SET THEORY

Set - Definition

- A set is an unordered collection of different elements.

- A set can be written explicitly by listing its elements using set bracket.
- If the order of the elements is changed or any element of a set is repeated, it does not make any changes in the set.

Some Example of Sets

- A set of all positive integers
- A set of all the planets in the solar system
- A set of all the states in India
- A set of all the lowercase letters of the alphabet

Representation of a Set

Sets can be represented in two ways –

- Roster or Tabular Form
- Set Builder Notation

Roster or Tabular Form

- The set is represented by listing all the elements comprising it.

- The elements are enclosed within braces and separated by commas.

Set of vowels in English A={a,e,i,o,u}

Set of even numbers less than 10, B={2,4,6,8}

Set Builder Notation

The set is defined by specifying a property that elements of the set have in common.

The set is described as $A=\{x:p(x)\}$

Example 1 – The set {a,e,i,o,u} is written as –

$A=\{x:x \text{ is a vowel in English alphabet}\}$

Example 2 – The set {1,3,5,7,9} is written as –

$B=\{x:1\le x<10 \text{ and } (x\%2)\neq 0\}$

If an element x is a member of any set S, it is denoted by $x\in S$ and if an element y is not a member of set S, it is denoted by $y\notin S$

Example – If $S=\{1,1.5,2,2.5\}$,

$1\in S$ but $3.5\notin S$

Cardinality of a Set

Cardinality of a set S, denoted by $|S|$ is the number of elements of the set.

The number is also referred as the cardinal number.
If a set has an infinite number of elements, its cardinality is ∞.

Example – $|\{1,4,3,5\}|=4$,

$|\{1,2,3,4,5,\ldots\}|=\infty$

Types of Sets

Sets can be classified into many types. Some of which are finite, infinite, subset, universal, proper, singleton set, etc.

Finite Set

A set which contains a definite number of elements is called a finite set.

Example – $S=\{x \mid x \in N \text{ and } 70>x>50\}$

Infinite Set

A set which contains infinite number of elements is called an infinite set.

Example – $S=\{x \mid x \in N \text{ and } x>10\}$

Subset

A set X is a subset of set Y (Written as X⊆Y or X ⊂ Y) if every element of X is an element of set Y.

Example 1 – Let, X={1,2,3,4,5,6} and Y={1,2}.

Here set Y is a subset of set X as all the elements of set Y is in set X.
Hence, **we can write Y ⊂ X.**

Example 2 – Let, X={1,2,3} and Y={1,2,3}.

Here set Y is a subset (Not a proper subset) of set X as all the elements of set Y is in set X. Hence, we can write Y⊆X.

Subset

Proper Subset

The term "proper subset" can be defined as "subset of but not equal to".

A Set X is a proper subset of set Y (Written as $X \subset Y$) if every element of X is an element of set Y and $|X|<|Y|$.

Example – Let, $X=\{1,2,3,4,5,6\}$ and $Y=\{1,2\}$

Here set $Y \subset X$ since all elements in Y are contained in X too and X has at least one element is more than set Y.

Empty Set or Null Set

- An empty set contains **no elements**. It is denoted by Ø.

- As the number of elements in an empty set is finite, empty set is a finite set.
- The cardinality of empty set or null set is zero.
- **Example** – $S=\{x \mid x \in N \text{ and } 7<x<8\}=\emptyset$

Singleton Set or Unit Set

Singleton set or unit set contains only one element.

A singleton set is denoted by $\{s\}$.

Example – $S=\{x \mid x \in N, 7<x<9\} = \{8\}$

Equal and Equivalent Set

Equal Set

If two sets contain the same elements they are said to be equal.

Example – If A={1,2,6} and B={6,1,2} they are equal as every element of set A is an element of set B and every element of set B is an element of set A.

Equivalent Set

If the cardinalities of two sets are same, they are called equivalent sets.

Example – If A={1,2,6} and B={16,17,22} they are equivalent as cardinality of A is equal to the cardinality of B. i.e. $|A|=|B|=3$

Overlapping and Disjoint Set

- Two sets that have at least one common element are called overlapping sets.

- **Example** – Let, A={1,2,6} and B={6,12,42}
- There is a common element '6', hence these sets are overlapping sets.
- Two sets A and B are called disjoint sets if they do not have even one element in common.
- **Example** – Let, A={1,2,6} and B={7,9,14},
- There is not a single common element, hence these sets are disjoint sets.

Venn Diagrams

Venn is a schematic diagram that shows all possible logical relations between different mathematical sets.

Set Operations

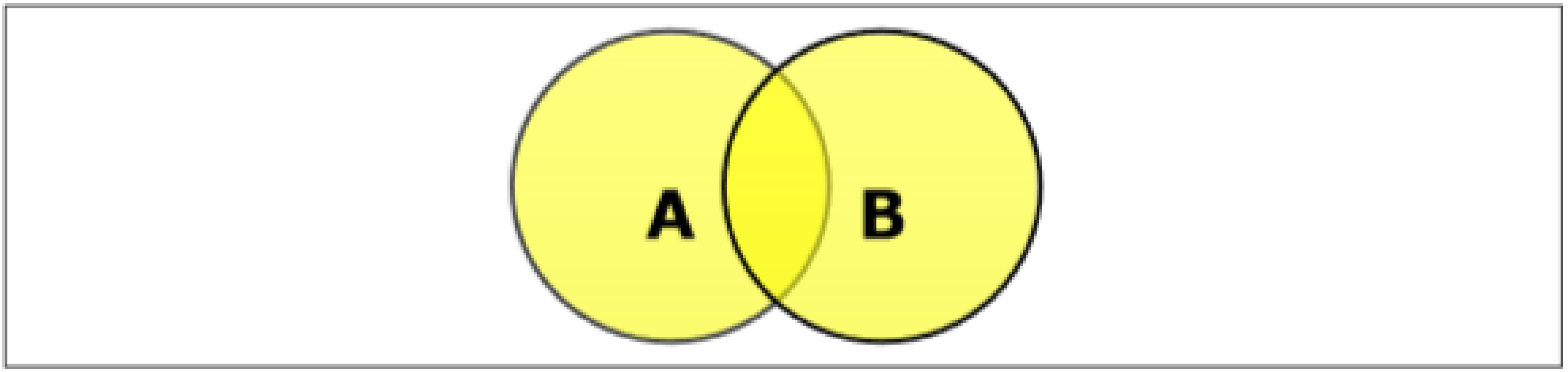
- Set Union (A U B)
- Set Intersection (A ∩ B)
- Set Difference (A-B or B-A)
- Complement of Set (A′)
- Cartesian Product.(A×B or B × A)

Set Union

The union of sets A and B (denoted by AUB) is the set of elements which are in A, in B, or in both A and B.

Hence, $A \cup B = \{x \mid x \in A \text{ OR } x \in B\}$.

Example –



A={5,6,7,8}
B={8,9,10} then

AUB={5,6,7,8,9,10}

(The common element occurs only once)

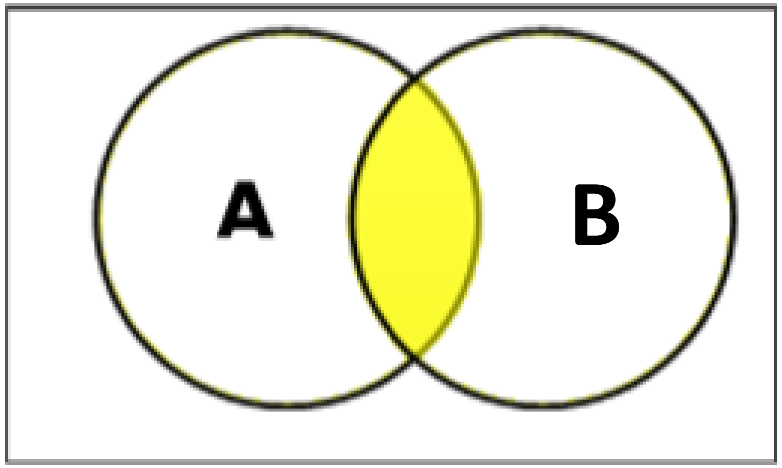
Set Intersection

The intersection of sets A and B (denoted by $A \cap B$) is the set of elements which are in both A and B.

Hence, $A \cap B = \{x \mid x \in A \text{ AND } x \in B\}$

Example –

If $A=\{11,12,13\}$ and $B=\{13,14,15\}$ then $A \cap B=\{13\}$



Set Difference/ Relative Complement

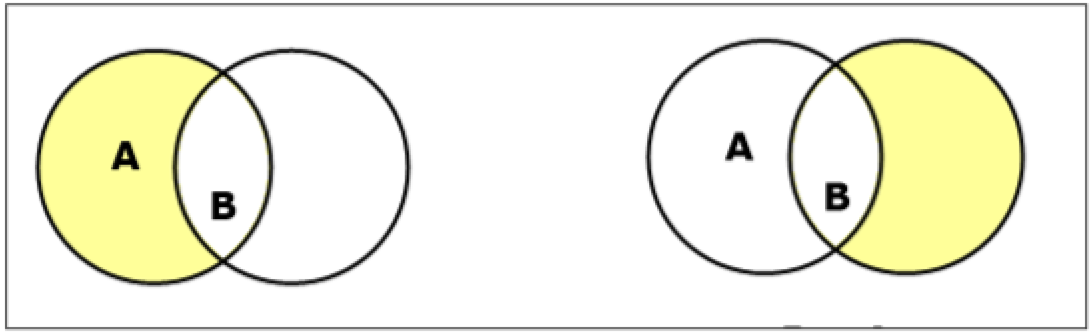
The set difference of sets A and B (denoted by **A–B**) is the set of elements which are only in A but not in B.

Hence, $A-B=\{x \mid x \in A \text{ AND } x \notin B\}$.

Example –
$A=\{21,22,23,24,25\}$ and $B=\{20,22,24,26\}$

then

$(A-B)=\{21,23,25\}$



$(B-A)=\{20,26\}$

Complement of a Set

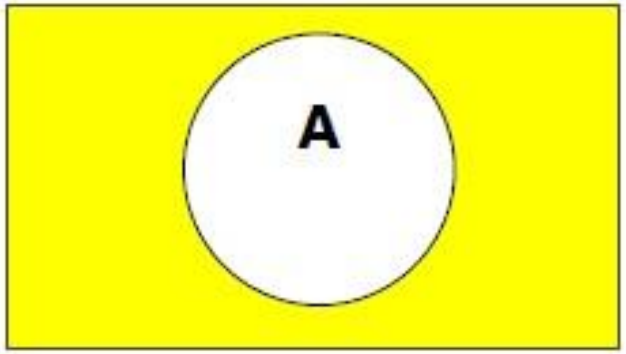
The complement of a set A (denoted by A') is the set of elements which are not in set A.

Hence, $A' = \{x \mid x \notin A\}$.

$A' = (U - A)$ U is a universal set which contains all objects.

If A={x|x belongs to set of odd integers} then

A′={y|y does not belong to set of odd integers}



Cartesian Product / Cross Product

The Cartesian product of n number of sets $A_1,A_2,\ldots A_n$ denoted as $A_1 \times A_2 \cdots \times A_n$ can be defined as all possible ordered pairs $(x_1,x_2,\ldots x_n)$ where $x_1 \in A_1, x_2 \in A_2, \ldots x_n \in A_n$

Example – If we take two sets $A=\{a,b\}$ and $B=\{1,2\}$

The Cartesian product of A and B is written as:

$A \times B=\{(a,1),(a,2),(b,1),(b,2)\}$

The Cartesian product of B and A is written as

$B \times A=\{(1,a),(1,b),(2,a),(2,b)\}$

Power Set

Power set of a set S is the set of all subsets of S including the empty set. The cardinality of a power set of a set S of cardinality n is 2^n.

Power set is denoted as P(S).

Example –

For a set S={a,b,c,d} Now calculate the subsets –

- Subsets with 0 elements – {Ø} (the empty set)
- Subsets with 1 element – {a},{b},{c},{d}
- Subsets with 2 elements – {a,b},{a,c},{a,d},{b,c},{b,d},{c,d}
- Subsets with 3 elements – {a,b,c},{a,b,d},{a,c,d},{b,c,d}
- Subsets with 4 elements – {a,b,c,d}{
- Hence, P(S)=

{{Ø},{a},{b},{c},{d},{a,b},{a,c},{a,d},{b,c},{b,d},{c,d},{a,b,c},{a,b,d},{a,c,d},{b,c,d},{a,b,c,d}}

|P(S)|=2^4=16

Inclusion-Exclusion

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$
- $n(A) = n(A - B) + n(A \cap B)$
- $n(B) = n(B - A) + n(A \cap B)$
- $n(A \cap B) = \emptyset$
- $n(A \cup B) = n(A) + n(B)$

Principle of Inclusion-Exclusion

- **For two sets:**
- $|A \cup B| = |A| + |B| - |A \cap B|$

- Example: In a discrete mathematics class every student is a major in computer science or mathematics or both. The number of students having computer science as a major (possibly along with mathematics) is 25; the number of students having mathematics as a major (possibly along with computer science) is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in the class?

Solution:

$$|A \cup B| = |A| + |B| - |A \cap B| = 25 + 13 - 8 = 30$$

For three sets:

- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

- Example: A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken a course in at least one of Spanish French and Russian, how many students have taken a course in all 3 languages.

- Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian.

$|S| = 1232$, $|F| = 879$, $|R| = 114$

$|S\cap F| = 103$, $|S\cap R| = 23$, $|F\cap R| = 14$

$|S\cup F\cup R| = 2092$.

Using the equation

$$|S\cup F\cup R| = |S| + |F| + |R| - |S\cap F| - |S\cap R| - |F\cap R| + |S\cap F\cap R|$$

we obtain $2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S\cap F\cap R|$.

Solving for $|S\cap F\cap R|$ yields 7.