Bancilal Ramnath Agarwal Charitable Trust's

Vishwakarma Institute of Technology, Pune 37.
DISCRETE MATHEMATICS ES1030

Set Theory ::

Set:: Unordered Collection of objects.
The objects in a set are called the elements, or members, of the set. A set is said to contain its elements.
If there are exactly n distrinct elements in a set S then S is said to be finite set and n is called as cardinality of S. The cardinality of a set S is denoted as $|S|$.
Sets are graphically represented by Veen diagram. In this diagram the universal set is represented by a rectangle. Inside this rectangle circles or oval shapes or other geometric shapes are used to represent sets.

Some inportant definitions and operations on set

1. **Null set/ Empty Set::** Set with no elements. Denoted as ϕ or $\{\}$.

2. **Subset::** If every element of a set A is also an element of a set B then A is said to be a subset of B. It is denoted as $A \subseteq B$. Empty set ϕ is a subset of every set and every set is a subset of itself. If a non empty set A is a subset of a set B but $A \neq B$ then A is said to be **proper subset** of B and is denoted as $A \subset B$.

3. **Equality of sets::** Two sets A and B are said to be equal $A = B$ if A is subset of B and B is subset of A, i.e., $A \subseteq B$ and $B \subseteq A$.

4. **Union ::** The union of two sets A and B is the set which contains elements of A or B or both. $A \cup B = \{x : (x \in A) \vee (x \in B)\}$.

5. **Intersection ::** The intersection of two sets A and B is the set which contains elements common to both the sets. $A \cap B = \{x : (x \in A) \wedge (x \in B)\}$.

6. **Disjoint Sets ::** Two sets A and B are said to be disjoint if $A \cap B = \phi$.

7. **Complement ::** The collection of all those elements in the universal set U which are not in A is known as the complement of set A and is denoted as $\bar{A}$. OR A^C OR $\neg A$.
$\bar{A} = \{x : (x \in U) \wedge (x \notin A)\}$.

8. **Relative Complement ::** The Relative complement of set B with respect to set A or the difference of A and B is the collection of all elements of set A which are not in set B.Thus $A - B = \{x : (x \in A) \wedge (x \notin B)\}$.

9. **Symmetric Difference or Ring sum ::** The symmetric Difference or Ring sum of two sets A and B is the union of relative complements $A - B$ and $B - A$. It is denoted as $A \Delta B$ OR $A \oplus B$. $A \Delta B = A \oplus B = (A - B) \vee (B - A) = \{x : (x \in A \vee B) \wedge (x \notin A \wedge B)\}$.

10. **Cartesian Product ::** The cartesian product of two sets A and B is the set of all ordered pairs of elements of A and B. $A \times B = \{(a, b) : (a \in A) \wedge (b \in B)\}$. The cartesian product is not commutative $A \times B \neq B \times A$.
The cartesian product of n sets $A_1, A_2, \cdots, A_n$ denoted as $A_1 \times A_2 \times \cdots \times A_n$ is a collection of ordered n tuples.
$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2 \cdots, a_n) : (a_1 \in A_1) \wedge (a_2 \in A_2) \wedge \cdots \wedge (a_n \in A_n)\}$.

11. **Power Set ::** The collection of all possible subsets of the set A is defined as the power set of a the set A. If the set A contains n elements then there are 2^n distinct subsets of the set A. Hence the power set of the set A contains 2^n elements.Elements of the power set are themselves sets.

Some standard Notations::

- $\mathbb{N}-$ The set of natural/ counting numbers
- $\mathbb{Z}-$ The set of integers
- $\mathbb{Q}-$ The set of rational numbers
- $\mathbb{R}-$ The set of real numbers
- $I-$ The set of positive integers

Laws of algebra of sets

Idempotent Law	$A \cup A = A$	$A \cap A = A$
Commutative Law	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative Law	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributive Law	$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$	$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
Identity Law	$A \cup \phi = A$	$A \cap U = A, U$ universal set
Domination Law	$A \cup U = U, U$ universal set	$A \cap \phi = \phi$
Complement Law	$A \cup \bar{A} = U, U^C = \phi$	$A \cap \bar{A} = \phi, \phi^C = U$
Involution Law	$(A^C)^C = A$	
De Morgans Law	$(A \cup B)^C = A^C \cap B^C$	$(A \cap B)^C = A^C \cup B^C$
Absorption Law	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$

Computer Representation of Sets :: Assume that the universal set U is finite with n elements.First, specify an arbitrary ordering of the elements of U, say $a_1, a_2, \cdots, a_n$. Represent a subset A of U with the bit string of length n, where the j^{th} bit in this string is 1 if a_j is a member of A and is 0 if a_j does not belong to A. Bitwise Boolean operations can be performed on the bit strings representing the two sets to obtain the bit string of union,intersection,complement, symmetric difference.
The Inclusion-Exclusion Principle ::

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i,j=1,\ i<j}^{n} |A_i \cap A_i| + \sum_{i,j,k=1,\ i<j<k}^{n} |A_i \cap A_i \cap A_k| + \cdots + (-1)^{n-1} \left|\bigcap_{i=1}^{n} A_i\right|$$

In particular,

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

Suppose out of given N objects
A_1:: denotes the set of objects possessing characteristic A
A_2:: denotes the set of objects possessing characteristic B
A_3:: denotes the set of objects possessing characteristic C then

- The number of objects possessing at least one of the characteristic $= |A_1 \cup A_2 \cup A_3|$
- The number of objects possessing none of the three characteristic $= \left|(A_1 \cup A_2 \cup A_3)^C\right|$
  $\left|(A_1 \cup A_2 \cup A_3)^C\right| = N - |A_1 \cup A_2 \cup A_3|$
- The number of objects possessing characteristic A only $= \left|A_1 \cap A_2^C \cap A_3^C\right|$
  $\left|A_1 \cap A_2^C \cap A_3^C\right| = |A_1| - |A_1 \cap A_2| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$
- The number of objects possessing characteristic B only $= \left|A_1^C \cap A_2 \cap A_3^C\right|$
  $\left|A_1^C \cap A_2 \cap A_3^C\right| = |A_2| - |A_1 \cap A_2| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$
- The number of objects possessing characteristic B only $= \left|A_1^C \cap A_2^C \cap A_3\right|$
  $\left|A_1^C \cap A_2^C \cap A_3\right| = |A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$
- The number of objects possessing exactly one of the three characteristic =
  $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - 2|A_1 \cap A_2| - 2|A_1 \cap A_3| - 2|A_2 \cap A_3| + 3|A_1 \cap A_2 \cap A_3|$
- The number of objects possessing characteristics A and B but not C $= \left|A_1 \cap A_2 \cap A_3^C\right|$
  $\left|A_1 \cap A_2 \cap A_3^C\right| = |A_1 \cap A_2| - |A_1 \cap A_2 \cap A_3|$
- The number of objects possessing characteristics A and C but not B $=\left|A_1 \cap A_2^C \cap A_3\right|$
  $\left|A_1 \cap A_2^C \cap A_3\right| = |A_1 \cap A_3| - |A_1 \cap A_2 \cap A_3|$
- The number of objects possessing characteristics B and C but not A $=\left|A_1^C \cap A_2 \cap A_3\right|$
  $\left|A_1^C \cap A_2 \cap A_3\right| = |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$

Principle of Mathematical Induction

1. First form of Induction
   Let $P(n)$ be a proposition defined on the set of natural numbers $\mathbb{N}$, i.e., $P(n)$ is either true or false for each $n \in \mathbb{N}$. Suppose
   (i) $P(1)$ is true.
   (ii) $P(k+1)$ is true whenever $P(k)$ is true.
   Then $P(n)$ is true for every positive integer n.
2. Second form of Induction
   Let $P(n)$ be a proposition defined on the set of natural numbers $\mathbb{N}$, i.e., $P(n)$ is either true or false for each $n \in \mathbb{N}$. Suppose

(i) $P(1)$ is true.

(ii) $P(1) \wedge P(2) \wedge P(3) \cdots \wedge P(k)$ is true then $P(k+1)$ is true.
Then $P(n)$ is true for every positive integer n.

Relation :: Let A and B be non empty sets. Any subset R of $A \times B$ is defined as a relation from set the A to the set B. If (a, b) is an element of a set R then it is denoted as aRb.
Domain of a relation $R = \{a \in A/(\exists\, b \in B)(aRb)\}$
Range of a relation $R = \{b \in B/(\exists\, a \in A)(aRb)\}$
Thus, $Domain(R) \subseteq A$ and $Range(R) \subseteq B$.
If $A = B$ then R is said to be relation on set A. The relation $A \times A$ is known as universal relation and empty set is defined as a void relation.
Operations on relation:: Let R and S be two relations defined on the A to the set B.

1. Union :: $a(R \cup S)b \Leftrightarrow (aRb) \vee (aSb)$
2. Intersection :: $a(R \cap S)b \Leftrightarrow (aRb) \wedge (aSb)$
3. Complement (denoted as $\overline{R}$): $a(\widetilde{R})b \Leftrightarrow a(\bar{R})b$
4. Relative complement :: $a(R - S)b \Leftrightarrow (aRb) \wedge (a\bar{S}b)$
5. Converse :: Given a relation $R : A \to B$ a relation $\widetilde{R}$ fron B to A is called converse of R where the elements of $\widetilde{R}$ are obtained by interchanging the order pairs in R.

Let R be a relation from a set A to a set B and S be a relation from a set B to a set C.
The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A, c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $R \circ S$.
Properties of relation :: Let R be a relation defined on a set A.

1. **Reflexive ::** R is said to be reflexive if $aRa\ \forall\ a \in A$.
   e.g. $\leq$, $\geq$, $=$, divisibility, integral multiplicity.
2. **Symmetry ::** R is said to be symmetric if whenever aRb then bRa for every $a, b \in A$.
   e.g. similarity,congruency of triangles, perpendicular lines.
3. **Transitivity ::** R is said to be a transitive relation if whenver aRb and bRc then aRc for all $a, b, c \in A$. e.g. similarity, congruency relation of triangles, parallel line relation for lines.
4. **Antisymmetry ::** R is said to be antisymmetric if for all $a, b \in A, aRb$ and bRa then $a = b$. e.g. $\leq, \geq$, divisibility,integral multiplicity, set inclusion, set exclusion.
5. **Irreflexive ::** R is said to be irreflexive if for every $a \in A, a$ is not related to itself. e.g. $<, >$, proper set inclusion, proper set exclusion.
6. **Asymmetric ::** R is said to be asymmetric if whenever aRb then b is not related to a.
   e.g. $<, >$

Relation Matrix :: R is a relation defined on a finite set $A = \{a_1, a_2. \cdots, a_n\}$ then relation matrix of R denoted as M_R is a $n \times n$ square matrix whose $(i,j)^{th}$ entry is 1 if a_i is related to a_j through R and is zero if a_i is not related to a_j.
Digraph :: Pictorial representation of a relation. Let R is a relation defined on set $A = \{a_1, a_2. \cdots, a_n\}$. The elements of set A are represented by points or circles called nodes or vertices.
If $a_i R a_j$ then draw an arrow directing from a_i to a_j.
If $a_i R a_i$ then draw an arc / a loop at a_i.
If $a_i R a_j$ and $a_j R a_i$ then draw two arrows between a_i and a_j pointing in both the directions.
Note that::

Properties	**Relation Matrix**	**Digraph**
Reflexive	all diagonal elements are 1	loop at each node
Symmetry	matrix is also symmetric	parallel edges
Antisymmetric	$(M_R)_{ij} = 1$ and $(M_R)_{ji} = 0$ for $i \neq j$.	No direct return path exist
Irreflexive	all diagonal elements are 0	no loop at each node

Equivalence Relation :: A relation R defined on a set A is called an equivalence relation if R is reflexive, symmetric and transitive. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation. E.g. Equality relation on set of real numbers, Similarity, congruency relations on set of triangles, parallel line relation on set of lines.
Equivalence Classes :: Let R be an equivalence relation defined on set A. For an element $a \in A$ the cllection of all elements $b \in A$ which are related to a is defined as equvalence class of a. Thus, $[a] = \bar{a} = \{b \in A / bRa\}$. The number of distinct equivalence classes is defined as **rank** of a relation.

Theorem 1. *R be an equivalence relation defined on a set A then the following results hold for equivalence classes.*

1. Every element lies in its equivalence class.
   Proof :: By definition of equivalence class, $[a] = \{x \in A / xRa\}$. As R is an equivalence relation,R is reflexive. Therefore, $aRa \; \forall \; a \in A \Rightarrow a \in [a]$.

2. If aRb then $[a] = [b]$.
   Proof :: $aRb \Rightarrow bRa$, as R is symmetric being defined as an equivalence relation. By definition of equivalence class of a, $b \in [a]$. Also, by definition of equivalence class of b, $[b] = \{x \in A / xRb\}$. But then by property (1),$b \in [b]$. This implies $[a] \subseteq [b]$. Similarly, we can prove that $[b] \subseteq [a]$. Therefore, $[a] = [b]$.

3. Any two equivalence classes are either identical or disjoint.
   Proof:: If $[a] = [b]$ then nothing to prove.
   If $[a] \neq [b]$ then to prove that $[a] \cap [b] = \phi$. Suppose, $[a] \cap [b] \neq \phi$ then there is some $c \in A$ such that $c \in [a] \cap [b], i.e., c \in [a]$ and $c \in [b]$.
   $c \in [a] \Rightarrow cRa$. As R is symmetric, $cRa \Rightarrow aRc$. Also, $c \in [b] \Rightarrow cRb$. Thus, aRc, cRb then

transitivity of R implies aRb. But then by property (2), $[a] = [b]$. This is a contradiction to our assumption. Hence, $[a] \cap [b] = \phi$.

Partion of a set:: Let S be a given set and $A = \{A_1, A_2, \cdots, A_m\}$, where $A_i \subseteq S, \quad i = 1, 2, \cdots, m$ be a family of subsets of S such that $S = \bigcup_{i=1}^{m} A_i$ then the family A is called a covering of S. If all $A_i's$ are mutually disjoint then the family A is called a partition of S and the subsets $A_i's$ are called **BLOCKS** of the partition. e.g. $S = \{a, b, c, d\}, A_1 = \{\{a, b\}, \{a, b, c\}, \{b, c, d\}\}, A_2 = \{\{a, c, d\}, \{a, b, c\}, \{b, c, d\}\}$ $A_3 = \{\{a, b\}, \{c, d\}\}$,
$A_4 = \{\{a, c\}, \{b, d\}\}, A_5 = \{\{a\}, \{b\}, \{c\}, \{d\}\}$.
A_1, A_2 are coverings of S while A_3, A_4, A_5 are partitions of S.
Note that:: Every equivalence relation R defined on a set A generates a unique partition of set A. The blocks of this partition correspond to the $R-$equivalence classes. Similarly, given a partition of a set A, it defines an equivalence relation on A.
If $C = \{C_1, C_2, \cdots, C_m\}$ is a partition then the equivalence relation corresponding to this relation is $R = (C_1 \times C_1) \cup (C_2 \times C_2) \cup \cdots \cup (C_m \times C_m)$.
Congruence relation :: Let $\mathbb{Z}$ denote the set of integers and m be a positive integer. A relation R defined on $\mathbb{Z}$ such that a is related to b if the difference $a - b$ is divisible by m is called a congruence relation. Thus aRb iff $m|a - b$, i. e. $a - b = mk$, where k is an integer.
Theorem :: R is an equivalence relation.
Proof :: (i)Since $a - a = 0$ which is divisible by m. $\therefore \; aRa \; \forall \; a \in \mathbb{Z}$. Hence R is reflexive.
(ii) To prove that R is symmetric.
$aRb \Rightarrow m|a - b \Rightarrow a - b = mk$ but then $b - a = (-k)m$. This implies bRa. Therefore R is symmetric.
(iii) To prove that R is transitive.
$aRb \Rightarrow m|a - b \Rightarrow a - b = mk_1$ and $bRc \Rightarrow m|b - c \Rightarrow b - c = mk_2, \; k_1, k_2 \in \mathbb{Z}$ then $a - c = (a - b) + (b - c) = mk_1 + mk_2 = (k_1 + k_2)m = km$, where $k = k_1 + k_2$ is an integer. This implies aRc. Therefore R is transitive.
To find equivalence classes.
For $n \in \mathbb{Z}$ the equivalence class of n,

$$\begin{aligned} [n] = \bar{n} &= \{x \in \mathbb{Z}/xRn\} \\ &= \{x \in \mathbb{Z}/m|x - n\} \\ &= \{x \in \mathbb{Z}/x - n = mk, \; k \in \mathbb{Z}\} \\ &= \{x \in \mathbb{Z}/x = mk + n, \; k \in \mathbb{Z}\} \end{aligned}$$

$$\text{Thus,} \;\; [0] = mk, [1] = mk + 1, [2] = mk + 2, \cdots\cdots\cdots, [m-1] = mk + (m-1)$$

are distinct equivalence classes.These classes partions the set of integers. These classes are also known as residue classes modulo positive integer m.
Sum and product of the partitions::
A partition π' is called a refinement of a partition π if every block of π' is contained in a block of π. Let R be an equivalence relation defined on a set A and π_1, π_2 be partitions of A corresponding to the equivalence relations R_1 and R_2 respectively. The **product** of π_1 and π_2 is a partition π such that

1. π refines both π_1 and π_2.

2. If $\pi^{'}$ refines both π_1 and π_2, then $\pi^{'}$ refines π.

The **sum** of π_1 and π_2 is a partition π such that

1. Both π_1 and π_2 refines π
2. If Both π_1 and π_2 refines $\pi^{'}$, then π refines $\pi^{'}$.

Note that :: If π_1 and π_2 are the partitions corresponding to the equivalence relations R_1 and R_2 respectively then the sum $\pi_1 + \pi_2$ is the partition corresponding to the equivalence relation transitive closure of $R_1 \cup R_2$ while the product $\pi_1 \cdot \pi_2$ is a partition corresponding to the equivalence relation $R_1 \cap R_2$.
Compatibility relation :: A relation R defined on a set A is said to be a compatibility relation if it is reflexive and symmetric.
Maximal Compatibility block :: Let R be a compatibility relation defined on a non empty set A. A subset $B \subseteq A$ is called a maximal compatibility block if any element of B is compatible to every other element of B and no element of $A - B$ is compatible to all the elements of B. e.g. Consider $A = \{1, 2, 3, 4\}$, and
$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 4), (3, 1), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$
R is a compatibility relation on A. The maximal compatibility blocks are $\{1, 2, 4\}$, $\{1, 3, 4\}$.
Partial order relation :: A binary relation R defined on set a P is said to be a partial order relation if R is reflexive, antisymmetric and transitive. Usually a partial order relation is denoted as $\leq$ and a set $(P, \leq)$ is called a partially order set or Poset.(Note that here $\leq$ is simply a notation) Examples are $\leq, \geq$, integral multiplicity, divisibility, set inclusion, set exclusion.
Chains and antichains ::
Let $(P, \leq)$ be a poset. If for every $x, y \in P$ either $x \leq y$ or $y \leq x$ then $\leq$ is called a simple ordering or a linear ordering and $(P, \leq)$ is called **a totally order or a simply ordered set or a linearly ordered set or a chain.** e.g. Set of naturals $\mathbb{N}$ is a chain with ' less equal' as well as ' greater equal' is a chain. A subset A of $(P, \leq)$ is called as **antichain** if no two distinct elements of A are related to each other.
e.g. $D = \{1, 2, 3, 4, 6, 8, 12\}$ and $\leq$: is a divisibility relation on D then $\{1, 2, 4, 8\}$, $\{1, 3, 6, 12\}$, $\{1, 2, 6, 12\}$ are chains and $\{2, 3\}$, $\{4, 6\}$, $\{8, 12\}$, $\{6, 8\}$ are antichains.
Two elements x and y of a poset $(P, \leq)$ are called **comparable** if either $x \leq y$ or $y \leq x$. If neither x is related to y OR y is related to x then x and y are said to be **incomparable**. In a partially ordered set $(P, \leq)$, an element $y \in P$ is said to be a **cover** of an element $x \in P$ if there is no element $z \in P$ such that $x \leq z$ and $z \leq y$.
Hasse Diagram :: Let $(P, \leq)$ be a poset. Diagramatic representation of a partial order relation $\leq$ is known as a Hasse diagram representation.In such a diagram, each element is represented by a samll circle or a dot. The circle for $x \in P$ is drawn below the circle for $y \in P$ if $x \leq y$ and a line is drawn between x and y if y covers x. If $x \leq y$ but y does not cover x, then x and y are not connected by a single line.
For a totally ordered set $(P, \leq)$, the Hasse diagram consists of circles, one below the other.
An element $a \in P$ of a poset (P, R) is called **maximal** if there is no $b \in P$ such that aRb. Similarly, An element $d \in P$ of a poset (P, R) is called **minimal** if there is no $c \in P$ such

that cRd. There can be more than one maximal and minimal elements. They are the "top" and "bottom" elements in the Hasse diagram respectively.
An element $a \in P$ of a poset (P, R) is called **the greatest element** if $bRa \ \forall \ b \in P$. An element $d \in P$ of a poset (P, R) is called **the least element** if $dRc \ \forall \ c \in P$. The greatest and the least elements are unique, if they exist.
Let (P, R) be a poset and $A \subseteq P$. An element $u \in P$ is such that $aRu \ \forall \ a \in A$ then u is called as **an upper bound**. If u is least among all the upper bounds then u is called as **the least upper bound(lub)**. An element $g \in P$ is such that $gRa \ \forall \ a \in A$ then g is called as **a lower bound**. If g is greatest among all the lower bounds then g is called as **the greatest lower bound(glb)**.
Lattice ::
A lattice is a poset $(L, \leq)$ in which every pair of elements $a, b \in L$ has a greatest lower bound and a least upper bound. The greatest lower bound GLB is called the meet or product denoted as $\wedge$ or $*$ and the least upper bound LUB is called the join or sum denoted as $\vee$ or $\oplus$. A lattice $(L, \leq)$ with meet $\wedge$ and join $\vee$ is denoted as $(L, \leq, \wedge, \vee)$ or $(L, \leq, *, \oplus)$.
e.g. 1. S is a non empty set and $P(S)$ power set of S. then $(P(S), \subseteq)$ is a lattice with meet operation as intersection and join operation as union of sets.
2. I^+ be a set of positive integers and $\leq$ divisibility relation defined on I^+, i.e., $a \leq b$ iff a divides b. $(I^+, \leq)$ is a lattice with meet as gcd and join as lcm.
Complete lattice :: A lattice is called complete if each of its nonempty subsets has a least upper bound and a greatest lower bound. The least and greatest elements of a lattice, if they exist, are called the bouhnds and are denoted as 0 and 1 respectively.
A lattice $(L, \leq, *, \oplus)$ which has both elements 0 and 1 is called a bounded lattice.
The bounds 0 and 1 satisfy $a \oplus 0 = a, \ a * 1 = a, \ a \oplus 1 = 1, \ a * 0 = 0, \ \forall a \in L$. 0 is the identity of $\oplus$ and 1 is the identity of $*$.
Complemented lattice :: In a bounded lattice $(L, \leq, *, \oplus, 0, 1)$ an element $b \in L$ is called is called a complement of an element a if $a * b = 0, \ a \oplus b = 1$. A lattice $(L, \leq, *, \oplus, 0, 1)$ is said to be a complemented lattice if every element of L has at least one complement. S a nonempty set then $(P(S), \subseteq, \cup, \cap)$ is a complemented lattice.
Distributive lattice:: A lattice $(L, \leq, *, \oplus)$ is called a distributive lattice if for any $a, b, c \in L, \ a * (b \oplus c) = (a * b) \oplus (a * c)$ and $a \oplus (b * c) = (a \oplus b) * (a \oplus c)$. S is a non empty set and $P(S)$ power set of S. $(P(S), \subseteq, \cup, \cap)$ is a distributive lattice.
Lexicographic Order :: Let (P_1, R_1) and (P_2, R_2) be two posets. The lexicographic ordering $\preceq$ on $P_1 \times P_2$ is defined as $(a_1, b_1) \preceq (a_2, b_2)$ if either $a_1 R_1 a_2$ OR if $a_1 = a_2$ then $b_1 R_2 b_2$.