Problem - 2

- moon lander with state $[h, v, m]^T$ to have the following dynamics

$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g + \frac{\alpha(t)}{m(t)} \\ \dot{m}(t) = -k\alpha(t) \end{cases}$$

- Here, h is the altitude, V is the velocity and m is the mass of moon langder, $\alpha(t) \in [0, 1]$

- minimize the total applied thrust before landing is as equal as the maximizing the mass of moon lander, which gives us the minimum fuel cosumption

$$\min_{\alpha(t)} \quad P(\alpha) = \int_0^T \alpha(t)\, dt$$

$$\min_{\alpha(t)} \int_0^T \alpha(t)\, dt = \frac{m_0 - m(T)}{k}$$

- in terms of general notations the state vector

$$f = \begin{bmatrix} v \\ -g + \frac{\alpha}{m} \\ -k\alpha \end{bmatrix} \qquad \text{Thrust } l = \alpha$$

→ Hamiltonian:

$$H = -l + \lambda^T f$$

$$= -\alpha + \lambda_1 \dot{x}_1 + \cancel{\lambda_2} \lambda_2 \dot{x}_2 + \lambda_3 \dot{x}_3$$

$$= -\alpha + \lambda_1 V + \lambda_2 \left(-g + \frac{\alpha}{m}\right) + \lambda_3 (-k\alpha)$$

– to find the optimal control policy we apply pontryagin's maximum principal.

$$\alpha^* = \arg\max(H) \quad \text{w.r.t. } \alpha \in [0, 1]$$

$$= \arg\max \left(-1 + \frac{\lambda_2}{m} - \lambda_3 k\right)\alpha + \lambda_1 V - \lambda_2 g$$

– $$\alpha^* = \begin{cases} 0 & b \le 0 \\ 1 & b > 0 \end{cases}$$

where $b = -1 + \frac{\lambda_2}{m} - \lambda_3 k$

– $$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \\ \frac{\partial H}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda_1 \\ \frac{\lambda_2 \alpha}{m^2} \end{bmatrix}$$

$$\dot{\lambda} = \begin{bmatrix} 0 \\ -\lambda 1 \\ \frac{\lambda_2 \alpha}{m^2} \end{bmatrix}$$

- $\dot{b} = \frac{\dot{\lambda}_2}{m} - \frac{\lambda_2 \dot{m}}{m^2} - \dot{\lambda}_3 k$

- now, substituting $\dot{\lambda}$ in $\dot{b}$

$$\dot{b} = -\frac{\lambda 1}{m} - \frac{\lambda 2(-k\alpha)}{m} - \left(\frac{\lambda 2 \alpha}{m^2}\right)k$$

$$\boxed{\dot{b} = \frac{-\lambda 1}{m}}$$

- mass is always greater than zero
  $m > 0$

- The value of $\lambda 1$ is always constant
  $\dot{\lambda}_1 = 0$

- if $\lambda 1$ is positive, then $\dot{b}$ is always negative
  if λ_1 is negative, then $\dot{b}$ is always Positive

- Thus, b is monotonic, because it's first derivative does not change sign

- so, b changes monotonicly, which means either b go from negative to Positive or Positive to negative

- and for this moon lander case, moonlander will turn the thrust on at a specific time

- with this we can say that optimal policy is

$$\alpha^* = \begin{cases} 0 & \text{if } b \leq 0 \quad \text{at} \quad t \in [0, t^*] \\ 1 & \text{if } b > 0 \quad \text{at} \quad t \in [t^*, \tau] \end{cases}$$