SOLVING LINEAR RECURRENCE RELATION

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LINEAR HOMOGENEOUS RECURRENCE RELATION

Definition: A linear homogeneous recurrence relation of degree k with constant coefficient is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Where $c_1, c_2, \ldots c_k$ are real numbers and $c_k \neq 0$.

EXAMPLE

Determine which of the following are linear homogeneous recurrence relation with constant coefficients. Also find the degree of those that are.

a. $a_n = a_{n-1} + 2a_{n-2}$

b. $a_n = a_{n-1}^2 + 2a_{n-2}$

c. $a_n = a_{n-1} + 2a_{n-3} + n - 2$

d. $a_n = 4a_{n-1} + 2a_{n-4} + 6a_{n-7}$

SOLUTION OF LINEAR HOMOGENEOUS RECURRENCE RELATION WITH CONSTANT COEFFICIENT

Consider, $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$

Let $a_n = r^n$ be the solution

$$\therefore r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

Assuming $r^n \neq 0$, we get

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} + \cdots + c_k = 0$$

Called as characteristic equation of the recurrence relation. The solution of this equation are called as characteristic roots.

THEOREM : REAL DISTINCT ROOTS

Let $c_1, c_2, \dots c_k$ be real numbers. Suppose the characteristic equation

$r^k - c_1 r^{k-1} - c_2 r^{k-2} + \cdots + c_k = 0$

has k distinct roots $r_1, r_2, \dots, r_k$. Then solution of recurrence relation is

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$ for $n = 0,1,2,3, \dots$

Where $\alpha_1, \alpha_{2,}, \dots, \alpha_k$ are constants.

REAL REPEATED ROOTS

If the roots are real and repeated,

For example: If the root r is repeated 2 times then the general solution will be of the form

$$a_n = (c_1 + c_2 n) r^n$$

If the root r is repeated 3 times then the general solution will be of the form

$$a_n = (c_1 + c_2 n + c_3 n^2) r^n$$

If the root r is repeated k times then the general solution will be of the form

$$a_n = \left(c_1 + c_2 n + c_3 n^2 + \cdots + c_k n^{k-1}\right) r^n$$

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COMPLEX PAIR OF ROOTS

If roots of characteristic equations are complex i. e. say

$\alpha + i\beta \;\; and \; \alpha - i\beta,$

Then solution corresponding to these roots is of the form

$$a_n = r^n(c_1 cosn\theta + c_2 sinn\theta)$$

$where \; r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

EXAMPLES

Solve the recurrence relation using given initial conditions

a. $a_n = a_{n-1} + 6a_{n-2}$, for $n \geq 2, a_0 = 3, a_1 = 6.$

b. $a_n = 7a_{n-1} - 10a_{n-2}$, for $n \geq 2, a_0 = 2, a_1 = 1.$

c. $a_n = 6a_{n-1} - 8a_{n-2}$, for $n \geq 2, a_0 = 4, a_1 = 10.$

d. $a_n = -6a_{n-1} - 9a_{n-2}$, for $n \geq 2, a_0 = 3, a_1 = -3.$

EXERCISE

Solve the recurrence relation using given initial conditions

a. $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3},\ a_0 = -5, a_1 = 4, a_2 = 88.$

b. $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3},\ a_0 = 5, a_1 = -9, a_2 = 15.$

SOLUTION OF NON HOMOGENEOUS RECURRENCE RELATION

A recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F_n$$

Where $c_1, c_2, \ldots c_k$ are real numbers and $c_k \neq 0.$ is called non-homogeneous recurrence relation.

Its solution is of the form $a_n = a_n^{(h)} + a_n^{(p)},$ where $a_n^{(h)}$ is a solution of

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ and $a_n^{(p)} is\ a\ solution\ of$

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F_n$ called as particular solution

METHOD OF FINDING PARTICULAR SOLUTION

If F_n is one of the following, then its particular solution can easily obtained

a. Polynomial in n.

b. Powers of constant.

S. No.	F_n	Assumed $a_n^{(p)}$
1	A constant C	A constant d
2	$c_0 + c_1 n$	$d_0 + d_1 n$
3	$c_0 + c_1 n + c_2 n^2$	$d_0 + d_1 n + d_2 n^2$
4	A polynomial of degree n	A polynomial of degree n
5	r^n	
Case I	When r is not the root of characteristic equation	Ar^n
Case II	When r is m times repeated root of characteristic equation	$An^m r^n$
6	$(c_0 + c_1 n + c_2 n^2) r^n$	$(d_0 + d_1 n + d_2 n^2) r^n$ When r is not the root of characteristic equation
		$n^m (d_0 + d_1 n + d_2 n^2) r^n$ When r is m times repeated root of characteristic equation

EXAMPLE

a. Consider the nonhomogeneous recurrence relation $a_n = 3a_{n-1} + 2^n$

   i. Show that $a_n = n2^n$ is the solution of this recurrence relation.

   ii. Find all the solution of this recurrence relation.

   iii. Find the solution with $a_o = 1$.

EXAMPLE

b. Consider the nonhomogeneous recurrence relation

$$a_n = 2a_{n-1} + 2^n$$

   i. Show that $a_n = n2^n$ is the solution of this recurrence relation.
   ii. Find all the solution of this recurrence relation.
   iii. Find the solution with $a_o = 1$.

EXAMPLE

What is the general solution of

$a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + F(n)$, where

a. $F(n) = n^2$

b. $F(n) = 2^n$

c. $F(n) = n2^n$

d. d. $F(n) = (-2)^n$

e. $F(n) = n^2 2^n$

f. $F(n) = n^2(-2)^n$

g. $F(n) = 3$