$$a_2 = 3a_1 + 1 = 3 \times 4 + 1 = 13 = \left(\frac{\quad}{2}\right)$$

$$a_3 = 3a_2 + 1 = 3 \times 13 + 1 = \left(\frac{3^{3+1} - 1}{2}\right)$$

$$\vdots$$

$$a_n = 3a_{n-1} + 1 = \left(\frac{3^{n+1} - 1}{2}\right).$$

Hence, $a_n = \left(\frac{3^{n+1} - 1}{2}\right)$ is the solution.

**Example 5.66** A person deposits Rs 1,000 in an account that yields 9% interest compounded yearly.

(i) Set up a recurrence relation for the amount in the account at the end of $n$ years.

(ii) Find an explicit formula for the amount in the account at the end of $n$ years.

(iii) How much money will the account contain after 100 years?

***Solution*** (i) Let $S_n$ denote the amount in the account after $n$ years.

But, the amount in the account after $n$ years

= the amount in the account after $(n - 1)$ years + interest for the $n^{\text{th}}$ year

i.e., $S_n = S_{n-1} + (0.09)\,S_{n-1}$, since the interest is 9% per year

i.e., $S_n = (1.09)S_{n-1}$

This is the required recurrence relation for the amount in the account at the end of $n$ years.

(ii) Explicit formula for $S_n$:

Now, $S_1 = (1.09)S_0$

$$S_2 = (1.09)S_1 = (1.09)^2 S_0$$
$$S_3 = (1.09)S_2 = (1.09)^3 S_0$$
$$\vdots$$
$$S_n = (1.09)S_{n-1} = (1.09)^n S_0$$

$\Rightarrow S_n = (1.09)^n S_0$

i.e., $S_n = (1.09)^n \times 1{,}000$, since $S_0 =$ Rs 1,000 (5.2)

Using mathematical induction, we can prove the validity of Eq. (5.2)

When $n = 0$, $S_0 = (1.09)^0 \times 1{,}000$

$= 1{,}000$

$\therefore$ The result (i) is true for $n = 0$.

We assume that $S_k = (1.09)^k \times 1{,}000$ is true.

We need to prove that $S_{k+1} = (1.09)^{k+1} \times 1{,}000$ is true.

From the recurrence relation, we have

$S_{k+1} = (1.09)S_k$

$= (1.09).\ (1.09)^k \times 1{,}000$ [by our assumption]

$\Rightarrow S_{k+1} = (1.09)^{k+1} \times 1{,}000$

$\Rightarrow S_{k+1}$ is true.

Thus, by the principle of mathematical induction, $S_n$ is true for all values of $n$.

$\therefore$ The explicit formula is $S_n = (1.09)^n.(1{,}000)$

(iii) When $n = 100$, we have

$S_{100} = (1.09)^{100} \times 1{,}000$

$=$ Rs $(1.09)^{100} \times 1{,}000$

$\therefore$ Money in the account after 100 years = Rs $1{,}000(1.09)^{100}$. ■

**Example 5.67** Suppose the number of bacteria in a colony triples every hour.

(i) Set up a recurrence relation for the number of bacteria after $n$ hours have elapsed.

(ii) If 100 bacteria are used to begin a new colony, how many bacteria will be there in the colony in 10 hours?

*Solution* (i) Let $a_n$ be the number of bacteria at the end of $n$ hours.

$\therefore$ $a_{n-1}$ is the number of bacteria at the end of $(n - 1)$ hours.

Since the number of bacteria in a colony triples every hour, $a_n = 3a_{n-1}$.

This is true whenever $n$ is a positive integer.

Hence, the recurrence relation for the number of bacteria after $n$ hours is (5.3)

$$a_n = 3a_{n-1}.$$

(ii) Let $a_0 = 100$. Then

$$a_1 = 3a_0 = 3 \times 100$$
$$a_2 = 3a_1 = 3^2 \times 100$$
$$a_3 = 3a_2 = 3^3 \times 100$$
$$\vdots$$
$$a_n = 3a_{n-1} = 3^n \times 100$$
$$\Rightarrow a_n = 3^n \times 100 \qquad (5.4)$$

We can prove the validity of Eq. (5.4) by using the induction principle.

When $n = 0$, $a_0 = 3^0 \times 100 = 100$. Therefore, $a_0$ is trivially true.

We assume that $a_k = 3^k \times 100$ is true.

We need to prove that $a_{k+1} = 3^{k+1} \times 100$ is true.

Now, $a_{k+1} = 3.a_k$ [from the recurrence relation given in Eq. (5.3)]

$= 3 \times 3^k \times 100$ [by our assumption]

$= 3^{k+1} \times 100$

$\Rightarrow a_{k+1} = 3^{k+1} \times 100$

$\Rightarrow a_{k+1}$ is true.

Hence, by the principle of mathematical induction, $a_n$ is true for every positive integer $n$.

Thus, the explicit formula is $a_n = 3^n \times 100$.

When $n = 10$, we have

$$a_{10} = 3^{10} \times 100$$
$$= 59,04,900$$

Therefore, the number of bacteria in the colony in 10 hours = 59,04,900. ■

*acci sequence* The sequence {1, 1, 2, 3, 5, 8, 13, .....} is called Fibonacci sequence.

e recurrence relation for the Fibonacci sequence is

$F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, $F_1 = 1$, $F_2 = 1$