f(x) = 10\,e^{-10(x-5)}$ for $x > 5$

a) $E(X) = \text{mean} = \int_5^\infty 10\,e^{-10(x-5)}\, x\, dx$

$$= 10\left[x\left(\frac{e^{-10(x-5)}}{-10}\right) - \left(\frac{e^{-10(x-5)}}{(-10)^2}\right)\right]_5^\infty$$

$$= 10\left[0 - \left(-\frac{1}{2} - \frac{1}{100}\right)\right] = 5.1$$

$$E(X^2) = \int_5^\infty 10\,e^{-10(x-5)}\, x^2\, dx$$

$$E(X^2) = 10\left[x^2\left(\frac{e^{-10(x-5)}}{-10}\right) - 2x\left(\frac{e^{-10(x-5)}}{(-10)^2}\right) + 2\left(\frac{e^{-10(x-5)}}{(-10)^3}\right)\right]_5^\infty$$

$$E(X^2) = 26.02$$

variance $\sigma^2 = E(X^2) - [E(X)]^2$

$$= 26.02 - (5.1)^2 = 0.01$$

b) $P(X > 5.1) = \int_{5.1}^\infty 10\,e^{-10(x-5)}\, dx$

$$= \cancel{10}\left[\frac{e^{-10(x-5)}}{-\cancel{10}}\right]_{5.1}^\infty$$

$$= -e^{-\infty} + e^{-1.0}$$

$$= 0 + e^{-1.0} = \frac{1}{e}$$

$$= 0.3678$$

Bernoulli's Integration

$$\int uv\,dx = uv' - u_1 v'' + u_2 v''' - u_3 v^{iv} + \cdots$$

where
I, II, III, IV ... Integration
1, 2, 3 ... differentiation

Continuous Uniform distribution

1. Suppose X has a continuous uniform distribution over the interval [1.5, 5.5].
(a) Determine the mean, variance, and standard deviation of X.
(b) What is $P(X < 2.5)$?(c) Determine the cumulative distribution function.

2. Suppose X has a continuous uniform distribution over the interval[-1,1].
(a) Determine the mean, variance, and standard deviation of X.
(b) Determine the value for x such that $P(-x<X<x)=0.90$.
(c) Determine the cumulative distribution function.

3. The net weight in pounds of a packaged chemical herbicide is uniform for $49.75 < x < 50.25$ pounds.
(a) Determine the mean and variance of the weight of packages.
(b) Determine the cumulative distribution function of the weight of packages.
(c) Determine $P(X < 50.1)$.

4. The thickness of a flange on an aircraft component is uniformly distributed between 0.95 and 1.05 millimeters.
(a) Determine the cumulative distribution function of flange thickness.
(b) Determine the proportion of flanges that exceeds 1.02 millimeters.
(c) What thickness is exceeded by 90% of the flanges?
(d) Determine the mean and variance of flange thickness.

5. Suppose the time it takes a data collection operator to fill out an electronic form for a database is uniformly between 1.5 and 2.2 minutes.
(a) What is the mean and variance of the time it takes an operator to fill out the form?
(b) What is the probability that it will take less than two minutes to fill out the form?
(c) Determine the cumulative distribution function of the time it takes to fill out the form.

6. The thickness of photo resist applied to wafers in semiconductor manufacturing at a particular location on the wafer is uniformly distributed between 0.2050 and 0.2150 micrometers.
(a) Determine the cumulative distribution function of photo resist thickness.
(b) Determine the proportion of wafers that exceeds 0.2125micrometers in photoresist thickness. (c) What thickness is exceeded by 10% of the wafers?
(d) Determine the mean and variance of photo resist thickness.

7. An adult can lose or gain two pounds of water in the course of a day. Assume that the changes in water weight is uniformly distributed between minus two and plus two pounds in a day. What is the standard deviation of your weight over a day?

8. A dolphin show is scheduled to a start at 9:00 A.M, 9:30 A.M and 10:00A.M. Once the show starts, the gate will be closed. A visitor will arrive at the gate at a time uniformly distributed between 8:30 A.M and 10:00 A.M. Determine
(a) The cumulative distribution functions of the time (in minutes) between arrival and 8:30A.M.
(b) The mean and variance of the distribution in the previous part.
(c) The probability that a visitor waits less than 10 minutes for a show.
(d) The probability that a visitor waits more than 20 minutes for a show.

9. Measurement error that is continuous and uniformly distributed from -3 to +3 milli volts is added to the true voltage of a circuit. Then the measurement is rounded to the nearest millivolt so that it becomes discrete. Suppose that the true voltage is 250 millivolts.
(a) What is the probability mass function of the measured voltage?
(b) What is the mean and variance of the measured voltage?

Continuous Uniform distribution

A continuous random variable X with pdf $f(x) = \frac{1}{b-a}$ $a \le x \le b$ is a continuous uniform random variable.

The mean and variance of continuous uniform distribution is

$$\boxed{\mu = E(X) = \frac{a+b}{2}, \quad \sigma^2 = V(X) = \frac{(b-a)^2}{12}}$$

The cumulative distribution of continuous uniform distribution of random variable X is

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x < b \\ 1 & b \le x. \end{cases}$$

Problems

① $[1.5, 5.5]$

a) $\mu = \frac{a+b}{2} = \frac{1.5+5.5}{2} = 3.5$

Variance $\sigma^2 = \frac{(b-a)^2}{12} = \frac{(5.5-1.5)^2}{12} = \frac{16}{12} = \frac{4}{3}$

$S.D = \sqrt{\text{Variance}} = \sqrt{\frac{4}{3}} = 1.15$

(c) $F(x) = \begin{cases} 0 & x < 1.5 \\ \frac{x-1.5}{5.5-1.5} & 1.5 \le x < 5.5 \\ 1 & 5.5 \le x. \end{cases}$

(b) $P(X < 2.5) = \int_{1.5}^{2.5} \frac{1}{b-a}\,dx = \int_{1.5}^{2.5} \frac{1}{5.5-1.5}\,dx = \left[\frac{x}{4}\right]_{1.5}^{2.5} = \frac{1}{4}$

(4) $[-1, 1]$

a) $\mu = \frac{a+b}{2} = \frac{-1+1}{2} = 0$, $\sigma^2 = \frac{(b-a)^2}{12} = \frac{(1+1)^2}{12} = \frac{4}{12} = \frac{1}{3}$

$S.D = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$

b) $P(-x < X < x) = 0.9 \Rightarrow \int_{-x}^{x} \frac{1}{b-a} dx = 0.9$

$\int_{-x}^{x} \frac{1}{1+1} dx = 0.7$

$\Rightarrow \frac{1}{2}\left[x\right]_{-x}^{x} = 0.9 \Rightarrow \frac{1}{2}\left[x+x\right] = 0.9 \Rightarrow \boxed{x = 0.9}$

(c) $F(x) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{2} & -1 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$ | $F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x < b \\ 1 & b \leq x \end{cases}$

(5) $[49.75, 50.25]$

a) $\mu = \frac{a+b}{2} = \frac{49.75 + 50.25}{2} = 50$

$\sigma^2 = \frac{(b-a)^2}{12} = \frac{(50.25 - 49.75)^2}{12} = \frac{(0.5)^2}{12} = \frac{0.25}{12}$

& $S.D = \sqrt{\text{variance}} = 0.14$

b) $F(x) = \begin{cases} 0 & x < 49.75 \\ \frac{x - 49.75}{0.5} & 49.75 \leq x < 50.25 \\ 1 & 50.25 \leq x \end{cases}$

c) $P(X < 50.1) = \int_{49.75}^{50.1} \frac{1}{0.5} dx = \frac{1}{0.5}\left[x\right]_{49.75}^{50.1} = 0.7$

4) $[0.95, 1.05]$

a) $F(x) = \begin{cases} 0 & x < 0.95 \\ \frac{x-0.95}{0.1} & 0.95 \le x < 1.05 \\ 1 & 1.05 \le x \end{cases}$

b) $P(X > 1.02) = \int_{1.02}^{1.05} \frac{1}{b-a}\,dx = \int_{1.02}^{1.05} \frac{1}{0.1}\,dx = 10\left[x\right]_{1.02}^{1.05} = 0.3$

c) $P(X > x) = 0.9$

$1 - P(X \le x) = 0.9 \Rightarrow P(X \le x) = 1 - 0.9$

$\int_{0.95}^{x} \frac{1}{0.1}\,dx = 0.1 \Rightarrow \frac{1}{0.1}\left[x\right]_{0.95}^{x} = 0.1$

$[x - 0.95] = 0.01 \Rightarrow \boxed{x = 0.96}$

d) $\mu = \frac{0.95 + 1.05}{2} = 1$, $\sigma^2 = \frac{(1.05 - 0.95)^2}{12}$

$\sigma^2 = 8.33 \times 10^{-4}$.

5) $[1.5, 2.2]$

a) $\mu = \frac{1.5 + 2.2}{2} = 1.85$, $\sigma^2 = \frac{(2.2 - 1.5)^2}{12} = 0.041$

a) $F(x) = \begin{cases} 0 & x < 1.5 \\ \frac{x-1.5}{0.7} & 1.5 \le x < 2.2 \\ 0 & 2.2 \le x \end{cases}$

b) $P(X \le 2) = \int_{1.5}^{2} \frac{1}{0.7}\,dx = \frac{1}{0.7}\left[x\right]_{1.5}^{2} = 0.71$

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6) $[0.2050, 0.2150]$

a) $$F(x) = \begin{cases} 0 & x < 0.2050 \\ \dfrac{x - 0.2050}{0.01}, & 0.2050 < x < 2150 \\ 1 & 0.2050 \leq x \end{cases}$$

b) $$F(X > 0.2125) = \int_{0.2125}^{0.2160} \frac{1}{0.01}\, dx = \left[\frac{x}{0.01}\right]_{0.2125}^{0.2160} = 0.25$$

c) $$F(X > x) = 0.1$$

$$1 - P(X \leq x) = 0.1$$

$$P(X \leq x) = 1 - 0.1$$

$$\int_{0.2050}^{x} \frac{1}{0.01}\, dx = 0.9 \Rightarrow \frac{1}{0.01}[x]_{0.2050}^{x} = 0.9$$

$$\Rightarrow [x - 0.2050] = 0.009$$

$$\boxed{x = 0.2140}$$

d) $$\mu = \frac{0.2050 + 0.2150}{2} = 0.2100$$

$$\sigma^2 = \frac{(0.2150 - 0.2050)^2}{12} = 8.33 \times 10^{-4}.$$

7) $[-2, 2]$

$$\sigma^2 = \frac{(2+2)^2}{12} = \frac{4}{3} \Rightarrow \boxed{\sigma = S.D = 1.15}$$

8) a) $F(x) = \frac{1}{90}$ $\quad 0 \leq x \leq 90$ (Minutes)

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{90} & 0 \leq x < 90 \\ 1 & 90 \leq x. \end{cases}$$

8 b) $\mu = \frac{0+90}{2} = 45$, $\sigma^2 = \frac{(90-0)^2}{12} = 675$

c) $P(X<10) = 3\int_0^{10} \frac{1}{90}\,dx$ (There are 3 shows) | $\int_{8\cdot50}^{9} \frac{1}{90}\,dx + \int_{9\cdot20}^{9\cdot30} \frac{1}{90} + \int_{9\cdot50}^{10} \frac{1}{90}$

$= \frac{3}{90}\left[x\right]_0^{10} = \frac{30}{90} = \frac{1}{3}$

d) $P(X>20) = 1-P(X\le 20)$

$= 1-3\int_0^{20}\frac{1}{90}\,dx = 1-\frac{3}{90}\left[x\right]_0^{20}$

$= 1-\frac{1}{30}[20] = 1-\frac{2}{3} = \frac{1}{3}$

9 Let x denotes the voltage in circuit

[247, 253]

$\mu = \frac{247+253}{2} = 250$

$\sigma^2 = \frac{(253-247)^2}{12} = 3$

Normal distribution:

A random variable x with pdf

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$ is a normal random variable with parameters μ where $-\infty < \mu < \infty$ & $\sigma > 0$. Also $E(X) = \mu$ & $V(X) = \sigma^2$. The mean & variance of x are equal to μ & σ^2.

Standard normal Random variable:

A normal random variable with $\mu = 0$ & $\sigma^2 = 1$ is called standard normal variable & is denoted by Z. The cdf of a standard normal Radom variable is denoted by

$$\Phi(z) = P(Z \le z)$$

Standardizing a Normal Random Variable.

If X is a normal random variable with $E(X) = \mu$ & $V(X) = \sigma^2$ the random variable $Z = \frac{X-\mu}{\sigma}$ is a normal random variable with $E(Z) = 0$ & $V(Z) = 1$. That is Z is a standard normal random variable.

Standardizing to Calculate probability:

Suppose X is a normal random variable with mean μ & variance σ^2 then

$$P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) = P(Z \le z)$$

where Z is a standard normal random variable & $\boxed{z = \frac{x-\mu}{\sigma}}$ is the z-value

Normal distribution

1. Assume Z has a standard normal distribution. Use Appendix Table III to determine the value for z that solves each of the following:
(a) $P(Z < z) = 0.9$ (b) $P(Z < z) = 0.5$ (c) $P(Z > z) = 0.1$ (d) $P(Z > z) = 0.9$ (e) $P(-1.24 < Z < z) = 0.8$ (f) $P(-z < Z < z) = 0.95$ (g) $P(-z < Z < z) = 0.99$
(h) $P(-z < Z < z) = 0.68$ (i) $P(-z < Z < z) = 0.9973$

2. Assume X is normally distributed with a mean of 10 and a standard deviation of 2. Determine the value for x that solves each of the following:
(a) $P(X > x) = 0.5$ (b) $P(X > x) = 0.95$ (c) $P(x < X < 10) = 0.2$ (d) $P(-x < X - 10 < x) = 0.95$
(e) $P(-x < X - 10 < x) = 0.99$

3. The compressive strength of samples of cement can be modeled by a normal distribution with a mean of 6000 kilograms per square centimeter and a standard deviation of 100kilograms per square centimeter.
(a) What is the probability that a sample's strength is less than 6250 Kg/cm^2?
(b) What is the probability that a sample's strength is between 5800 and 5900 Kg/cm^2?
(c) What strength is exceeded by 95% of the samples?

4. The time until recharge for a battery in a laptop computer under common conditions is normally distributed with mean of 260 minutes and a standard deviation of 50 minutes.
(a) What is the probability that a battery lasts more than four hours?
(b) What are the quartiles (the 25% and 75% values) of battery life?
(c) What value of life in minutes is exceeded with 95% probability?

5 An article in *Knee surgery sports traumatol Arthrose* " Effect of provider volume on resource utilization for surgical procedures," (2005,Vol.13,PP. 273-279) showed a mean time of 129 minutes and a standard deviation of 14 minutes for ACL reconstruction surgery at high-volume hospitals(with more than 300 such surgeries per year).
(a) What is the probability that your ACL surgery at a high-volume hospital requires a time more than two standard deviations above the mean?
(b) What is the probability that your ACL surgery at a high-volume hospital is completed in less than 100minutes?
(c) The probability of a completed ACL surgery at a high-volume hospital is equal to 95% at what time?
(d) If your surgery requires 199 minutes, what do you conclude about the volume of such surgeries at your hospital? Explain.

6.Cholesterol is a fatty substance that is an important part of the outer lining (membrane) of cells in the body of animals. Its normal range for an adult is 120-240 mg.dl. The food and Nutrition Institute of the Philippiness found that the total cholesterol level for Filipino adults has a mean of 159.2 mg/dl and 84.1% of adults have a cholesterol level below 200mg/dl. Suppose that the total cholesterol level is normally distributed.
(a) Determine the standard deviation of this distribution.
(b) What are the quartiles (the 25% and 75% values) of this distribution?
(c) What is the value of the cholesterol level that exceeds 90% of the population?
(d) An adult is at moderate risk if cholesterol level is more than one but less than two standard deviations above the mean. What percentage of the population is at moderate risk according to this criterion/

7. The line width for semiconductor manufacturing is assumed to be normally distributed with a mean of 0.5 micrometer and a standard deviation of 0.05 micrometer.
(a) What is the probability that a line width is greater than 0.62 micro meter?
(b) What is the probability that a line width is between 0.47 and 0.63 micrometer?
(c) The line width of 90% of samples is below what value?

8. The fill volume of an automated filling machine used for filling cans of carbonated beverage is normally distributed with a mean of 12.4 fluid ounces and a standard deviation of
0.1 fluid ounce.
(a) What is the probability a fill volume is less than 12 fluid ounces?
(b) If all cans less than 12.1 or greater than 12.6 ounces are scrapped, what proportion of cans is scrapped?
(c) Determine specifications that are symmetric about the mean that include 99% of all cans.

Normal Distribution (continued)

9. The reaction time of a driver to visual stimulus is normally distributed with a mean of 0.4 seconds and a standard deviation of 0.05 seconds.
(a) What is the probability that a reaction requires more than 0.5 seconds?
(b) What is the probability that a reaction requires between 0.4 and 0.5 seconds?
(c) What is the reaction time that is exceeded 90% of the time?

10. The speed of a file transfer from a server on campus to a personal computer at a student's home on a weekday evening is normally distributed with a mean of 60 kilobits per second and a standard deviation of 4 kilobits per second.
(a) What is the probability that the file will transfer at a speed of 70 kilobits per second or more?
(b) What is the probability that the file will transfer at a speed of less than 58 kilobits per second?
(c) If the file is 1 megabyte, what is the average time it will take to transfer the file? (Assume eight bits per byte.)

11. The average height of a woman aged 20-74 years is 64inches in 2002 with an increase of approximately one inch from 1960.Suppose the height of a woman is normally distributed with a standard deviation of 2 inches.
(a) What is the probability a randomly selected woman in this population is between 58 inches and 70 inches?
(b) What are the quartiles of this distribution?
(c) Determine the height that is symmetric about the mean that includes 90% of this population.
(d) What is the probability that five women selected at random from this population all exceed 68 inches?

12. In an accelerator center, an experiment needs a 1.41 cm thick aluminum cylinder Suppose that the thickness of a cylinder has a normal distribution with a mean 1.41cm and a standard deviation of 0.01 cm.
(a) What is the probability that a thickness is greater than 1.42cm?
(b) What thickness is exceeded by 95% of the samples?
(c) If the specifications require that the thickness is between 1.39cm and 1.43cm, what proportion of the samples meet specifications?

13. The demand for water use in phoenix in 2003 hit a high of about 442 million gallons per day on June 27,2003 .Water use in the summer is normally distributed with a mean of 310 million gallons per day and a standard deviation of 45 million gallons per day. City reservoirs have a combined storage capacity of nearly 350 million gallons.
(a) What is the probability that a day requires more water than is stored in city reservoirs?
(b) What reservoir capacity is needed so that the probability it is exceeded is 1%?
(c) What amount of water use is exceeded with 95% probability ?

14. The life of a semiconductor laser at a constant power is normally distributed with a mean of 7000 hours and a standard deviation of 600 hours.
(a) What is the probability that a laser fails before 5000 hours?
(b) What is the life in hours that 95% of the lasers exceed?
(c) If three lasers are used in a product and they are assumed to fail independently, what is the probability that all three are still operating after 7000 hours?

15. The diameter of the dot produced by a printer is normally distributed with a mean diameter of 0.002 inch and a standard deviation of 0.0004 inch.
(a) What is the probability that the diameter of a dot exceeds 0.0026 inch?
(b) What is the probability that a diameter is between 0.0014 and 0.0026 inch?

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$$

Table III Cumulative Standard Normal Distribution

z	−0.09	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.02	−0.01	−0.00
−3.9	0.000033	0.000034	0.000036	0.000037	0.000039	0.000041	0.000042	0.000044	0.000046	0.000048
−3.8	0.000050	0.000052	0.000054	0.000057	0.000059	0.000062	0.000064	0.000067	0.000069	0.000072
−3.7	0.000075	0.000078	0.000082	0.000085	0.000088	0.000092	0.000096	0.000100	0.000104	0.000108
−3.6	0.000112	0.000117	0.000121	0.000126	0.000131	0.000136	0.000142	0.000147	0.000153	0.000159
−3.5	0.000165	0.000172	0.000179	0.000185	0.000193	0.000200	0.000208	0.000216	0.000224	0.000233
−3.4	0.000242	0.000251	0.000260	0.000270	0.000280	0.000291	0.000302	0.000313	0.000325	0.000337
−3.3	0.000350	0.000362	0.000376	0.000390	0.000404	0.000419	0.000434	0.000450	0.000467	0.000483
−3.2	0.000501	0.000519	0.000538	0.000557	0.000577	0.000598	0.000619	0.000641	0.000664	0.000687
−3.1	0.000711	0.000736	0.000762	0.000789	0.000816	0.000845	0.000874	0.000904	0.000935	0.000968
−3.0	0.001001	0.001035	0.001070	0.001107	0.001144	0.001183	0.001223	0.001264	0.001306	0.001350
−2.9	0.001395	0.001441	0.001489	0.001538	0.001589	0.001641	0.001695	0.001750	0.001807	0.001866
−2.8	0.001926	0.001988	0.002052	0.002118	0.002186	0.002256	0.002327	0.002401	0.002477	0.002555
−2.7	0.002635	0.002718	0.002803	0.002890	0.002980	0.003072	0.003167	0.003264	0.003364	0.003467
−2.6	0.003573	0.003681	0.003793	0.003907	0.004025	0.004145	0.004269	0.004396	0.004527	0.004661
−2.5	0.004799	0.004940	0.005085	0.005234	0.005386	0.005543	0.005703	0.005868	0.006037	0.006210
−2.4	0.006387	0.006569	0.006756	0.006947	0.007143	0.007344	0.007549	0.007760	0.007976	0.008198
−2.3	0.008424	0.008656	0.008894	0.009137	0.009387	0.009642	0.009903	0.010170	0.010444	0.010724
−2.2	0.011011	0.011304	0.011604	0.011911	0.012224	0.012545	0.012874	0.013209	0.013553	0.013903
−2.1	0.014262	0.014629	0.015003	0.015386	0.015778	0.016177	0.016586	0.017003	0.017429	0.017864
−2.0	0.018309	0.018763	0.019226	0.019699	0.020182	0.020675	0.021178	0.021692	0.022216	0.022750
−1.9	0.023295	0.023852	0.024419	0.024998	0.025588	0.026190	0.026803	0.027429	0.028067	0.028717
−1.8	0.029379	0.030054	0.030742	0.031443	0.032157	0.032884	0.033625	0.034379	0.035148	0.035930
−1.7	0.036727	0.037538	0.038364	0.039204	0.040059	0.040929	0.041815	0.042716	0.043633	0.044565
−1.6	0.045514	0.046479	0.047460	0.048457	0.049471	0.050503	0.051551	0.052616	0.053699	0.054799
−1.5	0.055917	0.057053	0.058208	0.059380	0.060571	0.061780	0.063008	0.064256	0.065522	0.066807
−1.4	0.068112	0.069437	0.070781	0.072145	0.073529	0.074934	0.076359	0.077804	0.079270	0.080757
−1.3	0.082264	0.083793	0.085343	0.086915	0.088508	0.090123	0.091759	0.093418	0.095098	0.096801
−1.2	0.098525	0.100273	0.102042	0.103835	0.105650	0.107488	0.109349	0.111233	0.113140	0.115070
−1.1	0.117023	0.119000	0.121001	0.123024	0.125072	0.127143	0.129238	0.131357	0.133500	0.135666
−1.0	0.137857	0.140071	0.142310	0.144572	0.146859	0.149170	0.151505	0.153864	0.156248	0.158655
−0.9	0.161087	0.163543	0.166023	0.168528	0.171056	0.173609	0.176185	0.178786	0.181411	0.184060
−0.8	0.186733	0.189430	0.192150	0.194894	0.197662	0.200454	0.203269	0.206108	0.208970	0.211855
−0.7	0.214764	0.217695	0.220650	0.223627	0.226627	0.229650	0.232695	0.235762	0.238852	0.241964
−0.6	0.245097	0.248252	0.251429	0.254627	0.257846	0.261086	0.264347	0.267629	0.270931	0.274253
−0.5	0.277595	0.280957	0.284339	0.287740	0.291160	0.294599	0.298056	0.301532	0.305026	0.308538
−0.4	0.312067	0.315614	0.319178	0.322758	0.326355	0.329969	0.333598	0.337243	0.340903	0.344578
−0.3	0.348268	0.351973	0.355691	0.359424	0.363169	0.366928	0.370700	0.374484	0.378281	0.382089
−0.2	0.385908	0.389739	0.393580	0.397432	0.401294	0.405165	0.409046	0.412936	0.416834	0.420740
−0.1	0.424655	0.428576	0.432505	0.436441	0.440382	0.444330	0.448283	0.452242	0.456205	0.460172
0.0	0.464144	0.468119	0.472097	0.476078	0.480061	0.484047	0.488033	0.492022	0.496011	0.500000

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$$

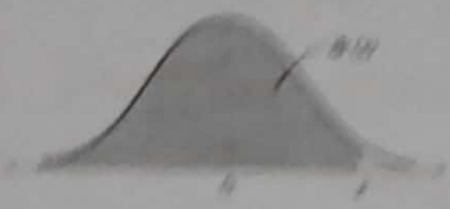
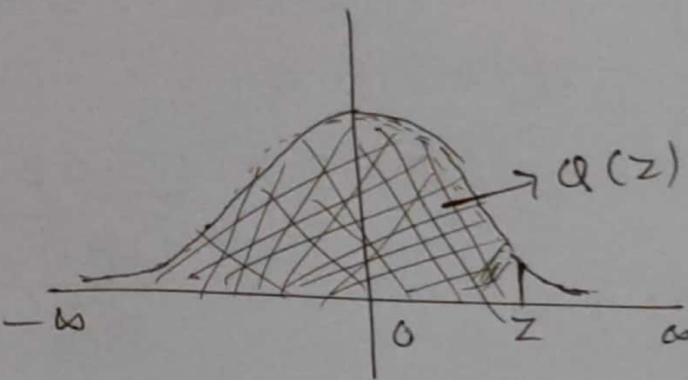


Table III Cumulative Standard Normal Distribution (continued)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.523922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555670	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499
3.3	0.999517	0.999534	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999822	0.999828	0.999835
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925
3.8	0.999928	0.999931	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.999950
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967

obtained by standardizing X.

The normal curve is bell shaped curve and is symmetrical about line $X = \mu$. The area under normal pdf from $-\infty < x < \infty$ is 1 & is symmetrical about the line $Z = 0$



$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

Problems on Normal distribution

1. a) $P(Z < z) = 0.9 \Rightarrow z = 1.28$

   b) $P(Z < z) = 0.5 \Rightarrow z = 0.$

   c) $P(Z > z) = 0.1$

   $1 - P(Z \le z) = 0.1$

   $P(Z \le z) = 1 - 0.1$

   $P(Z \le z) = 0.9 \Rightarrow z = 1.28$

   d) $P(Z > z) = 0.9$

   $1 - P(Z \le z) = 0.9$

   $P(Z \le z) = 0.1 \Rightarrow z = -1.28$

   e) $P(-1.24 < Z < z) = 0.8$

$P(Z < z) - P(-1.24) = 0.8$

$P(Z < z) - 0.107488 = 0.8$

$P(Z < z) = 0.907488 \quad \therefore \boxed{z = 1.32}$

I (f) $P(-z < Z < z) = 0.95$

$P(Z < z) - P(Z < -z) = 0.95$

$Q(z) - Q(-z) = 0.95$

$Q(z) - [1 - Q(z)] = 0.95$

$Q(z) - 1 + Q(z) = 0.95$

$2Q(z) = 1.95 \Rightarrow Q(z) = \frac{1.95}{2}$

$Q(z) = 0.975 = Q(1.96)$

$\boxed{z = 1.96}$

OR

$P(-z < Z < z) = 0.95$

$P(Z < z) = 0.95 + \frac{1 - 0.95}{2}$

$P(Z < z) = 0.975$

$\boxed{z = 1.96}$

(g) $P(-z < Z < z) = 0.99$

$P(Z < z) = 0.99 + \frac{1 - 0.99}{2}$

$P(Z < z) = 0.995$

$\boxed{z = 2.57}$

h) $P(-z < Z < z) = 0.68$

$P(Z < z) = 0.68 + \frac{1 - 0.68}{2}$

$P(Z < z) = 0.84 \Rightarrow \boxed{z = 1}$

i) $P(-z < Z < z) = 0.9973$

$P(Z < z) = 0.9973 + \frac{1 - 0.9973}{2}$

$P(Z < z) = 0.99865 \Rightarrow \boxed{z = 3}$

ii) $Q(z) + Q(-z) = 1$

$Q(-z) = 1 - Q(z)$

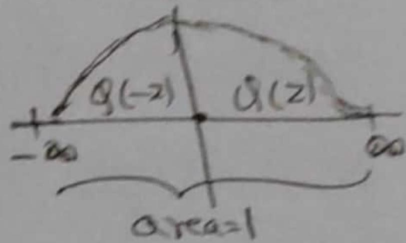
iii) $P(a < Z < b)$

$= P(Z < b) - P(Z < a)$

Note

$P(Z < z) = Q(z)$

$Q(z)$ from table.



Note

$P(-z < Z < z)$

$= 2Q(z) - 1$

(2) Mean $= \mu = 10$, S.D $= \sigma = 2$ (28)

(a) $P(X > x) = 0.5$

$P\left(\frac{X-10}{2} > \frac{x-10}{2}\right) = 0.5$

$1 - P\left(\frac{X-10}{2} \leq \frac{X-10}{2}\right) = 0.5$

$P\left(\frac{X-10}{2} \leq \frac{x-10}{2}\right) = 0.5$

$\frac{x-10}{2} = 0 \Rightarrow \boxed{x=10}$

$x=10$ is due to symmetry of Normal curve

$P\left(Z > \frac{x-10}{2}\right) = 0.5$

$\therefore Z = 0$ from table for 0.5

$\frac{x-10}{2} = 0$

$x - 10 = 0$

$\boxed{x=10}$

b) $P(X > x) = 0.95$

$P\left(\frac{X-10}{2} > \frac{x-10}{2}\right) = 0.95$

$1 - P\left(\frac{X-10}{2} < \frac{x-10}{2}\right) = 0.95$

$P\left(\frac{X-10}{2} < \frac{x-10}{2}\right) = 0.05$

$\frac{x-10}{2} = -1.64$

$x - 10 = -3.28$

$\boxed{x = 6.27}$

OR $P(X > x) = 0.95$

$P\left(Z > \frac{x-10}{2}\right) = 0.95$

$\Rightarrow 1 - P\left(Z < \frac{x-10}{2}\right) = 0.95$

$P\left(Z < \frac{x-10}{2}\right) = 0.05$

From table

$Z = -1.64$ for 0.05

$\frac{x-10}{2} = -1.64$

$x = 6.27$

c) $P(x < X < 10) = 0.2$

$P\left(\frac{x-10}{2} < \frac{X-10}{2} < \frac{10-10}{2}\right) = 0.2$

$P\left(\frac{x-10}{2} < \frac{X-10}{2} < 0\right) = 0.2$

$P(X=0) - P\left(\frac{X-10}{2} < \frac{X-10}{2}\right) = 0.2$

OR

$P(x < X < 10) = 0.2$

$P\left(\frac{x-10}{2} < Z < 0\right) = 0.2$

$0.5 - P\left(Z < \frac{x-10}{2}\right) = 0.2$

$P\left(Z < \frac{x-10}{2}\right) = 0.3$

$\frac{x-10}{2} = 0.52$

$\boxed{x = 8.96}$

$0.5 - P\left(\frac{X-10}{2} \leq \frac{x-10}{2}\right) = 0.2$

$P\left(\frac{X-10}{2} \leq \frac{x-10}{2}\right) = 0.3$

$\frac{x-10}{2} = -0.52$

$x = 10 - 1.04$

$\boxed{x = 8.96}$

(2) (d) $P(-x < X-10 < x) = 0.95$

$P\left(-\frac{x}{2} < \frac{X-10}{2} < \frac{x}{2}\right) = 0.95$

$P(-z < Z < z) = 0.95 \qquad | \, z = \frac{x}{2}$

$P(Z < z) = 0.95 + \frac{1-0.95}{2} = 0.975$

$\Rightarrow \frac{x}{2} = 1.96$

$\Rightarrow \boxed{x = 3.92}$

(e) $P(-x < X-10 < x) = 0.99$

$P\left(-\frac{x}{2} < \frac{X-10}{2} < \frac{x}{2}\right) = 0.99$

$P\left(\frac{X-10}{2} < \frac{x}{2}\right) = 0.99 + \frac{1-0.99}{2}$

$P\left(\frac{X-10}{2} < \frac{x}{2}\right) = 0.995$

$\frac{x}{2} = 2.57$

$\boxed{x = 5.14}$

3) $\mu = 6000 \quad \sigma = 100$

a) $P(X < 6250) = P\left(\frac{X-6000}{100} < \frac{6250-6000}{100}\right)$

$= P\left(\frac{X-6000}{100} < 2.5\right)$

$= P(Z < 2.5)$

$= \Phi(2.5)$

$= 0.9937$

$Z = \frac{X-\mu}{\sigma}$

Z is standard normal variate with c.d.f. 'Φ' given in table.

b) $P(5800 < X < 5900)$

$= P\left(\frac{5800-6000}{100} < \frac{X-6000}{100} < \frac{5900-6000}{100}\right)$

$= P(-2 < Z < -1)$

$= P(Z < -1) - P(Z < -2)$

$= 0.158655 - 0.022750$

$= 0.158655 - 0.022750$

$= 0.1359$

or

$P(-2 < Z < -1)$

$= P(1 < Z < 2)$

$= \Phi(2) - \Phi(1)$

$= 0.1359$

c) $P(X > x) = 0.95$

$P\left(\frac{X-6000}{100} > \frac{x-6000}{100}\right) = 0.95$

$1 - P\left(\frac{X-6000}{100} < \frac{x-6000}{100}\right) = 0.95$

$1 - P(Z < z) = 0.95 \quad \boxed{z = \frac{x-6000}{100}}$

$P(Z < z) = 0.05$

$z = -1.64$

$\frac{x-6000}{100} = -1.64$

$\boxed{x = 5836}$

4) $\mu = 260 \quad \sigma = 50$ (31)

a) $P(X > 240) = P\left(\frac{X-260}{50} > \frac{240-260}{50}\right)$

$= P(Z > -0.4)$

$= 1 - P(Z < -0.4)$

$= 1 - 0.312607$ $\boxed{Q(-0.4) = 0.312607}$

$= 0.687393$

b) x_1 & x_2 values for 25% & 75% values of Battery life

$P(X < x_1) = 0.25 \Rightarrow P\left(\frac{X-260}{50} < \frac{x_1-260}{50}\right) = 0.25$

$P\left(Z < \frac{x_1-260}{50}\right) = 0.25$

$\frac{x_1-260}{50} = -0.67 \Rightarrow x_1 - 260 = -33.5$

$\boxed{x_1 = 226.5}$

$P(X < x_2) = 0.75 \Rightarrow P\left(\frac{X-260}{50} < \frac{x_2-260}{50}\right) = 0.75$

$P\left(Z < \frac{x_2-260}{50}\right) = 0.75$

$\frac{x_2-260}{50} = 0.68$

$x - 260 = 34$

$\boxed{x = 294}$

c) $P(X > x_3) = 0.95 \Rightarrow P\left(\frac{X-260}{50} > \frac{x_3-260}{50}\right) = 0.95$

$1 - P\left(\frac{X-260}{50} < \frac{x_3-260}{50}\right) = 0.95$

$P\left(\frac{X-260}{50} < \frac{x_3-260}{50}\right) = 0.05$

$$\therefore \frac{x-260}{50} = -1.64$$

$$x - 260 = -82$$

$$\boxed{x = 178}$$

5 $\mu = 129 \quad \sigma = 14$

a) $P(X > (2 \times 14 + 129)) = P\left(\frac{X-129}{14} > \frac{2\times14+129-129}{14}\right)$

$$= P(Z > 2)$$

$$= 1 - P(Z \le 2) \qquad | \; \Phi(2) = 0.977250$$

$$= 1 - 0.977250$$

$$= 0.02275$$

b) $P(X < 100) = P\left(\frac{X-129}{14} < \frac{100-129}{14}\right)$

$$= P(Z < -2.071)$$

$$= \Phi(-2.071)$$

$$= 0.019226$$

c) $P(X < x) = 0.95$

$P\left(\frac{X-129}{14} < \frac{x-129}{14}\right) = 0.95 \Rightarrow P(Z < z) = 0.95$

$\frac{x-129}{14} = 1.65 \Rightarrow x - 129 = 23.1$

$$\boxed{x = 152.1}$$

d) $199 > 152.1$, So the volume at such surgery is very small $< 5\%$

(6) $\mu = 159.2 \quad \sigma = 6$

a) $P(X \le 200) = 0.841$

$$P\left(\frac{X-159.2}{\sigma} \le \frac{200-159.2}{\sigma}\right) = 0.841$$

$$P(Z \le \cancel{z}) = 0.841$$

$$\therefore \frac{200-159.2}{\sigma} = 1 \qquad Q(0.841) = 1$$

$$40.8 = \sigma \Rightarrow \boxed{\sigma = 40.8}$$

b) $P(X < x) = 0.25$

$$P\left(\frac{X-159.2}{40.8} < \frac{x-159.2}{40.8}\right) = 0.25$$

$$\frac{x-159.2}{40.8} = -0.67$$

$$x = 159.2 - 27.336$$

$$\boxed{x = 131.864}$$

$P(X < x) = 0.75$

$$P\left(\frac{X-159.2}{40.8} < \frac{x-159.2}{40.8}\right) = 0.75$$

$$\frac{x-159.2}{40.8} = 0.68$$

$$x = 159.2 + 27.744$$

$$\boxed{x = 186.944}$$

c) $P(X > x) = 0.9$

$$1 - P(X \le x) = 0.9$$

$$1 - P\left(\frac{X-159.2}{40.8} \le \frac{x-159.2}{40.8}\right) = 0.9$$

$$P\left(\frac{X-159.2}{40.8} \le \frac{x-159.2}{40.8}\right) = 0.1$$

$$\frac{x-159.2}{40.8} = -1.28$$

$$x = 159.2 - 52.224$$

$$\boxed{x = 106.976}$$

(6) d) $P(200 < X < 240.8) = P(X < 240.8) - P(X < 200)$

$= P\left(\frac{X-159.2}{40.8} < \frac{240.8-159.2}{40.8}\right) - P\left(\frac{X-159.2}{40.8} < \frac{200-159.2}{40.8}\right)$

$= P(Z < 2) - P(Z < 1)$

$= 0.977250 - 0.841345$

$= 0.1319$

$Q(2) = 0.977250$

$Q(1) = 0.841345$

(7) a) $\mu = 0.5$ $\sigma = 0.05$

$P(X > 0.62) = 1 - P(X \leq 0.62)$

$= 1 - P\left(\frac{X-0.5}{0.05} \leq \frac{0.62-0.5}{0.05}\right)$

$= 1 - P(Z \leq 2.4)$

$= 1 - Q(2.4)$

$= 1 - 0.991802$

$= 0.008192$

b) $P(0.47 < X < 0.63) = P\left(\frac{0.47-0.5}{0.05} < \frac{X-0.5}{0.05} < \frac{0.63-0.5}{0.05}\right)$

$= P(-0.62 < Z < +2.6)$

$= P(Z < 2.6) - P(Z < -0.62)$

$= Q(2.6) - Q(-0.62)$

$= 0.995339 - 0.274253$

$= 0.721086$

c) $P(X < x) = 0.9 \implies P\left(\frac{X-0.5}{0.05} < \frac{x-0.5}{0.05}\right) = 0.9$

$\frac{x-0.5}{0.05} = 1.29 \implies x - 0.5 = 0.0645$

$\implies \boxed{x = 0.5645}$

8 a) $\mu = 12.4 \quad \sigma = 0.1$

$P(X<12) = P\left(\frac{X-12.4}{0.1} < \frac{12-12.4}{0.1}\right)$

$= P(Z<-4) = 0$

b) $P(X<12.1) + P(X>12.6) =$

$= P\left(\frac{X-12.4}{0.1} < \frac{12.1-12.4}{0.1}\right) + P\left(\frac{X-12.4}{0.1} > \frac{12.6-12.4}{0.1}\right)$

$= P(Z<-3) + P(Z>2)$

$= P(Z<-3) + [1-P(Z\leq 2)]$

$= 0.001350 + 1 - 0.977250$

$= 0.0241$ (2.41% of cans are scrapped)

c) The slecitification that are symmetric about mean that include 99% of cans are $P(-x < X-\mu < x) = 0.99$

$P\left(-\frac{x}{0.1} < \frac{X-\mu}{\sigma} < \frac{x}{0.1}\right) = 0.99$

$Q(10x) - 1 + Q(10x) = 0.99$

$2Q(10x) = 1.99$

$Q(10x) = 0.995$

$10x = 2.58$

$\boxed{x = 0.258}$

$P(12.4 - x \leq X \leq 12.4 + x)$

$= P\left(\frac{-x}{0.1} \leq Z \leq \frac{x}{0.1}\right)$

$= 2Q\left(\frac{x}{0.1}\right) - 1$

The interval in question is
[12.142, 12658]

(9) a) $\mu = 0.4 \quad \sigma = 0.05$ (36)

$$P(X>0.5) = 1 - P(X \leq 0.5)$$
$$= 1 - P\left(\frac{X-0.4}{0.05} \leq \frac{0.5-0.4}{0.05}\right)$$
$$= 1 - P(Z \leq 2)$$
$$= 1 - 0.97725 = 0.02275$$

b) $P(0.4 < X < 0.5) = P(X<0.5) - P(X<0.4)$
$$= P\left(\frac{X-0.4}{0.05} < \frac{0.5-0.4}{0.05}\right) - P\left(\frac{X-0.4}{0.05} < \frac{0.4-0.4}{0.05}\right)$$
$$= P(Z<2) - P(Z<0)$$
$$= 0.97725 - 0.5 = 0.47725$$

c) $P(X>x) = 0.9$
$$1 - P(X<x) = 0.9$$
$$1 - P\left(\frac{X-0.4}{0.05} < \frac{x-0.4}{0.05}\right) = 0.9$$
$$P\left(Z < \frac{x-0.4}{0.05}\right) = 0.1$$
$$\frac{x-0.4}{0.05} = -1.28 \implies \boxed{x = 0.336}$$

(10) a) $\mu = 60 \quad \sigma = 4$
$$P(X>70) = P\left(\frac{X-60}{4} > \frac{70-60}{4}\right)$$
$$= P\left(\frac{X-60}{4} > 2.5\right)$$
$$= 1 - P\left(\frac{X-60}{4} < 2.5\right) = 1 - P(Z<2.5)$$
$$= 1 - 0.993790$$
$$= 0.00621$$

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(10) (b) $P(X < 58) = P\left(\frac{X-60}{4} < \frac{58-60}{4}\right)$ (37)

$= P(Z < -0.5)$

$= 0.308538$

(c) 1 megabyte $= 1000000 \times 8$

$= \frac{8000000 \text{ bits}}{60000 \text{ bits/sec}}$

$= 133.33$ sec

(11)

$\mu = 64 \quad \sigma = 2$

a) $P(58 < X < 70) = P\left(\frac{58-64}{2} < \frac{X-64}{2} < \frac{70-64}{2}\right)$

$= P(-3 < Z < 3) = P(Z < 3) - P(Z < -3)$

$= 0.99865 - 0.00135 = 0.9973$

b) $P(X \leq x) = 0.25$

$P\left(\frac{X-64}{2} \leq \frac{x-64}{2}\right) = 0.25$

$P\left(Z \leq \frac{x-64}{2}\right) = 0.25$

$\frac{x-64}{2} = -0.67 \Rightarrow \boxed{x = 62.66}$

$P(X \leq x) \leq 0.75$

$P\left(\frac{X-64}{2} \leq \frac{x-64}{2}\right) \leq 0.75$

$P\left(Z \leq \frac{x-64}{2}\right) \leq 0.75$

$\frac{x-64}{2} = 0.68$

$\therefore \boxed{x = 65.36}$

(11) (c) $P\left(-\frac{x}{2} \le \frac{X-\mu}{\sigma} \le \frac{x}{2}\right) = 0.9$ $\quad \boxed{P(\mu - x \le X \le \mu + x) = 0.9}$

$P\left(-z \le \frac{X-64}{2} \le z\right) = 0.9 \quad \Big|\ z = \frac{x}{2}$

(38)

$P(-z \le Z \le z) = 0.9$

$P(Z \le z) = 0.9 + \frac{1-0.9}{2}$

$P(Z \le z) = 0.95$

$z = 1.65$

$\frac{x}{2} = 1.65 \Rightarrow \boxed{x = 3.3}$

(d) $P(X > 68) = 1 - P(X \le 68)$

$= 1 - P\left(\frac{X-64}{2} \le \frac{68-64}{2}\right)$

$= 1 - P(Z \le 2)$

$= 1 - 0.97725$

$= 0.02275$

$\therefore$ for 5 women $= (0.02275)^5 = 6.09 \times 10^{-9}$

($\because$ Independent)

(12) $\mu = 1.41 \quad \sigma = 0.01$

a) $P(X > 1.42) = 1 - P(X \le 1.42)$

$= 1 - P\left(\frac{X-1.41}{0.01} \le \frac{1.42-1.41}{0.01}\right)$

$= 1 - P(Z \le 1)$

$= 1 - 0.841345$

$= 0.158655$

(12) b) $P(X \geq x) = 0.95$

$$1 - P(X \leq x) = 0.95$$

$$1 - P\left(\frac{x - 1.41}{0.01} \leq \frac{x - 1.41}{0.01}\right) = 0.95$$

$$1 - P\left(z \leq \frac{x - 1.41}{0.01}\right) = 0.95$$

$$P\left(z \leq \frac{x - 1.41}{0.01}\right) = 0.05$$

$$\frac{x - 1.41}{0.01} = -1.64 \rightarrow \boxed{x = 1.3936}$$

(c) $P(1.39 < X < 1.43) = P\left(\frac{1.39 - 1.41}{0.01} < \frac{X - 1.41}{0.01} < \frac{1.43 - 1.41}{0.01}\right)$

$$= P(-2 \leq z \leq 2)$$

$$= P(z \leq 2) - P(z \leq -2)$$

$$= 0.97725 - 0.02275 = 0.9547$$

(13) $\mu = 310 \quad \sigma = 45$

a) $P(X > 350) = 1 - P(X \leq 350)$

$$= 1 - P\left(\frac{X - 310}{45} \leq \frac{350 - 310}{45}\right)$$

$$= 1 - P(z \leq 0.88)$$

$$= 1 - 0.813267 = 0.186733$$

b) $P(X > x) = 0.01$

$$\Rightarrow 1 - P(X < x) = 0.01 \Rightarrow 1 - P\left(\frac{X - 310}{45} < \frac{x - 310}{45}\right) = 0.01$$

$$1 - P\left(z < \frac{x - 310}{45}\right) = 0.01$$

$$P\left(z < \frac{x - 310}{45}\right) = 0.99$$

$$\frac{x - 310}{45} = 2.33 \Rightarrow \boxed{x = 414.85}$$

(13) (c) $P(X > x) = 0.95 \Rightarrow 1 - P(X < x) = 0.95$ (10)

$$\Rightarrow 1 - P\left(\frac{X-310}{45} < \frac{x-310}{45}\right) = 0.95$$

$$P\left(Z \le \frac{x-310}{45}\right) = 0.05$$

$$\frac{x-310}{45} = -1.64 \Rightarrow x = 310 - 73.8$$

$$\Rightarrow \boxed{x = 236.2}$$

(14) $\mu = 7000 \quad \sigma = 600$

a) $P(X > 5000) = 1 - P(X \le 5000)$

$$= 1 - P\left(\frac{X-7000}{600} \le \frac{5000-7000}{600}\right)$$

$$= 1 - P(Z \le 3.33)$$

$$= 1 - 0.999517 = 0.000403$$

b) $P(X > x) = 0.95$

$$1 - P(X \le x) = 0.95 \Rightarrow P\left(\frac{X-7000}{600} \le \frac{x-7000}{600}\right) = 0.05$$

$$P\left(Z \le \frac{x-7000}{600}\right) = 0.05$$

$$\frac{x-7000}{600} = -1.64 \Rightarrow \boxed{x = 6016}$$

c) $P(X > 7000) = 1 - P(X \le 7000)$

$$= 1 - P\left(\frac{X-7000}{600} \le \frac{7000-7000}{600}\right)$$

$$= 1 - P(Z \le 0)$$

$$= 1 - 0.5 = 0.5 \text{ (for one laser)}$$

P of 3 lasers $= (0.5)^3 = 0.125$

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(15) $\mu = 0.002 \quad \sigma = 0.0004$

a) $P(X > 0.0026) = 1 - P(X \leq 0.0026)$

$= 1 - P\left(\frac{X - 0.002}{0.0004} \leq \frac{0.0026 - 0.002}{0.0004}\right)$

$= 1 - P(Z \leq 1.5)$

$= 1 - 0.933193 = 0.066807$

b) $P(0.0014 < X < 0.0026)$

$= P\left(\frac{0.0014 - 0.002}{0.0004} < \frac{X - 0.002}{0.0004} < \frac{0.0026 - 0.002}{0.0004}\right)$

$= P(-1.5 \leq Z \leq 1.5)$

$= P(Z \leq 1.5) - P(Z \leq -1.5)$

$= 0.933193 - 0.066807$

$= 0.866386.$

Normal approximation to Binomial distribution

1- Suppose that X is a binomial random variable with $n=200$ and $p=0.4$

(a) Approximate the probability that X is less than or equal to 70.

(b) Approximate the probability that X is greater than 70 and less than 90.

(c) Approximate the probability that $X=80$.

2- The manufacturing of semiconductor chips produces 2% defective chips. Assume the chips are independent and that a lot contains 1000 chips.

(a) Approximate the probability that more than 25 chips are defective.

(b) Approximate the probability that between 20 and 30 chips are defective.

3. There were 49.7 million people with some type of long-lasting condition or disability living in the United States in 2000. This represented 19.3 percent of the majority of civilians aged five and over. A sample of 1000 persons is selected at random.

(a) Approximate the probability that more than 200 persons in the sample have a disability.

(b) Approximate the probability that between 180 and 300 people in the sample have a disability.

4. Phoenix water is provided to approximately 1.4 million people Who are served through more than 362,000 accounts. All accounts are metered and billed monthly. The probability that an account has an error in a month is 0.001, and accounts can be assumed to be independent.

(a) What is the mean and standard deviation of the number of account errors each month?

(b) Approximate the probability of fewer than 350 errors in a month.

(c) Approximate a value so that the probability that the number of errors exceeding this value is 0.05.

(d) Approximate the probability of more than 400 errors per month in the next two months. Assume that results between months are independent.

5. An electronic office product contains 5000 electronic components. Assume that the probability that each component operates without failure during the useful life of the product is 0.999, and assume that the components fail independently. Approximate the probability that 10 or more of the original 5000 components fail during the useful life of the product.

6. A corporate Web site contains errors on 50 of 1000 pages. If 100 pages are sampled randomly, without replacement, approximate the probability that at least 1 of the pages in error are in the sample.

Normal approximation to Binomial distribution

If X is a binomial random variable with parameters n & p

$$Z = \frac{X - \mu}{\sigma} = \frac{X - np}{\sqrt{npq}}$$ is approximately standard normal variable. To approximate to Binomial distribution to normal Binomial distribution a continuity correction is applied as follows

Mean $\mu = np$, S.D $= \sqrt{npq}$ in Binomial dist.

$$P(X \le x) = P(X \le x + 0.5) \quad \& \quad P(X \ge x) = P(X \ge x - 0.5)$$

$$= P\left(Z \le \frac{x + 0.5 - np}{\sqrt{npq}}\right)$$

$$= P\left(Z \le \frac{x + 0.5 - \mu}{\sigma}\right)$$

This approximation is for $np > 5$, $nq > 5$, (or) $n(1-p) > 5$ $\quad \because p + q = 1$, $q = 1 - p$

Note: 1) $P(X \le x) = P(X \le x + 0.5)$

2) $P(X < x) = P(X < x - 1 + 0.5) = P(X \le x - 0.5)$

3) $P(x_1 < X < x_2) = P(x_1 + 0.5 \le X \le x_2 - 0.5)$

4) $P(X > x) = 1 - P(X \le x)$

$$= 1 - P(X \le x + 0.5)$$

Problems

(1) $n = 200 \quad p = 0.4$

Continuity correction factor

$\mu = np = 80$

$\sigma = \sqrt{npq}$

$= \sqrt{48} = 6.92$

a) $P(X \leq 70) = P(X \leq 70 + 0.5) \rightarrow$

$= P(X \leq 70.5)$

$= P\left(Z \leq \frac{70.5 - 80}{6.92}\right)$

$= P(Z \leq -1.37)$

$= 0.085343$

b) $P(70 < X < 90) = P(70.5 \leq X \leq 89.5)$

$= P(X \leq 89.5) - P(X \leq 70.5)$

$= P\left(Z \leq \frac{89.5 - 80}{6.92}\right) - P\left(Z \leq \frac{70.5 - 80}{6.92}\right)$

$= P(Z \leq 1.372) - P(Z \leq -1.372)$

$= 0.914657 - 0.085343$

$= 0.8293$

c) $P(X = 80) = P(79.5 \leq X \leq 80.5)$

$= P(X \leq 80.5) - P(X \leq 79.5)$

$= P\left(Z \leq \frac{80.5 - 80}{6.92}\right) - P\left(Z \leq \frac{79.5 - 80}{6.92}\right)$

$= P(Z \leq 0.072) - P(Z \leq -0.072)$

$= 0.5279 - 0.4721$

$= 0.0558.$

(b) $P(70 < X < 90) = P(71 \le X \le 89)$

$= P(70.5 \le X \le 89.5)$

$= P\left(\frac{70.5 - \mu}{\sigma} \le X \le \frac{89.5 - \mu}{\sigma}\right)$

(c) $P(X = 80) = P(79.5 \le X \le 80.5)$

$= P\left(\frac{79.5 - \mu}{\sigma} \le Z \le \frac{80.5 - \mu}{\sigma}\right)$

$= P(-0.072 \le Z \le 0.072)$

$= 2\,\phi(0.072) - 1$

② $p = 0.02$ $n = 1000$, $q = 1 - p = 1 - 0.02 = 0.98$

$\mu = np = 0.02 \times 1000 = 20$, $\sigma = \sqrt{npq}$

$\sigma = \sqrt{19.6} = 4.43$

a) $P(X > 25) = 1 - P(X \leq 25)$

$= 1 - P(X \leq 25.5)$

$= 1 - P\left(Z \leq \dfrac{25.5 - 20}{4.43}\right)$

$= 1 - P(Z \leq 1.24)$

$= 1 - 0.892512$

$= 0.107488$

b) $P(20 < X < 30) = P(20.5 \leq X \leq 29.5)$

$= P(X \leq 29.5) - P(X \leq 20.5)$

$= P\left(Z \leq \dfrac{29.5 - 20}{4.43}\right) - P\left(Z \leq \dfrac{20.5 - 20}{4.43}\right)$

$= P(Z \leq 2.14) - P(Z \leq 0.11)$

$= 0.983823 - 0.543795$

$= 0.44$

③ a $p = 0.193$ $n = 1000$, $q = 1 - p = 0.807$

$\mu = np = 193$ $\sigma = \sqrt{npq} = 12.48$

a) $P(X > 200) = 1 - P(X \leq 200)$

$= 1 - P(X \leq 200.5)$

$= 1 - P\left(Z \leq \dfrac{200.5 - 193}{12.48}\right)$

(8) a) $P(X > 300) = 1 - P(Z \le 0.6)$
$= 1 - 0.725747$
$= 0.274253$

b) $P(180 < X < 300) = P(180.5 \le X \le 299.5)$
$= P(X \le 299.5) - P(X \le 180.5)$
$= P\left(Z \le \frac{299.5 - 193}{12.48}\right) - P\left(Z \le \frac{180.5 - 193}{12.48}\right)$
$= P(Z \le 8.55) - P(Z \le -1)$
$= 1 - 0.158655 = 0.841345$

(9) $p = 0.001 \quad n = 362000 \quad q = 0.999$

a) $\mu = np = 362, \; \sigma = \sqrt{npq} = 19.02$

b) $P(X < 350) = P(X \le 349)$
$= P(X \le 349.5)$
$= P\left(Z \le \frac{349.5 - 362}{19.02}\right)$
$= P(Z \le -0.66)$
$= 0.254627$

c) $P(X > x) = 0.05$
$1 - P(X \le x) = 0.05$
$P(X \le x + 0.5) = 0.95$
$P\left(Z \le \frac{x + 0.5 - 362}{19.02}\right) = 0.95$

(47)

$$\frac{x+0.5-362}{19.02}=1.65$$

$$x+0.5-362=31.383$$

$$x=362+31.383-0.5$$

$$\boxed{x=392.883}$$

(4) (a) $P(X>400)=1-P(X\leq 400)$

$$=1-P(X\leq 400.5)$$

$$=1-P\left(z\leq\frac{400.5-362}{19.02}\right)$$

$$=1-P(z\leq 2.02)$$

$$=1-0.978308=0.021692$$

(5) $p=0.999 \quad n=5000 \quad q=1-p$

$\mu=np=4950 \quad \sigma=\sqrt{npq}=2.22$

Here $p=0.001$

$\therefore np=5 \quad \sigma=\sqrt{npq}=2.24$

$$P(X\geq 10)=1-P(X<10)=1-P(X\leq 9)$$

$$=1-P(X\leq 9.5)$$

$$=1-P\left(z\leq\frac{9.5-5}{2.24}\right)$$

$$=1-P(z\leq 2.01)$$

$$=1-0.977784$$

$$=0.022$$

(c) $p = \frac{50}{1000} = 0.05 \quad n = 100$

$\mu = np = 5 \quad \sigma = \sqrt{npq} = 2.18$

$P(X \geq 1) = 1 - P(X < 1) = 1 - P(X \leq 0)$

$= 1 - P(X \leq 0.5)$

$= 1 - P\left(Z \leq \frac{0.5 - 5.0}{2.18}\right)$

$= 1 - P(Z \leq -2.06)$

$= 1 - 0.019699$

$= 0.98$

Normal approximation to poisson distribution

1. Suppose that X is a Poisson random variable with $\lambda = 6$.

(a) Compute the exact probability that X is less than 4.

(b) Approximate the probability that X is less than 4 and compare to the result in part (a).

(c) Approximate the probability that $8 < X < 12$.

2.Suppose that X has a Poisson distribution with a mean of 64. Approximate the following probabilities:

(a) $P(X > 72)$ (b) $P(X < 64)$ (c) $P(60 < X \leq 68)$

3.Suppose that the number of asbestos particles in a sample of 1 squared centimeter of dust is a Poisson random variable with a mean of 1000. What is the probability that 10 squared centimeters of dust contains more than 10,000 particles?

4. A high-volume printer produces minor print-quality errors on a test pattern of 1000 pages of text according to a Poisson distribution with a mean of 0.4 per page.

(a) Why are the number of errors on each page independent random variables?

(b) What is the mean number of pages with errors (one or more)?

(c) Approximate the probability that more than 350 pages contain errors (one or more).

5. Hits to a high-volume Web site are assumed to follow a Poisson distribution with a mean of 10,000 per day. Approximate each of the following:

(a) The probability of more than 20,000 hits in a day

(b) The probability of less than 9900 hits in a day

(c) The value such that the probability that the number of hits in a day exceed the value is 0.01.

Normal Approximation to Poisson Distribution

If 'X' is a poisson random variable then Mean $= E(X) = \lambda$ and S.D $= \sigma = \sqrt{\lambda}$ then $z = \frac{x-\mu}{\sigma} = \frac{x-\lambda}{\sqrt{\lambda}}$ is approximately a standard normal variable.

The approximate poisson probability with normal distribution, a continuity correction is applied as follows

$$P(X \le r) = P(X \le x + 0.5) = P\left(Z \le \frac{x+0.5-\lambda}{\sqrt{\lambda}}\right)$$

Note: $P(x) = \frac{e^{-\lambda}\lambda^{x}}{\lfloor x}$ (Poission Distribution)

I a) $\lambda = 6$

$$P(X<4) = P(0) + P(1) + P(2) + P(3)$$

$$= \frac{e^{-6}6^{0}}{\lfloor 0} + \frac{e^{-6}6^{1}}{\lfloor 1} + \frac{e^{-6}6^{2}}{2!} + \frac{e^{-6}6^{3}}{3!}$$

$$\boxed{P(X<4) = 0.151}$$

b) $P(X<4) = P(X \le 3) = P\left(Z \le \frac{3+0.5-6}{\sqrt{6}}\right)$

$$= P(Z \le -1.02) = 0.1538$$

c) $P(8 < X < 12) = P\left(\frac{8+0.5-6}{\sqrt{6}} \le Z \le \frac{11.5-6}{\sqrt{6}}\right)$

$$= P(1.021 \le Z \le 2.2450)$$

$$= P(Z \le 2.245) - P(Z \le 1.021)$$

$$= 0.9874 - 0.8461 = 0.1413$$

$P(8 < X < 12)$

$= P(9 \le X \le 11)$

$= P(8.5 \le X \le 11.5)$

(2) $\lambda = 64$

a) $P(X>72) = 1 - P(X \le 72)$

$= 1 - P\left(Z \le \frac{72+0.5-64}{\sqrt{64}}\right)$

$= 1 - P(Z \le 1.061)$

$= 1 - 0.855428$

$= 0.144571$

b) $P(X<64) = P(X \le 63)$

$= P\left(Z \le \frac{63+0.5-64}{6}\right)$

$= P(Z \le -0.06) = 0.476078$

(c) $P(60 < X \le 68) = P\left(\frac{60+0.5-64}{\sqrt{64}} \le Z \le \frac{68+0.5-64}{\sqrt{64}}\right)$

$= P(Z \le 0.56) - P(Z \le -0.44)$

$= 0.71226 - 0.329969$

$= 0.382291$

(3) Let X denote the no of Particles in 10 cm² dust Then 'X' is poisson Random variable with $\boxed{\lambda = 10 \times 1000 = 10,000}$, $(\lambda = 1000)$

$P(X > 10000) = 1 - P(X \le 10000)$

$= 1 - P\left(Z \le \frac{10000.5 - 10000}{\sqrt{10000}}\right)$

$= 1 - P(Z \le 0.005)$

$= 1 - 0.5$

$= 0.5$

(4) $\lambda = 0.4$

a) Number of events in disjoint intervals are independent, the pages are disjoint intervals and consequently errors counts per page are independent

b) $P(X \geq 1) = 1 - P(X < 1)$ $\qquad P(x) = \frac{e^{-\lambda} \lambda^{x}}{\lfloor x}$

$= 1 - P(X = 0)$

$= 1 - \frac{e^{-0.4} (0.4)^0}{\lfloor 0}$

$= 1 - 0.670 = 0.330$

The mean number of pages with one or more errors is $\mu = np = 0.33 \times 1000 = 330$ pages.

c) $\mu = np = 330$ $\qquad \sqrt{npq} = 14.87$

Let Y be the no of pages with errors

$p = 0.33$
$q = 0.67$

$P(Y > 350) = 1 - P(Y \leq 350)$

$= 1 - P\left(Z \leq \frac{350 + 0.5 - 330}{14.87}\right)$

$= 1 - P(Z \leq 1.38)$

$= 1 - 0.916207$

$= 0.083793$

(5) $\lambda = 10000$ (52)

a) $P(X > 20000) = 1 - P(X \leq 20000)$

$$= 1 - P\left(z \leq \frac{20000.5 - 10000}{100}\right)$$

$$= 1 - P\left(z \leq \frac{10000.5}{100}\right)$$

$$= 1 - P(z \leq 100) = 1 - 1 = 0$$

b) $P(X < 9900) = P\left(z \leq \frac{9900 + 0.5 - 10000}{100}\right)$

$$= P(z \leq -1.01)$$

$$= 0.1586$$

c) $P(X > x) = 0.01$

$1 - P(X \leq x) = 0.01$

$P(X \leq x) = 0.99 \Rightarrow P\left(X \leq \frac{x + 0.5 - 10000}{100}\right) = 0.99$

$P(z \leq Z) = 0.99$

$Z = 2.33$ $\quad\Big|\quad \therefore Z = \frac{x + 0.5 - 10000}{100}$

$$\frac{x + 0.5 - 10000}{100} = 2.33$$

$x + 0.5 - 10000 = 233$

$x = 233 + 10000 - 0.5$

$$\boxed{x = 10233}$$

Exponential distribution

1. Suppose X has an exponential distribution with mean equal to 10. Determine the following
(a) $P(X > 10)$ (b) $P(X > 20)$ (c) $P(X < 30)$
(d) Find the value of x such that $P(X < x) = 0.95$

2. Suppose the counts recorded by a Geiger counter follow a Poisson process with an average of two counts per minute.
(a) What is the probability that there are no counts in a 30-second interval?
(b) What is the probability that the first count occurs in less than 10 seconds?
(c) What is the probability that the first count occurs between 1 and 2 minutes after start-up?

3. Suppose that the log-ons to a computer network follow a Poisson process with an average of 3 counts per minute.
(a) What is the mean time between counts?
(b) What is the standard deviation of the time between counts?
(c) Determine x such that the probability that at least one count occurs before time x minutes is 0.95.

4. The time between calls to a plumbing supply business is exponentially distributed with a mean time between calls of 15 minutes.
(a) What is the probability that there are no calls within a 30-minute interval?
(b) What is the probability that at least one call arrives within a 10-minute interval?
(c) What is the probability that the first call arrives within 5 and 10 minutes after opening?
(d) Determine the length of an interval of time such that the probability of at least one call in the interval is 0.90.

5. The life of automobile voltage regulators has an exponential distribution with a mean life of six years. You purchase an automobile that is six years old, with a working voltage
Regulator, and plan to own it for six years.
(a) What is the probability that the voltage regulator fails during your ownership?
(b) If your regulator fails after you own the automobile three years and it is replaced, what is the mean time until the next failure?

6. Suppose that the time to failure (in hours) of fans in a personal computer can be modeled by an exponential distribution with $\lambda = 0.0003$
(a) What proportion of the fans will last at least 10,000 hours?
(b) What proportion of the fans will last at most 7000 hours?

7. The time between the arrival of electronic messages at your computer is exponentially distributed with a mean of two hours.
(a) What is the probability that you do not receive a message during a two-hour period?
(b) If you have not had a message in the last four hours, what is the probability that you do not receive a message in the next two hours?
(c) What is the expected time between your fifth and sixth messages?

8. The time between arrivals of taxis at a busy intersection is exponentially distributed with a mean of 10 minutes.
(a) What is the probability that you wait longer than one hour for a taxi?
(b) Suppose you have already been waiting for one hour for a taxi, what is the probability that one arrives within the next 10 minutes?
(c) Determine x such that the probability that you wait more than x minutes is 0.10.
(d) Determine x such that the probability that you wait less than x minutes is 0.90.
(e) Determine x such that the probability that you wait less than x minutes is 0.50.

9. The number of stork sightings on a route in south Carolina follows a Poisson process with a mean of 2.3 per year.
(a) What is the mean time between sightings?
(b) What is the probability that there are no sightings within three months (0.25 years)?
(c) What is the probability that the time until the first sighting exceed six months ?
(d) What is the probability of no sighting within three years?

Exponential distribution (65)

The random variable X that equals the distance between successive events of Poisson's process with mean $\lambda > 0$ is an exponential random variable with parameters λ, the pdf of X is given by

$$f(x) = \lambda e^{-\lambda x} \quad 0 < x < \infty$$

$$\text{Mean} = \frac{1}{\lambda} \text{ and variance} = \frac{1}{\lambda^2}.$$

Problems

① a) $\mu = 10 = \frac{1}{\lambda}$, $f(x) = \lambda e^{-\lambda x} = \frac{e^{-\frac{x}{10}}}{10}$

$$P(X > 10) = \int_{10}^{\infty} f(x)\,dx = \int_{10}^{\infty} \frac{e^{-\frac{x}{10}}}{10}\,dx$$

$$= \frac{1}{10}\left[\frac{-e^{-\frac{x}{10}}}{\frac{1}{10}}\right]_{10}^{\infty} = -e^{-\infty} + e^{-1} = 0 + \frac{1}{e}$$

$$= 0.368$$

b) $P(X > 20) = \int_{20}^{\infty} f(x)\,dx = \int_{20}^{\infty} \frac{e^{-\frac{x}{10}}}{10}\,dx$

$$= \frac{1}{10}\left[\frac{-e^{-\frac{x}{10}}}{\frac{1}{10}}\right]_{20}^{\infty} = -e^{-\infty} + e^{-2} = \frac{1}{e^2} = 0.135$$

c) $P(X < 30) = \int_{0}^{30} \frac{e^{-\frac{x}{10}}}{10}\,dx = \left[-e^{-\frac{x}{10}}\right]_{0}^{30}$

$$= -e^{-3} + e^{0} = 1 - \frac{1}{e^3} = 0.95.$$

1 a) $P(X < x) = 0.95$

$$\int_0^x \frac{e^{-\frac{x}{10}}}{10}\,dx = 0.95 \Rightarrow \frac{1}{10}\left[-\frac{e^{-\frac{x}{10}}}{\frac{1}{10}}\right]_0^x = 0.95$$

$$-e^{-\frac{x}{10}} + e^{0} = 0.95 \Rightarrow e^{-\frac{x}{10}} = 1 - 0.95$$

$$e^{-\frac{x}{10}} = 0.05 \Rightarrow \ln\left(e^{-\frac{x}{10}}\right) = \ln(0.05)$$

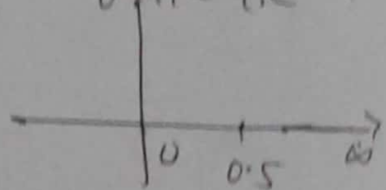
$$-\frac{x}{10} = \ln(0.05) \Rightarrow -\frac{x}{10} = -2.996$$

$$\boxed{x = 29.96}$$

2 a) Mean $\lambda = 2$

X: Time in minute until first count

$P(X > 30\,sec) = P(X > 0.5\ minutes)$ $\quad f(x) = \lambda e^{-\lambda x}$



$$= \int_{0.5}^{\infty} 2e^{-2x}\,dx$$

$$= 2\left[-\frac{e^{-2x}}{2}\right]_{0.5}^{\infty} = -e^{-\infty} + e^{-1}$$

$$= 0 + e = 0.368$$

b) $P(X < 10\,sec) = P(X < 0.167\ minutes)$

$$= \int_0^{0.167} 2e^{-2x}\,dx$$

$$= \cancel{2}\left[-\frac{e^{-2x}}{\cancel{2}}\right]_0^{0.167} = -e^{-0.334} + e^{0} = 0.284$$

c) $P(1 < X < 2) = \int_1^2 2e^{-2x}\,dx = \left[-e^{-2x}\right]_1^2$

$$= -e^{-4} + e^{-2} = 0.117$$

3 $\lambda = 3$

a) $\mu = \frac{1}{\lambda} = \frac{1}{3} = 20$ seconds

(b) $\sigma^2 = \frac{1}{\lambda^2} = \frac{1}{9} \Rightarrow \sigma = \frac{1}{3} = 20$ seconds

(c) $P(X < x) = 0.95$

$$\int_0^x 3e^{-3x}\,dx = 0.95 \Rightarrow \left[\cancel{3}\frac{e^{-3x}}{-\cancel{3}}\right]_0^x = 0.95$$

$$-e^{-3x} + e^0 = 0.95 \Rightarrow e^{-3x} = 1 - 0.95$$

$$e^{-3x} = 0.05 \Rightarrow -3x = \ln(0.05)$$

$$-3x = -2.996 \Rightarrow x = 0.998 \cong 1 \text{ minute}$$

4 $\mu = 15 = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{15}$ $\boxed{f(x) = \frac{e^{-\frac{x}{15}}}{15}}$

a) $$P(X > 30) = \int_{30}^{\infty} \frac{e^{-\frac{x}{15}}}{15}\,dx = \frac{1}{15}\left[\frac{-e^{-\frac{x}{15}}}{\cancel{1}/15}\right]_{30}^{\infty}$$

$$= \cancel{-e^{-\infty}}^{\to 0} + e^{-2} = \frac{1}{e^2} = 0.135$$

b) $$P(X < 10) = \int_0^{10} \frac{e^{-\frac{x}{15}}}{15}\,dx = \left[-e^{-\frac{x}{15}}\right]_0^{10}$$

$$= -e^{-0.66} + e^0 = 0.49$$

c) $$P(5 < X < 10) = \int_5^{10} \frac{e^{-\frac{x}{15}}}{15}\,dx = \left[\frac{1}{15}\frac{e^{-\frac{x}{15}}}{\left(-\frac{1}{15}\right)}\right]_5^{10}$$

$$= \left[-e^{-\frac{x}{15}}\right]_5^{10} = -e^{-0.66} + e^{-0.33}$$

$$= 0.2057$$

(56)

(4) d) $P(X < x) = 0.9$

$$\int_0^x \frac{e^{-\frac{x}{15}}}{15} dx = 0.9$$

$$\left[-e^{-\frac{x}{15}}\right]_0^x = 0.9 \Rightarrow -e^{-\frac{x}{15}} + e^0 = 0.9$$

$$e^{-\frac{x}{15}} = 1 - 0.9 \Rightarrow e^{-\frac{x}{15}} = 0.1$$

$$-\frac{x}{15} = \ln(0.1) \Rightarrow -\frac{x}{15} = -2.30$$

$$\boxed{x = 34.54}$$

(5) a) $\mu = 6 = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{6}$ X: Life of regulator

$$\boxed{f(x) = \frac{e^{-\frac{x}{6}}}{6}}$$

$$P(X < 6) = \int_0^6 f(x)\, dx$$

$$= \int_0^6 \frac{e^{-\frac{x}{6}}}{6} dx = \frac{1}{6}\left[\frac{-e^{-\frac{x}{6}}}{\frac{1}{6}}\right]_0^6$$

$$= \left[-e^{-\frac{x}{6}}\right]_0^6 = -e^{-1} + e^0 = 1 - \frac{1}{e}$$

$$= 1 - 0.368 = 0.632$$

b) Mean = 6 years

6 $\lambda = 0.0003$ (The time to failure in hours)

$$f(x) = 0.0003\, e^{-0.0003x}$$

a) $P(X > 10000) = \int_{10000}^{\infty} 0.0003\, e^{-0.0003x}\, dx$

$$= \left[\cancel{(0.0003)} \frac{e^{-0.0003x}}{(-\cancel{0.0003})} \right]_{10000}^{\infty}$$

$$= \left[-e^{(-0.0003)x} \right]_{10000}^{\infty}$$

$$= -e^{-\infty} + e^{-3} = 0 + \frac{1}{e^{3}} = 0.049$$

b) $P(X < 7000) = \int_{0}^{7000} 0.0003\, e^{-0.0003x}\, dx$

$$= \left[-e^{-0.0003x} \right]_{0}^{7000} = -e^{-2.1} + e^{0}$$

$$= 1 - e^{-2.1} = 1 - 0.122 = 0.877$$

7 $\mu = 2 = \frac{1}{\lambda}$, X: Time until message is received.

$\therefore \boxed{\lambda = \frac{1}{2}}$ $\qquad f(x) = \frac{e^{-\frac{x}{2}}}{2}$

a) $P(X > 2) = \int_{2}^{\infty} \frac{e^{-\frac{x}{2}}}{2}\, dx = \frac{1}{2} \left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}} \right]_{2}^{\infty}$

$$= \left[-e^{-\frac{x}{2}} \right]_{2}^{\infty} = -e^{-\infty} + e^{-1} = .$$

$$= 0 + \frac{1}{e} = 0.367$$

7 (b) $P(X>4+2/X>4) = P(X>2) = 0.362$

(By lack of memory property)

$0 \quad 4 \quad 2$
$\quad\; 0 \quad 2$

(c) The expected time between any two consecutive message is mean of X

$\therefore E(X) = 2$ hours

8 $\boxed{\mu = 10 = \frac{1}{\lambda}}$ $\qquad f(x) = \dfrac{e^{-\frac{x}{10}}}{10}$ ($\mu = 10$ minutes)

a) $P(X > 1\text{ hr}) = P(X > 60\text{ min})$

$$= \int_{60}^{\infty} \frac{e^{-\frac{x}{10}}}{10}\,dx = \frac{1}{10}\left[\frac{-e^{-\frac{x}{10}}}{\frac{1}{10}}\right]_{60}^{\infty}$$

$$= \left[-e^{-\frac{x}{10}}\right]_{60}^{\infty} = -e^{-\infty} + e^{-6} = 0 + e^{-6}$$

$$= 2.478 \times 10^{-3}$$

b) $P(X<10) = \int_{0}^{10} \dfrac{e^{-\frac{x}{10}}}{10}\,dx = \left[-e^{-\frac{x}{10}}\right]_{0}^{10}$

$$= -e^{-1} + e^{0} = 1 - \frac{1}{e} = 0.633$$

[Employ the lack of memory property
$P(X>70/X<60) = P(X<10)$]

c) $P(X>x) = 0.1$

$$\int_{x}^{\infty} \frac{e^{-\frac{x}{10}}}{10}\,dx = 0.1 \Rightarrow \left[-e^{-\frac{x}{10}}\right]_{x}^{\infty} = 0.1$$

$\cancel{-e^{-\infty}} + e^{-\frac{x}{10}} = 0.1 \Rightarrow e^{-\frac{x}{10}} = 0.1$

$-\frac{x}{10} = \ln(0.1) \Rightarrow -\frac{x}{10} = -2.303$

$\Rightarrow \boxed{x = 23.03}$ minutes

(8) d) $P(X < x) = 0.9 \Rightarrow \int_0^x \frac{e^{-\frac{x}{10}}}{10} dx = 0.9$

$\left[-e^{-\frac{x}{10}}\right]_0^x = 0.9$

$\Rightarrow -e^{-\frac{x}{10}} + e^0 = 0.9 \Rightarrow e^{-\frac{x}{10}} = 1 - 0.9$

$e^{-\frac{x}{10}} = 0.1 \Rightarrow -\frac{x}{10} = \ln(0.1)$

$-\frac{x}{10} = -2.303 \Rightarrow \boxed{x = 23.03}$ minutes

e) $P(X < x) = 0.5$

$\int_0^x \frac{e^{-\frac{x}{10}}}{10} = 0.5 \Rightarrow \left[-e^{-\frac{x}{10}}\right]_0^x = 0.5$

$-e^{-\frac{x}{10}} + e^0 = 0.5 \Rightarrow e^{-\frac{x}{10}} = 0.5$

$\Rightarrow -\frac{x}{10} = \ln(0.5) \Rightarrow -\frac{x}{10} = -0.693$

$\Rightarrow x = 6.93 \simeq 7$ minutes.

9 $\lambda = 2.3$ X: Time before Shitting

a) $\mu = \frac{1}{\lambda} = \frac{1}{2.3} = 0.435$

Let X be the Exponential random Variable arising from the Poisson process with the rate $\lambda = 2.3$ sightings per year. Its cdf is then given by $F(x) = 1 - e^{-2.3x}$ $x \geq 0$

(b) $P(X > 0.25 \text{ years}) = 1 - F(0.25)$

$= 1 - [1 - e^{-2.3(0.25)}]$

$= e^{-2.3 \times 0.25} = 0.56$

(c) Six months is 0.5 years, so we can calculate

$P(X > 0.5 \text{ years}) = 1 - F(0.5)$

$= e^{-2.3 \times 0.5} = 0.32$

(d) $P(X > 3)$ years $= 1 - F(3)$

$= e^{-2.3 \times 3} = 0.001$

9 OR (b) $P(X > 0.25) = \int_{0.25}^{\infty} 2.3 e^{-2.3x} dx = \left[(2.3) \frac{e^{-2.3x}}{-2.3}\right]_{0.25}^{\infty}$

$= \left[-e^{-2.3x}\right]_{0.25}^{\infty} = -e^{-\infty} + e^{-2.3(0.25)} = 0.56$

9 (c) $P(X > 0.5) = \int_{0.5}^{\infty} 2.3 e^{-2.3x} dx = \left[-e^{-2.3x}\right]_{0.5}^{\infty}$

$= -e^{-\infty} + e^{-2.3 \times 0.5} = 0.317$

9) d) $P(X>3) = \int_3^{\infty} 2.3 e^{-2.3x} dx = \left[-e^{-2.3x}\right]_3^{\infty}$

$= -e^{-\infty} + e^{-2.3(3)} = 1.01 \times 10^{-3}$.

10) $\lambda = \frac{14.4}{225} = 0.064$, X = no of insect fragments per gram

a) $\mu = \frac{1}{\lambda} = \frac{1}{0.064} = 15.625$ grms

b) $P(X > 28.35\ gm) = \int_{28.35}^{\infty} 0.064\, e^{-0.064x} dx$

$= 0.064 \left[\frac{-e^{-0.064x}}{0.064}\right]_{28.35}^{\infty} = \left[-e^{-0.064x}\right]_{28.35}^{\infty}$

$= -e^{-\infty} + e^{-(0.064)(28.35)} = 0.163$

c) $P(X > 28.35 \times 7\ grm) = P(X > 198.45)$

$= \int_{198.45}^{\infty} 0.064\, e^{-0.064x}\, dx$

$= \left[-e^{-0.064x}\right]_{198.45}^{\infty}$

$= -e^{-\infty} + e^{-(0.064)(198.45)}$

$= 3 \times 10^{-6}$.

11) $\mu = 5 = \frac{1}{\lambda} \rightarrow \lambda = \frac{1}{5} = 0.2$ miles

a) $P(X > 10 \text{ miles}) = \int_{10}^{\infty} 0.2\, e^{-0.2x}\, dx$

$$= \left[0.2 \frac{e^{-0.2x}}{(-0.2)}\right]_{10}^{\infty} = \left[-e^{-0.2x}\right]_{10}^{\infty}$$

$$= -e^{-\infty} + e^{-2} = 0 + 0.135$$

$$= 0.135$$

b) Y : no of cracks in miles of highway

$\lambda = 10(0.2) = 2$

$P(Y=2) = \frac{e^{-2} 2^2}{\lfloor 2} = 0.2707$ (Possian Random Variable)

c) $\sigma_x = \sqrt{\frac{1}{\lambda^2}} = \frac{1}{0.2} = 5$ miles.