SET

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SETS

Sets are used to group the objects together

Most of the time the objects in a set have similar properties

For Example:

1. All enrolled Students of VIT
2. All the students taking a course in discrete Mathematics
3. Set of variables in an expression

Language of a sets is a means to study such collections in organized fashion

DEFINITION

A set is a unordered collection of objects

Objects of the set are called the elements, or members, of a set. A set is said to contain its elements

Note: Lowercase letters are usually used to denote elements of a set.

BASIC PROPERTIES OF SETS

Sets are inherently *unordered*:

- No matter what objects a, b, and c denote, $\{a, b, c\} = \{a, c, b\} = \{b, a, c\} = \{b, c, a\} = \{c, a, b\} = \{c, b, a\}$.

All elements are *distinct* (unequal); multiple listings make no difference!

- $\{a, b, c\} = \{a, a, b, a, b, c, c, c, c\}$.
- This set contains at most 3 elements!

METHODS OF SET NOTATION

Listing Method: List all members of a set between braces.

Examples:

a) Set of all vowels in the English alphabets $=\{a, e, i, o, u\}$

b) Set of all even integers less than 15 $=\{2,4,6,8,10, 12,14\}$

c) Set of all positive integers less than 100 $=\{1,2,3, \dots, 99\}$

d) Empty Set or Null Set denoted by $\varnothing = \{\} = \{x \mid \textbf{False}\}$

$\neg\exists x\colon x \in \varnothing.$

e) Singleton set: Set containing only one element

Set builder notation

Set builder notation: We characterize all those elements in the set by stating the property or properties so that they become the member of the set.

For any proposition $P(x)$ over any universe of discourse, $\{x \mid P(x)\}$ is *the set of all x such that P(x)*.

Example: Use set builder notation to to describe the set

1. $\{0,3,6,9,12\} = \{3\mathrm{n} |\ \mathrm{n}\epsilon Z, 0 \leq n \leq 4\}$
2. $\{-3,-2,-1,0,1,2,3\} = \{x | x\epsilon Z, -3 \leq x \leq 3\}$
3. $\{m, n, o, p\} = \{x | x\ is\ a\ set\ of\ all\ english\ alhphabets\ between\ l\ and\ q\}$

EXAMPLE

Example: List the members of the set

1. $\{x|x\ is\ a\ real\ number\ such\ that\ x^2 = 1\} = \{-1,1\}$
2. $\{x|x\ is\ a\ postive\ intger\ less\ than\ 12\} = \{1,2,3,4, \dots, 11\}$

COMPUTER REPRESENTATION OF A SET

Let U be a finite universal set. First specify an arbitrary ordering of an elements of U, for example $a_1 a_2 a_3 \dots a_n$ represent a subset A of U with bit string of length n, where ith bit is given by

$$a_i = \begin{cases} 1, i \in A \\ 0, i \notin A \end{cases}$$

EXAMPLE

1. Suppose that the universal set is $U = \{1,2,3,\ldots,10\}$. Express each of these sets with bit strings.

A) {3,4,5}

B) {1,3,6,10}

C) {1,3,5,9}

2. Using the same universal ser as in previous problem, find a set specified by each of these bit strings

A) 1111000110

B) 1010010110

D) 0110011101

AXIOM OF EXTENSIONALITY

Let A and B be two sets. A and B are equal if and only if they have **same members.**

For Example:

1. Consider the sets $A = \{a, b, c\}, B = \{1,2,3\}, C = \{c, b, a\}, D = \{3,1,2,2\},\ F = \{1,3,3)\}$

Ans: $A = C, B = D$

2. Determine whether each of the pair of sets equal

a) $\{1,3,3,5,5,5, 5\}, \{1,3,5\}$

b) $\{\{2\}\}, \{2, \{2\}\}$

c) $\phi, \{\phi\}$

EXAMPLE

1. Let $A = \{a, b, \{a, b\}, \{\{a, b\}\}\}$. Identify each of the statement true or false. Justify your answer.
   a. $a \in A$
   b. $\{a\} \in A$
   c. $\{a, b\} \in A$
   d. $\{\{a, b\}\} \subseteq A$
   e. $\{a, b\} \subseteq A$
   f. $\{a, \{b\}\} \subseteq A$.

2. Determine whether each of the following statement is true or false for arbitrary sets A,B,C. Justify your answer.

a. If $A \in B\ and\ B \subseteq C, then\ A \in C$.

b. If $A \in B\ and\ B \subseteq C, then\ A \subseteq C$.

c. If $A \subseteq B\ and\ B \in C, then\ A \in C$.

d. If $A \subseteq B\ and\ B \in C, then\ A \subseteq C$.

Counter example for c and d.

Let $A = \{a\}, B = \{a, b\}, C = \{\{a, b\}\}$.

Example: Which of the following propositions are true?

Here $S = \{2, a, 3,4\}, R = \{a, 3,4,1\},$
$and\ E\ is\ the\ universal\ set.$

$a.\ \ a \in S$ b. $a \in R$ c. $R = S$ d. $\phi \subset R$

e. $\{a\} \subseteq S$ f. $\{a\} \in S$ g. $\phi \subseteq R$ h. $R \subseteq S$

Example: For each of the following sets, determine whether 2 is an element of that set

a) $\{x \in R: x\ is\ an\ integer\ greter\ than\ 1\}$

b) $\{x \in R: x \text{ is the square of an integer}\}$

c) $\{2, \{2\}\}$

d) $\{\{2\}, \{2, \{2\}\}\}$

e) $\{\{\{2\}\}\}$

SOME SPECIAL SETS

$\mathbb{N} - Set\ of\ natural\ number = \{1,2,3, \ldots\}$

$\mathbb{Z} - Set\ of\ integers = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$

$\mathbb{Z}^{+} - Set\ of\ positive\ integers = \{0,1,2,3, \ldots\}$

$\mathbb{Q} - set\ of\ rational\ numbers = \left\{\frac{p}{q} \middle|\ p, q \in \mathbb{Z}, q \neq 0\right\}$

$\mathbb{Q}^{+} - set\ of\ positive\ rational\ numbers$

$\mathbb{R} - set\ of\ all\ real\ numbers$

$\mathbb{R}^{+} - set\ of\ all\ positive\ real\ numbers$

$\mathbb{C}$ –Set of complex numbers

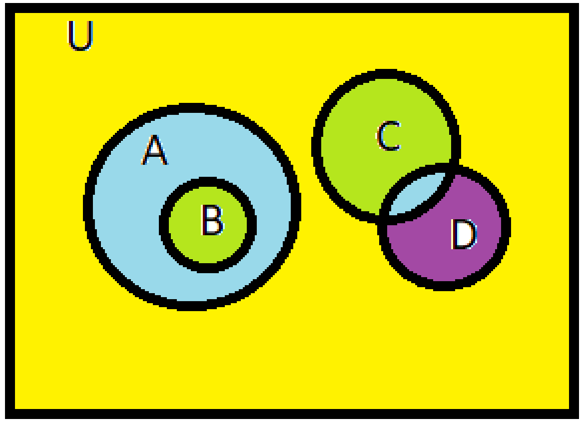
UNIVERSAL SET

In any application of the set theory, the members of the set under consideration are belongs to some large set called the universal set.

VENN DIAGRAM

Venn diagram is a pictorial representation of sets in which sets are represented by enclosed areas in the plane.

Universal U is represented by interior of rectangle, and other sets by discs lying within rectangle



ARGUMENT AND VENN DIAGRAM

Many verbal statements are essentially about sets and can therefore described by Venn diagrams.

Hence Venn diagram can sometimes be used to determine whether or not argument is valid.

EXAMPLE:

Is the following argument valid?

1. My Saucepans are the only things that are made of tin.
2. I find all your presents are very useful.
3. None of my saucepans is of the slightest use

Your presents to me are not made of tin

EXAMPLE

Is the following argument valid?

1. Babies are illogical
2. Nobody is despised who can manage a crocodile
3. Illogical people are despised.

Babies cannot manage crocodile

EXAMPLE

Determine the validity of the following statement

1. All my friends are musicians
2. John is my friend
3. None of my neighbors are musicians

John is not my neighbor

EXAMPLE

Is the following argument valid?

1. All dictionaries are useful
2. Mary owns only romance novels
3. No romance novel is useful

Determine the validity of each of the conclusions

a. Romance novels are not dictionaries
b. Mary does not own a dictionary
c. All useful books are dictionaries

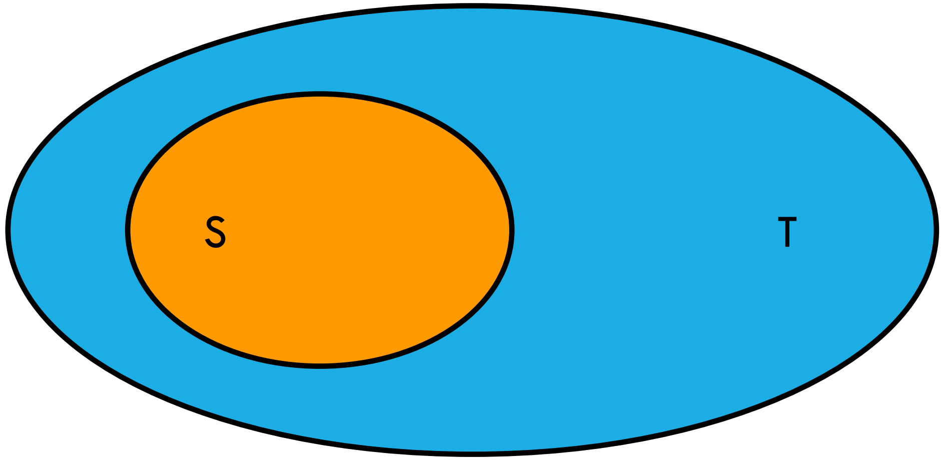
DEFINITION

Subset: The set A is a subset of set B if and only if every element of A is also a element of B denoted by $A \subseteq B$.

$A \subseteq B$ if and only if $\forall x(x \in A \rightarrow x \in B)$.

Theorem: $(\mathrm{i})\;\; \emptyset \subseteq S \; and \;\; (ii)\;\; S \subseteq S$

Proper Subset: Set A is a proper subset of set B if and only if A is a subset of B and $A \neq B\; denoted\; by\; A \subset B$.



Venn Diagram equivalent of $S \subset T$

$S \supseteq T$ ("S is a superset of T") means $T \subseteq S$.

Note $S=T \Leftrightarrow S \subseteq T \wedge S \supseteq T$.

means $\neg(S \subseteq T)$, *i.e.* $\exists x(x \in S \wedge x \notin T)$

The objects that are elements of a set may *themselves* be sets.

E.g. let $S=\{x \mid x \subseteq \{1,2,3\}\}$

then $S=\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Note that $1 \neq \{1\} \neq \{\{1\}\}$!!!!

ORDERED *N*-TUPLES

For $n \in \boldsymbol{N}$, an *ordered n-tuple* or a *sequence of length n* is written $(a_1, a_2, \ldots, a_n)$. The *first* element is a_1, *etc.*

These are like sets, except that duplicates matter, and the order makes a difference.

Note $(1, 2) \neq (2, 1) \neq (2, 1, 1)$.

Empty sequence, singlets, pairs, triples, quadruples, quintuples, …, *n*-tuples.

SET OPERATIONS
COMPLEMENT OF A SET

Let A be a given set. Complement of A denoted by $\bar{A}$ or by A' is defined as

$$\bar{A} = \{\mathrm{x} | \mathrm{x} \notin A\}$$

Properties of a complement of a set:

- $\overline{\emptyset} = U$
- $\overline{U} = \emptyset$
- $\bar{\bar{A}} = A$

EXAMPLES

$A = \{x \mid x \in \mathbb{R} \ \ and \ x \leq 9\}, then \ \bar{A} =?$

If $U = \mathbb{N} \ \ and \ \ E = \{2,4,6, \ldots\} \ then \ \bar{E} \ =?$

UNION OF SETS

For sets A, B, their *union* $A \cup B$ is the set containing all elements that are either in A, **or** ("$\vee$") in B (or, of course, in both).

Formally, $\forall A,B$: $A \cup B = \{x \mid x \in A \vee x \in B\}$.

Note that $A \cup B$ contains all the elements of A or it contains all the elements of B:
$\forall A, B$: $(A \cup B \supseteq A) \wedge (A \cup B \supseteq B)$

Properties of union of sets

$A \cup \emptyset = A$

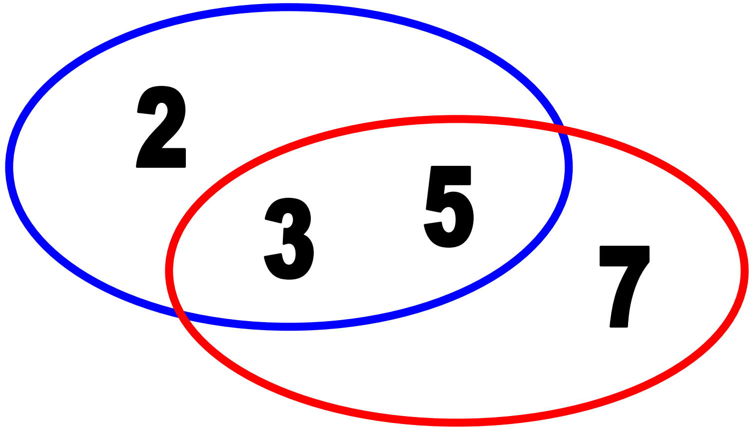
$A \cup U = U$

$A \cup \bar{A} = U$

EXAMPLE

$\{a,b,c\} \cup \{2,3\} = \{a,b,c,2,3\}$

$\{2,3,5\} \cup \{3,5,7\} = \{2,3,5,3,5,7\} = \{2,3,5,7\}$



INTERSECTION OF SETS

For sets A, B, their *intersection* $A\cap B$ is the set containing all elements that are simultaneously in A **and** ("$\wedge$") in B.

Formally, $\forall A,B$: $A\cap B \equiv \{x \mid x\in A \wedge x\in B\}$.

Note that $A\cap B$ is a subset of A **and** it is a subset of B:
$\forall A, B$: $(A\cap B \subseteq A) \wedge (A\cap B \subseteq B)$

Properties of intersection of Sets

$A \cap \emptyset = \emptyset$

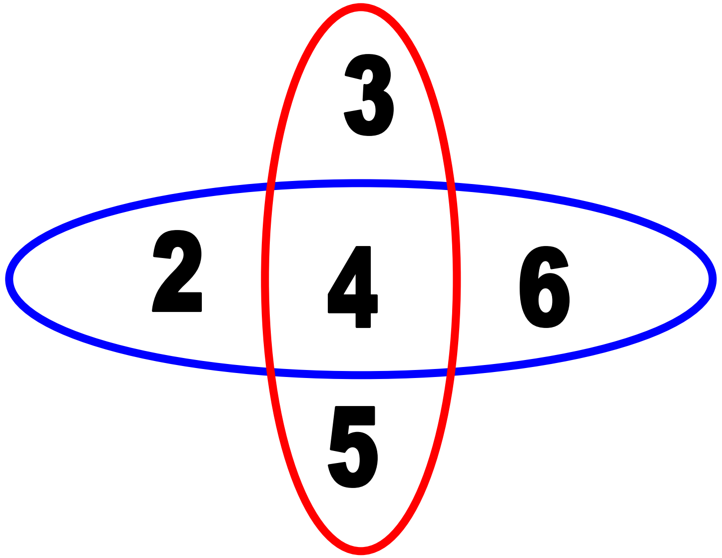
$A \cap U = A$

$A \cap \bar{A} = \emptyset$

EXAMPLES

$\{a,b,c\} \cap \{2,3\} =$ ___

$\{2,4,6\} \cap \{3,4,5\} =$ ______



SET DIFFERENCE

For sets A, B, the *difference of A and B*, written $A-B$, is the set of all elements that are in A but not B.

$$
\begin{aligned}
A - B :\equiv& \{x \mid x \in A \land x \notin B\} \\
=& \{x \mid \neg(x \in A \rightarrow x \in B) \}
\end{aligned}
$$

Also called:
The *complement of B with respect to A*

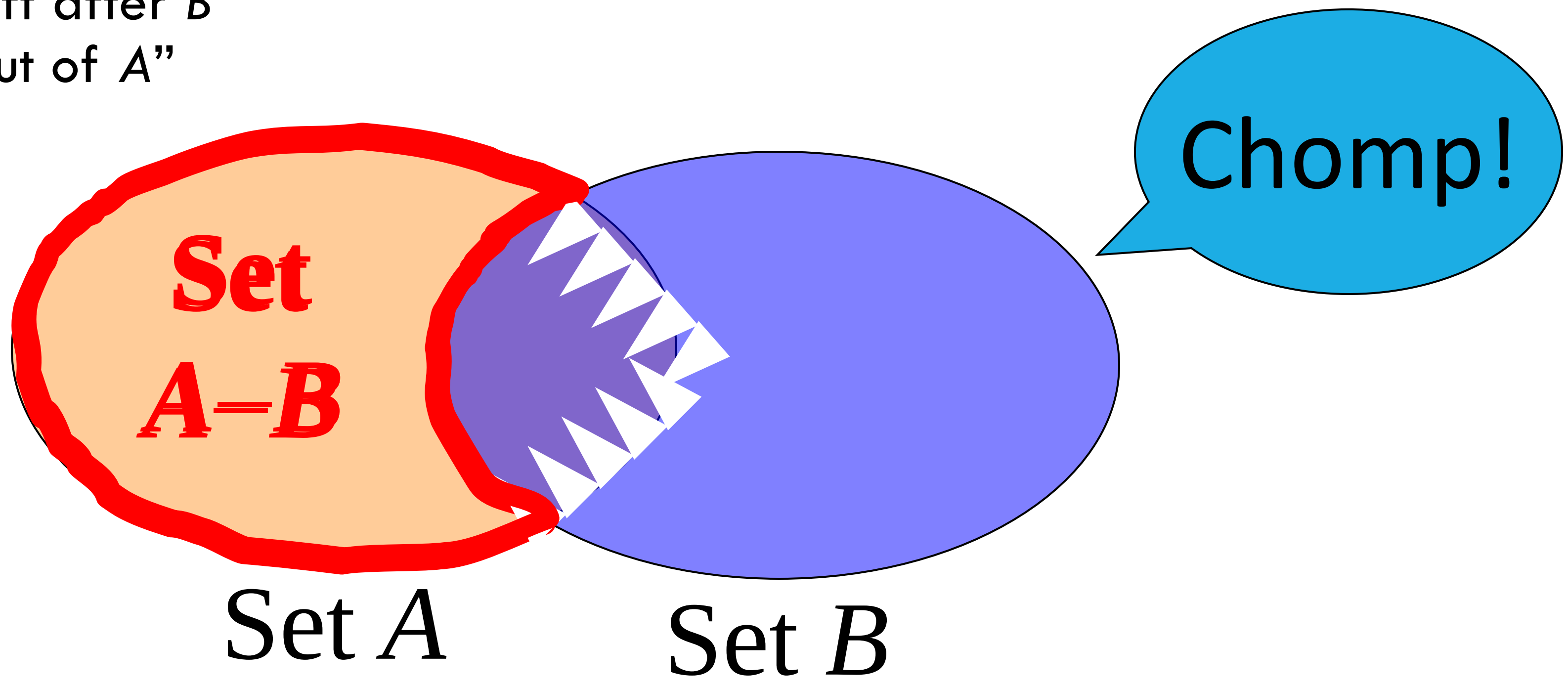
SET DIFFERENCE EXAMPLES

$\{1,2,3,4,5,6\} - \{2,3,5,7,9,11\} = \{1, 4, 6\}$

$\mathbf{Z} - \mathbf{N} = \{\dots, -1, 0, 1, 2, \dots\} - \{0, 1, \dots\}$
$= \{x \mid x \text{ is an integer but not a nat. \#}\}$
$= \{x \mid x \text{ is a negative integer}\}$
$= \{\dots, -3, -2, -1\}$

SET DIFFERENCE - VENN DIAGRAM

A-*B* is what's left after *B* "takes a bite out of *A*"



PROPERTIES OF SET DIFFERENCE

$\bar{A} = U - A$

$A - A = \emptyset$

$A - \bar{A} = A \; and \; \overline{A} - A = \bar{A}$

$A - B = A \cap \bar{B}$

$A - B = B - A \; iff \; A = B$

$A - B = A \;\; iff \; A \cap B = \emptyset$

$A - B = \emptyset$ iff $A \subseteq B$

SYMMETRIC DIFFERENCE

The symmetric difference of two sets A and B, denoted by

$A \oplus B$, *is defined as*

$A \oplus B = \{x | x \in A - B \ \ or \ \ x \in B - A\} = (A - B) \cup (B - A)$

Properties:

1. $A \oplus A = \emptyset$ 2. $A \oplus \emptyset = A$ 3. $A \oplus U = \bar{A}$ 4. $A \oplus \bar{A} = U$

5. $A \oplus B = (A - B) \cup (B - A)$

EXAMPLES:

If $A = (a, b, e, d\}, \ B = \{a, e, f, g\}, then$ A $\oplus$ B=?

If $A = \{2,4,5,9\}, \ B = \{x \in \mathbb{Z}^{+} | x^{2} < 25\}, then$ A $\oplus$ B=?

If $A = \{\emptyset\}, B = \{a, \emptyset, \{\emptyset\}\}, then$ A $\oplus$ B=?

SET IDENTITIES

Identity: $A \cup \varnothing = A \quad A \cap U = A$

Domination: $A \cup U = U \quad A \cap \varnothing = \varnothing$

Idempotent: $A \cup A = A = A \cap A$

Double complement: $\bar{\bar{A}} = A$

Commutative: $A \cup B = B \cup A \quad A \cap B = B \cap A$

Associative: $A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

DE MORGAN'S LAW FOR SETS

Exactly analogous to (and derivable from) DeMorgan's Law for propositions.

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

PROVING SET IDENTITIES

To prove statements about sets, of the form $E_1 = E_2$ (where Es are set expressions), here are three useful techniques:

Prove $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ separately.

Use logical equivalences.

Use a *membership table*.

METHOD 1: MUTUAL SUBSETS

Example: Show $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$.

Show $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.

- Assume $x\in A\cap(B\cup C)$, & show $x\in(A\cap B)\cup(A\cap C)$.
- We know that $x\in A$, and either $x\in B$ or $x\in C$.
  - Case 1: $x\in B$. Then $x\in A\cap B$, so $x\in(A\cap B)\cup(A\cap C)$.
  - Case 2: $x\in C$. Then $x\in A\cap C$, so $x\in(A\cap B)\cup(A\cap C)$.
- Therefore, $x\in(A\cap B)\cup(A\cap C)$.
- Therefore, $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.

Show $(A\cap B)\cup(A\cap C) \subseteq A\cap(B\cup C)$. …

METHOD 3: MEMBERSHIP TABLES

Just like truth tables for propositional logic.

Columns for different set expressions.

Rows for all combinations of memberships in constituent sets.

Use “1” to indicate membership in the derived set, “0” for non-membership.

Prove equivalence with identical columns.

MEMBERSHIP TABLE EXAMPLE

Prove $(A \cup B) - B = A - B$.

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
0	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

MEMBERSHIP TABLE EXERCISE

Prove $(A \cup B) - C = (A - C) \cup (B - C)$.

A	B	C	$A \cup B$	$(A \cup B) - C$	$A - C$	$B - C$	$(A - C) \cup (B - C)$
0	0	0	0	0	0	0	0
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					
1	1	0					
1	1	1					

POWER SET

Power Set: The collection of all subsets of a set A is called power set of A and is denoted by 2^A $or\ P(A)$.

Cardinality of $= 2^{|A|} = 2^n\ \ where\ n\ is\ number\ of\ elemnts\ in\ set\ A$.

Example:

1. Find the power set of $A = \{a, b, c\}$.

$\rightarrow P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

2. Find the power set of $A = \{a, \{b\}\}$.

$\rightarrow P(A) = \{\emptyset, \{a\}, \{\{b\}\}, \{a, \{b\}\}\}$

EXAMPLE

1. If $A = \{\emptyset, a\}, then\ constuct\ the\ sets\ A \cup P(A)\ and\ A \cap P(A)$.

$\rightarrow P(A) = \big\{\emptyset, \{\emptyset\}, \{a\}, \{\emptyset, a\}\big\}$

$A \cup P(A) = \big\{\emptyset, a, \{\emptyset\}, \{a\}, \{\emptyset, a\}\big\}$

$A \cap P(A) = \{\emptyset\}$.

2. Let $A = \{\emptyset\},\ B = P\big(P(A)\big)$

a. Is $\emptyset \in B?\quad Is\ \emptyset \subseteq B?$

b. Is $\{\emptyset\} \in B?\quad Is\ \{\emptyset\} \subseteq B?$

c. Is $\{\{\emptyset\}\} \in B?\quad Is\ \big\{\{\emptyset\}\big\} \subseteq B?$

EXAMPLE

3. Let A and B are any two arbitrary sets.

a. Show that $P(A \cap B) = P(A) \cap P(B)$ *or give a counter example*.
b. Show that $P(A \cup B) = P(A) \cup P(B)$ *or give a counter example*.

EXERCISE

1. Determine whether the following statements are true or false. Justify your answer.
   a. $\{a, \emptyset\} \in \{a, \{a, \emptyset\}\}$
   b. $\{a, b\} \in \{a, b, \{a, b\}\}$
   c. $\{a, b\} \subseteq \{a, b, \{a, b\}\}$
   d. $\{a, c\} \in \{a, b, c, \{a, b, c\}\}$

EXERCISE

2. If $\mathrm{A} = \{a, b, \{a, c\}, \emptyset\}$, determine the following sets

a. $A - \{a, c\}$

b. $\{\{a, c\}\} - A$

c. $A - \{a, b\}$

d. $\{a, c\} - A$

EXERCISE

3. If $U = \{n|n \in \mathbb{N}, n \leq 20\}$, $A = \{n|n \in \mathbb{N}, 4 < n \leq 15\}$,

$\mathrm{B} = \{n|n \in \mathbb{N}, 10 < 17\}$, $\mathrm{C} = \{n|n \in \mathbb{N}, 7 < n < 16\}$.

Find $\bar{A} - \bar{B}$ and $\overline{C} - \bar{A}$.

EXERCISE

4.

Let A, B, and C be the subsets of universal set U. given that $A \cap B = A \cap C$ and $\bar{A} \cap B = \bar{A} \cap C, is\ it\ necessary\ that\ B = C?\ \ Justify\ your\ answer.$

$\Longrightarrow B = B \cap U = B \cap (A \cup \bar{A}) = (B \cap A) \cup (B \cup \bar{A}\ )$

$= (C \cap A) \cup (C \cup \bar{A}\ ) = C \cap (A \cup \bar{A}) = C$

EXERCISE

5. (i) given that $A \cap B = A \cap C, is\ it\ necessary\ that\ B = C?$
$Justify\ your\ answer.$
$\Longrightarrow Let\ \ A = \{a, b, c\}, B = \{c\}, C = \{c, d\}$

(ii) given that $A \cup B = A \cup C, is\ it\ necessary\ that\ B = C?$
$Justify\ your\ answer.$

EXERCISE

6. Using rules of set operations, simplify the following

a. $(\overline{A \cup B}) \cup (\bar{A} \cap B)$

b. $[(A \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap B)] \cap B$

c. $((A \cup B) \cap \bar{A}) \cup \overline{(B \cap A)}$

d. $\overline{[(A \cap B) \cup C]} \cap \bar{B}$