COUNTING

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COMBINATORICS

- Count the number of ways to put things together into various combinations.

    e.g. If a password is 6-8 letters and/or digits, how many passwords can there be?

- Two main rules:
    - Sum rule
    - Product rule

BASIC COUNTING PRINCIPLES

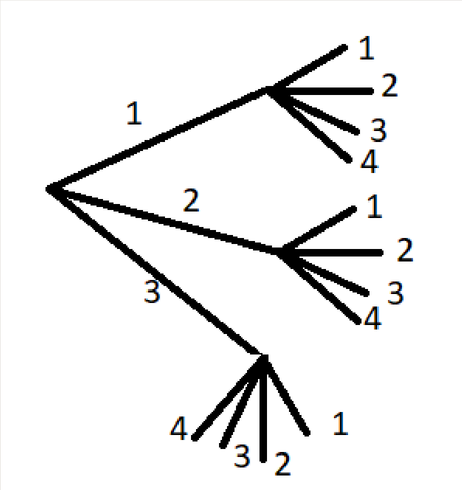
- Let us consider two tasks:
  - m is the number of ways to do **task 1**
  - n is the number of ways to do **task 2**
  - Tasks are <u>independent</u> of each other, i.e.,
    - Performing **task 1** does not accomplish **task 2** and vice versa.
- *<u>Sum rule</u>*: the number of ways that "**either** task 1 **or** task 2 can be done, but **not both**", is $m+n$.
- Generalizes to multiple tasks ...

EXAMPLE

A student can choose a computer project from one of the three lists. The three lists contains 23, 42, and 15 possible projects, respectively. No project is on more than one list. How many projects are there to choose from?

BASIC COUNTING PRINCIPLES

- **The Product Rule:** Suppose that a procedure can be broken into a sequence of two tasks. If there are n ways to do the first task and for each of these ways of doing first task, there are m ways, then there $m \cdot n$ ways to do the procedure.
- Here $n = 3, m = 4$



EXAMPLE

The chairs of an auditorium are to be labeled with a letter and a positive integer not to exceed 100. What is the largest number of chairs that can be labeled differently?

EXAMPLE

How many different license plates are available if each plate contains a sequence of two letters followed by four digits.

EXAMPLE

How many one to one functions are there form a set with m elements to set with n elements

EXAMPLE

How many different bit strings are there of length eight?

EXAMPLE

How many bit strings of length 8 either start with bit 1 or end with the two bits 00?

EXAMPLE

- Suppose a password for the computer system must have at least 8, but not more than 12 characters, where each character in the password is a lowercase English alphabet, uppercase English alphabet, a digit, or one of the six special characters *, <,>, !, +, and =.

1. How many different passwords are available?

CONTINUED

2. How many of these passwords contain at least one of the six special character?

3. If it takes nanosecond for a hacker to check whether each possible password is your password, how long would it take this hacker to try every possible password?

EXAMPLE

Each user on a computer system has a password, which is six to eight characters long where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

TREE DIAGRAM

Tree diagram can be used to solve counting problems

EXAMPLE

How many bit strings of length 4 do not have two consecutive 1s?

EXAMPLE

A Playoffs between two teams consists of at most five games. The first team that wins three games wins the playoff. How many different ways a playoff to occur.

EXAMPLE

Suppose that a popular style shoes is available for both man and woman. The woman shoe come in sizes 6, 7, 8 and 9, and the man's shoe come in size 8,9,10,11, and 12. the man's shoe come in black and white and woman shoe come in white, red and black, Determine the number of different shoes the store has to stock to have at least one pair of shoes for all available sizes and colors for both man and woman

PERMUTATIONS

A **permutation** of a set of distinct objects is an **ordered arrangement** of these objects.

We also interested in ordered arrangements of some of the elements of a set. The ordered arrangements of r elements is called **r-permutation**.

The number of r-permutations of a set containing n elements is denoted by $P(n, r)$ $or\ by$

THEOREM

- If n is a positive integer and r is an integer $1 \leq r \leq n$, then

$$P(n,\ r) = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

- $P(n, 0) = 1.$
- $P(n, n) = n!.$

EXAMPLE

- Find the permutation of the set $A = \{1,2,3,4\}$, taking the three elements at a time

EXAMPLE

- Four persons enter in a bus in which there are six vacant seats. In how many ways can they take there places?

EXAMPLE

a. Suppose that **repetitions are not permitted**, how many 4 digit numbers can be formed from the six digits 1,2,3,5,7,8?
b. How many such numbers are less than 4000?
c. How many numbers in (a) are even?
d. How many numbers in (a) are odd?
e. How many numbers in (a) contain both the digits 3 and 5?

EXAMPLE

Six different mathematics books, four different Discrete Structure books and three different computer books are to be arranged on a shelf. How many different arrangement are possible if

a. The books in each subject must be together?

b. Only Discrete Structure books are together?

EXAMPLE

- How many permutations of the letters A,B,C,D,E,F,G, and H contain
  - The string CDE?
  - The string AB and FG?
  - The String ABC and CDE?
  - The string ABC and BED?

EXAMPLE

- A menu card in a restaurant displays 4 soups, five main courses, three deserts, and 5 beverages. How many different menus can a customer select if
  - He select one item from each group
  - He chooses to omit beverages, but select one each from the other groups.
  - He chooses to omit desserts, but select one each from the other groups.

EXAMPLE

- A terrorist has planted an armed nuclear bomb in your city, and it is your job to disable it by cutting wires to the trigger device.
- There are 10 wires to the device.
- If you cut exactly the right three wires, in exactly the right order, you will disable the bomb, otherwise it will explode!
- If the wires all look the same, what are your chances of survival?

COMBINATIONS

- Now consider the counting of unordered selection of objects
- r-combinations of elements of a set with n distinct elements is unordered selection of r objects from the set, denoted by

  $C(n,r) = \binom{n}{r} =^n C_r = \frac{n!}{r!(n-r)!}$
- $C(n,r) = C(n, n-r)$
- $C(n,0) = C(n,n) = 1$

EXAMPLE

Find the permutation and combination of 3 objects selected from a set $A = \{a, b, c, d\}$

combination	permutation
abc	abc, acb, bac, bca, cab, cba
abd	abd, adb, bad, bda, dab, dba
acd	acd, adc, cad, cda, dac, dca
bcd	bcd, bdc, cbd, cdb, dbc, dcb

EXAMPLE

- A club has 25 members.
  - How many ways are there to choose 4 members of the club to serve as an executive committee?
  - How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than two office?

EXAMPLE

	Answer
• How many bit string of length 8 contain	
• Exactly 5 1s?	56
• An equal number of 0s and 1s?	70
• At least four 1s?	163
• At least three 1s and at least three 0s?	182

EXAMPLE

- Suppose a department consists of 8 men and 9 women. In how many ways can we a select a committee of
  - Three men and 4 women?
  - Four persons that has at least one woman?
  - Four persons that has at most one man?
  - Four persons that has persons of both sexes?
  - Four persons so that 2 specific persons are not included?

BINOMIAL COEFFICIENTS

Binomial Theorem: Let x and y be variables, let n be non negative integer, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

COROLLARIES

1. $2^n = \sum_{k=0}^{n} \binom{n}{k}$
2. $0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$
3. $3^n = \sum_{k=0}^{n} 2^k \binom{n}{k}$

PASCAL'S IDENTITY AND TRIANGLE

Theorem: Let n and be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$
$$\cdots$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$
$$\cdots$$

VANDERMONDE'S IDENTITY

Let m, n, r be positive integers with r less than m and n. Then

$$\binom{m+n}{r} = \sum_{j=0}^{r} \binom{m}{r-j} \binom{n}{j}$$

EXAMPLE

What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x - 3y)^{200}$

EXAMPLE

1. Give a formula for the coefficient of x^k in the expansion of

$\left(x+\frac{1}{x}\right)^{100}, where\ k\ is\ an\ integer.$

1. Give a formula for the coefficient of x^k in the expansion of

$\left(x^2-\frac{1}{x}\right)^{100}, where\ k\ is\ an\ integer.$

GENERALIZED PERMUTATIONS AND COMBINATIONS

- How to solve counting problems where <u>elements may be used more than once</u>?
- How to solve counting problems in which <u>some elements are not distinguishable</u>?
- How to solve problems involving counting the ways we to <u>place distinguishable elements in distinguishable boxes</u>?

PERMUTATIONS WITH REPETITION

- The number of *r*-permutations of a set of *n* objects with **repetition** allowed is n^r.
- Example: How many strings of length *n* can be formed from the English alphabet?

PERMUTATIONS AND COMBINATIONS WITH AND WITHOUT REPETITION

TABLE 1 Combinations and Permutations with and without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

PERMUTATIONS WITH NON-DISTINGUISHABLE OBJECTS

- The number of different permutations of *n* objects, where there are n_1 non-distinguishable objects of type 1, n_2 non-distinguishable objects of type 2, …, and n_k non-distinguishable objects of type *k*, is

$$P(n: n_1, n_2, \dots, n_k) = \frac{n!}{n_1! n_2! \dots n_k!},$$

$$where\ n_1 + n_2 + \cdots + n_k = n.$$

EXAMPLE

- How many different strings can be made by reordering the letters of the word

1. SUCCESS.
2. MISSISSIPPI
3. ABRAKADABRA

COMBINATIONS WITH REPETITION
(INDISTINGUISHABLE OBJECTS AND DISTINGUISHABLE BOXES)

The number of *r*-combinations from a set with *n* elements when **repetition** of elements is allowed are $C(n+r-1, r)$.

EXAMPLE

There are boxes of identical red, red white and blue balls, where each box contain at least 10 balls. How many ways are there to select 10 balls if

a. There is no restriction?

b. At least one white ball must selected?

CONTINUED

c. At least one red ball, at least 2 blue balls and at least 3 white balls must be selected?
d. Exactly one red ball and at least one blue ball must be selected?
e. At most one white ball must be selected?
f. Twice as many red balls as white balls must be selected?

EXAMPLE:

- How many ways are there to select 5 bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills? Assume that the order in which bills are chosen does not matter and there are at least 5 bills of each type.

EXAMPLE

How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29$$

Where $x_i,\ i = 1,2,3,4,5,6$ is a nonnegative integer such that

a. $x_i > 1\ for\ i = 1,2,3,4,5,6.$

b. $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, x_4 \geq 4, x_5 \geq 5, and\ x_6 \geq 6.$

c. $x_1 \leq 5?$

d. $x_1 < 8\ and\ x_2 > 8.$

EXAMPLE

How many positive integer less than 10, 00,000 have the sum of their digits equals to 19.

EXAMPLE (INDISTINGUISHABLE OBJECTS AND DISTINGUISHABLE BOXES)

How many ways are there to place 10 non-distinguishable balls into 8 distinguishable bins?

Solution: $C(8+10-1, 10) = C(17,10) = 19448$.

PERMUTATIONS AND COMBINATIONS WITH AND WITHOUT REPETITION

TABLE 1 Combinations and Permutations with and without Repetition.

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r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

DISTINGUISHABLE OBJECTS INTO BOXES

PERMUTATIONS WITH INDISTINGUISHABLE OBJECTS
DISTRIBUTING DISTINGUISHABLE OBJECTS INTO DISTINGUISHABLE BOXES

The number of ways to distribute *n* distinguishable objects into *k* distinguishable boxes so that objects are placed into box *i*, *i=1,2,…,k. is given by*

$$P(n: n_1, n_2, \dots, n_k) = \frac{n!}{n_1! n_2! \dots n_k!},$$

EXAMPLE

- How many ways are there to distribute hands of 5 cards to each of 4 players from the standard deck of 52 cards?

INDISTINGUISHABLE OBJECTS AND DISTINGUISHABLE BOXES

- The number of ways of putting r indistinguishable objects into n boxes

$=C(n+r-1,r)$

DISTINGUISHABLE OBJECTS AND INDISTINGUISHABLE BOXES

How many ways are there to put four different employees into three distinguishable offices, where each office can contain any number of employees.

STERLING FORMULA

$\mathrm{S(n,j)}$ = number of ways to distribute n distinguishable objects into k indistinguishable boxes so that no box is empty is given by

$$\sum_{j=1}^{k} S(n,j) = \sum_{j=1}^{k} \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j-i)^n$$

Where $\mathrm{S(n,j)} = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i \binom{j}{i} (j-i)^n$ is called Sterling numbers of second kind.

EXAMPLE

Five balls are to be placed in three boxes. Each box can hold all five balls. In how many ways can we place the balls so that no box is empty, if

a. Balls and boxes are different?
b. Balls are identical and boxes are different?
c. Balls are different and boxes are identical?
d. Balls and boxes are identical?

EXAMPLE

A professor packs her collection of 40 issues of mathematics journal in four boxes with 10 issue per box. How many ways can she distribute the journals if

a. Each box is numbered so that they are distinguishable?

b. The boxes are identical, so that they can not be distinguished?