$$= \frac{25!}{4!(21)!} = \frac{25 \times 24 \times 23 \times 22}{4 \times 3 \times 2 \times 1}$$

$$= 12{,}650$$

Example 5.21 How many bit strings of length 8 contain

(i) exactly five 1s?
(ii) an equal number of 0s and 1s?
(iii) at least four 1s?
(iv) at least three 1s and at least three 0s?

Solution (i) A bit string of length 8 can be considered to have eight positions. These eight positions can be filled up with exactly five 1s and three 0s. Therefore, the required number of bit strings

$$= {}_8C_5 = \frac{8!}{5!.3!}$$

$$= \frac{8 \times 7 \times 6}{3 \times 2 \times 1}$$

$$= 56$$

(ii) These eight positions can be filled up with four 1s and four 0s. Therefore, the required number of bit strings

$$= \frac{8!}{4!.4!}$$

$$= \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1}$$

$$= 70$$

(iii) These eight positions can be filled up with four 1s and four 0s, five 1s and three 0s, six 1s and two 1s, seven 1s and one 0 or eight 1s and no 0s. Therefore, the required number of bit strings

$$= {}_8C_4 + {}_8C_5 + {}_8C_6 + {}_8C_7 + {}_8C_8$$

$$= \frac{8!}{4!.4!} + \frac{8!}{5!.3!} + \frac{8!}{6!.2!} + \frac{8!}{7!.1!} + \frac{8!}{8!.0!}$$

$$= 163$$

+ = OR.
* = and.
Here it's OR.

(iv) These eight positions can be filled up with three 1s and five 0s, four 1s and four 0s or five 1s and three 0s. Therefore, the required number of bit strings

$$= \frac{8!}{3!.5!} + \frac{8!}{4!.4!} + \frac{8!}{5!.3!}$$

$$= \frac{8\times7\times6}{3\times2} + \frac{8\times7\times6\times5}{4\times3\times2\times1} + \frac{8\times7\times6}{3\times2}$$
$$= 56 + 70 + 56$$
$$= 182$$

Example 5.22 Suppose a department consists of 10 men and 15 women. How many ways are there to form a committee with six members if it must have three men and three women?

Solution Number of men in the department, $n(H) = 10$
Number of women in the department, $n(W) = 15$

We need six members to form a committee in which the number of men must be equal to the number of women. Hence, the required number of ways $= {}_{10}C_3 \times {}_{15}C_3$

$$= \frac{10\times9\times8}{3\times2} \times \frac{15\times14\times13}{3\times2}$$
$$= 120 \times 455$$
$$= 54,600$$

Thus, there are $4 \times 3 \times 2 \times 3 = 72$ ways to travel round trip ... line more than once.

Example 5.17 Suppose repetitions are not permitted, answer the following questions:

(i) How many three-digit numbers can be formed from the six digits 2, 3, 5, 6, 7 and 9?
(ii) How many of these numbers are less than 400?
(iii) How many of these are even?

Solution In each case, we draw three boxes to represent an arbitrary number, and then write in each box the number of digits that can be placed there.

(i) The first box can be filled in six ways, the second one can be filled in five ways and the last box can be filled in four ways (see Fig. 5.1).

$$\boxed{}\ \boxed{}\ \boxed{}$$

$$\boxed{6}\ \boxed{5}\ \boxed{4}$$

Figure 5.1 Formation of three-digit numbers from given six-digits

Thus, there are $6 \times 5 \times 4 = 120$ numbers.

(ii) The first box can be filled in only two ways, by 2 or 3, since each number must be less than 400. The second box can be filled in five ways and the last box can be filled in four ways (see Fig. 5.2).

$$\boxed{2}\ \boxed{5}\ \boxed{4}$$

Figure 5.2 Formation of three-digit numbers less than 400

Thus, there are $2 \times 5 \times 4 = 40$ numbers.

(iii) The third box can be filled in only two ways, by 2 or 6, since the number must be even. The first box can be filled in five ways and the second box can be filled in four ways (see Fig. 5.3).

$$\boxed{5}\ \boxed{4}\ \boxed{2}$$

Figure 5.3 Formation of three-digit even numbers

Thus, there are $5 \times 4 \times 2 = 40$ numbers.

Example 5.18 In how many ways can four mathematics books, three history books, three chemistry books and two sociology books be arranged on a shelf so that all books of the same subject are together?

Solution First, the books must be arranged on the shelf in four units according to subject matter (see Fig. 5.4).

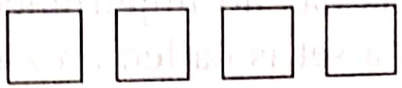

Figure 5.4 Arrangement of books

The first box can be filled by one of the four subjects, the next by any three remaining subjects, the next by any two remaining subjects and the last box can be filled by the last subject (see Fig. 5.5).

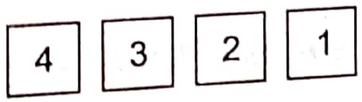

Figure 5.5 Subject-wise arrangement

Thus, there are $4 \times 3 \times 2 \times 1 = 4!$ ways of arranging the books on the shelf according to subject matter. Now, in each of the above cases, the mathematics books can be arranged in $4!$ ways, the history books in $3!$ ways, the chemistry books in $3!$ ways and the sociology books in $2!$ ways.

Thus, there are $4!4!3!3!2! = 41{,}472$ arrangements.

$$= {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \cdots + {}_nC_ny^n$$

Proof We can prove the above theorem by using the combinatorial concept.

When we expand $(x + y)^n$, the terms in the product are of the form $x^{n-i}y^i$, for $i = 0, 1, 2, \ldots, n$. To count the number of terms of the form $x^{n-i}y^i$, it is enough to choose $(n - i)$ x's from the sum n. Therefore, the coefficient of $x^{n-i}y^i$ is ${}_nC_{n-i}$

$$= {}_nC_i$$

Thus, $(x + y)^n = \sum_{i=0}^{n} {}_nC_ix^{n-i}y^i$

Example 5.24 Find the expansion of $(x + y)^6$.

Solution From binomial theorem, we have

$$(x + y)^6 = \sum_{i=0}^{n} {}_nC_ix^{6-i}y^i$$

$$= {}_6C_0x^6 + {}_6C_1x^5y + {}_6C_2x^4y^2 + {}_6C_3x^3y^3 + {}_6C_4x^2y^4 + {}_6C_5xy^5 + {}_6C_6y^6$$

$$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

Example 5.25 Find the coefficient of x^5y^8 in $(x + y)^{13}$.

Solution From binomial theorem, the coefficient of x^5y^8 is

$${}_{13}C_5 = \frac{13!}{5!.8!}$$

$$= \frac{13 \times 12 \times 11 \times 10 \times 9}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= 1{,}287$$

Example 5.26 What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x - 3y)^{200}$?

Solution From binomial theorem, we can determine the coefficient of $x^{101}y^{99}$ as

$$(2x - 3y)^{200} = [2x + (-3y)]^{200}$$

$$= \sum_{i=0}^{200} {}_{200}C_i(2x)^i(-3y)^{200-i}$$

Substituting $i = 101$, we get the coefficient of $x^{101}y^{99}$ in the expansion as

$$(2x - 3y)^{200} = {}_{200}C_{101}(2)^{101}(-3)^{99}$$

$$= -{}_{200}C_{101}2^{101}3^{99}$$

$$= \frac{-200!}{101!99!}2^{101}3^{99}$$