MATHEMATICAL INDUCTION

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MATHEMATICAL INDUCTION

It is a powerful technique for establishing many properties of natural numbers.

PRINCIPLE OF MATHEMATICAL INDUCTION

Let P(n) be a statement involving a natural number n.

1. If P(n) is true for $n = n_0$

2. Assuming P(k) is true for $k \geq n_0$ we prove P(k+1) is also true,

Then P(n) is true for all natural number $n \geq n_0$.

Step (1) is called basis of induction

Step (2) is called induction step. The assumption that P(n) is true for all n=k is called the induction hypothesis.

EXAMPLES

1. Prove that $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$

Solution: Let P(n): $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$

For $n=1, P(1)$: $\frac{1}{1.4}=\frac{1}{4}$.

Hence $P(1)$ is true

To prove $P(K) \Longrightarrow P(k+1)$

Consider $\mathrm{P(k+1)}$: $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots+\frac{1}{(3k-2)(3k+1)}+$

$$\frac{1}{(3k+1)(3k+4)}=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{k(3k+4)+1}{(3k+1)(3k+4}=\frac{\left(3k^2+4k+1\right)}{(3k+1)(3k+4)}=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}=\frac{k+1}{3(k+1)+1}$$

Hence assuming $\mathrm{P(k)}$ is true, $\mathrm{P(k+1)}$ is also true.

$\therefore P(n)\ \ is\ true\ for\ all\ n \geq 1$

EXAMPLE

Find a formula for $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{n(n+1)}$ and prove it

Solution: $\frac{1}{1.2}=\frac{1}{2}$

$$\frac{1}{1.2}+\frac{1}{2.3}=\frac{4}{2.3}=\frac{2}{3}$$

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}=\frac{9}{3.4}=\frac{3}{4}$$

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Let $\mathrm{P(n)}$: $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

For $n=1$: $P(1)$: $\frac{1}{1.2}=\frac{1}{2}$ is true

Let $\mathrm{P(n)}$ is true for some positive integer k.

To prove $\mathrm{P(k)} \rightarrow \mathrm{P(k+1)}$

Consider $\mathrm{P(k+1)}$: $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

Hence the result

EXAMPLE

Formulate and prove by induction, a general formula stemming from the observation

$1^3 = 1$

$2^3 = 3 + 5$

$3^3 = 7 + 9 + 11$

$4^3 = 13 + 15 + 17 + 19$

Note that

$3 = 2.1 + 1$

$7 = 3.2 + 1$

$13 = 4.3 + 1 \ldots$

Therefore first term of n^3 is $n(n-1)+1$.

$$n^3 = [n(n-1)+1] + [n(n-1)+3] + \cdots . + [n(n-1)+2n-1]$$

$$= \sum_{i=1}^{n}\{ n(n-1) + (2i-1)\}$$

Let $P(n)$: $n^3 = \sum_{i=1}^{n}\{ n(n-1) + (2i-1)\}$

Let us verify the formula

$for\ n = 1{:}\ \ 1^3 = 1(1-1) + 2 - 1 = 1$

$for\ n = 2{:}\ \ 2^3 = [2(2-1) + 2 - 1] + [2(2-1) + 4 - 1] = 3 + 5$

Now assume the result for $n = k$ and prove it for $n = k + 1$

$P(k)$: $k^3 = \sum_{i=1}^{k}\{k(k-1) + (2i-1)\}$

Consider

$\sum_{i=1}^{k+1}\{(k+1)k + (2i-1)\} = \sum_{i=1}^{k}\{k(k+1) + (2i-1)\} + k(k+1) + 2(k+1) - 1$

$= \sum_{i=1}^{k}\{k(k-1) + (2i-1)\} + \sum_{i=1}^{k} 2k + k(k+1) + 2(k+1) - 1$

$= k^3 + 3k^2 + 3k + 1$

$= (k+1)^3$

Hence the result

EXAMPLE

Use Mathematical induction to prove that $n^3 + 2n$ is divisible by 3.

Solution: Let $P(n)$: $n^3 + 2n$ is divisible by 3

Consider $P(1)$: $1^3 + 2 = 3$ which is divisible by 3.

Assume $P(k)$ is true

To prove $P(k) \rightarrow P(k+1)$

Consider $(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + 3(k^2 + k + 1)$

Each of the term $(k^3 + 2k)$ and $3(k^2 + k + 1)$ is divisible by 3, hence

$(k+1)^3 + 2(k+1)$ is divisible by 3.

Hence the result

EXAMPLE

Use Mathematical induction to prove that $3^n + 7^n - 2$ is divisible by 8 for $n \geq 1$.

Solution: Let $P(n): 3^n + 7^n - 2$ is divisible by 8 for $n \geq 1$.

Consider $P(1): 3^1 + 7^1 - 2 = 8$ which is divisible by 8.

Assume $P(k)$ is true

To prove $P(k) \rightarrow P(k+1)$

Consider $3^{k+1} + 7^{k+1} - 2 = 33^k + 77^k - 2 = 3\left(3^k + 7^k - 2\right) + 4(7^k + 1)$

Each of the term $3^n + 7^n - 2$ and $4(7^k + 1)$ is divisible by 8, as $(7^k + 1)$ is even

$3^{k+1} + 7^{k+1} - 2$ is divisible by 8.

Hence the result

EXAMPLE

Show that for any integer $n > 1$, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$

EXAMPLE

Let n be a positive integer. Show that $2^n \times 2^n$ chessboard with one square removed can be covered using L-shaped pieces, where each piece covers three squares at a time.

EXAMPLE: SOLITAIRE GAME PROBLEM

For every integer i, there is an unlimited supply of balls mark with number i. initially a tray of balls is given and the balls are thrown one at a time. If a ball marked with i is thrown away it is replaced by any finite number of balls marked 1,2,3,… ,i-1. There is no replacement for a ball marked 1. game ends when the tray is empty. Show that the game is always terminates after finite number of moves.

PRINCIPLE OF STRONG MATHEMATICAL INDUCTION

To show that P(n) is true for all positive integers n, we must verify following two steps

(i) Basic step: The proposition P(1) is shown to be true.

(ii) Inductive step: $[P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$ is shown to be true for every positive integer k.

EXAMPLE

Use mathematical induction to prove that if n is an integer greater than 1, then n can be written as product of primes.

Solution: Let P(n): if n is an integer greater than 1, then n can be written as product of primes.

Basic Step: P(2) is true as $2 = 2 \times 1$

Inductive step: Let $P(1) \wedge P(2) \wedge \ldots \wedge P(k)$ be true for all positive integer k.

To prove: $P(k+1)$ is true

- if k+1 is prime, then $P(k+1)$ is true
- if k+1 is not a prime, then $k + 1 = m \times n, where\ 2 \leq m \leq n < k + 1$

Thus by our assumption m and n can be expressed as product of primes

$\therefore P(k+1)$ is true,

Hence by principle of Mathematical induction P(n) is true for all n >1.

EXAMPLE: JIGSAW PUZZLE

Show that for a Jigsaw puzzle with n pieces, it will always take n-1 moves to solve the problem.

EXAMPLE

Prove that a set with n elements has $\frac{n(n-1)}{2}$ subsets containing exactly 2 elements.

EXAMPLE

Which amount of money can be formed just using 2 Rs and 5 Rs notes. Prove your answer using strong induction.