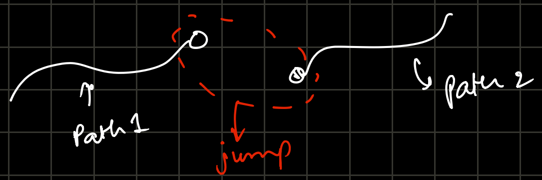
# Topological spaces :

At the large level : spacetime is a set

But it is not enough to talk about continuity of maps

So, when we are talking about classical physics, we say there are not jumps in the trajectory of the particle




⇒ We do not want that

weakest structure that can be established on a set which allows definition of continuity of maps.

↳ a topology

Def : let $M$ be a set. A topology $\mathcal{O}$ is a subset $\mathcal{O} \subseteq P(M)$

↳ Power set of $M$, set of all subsets of $M$.

# Topology is a subset satisfying the 3 axioms.

i) $\phi \in \mathcal{O}$ (because $\phi \in M$), $M \in \mathcal{O}$

ii) $U \in \mathcal{O}$, $V \in \mathcal{O}$ $\Rightarrow$ $U \cap V \in \mathcal{O}$

iii) $U_{\alpha} \in \mathcal{O}$ $\Rightarrow \left( \bigcup\limits_{\alpha \in A} U_{\alpha} \right) \in \mathcal{O}$

$\alpha \in A$

Examples :

$M = \{1,2,3\}$

a) $\mathcal{O}_1 := \{\phi, \{1,2,3\}\}$, is $\mathcal{O}_1$ a topology ? (Ans is yes)

b) $\mathcal{O}_2 := \{\phi, \{1\}, \{2\}, \{1,2,3\}\}$

↳ First axiom is satisfied

ii) $\phi \cap \{1\} \in \mathcal{O}$, $\phi \cap \{2\} \in \mathcal{O}$ $\phi \cap \{1,2,3\} \in \mathcal{O}$

$\{1\} \cap \{2\} \in M$, $\{1\} \cap \{1,2,3\} \in \mathcal{O}$

$\{2\} \cap \{1,2,3\} \in \mathcal{O}$

(iii) $\phi \cup \{1\} \cup \{2\} \cup \{1,2,3\} \in M$ but but but ... $\{1\} \cup \{2\} \notin \mathcal{O}$

$\therefore$ It's not a topology.

M be any set

(a) $\mathcal{O} := \{\emptyset, M\}$

this type of topology is called 'Chaotic' topology

(b) $\mathcal{O} := \mathcal{P}(M)$

this is called a 'discreate' topology

(c) $M = \mathbb{R}^d = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$

$= \{(p_1, \ldots, p_d) \mid p_i \in \mathbb{R}\}$

<u>def</u> $\mathcal{O}_{standard} \subseteq \mathcal{P}(\mathbb{R}^d)$

Soft ball : $B_r(p) := \{(q_1, \ldots, q_d) \mid \sum_{i=1}^{d} (q_i - p_i)^2 < r^2\}$

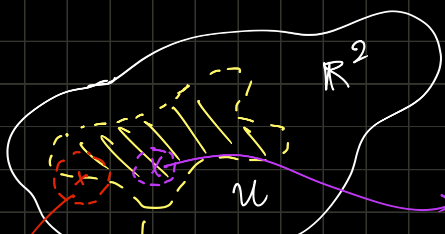
$r \in \mathbb{R}^+$ (Radius of the ball), $p \in \mathbb{R}^d$, $(q_1, \ldots, q_d) \in \mathbb{R}^d$

$\mathcal{U} \in \mathcal{O}_{standard}$

$\Updownarrow$

$\forall p \in \mathcal{U} \; \exists r \in \mathbb{R}^+ : B_r(p) \subseteq \mathcal{U}$

In words, for every point p in $\mathcal{U}$, there exist a r in $\mathbb{R}^+$ for which $B_r(p)$ lies entirely in $\mathcal{U}$.




consider any point in $\mathcal{U}$, you can make a soft ball.

but if we would have consider boundary then,

even if we take a very small radius, there would be points lying out of $\mathcal{U}$.

boundary doesn't belong to the set

<u>Terminology:</u>

M → set

$\mathcal{O}$ → topology := set of open sets

$(M, \mathcal{O})$ → topological space

$\mathcal{U} \in \mathcal{O}$ call $\mathcal{U}$ an <u>open set</u>

$M \setminus A \in \mathcal{O} \Rightarrow$ call $A \subseteq M$ closed set

compliment

is closed set opposite of open set ?

NO!!

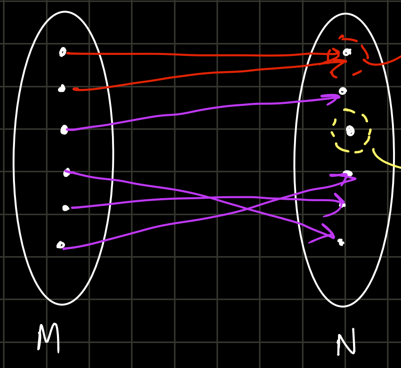
$\phi$, $\phi \in \mathcal{O} \Rightarrow \phi$ is an open set

$M / \phi = M \in \mathcal{O} \Rightarrow \phi$ is a closed set

## 2. Continuous maps :

A map $f : \underset{\text{set}}{M} \rightarrow \underset{\text{set}}{N}$

(Domain) (Target)




If this point exist then the map is not injective

if this point exist then then map is not surjective

Is this map continuous ??

It depends on which topologies are chosen on $M$ & $N$

**Def:** Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological spaces, then a map

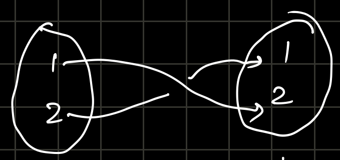
$f : M \rightarrow N$ is called continuous (wrt $\mathcal{O}_M$ & $\mathcal{O}_N$) if $\forall V \in \mathcal{O}_N : \text{preim}_f(V) \in \mathcal{O}_M$

$m \mapsto f(m)$ $\Downarrow$ (pre-image)

Ex : $M = \{1, 2\}$ $\quad \mathcal{O}_M = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

$N = \{1, 2\}$ $\quad \mathcal{O}_N = \{\phi, \{1,2\}\}$

(a)

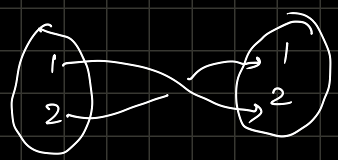



$f : M \longrightarrow N$

$\therefore \text{preim}_f(\phi) = \phi \in \mathcal{O}_M$

$\text{preim}_f(\{1,2\}) = M \in \mathcal{O}_M$

$\therefore f$ is continuous.




$g : N \longrightarrow M$

$\therefore \text{preim}_g(\phi) = \phi \in \mathcal{O}_N$

$\text{preim}_g(\{1\}) = \{2\} \notin \mathcal{O}_N$

$\therefore g$ is not continuous

$\hookrightarrow g = f^{-1}$ basically.

## 3. Composition of continuous maps :

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & N \arrow[r, "g"] & P \end{tikzcd}$$

if $f$ is continuous, $g$ is continuous

then $g \circ f$ is also continuous.

$g \circ f : M \rightarrow P$

$m \mapsto (g \circ f)(m) := g(f(m))$

$\Rightarrow$ Proof: let $V \in \mathcal{O}_P$

$\text{preim}_{g \circ f}(V) := \{m \in M \mid (g \circ f)(m) \in V\}$

$= \{m \in M \mid f(m) \in \text{prim}_g(V)\}$

$= \text{preim}_f(\underbrace{\underbrace{\text{preim}_g(V)}_{\in \mathcal{O}_N}}_{\in \mathcal{O}_M})$

4. <u>Inheriting a topology:</u>

Many useful way to inherit a topology from some given topological space (s)

Important for spacetime physics

$S \subseteq \mathcal{O}_M \longrightarrow M$ (topological space)

Can we construct on $S$ a topology from $\mathcal{O}_m$ on $M$.

Yes, we can.

Def $\mathcal{O}|_S \subseteq P(S)$ "Subset topology"

$$\mathcal{O}|_S := \{ U \cap S \mid U \in \mathcal{O}_M \}$$

<u>Claim</u>: ① $\phi = \underset{\mathcal{O}_M}{\underset{\cap}{\phi}} \cap S \Rightarrow \phi \in \mathcal{O}|_S$

$S = \underset{\mathcal{O}_M}{\underset{\cap}{M}} \cap S \Rightarrow S \in \mathcal{O}|_S$

② $A \in \mathcal{O}_S$, $B \in \mathcal{O}_S \Rightarrow \exists\ \tilde{A} \in \mathcal{O}_M$, $\tilde{B} \in \mathcal{O}_M$

$A = \tilde{A} \cap S$

$B = \tilde{B} \cap S$

$$A \cap B = (\tilde{A} \cap S) \cap (\tilde{B} \cap S) = (\underbrace{\tilde{A} \cap \tilde{B}}_{\mathcal{O}_M} \cap S)$$

$\therefore A \cap B = \mathcal{O}_M \cap S$