Ans

Q1) Solve the Given recurrence rel^n

$a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3$, $a_1 = 2$, $a_2 = 6$ & $a_3 = 8$

Now, The recurrence rel^n is,

$a_n - 5a_{n-2} + 4a_{n-4} = 0$ — (*) ~~Degree of~~

order of this eq^n is 4

$\Rightarrow$ ~~Characteristic eqn of (i) is,~~

~~$r^4 + 0 r^3$.~~

i.e. eq^n (i) is,

$a_n + (0)a_{n-1} + (-5)a_{n-2} + (0)a_{n-3} + 4a_{n-4} = 0$

Now, order of this eq^n 4

$\Rightarrow$ degree of characteristic eq^n is 4

So, Characteristic eq^n is,

$r^4 + (0)r^3 + (-5)r^2 + (0)r + 4 = 0$

i.e. $r^4 - 5r^2 + 4 = 0$

$\Rightarrow (r^2 - 4)(r^2 - 1) = 0$

$\Rightarrow$ ~~$r = -2, 1$~~, $r = \pm 2, \pm 1$

So, $a_n = \alpha_1(-2)^n + \alpha_2(+2)^n + \alpha_3(-1)^n + \alpha_4(1)^n$, $\forall n \geq 0$ ~~is soln~~ — (**)

Now, Put $n=0$, $n=1$, $n=2$, $n=3$ in this eq^n,

$\Rightarrow 3 = a_0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ — (i)

$\Rightarrow$ $2 = a_1 = -2\alpha_1 + 2\alpha_2 - \alpha_3 + \alpha_4$ — (ii)

$6 = a_2 = 4\alpha_1 + 4\alpha_2 + \alpha_3 + \alpha_4$ — (iii)

$8 = a_3 = -8\alpha_1 + 8\alpha_2 - \alpha_3 + \alpha_4$ — (iv)

Hence, After solving this system of linear eqn we will get unique ~~soln for~~ values for unknown $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

& Put this values in (**)

Q2) Solve the Given Non homogeneous recurrence rel^n

$a_n = 8a_{n-2} - 16a_{n-4} + \underline{\underline{n^2 4^n}}$ — (*)

Sol^n Now, ~~Associated~~ total sol^n of this eq^n (*) is

$a_n = a_n^{(H)} + a_n^{(P)}$

where $a_n^{(H)}$ is sol^n of Associated Homogeneous eqn

Now Associated homogeneous eqn is

$a_n = 8a_{n-2} - 16a_{n-4}$

$\Rightarrow a_n + (0).a_{n-1} - 8a_{n-2} + (0).a_{n-3} + (-16)a_{n-4} = 0$

$\Rightarrow$ The characteristic eqn is,

$r^4 + (0).r^3 - 8r^2 + (0).r + 16 = 0$

i.e. $r^4 - 8r^2 + 16 = 0$

$\Rightarrow (r^2-4)^2 = 0$

$\Rightarrow r^2 = 4, 4$

$\Rightarrow r = \pm 2, \pm 2,$

$\Rightarrow a_n^{(H)} = (\alpha_1 + \alpha_2 n)(-2)^n + (\alpha_3 + \alpha_4 n)(2)^n$

($\because$ -2 & 2 are roots of multiplicity 1)

— (1)

Now, $F(n) = n^2 \cdot 4^n$

from Given recurence relⁿ

~~So~~ Now, 4 is Not the root of Characteristic eqn ~~multiplicity~~

$\therefore$ Hence Particular solⁿ is of the form,

$\Rightarrow a_n^{(p)} = (\alpha_0 + \alpha_1 n + \alpha_2 n^2).4^n$ — (ii)

Put this eqn in given recurrence relⁿ (*)

$$(\alpha_0 + \alpha_1 n + \alpha_2 n^2)4^n = 8.(\alpha_0 + \alpha_1(n-2) + \alpha_2(n-2)^2).4^{n-2} - 16(\alpha_0 + \alpha_1(n-4) + \alpha_2(n-4)^2)4^{n-4} + n^2 4^n$$

$$\Rightarrow (\alpha_0 + \alpha_1 n + \alpha_2 n^2)4^n = \frac{1}{2}\left[(\alpha_0 + (-2)\alpha_1 + 4\alpha_2) + (\alpha_1 - 4\alpha_2)n + \alpha_2 n^2\right]4^n - \frac{1}{16}\left[(\alpha_0 - 4\alpha_1 + 16\alpha_2) + (\alpha_1 - 8\alpha_2)n + \alpha_2 n^2\right].4^n + n^2 4^n$$

$$\Rightarrow (\alpha_0 + \alpha_1 n + \alpha_2 n^2)4^n = \left\{\left[\frac{\alpha_0 - 2\alpha_1 + 4\alpha_2}{2} - \frac{\alpha_0 - 4\alpha_1 + 16\alpha_2}{16}\right] + \left[\frac{-2\alpha_1 + 4\alpha_2}{2} - \frac{\alpha_1 - 8\alpha_2}{16}\right]n + \left[\frac{\alpha_2}{2} - \frac{1\alpha_2}{16} + 1\right]n^2\right\}4^n$$

$\Rightarrow \quad 7\alpha_0 + 12\alpha_1 + 16\alpha_2 = 16\alpha_0 \Rightarrow \alpha_0 + 2\alpha_1 - 4\alpha_2 = 0$ — (i)

$\Rightarrow \quad -17\alpha_1 + 40\alpha_2 = 16\alpha_1$ — (ii)

$\Rightarrow \quad 7\alpha_2 + 16 = 16\alpha_2$ — (iii)

eq^n (i) $\Rightarrow$

$$9\alpha_0 + 12\alpha_1 - 16\alpha_2 = 0$$

eq^n (ii) $\Rightarrow \; 33\alpha_1 - 40\alpha_2 = 0$

eq^n (iii) $\Rightarrow \; 9\alpha_2 = 16 \Rightarrow \boxed{\alpha_2 = \frac{16}{9}}$

so $\quad 33\alpha_1 - 40\alpha_2 = 0 \Rightarrow \alpha_1 = \frac{40}{30}\left(\frac{16}{9}\right)$

$$\Rightarrow \boxed{\alpha_1 = \frac{64}{27}}$$

& $9\alpha_0 + 12\alpha_1 - 16\alpha_2 = 0 \Rightarrow \alpha_0 = \left[\frac{(16)^2}{9} - (4)\frac{64}{9}\right]\frac{1}{9}$

$$\Rightarrow \alpha_0 = \frac{1}{81}[0]$$

$$\Rightarrow \alpha_0 = 0$$

eq^n (ii) $\Rightarrow \quad a_n^{(p)} = \left(\frac{64}{27}n + \frac{16}{9}n^2\right) \cdot 4^n$ — (iii)

so (i) & (iii) $\Rightarrow$

$$a_n = a_n^{(h)} + a_n^{(p)} = (\alpha_1 + \alpha_2 \cdot n)(-2)^n + (\alpha_3 + \alpha_4 n)(2^n)$$
$$+ \left(\frac{64}{27}n + \frac{16}{9}n^2\right)4^n$$

Q3) What is the general form of ~~recurrence~~ perticular solution of the linear ~~&~~ non Homogeneous ~~eq^n~~ recurence rel^n

$$a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + f(n)$$

if

a) $f(n) = n^2$ b) $f(n) = 2^n$

c) $f(n) = n \cdot 2^n$ d) $f(n) = (-2)^n$

e) $f(n) = n^2 2^n$ f) $f(n) = n^3(-2)^n$

g) $f(n) = 3$

Sol^n: Now, Associated Linear Homogeneous eq^n is
$$a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$$
$\Rightarrow$ Char. eq^n is, $r^3 - 6r^2 + 12r - 8 = 0$

Now, $r = 2$ is root of this eq^n

$\Rightarrow$ $r-2$ is factor of $r^3 - 6r^2 + 12r - 8$

Now,
$$
\begin{array}{r|l}
 & r^2 - 4r + 4 \\
r-2 & r^3 - 6r^2 + 12r - 8 \\
 & -r^3 + 2r^2 \\
\hline
 & -4r^2 + 12r - 8 \\
 & 4r^2 - 8r \\
\hline
 & +4r - 8 \\
 & -4r + 8 \\
\hline
 & 0
\end{array}
$$

$\Rightarrow$ $r^3 - 6r^2 + 2r - 8 = (r-2)(r^2 - 4r + 4)$

$r^3 - 6r^2 + 2r - 8 = (r-2)^3$

Hence, $r = 2, 2, 2$

i.e. 2 is the root of multiplicity 3

Now

a) $f(n) = n^2$

i.e. $f(n) = n^2(1^n)$

& 1 is not the root of char. eq^n so
$$a_n^{(p)} = (b_0 + b_1 n + b_2 n^2).1^n$$

b) $f(n) = 2^n$

i.e. $f(n) = (1.)2^n$

& 2 is root of char. eq^n of multiplicity 3

$\Rightarrow$
$$a_n^{(p)} = (b_0).n^3.2^n$$

c) $f(n) = (n) 2^n$

so, α 2 is root of char. eqn of multiplicity 3

$\Rightarrow$, $a_n^{(p)} = (b_0 + b_1 n) n^3 \cdot 2^n$

d) $f(n) = (1)(-2)^n$

-2 is Not root of char. eqn.

$\Rightarrow$ $a_n^{(p)} = (b_0)(-2)^n$

e) $f(n) = (n^2) 2^n$

Now, 2 is root of char. eqn of Associated homogeneous eqn of multiplicity 3

Ans: $\Rightarrow$ ~~f(n)~~ = $a_n^{(p)} = (b_0 + b_1 n + b_2 n^2) n^3 \cdot 2^n$

is the particular soln for Given recurence reln with $f(n) = n^2 2^n$

f) $f(n) = n^3 (-2)^n$

Now, -2 is Not the root of characteristic eqn so,

$a_n^{(p)} = (b_0 + b_1 n + b_2 n^2 + b_3 n^3)(-2)^n$

g) ~~f(3)~~ $f(n) = 3$ i.e. $f(n) = (3) 1^n$

Now 1 is not the root of char. eqn.

$\Rightarrow$ $a_n^{(p)} = (b_0) 1^n$

i.e. $a_n^{(p)} = b_0$

Q4) Find all solns of the recurence reln

$a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$ — (*)

with $a_0 = 1$ & $a_1 = 4$

soln Now, Associated Homog. reccurence reln is

$a_n - 4a_{n-1} + 3a_{n-2} = 0$

$\Rightarrow$ char. eqn is $r^2 - 4r + 3 = 0$

$\Rightarrow (r-3)(r-1) = 0$

$\Rightarrow r = 3, 1$

So $a_n^{h} = \alpha_1 (3)^n + \alpha_2 (1)^n$

Now, $a_0 = \alpha_1 + \alpha_2 \Rightarrow 1 = \alpha_1 + \alpha_2$ — (i)

$a_1 = 3\alpha_1 + \alpha_2 \Rightarrow 4 = 3\alpha_1 + \alpha_2$ — (ii)

eqn (i) − eqn (ii),

$\Rightarrow \cancel{-3 = -2\alpha_1} \quad -3 = -2\alpha_1$

$\Rightarrow \boxed{\alpha_1 = \frac{3}{2}}$

eqn (i) $\Rightarrow \boxed{\alpha_2 = -\frac{1}{2}}$

Hence; $a_n^{(h)} = \left(\frac{3}{2}\right)(3)^n + \left(-\frac{1}{2}\right)\cdot(1)^n$ — (iii)

Now,

$\cancel{a_n^{(p)}} \quad f(n) = 2^n + n + 3$

$\Rightarrow$

Now, corrosponding to $(1).2^n$ perticular soln is $(b_0)2^n$

& corrospnding to $(n+3)(1)^n$ perticular soln is $(b_1 + b_2 n)n^1(1)^n$

becoz 1 is root of multiplicity 1 of char. eqn

$\Rightarrow a_n^{(p)} = b_0 . 2^n + (b_1 + b_2 n).n$ is perticular soln

Put this in ~~eqn~~ Given recurence reln ⊛,

$\Rightarrow$

$b_0 . 2^n + (b_1 + b_2 n)n = 4.\left(b_0 2^{n-1} + [b_1 + b_2(n-1)](n-1)\right)$
$-3\left[b_0 2^{n-2} + (b_1 + b_2(n-2))(n-2)\right] + 2^n + n + 3$

$\Rightarrow b_0 2^n + b_2 n^2 + b_1 n = \left(2b_0 - \frac{3}{4}b_0 + 1\right)2^n$
$+ (4b_2 - 3b_2)n^2 + (-8b_2 + 4b_1 + 12b_2 - 3b_1 + 1)n$
$+ (-4b_1 + 6b_1 + 3)$

$-\frac{5}{4}b_0 + 1 = b_0 \Rightarrow \boxed{b_0 = \frac{4}{9}}$

$b_2 = b_2$

$b_1 + 4b_2 + 1 = b_1 \Rightarrow \boxed{b_2 = -\frac{1}{4}}$

$\cancel{(-4)} + 2b_1 + 3 = 0 \Rightarrow \boxed{b_1 = \frac{-3}{2}}$

$$\Rightarrow a_n^{(p)} = \left(\frac{4}{9}\right) \cdot 2^n + \left(-\frac{3}{2} - \frac{1}{4}n\right) \cdot n \quad \text{— (iv)}$$

is perticular soln

$\Rightarrow$

Ans: $a_n = a_n^{(H)} + a_n^{(p)} = \left(\frac{3}{2}\right)(3^n) - \frac{1}{2}$

$+ \left(\frac{4}{9}\right) \cdot 2^n + \left(\frac{-3}{2} - \frac{1}{4}n\right) \cdot n$

$, \forall n \geq 0$