RECURRENCE RELATION

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SEQUENCE

Definition: A function whose domain is set of natural numbers and whose codomain is set of real numbers is called a sequence. Sequence is denoted by $\{a_n\}_{n=1}^{\infty}. = \{a_1, a_2, \ldots\}$.

i. e. $f: \mathbb{N} \rightarrow \mathbb{R}$ is called a sequence.

EXAMPLES

1. $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$
2. $a, ar^2, ar^3, \dots$
3. $a, a + r, a + 2r, \dots, a + (n - 1)r, \dots$

CONVERGENCE OF AN INFINITE SEQUENCE

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be convergent if $\lim_{n\to\infty} a_n$ exists.

If $\lim_{n\to\infty} a_n$ does not exists then $\{a_n\}_{n=1}^{\infty}$ is said to be divergent

EXAMPLE

1. Is $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$ convergent or divergent
2. Is $1, -\frac{1}{2}, \frac{1}{3}, \dots, (-1)^n \frac{1}{n},$ convergent or divergent
3. Is $\left(\frac{1+n^2}{3n^2+5}\right)_{n=0}^{\infty}$, convergent or divergent.

INFINITE SERIES

Let $\{a_n\}_{n=1}^{\infty}$ be an infinite sequence, then $\sum_{n=0}^{\infty} a_n$ is called an infinite series.

SEQUENCE OF PARTIAL SUM

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series, then

Let $s_1 = a_1$

$s_2 = a_1 + a_2;$

………………..

$s_n = a_1 + a_2 + \cdots . + a_n;$

Then $\{s_n\}_{n=1}^{\infty}$ *is called sequence of partial sum*.

CONVERGENCE OF AN INFINITE SERIES

$\sum_{n=0}^{\infty} a_n$ infinite series is convergent if the sequence of partial sum is convergent.

i. e. $\lim_{n \to \infty} S_n$ exists.

STANDARD SERIES AND THEIR CONVERGENCE

Auxiliary Series: A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called an auxiliary series.

It is convergent if $p > 1$.

And it is divergent if $p \leq 1$.

GEOMETRIC SERIES

Geometric Series: A series of the form $\sum_{n=1}^{\infty} a^n$ is called a geometric series.

And it is convergent if $|a| < 1$.

And divergent if $|a| \geq 1$

SUM OF GEOMETRIC SERIES

- $\sum_{n=0}^{k} a^n = \frac{1-a^{k+1}}{1-a}\ ,\ a \neq 1$
- $\sum_{n=0}^{k} a^n = k+1\ ,\quad a = 1$
- $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

TEST FOR DIVERGENCE

If $\sum_{n=0}^{\infty} a_n$ infinite series is convergent, then $\lim_{n\to\infty} a_n = 0$.

But converse is not true, that is if $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=0}^{\infty} a_n$ may or may not be convergent.

But if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ is divergent.

EXAMPLE

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{3n^2+8}$ is divergent.

Solution: Consider $\lim_{n\to\infty} \frac{n^2}{3n^2+8} = \frac{1}{3}$

Therefore given series is divergent.

TESTS FOR CONVERGENCE COMPARISON TEST

If two positive terms series be such that $\sum u_n \;\; and \;\; \sum v_n$ be such that

1. (i) $\sum v_n$ is convergent (ii) $u_n \leq v_n$, then $\sum u_n$ is convergent
2. (i) $\sum v_n$ is divergent (ii) $u_n \geq v_n$, then $\sum u_n$ is divergent

Limit form: If two positive terms be such that $\sum u_n \;\; or \;\; \sum v_n$ be such that $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) \; is \; finite \; (\neq 0), then \; \sum u_n \;\; and \;\; \sum v_n$ both converges or both diverges.

EXAMPLE

Test for convergence the series

1. $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\ldots$
2. $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\cdots$
3. $\sum\frac{1}{n}sin\left(\frac{1}{n}\right)$

D'ALEMBERT'S RATIO TEST

Let $\sum u_n$ be series of positive terms and let $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$

1. If $L < 1$, then series converges.
2. If $L > 1$, then series diverges.
3. If $L = 1$, then test is inconclusive

EXAMPLES

- $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$
- $\sum_{n=1}^{\infty} \frac{sin4n}{4^n}$
- $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n^2 2^n}{n!}$

RECURRENCE RELATIONS

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, $a_0, a_1, a_2, \ldots$ for all integers n with $n \geq n_0$ where n_0 is nonnegative integer. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

EXAMPLE

Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions

a. $a_n = a_{n-1} - a_{n-2},\ a_0 = 2, a_1 = -1.$

b. $a_n = 3a_{n-1}^2,\ a_0 = 1.$

c. $a_n = na_{n-1} + a_{n-2}^2, a_0 = -1, a_1 = 0.$

EXAMPLE

Is the sequence $\{a_n\}$ a solution of the recurrence relation

$a_n = 8a_{n-1} - 16a_{n-2}$ if

a. $a_n = 0$

b. $a_n = 2^n$

c. $a_n = n4^n$

d. $a_n = 2 \cdot 4^n + 3n4^n$

e. $a_n = n^2 4^n$

MODELLING OF RECURRENCE RELATION

EXAMPLE: COMPOUND INTEREST

A person deposits Rs. 1000 in an account that yields 9% interest compounded annually.

a. Set up a recurrence relation for the amount in the account at the end of n years.
b. Find an explicit formula for the amount in the account at the account at the end of n years.
c. How much money will the account contain after 100 years?

EXAMPLE: RABBITS AND FIBONACCI NUMBERS

By Leonardo Pisano

A young pair of rabbits (One of each sex) is placed on an Iceland.

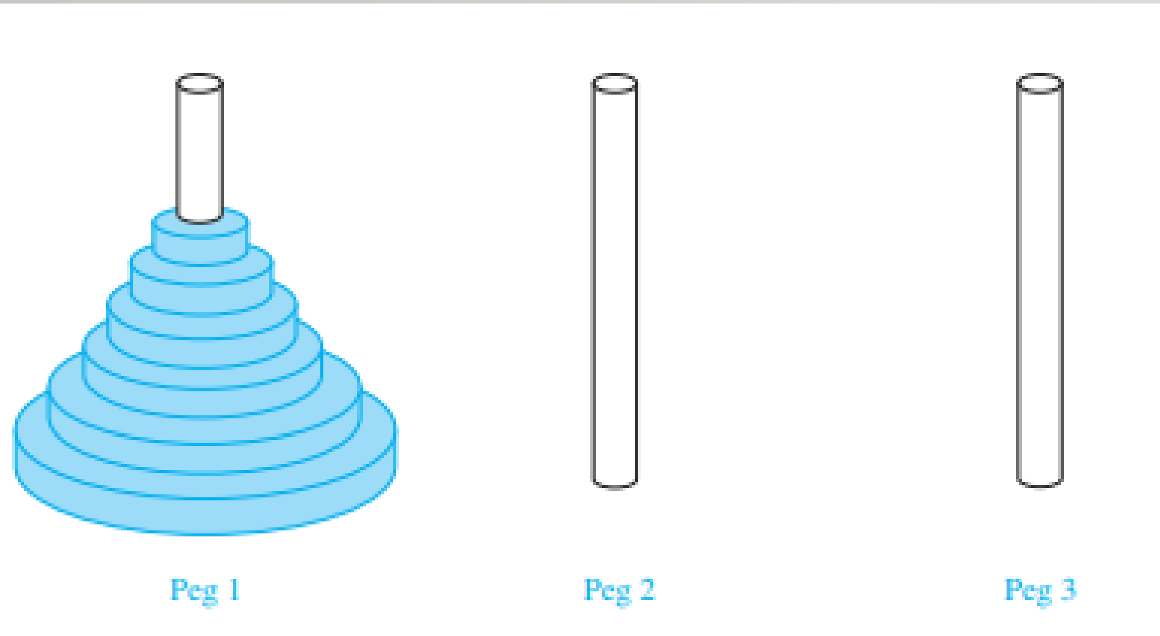
A pair of rabbits does not breed until they are 2 months old. After they are two months old, each pair of rabbits produces another pair in each month, as shown in figure 1. find an recurrence relation for the number of pairs of rabbits on the Iceland after n months, assuming that no rabbits ever die.

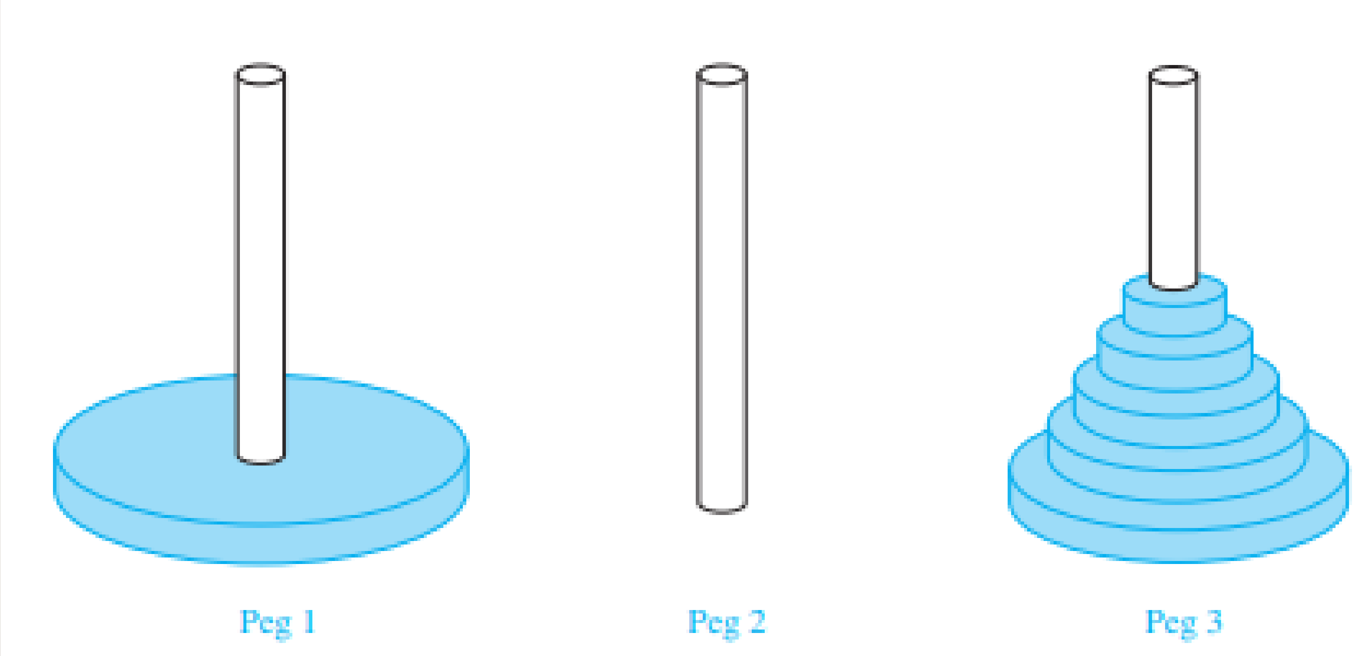
Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

THE TOWER OF HANOI

This puzzle is due to French Mathematician Edouard Lucas

It consists of 3 pegs mounted on a board together with discs of different sizes. initially these disks are placed on the first peg in order of size, with largest on the bottom. The rules of the puzzle allow disks to be moved one at a time from 1 peg to another as long as a disc is never on top of smaller disk. The goal of the puzzle is to have all the disks on the second peg in order of size, with the largest disk at the bottom.


Peg 1
Peg 2
Peg 3


Peg 1
Peg 2
Peg 3

EXAMPLE

Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive zeros. How many bit strings are there of length five.

EXAMPLE

1. Find a recurrence relation and give initial condition conditions for the number of bit strings of length n that contains a pair of consecutive zeros.
2. What are the initial conditions?
3. How many bit strings of length seven contain two consecutive zeros?

EXAMPLE: CODEWORD ENUMERATION

A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. Let a_n be the n-digit valid codewords. Find a recurrence relation for a_n.

EXAMPLE

1. Find a recurrence relation for a number of ways to climb n stairs if the person climbing stairs can take one stair or two stair at a time.
2. What are the initial conditions?
3. How many ways can this person climb a flight of eight stairs?