5.7.4 Solution of Recurrence Relation Using Generating Function

We can solve recurrence relations using generating functions as illustrated in the following examples

Example 5.91 Use generating functions to solve the recurrence relation $a_n = 3a_{n-1} + 2, n \geq 1$ with $a_0 = 1$.

Solution Let the generating function of the sequence $\{a_n\}$ be $G(z) = \sum_{n=0}^{\infty} a_n z^n$

$$\therefore G(z) = \frac{1}{1-3z} + \frac{2}{1-z} \quad (5.48)$$

Given recurrence relation is $a_n = 3a_{n-1} + 2$. (5.49)

Multiplying both sides of Eq. (5.49) by z^n and summing over all $n \geq 1$, we have

$$\sum_{n\geq 1}^{\infty} a_n z^n = 3\sum_{n\geq 1} a_{n-1} z^n + 2\sum_{n\geq 1} z^n$$

$$= 3z\sum_{n\geq 1} a_{n-1} z^{n-1} + 2z\sum_{n\geq 1} z^{n-1}$$

$$\Rightarrow G(z) - a_0 = 3zG(z) + 2z[1 + z + z^2 + \cdots]$$

$$\Rightarrow (1-3z)G(z) = \frac{2z}{(1-z)} \quad [\text{since } a_0 = 1 \text{ and } [1 + z + z^2 + \cdots] = (1-z)^{-1}]$$

$$\Rightarrow (1-3z)G(z) = \frac{1+z}{1-z}$$

$$\Rightarrow G(z) = \frac{(1+z)}{(1-z)(1-3z)}$$

$$\text{Let } \frac{(1+z)}{(1-z)(1-3z)} = \frac{A}{(1-z)} + \frac{B}{(1-3z)}$$

Equating the numerators on both sides, we get

$$(1 + z) = A(1 - 3z) + B(1 - z)$$

$\Rightarrow A = -1$

and $B = 2$

$$\therefore G(z) = \frac{2}{(1-3z)} - \frac{1}{(1-z)}$$

Therefore, the required solution is $a_n = 2(3^n) - 1$.

Example 5.92 Use the method of generating function to solve the following recurrence relation:

$$a_n - 2a_{n-1} - 3a_{n-2} = 0, n \geq 2 \text{ with } a_0 = 3\ a_1 = 1$$

Solution Let the generating function of the sequence $\{a_n\}$ be $G(z)$.

$$\therefore G(z) = \sum_{n=0}^{\infty} a_n z^n \quad (5.50)$$

The given recurrence relation is $a_n - 2a_{n-1} - 3a_{n-2} = 0$.

Multiplying both sides of Eq. (5.51) by z^n and adding the sum over all $n \geq 2$, we get

$$\sum_{n\geq 2} a_n z^n - 2\sum_{n\geq 2} a_{n-1} z^n - 3\sum_{n\geq 2} a_{n-2} z^n = 0$$

$$\Rightarrow \sum_{n\geq 2} a_n z^n - 2z\sum_{n\geq 2} a_{n-1} z^{n-1} - 3z^2 \sum_{n\geq 2} a_{n-2} z^{n-2} = 0$$

$$\Rightarrow [G(z) - a_0 - a_1 z] - 2z\,[G(z) - a_0] - 3z^2\, G(z) = 0$$

$$\Rightarrow (1 - 2z - 3z^2)\, G(z) - 3 - z - 2z(-3) = 0 \qquad \text{[given that } a_0 = 3, a_1 =$$

$$\Rightarrow (1 - 2z - 3z^2)\, G(z) = 3 - 5z$$

$$\Rightarrow G(z) = \frac{(3-5z)}{(1-2z-3z^2)}$$

$$= \frac{(3-5z)}{(1-3z)(1+3z)}$$

Let $\dfrac{(3-5z)}{(1-3z)(1+z)} = \dfrac{A}{(1-3z)} + \dfrac{B}{1+z}$

Equating the numerators on both sides, we get

$$3 - 5z = A(1 + z) + B(1 - 3z)$$

From this, we get $A = 1$ and $B = 2$

$$\therefore G(z) = \frac{1}{1+3z} + \frac{2}{1+z}$$

$$= 1(3^n) + 2(-1)^n$$

Thus, the required solution is $a_n = 1\,(3^n) + 2\,(-1)_n$.

Example 5.93 Use the method of generating f... relati...

$\Rightarrow a_n = 4^{n-1} - (n+1)2^{n+1}$

Example 5.94 Use the method of generating function to solve the recurrence relatio
$+ 2(n-1)$ with $a_0 = 3, n \geq 1$.

Solution Let the generating function for the sequence $\{a_n\}$ be given by $G(z)$.

$$\therefore G(z) = \sum_{n=0}^{\infty} a_n z^n$$

The given recurrence relation is $a_n = a_{n-1} + 2n - 2, n \geq 1$.

Multiplying both sides of Eq. (5.55) by z^n and summing over all $n \geq 1$, we get

$$\sum_{n\geq 1} a_n z^n = \sum_{n\geq 1} a_{n-1} z^n + \sum_{n\geq 1} (2n-2) z^n$$

$$\Rightarrow \sum_{n\ge1} a_n z^n = z\sum_{n\ge1} a_{n-1} z^{n-1} + 2\sum_{n\ge1} nz^n - 2\sum_{n\ge1} z^n$$

$$\Rightarrow [G(z) - a_0] = zG(z) + 2\,[z + 2z^2 + 3z^3 + \cdots + (n+1)\,z^{n+1} + \cdots] - 2[z + z^2 + z^3 + \cdots]$$

$$\Rightarrow (1-z)\,G(z) = 3 + 2z\,[1 + 2z + 3z^2 + \cdots + (n+1)\,z^n + \cdots] - 2\,[(1-z)^{-1} - 1$$

$$= 3 + 2z(1-z)^{-2} - \frac{2}{1-z} + 2$$

$$= 5 + \frac{2z}{(1-z)^2} - \frac{2}{1-z}$$

$$\Rightarrow G(z) = \frac{5}{(1-z)} - \frac{2}{(1-z)^2} - \frac{2z}{(1-z)^3}$$

$$G(z) = 5\,[1 + z + z^2 + \cdots + z^n] - 2\,[1 + 2z + 3z^2 + \cdots + (n+1)\,z^n] + 2z\,[1\times2 + 2\times3\times z + 3\times4z^2 + \cdots + (n+1)(n+2)\,z^n + \cdots]$$

Therefore, the required solution is written as

$$a_n = 5(1^n) - 2(n+1) + 2n\,(n+1)$$
$$= 5 - 2n - 2 + 2n^2 + 2n$$
$$= 2n^2 + 3$$

Example 5.85 Use the generating function to solve th