RELATIONS

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CARTESIAN PRODUCT

Let A and B be any two sets, then the Cartesian product of A and B denoted by

$A \times B \; is \; defined \; as$

$A \times B = \{(a, b)|a \in A \;\; and \;\; b \in B\}$

Example: Let $A = \{1,2,3\},\;\; B = \{a, b\}, then$

$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

$B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)$

Note: In general $A \times B \neq B \times A$

PROPERTIES OF CARTESIAN PRODUCT

For any three sets A, B, and C

1. $(A \times B) \times C \neq A \times (B \times C)$

Note:

As $(A \times B) \times C = \left\{\left((a,b),c\right) \middle| (a,b) \in A \times B, and\ c \in C\right\}$

$A \times (B \times C) = \left\{\left(a,(b,c)\right) \middle| a \in A, (b,c) \in B \times C\right\}$

$A \times B \times C = \{(a,b,c)|$a ∈A, b∈B, c∈C}.

2. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

3. $A \times (B \cap C) = (A \times B) \cap (A \times C)$

EXERCISE

1. If $A \times B = \{(6,2),(2,1),(6,1),(3,5),(6,4),(6,5),(2,2),(2,4),(2,5),\}$

Find A and B.

2.Determine x and y from the following

(a) $(5x, y+2) = (3x-5, 2y+7)$

(b) $(y^2, 4) = (2y-1, x-3)$

RELATION

Relationships between elements of sets occur very often.

- (Employee, Salary)
- (Students, Courses, GPA)

We use ordered pairs (or *n-tuples*) of elements from the sets to represent relationships.

BINARY RELATIONS

A *binary relation R* from set *A* to set *B*, written $R{:}A \leftrightarrow B$, is a subset of $A \times B$.

- *E.g.*, let $\mathbf{<} : \mathbf{N} \leftrightarrow \mathbf{N} :\equiv \{(n, m) \mid n < m\}$

The notation *aRb* means $(a, b) \in R$.

- *E.g.*, $a \mathbf{<} b$ means $(a, b) \in \mathbf{<}$

If *aRb* we may say

"*a* is related to *b* (by relation *R*)"

EXAMPLE

A: {students at VIT}, B: {courses offered at VIT}

R: “relation of students enrolled in courses”

(Pranav, CS365), (Shruti, CS201) are in R

If Mary does not take CS365, then (Mary, CS365) is not in R!

If CS480 is not being offered, then (Jason, CS480), (Mary, CS480) are not in R!

DOMAIN AND RANGE OF A RELATION

- Let S be a binary relation. The domain of the relation S

$=D(S) = \{x \mid (\exists y)((x, y) \in S)\}$.

- The range of the relation S $=\mathrm{R}(S) = \{y \mid (\exists x)((x, y) \in S)\}$.

Example: Consider $S = \{(1, a), (2, b), (c, 2)\}$.

$Find\ domain\ and\ range\ of\ relation\ S.$

$\rightarrow D(S) = \{1,2, c)\ \ and\ \ R(S) = \{a, b, 2)$

RELATION MATRIX

Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$ be finite sets containing m and n elements. Let R be the relation from set A to set B, then R can be represented by a $m \times n$ matrix $M_R = [r_{ij}]$ $defined\ as$

$$r_{ij} = \begin{cases} 1, & if\ (a_i, b_j) \in R \\ 0, & if (a_i, b_j) \notin R \end{cases}$$

EXAMPLE

Let $A = \{1,2,3,4\}$ and $B = \{a, b, c\}$.

$Let\ R\ be\ relation\ from\ set\ A\ to\ set\ B\ given\ by$

$R = \{(1, b), (2, a), (2, c), (3, c), (4, b), (4, c)\}$.

$Find\ the\ matrix\ associated\ with\ this\ relation$

$$\rightarrow M_R = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

EXAMPLE

Let $A = \{1,2,3,4\}. \mathit{Find\ the\ relation\ R\ determined\ by\ the\ matrix}$

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$\rightarrow R = \{(1,2), (1,4), (2,1), (2,3), (2,4), (3,3), (4,2), (4,3)\}$

JOIN AND MEET OF THE ZERO-ONE MATRIX

Example: Find join and meet of $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, B= $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

BOOLEAN PRODUCT OF MATRICES

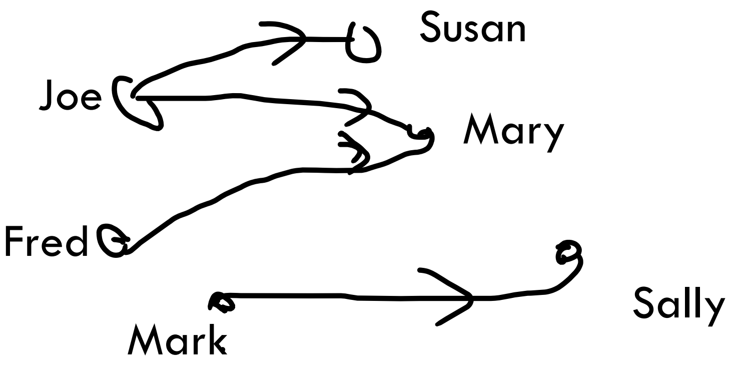
Let $A = [a_{ij}]$ be an $m \times k$ zero one matrix and B= $[b_{ij}]$ be an k× n zero one matrix. Then Boolean product of $A\ and\ B, denoted\ by\ A \odot B$, is an m× n matrix with ijth entry c_{ij} is given by

$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \dots \vee (a_{ik} \wedge b_{kj})$

GRAPH OF A RELATION

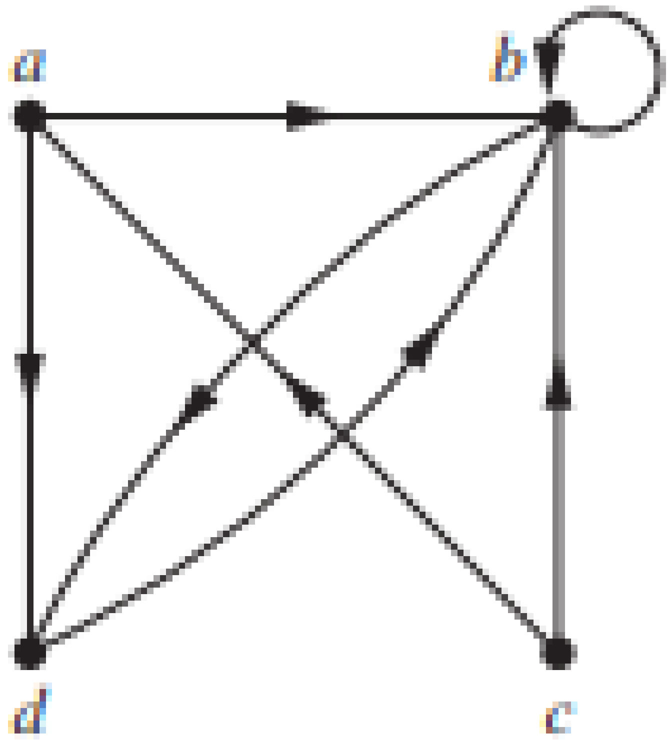
A *directed graph* or *digraph* $G=(V_G,E_G)$ is a set V_G of *vertices (nodes)* with a set $E_G \subseteq V_G \times V_G$ of *edges (arcs, links)*. Visually represented using dots for nodes, and arrows for edges. Notice that a relation $R{:}A \leftrightarrow B$ can be represented as a graph $G_R=(V_G=A \cup B,\ E_G=R)$.

$$\begin{array}{r c} & \begin{array}{ccc} \text{Susan} & \text{Mary} & \text{Sally} \end{array} \\ \begin{array}{r} \text{Joe} \\ \text{Fred} \\ \text{Mark} \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$



EXAMPLE-GRAPH OF A RELATION

Graph of a relation with vertices a, b, c and d and edges (a, b), (a, d), (b, b), (b, d), (c, a), (



COMPLEMENTARY RELATIONS

Let $R{:}A \leftrightarrow B$ be any binary relation.

Then, $R{:}A \leftrightarrow B$, the *complement* of R, is the binary relation defined by

$$\bar{R} :\equiv \{(a, b) \mid (a, b) \notin R\} = (A \times B) - R$$

Note this is just R if the universe of discourse is $U = A \times B$; thus the name *complement*.

Note the complement of $\bar{R}$ is R.

Example: Complement of $< \; = \{(a, b) \mid (a, b) \notin \, <\}$

$$= \{(a, b) \mid \neg a < b\} = \; \geq$$

INVERSE RELATIONS

Any binary relation $R{:}A\leftrightarrow B$ has an *inverse* relation $R^{-1}{:}B\leftrightarrow A$, defined by

$$R^{-1} :\equiv \{(b, a) \mid (a, b)\in R\}.$$

E.g., $<^{-1} = \{(b, a) \mid a<b\} = \{(b, a) \mid b>a\} = >$.

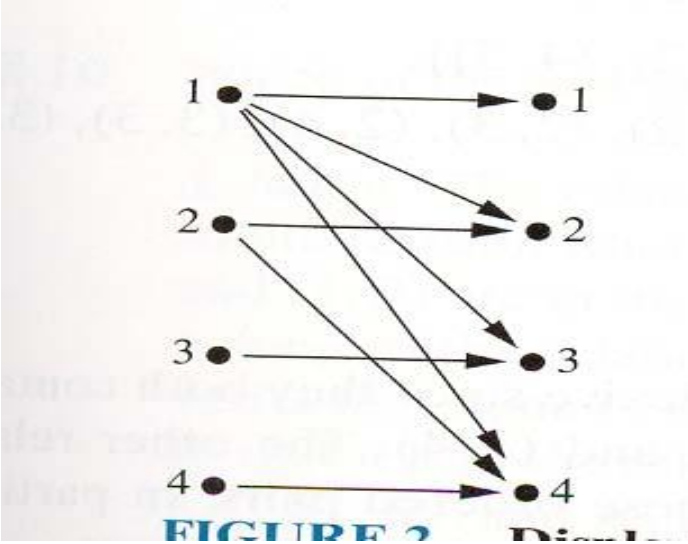
Note: Inverse relation is also called as converse relation

FUNCTIONS AS RELATIONS

A function f:A→B is a relation from A to B

A relation from A to B is not always a function f:A→B (e.g., relations could be one-to-many)

Relations are generalizations of functions!



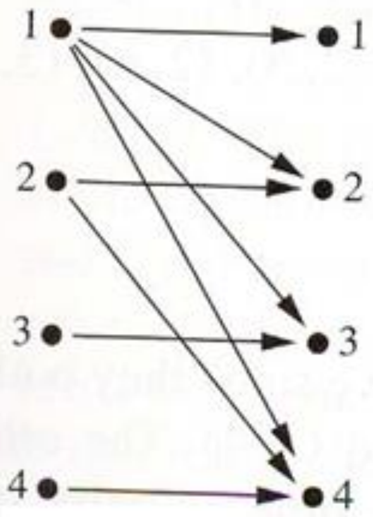
RELATIONS ON A SET

A (binary) relation from a set *A* to itself is called a relation on the set *A*.

For Example:

A: {1,2,3,4}

R={(a, b) | a divides b}



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Displaying the Ordered Pairs in the Relation *R* from Example

EXAMPLE

How many relations are there on a set A with n elements?

→ The maximum number of elements in a binary relation on a set A with n elements = Number of elements in $A \times A = n^2$

Each element has two choices, either to appear on a binary relation or doesn't appear on a binary relation.

∴ Number of binary relations $= 2^{n^2}$

OPERATIONS ON RELATION

If R and S are any two relations, then $R \cup S\ defines\ a\ relation\ such\ that$

$\mathrm{x}(R \cup S)y \Leftrightarrow xRy\ or\ xSy$

Similarly

$\mathrm{x}(R \cap S)y \Leftrightarrow xRy\ and\ xSy$

$\mathrm{x}(R - S)y \Leftrightarrow (x, y) \in R\ and\ (x, y) \notin S$

COMPOSITION OF RELATION

Let $R{:}A \leftrightarrow B$, and $S{:}B \leftrightarrow C$. Then the *composite* $S \circ R$ of R and S is defined as:

$$S \circ R = \{(a, c) \mid aRb \wedge bSc\}$$

Function composition $f \circ g$ is an example.

The n^{th} power R^n of a relation R on a set A can be defined recursively by:

$$R^1 :\equiv \mathrm{R}\,; \qquad R^{n+1} :\equiv R^n \circ R \quad \text{for all } n \geq 0.$$

EXAMPLE

Define $R = \{(1,2),(3,4),(2,2)\}$ *and* $S = \{(4,2),(3,5),(3,1),(1,3)\}$

1. Find $R \circ S, S \circ R$.

2. Obtain relation Matrices for $R \circ S, S \circ R$.

Solution: $R \circ S = \{(4,2),(3,2),(1,4)\}$

$S \circ R = \{(3,2)\}$

$$\mathrm{M_R} \circ M_s = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} & & 1 & & \\ & & & & \\ 1 & & & & 1 \\ & 1 & & & \\ & & & & \\ & & & & \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad & \quad \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

THEOREMS ON COMPOSITION

Theorem 1: Let
R_1, R_2 *and* R_3 *be relations from* A *to* B, B *to* C *and* C *to* D *respectively*.
Then

$$R_1(R_2R_3) = (R_1R_2)R_3.$$

Theorem 2: Let
R_1, R_2 *be relations from* A *to* B, B *to* C *respectively, then*

$$(R_1R_2)^{-1} = R_2^{-1}\, R_1^{-1}$$

EXAMPLE

Let $A = \{1,2,3,4\}, B = \{a, b, c, d\}, and\ C = \{x, y, z\}$.

$Let\ R = \{(1, a), (2, d),\ \ (3, a), (3, b), (3, d)\}\ \ and$

$S = \{(b, x), (b, z), (c, y), (d, z)\}$.

Find $S \circ R$.

Solution:$S \circ R = \{(2, z), (3, x), (3, z)\}$.

EXERCISE

1. Let R be the relation from $X = \{2,3,4,5\}$ and $Y = (3,6,7,10\}$ $defind\ by$ $"\text{x divides y}". Then$

(a) write R as Set of order pairs.

(b) Find the Range and domain of R.

2. $If\ R = \{(x, y)|x, y \in N, x + 3y = 12\}$

(a) write R as Set of order pairs.

(b) Find the Range and domain of R.

3. Let $X = \{1,2,3,4\}. Given$

$R = \{(x, y)|x, y \in X\ and\ (x - y) is\ non\ zero\ multiple\ of\ 2\}$

$S = \{(x, y)|x, y \in X\ and\ (x - y) is\ non\ zero\ multiple\ of\ 3\}$

Find $R \cup S\ \ and\ \ R \cap S.$

PROPERTIES OF BINARY RELATION ON A SET REFLEXIVE RELATION

A relation R on A is *reflexive* if $\forall a \in A$, aRa or $(a, a) \in R$.

Examples:

- *E.g.*, the relation $\geq :\equiv \{(a, b) \mid a \geq b\}$ is reflexive.
- The relation of inclusion is reflexive in the family of all subsets of a universal set.
- Is the "divide" relation on the set of positive integers reflexive?

EXAMPLES

1. Consider the following five relations on the set $A = \{1,2,3,4\}, which\ of\ the\ following\ are\ reflexive\ relations$

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4), (4,1), (3,2)\}$

$R_3 = \{(1,4), (2,1), (2,3)\}$

$R_4 = \emptyset$

$R_5 = A \times A$

EXAMPLE

2. Determine which of the relation reflexive.

1. Relation “$\leq$ " on the set of integers
2. Set inclusion " $\subseteq$” on a collection of sets
3. Relation “perpendicular” on the set of lines in the plane
4. Relation “parallel” on the set of lines in a plane
5. Relation | divisibility on the set of positive integer.

IRREFLEXIVE RELATION

A relation R on A is *irreflexive* if $\forall a \in A$, $(a, a) \notin R$.

Note: A relation which is not reflexive is not necessarily irreflexive

Examples:

- A relation $<$ is irreflexive
- Let $A = \{a, b, c\}$, Let $R = \{(a, a), (a, b), (a, c)\}$ is neither reflexive nor irreflexive.

SYMMETRIC RELATION

A binary relation R on A is symmetric if and only if $R = R^{-1}$, that is, if

$(a, b)\in R \leftrightarrow (b, a)\in R$.

Examples:

- *E.g.*, **=** (equality) is symmetric.
- **<** is not symmetric relation.
- "is married to" is symmetric.
- "Similarity of triangle" in a plane is symmetric relation
- The relation of "being a brother" in in a set of people is not symmetric
- The relation of "being a brother" in in a set of males is symmetric
- "likes" is not a symmetric.

ASYMMETRIC RELATION

A binary relation *R* on *A* is asymmetric if and only if *whenever* $(a, b) \in R, then\ (b, a) \notin R$, that is, if

$(a, b) \in R \rightarrow (b, a) \notin R$.

Example: Let < be relation defined on set of real numbers is asymmetric.

Example: Let $A = \{2,3,4\}$ and let R be relation " is a divisor of" defined by $R = \{(2,2), (2,4), (3,3), (4,4)\}$

R is not asymmetric relation

ANTISYMMETRIC RELATION

A binary relation R is *antisymmetric* if $(a, b) \in R \rightarrow (b, a) \notin R$.

Examples:

- $<$ is antisymmetric relation,
- “likes” is not.

TRANSITIVE RELATION

A relation R is *transitive* if and only if (for all a, b, c)

$$(a, b) \in R \land (b, c) \in R \rightarrow (a, c) \in R.$$

Examples:

- "is an ancestor of" is transitive.
- The relations " $<, \leq, >, \geq, =$ " $are\ transitive$ relations
- The relations " $\subseteq, \subset, \supseteq, \supset$ " $are\ transitive$ relations
- "Similarity of triangle" in a plane is transitive relation
- Is the "divides" relation on the set of positive integers transitive?

EXERCISE

1. Give an example of a relation that is both symmetric and antisymmetric.
2. Give an example of a relation that is both irreflexive and transitive.
3. Let $S = \{1,2, \ldots, 10\}, and\ a\ relation\ R\ on\ S\ is\ defined\ as$

   $R = \{(x, y) | x + y = 11\}$ describe the properties of R.

Note: The property of transitivity can be expressed In terms of the composition of relations. For a given relation R on A we define

$$R \circ R = R^2 \ \ and\ R^{n-1} \circ R = R^n$$

Theorem: A relation R on a set is transitive if and only if $R^n \subseteq R \ \ for\ n \geq 1$.

RELATION MATRIX OPERATIONS

Matrix for composition of relation:

Let M_R and M_S be the matrices of the relations R and S respectively.

Then Matrix for the relation $R \circ S$ is $M_R M_S$.

PROPERTIES OF RELATION ON A SET

1. A relation is reflexive if and only if all diagonal entries in relation matrix is 1
2. A relation is symmetric if and only if relation matrix M_R is symmetric.

   i.e. $M_R = M_R^T$.

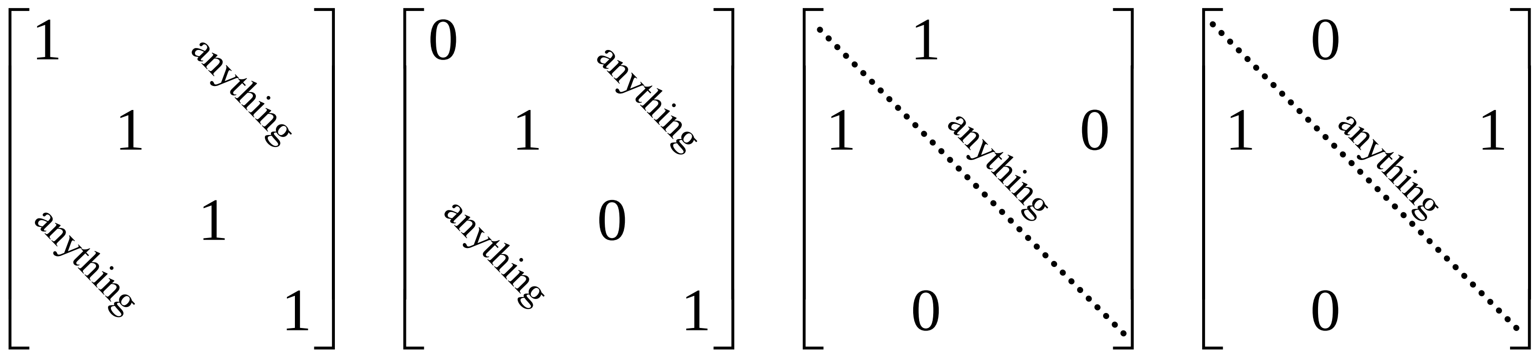
3. If a Relation is antisymmetric, then its matrix M_R is such that

$$if\ \ r_{ij} = 1\ then\ r_{ji} = 0\ .$$

ZERO-ONE REFLEXIVE, SYMMETRIC

Terms: *Reflexive, non-Reflexive, symmetric, and antisymmetric.*

- These relation characteristics are very easy to recognize by inspection of the zero-one matrix.



Reflexive: all 1's on diagonal

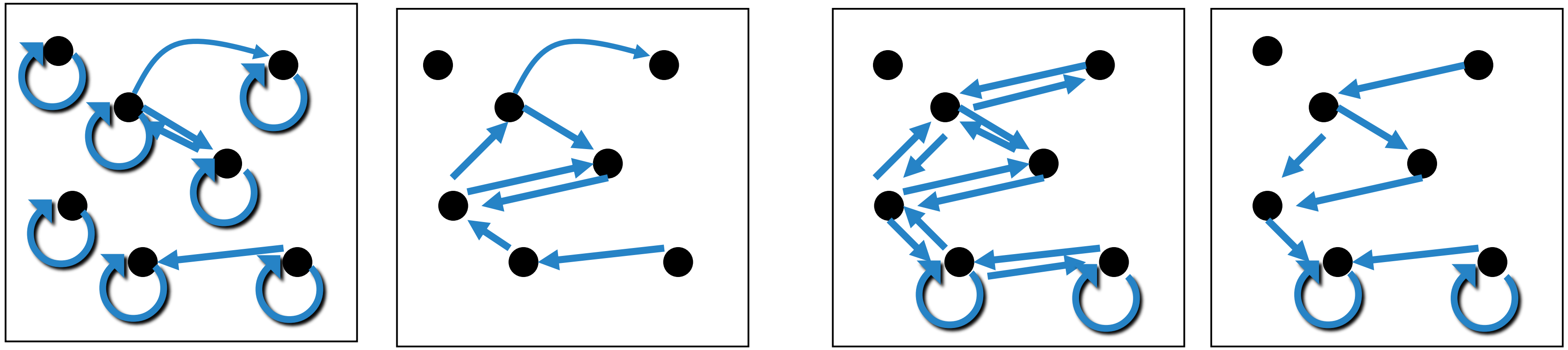
Non-reflexive: some 0's on diagonal

Symmetric: all identical across diagonal

Antisymmetric: all 1's are across from 0's

DIGRAPH REFLEXIVE, SYMMETRIC

It is extremely easy to recognize the reflexive/irreflexive/symmetric/antisymmetric properties by graph inspection.



Reflexive: Every node has a self-loop	Irreflexive: No node links to itself	Symmetric: Every link is bidirectional	Antisymmetric: No link is bidirectional

Asymmetric, non-antisymmetric

Non-reflexive, non-irreflexive

EXERCISE

1. Let $A = \{1,2,3,4\} and\ B = \{a, b, c\}. Let\ R\ be\ relation\ from\ set\ A\ to\ Set\ B$:

   $R = \{(1, a), (1, c), (3, b), (4, a), (4, c)\}$

a. Find the matrix of the relation

b. Draw the graph of a relation

c. Find the inverse relation

d. Determine domain and range of a relation

EXERCISE

2. Let $A = \{1,2,3,4,6\}, and\ Let\ R\ be\ the\ relation\ on\ A\ defined\ by$ x devides y.

a. Draw R as a set of order pairs

b. Draw its directed graph

c. Find the inverse relation R^{-1} of R. Can R^{-1} be describe in words.

EXERCISE

3. Each of the following defines a relation on the positive integer N

a. " x is greater than y"
b. "xy is a square of a integer"
c. $x+y=10$
d. $X+4y=10$

Determine which relations are (a) reflexive (b) symmetric (c) antisymmetric (d) transitive.

CLOSURES OF RELATIONS

For any property X, the "X closure" of a set A is defined as the "smallest" superset of A that has the given property.

The ***reflexive closure*** of a relation R on A is obtained by adding (a, a) to R for each $a \in A$. *i.e.*, it is $R \cup I_A$

The ***symmetric closure*** of R is obtained by adding (b, a) to R for each (a, b) in R. *I.e.*, it is $R \cup R^{-1}$

The ***transitive closure*** or *connectivity relation* of *R* is obtained by repeatedly adding (*a*, c) to *R* for each (*a*, *b*), (*b*, c) in *R*.

$$R^* = \bigcup_{n \in \mathbf{Z}^+} R^n$$

EXAMPLES

1. Let $R = \{(1,1),(1,3),(2,4),(3,1),(3,3),(4,3)\}\ be\ a\ relation\ on$

$A = \{1,2,3,4\}$.

Find reflexive closure and symmetric closure of R.

$\rightarrow\ Reflexive(R) = R \cup I_A = \{(1,1),(1,3),(2,4),(3,1),(3,3),(4,3),(2,2),(4,4)\}$

$R^{-1} = \{(4,2),(3,4)\}$

$Symmetric(R) = R \cup R^{-1}$

2. Consider the following relation $R = \{(1,2),(2,3),(3,3)\}\ on\ A = \{1,2,3\}$

$\rightarrow R^2 = R \circ R = \{(1,3),(2,3),(3,3)\}$

$R^3 = R^2 \circ R = \{(1,3),(2,3),(3,3)\}$

Transitive(R) $= R \cup R^2 \cup R^3 = \{(1,2),(2,3),(3,3),(1,3)\}$

3. consider the relation $R = \{(a,a),(a,b),(b,c),(c,c)\}\ on\ set\ A = (a,b,c\}$

Find (a) Reflexive(R) (b) Symmetric(R) (c) Transitive(R)

EXERCISE

1. Consider the relations on the set of integers
   a. $R_1 = \{(a, b)|a \leq b\}$
   b. $R_2 = \{(a, b)|a > b\}$
   c. $R_3 = \{(a, b)|a = |b|\}$
   d. $R_4 = \{(a, b)|a = b\}$
   e. $R_5 = \{(a, b)|a = b + 1\}$
   f. $R_6 = \{(a, b)|a + b \leq 3\}$
   - Which of these relation contain each of the pair $(1,1), (1,2), (2,1), (1, -1), and\ (2,2)$

- Which of the relations are reflexive
- Which of the relations are symmetric
- Which of the relations are antisymmetric
- Which of the relations are transitive

EXAMPLE 2

How many relations are their on a set with n elements that are

- reflexive: $2^{n(n-1)}$
- Symmetric: $2^{\frac{n(n+1)}{2}}$
- Antisymmetric: $2^n 3^{\frac{n(n-1)}{2}}$
- Asymmetric: $3^{\frac{n(n-1)}{2}}$
- Irrreflexive: $2^{n(n-1)}$
- Reflexive and symmetric: $2^{\frac{n(n-1)}{2}}$
- Neither reflexive nor symmetric: $n^2 - 2 \cdot 2^{n(n-1)}$

EXAMPLE 3

Is divides relation on the set of positive integer

Reflexive?

Symmetric?

Antisymmetric?

Transitive ?

EXAMPLE 4

Let R_1 and R_2 be the "divides" and "is a multiple of" relations on the set of positive integers. That is $R_1 = \{(a, b)|\ a|b\ \}\ and\ R_2 = \{(a, b)|a = kb, k \in \mathbb{Z}\}$.

Find

1. $R_1 \cup R_2$
2. $R_1 \cap R_2$
3. $R_1 - R_2$
4. $R_2 - R_1$
5. $R_1 \oplus R_2$

EXAMPLE 5

Let R_1 and R_2 be the "congruent modulo 3" and "congruent modulo 4" relations on the set of positive integers. That is

$$R_1 = \{(a, b) |\ a \equiv b(mod3)\}\ and\ R_2 = \{(a, b)| a \equiv b(mod4)).$$

Find

1. $R_1 \cup R_2$
2. $R_1 \cap R_2$
3. $R_1 - R_2$
4. $R_2 - R_1$
5. $R_1 \oplus R_2$