GENERATING FUNCTION

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GENERATING FUNCTION

Let $\{a_n\}_{n=0}^{\infty}$ be any sequence. Then generating function of $\{a_n\}_{n=0}^{\infty}$ is defined as

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

POWER SERIES

- $\frac{1}{1-x}$ *is the generating function of the sequence 1, 1, 1, 1, ...*

Because $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

- $\frac{1}{1-ax}$ *is the generating function of the sequence* $1, a, a^2, a^3, \ldots$

Because $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

OPERATIONS ON POWER SERIES

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k \quad and \quad g(x) = \sum_{k=0}^{\infty} b_k x^k\,, then$

1. $f(x) + g(x) = \sum_{k=0}^{\infty}(a_k + b_k)x^k$

2. $f(x)g(x) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j b_{k-j}\right)x^k$

EXAMPLE

1. If $f(x) = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} a_n x^n$, find a_n
2. If $f(x) = \frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} a_n x^n$, find a_n

EXTENDED BINOMIAL COEFFICIENT

Let u be nay real number and k be any positive integer, the extended binomial coefficient $\binom{u}{k}$ is defined as

$$\binom{u}{k} = \begin{cases} u(u-1)\ldots\dfrac{u-k+1}{k!} & if \quad k > 0, \\ 1 & if \;\; k = 0 \end{cases}$$

EXAMPLE

1. Find extended binomial coefficient of $\binom{-3}{4}$ and $\binom{\frac{1}{2}}{4}$.
2. Show that $\binom{-n}{k} = (-1)^k C(n+k-1, k)$.

THE EXTENDED BINOMIAL THEOREM

Let x be a real number with $|x| < 1$ and u is any real number.

Then

$$(1 + x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

EXAMPLE

1. Find the generating functions for $(1 + x)^{-n}$ and $(1 - x)^{-n}$, $where$ n is a positive integer using extended binomial theorem.
2. Find the coefficient of x^{10} in the power series of each of these functions.

   a. $\frac{1}{(1+x)^2}$

   b. $\frac{1}{(1+2x)^4}$

EXAMPLE: SOLUTION OF RECURRENCE RELATION USING GENERATING FUNCTION

1. Use generating function to solve the recurrence relation

   a. $a_n = 7a_{n-1},\ where\ a_0 = 5.$

   b. $a_n = 3a_{n-1} + 4^{n-1},\ where\ a_0 = 1.$

   c. $a_n = a_{n-1} + 2a_{n-2} + 2^n,\ where\ a_0 = 4\ and\ \ a_1 = 12.$

   d. $a_n = 2a_{n-1} + 3a_{n-2} + 4^n + 6,\ where\ a_0 = 20$

   $and\ \ a_1 = 60.$

EXAMPLE

1. Find the generating function for the sequence 3, 3, 3, 3, 3, 3, 3.
2. Find the generating function for the sequence 1, 4, 16, 64, 256.

EXAMPLE

Find the closed form for the generating function for each of these sequence

a. 0, 2, 2,2,2,2,2,2,2,0,0,0,…

b. 0,0,0,0,1,1,1,1,1,1,1,1,1,0,0,0,0,…

c. $\binom{7}{0}, \binom{7}{1}, \binom{7}{2}, \ldots, \binom{7}{7}, 0,0,0, \ldots$

d. $3, -3, 3, -3, 3, -3, \ldots$

e. 1, 1, 0,1,1,1,1,1…

EXAMPLE

Find the closed form for the generating function for each of these sequence.

a. $a_n = 4, for\ all\ n = 0,1,2,,3, \ldots$

b. $a_n = 2^n, for\ all\ n = 0,1,2,,3, \ldots$

c. $a_n = 2n + 3, for\ all\ n = 0,1,2,,3, \ldots$

d. $a_n = \binom{n+4}{n}, for\ all\ n = 0,1,2,,3, \ldots$

EXAMPLE

For each of these generating functions, find the closed formula for the sequence it determines.

1. $(x^2 + 1)^3$ 2. $\frac{1}{1-2x^2}$ 3. $\frac{x^2}{(1-x)^3}$

2. $\frac{x}{1+x+x^2}$ 3. $e^{3x^2} - 1$ 4. $\frac{1+x^3}{(1+x)^3}$

EXAMPLE

Find the closed form for the generating function for each of these sequence.

a. $a_n = \frac{1}{(n+1)!}, for\ all\ n = 0,1,2,,3,\ldots$

b. $a_n = \binom{n}{2}, for\ all\ n = 0,1,2,,3,\ldots$

c. $a_n = \binom{10}{n+1}, for\ all\ n = 0,1,2,,3,\ldots$