

Radiative Transfer in  
Astrophysics

## # Radiative processes in astrophysics:

Light is an electromagnetic wave where  $\vec{E}$  and  $\vec{B}$  are perpendicular to each other as well as the direction of propagation  $\vec{k}$ .

Now, light is a ray, a particle interacting with matter or a wave ??

When the energy of light  $\frac{hc}{\lambda} \ll k_B T \rightarrow$  temp. of the matter with which wave light is interacting.

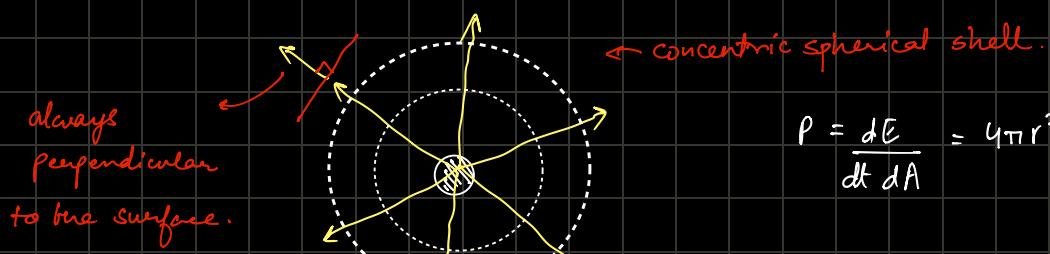
At higher energies i.e.  $\frac{hc}{\lambda} > k_B T$  it would interact as photon.

And lastly, when  $\lambda$  (wavelength)  $\ll$  the size of the medium, light is assumed as a ray.

## # Radiative transfer:

We are all familiar with the term flux, it Energy flow perpendicular to the surface per unit area per unit time.  
(second)

Now, assume a star as a isotropic source of total power  $P$

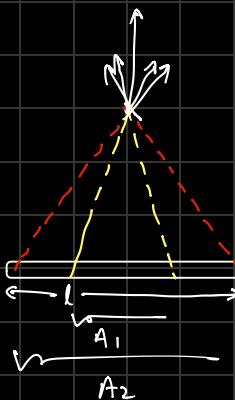


$$P = \frac{dE}{dt dA} = 4\pi r^2 \cdot F \Rightarrow F \propto \frac{1}{r^2}$$

(Inverse square law)

but now consider an extended object.

The total amount of photons going in the direction  $\hat{n}$  increases as the emitting surface under consideration goes from  $A_1$  to  $A_2$ .



∴ Now we slightly change our definition, we define flux in a direction that is per unit solid angle

$$\left\{ \mathcal{I} = \frac{dE}{dt dA} \times \frac{1}{d\Omega} \right\}$$

∴ But we have one more issue, frequency what if energy flows from one frequency to another one.

So, finally the most general equation we get is.

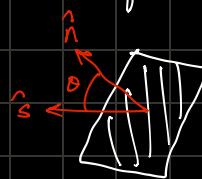
$$\left\{ I_\nu = \frac{dE}{dt dA d\Omega} \times \frac{1}{d\nu} \right\}$$

This is known as the specific intensity or "Brightness"

Imagine a unit surface  $\hat{s}$  that makes an angle  $\theta$  with the ray, so the energy flow per unit area per unit time  $\perp$  to the surface is.

$$F_\nu = \int_{\Omega} I_\nu \hat{n} \cdot \hat{s} d\Omega$$

$$= \int_{\Omega} I_\nu \cos\theta d\Omega$$



$$\text{Total flux is } F = \int F_\nu d\nu$$

# Energy density and radiation pressure:

$$u'_\nu (\text{Energy density}) = \frac{I_\nu}{c}$$

In a particular direction.

In many cases, we deal with a roughly isotropic radiation field. In such cases, it is justified to define angle avg. specific intensity

$$J_\nu = \frac{\int I_\nu d\Omega}{\int d\Omega} = \frac{1}{4\pi} \int I_\nu d\Omega$$

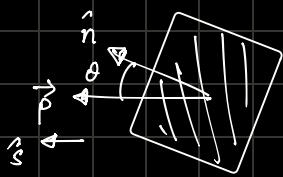
$$\therefore u'_\nu (\text{Energy density at any location}) = \int u'_\nu d\Omega = \int \frac{I_\nu}{c} d\Omega = \frac{1}{c} \int I_\nu d\Omega$$

$$u_\nu = \frac{4\pi}{c} J_\nu \Rightarrow \text{Total energy density}$$

$$u = \int u_\nu d\nu = \frac{4\pi}{c} \int J_\nu d\nu$$

Now, to define radiation pressure, we note that the number flux of photons through the unit surface  $\hat{s}$  is

$$N_{ph} = \int \frac{I_\nu}{h\nu} \cos\theta d\Omega$$



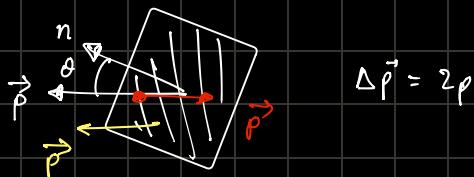
momentum perpendicular to the unit surface is  $\vec{p} \cdot \hat{n} = p \cos \theta$

$$= \left( \frac{h\nu}{c} \right) \cos \theta$$

$$\therefore \vec{P}_v \text{ (momentum flux along } \hat{z}) = \int n_{ph} \cdot \vec{p} \cdot \cos \theta d\Omega = \int \frac{I_v \cos \theta}{h\nu} \cdot \frac{h\nu}{c} \cos \theta \cdot d\Omega$$

$$= \frac{1}{c} \int I_v \cos^2 \theta \cdot d\Omega$$

And of course  $p = \int \vec{p}_v \cdot dv$



$$\therefore \text{Rad. pressure} = 2 \int p_v \cdot dv = \frac{2}{c} \int_V \int_{\Omega} I_v \cos^2 \theta \cdot d\Omega \cdot dv$$

$$= \frac{2}{c} \int I_v dv \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos^2 \theta \cdot \sin \theta \cdot d\theta \quad \begin{aligned} \cos \theta &= t \\ -\sin \theta d\theta &= dt \\ \Rightarrow & \end{aligned}$$

$$= \frac{4\pi}{c} \int I_v dv \int_1^0 (-t^2) \cdot dt$$

$$= \frac{4\pi}{c} \underbrace{\int J_v dv}_{U} \cdot \left( \frac{1}{3} \right) \quad \begin{aligned} &\text{In this step, we have assumed an} \\ &\text{isotropic radiation field such that} \\ &I_v = J_v \end{aligned}$$

$$= \frac{U}{3}$$

$$\therefore P_{rad} = \frac{U}{3}$$

# Force due to radiation:

Number of photons absorbed per unit volume per unit time is  $N_{abs} = n \cdot n_{ph} \sigma_v c$   
and the intensity  $I_v = n_{ph} c \cdot h\nu$

$\therefore$  Total number of photons absorbed per unit volume per sec per unit sr is

$$\dot{N}_{abs} = n \sigma_v \frac{I_v}{h\nu}$$

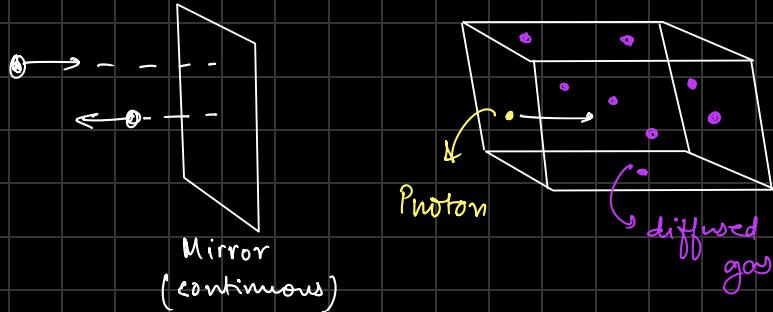
Each photon carries:  $\frac{h\nu}{c}$  momentum

$$F_v = \int N_{\text{abs}} \frac{h\nu}{c} \cos\theta d\Omega ; \text{ Rate of momentum absorbed in all directions}$$

$$\frac{n\sigma_v}{c} \int I_v \cos\theta d\Omega \Rightarrow \int \frac{m\sigma_v}{c} F_v dv = \int \frac{\epsilon_v}{c} F_v dv$$

$$k_v = \frac{\sigma_v}{m} ; \text{ mass-absorption co-efficient or opacity.}$$

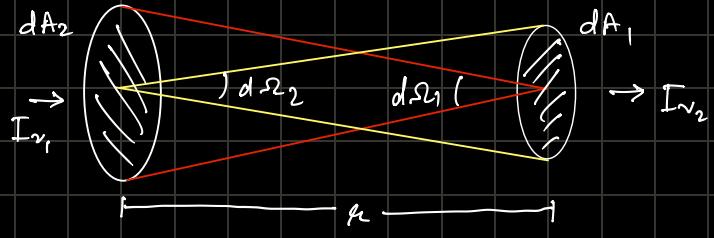
$$\left\{ f = \frac{1}{c} \int k_v F_v dv ; \text{ acceleration due to radiation} \right\}$$



here photon will have a probability of getting absorbed or scattered.

if it is  $\sigma_v$   $\rightarrow$  number density.  
 $\downarrow$  cross-section

# Constancy of  $I_v$  in free space :



$$F_1 = I_{v_1} dA_1 dt d\Omega_1$$

$$F_2 = I_{v_2} dA_2 dt d\Omega_2$$

$$F_1 = F_2 \text{ and } d\Omega_1 = \frac{dA_2}{r^2}$$

$$\therefore I_{v_1} = I_{v_2} \quad \therefore \text{In free space, specific intensity is distance independent.}$$

Radiative transfer equation :

Now, if the medium b/w these two surfaces is emitting then  $I_v$  should increase so let's say  $\epsilon_v$  is the emissivity of the medium.

$\therefore$  A unit volume of the medium emits  $j_v = \frac{\epsilon_v}{4\pi}$  energy per unit solid angle.

$$I_{v_1} \xrightarrow{\dots} \rightarrow I_{v_2} \quad dI_v = j_v ds ; I_v \text{ should increase.}$$

Rate of absorption per unit volume is  $\dot{N}_{abs} = n \cdot n_{ph} \cdot \sigma_v \cdot c$   
 following the specific intensity  $I_v = n_{ph} h v \cdot c$

$$\therefore \dot{N}_{abs} = \frac{n \sigma_v I_v}{h v}$$

$$E_{abs} = \dot{N}_{abs} h v = n \sigma_v I_v$$

$$\therefore dI_v = -n \sigma_v I_v ds$$

↓ negative sign because energy is absorbed.

$$\frac{dI_v}{ds} = j_v - \frac{n \sigma_v}{\alpha_v} I_v \Rightarrow \text{This is the radiative transfer equation.}$$

$$\alpha_v = n \sigma_v \quad \text{absorption coefficient}$$

$$\frac{dI}{ds} = j - \alpha_v I \Rightarrow \frac{d}{\alpha_v ds} \cdot I = \frac{j}{\alpha_v} - \frac{I}{\alpha_v}$$

$$dT = \alpha_v ds = (n \sigma_v ds)$$

$$S_v = \frac{j}{\alpha_v} ; \text{Source term.}$$

$$\therefore \frac{dI}{dT} = S_v - I$$

Before moving forward let's have a discussion for this  $dT$  term

$$dT = n \sigma_v ds, \quad \text{we know mean free path is } l = \frac{1}{n \sigma_v} \Rightarrow [n \sigma_v l = 1]$$

↳ This same sort of probability if photon gonna get absorbed after travelling  $ds$

↳ we can say it tell us about the depth of the medium that photon can travel

$\therefore T$  is famously known as the optical depth.

$$\frac{dI}{dT} + I = S \quad (\text{we are dropping } v \text{ subscript})$$

$$e^{\tau} \frac{dI}{dT}, \quad e^{\tau} I = S e^{\tau}$$

$$\frac{d(I e^{\tau})}{dT} = S e^{\tau} \Rightarrow \int(I e^{\tau}) = \int S e^{\tau} d\tau'$$

$$I(\tau) \cdot e^{\tau} = I(\tau_0) + \int S e^{\tau} d\tau'$$

$$I(\tau) = I(0) e^{-\tau} + \int s e^{(\tau' - \tau)} d\tau'$$

Now assuming source function to be constant ( $s = \text{constant}$ )

$$I(\tau) = I(0) e^{-\tau} + s e^{-\tau} (e^{\tau'})^{\tau}$$

$$= I(0) e^{-\tau} + s e^{-\tau} (e^{\tau} - 1)$$

$$= I(0) e^{-\tau} + s (1 - e^{-\tau})$$

### # Blackbody radiation:

Till this point we learned about emissivity & absorption (basic things) now, let's look at the problem from thermodynamic perspective, the only quantity that matters now is  $T$  (temperature).

$\therefore$  Everything emissivity, absorption will now depend only on  $T$ .

since we are talking only about  $T$ , the matter must be in thermodynamical equilibrium.

Matter must emit whatever it absorbs in any local volume  $\Rightarrow \frac{dI}{dt} = 0$

$$\therefore \dot{j}_v = \alpha B_v$$

$\hookrightarrow$  Specific intensity

This is known as Kirchhoff's law.

$B_v$  was known as a blackbody radiation, its theoretical form was unknown. It was only after Planck (1909) that this form was found.

$$B_v(T) = \frac{2v^2}{c^2} \frac{hv}{\left(\exp\left(\frac{hv}{k_B T}\right) - 1\right)}$$

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{\left(\exp\left(\frac{hc}{k_B T \lambda}\right) - 1\right)}$$

$\therefore v \rightarrow T \Rightarrow B_v(v) dv = B_\lambda(\lambda) d\lambda \Rightarrow$  This was true first the energy was connected with frequency of radiation

$$E = hv$$

Let us try to derive blackbody radiation:

Let's consider a 'radiation gas' at temp T and that the photons in this gas have a phase space distribution given by  $n_{ph}(\vec{r}, \vec{p})$

$$\therefore E = n_{ph} h\nu d^3x d^3p \quad d^3x = c dt \cdot dA$$

$$d^3\vec{p} = p^2 dp \cdot d\Omega \quad p = \frac{h\nu}{c} \Rightarrow dp = \frac{h}{c} dv$$

$$\therefore d^3\vec{p} = \frac{h^2 v^2}{c^2} \cdot \frac{h}{c} dv \approx \frac{h^3 v^2}{c^3} dv$$

$$\therefore E = n_{ph} \frac{h^4 v^3}{c^2} dt dA dv d\Omega$$

#  $I_v = \frac{h^4 v^3}{c^2} n_{ph}(\vec{r}, \vec{p})$

For a quantum system  $\Delta x \cdot \Delta p \geq \hbar$   $\therefore (\Delta x)^3 \cdot (\Delta p)^3 \geq \hbar^3$   
and since photon has 2 polarization  $v = \lambda^2 / 2$

$$\therefore \text{Number of photons } (n_r) = (n_{ph}) \frac{\hbar^3}{2}$$

$$I_v = \frac{2n_r h\nu^3}{c^2} ; \quad n_r \equiv n_r(\vec{r}, \vec{p}) \quad \text{in equilibrium we can take average of photon number}$$

In statistical equilibrium, the probability that n-photons will be in a given state would be

$$P(n) \propto e^{-\beta E_n} \\ \propto e^{-\beta nh\nu n} \quad \beta = \frac{1}{k_B T}$$

$$\therefore \text{Avg. no of photons} = \frac{\sum_{n=0}^{\infty} n e^{-\beta nh\nu n}}{\sum_{n=0}^{\infty} e^{-\beta nh\nu n}} = \frac{1}{(e^{\frac{h\nu}{k_B T}} - 1)}$$

$$\therefore B_v = \frac{2v^2}{c^2} \frac{h\nu}{\left(\exp\left(\frac{h\nu}{kT}\right) - 1\right)}$$

when  $kT \gg h\nu$  (large wavelength)

$$I_\nu(T) = \frac{2\nu^2}{c^2} \cdot kT$$

$$\exp\left(\frac{h\nu}{kT}\right) - 1 \approx \frac{h\nu}{kT}$$

↳ "Rayleigh-Jeans" formula for blackbody emission.

$$1 + \frac{h\nu}{kT} + O\left(\frac{h\nu}{kT}\right)^2$$

↓  
ignored.

↳ This formula is widely used by Radio astronomers.

$$T_b = \frac{c^2}{2\nu^2 k_B} I_{\text{measured}}$$

↳ This is known as the brightness temperature.

Does represent the actual temp.  
good estimate of the medium. (For a source not emitting  
for a source that is in thermal equilibrium thermally)

At low wavelength ( $h\nu \gg kT$ )

$$I_\nu(T) = \frac{2h\nu^3}{c^2} e^{-h\nu/kT}$$

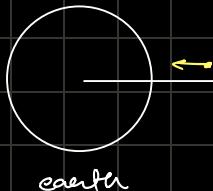
"Wein's formula". (Medium should be in thermal equilibrium)

$\lambda_{\text{max}} \cdot T = 0.29 \text{ cm K}$  or  $h\nu_{\text{max}} = 2.82 k_B T$   $\Rightarrow$  This is known as Wein's displacement law.

This equation gives a statement about statistical equilibrium b/w the photons and the matter particles.

In a gas, ideally  $E = \frac{3}{2} k_B T$   $\therefore$  In equilibrium  $\bar{\nu}_{\text{photon}} = h\nu = \frac{3}{2} k_B T = 1.5 k_B T$   
but it is actually  $2.82 k_B T$ . This fact arises because photons have slightly different adiabatic index.

Now by now, you should be thinking that what is the advantage of this Wein's displacement law!?



$$d_* = ??$$



How to find  $d_A$ ?

We can find the spectrum at Earth, the normalised value will be obviously different when measured at Earth, but but but ... the shape would be same  $\therefore d_{\text{max}}$  would be same.

$d_{\text{max}}$  will lead us to  $T$ , and from the  $T$  we can find the intrinsic brightness  $B_*$  and  $a = R_*$  (surface)

$$F_{\text{Earth}} = \pi B_* \left( \frac{R_*}{d_*} \right)^2 \quad \therefore \left\{ d_* = \sqrt{\frac{F_{\text{Earth}}}{\pi B_*}} \cdot R_* \right\}$$

EXTENSIVELY  
I have used this in my first research paper.

# Flux and Energy density:

Flux on a black body surface

$$F(T) = \int \pi B_v(T) d\tau = \frac{2\pi^5 k_B^4}{15 c^2 h^3} T^4 = \sigma T^4$$

$\hookrightarrow$  This is the Stefan-Boltzmann constant

$$\text{Energy density } u = \int \frac{4\pi}{c} B_v(T) d\nu = \frac{4}{c} F(T) = \frac{4\sigma}{c} T^4$$

$$\frac{4\sigma}{c} = a \text{ (Another form of constant)} \quad \therefore u = \sigma T^4$$

# Adiabatic index of black body radiation:

$$0 = dU + PdV$$

$$d(UV) = -PdV \quad P = \frac{V}{3} \leftrightarrow 3dP = dV$$

$$\therefore d_U \cdot V + u d_V = -P d_V$$

$$3dP \cdot V + 3P d_V = -P d_V$$

$$3dP \cdot V = -4P d_V$$

$$\frac{dp}{dv} = -\frac{4}{3} \frac{P}{V}$$

$$PV^\gamma = \text{constant}$$
$$dP \cdot V^\gamma + \gamma P V^{\gamma-1} dV = 0$$

$$dp + \frac{\gamma P}{V} dV = 0$$

$$\therefore \frac{dp}{dv} = -\gamma \frac{P}{V}$$

$\therefore \gamma = 4/3$  (Adiabatic index for the radiation gas)

$P = (\gamma_{\text{rad}} - 1) u \quad \therefore \text{From here also we get } [\gamma_{\text{rad}} = 4/3]$

## # Scattering :

Till now, we have only considered absorption and emission of photons due to the atomic processes. However now we put light to scattering.

The idea of scattering is that it only re-directs the photons & does not accumulate anything. So, the scattering emissivity  $E_{\text{scatt}}$  from a unit volume is simply the energy that the unit volume has absorbed.

$$E_{\text{scatt}} = \alpha_{\text{scatt}} \int I d\Omega ; \quad \alpha_{\text{scatt}} \text{ is the absorption co-efficient}$$

$$\frac{E_{\text{scatt}}}{4\pi} = \frac{\alpha_{\text{scatt}}}{4\pi} \int I d\Omega \Rightarrow j_{\text{scatt}} = \alpha_{\text{scatt}} J$$

In scattering, the propagation of a photon does not happen in a straight line but scatter in random direction after every scattering.  
Very similar to random walk.

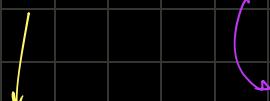
$$l = \sqrt{N} l_{\text{mfp}}$$

(mean free path)

$$\frac{dI}{ds} = j_{\text{emis}} - \alpha_{\text{abs}} I + j_{\text{scatt}} - \alpha_{\text{scatt}} I$$

$$\frac{dI}{ds} = \frac{j_{\text{emis}} + (\alpha_{\text{scatt}}/\alpha_{\text{tot}}) J}{\alpha_{\text{tot}}} - I \quad ; \quad \left\{ \alpha_{\text{tot}} = \alpha_{\text{scatt}} + \alpha_{\text{abs}} \right\}$$

$\left\{ d\tau = \alpha_{\text{tot}} ds \right\}$  Total optical depth.



Represents the fraction of the scattered light to the total absorbed light, it is known as "single scattering albedo."

To solve this eqn  
we need to know

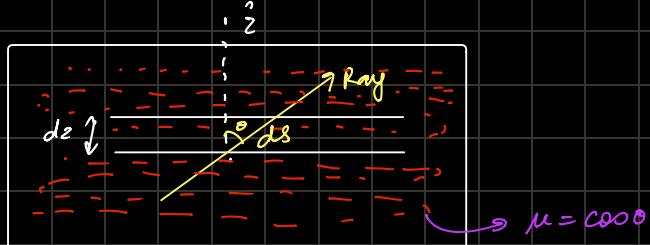
$$J = \frac{1}{4\pi} \int I d\Omega ; \quad \text{knowledge of } I \text{ in all directions.}$$

It's like to know different you should know the integral but to know integral you should know the differential. (Chicken-egg problem)

Such eq's are known as Integro-differential equations and are very difficult to solve.

A general solution RTE is obtained after some approximation.

Let's consider a slab of matter through which radiation passes. The slab properties are assumed to vary only in the direction of  $z$ .



we are going to solve RTE in  $\hat{z}$

$$ds = \frac{dz}{\cos \theta} = \frac{dz}{\mu}$$

$$\begin{aligned} J &= \int \frac{I ds}{4\pi} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} I \sin\theta \cdot d\theta \\ &= \frac{2\pi}{4\pi} \int_{-1}^1 I d\mu \end{aligned}$$

$$\begin{aligned} \mu &= \cos \theta \\ d\mu &= -\sin \theta \cdot d\theta \end{aligned}$$

$$\therefore J = \frac{1}{2} \int_{-1}^{+1} I d\mu$$

$$\begin{aligned} \frac{dI}{ds} &= j_{emis} + \alpha_{scatt} J - (\alpha_{abs} + \alpha_{scatt}) I \\ \left( \frac{\mu dI}{dz} \right) &= j_{emis} + \alpha_{scatt} J - (\alpha_{abs} + \alpha_{scatt}) I \end{aligned}$$

To solve, we are going to use moment moment (used for scattering down.) systems

$$H = \frac{1}{2} \int_{-1}^{+1} I \mu d\mu ; \quad K = \frac{1}{2} \int_{-1}^{+1} I \mu^2 d\mu \quad (\text{first and second moments of intensity})$$

$$\frac{d}{dz} \left[ \frac{1}{2} \int_{-1}^1 I \mu d\mu \right] = \frac{1}{2} \int_{-1}^1 \left( j_{emis} + \alpha_{scatt} J - (\alpha_{abs} + \alpha_{scatt}) I \right) d\mu$$

$$\begin{aligned} \frac{dH}{dz} &= \int_{-1}^1 \frac{j_{emis}}{2} d\mu + \int_{-1}^1 \frac{\alpha_{scatt}}{2} J d\mu - \left( \frac{\alpha_{abs} + \alpha_{scatt}}{2} \right) \int_{-1}^1 I d\mu \end{aligned}$$

$$= j_{emis} + \alpha_{scatt} \cdot J - (\alpha_{abs} + \alpha_{scatt}) J$$

$$\therefore \boxed{\frac{dH}{dz} = j_{emis} - \alpha_{abs} J}$$

$$\frac{d}{dz} \cdot \frac{1}{2} \int_{-1}^1 I \mu d\mu = (j_{emis} + \alpha_{scatt} J) \frac{1}{2} \int_{-1}^1 \mu d\mu - \frac{(\alpha_{abs} + \alpha_{scatt})}{2} \int_{-1}^1 I \mu d\mu$$

$$\therefore \frac{dk}{dz} = \frac{(j_{emis} + \alpha_{scatt} J)}{2} \left[ \frac{1-1}{4} \right] - (\alpha_{abs} + \alpha_{scatt}) H$$

$$\therefore \frac{dk}{dz} = -(\alpha_{abs} + \alpha_{scatt}) H$$

$$\left[ \frac{d^2k}{dz^2} = -(\alpha_{abs} + \alpha_{scatt}) \frac{dH}{dz} = -(\alpha_{abs} + \alpha_{scatt})(j_{emis} - \alpha_{abs} J) \right]$$

we can solve these integro-differential equations if we make the right assumption about intensity.

Eddington's approximation.

It assumes the specific intensity to be nearly isotropic and angular dependence is only linear in  $\mu$ .

$$I(\mu, \tau) = a(\tau) + b(\tau)\mu$$

$$\therefore J = \frac{1}{2} \int_{-1}^1 I d\mu = a(\tau)$$

$$H = \frac{1}{2} \int_{-1}^1 I \mu d\mu = \frac{b}{3}(\tau)$$

$$K = \frac{1}{2} \int_{-1}^1 I \mu^2 d\mu = \frac{a}{3}(\tau)$$

$$\therefore K = \frac{1}{3} J$$

$$\left[ \frac{d^2k}{dz^2} = -(\alpha_{abs} + \alpha_{scatt}) \frac{dH}{dz} = -(\alpha_{abs} + \alpha_{scatt})(j_{emis} - \alpha_{abs} J) \right]$$

⇒ The above equation will be now.

$$\frac{1}{3} \frac{d^2 J}{dz^2} = -\alpha_{tot} (j_{emi} - \alpha_{abs} J)$$

$$\therefore \frac{d^2 J}{dz^2} = -\frac{3 j_{em}}{\alpha_{tot}} - \frac{3 \alpha_{abs}}{\alpha_{tot}} \cdot J$$

If we consider asymmetric pressure then we would substitute  $k = DJ$  where  $D$  is the eddington tensor.

$$\therefore D \cdot \frac{d^2 J}{d\tau^2} = \frac{\alpha_{abs}}{\alpha_{tot}} [J - s] \Leftrightarrow \text{this is the radiative diffusion equation.}$$

## II. Discrete Emission

Now, we are going to move our discussion to emission from the astrophysical plasma. There are mainly three types of emission

- Bound-Bound emission      • free-free emission      • free-bound emission

① Bound-Bound emission : This is due to the downward transition of an excited atom. Such transitions produce discrete line emissions corresponding to the energy difference between the energy levels.

② Free-Free emission : Name itself is suggesting that the electron remains free before and after the emission process. The emission happens due to deacceleration of electron.  
eg) cyclotron, synchrotron emission, inverse compton scattering.

But to understand these emission, we have to understand the emitting systems and these systems are categorised broadly.

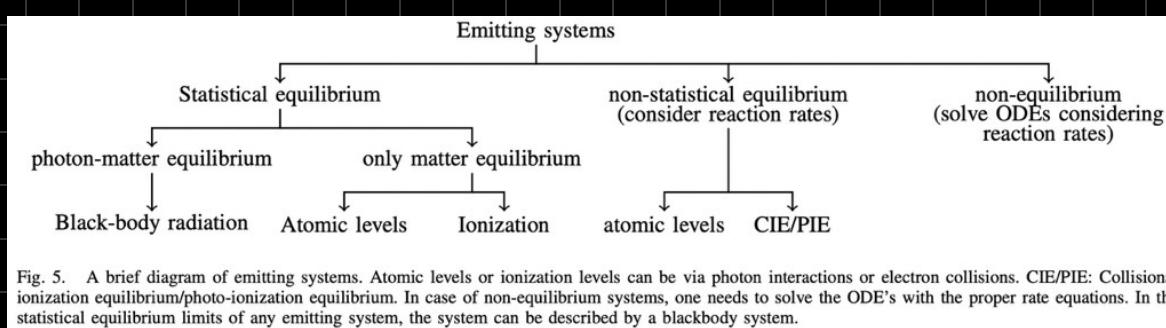


Fig. 5. A brief diagram of emitting systems. Atomic levels or ionization levels can be via photon interactions or electron collisions. CIE/PIE: Collisional ionization equilibrium/photo-ionization equilibrium. In case of non-equilibrium systems, one needs to solve the ODE's with the proper rate equations. In the statistical equilibrium limits of any emitting system, the system can be described by a blackbody system.

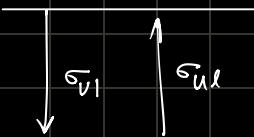
Let's start with statistical equilibrium

In single electron system, energy levels only depend on the principle quantum number, however for a multi-electron system it depends on both principle quantum number 'n' and the orbital quantum number 'l' due to spin orbit coupling.

∴ In statistical limit, we can write the density of atoms in upper and lower level as

$$\frac{n_u}{n_e} = \frac{g_u}{g_e} e^{-E_{ul}/k_B T} ; E_{ul} = E_u - E_l , g's$$

(1)



This however breaks down if there are not enough interaction b/w particles to achieve such equilibrium.

• Equilibrium with electrons:

In this particular case, electron collisions are responsible for the excitation and de-excitation of the energy thus maintain the equilibrium.

$$n_e n_e \langle \sigma_{eu} v \rangle = n_u n_e \langle \sigma_{eu} v \rangle \quad \text{for a two-level system.}$$

$$\therefore \frac{n_u}{n_e} = \frac{\langle \sigma_{eu} v \rangle}{\langle \sigma_{eu} v \rangle} = \frac{g_u}{g_e} e^{-E_{ul}/k_B T}$$

• Equilibrium with photons:

In this case, excitation is consider only by the absorption of photon by the direct absorption. De-excitation of the levels can happen either spontaneously or due to interactions with photons.

$$\frac{d}{dt} n_e = \frac{d}{dt} n_u$$

$$- n_e u_v B_{lu} = - n_u u_v B_{ul} - n_u A_{ul}$$

$u_v \rightarrow$  energy density

$v \rightarrow$  corresponding energy difference

$B_{lu}, B_{ul}$  represents the cross-section

$\frac{1}{A_{ul}}$  is the time scale for spontaneous  $u \rightarrow l$  transitions

Note that the normal reaction rate with photons could be given as  $R = \frac{dn_u}{dt}$

$$= n_u n_{ph} \epsilon_{ul} c = n_u \left( \frac{U_v}{h\nu} \right) \epsilon_{ul} \cdot c$$

$\therefore \boxed{B_{ul} = \frac{\epsilon_{ul} c}{h\nu}}$ ; A and B's are known as the Einstein coefficients

$$\left[ \frac{n_u}{n_e} = \frac{B_{lu}}{B_{ul} + A_{ul}} \right]$$

It can be argued that these co-efficients are the intrinsic properties of the atom. That means the probability of getting excited and deexcited is fixed.

The same atom in the local thermodynamic equilibrium must be able to produce black-body radiation.

$$U_v = \frac{4\pi}{c} B_v(T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{(e^{\frac{k_B T}{h\nu}} - 1)}$$

$\Rightarrow$  From the ratio of  $n_u$  &  $n_e$ , we get

$$U_v = \frac{A_{ul}/B_{ul}}{\left(\frac{n_e}{n_u}\right)\left(\frac{B_{lu}}{B_{ul}}\right) - 1}$$

$$\therefore \frac{A_{ul}}{B_{ul}} = \frac{8\pi h\nu^3}{c^3}; \quad \frac{n_e}{n_u} \left( \frac{B_{lu}}{B_{ul}} \right) = e^{\frac{k_B T}{h\nu}}$$

( )

$$B_{ul} = \frac{c^3 A_{ul}}{8\pi h\nu^3} \quad \frac{B_{lu}}{B_{ul}} = e^{\frac{h\nu/k_B T}{n_u/n_e}}$$

$\Rightarrow$  In thermal equilibrium we know  $\frac{n_u}{n_e} = \frac{g_u}{g_e} e^{-E_{ul}/k_B T}$

$$\therefore \frac{B_{\text{lu}}}{B_{\text{ul}}} = \frac{g_u}{g_e} e^{(\hbar\nu - E_{\text{ul}})/kT}$$

$$\therefore \left[ B_{\text{lu}} = \frac{g_u}{g_e} B_{\text{ul}} = \frac{g_u}{g_e} \cdot \frac{c^3}{8\pi h\nu^3} A_{\text{ul}} \right]$$

$$B_{\text{ul}} = \frac{\epsilon_{\text{ul}} \cdot c}{h\nu} \Rightarrow \left[ \epsilon_{\text{ul}} = \frac{g_u}{g_e} \cdot \frac{c^2}{8\pi\nu^2} A_{\text{ul}} \right]$$

To find this, we need to do some quantum theory calculations which we can't do !!

Question: What if the field is not black body, will the values of A, B will change ??

### • Equilibrium in ISM condition

$\frac{B_{\text{lu}}}{A_{\text{ul}}}$  is only a function of frequency

$$\bar{n}_\gamma = \left( \frac{B_{\text{lu}}}{A_{\text{ul}}} \right) V_\nu = \frac{c^3}{8\pi h\nu^3} V_\nu$$

$\therefore$  The total rate of downward transition in equation is  $\frac{dn_u}{dt} = (1 + \bar{n}_\gamma) A_{\text{ul}}$   
Hence at low values of the radiation field, the stimulated emission becomes unimportant

In case of ISM, where we do not have local thermal equilibrium & hence we only have excitation by collisions and de-excitation due to spontaneous emission. There, in the ISM

$$\frac{dn_u}{dt} = n_e n_e \langle \sigma_{\text{lu}} v \rangle - n_u n_e \langle \sigma_{\text{ul}} v \rangle - n_u A_{\text{ul}}$$

$\hookrightarrow$  Steady state :  $\frac{dn_u}{dt} = 0$

$$\therefore \frac{n_u}{n_e} = \frac{\langle \sigma_{\text{lu}} v \rangle}{\langle \sigma_{\text{ul}} v \rangle + \frac{A_{\text{ul}}}{n_e}}$$

This tell us  $\langle \sigma_{ul} v \rangle \sim \frac{A_{ul}}{n_e} \Rightarrow n_e \sim \frac{A_{ul}}{\langle \sigma_{ul} v \rangle}$  (This is known as the critical density)

$$T_{\text{gas}} \sim 10^6 \text{ K}, n_{\text{crit}} \gtrsim 10^3 \text{ cm}^{-3}$$

for ISM ( $n \ll n_{\text{crit}}$ )

$$\hookrightarrow n > n_{\text{crit}}$$

$\Rightarrow$  collision de-excitation dominates  
 $\Rightarrow$  population approach LTE

$$\hookrightarrow n \ll n_{\text{crit}}$$

$\Rightarrow$  Radiative decay dominates

$\Rightarrow$  Atom decay radiatively before another collision another collision can de-excite them

By considering both stimulated and spontaneous de-excitation

$$n_{e,\text{crit}} = \frac{(1 + \bar{n}_r) A_{ul}}{\langle \sigma_{ul} v \rangle}$$

$\Rightarrow$  below this density, the collision de-excitation is irrelevant and de-excitation happens only via spontaneous emission.

Strength of atomic lines:

$$P_{ul} = \frac{dn_{ul}}{dt} = \frac{dn_u}{dt} \times \frac{A_{ul}}{\sum_l A_{ul}} \times E_{ul}$$

$\underbrace{\quad}_{l < u} \rightarrow$  This whole term signifies how many of the total de-excitations are happening to level  $l$  in comparison to other lower levels.

$\frac{dn_u}{dt}$  can be obtained by considering that in equilibrium ( $\text{No.}_{\text{de-excitation}} = \text{No.}_{\text{excitation}}$ )

[from all the lower level]

$$\frac{dn_u}{dt} = \sum_{l < u} n_l n_e \langle \sigma_{l,u} v \rangle \approx n_0 n_e \langle \sigma_{u,0} v \rangle$$

$l_i < u \rightarrow \text{level } u$

$$= n_0 n_e \langle \sigma_{u,0} v \rangle \frac{g_u}{g_0} e^{-hv/k_B T}$$

$$\therefore \left\{ P_{ul} \equiv \varepsilon_v^{ul} = n_0 n_e \langle \sigma_{u,0} v \rangle \frac{g_u}{g_0} \frac{E_u A_{ul}}{\sum_l A_{ul}} e^{-hv/k_B T} \right\}$$

## Emission line notations:

Traditionally, any emission line from a neutral atom is denoted with Roman symbol I.

e.g.) Emission lines from neutral hydrogen ( $H^0$ ) are denoted as H I lines and emission lines from double ionized C ( $C^{2+}$ ) are denoted as C III lines

Allowed transitions:  $A \sim 10^8 \text{ s}^{-1}$

Semi-forbidden transitions: These transitions follow all the rules except the rule of spin, i.e. in the semi-forbidden transitions  $\Delta S \neq 0$ . Their co-efficients are almost  $10^6$  times lower than the allowed ones, notation '] $J$ '.

e.g.) N II ] 2143.2 Å

Forbidden lines: These transitions break multiple rules and have very low probability of transition, their co-efficients are also  $10^8 - 10^{12}$  times lower than the allowed ones, these are marked by '[ $J$ ]'. e.g.) [O II] 7321 Å

## Line broadening:

Till this point we have assumed that energy of  $n \rightarrow l$  transition is emitted at a frequency  $v_{nl}$ , but actually it is emitted observed at broad range frequencies around the central frequency  $v_{nl}$ .

First factor is the random motion of particles inside the thermal medium. This is known as 'thermal broadening' or 'Doppler broadening'.

### ① Doppler broadening:

An ion/atom moves while emitting the emission at a frequency  $v_0$ . This line is observed at a different frequency.

$$\frac{\Delta v}{v_0} = \frac{v_x}{c} \quad (\text{z-axis is the line of sight})$$

$$\therefore \left\{ \frac{\Delta v}{v_0} \cdot c = v_x \right\} ; \Delta v = v - v_0$$

For a thermal gas, no. of particle in a velocity range  $v_x$ ,  $v_x + dv_m$  is  $\propto \exp\left(-\frac{m}{2k_B T} v_x^2\right) dv$

$\therefore \phi(v) \propto \exp\left[-\frac{(v-v_0)^2}{\Delta v_0^2}\right] dv$ ; Probability of line being observed at a frequency  $v$ .

$$\int \phi(v) \cdot dv = 1 \Rightarrow \phi(v) = \frac{1}{\Delta v_0 \cdot \sqrt{\pi}} \exp\left(-\frac{(v-v_0)^2}{\Delta v_0^2}\right) ; \left\{ \Delta v_0 = \frac{v_0}{c \sqrt{2k_B T/m}} \right\}$$

↳ Gaussian. [why it is expected?]

## ② Natural line broadening :

$\Delta E \cdot \Delta t \geq h$ ; Heisenberg uncertainty principle

This is known as intrinsic line broadening

$$\frac{dn_u}{dt} = -n_u \sum_{\lambda < u} A_{u\lambda} = -n_u \gamma \quad \rightarrow \quad n_u \propto \exp[-\gamma t]$$

$$\frac{dn_u}{dt} = -\gamma \exp[-\gamma t]$$

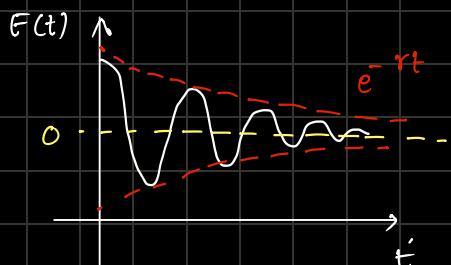
$P \propto -\frac{dn_u}{dt} \Rightarrow P \propto \gamma \exp[-\gamma t] \Rightarrow$  Electric field can be written as

$$E(t) \propto \overbrace{P(t)}^{} \propto \exp\left[-\frac{\gamma t}{2}\right] \cos(\omega t)$$

Power spectrum of the  $E(t)$  can be obtained by simply using F.T. i.e.

$$\hat{E}(\omega) = \int_0^\infty \frac{1}{2\pi} E(t) \cdot e^{i\omega t} dt$$

$$= \frac{E_0 \gamma^{1/2}}{4\pi} \int_0^\infty \left[ e^{-\left(\frac{\gamma}{2} - i(\omega + \omega_0)\right)t} + e^{-\left(\frac{\gamma}{2} - i(\omega - \omega_0)\right)t} \right] dt$$



$$= \frac{E_0 \gamma^{1/2}}{4\pi} \left[ \frac{1}{\frac{\gamma}{2} - i(\omega + \omega_0)} + \frac{1}{\frac{\gamma}{2} - i(\omega - \omega_0)} \right] \quad \therefore \text{We'll only consider this.}$$

↳ This says peak value at  $\omega = -\omega_0$  (Not possible)

$$P(\omega) = |\hat{E}(\omega)|^2$$

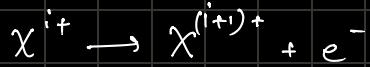
$$P(v) = \frac{E_0^2}{4\pi^2} \frac{r}{\gamma^2 + (v - v_0)^2} \quad \rightarrow \quad \phi(v) = \frac{4\gamma}{16\pi^2 \cdot (v - v_0)^2 + \gamma^2} \quad (\Delta v)_{\text{full}} = \frac{\gamma_{n1}}{2\pi}$$

↳ This is Lorentzian profile and generally it is much narrow at  $v \sim v_0$  compare to thermal broadening.

### Ionization state:

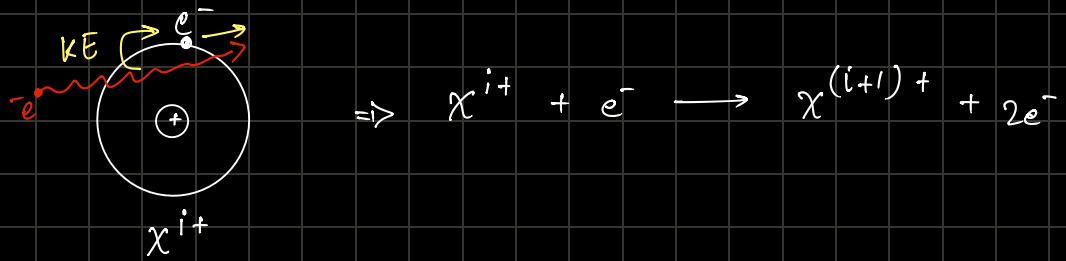
#### Types of ionization.

The process by which an atom or an ion loses an electron and becomes positively charged is known as ionization.



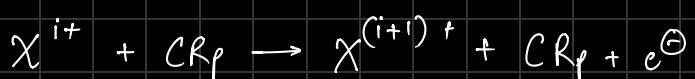
↳ To have this process, there has to be an agent or perturber.

When it is done by an electron, it is known as collisional ionization.

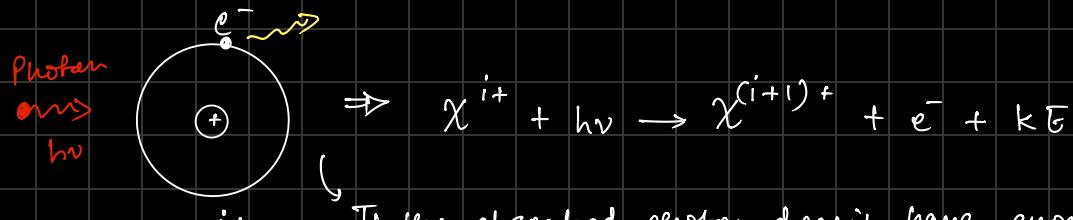


There is collisional ionization because of protons also but generally they have velocities much smaller compared to the electrons.

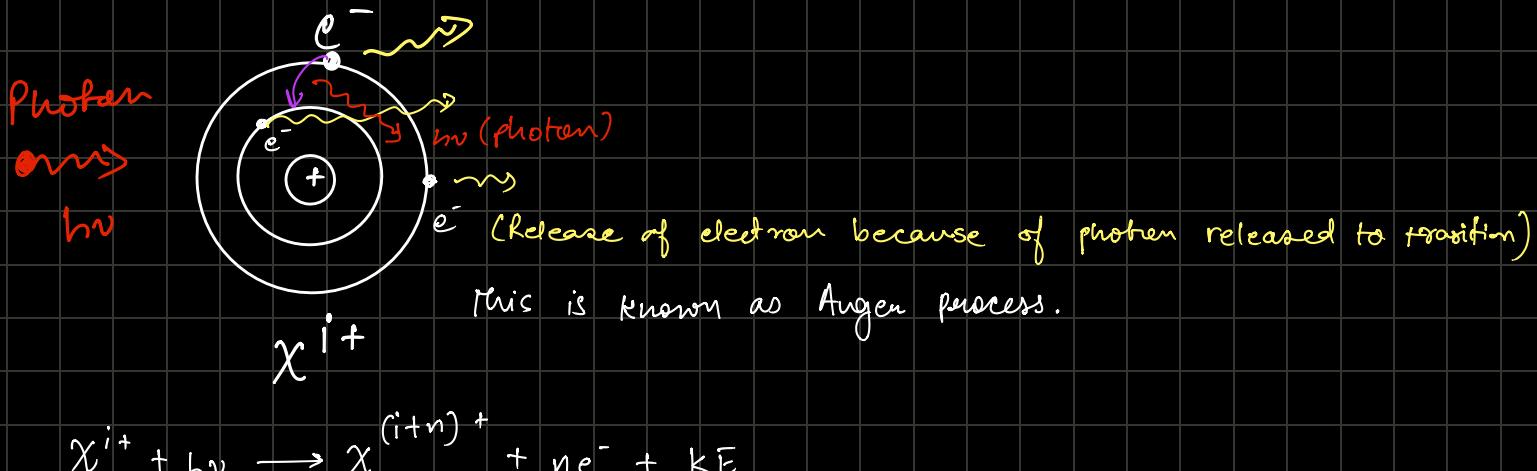
An example would be the ionization by cosmic ray proton ( $v_{\text{protons}}$  are high) which is known as cosmic ray ionization.



When this process is done by a photon then it's called photo-ionization.



If the absorbed photon doesn't have enough energy then atom goes to an excited state.



Photon will have energy  $h\nu$ , part of it will be used by ionization potential rest of it goes to the KE of the ejected electron which interacts to the medium via Coulomb interaction and increases the average KE of particles in the plasma. (Basically it heats up the medium).

There is also something called as  $X + Y^+ \rightarrow X^+ + Y$ , transfer of electron amongst 2 atoms/ions. For normal ISM cases, we will mostly focus on the collisional ionization and the photo-ionization.

### Statistical equilibrium in ions : Saha equation

Saha ionization describes a way to estimate the ionization balance in a thermal gas. For deriving the equation the important assumption is ions and electrons are part of an equilibrium statistical ensemble at a given temperature.

By definition, the system is in equilibrium, not losing energy  $\therefore$  Radiation produced by atoms must be absorbed by the gas itself so that the temp does not change at all.

$$\therefore \frac{N_A}{N_B} = \frac{g_A}{g_B} e^{-\frac{(E_A - E_B)}{k_B T}} ; \text{ Ratio of particles in two energy states.}$$



Electron ion pair has 2 energy levels  $\rightarrow$  one where electron is free

$$\therefore E_A = \frac{p^2}{2m}$$

$$\hookrightarrow E_B = -I_p, \text{ when it's bounded.}$$

$\Downarrow$  Ionization potential.

$$\therefore \frac{N_{i+1}}{N_i} = \frac{g_{i+1} \cdot g_e}{g_i} e^{-\left(\frac{p^2}{2m} + I_p\right)/k_B T}$$

$g_e$   $\equiv$  degeneracy of the A state consist of an ion and a free electron.

$$\begin{aligned} \because g_A = g_{i+1} \cdot g_e &\Rightarrow g_e = \text{no. of available states in phase space for an electron} \\ &= \text{no. of available states in space} \times \text{no. of states in momentum} \\ &= \frac{2 \cdot V_{\text{space}} V_{\text{mom.}}}{h^3} \end{aligned}$$

$$V_{\text{space}} = \frac{1}{n_e} \quad \Rightarrow \quad V_{\text{mom.}} = 4\pi p^2 \cdot dp$$

$\leftarrow$  number density

$$g_e = \frac{2}{h^3} \cdot \frac{1}{n_e} \cdot 4\pi p^2 \cdot dp = \frac{2}{n_e h^3} \cdot 4\pi p^2 \cdot dp$$

$$\frac{N_{i+1}}{N_i} = \frac{g_{i+1}}{g_i} \cdot \frac{2}{n_e h^3} \int_0^\infty e^{-\left(\frac{p^2}{2m} + I_p\right)/k_B T} \cdot 4\pi p^2 dp \Rightarrow \boxed{\frac{N_{i+1}}{N_i} = \frac{2 g_{i+1}}{g_i} \cdot \frac{(2\pi m_e k_T)^{3/2}}{n_e h^3} \exp\left(-\frac{I_p}{k_T}\right)}$$

$\hookrightarrow$  This equation is for states in statistical equilibrium b/w free photons, ions and free electrons

$\hookrightarrow$  Such an equilibrium is known as Local Thermodynamic Equilibrium (LTE)

$\hookrightarrow$  Such an equilibrium does not hold for ISM since photon can easily escape.

## Photo-ionization and recombination :

Photo-ionization is due to a photon interacting an electron.

$$R_{pi} = n_i n_{ph} \sigma_{pi} C = n_i \frac{4\pi J_N}{h\nu} \sigma_{pi}; \text{ Photo-ionization rate per unit vol.}$$

$$R_{pi} = \int_{h\nu_{min}}^{\infty} n_i \frac{4\pi J_N}{h\nu} \sigma_{pi}(\nu) d\nu = n_i \int_{h\nu_{min}}^{\infty} \sigma_{pi}(\nu) d\nu$$

$\hookrightarrow$  photo-ionization rate w-eff.

# We won't do any calculations here for  $\sigma_{pi}(\nu)$  since it's an elaborate QM cal.

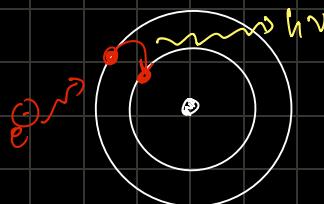
As discussed previously, every photon ionization is responsible for heating the medium known as photo-heating.

$$(H_{photo} = n_i \int_{h\nu_{min}}^{\infty} \frac{4\pi J_N}{h\nu} \sigma_{pi}(\nu) (h\nu - h\nu_{min}) d\nu)$$

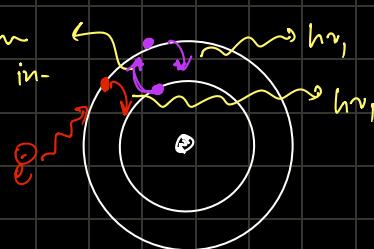
$\nu \longleftarrow$

Excess-energy after ionization, which goes to increase the KE of particles  $\therefore$  Increasing the temperature.

The opposite process of photo-ionization is known as recombination.



(Radiative recombination)



The rate of recombination per unit vol is given as

$$R_{rec} = n_i n_e \langle \sigma_{rec, nl} \nu \rangle$$

$$= n_i n_e \int \sigma_{rec, nl}(\nu) \cdot \nu f_{mb}(\nu) d\nu$$

$$= n_i n_e \cdot \frac{4(k_B T)^{3/2}}{\sqrt{2\pi m_e}} \int \sigma_{rec, nl}(x) \cdot x e^{-x} dx = n_i n_e \alpha_{rec, nl}(T)$$

Cross-section for the electron to recombine to a particular shell with principle and orbital quantum number  $(n, l)$

$$x = \frac{E}{kT}$$

The value of  $\sigma_{\text{rec},nl}$  can be obtained from  $\sigma_{pi}$  because both are inverse process

This is known as Einstein-Milne Relation

Einstein-Milne relation:



$$R_{\text{rec}} = R_{pi}$$

$$\therefore n_{i+1} \cdot n_e \int \sigma_{\text{rec},nl}(v) \cdot v f_{mb}(v) dv = n_i \int_{hv_{\min}}^{\infty} \frac{f_v}{hv} \sigma_{pi}(v) dv$$

$$\frac{1}{2} m_e v^2 = hv - I_b$$

$$\therefore n_{i+1} \cdot n_e \int \sigma_{\text{rec},nl}(v) \cdot v f_{mb}(v) dv = n_i \frac{4\pi J_v}{hv} \sigma_{pi}(v) dv$$

In statistical equilibrium b/w matter and photons, the radiation field would give us Blackbody radiation  $J_v = B_v$

$$\left( \frac{\sigma_{pi}(v)}{\sigma_{\text{rec}}(v)} = \frac{g_{i+1}}{g_i} \frac{m_e v^2 c^2}{h^2 v^2} \right) \Rightarrow \boxed{\text{Einstein-Milne relation}}$$

(  
The relation connects the cross-section for the polarization with the recombination rate for a particular energy level ( $n, l$ ).

$$\therefore \left[ \alpha_{\text{rec},nl}(T) = \int \sigma_{\text{rec},nl}(v) \cdot v f_{mb}(v) dv \right]$$

## Collisional ionization:

### ① Collisional ionization equilibrium:

This equilibrium occurs when the collisional ionization rate of an ion  $i$  to  $i+1$  becomes equal to the recombination rate of ion  $i+1$  to ion  $i$  i.e.

$$n_i n_e \langle \sigma_{ci} v \rangle = n_{i+1} \cdot n_e \alpha_{i+1}(\tau)$$

→ also denoted by  $\bar{\sigma}_{ci}(\tau)$ .

$$\therefore \frac{n_{i+1}}{n_i} = \frac{\langle \sigma_{ci} v \rangle}{\alpha_{i+1}(\tau)} \Rightarrow \langle \sigma_{ci} v \rangle = \int_0^\infty \bar{\sigma}_{ci}(v) v f_{mb}(v) dv$$

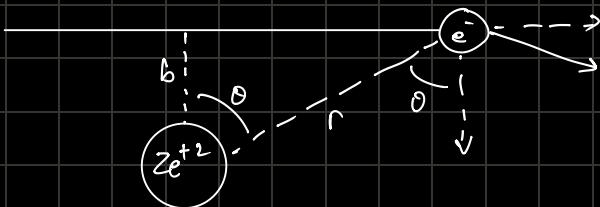
↳ This is true for the system that has only two ionization states.

### ② Collisional ionization cross-section:

Let's try to calculate  $\bar{\sigma}_{ci}(v)$

$$f(r) = -\frac{Ze^2 r}{r^2}$$

because of this force electron changes its direction.



Fixed i.e. stationary

$$F_\perp = F(r) \cos \theta = \frac{Ze^2 \cos \theta}{r^2} = \frac{Ze^2}{b^2} \cos^3 \theta$$

$$\therefore \frac{d \Delta p_\perp}{dt} = \frac{Ze^2}{b^2} \cos^3 \theta \Rightarrow \int_{-\infty}^{+\infty} F_\perp \cdot dt = \int_{-\infty}^{+\infty} f_\perp \frac{d(b \tan \theta)}{v}$$

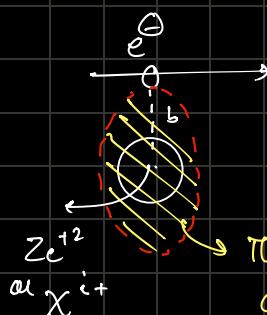
$$\Delta p_\perp \Rightarrow \frac{Ze^2}{b v} \int_{-\pi/2}^{\pi/2} \cos \theta \cdot d\theta \Rightarrow \frac{2Ze^2}{b v}$$

$$\therefore \Delta E = \frac{(\Delta p_\perp)^2}{2m} = \frac{2Z^2 e^4}{m_e b^2 v^2}$$

Now, we want that  $\Delta E > I_p$

$$\therefore \frac{2Z^2 e^4}{m_e b^2 v^2} > I_p$$

$$\therefore b_{\max} = \frac{2c^2}{v} \left( \frac{2}{m_e I_p} \right)^{1/2}$$



→ This should be cross-section.

$$\therefore \left\{ \sigma_{ci} = \sigma_{ci}(v) = \pi b_{max}^2 = \frac{2\pi z^2 e^4}{m_e v^2 I_p} \right\}$$

$$KE_{ele} > I_p \Rightarrow v_{min} = \sqrt{\frac{2I_p}{m}} \text{ for ionization}$$

$$\therefore \langle \sigma_{ci} v \rangle = \int_{v_{min}}^{\infty} \sigma_{ci}(v) \cdot v f_{mb}(v) dv = z^2 \left( \frac{8\pi}{m_e k_B T} \right)^{1/2} \frac{e^4}{I_p} e^{-I_p/k_B T} = \xi_{ci}(T)$$

$\Rightarrow$  If the velocity of ion 2 is modified significantly. This happens when  $I < E_e \leq 3I$ , in such cases the rate co-efficient is modified to

$$\left\{ \xi_{ci}(T) = C n a_0^2 \left( \frac{8kT}{\pi m_e} \right)^{1/2} e^{-I_p/kT} \right\}$$

where  $C \approx 1$   $a_0 = \frac{\hbar^2}{m_e e^2} \leftarrow$  Classical Bohr radius

When an ion travels very close to the neutral atom, such a close encounter creates a dipole, the force is given as

$$F_r = -\frac{2\alpha_N Z^2 e^2}{r^2}; \quad \alpha_N \text{ is the polarizability.}$$

Ionized region around a star:

Photo-ionization becomes very important with a high radiation field  $F_r$ . Usually this happens around radiation sources such as star. Radiation energy  $>> 13.6 \text{ eV} \therefore$  It ionize the hydrogen gas. Such ionized regions around the stars are known as Stromgren sphere.

Consider star emitting photon ( $E > 13.6 \text{ eV}$ ) at rate  $Q$  into the surrounding neutral medium that has hydrogen density of  $n$  particles  $\text{cm}^{-3}$ .

$$\therefore \text{After } t \text{ time, } n_{ph} = Qt$$

$$\therefore \text{Size of ionized region would be } \frac{4\pi}{3} r_{if}^3 n = Qt$$

$$\therefore r_{if} (\text{ionized region aka}) = \sqrt[3]{\frac{3Qt}{4\pi n}}$$

Now once these  $H^+$  ions starts recombining, it will slow down the process.

To process will stop if all free  $H^+$  gets recombined.

Time scale over which all ions recombine is .

$$\frac{dn_i}{dt} = R_{rec} = n_i n_e \alpha_{i+1}(r)$$

$$\therefore \tau_{rec} \sim \frac{1}{n_e \alpha_{i+1}(r)} \quad \text{Recombination time scale.}$$

$$r_{st} = \left( \frac{3 Q \tau_{rec}}{4 \pi n} \right)^{1/3} = \left( \frac{3 Q}{4 \pi n^2 \alpha_{H^+}(r)} \right)^{1/3} = 93 \cdot 3 \text{ pc} \quad Q_{ua}^{1/3} - n^{-2/3} \alpha_{-13}^{-1/3}$$

$$Q_{ua} = Q / 10^{19}$$

$$\alpha_{-13} = \alpha / 10^{-13} \text{ cm}^3$$

This radius is known as the Strongem radius.