PARTIAL ORDER RELATION

PROF. DR. SHAILENDRA BANDEWAR

PARTIAL ORDER RELATION

- A binary relation R on a set S is called a **partial ordering**, or partial order if and only if it is:
  - Reflexive
  - Antisymmetric
  - Transitive

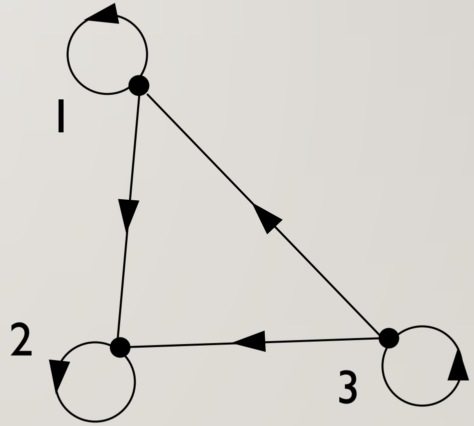
POSET

- A set S together with partial ordering R is called a **partially ordered set**, or *poset*, denoted: $(S, \preccurlyeq)$

- we could denote a poset as (S, R) where R is some relation

EXAMPLE (1)

- Let S = {1, 2, 3} and
- let R = {(1,1), (2,2), (3,3), (1, 2), (3,1), (3,2)}
- $(S, \leq)\ is\ a\ partial\ order\ set$



EXAMPLE (2)

- Show that $\geq$ is a partial order on the set of integers
  - It is reflexive: $a \geq a$ for all $a \in \mathbf{Z}$
  - It is antisymmetric: if $a \geq b$ then the only way that $b \geq a$ is when $b = a$
  - It is transitive: if $a \geq b$ and $b \geq c$, then $a \geq c$
- Note that $\geq$ is the partial ordering on the set of integers
- $(\mathbf{Z}, \geq)$ is the partially ordered set, or poset

EXAMPLE (3)

$$R = \{(a,b) \in A \times A \mid a \mid b\} \quad A \in \mathbb{Z}$$

Reflexive: if for all $a \in A$, $(a,a) \in R$	$a \mid a$ and $a \in A$
Antisymmetric: $if\ (a,b) \in R\ and\ (b,a) \in R\ then\ a = b$	$a \mid b$ then $b \mid a$ only if $a = b$
Transitive: $if\ (a,b) \in R\ and\ (b,c) \in R\ then\ (a,c) \in R$	$a \mid b$ and $b \mid c$ then $a \mid c$

EXAMPLE (4)

- Consider the power set of $\{a, b, c\}$ and the subset relation.

$$(P(\{a, b, c\}), \subseteq)$$

- Draw a graph of this relation.

COMPARABILITY

- **Comparability** means that the elements a and b of a poset $(S, \preccurlyeq)$ are comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$. In other words, task a must come before b or task b must come before a.

EXAMPLE

- Consider the power set of $\{a, b, c\}$ and the subset relation.
  $(P(\{a, b, c\}), \subseteq)$

  $\{a, c\} \not\subseteq \{a, b\}$ and $\{a, b\} \not\subseteq \{a, c\}$

  So, $\{a,c\}$ and $\{a,b\}$ are *incomparable*

TOTALLY ORDERED SET

If (S, ≼) is a poset and every two elements of S are comparable, then S is called a **totally ordered set**, sometimes called a linearly ordered set or chain

For example, the set of integers over the relation “less than or equal to” is a totally ordered set because for every element a and b in the set of integers, either a ≼ b or b ≼ a, thus showing order.

EXAMPLE

- In the poset $(\mathbf{Z}^+, \leq)$, are the integers 3 and 9 comparable?
  - Yes, as $3 \leq 9$
- Are 7 and 5 comparable?
  - Yes, as $5 \leq 7$
- As all pairs of elements in $\mathbf{Z}^+$ are comparable, the poset $(\mathbf{Z}^+, \leq)$ is a total order
  - totally ordered poset, linear order, or chain

EXAMPLE

- In the poset $(\mathbf{Z}^+, |)$ with “divides” operator $|$, are the integers 3 and 9 comparable?

$\nmid$

  - Yes, as $3 \mid 9$
- Are 7 and 5 comparable?
  - No, as $7 \nmid 5$ and $5 \nmid 7$ Thus, as there are pairs of elements in $\mathbf{Z}^+$ that are not comparable, the poset $(\mathbf{Z}^+, |)$ is a partial order. It is not a chain.

Definition: Let R be a total order on A and suppose $S \subseteq A$. An element s in S is a *least element* of S iff sRb for every b in S.

Similarly for *greatest* element.

Note: this implies that $<a, s>$ is not in R for any a unless $a = s$. (There is nothing smaller than s under the order R).

WELL-ORDERED SET

(S, $\preccurlyeq$) is a **well-ordered set** if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

- For example, the set of natural numbers over the relation "less than or equal to" is well-ordered because for every element a and b in the set of natural numbers, either $a \preccurlyeq b$ or $b \preccurlyeq a$, thus showing a totally ordered set.
- Additionally, I want you to notice that this total ordering has a "least element," namely the value of 1, as this is the smallest (least) natural number in the chain.

EXAMPLE: LEXICOGRAPHIC ORDERING

- Example: Consider the ordered pairs of positive integers,

  $Z^+ \times Z^+$ where

  $$(a_1, a_2) \preccurlyeq (b_1, b_2) \text{ if } a_1 < b_1, \text{ or if } a_1 = b_1 \text{ and } a_2 \leq b_2$$

EXAMPLE

- Example: $(\mathbf{Z}, \leq)$
  - Is a total ordered poset (every element is comparable to every other element)
  - It has no least element
  - Thus, it is not a well-ordered set

EXAMPLE

- Example: $(S, \leq)$ where $S = \{ 1, 2, 3, 4, 5 \}$
  - Is a total ordered poset (every element is comparable to every other element)
  - Has a least element (1)
  - Thus, it is a well-ordered set

HASSE DIAGRAM

- A Hasse diagram is a graph for a partial ordering that does not have loops or arcs that imply transitivity and is drawn upward, thus, eliminating the need for directional arrows.

HOW TO DRAW A HASSE DIAGRAM

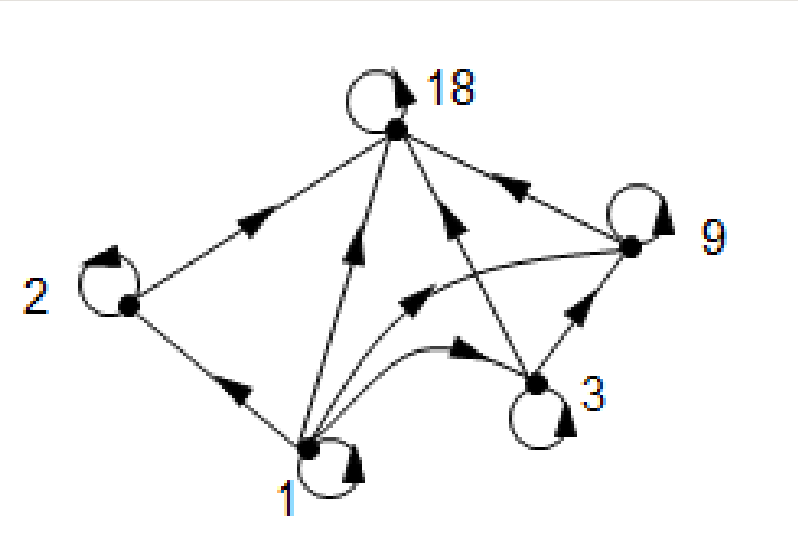
- Start with a directed graph of the relation in which all arrows point upward. Then eliminate:

1. the loops at all the vertices,
2. all arrows whose existence is implied by the transitive property,
3. the direction indicators on the arrows.

EXAMPLE

- Let A = {1, 2, 3, 9, 19} and consider the "divides" relation on A:

For all $a, b \in A,\ a \mid b \Leftrightarrow b = ka$ for some integer k.

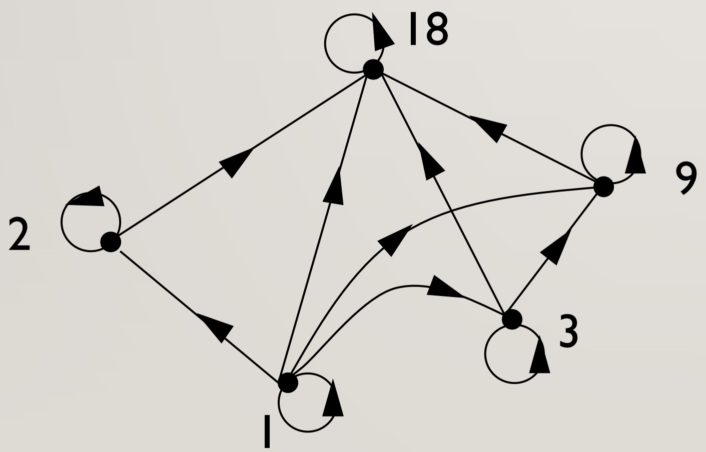


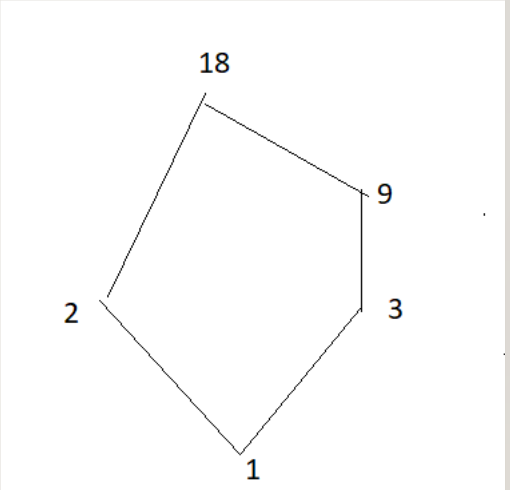
21 EXAMPLE

Eliminate the loops at all the vertices.

Eliminate all arrows whose existence is implied by the transitive property.

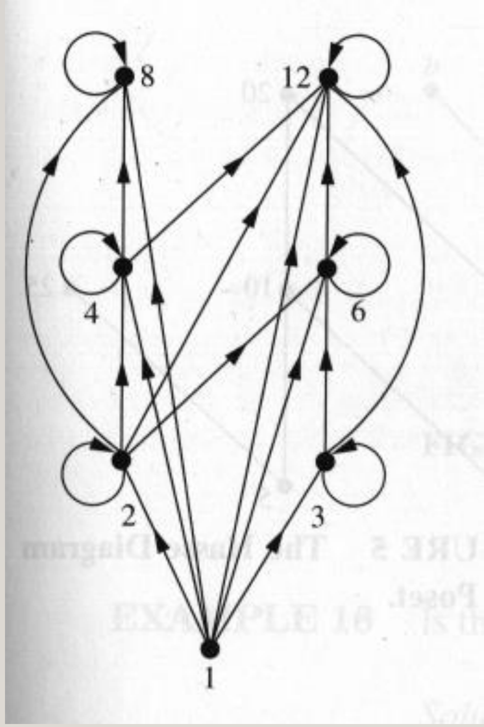
Eliminate the direction indicators on the arrows.

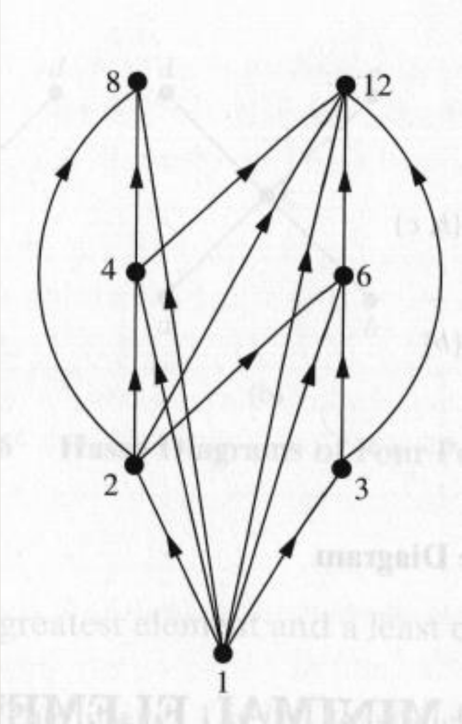


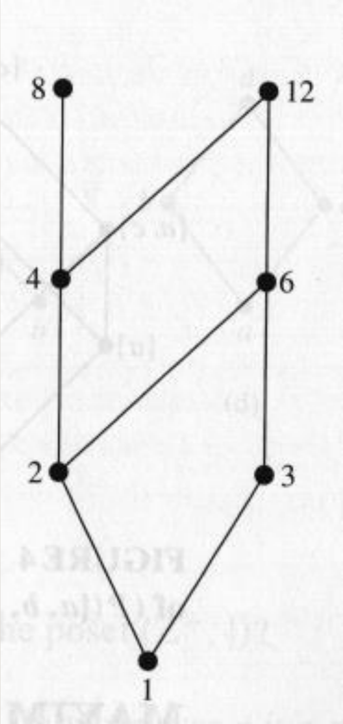


HASSE DIAGRAM

- For the poset ({1,2,3,4,6,8,12}, |)







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Construct the Hasse diagram of $(P(\{a, b, c\}), \subseteq)$.

The elements of $P(\{a, b, c\})$ are

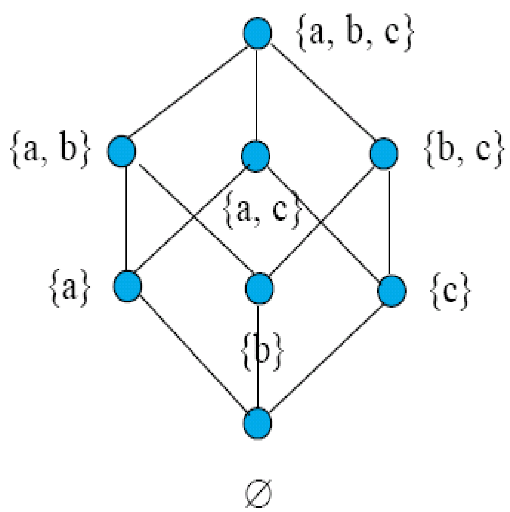
$$\emptyset$$

$$\{a\}, \{b\}, \{c\}$$

$$\{a, b\}, \{a, c\}, \{b, c\}$$

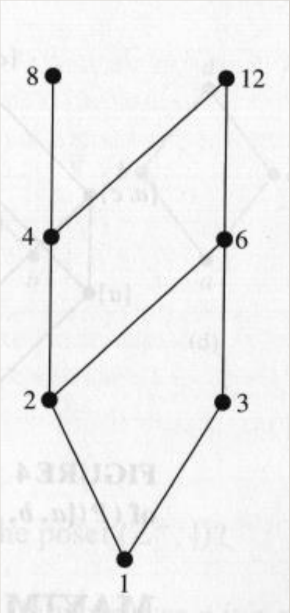
$$\{a, b, c\}$$

The digraph is

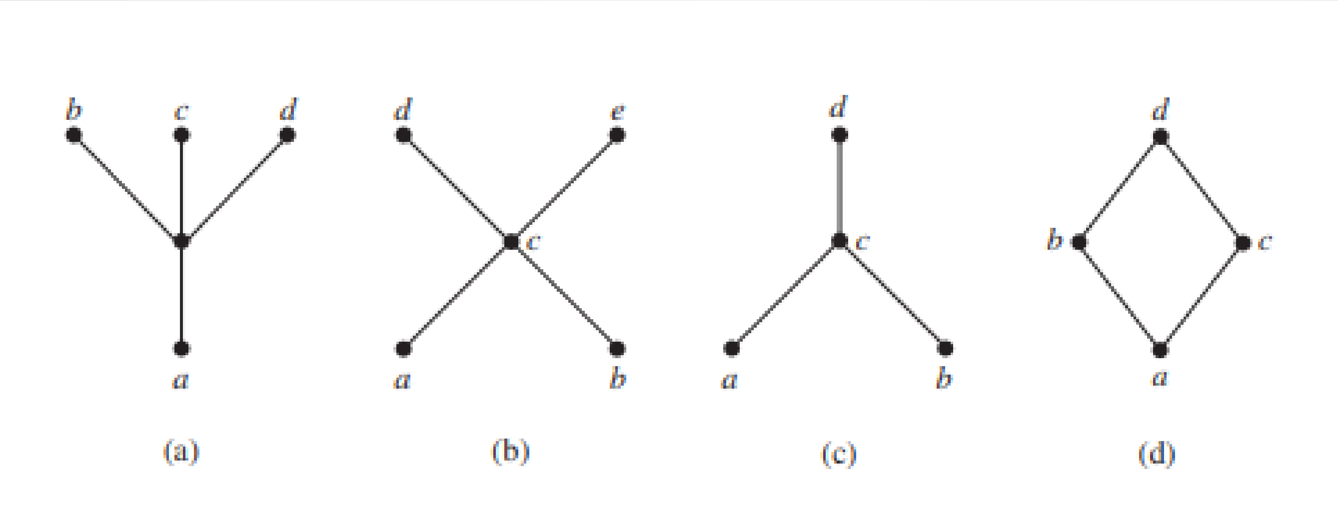


24 GREATEST ELEMENT
LEAST ELEMENT

a is the *greatest element* in the poset (S, $\preccurlyeq$) if $b \preccurlyeq a$ for all $b \in S$. Similarly, an element of a poset is called the *least element* if it is less or equal than all other elements in the poset. That is, a is the *least element* if $a \preccurlyeq b$ for all $b \in S$



Example: Determine whether the posets represented by the Hess diagrams shown in figure have a least element and greatest element



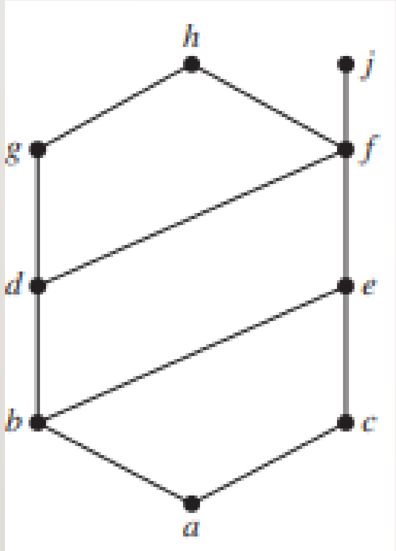
UPPER BOUND, LOWER BOUND

Sometimes it is possible to find an element that is greater than all the elements in a subset A of a poset $(S, \preccurlyeq)$. If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called an *upper bound* of A. Likewise, there may be an element less than all the elements in A. If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called a *lower bound* of A.

Examples 18, p. 574 in Rosen.

EXAMPLE

- Find the lower and upper bounds of the subsets $\{a, b, c\}, \{j, h\}, \{a, c, d, f\}$ in the poset with the Hasse diagram shown in figure



LEAST UPPER BOUND, GREATEST LOWER BOUND

The element x is called the *least upper bound* (lub) or infimum of the subset A if x is an upper bound that is less than every other upper bound of A.

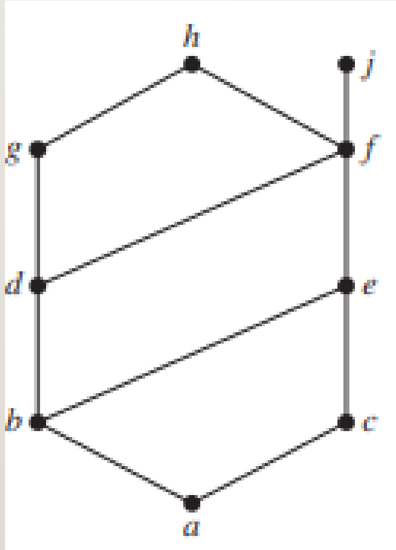
The element y is called the *greatest lower bound* (glb) or supremum of A if y is a lower bound of A and $z \preccurlyeq y$ whenever z is a lower bound of A.

- In the poset $(P(S), \subseteq)$, $\mathrm{lub}(A, B) = A \cup B$. What is the $\mathrm{glb}(A, B)$?

Examples 19 and 20, p. 574 in Rosen.

EXAMPLE

- Find the greatest lower bound and least upper bound of $\{b, d, g\}$, if they exists in the posets shown in figure



EXAMPLE

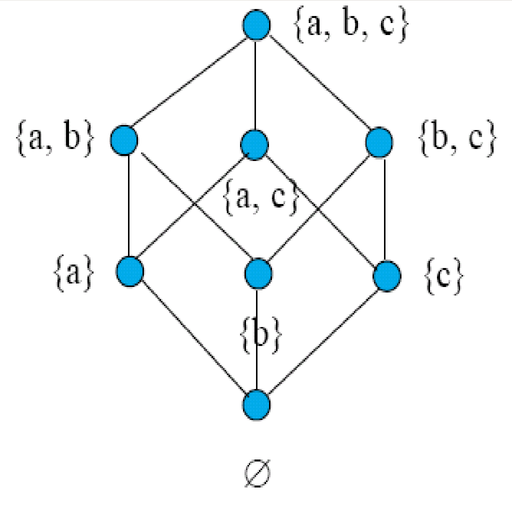
Find the LUB and GLB of the sets $\{3,9,12]$ and $\{1, 2, 4, 5,10\}$,

if they exists, in the posets $(Z^+, |)$.

31 LATTICES

A partially ordered set in which *every pair* of elements has both a least upper bound and a greatest lower bound is called a *lattice*.

$(P(\{a, b, c\}), \subseteq)$



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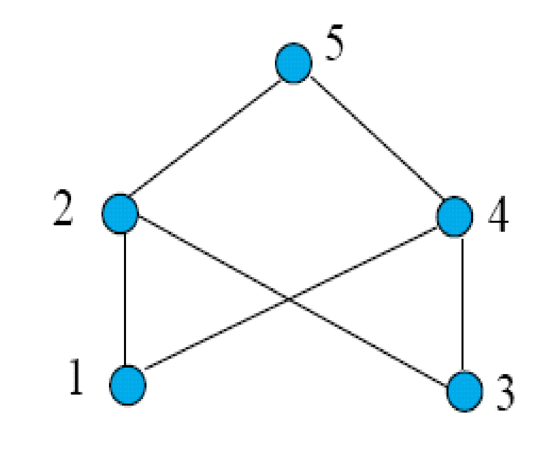
Consider the elements 1 and 3.

- Upper bounds of 1 are 1, 2, 4 and 5.
- Upper bounds of 3 are 3, 2, 4 and 5.
- 2, 4 and 5 are upper bounds for the pair 1 and 3.
- There is no lub since
  - 2 is not related to 4
  - 4 is not related to 2
  - 2 and 4 are both related to 5.
- There is no glb either.

The poset is not a lattice.



Examples 21 and 22, p. 575 in Rosen.