Image Processing and Computer Vision

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Multi-Resolution Analysis (MRA)

FFT Vs Wavelet

- FFT, basis functions: sinusoids
- Wavelet transforms: small waves, called wavelet
- FFT can only offer frequency information
- Wavelet: frequency + temporal information
- Fourier analysis doesn’t work well on discontinuous, “bursty” data
  - music, video, power, earthquakes,…

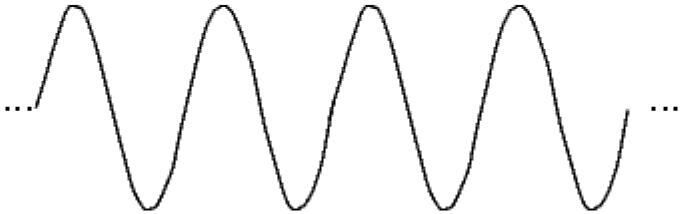
Fourier versus Wavelets

- Fourier
  - Loses time (location) coordinate completely
  - Analyses the ***whole*** signal
  - Short pieces lose “frequency” meaning
- Wavelets
  - Localized time-frequency analysis
  - Short signal pieces also have significance
  - *Scale = Frequency band*

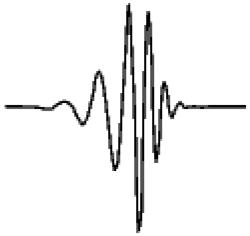
Wavelet Definition

"The wavelet transform is a tool that cuts up data, functions or operators into different frequency components, and then studies each component with a resolution matched to its scale"

Dr. Ingrid Daubechies, Lucent, Princeton U



Sine Wave

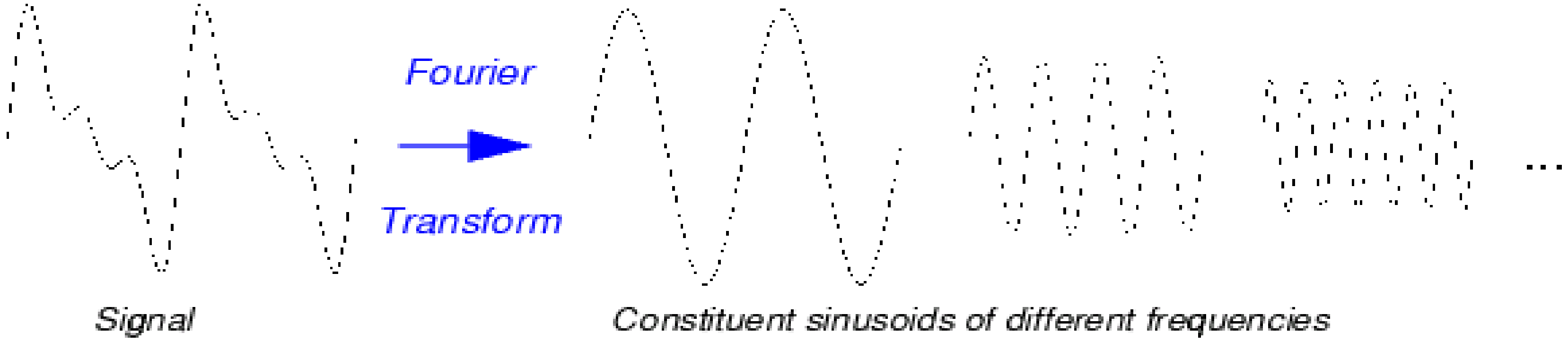


Wavelet (db10)

Fourier transform

Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$



Continuous Wavelet transform

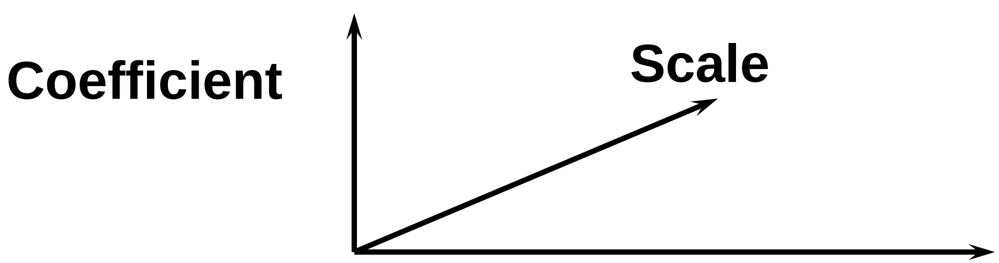
for each **S**cale

  for each **P**osition

    $$\text{Coefficient}(S,P) = \int_{\text{all time}} \text{Signal} \times \text{Wavelet}(S,P)$$

  end

end



Wavelet Transform

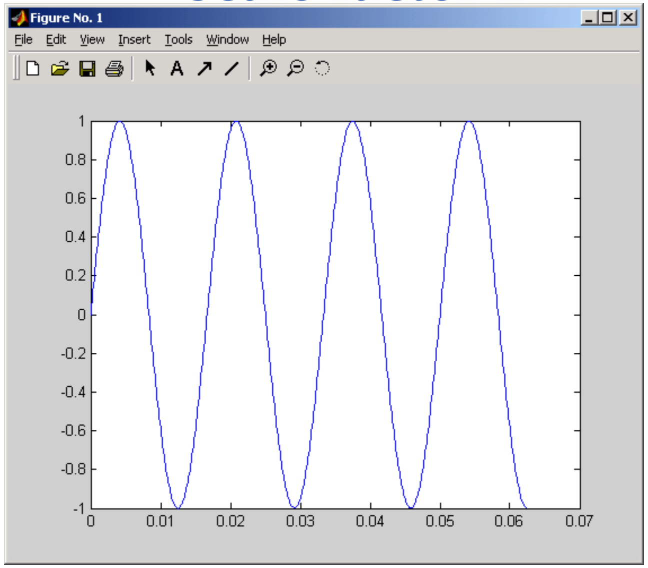
- Scale and shift original waveform
- Compare to a wavelet
- Assign a coefficient of similarity

Scaling-- value of "stretch"

- Scaling a wavelet simply means stretching (or compressing) it.

f(t) = sin(2t)

scale factor 2

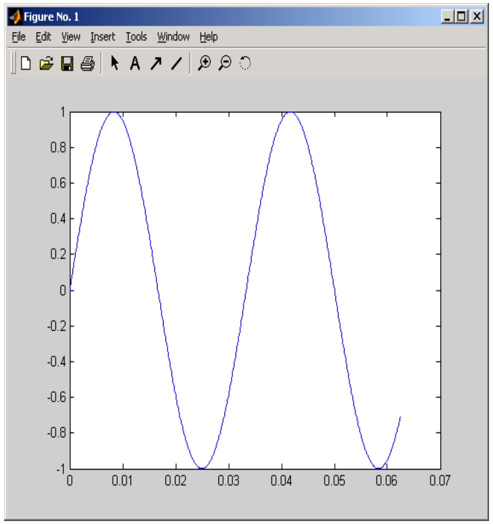


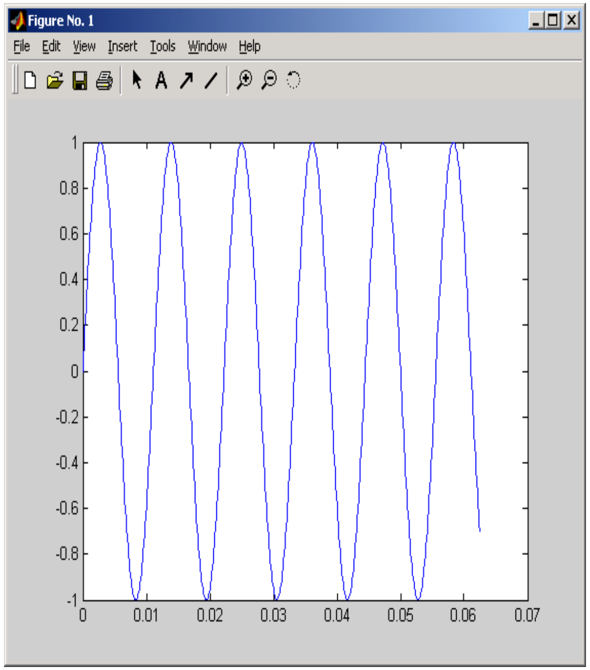
f(t) = sin(3t)

scale factor 3

f(t) = sin(t)

scale factor1

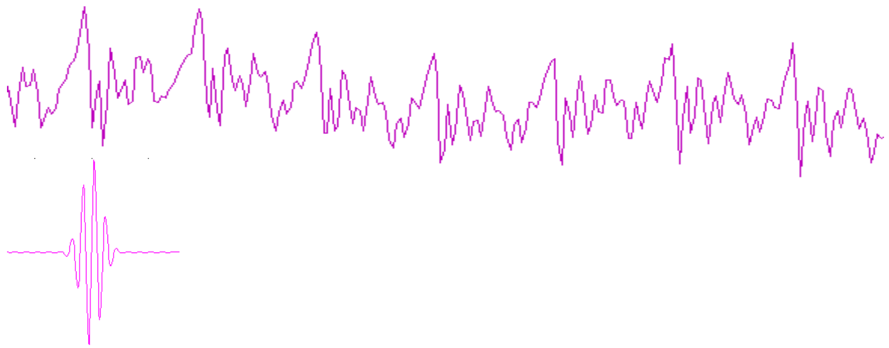




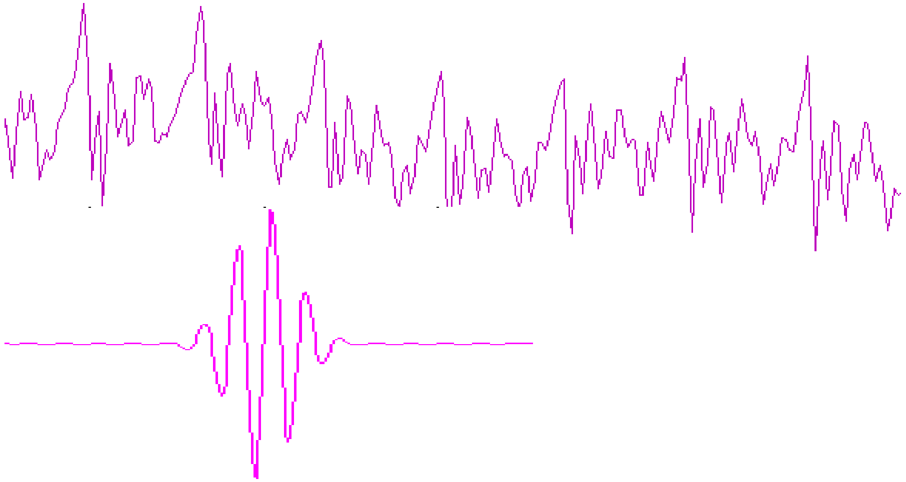
More on scaling

- It lets you either narrow down the frequency band of interest, or determine the frequency content in a narrower time interval
- Scaling = frequency band
- Good for non-stationary data
- Low scale→*a* Compressed wavelet→ Rapidly changing details→High frequency .
- High scale →*a* Stretched wavelet → Slowly changing, coarse features → Low frequency

Scale is (sort of) like frequency

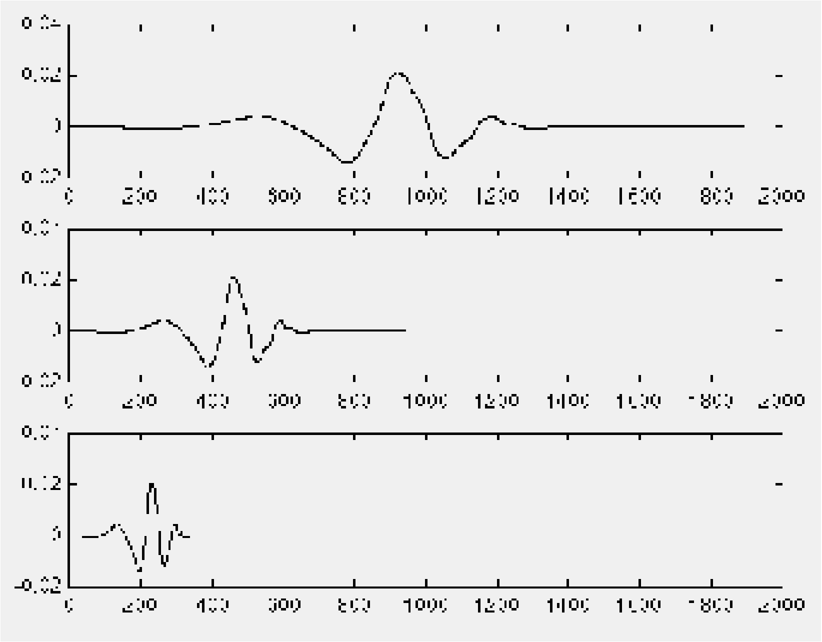


Small scale
-Rapidly changing details,
-Like high frequency



Large scale
-Slowly changing
 details
-Like low frequency

Scale is (sort of) like frequency



$$f(t) = \psi(t) \quad ; \quad a = 1$$

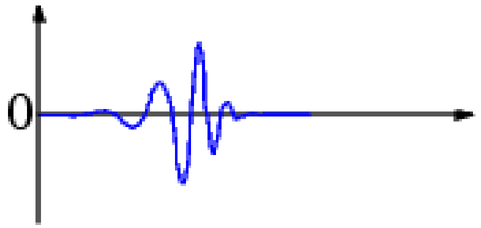
$$f(t) = \psi(2t) \; ; \quad a = \frac{1}{2}$$

$$f(t) = \psi(4t) \; ; \quad a = \frac{1}{4}$$

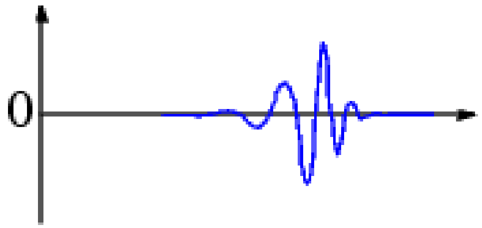
The scale factor works exactly the same with wavelets. The smaller the scale factor, the more "compressed" the wavelet.

Shifting

Shifting a wavelet simply means delaying (or hastening) its onset. Mathematically, delaying a function f(t) by *k* is represented by f(t-k)

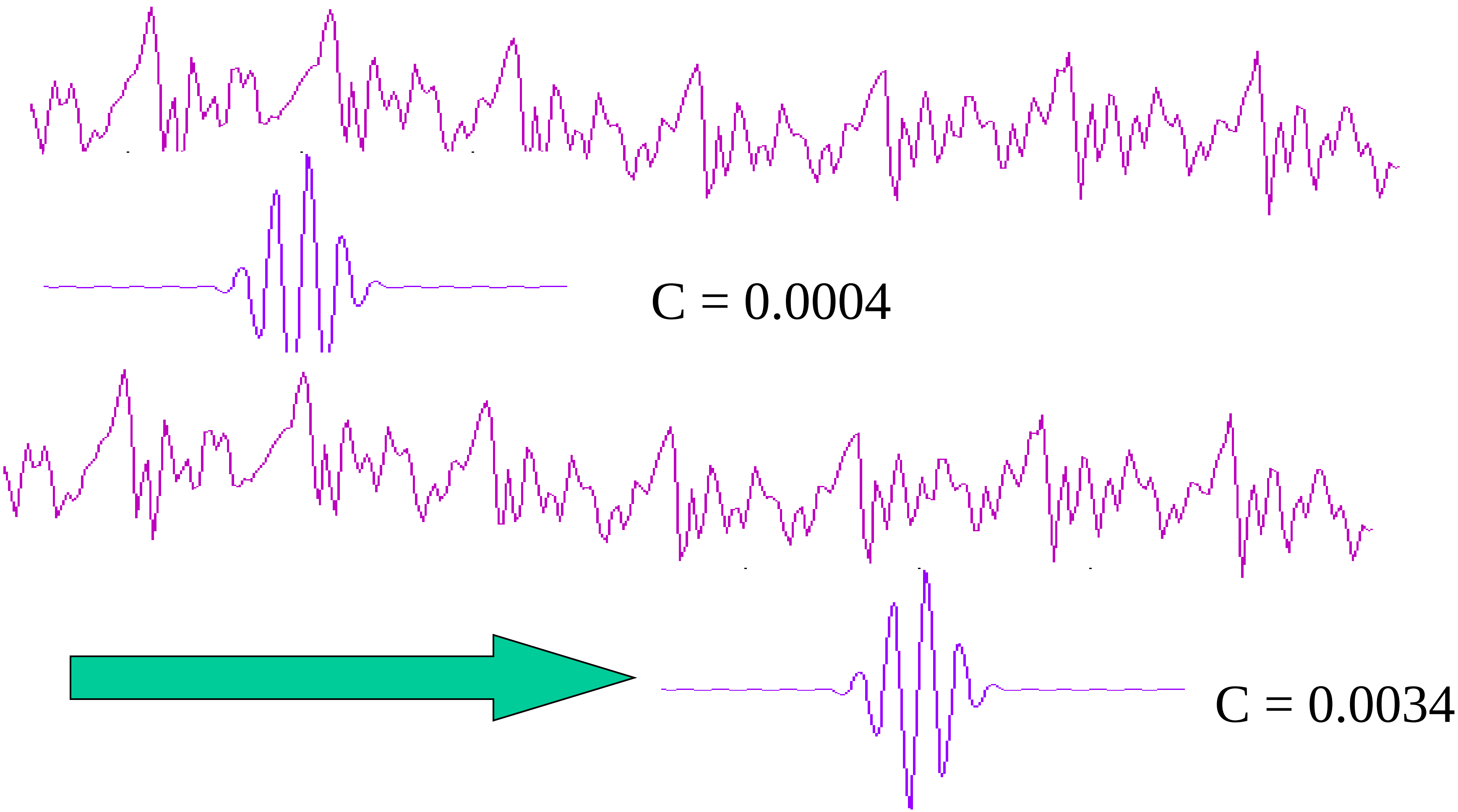


Wavelet function
$\psi(t)$



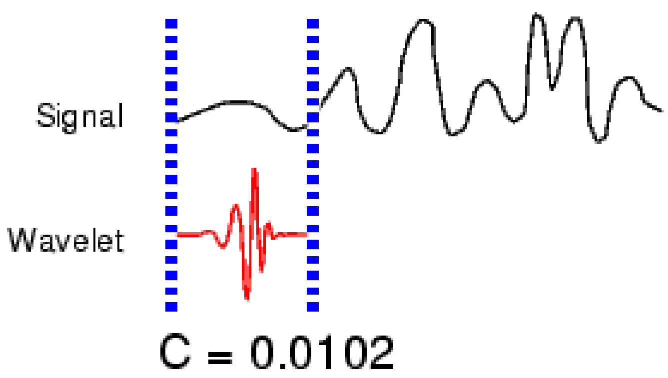
Shifted wavelet function
$\psi(t-k)$

Shifting



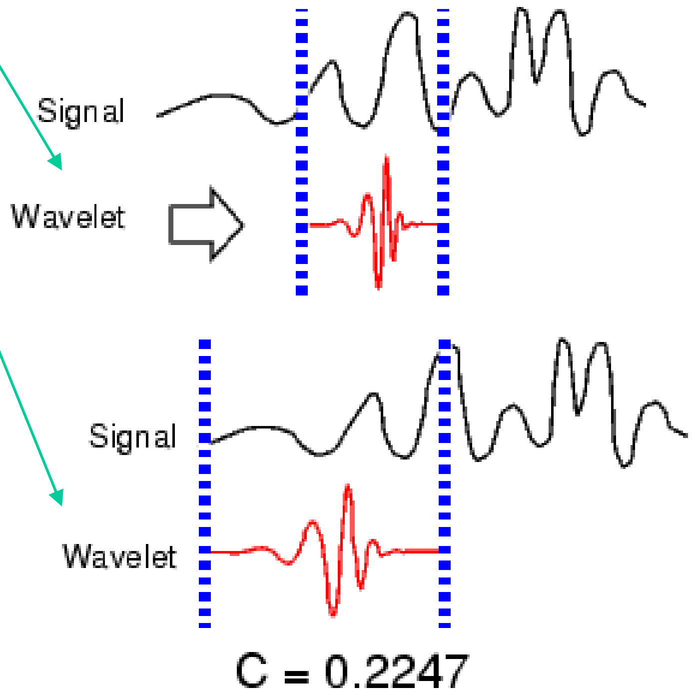
Five Easy Steps to a Continuous Wavelet Transform

1. Take a wavelet and compare it to a section at the start of the original signal.
2. Calculate a correlation coefficient c

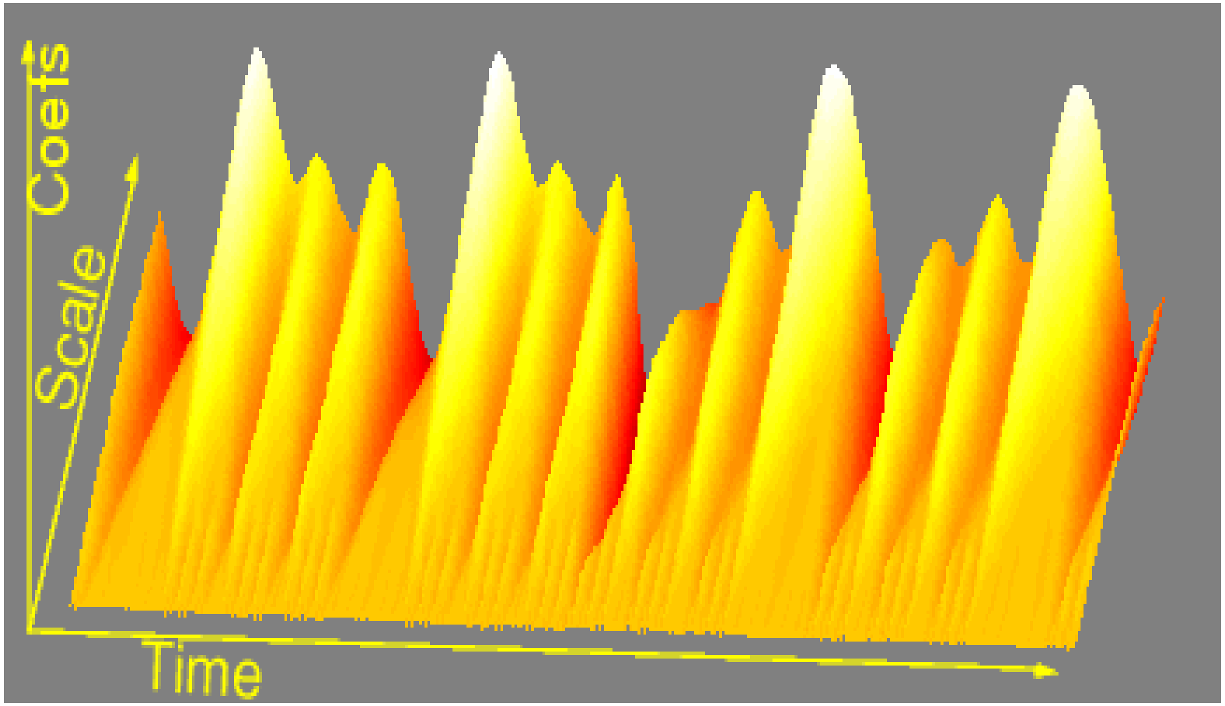


Five Easy Steps to a Continuous Wavelet Transform

3. Shift the wavelet to the right and repeat steps 1 and 2 until you've covered the whole signal.
4. Scale (stretch) the wavelet and repeat steps 1 through 3.
5. Repeat steps 1 through 4 for all scales.



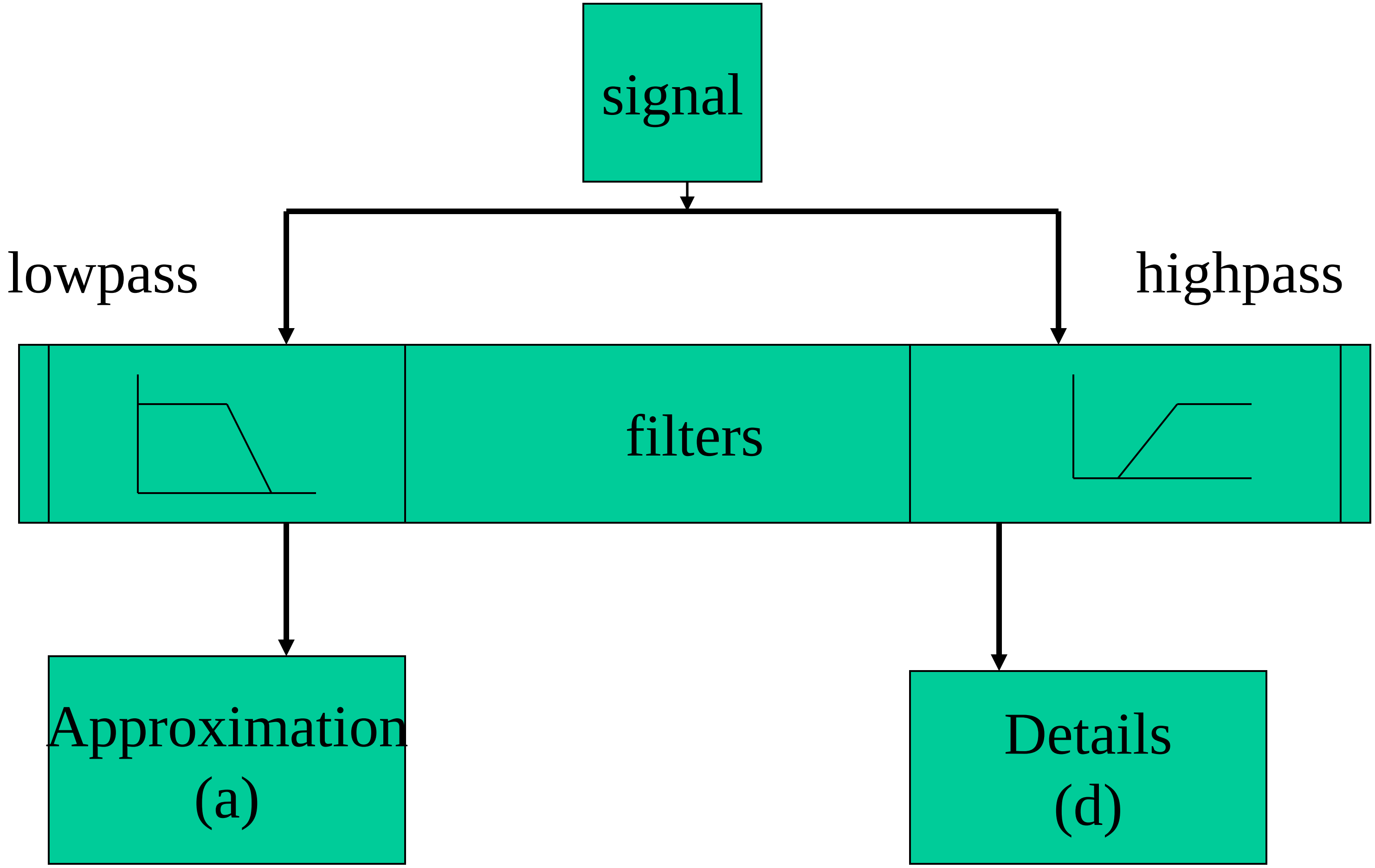
Coefficient Plots



Discrete Wavelet Transform

- "Subset" of scale and position based on power of two
  - rather than every "possible" set of scale and position in continuous wavelet transform
- Behaves like a filter bank: signal in, coefficients out
- Down-sampling necessary (twice as much data as original signal)

Discrete Wavelet transform

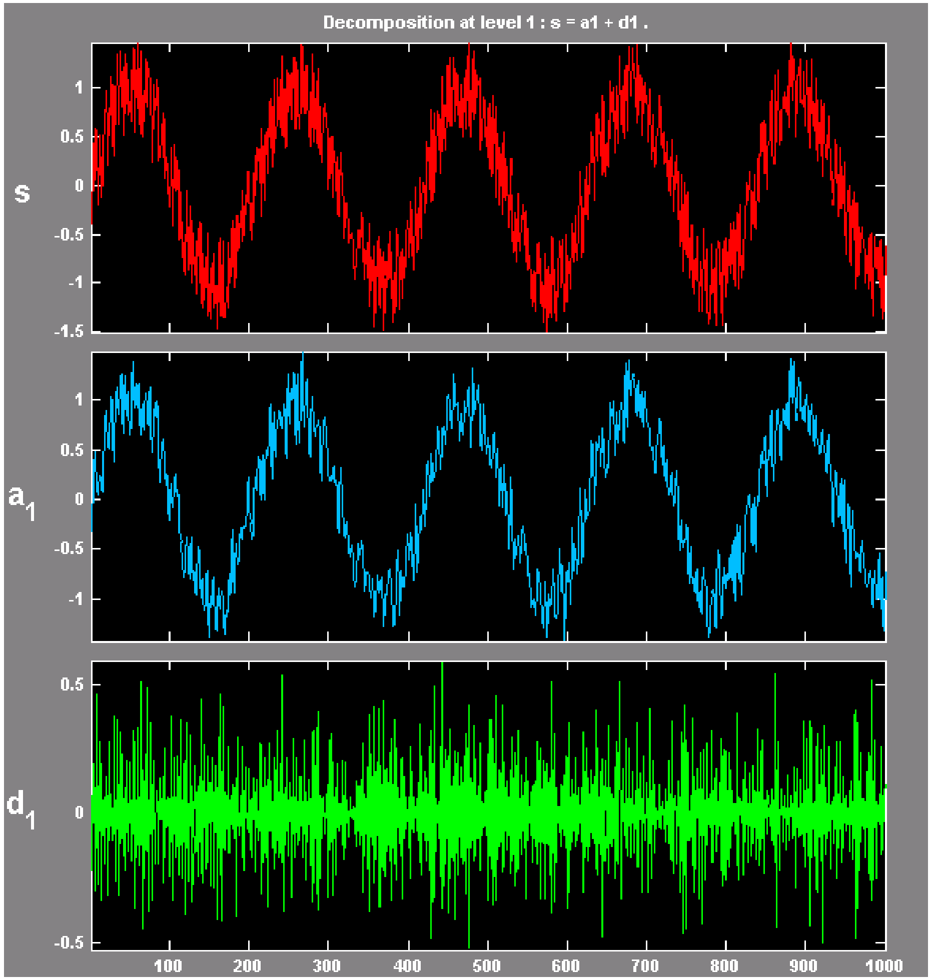


Results of wavelet transform: approximation and details

- Low frequency:
  - approximation (a)

- High frequency
  - Details (d)

- "Decomposition" can be performed iteratively

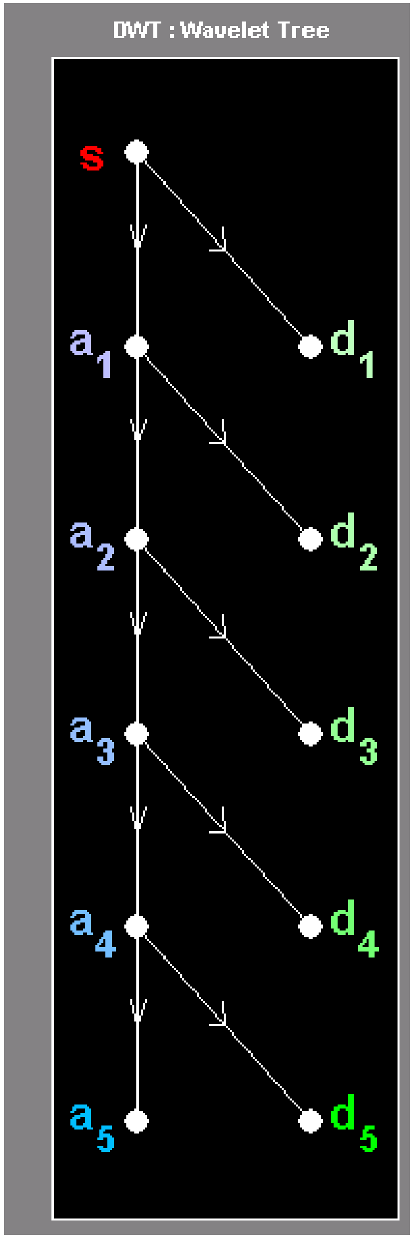


Levels of decomposition

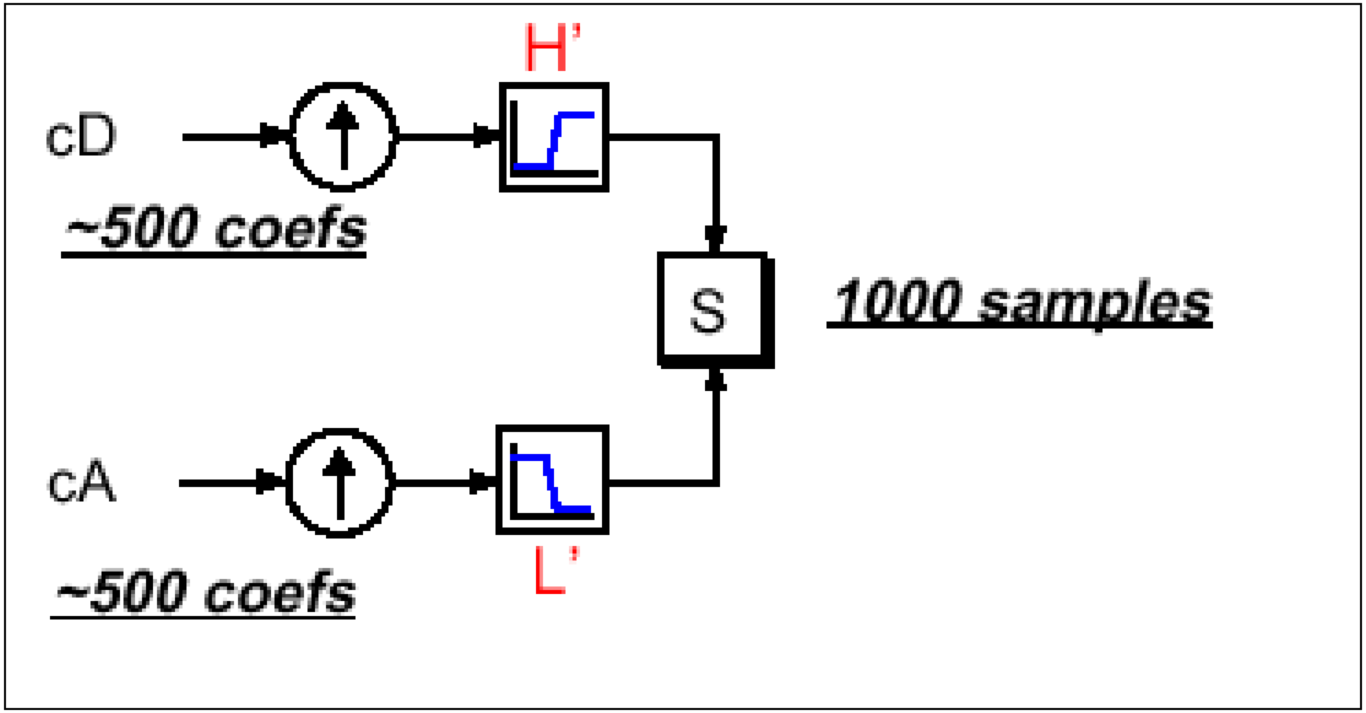
- Successively decompose the approximation

- Level 5 decomposition =
  a5 + d5 + d4 + d3 + d2 + d1

- No limit to the number of decompositions performed



Wavelet synthesis



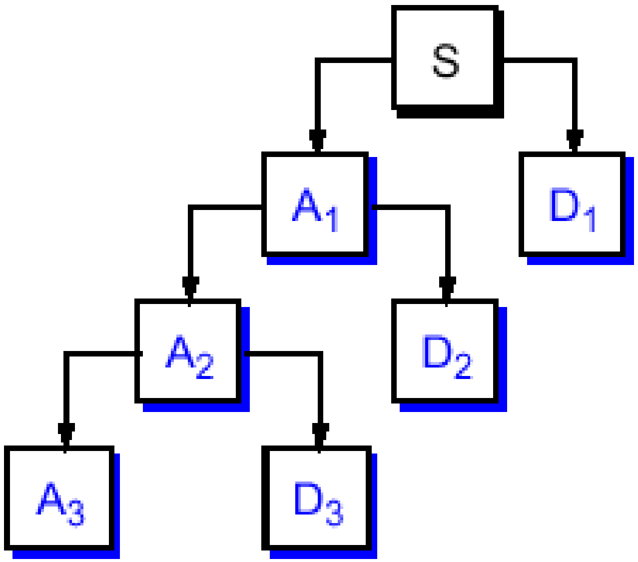
•Re-creates signal from coefficients

•Up-sampling required

Multi-level Wavelet Analysis

Multi-level wavelet decomposition tree

Reassembling original signal



$$S = A_1 + D_1$$
$$= A_2 + D_2 + D_1$$
$$= A_3 + D_3 + D_2 + D_1$$

Non-stationary Property of Natural Image

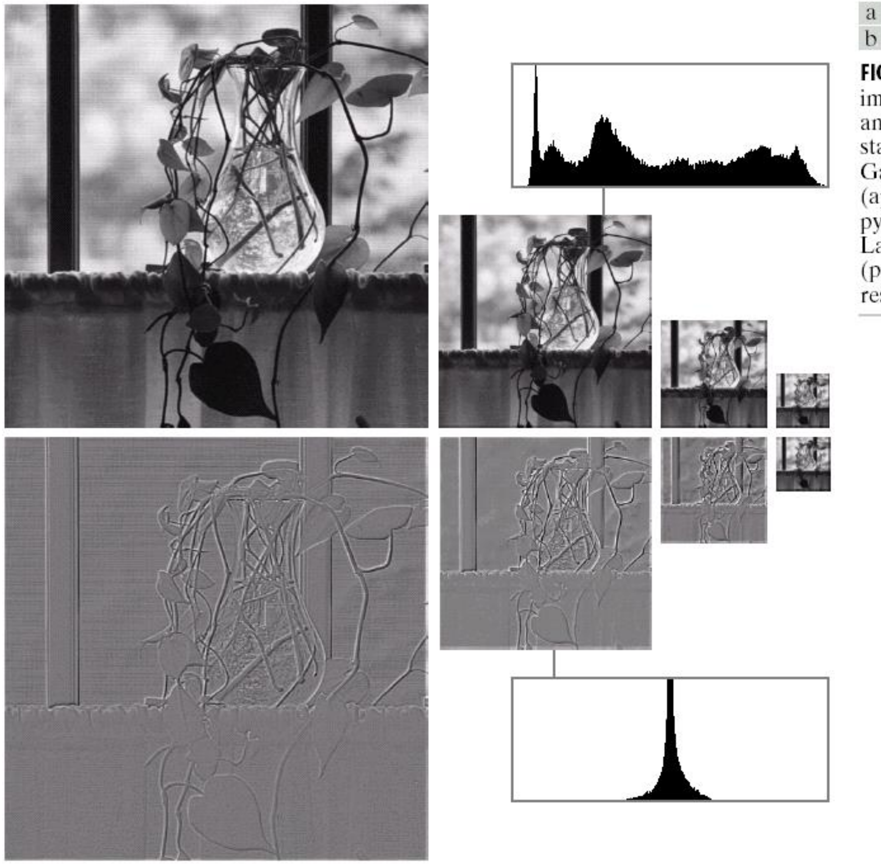
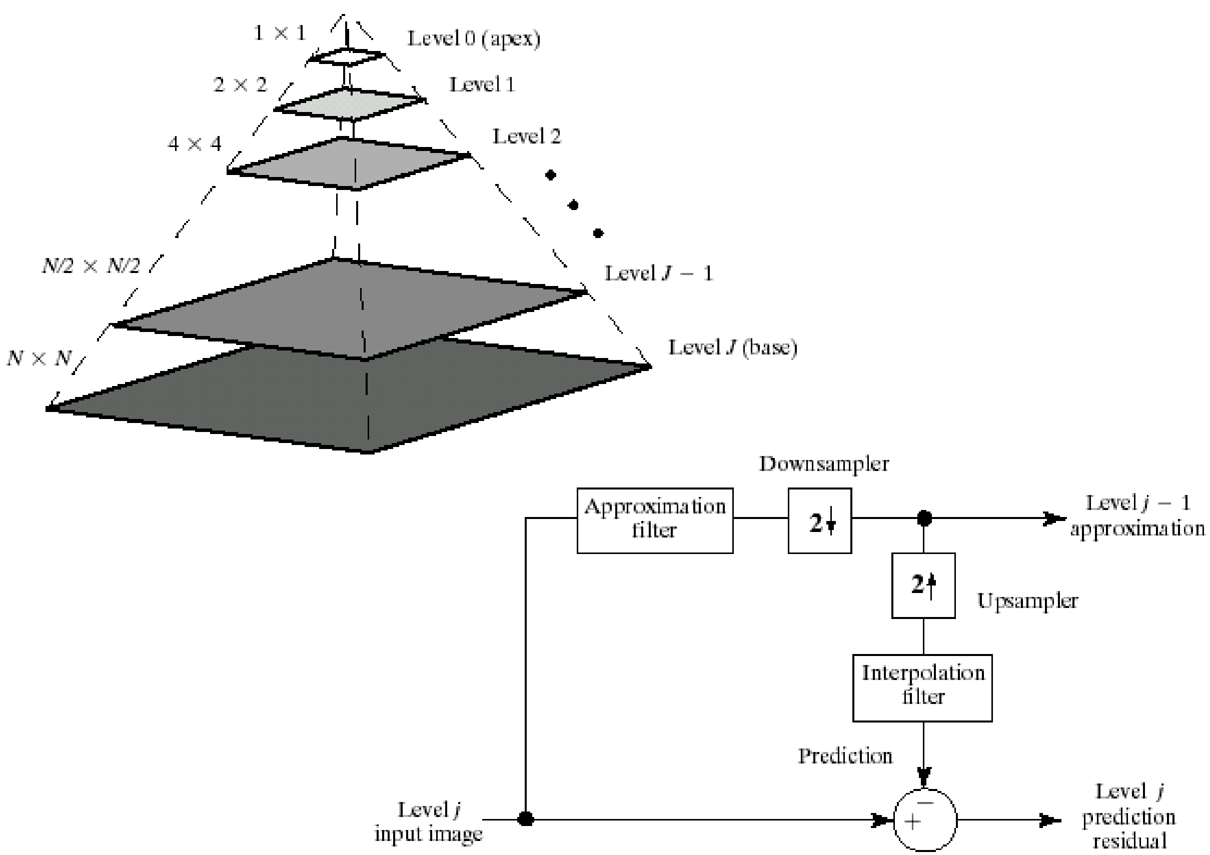


FIGURE 7.3 Two image pyramids and their statistics: (a) a Gaussian (approximation) pyramid and (b) a Laplacian (prediction residual) pyramid.

Pyramidal Image Structure



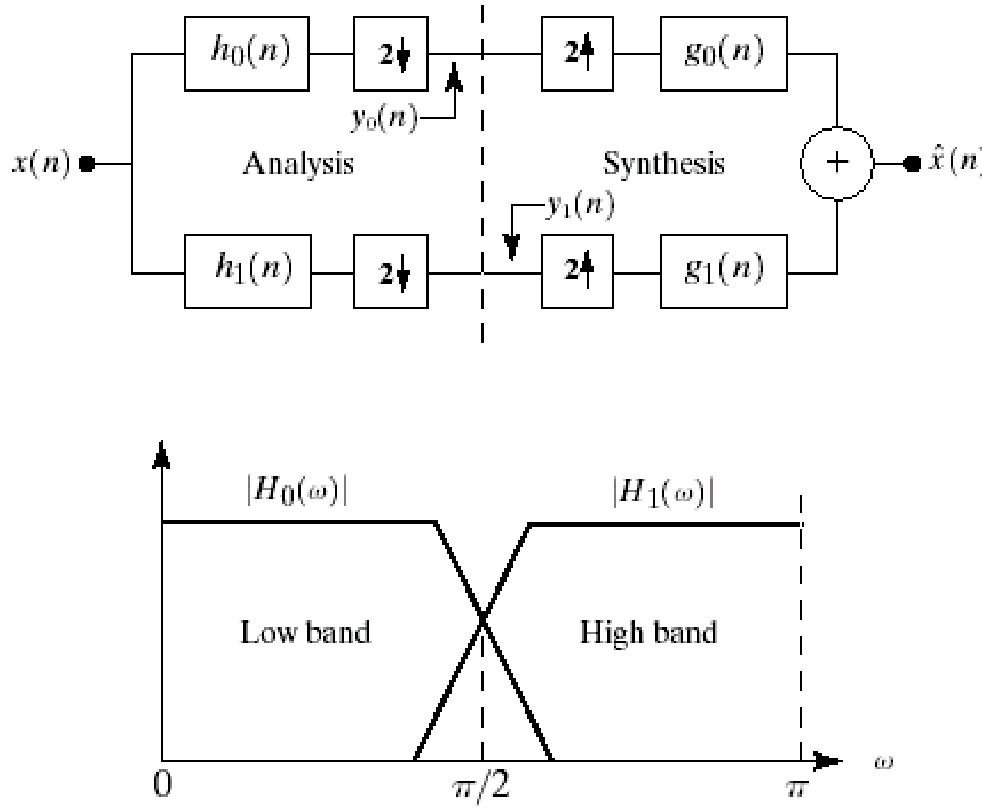
a b

FIGURE 7.2 (a) A pyramidal image structure and (b) system block diagram for creating it.

Image Pyramids

- Original image, the base of the pyramid, in the level J $=\log_2 N$, Normally truncated to P+1 levels.
- Approximation pyramids, predication residual pyramids
- Steps: .1. Compute a reduced-resolution approximation (from j to j-1 level) by downsampling; 2. Upsample the output of step1, get predication image; 3. Difference between the predication of step 2 and the input of step1.

Subband Coding



a
b

FIGURE 7.4 (a) A two-band filter bank for one-dimensional subband coding and decoding, and (b) its spectrum splitting properties.

Subband Coding

- Filters $h_1(n)$ and $h_2(n)$ are half-band digital filters, their transfer characteristics H_0-low pass filter, output is an approximation of $x(n)$ and H_1-high pass filter, output is the high frequency or detail part of $x(n)$
- Criteria: $h_0(n)$, $h_1(n)$, $g_0(n)$, $g_1(n)$ are selected to reconstruct the input perfectly.

Z-transform

- Z- transform a generalization of the discrete Fourier transform
- The Z-transform is also the discrete time version of Laplace transform
- Given a sequence{x(n)}, its z-transform is
- X(z) =

$$\sum_{-\infty}^{\infty} x(n) z^{-n}$$

Subband Coding

$$\hat{X}(z) = \frac{1}{2}[H_0(z)G_0(z) + H_1(z)G_1(z)]X(z) + \frac{1}{2}[H_0(-z)G_0(z) + H_1(-z)G_1(z)]X(-z)$$

$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$$
$$H_0(z)G_0(z) + H_1(z)G_1(z) = 2$$

Filter	QMF	CQF	Orthonormal
$H_0(z)$	$H_0^2(z) - H_0^2(-z) = 2$	$H_0(z)H_0(z^{-1}) + H_0^2(-z)H_0(-z^{-1}) = 2$	$G_0(z^{-1})$
$H_1(z)$	$H_0(-z)$	$z^{-1}H_0(-z^{-1})$	$G_1(z^{-1})$
$G_0(z)$	$H_0(z)$	$H_0(z^{-1})$	$G_0(z)G_0(z^{-1}) + G_0(-z)G_0(-z^{-1}) = 2$
$G_1(z)$	$-H_0(-z)$	$zH_0(-z)$	$-z^{-2K+1}G_0(-z^{-1})$

TABLE 7.1 Perfect reconstruction filter families.

2-D 4-band filter bank

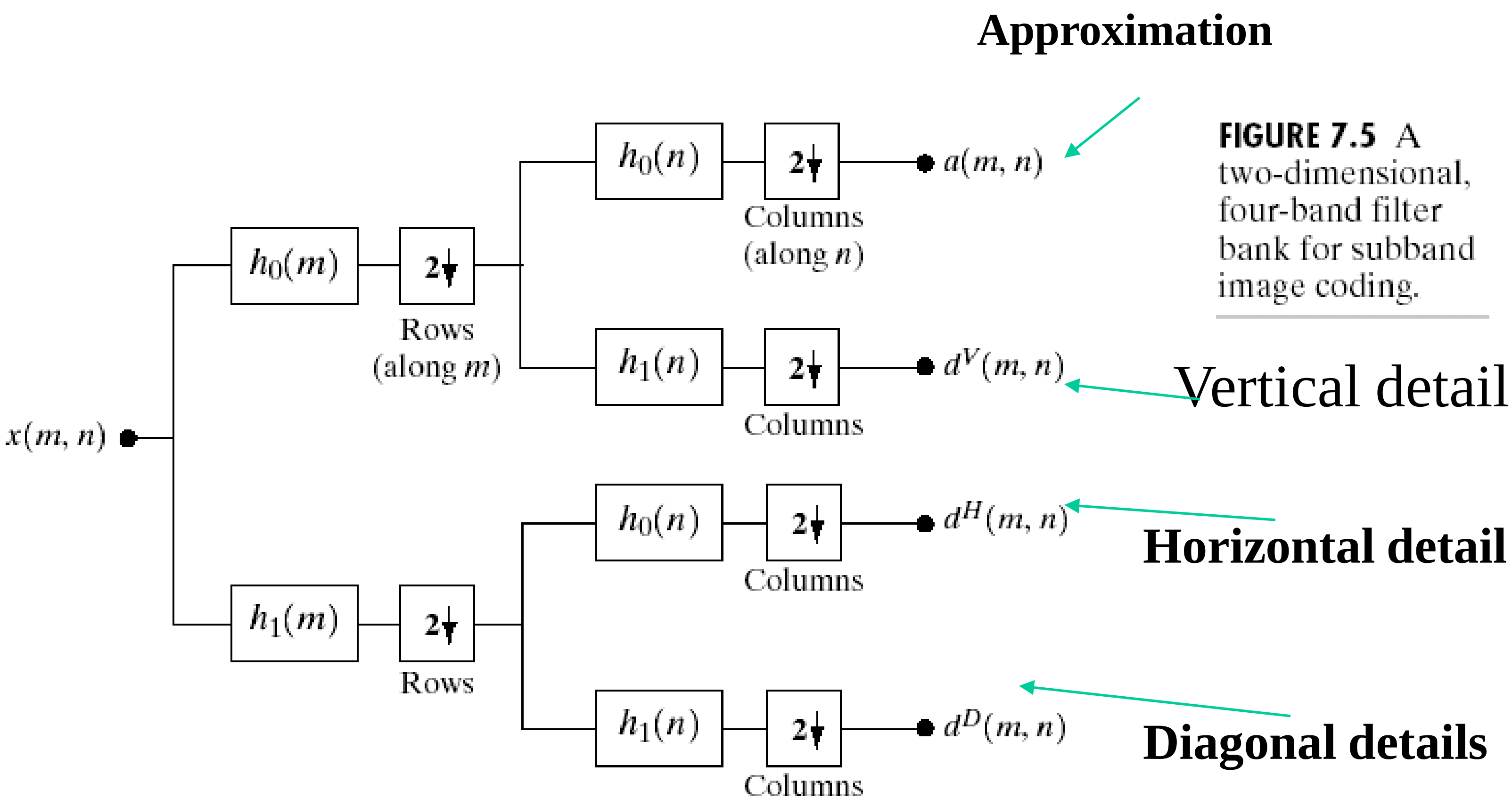


FIGURE 7.5 A two-dimensional, four-band filter bank for subband image coding.

Subband Example

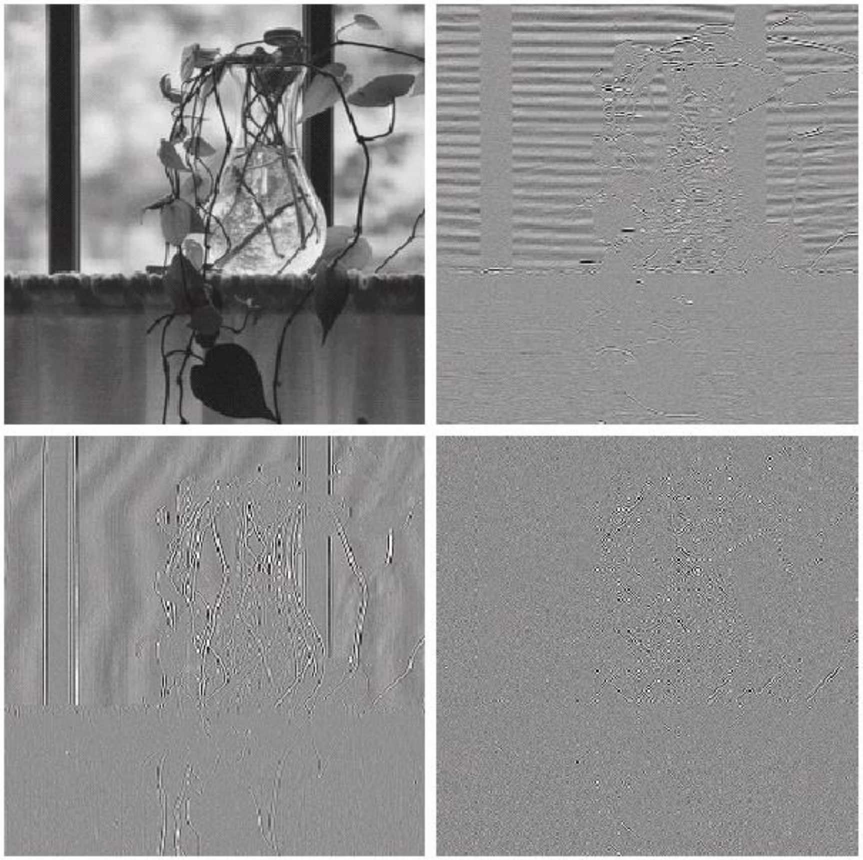


FIGURE 7.7 A four-band split of the vase in Fig. 7.1 using the subband coding system of Fig. 7.5.

Haar Transform

Haar transform, separable and symmetric

T = HFH, where F is an N×N image matrix

H is N×N transformation matrix, H contains the Haar basis functions, $h_k(z)$

$H_0(t) = 1$ for $0 \le t < 1$

$$H_1(t) = \begin{cases} 1 \, if \; 0 \le t < \frac{1}{2} \\ -1 \, if \; \frac{1}{2} \end{cases}$$

$$H_{2^p+n}(t) = \begin{cases} \sqrt{2^p} \; for \frac{n}{2^p} \le t < \frac{(n+0.5)}{2^p} \\ -\sqrt{2^p} \; for \frac{(n+0.5)}{2^p} \le t < \frac{(n+1)}{2^p} \\ 0 \, otherwise \end{cases}$$

$$where P = 1,2,3,\ldots and n = 0,1,\ldots,2^p - 1$$

Haar Transform

$$H_1(t) = \begin{cases} 1 if\ 0 \le t < \frac{1}{2} \\ -1 if\ \frac{1}{2} \end{cases}$$

$$H_{2^p+n}(t) = \begin{cases} \sqrt{2^p}\ for\ \frac{n}{2^p} \le t < \frac{(n+0.5)}{2^p} \\ -\sqrt{2^p}\ for\ \frac{(n+0.5)}{2^p} \le t < \frac{(n+1)}{2^p} \\ 0 otherwise \end{cases}$$

$$where P = 1,2,3,\ldots and n = 0,1,\ldots,2^p - 1$$

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Series Expansion

- In MRA, scaling function to create a series of approximations of a function or image, wavelet to encode the difference in information between different approximations
- A signal or function f(x) can be analyzed as a linear combination of expansion functions

$$f(x) = \sum_k \alpha_k \varphi_k$$

$$\alpha_k - \exp ansioncoefficients$$

$$\varphi_k(x) - basisfunction$$

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Scaling Function

Set{$\varphi_{j,k}(x)$} where,

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)\, for all j, k \in Z$$

K determines the position of $\varphi_{j,k}(x)$ along the x-axis, j -- $\varphi_{j,k}(x)$ width, and $2^{j/2}$—height or amplitude

The shape of $\varphi_{j,k}(x)$ change with j, $\varphi(x)$ is called **scaling function**

Haar scaling function

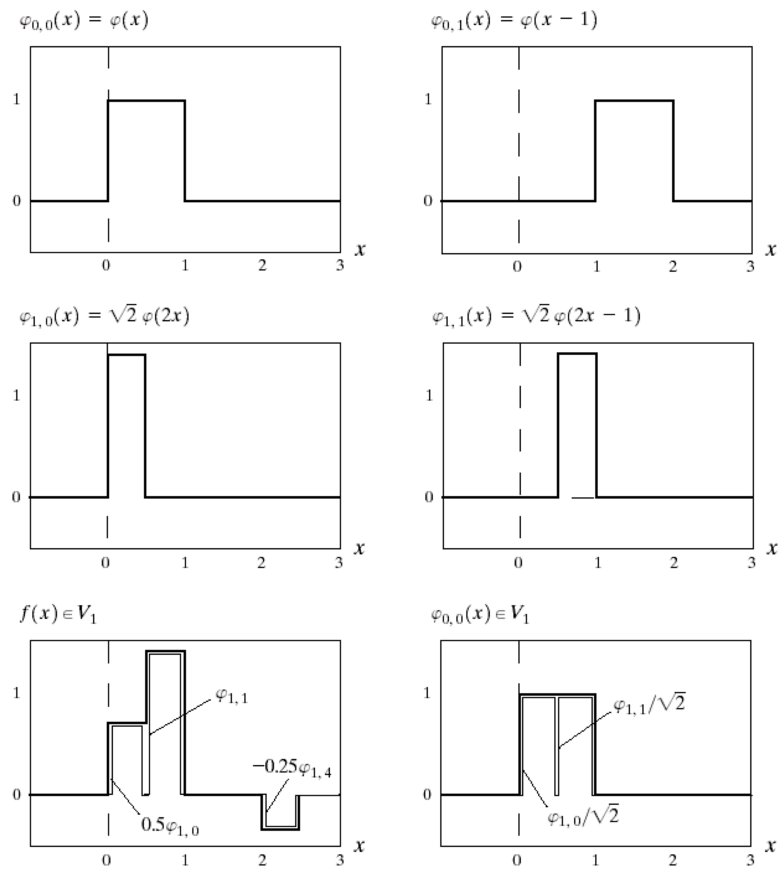

FIGURE 7.9 Haar scaling functions in V_0 in V_1.

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Fundamental Requirements of MRA

- The scaling function is orthogonal to its integer translate
- The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales
- The only function that is common to all V_j is $f(x) = 0$
- Any function can be represented with arbitrary precision

Refinement Equation

$$\varphi(x) = \varphi_{0,0}(x)$$

$$\varphi(x) = \sum_{n} h_{\varphi}(n)\sqrt{2}\varphi(2x-n)$$

$h_{\varphi}(x)$ coefficient –scaling function coefficient

$h_{\varphi}(x)$ – scaling vector

The expansion functions of any subspace can built from the next higher resolution space

Wavelet Functions

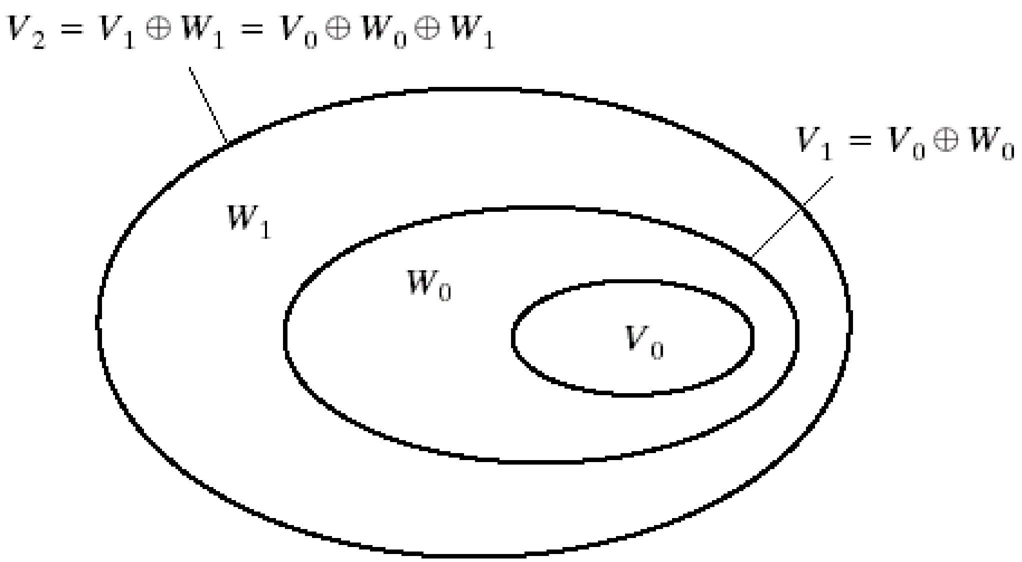
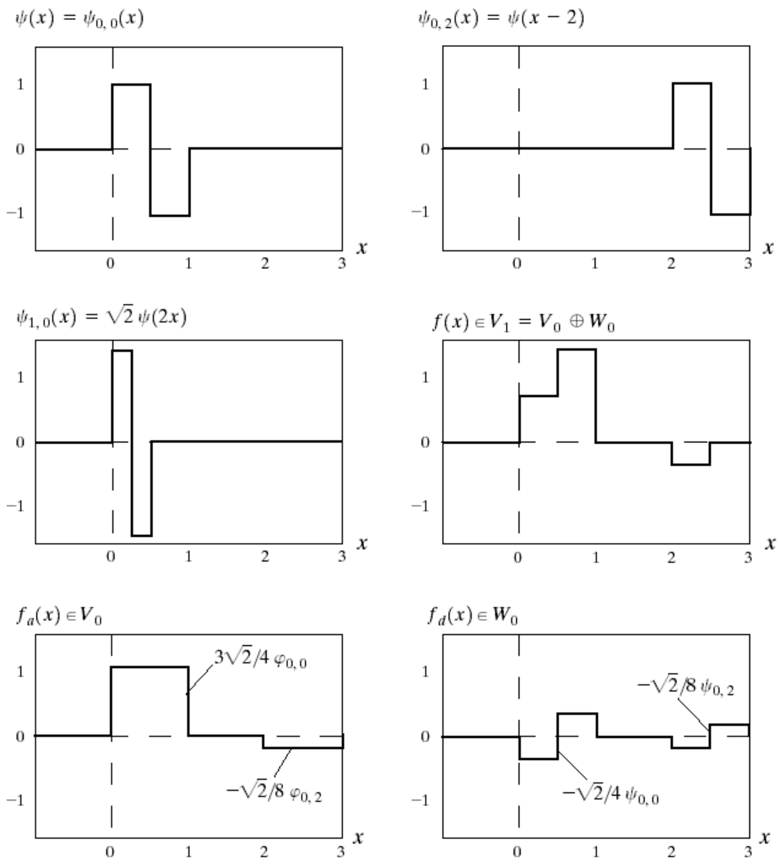


FIGURE 7.11 The relationship between scaling and wavelet function spaces.

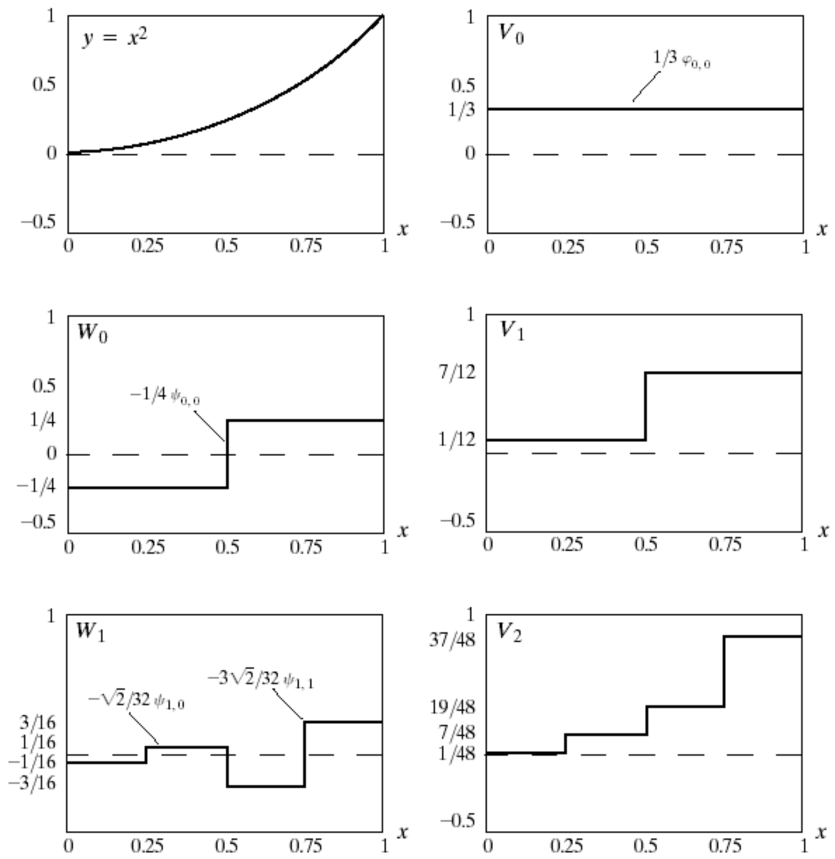
Wavelet Functions



a b
c d
e f

FIGURE 7.12 Haar wavelet functions in W_0 and W_1.

Wavelet Function



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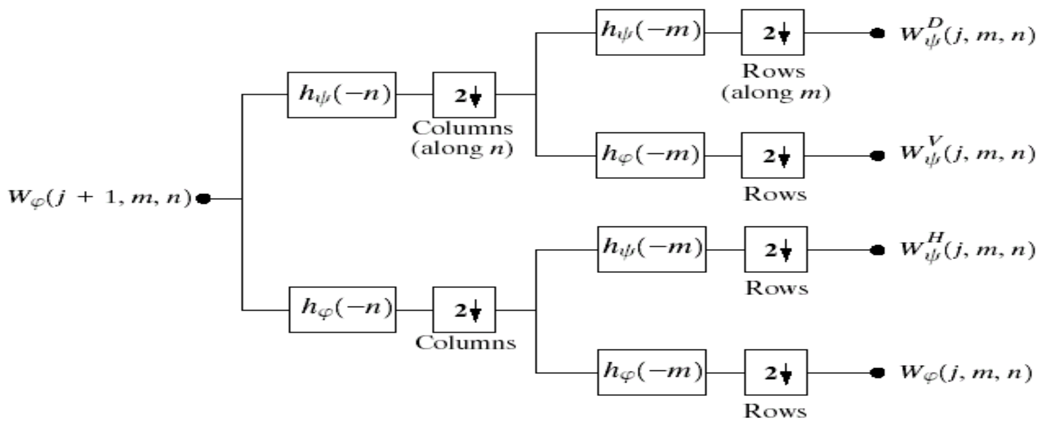
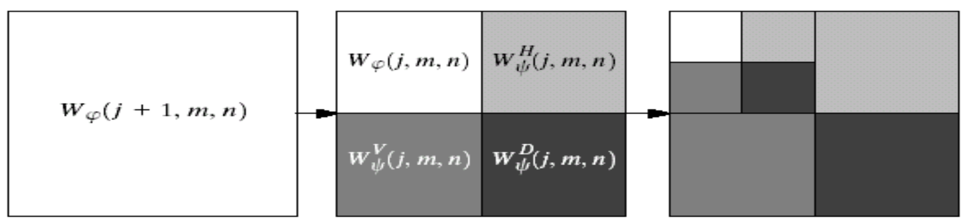
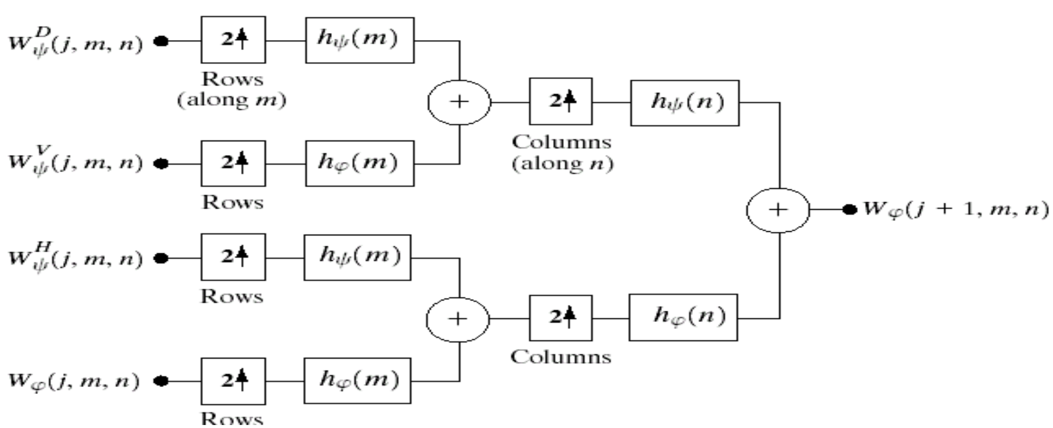
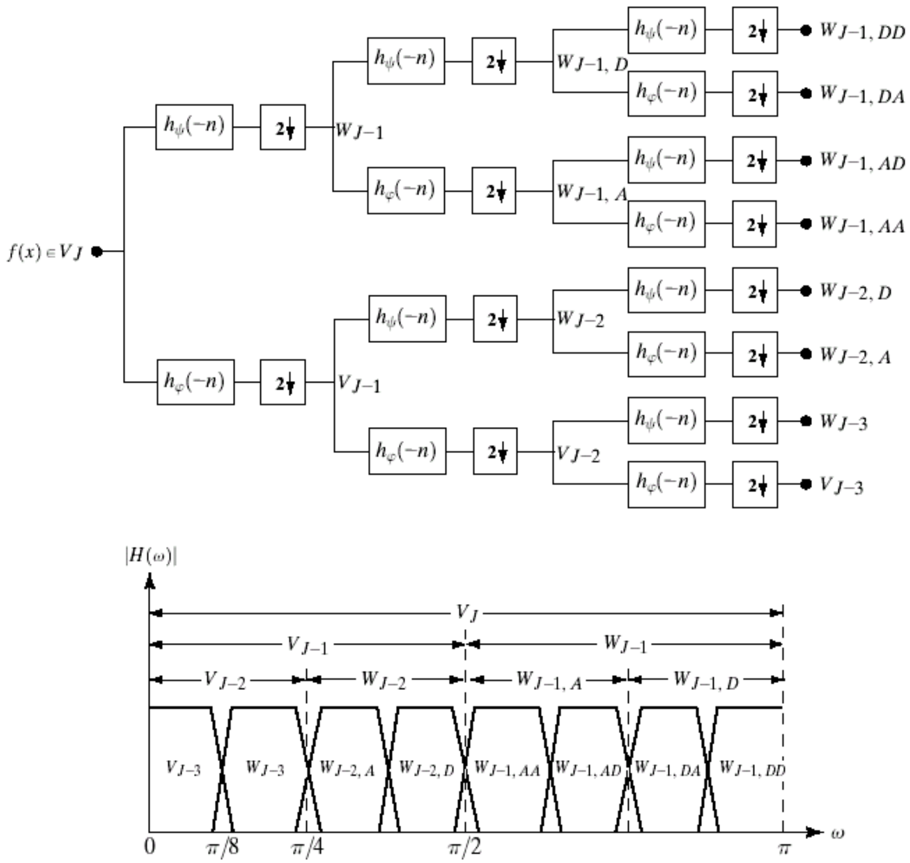
2-D Wavelet Transform







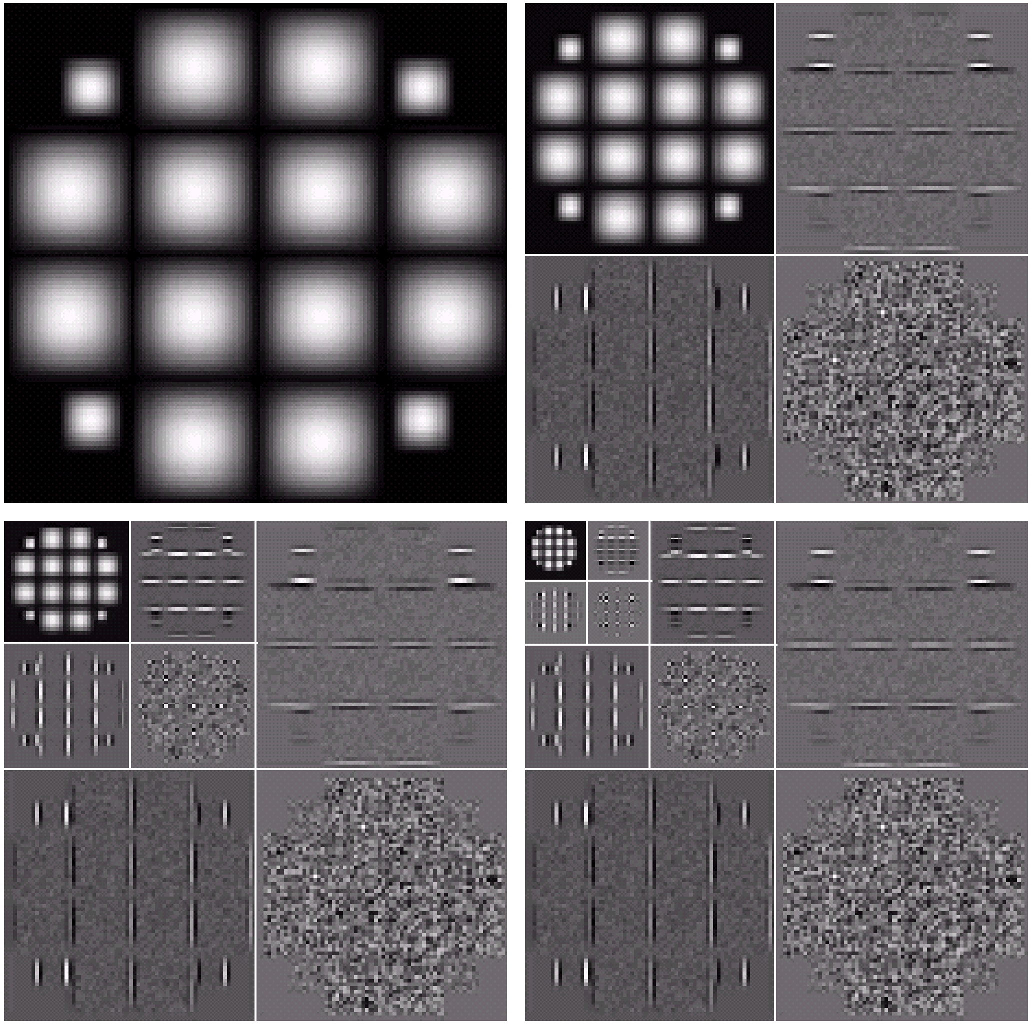
Wavelet Packets



a
b

FIGURE 7.30 The (a) filter bank and (b) spectrum splitting characteristics of a three-scale full wavelet packet analysis tree.

2-D Wavelets



a b
c d

FIGURE 7.23 A three-scale FWT.

Applications of wavelets

- Pattern recognition
  - Biotech: to distinguish the normal from the pathological membranes
  - Biometrics: facial/corneal/fingerprint recognition
- Feature extraction
  - Metallurgy: characterization of rough surfaces
- Trend detection:
  - Finance: exploring variation of stock prices
- Perfect reconstruction
  - Communications: wireless channel signals
- Video compression – JPEG 2000