EQUIVALENCE RELATION

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EQUIVALENCE RELATIONS

- An *equivalence relation* (e. r.) on a set *A* is any binary relation on *A* that is reflexive, symmetric, and transitive.
- R is an equivalence relation on A if at has three properties
  - For every $a \in A,\ (a, a) \in R$
  - If $(a, b) \in R, then\ (b, a) \in R$
  - If $(a, b) \in R\ and\ (b, c) \in R, then\ (a, c) \in R$

EQUIVALENCE RELATION EXAMPLES

- "Strings a and b are the same length."
- "Integers a and b have the same absolute value."
- "Real numbers a and b have the same fractional part (*i.e.*, $a - b \in \mathbf{Z}$)."

MORE EXAMPLES

- The classification of animals by species, that is, the relation "is of the same species as" is an equivalence relation on the set of animals.
- Let m be a fixed positive integer. Two integers a and b are said to be congruent modulo m, written as $a \equiv b(modm)$ if m divides $a - b$. This relation of congruence modulo m is an equivalence relation
- For example: 2$\equiv 7(mod5), 22 \equiv 4(mod6)$

MORE EXAMPLES

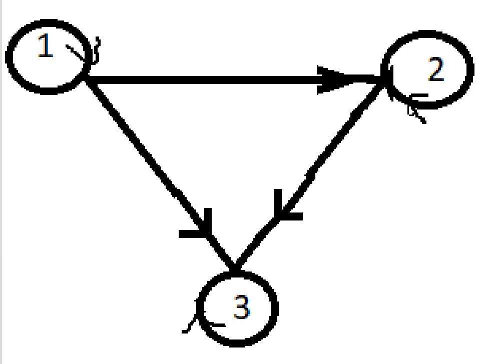
- Let $A = \{a, b, c, d\}, let\ R = \{(a, a), (b, a), (b, b), (c, a), (d, d), (d, c)\}$ Determine whether R is an equivalence relation.

- Let $A = \{a, b, c, d\}$ and Let $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

  $Determine\ whether\ R\ is\ an\ equivalence\ relation\ or\ not.$

- $Determine\ whether\ R\ is\ an\ equivalence\ relation\ or\ not.$

Shown in the digraph



THEOREM

Let R and S be relations on A, then following hold true

(i) If R and S are reflexive, then $R \cup S$ is reflexive

(ii) If R and S are reflexive, then $R \cap S$ is reflexive

(iii) If R and S are symmetric, then $R \cup S$ and $R \cap S$ are symmetric

(iv) If R and S are transitive, $then\ R \cap S\ is\ transitive.$

(vi) If R and S are equivalence relation, then $R \cap S$ is an equivalence relation

THEOREM

If R is an equivalence relation on A, then R^{-1} is also an equivalence relation on A.

MORE EXAMPLES

- Let us assume that R be a relation on the set of ordered pairs of positive integers such that ((a, b), (c, d)) ∈R if and only if $ad = bc$. Is R an equivalence relation?

MORE EXAMPLES

- Let assume that F is a relation on the set R real numbers defined by xRy if and only if x-y is an integer. Prove that F is an equivalence relation on R.

EQUIVALENCE CLASSES

- Let R be any equivalence relation on a set A.
- The *equivalence class* of a,

  $[a]_R :\equiv \{ b \mid aRb \}$ (optional subscript R)

  - It is the set of all elements of A that are "equivalent" to a according to the equivalence relation R.
  - Each such b (including a itself) is called a *representative* of $[a]_R$.

RANK OF A RELATION R

- The rank of a relation R is the number of distinct equivalence classes of R if number of classes are finite, otherwise rank is said to be infinite.

EXAMPLES

- Let $A = \{a, b, c\}$ $and\ let\ R = \{(a,a),(b,b),(c,c),(a,b),(b,a)\}$

$is\ equivalence\ relation\ on\ A.$ Find all equivalence classes of elements of A.

$\rightarrow [a] = \{a, b\}.$

$[b] = \{b, a\}.$

$[c] = \{c\}.$

EXAMPLE

- Let $A = \{1,2,3,4\}$ and

  $R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1), (2,3), (3,2), (3,3), (4,4)\}$

1. Show that R is an equivalence relation on A
2. Find equivalence classes of A.

EXAMPLE

- Let Z denote the set of integers. Let n be any positive integer and define a relation R on Z by setting

$(a, b) \in R \ iff \ n|(a - b)$.

1. Show that R is an equivalence relation on Z.
2. Determine its equivalence classes

MORE EXAMPLES

- "Strings a and b are the same length."
  - $[a]$ = the set of all strings of the same length as a.
- "Integers a and b have the same absolute value."
  - $[a]$ = the set $\{a, -a\}$
- "Real numbers a and b have the same fractional part (*i.e.*, $a - b \in \mathbf{Z}$)."
  - $[a]$ = the set $\{\ldots, a-2, a-1, a, a+1, a+2, \ldots\}$

EXAMPLE

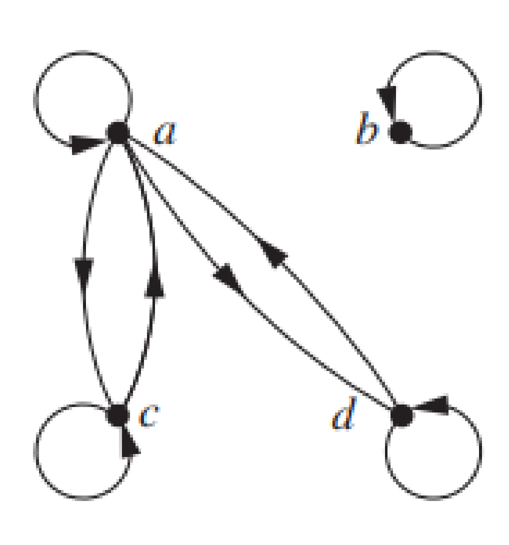
- Show that the relation consisting of all pairs (x, y) such that x and y are bit strings of length 3 or more that agree in their first three bits is an equivalence relation on the set of all bit strings of length three or more.
  - What are the equivalence classes for the bit stings

    (i) 010 (ii) 1010 (iii) 1111 (iv) 01010101

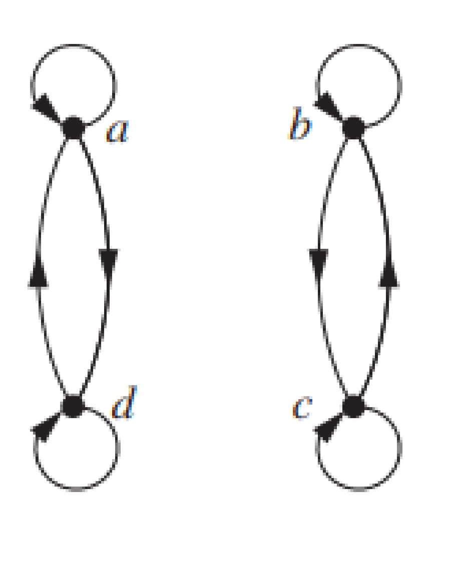
EXAMPLE

- Determine the relation with the directed graph is equivalence relation or not

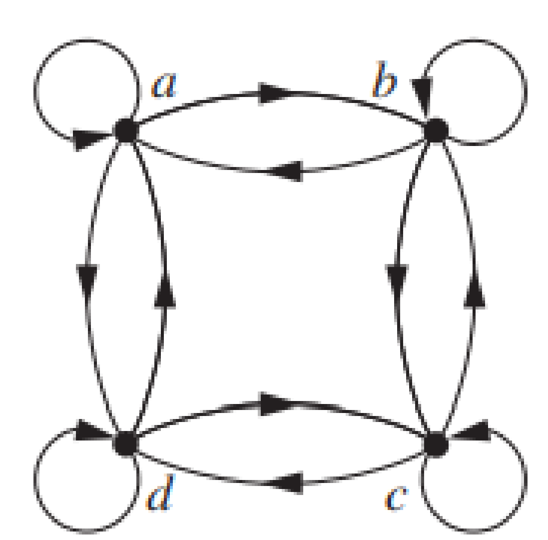
a.



b.



c.



EXAMPLE

Determine whether the relations represented by these matrices are equivalence relations.

a. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ b. $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

PARTITIONS

- A partition of a non empty set A is a collection of subsets $\{A_1, A_2, \dots, A_n\}$ *such that*

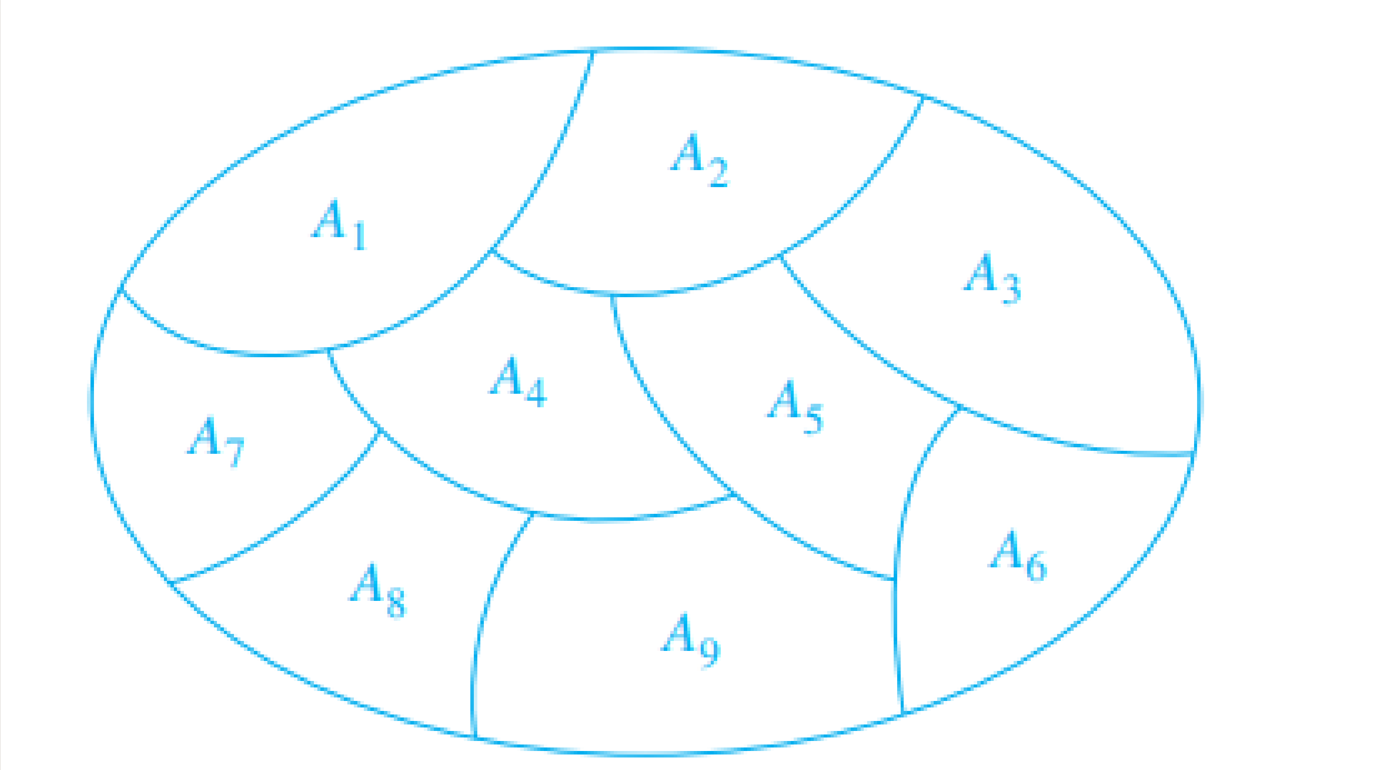
  (I) $A = \bigcup_{i=1}^{n} A_i$

  (ii) $A_i \cap A_j = \emptyset,\ for\ all\ i \neq j$.

Theorem: A *partition* of a set *A* is the set of all the equivalence classes $\{A_1, A_2, \ldots\}$ for some equivalence relation on *A*.

PARTITIONS

- partitions



EXAMPLE

- Let R be the equivalence relation on the set $A = \{1,2,3,4,5,6\}$:

  where

$R = \{(1,1), (1,5), (2,2), (2,3), (2,6), (3,2), (3,3), (3,6), (4,4), (5,1), (5,5)$

$(6,2), (6,3), (6,6)\}.$

Find the partition of A induced by R.

EXAMPLES

1. Let $A = \{a, b, c, d\}, \pi = \{[a, b], [c], [d]\}$

$be\ a\ partion\ of\ A. Find\ the\ equivalence\ relation\ induced\ by\ \pi.$

2. Let $A = \{1,2,3,5\}, \pi = \{[1,2], [3], [4, 5]\}$

$be\ a\ partion\ of\ A. Find\ the\ equivalence\ relation\ induced\ by\ \pi.$