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Remark: In General If $\underset{n\times n}{A}x = b$ is Given system.

& x_0 is one of the perticular soln of $Ax=b$

(ie. $Ax_0 = b$)

Then soln space of $Ax=b$ is $\{ v + x_0 \,;\, v \in Null(A) \}$

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For example,

Let, $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ & let v_1 & v_2 be eigenvectors corresponding to Eigenvalues 1 & 2 resp.

Then Find ~~&~~ solⁿ space (i.e. all solⁿs) of $A.X = 5v_1 - 3v_2$

Solⁿ: ~~Then~~ As v_1, v_2 are eigenvectors corresponding to Eigenvalues 1 & 2 resp.

$\Rightarrow A v_1 = 1 \cdot v_1$

$A v_2 = 2 v_2$

$\Rightarrow A(5v_1) = \cancel{5v_1}\; 5 \cdot A(v_1) = 5(1 v_1) = 5v_1$

$A\left(-\frac{3}{2} v_2\right) = -\frac{3}{2} \cdot A v_2 = -3v_2$

$\Rightarrow A \cdot \left[5v_1 - \frac{3}{2} v_2\right] = A \cdot (5v_1) + A\left(-\frac{3}{2} \cdot v_2\right)$

$A \cdot \left[5v_1 - \frac{3}{2} v_2\right] = 5v_1 - 3v_2$

$\Rightarrow \boxed{X_0 = 5v_1 - \frac{3}{2} v_2}$ is ~~the~~ one of the solⁿs of $Ax = 5v_1 - 3v_2$

$\Rightarrow$ Null (A) is solⁿ space of $Ax = 0$

i.e. $\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

~~Apply $R_1 \leftrightarrow R_2$~~

Now, echelon form of A is $\sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

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Let, $\boxed{x_3 = K}$ free variable. $K \in \mathbb{R}$

$x_1 - 2x_3 = 0 \Rightarrow x_1 = 2K$

$x_2 + 3x_3 = 0 \Rightarrow x_2 = -3.K$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2K \\ -3K \\ K \end{bmatrix} = K \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$Null(A) = \left\{ (K) \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} ; K \in \mathbb{R} \right\}$$

so that soln space of $AX = 5v_1 - 3v_2$ is ~~Null(A)~~

$$\left\{ v + x_0 ; v \in Null(A) \right\}$$

$$= \left\{ (K) \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + 5v_1 - \frac{3}{2} v_2 ; K \in \mathbb{R} \right\}$$

$\Uparrow$

This is soln space of $AX = 5v_1 - 3v_2$

In Gene...

This is also important example.

Q.) If A is some $n\times n$ matrix.
v_1 is Eigenvector Associated to Eigenvalue $\lambda_1=2$
v_2 is Eigenvector Associated to Eigenvalue $\lambda_2=5$

Then Find soln space of $A.X = 11v_1+17v_2$

Now, soln space $= \{X\in\mathbb{R}^n ;\ AX = 11v_1+17v_2\}$

Now, 1st we will Find some perticular one soln to this ~~soln space =~~

As $\left.\begin{aligned} Av_1 &= 2v_1 \\ Av_2 &= 5v_2 \end{aligned}\right\} \Rightarrow$

$\Rightarrow \cancel{A\left(\frac{2}{11}v_1\right)=}\ A\left(\frac{11}{2}v_1\right) = \frac{11}{2}(2v_1) = 11v_1$

$A.\left(\frac{17}{5}v_2\right) = \frac{17}{5}.(5v_2) = 17v_2$

$\Rightarrow A.\left(\frac{11}{2}v_1+\frac{17}{5}v_2\right) = A.\left(\frac{11}{2}v_1\right)+A.\left(\frac{17}{5}v_2\right)$

$\Rightarrow A.\left(\frac{11}{2}v_1+\frac{17}{5}v_2\right) = 11v_1+17v_2$

$\Rightarrow X_0 = \frac{11}{2}v_1+\frac{17}{5}v_2$ is one soln of $AX=11v_1+17v_2$

Now, let Y be any soln of $AX=11v_1+17v_2$
ie. $AY = 11v_1+17v_2$

$\Rightarrow A(Y-X_0) = AY-AX_0 = 11v_1+17v_2-(11v_1+17v_2)$

$\Rightarrow A(Y-X_0)=0$

$\Rightarrow Y - X_0 \in$ Null(A)

$\Rightarrow Y \in \text{Null}(A) + X_0$

As Y is arbitrary soln of $AX = 11v_1 + 17v_2$

$\Rightarrow$ soln space of $Ax = 11v_1 + 17v_2$ $= \{ Y = v + X_0 \; ; \; v \in \text{Null}(A) \}$

so In General result is

If A is nxn matrix

$0 \neq \lambda_1$ is Eigenvalue of A

$0 \neq \lambda_2$ is Eigenvalue of A

& v_1, v_2 are Eigenvectors associated to λ_1, λ_2 resp.

$\Rightarrow$ ~~Av_1~~ Then soln space of

$AX =$ ~~$a\lambda_1 + b\lambda_2$~~ $av_1 + bv_2$

will be $\left\{ v + \frac{av_1}{\lambda_1} + \frac{bv_2}{\lambda_2} \; ; \; v \in \text{Null}(A) \right\}$

<u>OR</u>

In General If $\underset{n\times n}{A}X = b$ is Given system

~~Then~~

~~& If b is X_0 is~~ perticular ~~soln of~~

& X_0 is one of the perticular soln of $AX = b$

(i.e. $AX_0 = b$)

Then,

soln space of $AX = b$ is $\{ Y + X_0 \; ; \;$ ~~where~~ $Y \in \text{Null}(A) \}$

Q consider $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

i) write the characteristic Eqn of A

ii) Find Eigenvalues & Eigenvectors of A

iii) State Algebraic & Geometric multiplicities of Each EigenValue.

iv) Find Eigenvalues of $A^2+5A-3I$

v) Is A diagonalizable ? why ?

vi) If A is diagonalizable Find the spectral matrix D and modal matrix P

Soln: i) Now, char eqn of A is $|\lambda I_{3\times3} - A| = 0$

$$\Rightarrow \begin{vmatrix} \lambda-3 & -2 & -4 \\ -2 & \lambda & -2 \\ -4 & -2 & \lambda-3 \end{vmatrix}$$

(compute char eqn carefully becoz if char eqn is wrong then whole soln will be wrong)

ie $\Rightarrow (\lambda-3)[\lambda(\lambda-3)-4] + 2[-2(\lambda-3) - 8] - 4[4+4\lambda] = 0$

$\Rightarrow (\lambda-3)[\lambda^2-3\lambda-4] + 2[-2\lambda-2] - 16[1+\lambda] = 0$

$\Rightarrow (\lambda-3)[(\lambda-4)(\lambda+1)] - 4[\lambda+1] - 16(1+\lambda) = 0$

$\Rightarrow \cancel{(\lambda-3)}\ (\lambda-3)[(\lambda-4)(\lambda+1)] - 20(1+\lambda) = 0$

(observe here I take common $\lambda+1$) $\Rightarrow (\lambda+1).\left[(\lambda-3).(\lambda-4) \quad -20\right] = 0$

$\Rightarrow (\lambda+1).\left[\lambda^2 - 7\lambda - 8\right] = 0$

(char eqn of A) $\Rightarrow$ ~~$(\lambda+1)$~~

(I will suggest you to find Eigenvalues 1st Find λ Characteristic eqn by ~~so~~ Finding $|\lambda I - A| = \det(\lambda I - A)$ because If we Find the char eqn in this way you can take common λ one of the factor of that eqn so you will ~~get~~ ~~the char~~ easily Find then roots of that eqn.)

char eqn is $(\lambda+1)(\lambda^2 - 7\lambda - 8) = 0$

ii) $\Rightarrow \lambda = -1, \quad \lambda = \dfrac{7 \pm \sqrt{49+32}}{2} = \dfrac{7 \pm \sqrt{81}}{2}$

$\Rightarrow \lambda = -1, \quad \lambda = \dfrac{7 \pm 9}{2}$

$\Rightarrow \lambda = -1, \quad \lambda = 8, -1$

$\Rightarrow \lambda = -1, -1, 8$

Now, This are the Eigen values of A

Now, $\lambda_1 = -1$ & $\lambda_2 = 8$

& $\boxed{\text{A.M. of } (\lambda_1 = -1) = 2}$, $\boxed{\text{A.M. of } (\lambda_2 = 8) = 1}$

iii) Now we will Find Eigenvectors corresponding to Eigenvalues $\lambda_1 = -1$ & $\lambda_2 = 8$

ie. we Find Eigenspaces $E_{\lambda_1 = -1}$ & $E_{\lambda_2 = 8}$

Now, $E_{\lambda_1 = -1} =$ ~~$\text{Null}[A - (1)I]$~~ or $\text{Null}[(-1)I - A]$ or $\text{Null}[A - (-1)I_{3\times3}]$

Date ~~8th~~
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Null $[A + I_{3\times3}]$ $\Downarrow$

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$$\Rightarrow E_{\lambda_1=-1} = \text{Null}\begin{bmatrix} 4 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 4 \end{bmatrix} \leftarrow B$$

(Now ~~to find~~ Null(B) means soln space of $BX = 0$)

Now, reduce this matrix B to echelon form of B

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~~Now, To reduce~~

Now, Apply, $R_2 \to R_2 - \frac{1}{2} R_1$

& $R_3 \to R_3 - R_1$

$$\Rightarrow \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 4 & 2 & 4 \\ \cancel{2} & \cancel{0} & \cancel{2} \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Now, x_3 is free Variable.

$\Rightarrow \boxed{x_3 = k}$, $k \in \mathbb{R}$

$\Rightarrow 4x_1 + 2x_2 + 4x_3 = 0$ — (i)

$-x_2 = 0 \Rightarrow \boxed{x_2 = 0}$

eqn (i) $\Rightarrow \boxed{x_1 = -k}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix}, \forall k \in \mathbb{R}$$

$$\Rightarrow E_{\lambda_1 = -1} = \left\{ (k) \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} ; k \in \mathbb{R} \right\}$$

$\Rightarrow$ Basis of $E_{\lambda_1=-1}$ is $\mathcal{B}_1 = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\Rightarrow \boxed{G.M.(\lambda_1 = -1) = 1}$ *

~~Now,~~ Hence, $(k)\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\forall k \in \mathbb{R}$, $k \neq 0$ are the Eigenvectors corresponds to Eigenvalue $\lambda_1 = -1$

Now, To find Eigenspace of $(\lambda_2 = 8) = E_{\lambda_2 = 8}$

$= \text{Null}[A - 8I_{3\times3}]$

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$$E_{\lambda_2=8} = \text{Null}\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \leftarrow B$$

Now Apply, $R_1 \to R_1 + R_3$, ~~R_3~~

$$\Rightarrow \sim \begin{bmatrix} -1 & 4 & -1 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix}$$

Apply $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + 4R_1$

$$\Rightarrow \sim \begin{bmatrix} -1 & 4 & -1 \\ 0 & 0 & 0 \\ 0 & 18 & -9 \end{bmatrix}$$

Apply, $R_3 \leftrightarrow R_2$

$$\Rightarrow \sim \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} -1 & 4 & -1 \\ 0 & 18 & -9 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Now, $\boxed{x_3 = k}$ k being Parameter $\in \mathbb{R}$

$$-x_1 + 4x_2 - x_3 = 0 \quad \text{—(i)}$$

$$18x_2 - 9x_3 = 0 \Rightarrow x_2 = \frac{1}{2}\cdot k$$

eqn (i) $\Rightarrow$ $x_1 = 2k - k \Rightarrow \boxed{x_1 = k}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k/2 \\ k \end{bmatrix}, \forall k \in \mathbb{R}$$

so Eigenvectors corresponds to Eigenvalue $\lambda_2 = 8$ are $(k)\cdot\begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$, $\forall 0 \neq k \in \mathbb{R}$

* & $E_{\lambda_2=8} = \left\{ (k)\cdot\begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} ; k \in \mathbb{R} \right\}$ is Eigenspace of $(\lambda_2 = 8)$

Basis of $E_{\lambda_2=8}$ is $B_2 = \left\{ \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} \right\}$ 11th page ~~8th page~~

* ~~Also~~ $\Rightarrow$ $\boxed{G.M. \text{ of } (\lambda_2 = 8) = \dim(E_{\lambda_2=8}) = 1}$

(You ~~can also~~ dont need Basis to Find G.M. ie ~~G.M. = dim~~

As. G.M of $\lambda_i = \dim(\text{Null}[A - \lambda_i I])$

$= \text{Nullity}[A - \lambda_i I]$

$= \text{No. of columns} - \text{Rank}[A - \lambda_i I]$

So you have to Find only Rank$[A - \lambda_i I]$ To Find G.M. of λ_i)

iv) Now, A.M. of $(\lambda_1 = -1) = 2 \neq 1 =$ G.M. of $(\lambda_1 = -1)$

$\Rightarrow$ A is Not Diagonalizable

(Because for Diagonalizable.

This is Nesscessary & sufficient condition for Diagonalizable.

i) char eq^n of A does Not have complex root

ii) For All Eigenvalues A.M. = G.M.

So You can not Find spectral & modal matrix

or $\nexists$ spectral & modal matrix.

iv) Now, $\lambda_1 = -1$ & $\lambda_2 = 8$ are Eigenvalues of A

$\Rightarrow$ Now, let $p(A) = A^2 + 5A - 3I$

$\Rightarrow p(t) = t^2 + 5t - 3$

Then As λ_i is Eigenvalue of A

$\Rightarrow p(\lambda_i)$ is Eigenvalue of $P(A)$

$\Rightarrow$ $p(-1)$ & $p(8)$ are Eigenvalues of $P(A) = A^2 + 5A - 3I$

ie $\Rightarrow$ -7 & 101 are two of the Eigenvalues of $P(A) = A^2 + 5A - 3I$