INCLUSION – EXCLUSION PRINCIPLE

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Finite Set: A finite set is a set with a finite number of elements and is countable.
Infinite Set: An infinite set, on the other hand, has an infinite number of elements, and an infinite set may be countable or uncountable. Yes, finite and infinite sets don't mean that countable and uncountable. There is a difference. For example, sets like N (natural numbers) and Z (integers) are countable though they are infinite because it is possible to list them.

Countable Set: A set having one-to-one correspondence (bijection) from each of these sets to the set of natural numbers N, and hence they are countable.
On the other hand, the set of all real numbers R is uncountable as we cannot list its elements and hence there can't be a bijection from R to N.

CARDINALITY OF A SET

Let S be a finite set containing n distinct element, then cardinality of set S is 'n' denoted by |S|=n.

Example:

1. S=Set of all odd positive integer less than 15, then |S|=7
2. $|\emptyset|=0$
3. $|\{\emptyset\}|=1$

Theorem: If $A \subseteq B, then\ |A| \leq |B|, where\ B\ is\ a\ finite\ set$

Cardinality of an infinite Set: cardinality of a countably infinite set (by its definition mentioned above) is n(N) and we use a letter from the Hebrew language called "aleph null" which is denoted by $\aleph_0$(it is used to represent the smallest infinite number) to denote n(N).

- The cardinality of any countable infinite set is $\aleph_0$.
- The cardinality of an uncountable set is greater than $\aleph_0$.

Theorem: If A and B are finite disjoint sets, then

$$|A \cup B| = |A| + |B| \ .$$

Theorem: Let A be finite set and B be any set, Then

$$|A - B| = |A| - |A \cap B|$$

Theorem: If A and B are finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Theorem: If A, B and C are finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Theorem: If A, B, C and D are finite sets, then

$$|A \cup B \cup C \cup D| =$$
$$= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D|$$
$$-|C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$$
$$-|A \cap B \cap C \cap D|.$$

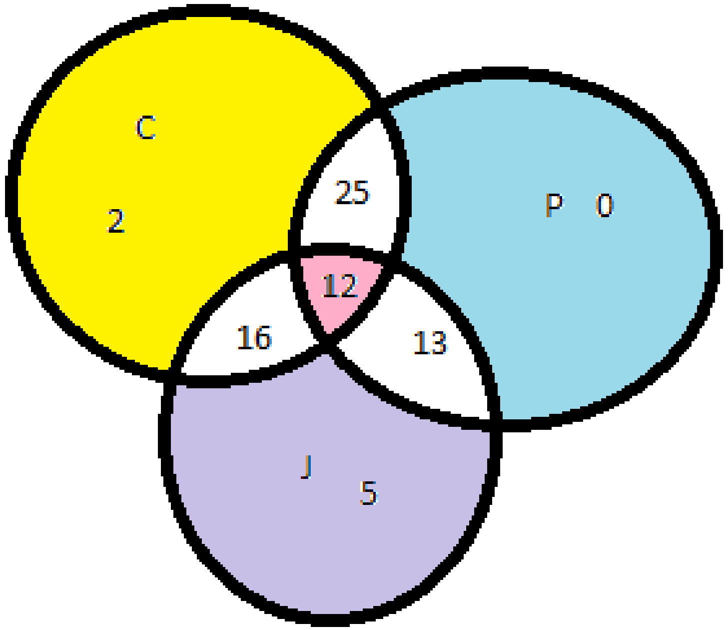
EXAMPLES

1. It was found that in first year 80 computer science students 50 know Python. 55 know C, 46 know Jawa. 37 knows C and Python, 28 knows C and Jawa, 25 knows Python and Jawa. 7 students did not know any of the language. Find

(i) How many know all three languages?

(ii) How many knows exactly two languages?

(iii) How may knows exactly one language?



2. Among the integers 1 to 1000.

a. How many of them are not divisible by 3, nor by 5, nor by 7?

b. How many of them are not divisible by 5 and 7 but divisible by 3?

Ans: a. 457, b. 229

3. How many integers between 1-1000 are divisible by 2,3,5,and 7

Ans: 772

3. Determine the number of primes not exceeding 100 and not divisible by 2,3,5, or 7.

Solution: Let P_1 be the property that an integer is divisible by 2

Let P_2 be the property that an integer is divisible by 3

Let P_3 be the property that an integer is divisible by 5

Let P_4 be the property that an integer is divisible by 7

Thus, number of positive integers not exceeding 100 that are not divisible by 2,3,5, and 7 is $= N(P_1'P_2'P_3'P_4') = k - N(P_1 \cup P_2 \cup P_3 \cup P_4)$, here k=99

Where

Note: $N(P_1'P_2'P_3'P_4') = N(P_1' \cap P_2' \cap P_3' \cap P_4')$

$$N(P_1 \cup P_2 \cup P_3 \cup P_4)$$
$$= N(P_1) + N(P_2) + N(P_3) + N(P_4) - N(P_1P_2) - N(P_1P_3) - N(P_1P_4)$$
$$-N(P_2P_3) - N(P_2P_4) - N(P_3P_4) + N(P_1P_2P_3) + N(P_1P_2P_4) + N(P_1P_3P_4)$$
$$+ N(P_2P_3P_4) - N(P_1P_2P_3P_4)$$
$$=\left\lfloor\frac{100}{2}\right\rfloor + \left\lfloor\frac{100}{3}\right\rfloor + \left\lfloor\frac{100}{5}\right\rfloor + \left\lfloor\frac{100}{7}\right\rfloor - \left\lfloor\frac{100}{6}\right\rfloor - \left\lfloor\frac{100}{10}\right\rfloor - \left\lfloor\frac{100}{14}\right\rfloor - \left\lfloor\frac{100}{15}\right\rfloor - \left\lfloor\frac{100}{21}\right\rfloor - \left\lfloor\frac{100}{35}\right\rfloor$$
$$+ \left\lfloor\frac{100}{30}\right\rfloor + \left\lfloor\frac{100}{42}\right\rfloor + \left\lfloor\frac{100}{70}\right\rfloor + \left\lfloor\frac{100}{105}\right\rfloor - \left\lfloor\frac{100}{210}\right\rfloor$$

$$= 50 + 33 + 20 + 14 - 16 - 10 - 7 - 6 - 4 - 2 + 3 + 2 + 1 - 0 + 0$$
$$= 78$$
$$N(P_1'P_2'P_3'P_4') = N - N(P_1 \cup P_2 \cup P_3 \cup P_4) = 99 - 78 = 21$$

Hence, Number of prime numbers not exceeding 100
=4+$N(P_1'P_2'P_3'P_4')$=4+21=25

4. How many solutions does $x_1 + x_2 + x_3 = 11\ have$? Where $x_1, x_2, and\ x_3$ non-negative integers with $x_1 \leq 3, x_2 \leq 4, and\ x_3$ ≤6.

Note: Number of integer solutions of an equation with n-variables, r-sum of n variables

$$= C(n + r - 1, r) = combination.$$

⟶Let $P_1\ denote\ the\ property\ x_1 > 3, P_2\ denote\ the\ property$

$$x_1 > 4, and\ P_3\ denote\ the\ property\ x_3 > 6.$$

The number of solutions satisfying $x_1 \leq 3, x_2 \leq 4, and\ x_3$ ≤6 is

$N(P_1' \cup P_2' \cup P_3') = N - N(P_1 \cup P_2 \cup P_3)$

Where $N = C(3 + 11 - 1, 11) = 78$

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$N(P_1 \cup P_2 \cup P_3) = N(P_1) + \mathrm{N}(\mathrm{P}_2) + \mathrm{N}(\mathrm{P}_3) - N(P_1P_2) - N(P_1P_3) -$
$N(P_2P_3) + +N(P_1P_2P_3)$
Where $N(P_1) = C(3 + 7 - 1, 7) = 36$

$N(P_2) = C(3 + 6 - 1, 6) = 28.$
$N(P_3) = C(3 + 4 - 1, 4) = 15.$

$N(P_1P_2) =$ number of solutions with $x_1 \geq 4\ and\ x_2 \geq 5$
$= C(3 + 2 - 1, 2) = 6.$
$N(P_1P_3) =$ number of solutions with $x_1 \geq 4\ and\ x_3 \geq 7$
$= C(3 + 0 - 1, 0) = 1.$
$N(P_2P_3) =$ number of solutions with $x_2 \geq 5\ and\ x_3 \geq 7 = 0$
$N(P_1P_2P_3) = 0.$
$N(P_1' \cup P_2' \cup P_3') = 6.$

Example: How many solutions in positive integers are there to the equation

$$x + y + z = 20; 2 < x < 6, 6 < y < 10, and\ 0 < z < 5?$$

Ans: 26

Example: Among 130 students, 60 study Mathematics, 51 study physics, and 30 study both mathematics and physics. Out of 54 students studying chemistry, 26 study mathematics, 21 study physics and 12 study both mathematics and physics. All the students studying neither mathematics nor physics study biology.

Find:

1. How many are studying Biology? Ans: 49
2. How many not studying chemistry are studying mathematics but not physics? Ans: 16
3. How many students are studying neither mathematics nor physics nor chemistry? Ans: 30