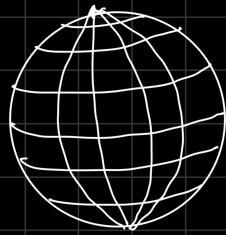
Curved Manifolds

Manifolds:

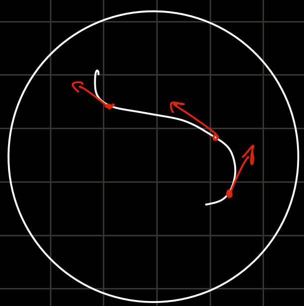
A manifold is essentially a continuous space which looks locally like Euclidean space.

You can imagine surface of sphere to be a manifold



Since the surface of sphere is a 2D manifold therefore we can parameterize it by 2 co-ordinates θ, ϕ.

Now, since we would like to study on these manifolds $\therefore$ We would it be continuous and differentiable, so if a manifold follow this then we can define vectors and co-vectors (one-forms)



then we would be also able to define a curve on the manifold, every curve has a tangent vector $\vec{v}$ defined as the linear function that takes the one-form $\tilde{d}\phi$ into the derivate of ϕ along the curve.

$$\langle \tilde{d}\phi, \vec{v} \rangle = \vec{v}(\tilde{d}\phi) = \nabla_{\vec{v}} \phi = \frac{d\phi}{d\lambda}$$

Let's define things properly and mathematically.

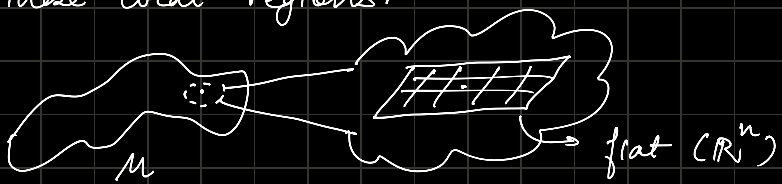
Let's again ask your-self, what is manifold?

We are all used to the properties of n-dimentional Euclidean space $\mathbb{R}^n$ but what if I want to describe a sphere? or any "curved" surface.

To address this problem we invent the notition of manifold, in general I don't care how space looks but local regions looks just like $\mathbb{R}^n$.
Now this does not mean that $g_{\mu\nu}(\text{space}) = \eta_{\mu\nu}$, our meaning was primitive notitions like functions and coordinates work in a similar way.

The entire manifold can be imagined as a fabric which is sewed together by these local regions.

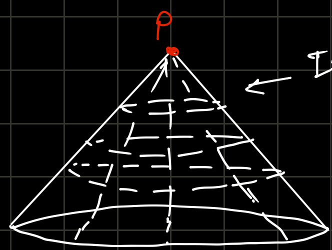
eg)



Direct product of two manifolds is a manifold.

e.g. M is of dimension n
M' is of dimension n' } $M \times M'$ will be a manifold of dimension $n+n'$, consisting of ordered pairs (p, p')
$p \in M$, $p' \in M'$.

Now tell is this a manifold



Is this a manifold?

and this is where we will also start talking about the "differentiable manifold". It is not differentiable at p.

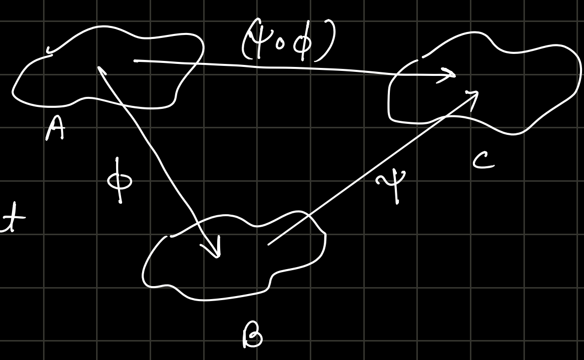
Now, it's time to make a rigorous definition.

Let's make some preliminary definitions.

- Given two sets M and N, a map $\phi : M \to N$ is a relationship that assigns to each element of M exactly one element of N. $\therefore$ A map is a generalization of function.

Now given two maps $\phi : A \to B$ & $\psi : B \to C$ we can define "composition" $\psi \circ \phi : A \to C$ by the operation $(\psi \circ \phi)(a) = \psi(\phi(a))$

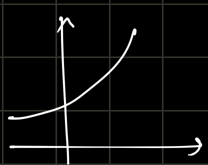
$a \in A$, $\phi(a) \in B$, $\psi(\phi(a)) \in C$



ϕ is called one-one or injective if each element of B has atmost one element of M mapped into it.

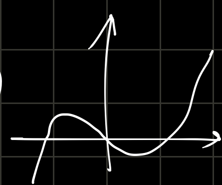
ϕ will called onto if B has atleast one element in A.

$\phi(x) = e^x$



one-one not onto

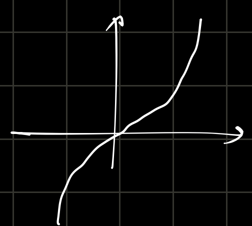
$\phi(x) = x(x-1)(x+1)$



$\phi(0), \phi(1), \phi(-1) \to 0$

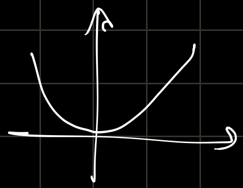
$\therefore$ onto but not one-one.

$\phi(x) = x^3$

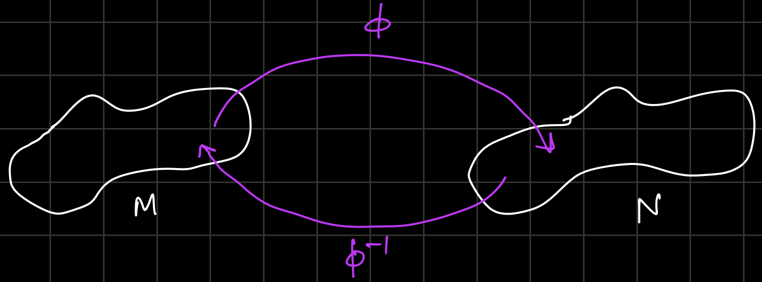


both

$\phi(x) = x^2$

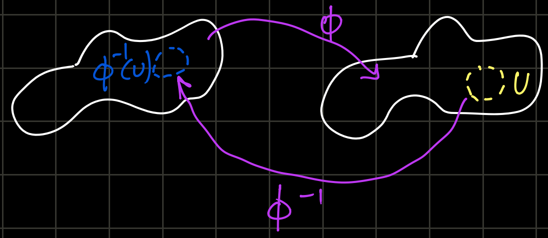


$\phi(x) : R \longrightarrow R^+$ $\quad R^+ \subset R$ $\quad \therefore \phi(x) = x^2$ is neighter one-one nor onto.



M is the "domain" of the map ϕ and set of points in N that M gets mapped into is called the "image" of ϕ.

For any subset $U \subset N$, the set of elements of M that get mapped to U is called the pre-image of U or $\phi^{-1}(U)$



A map that is both one-one and onto is known as invertible or bijective

In this case the inverse map in $\phi^{-1} : N \to M$ by $(\phi^{-1} \circ \phi)(a) = a$

Okay, now lets talk about the continuity and differentiability.

If a map $\phi : R^m \longrightarrow R^N$ $\quad R^m \equiv (x^1, x^2, \dots, x^m)$

$R^N \equiv (y^1, y^2, \dots, y^n)$

$\therefore y^1 = \phi^1(x^1, x^2, \dots, x^m) \dashrightarrow y^p = \phi^p(x^1, x^2, \dots, x^m)$ $\quad p = \{1, 2, \dots, n\}$

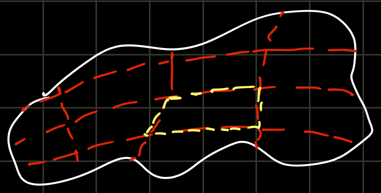
If for y^p p^{th} derivative exist and is continuous then we say

$\phi : R^m \to R^n$ as C^p, so C^0 map will be continuous by not necessarily differentiable, C^∞ map is continuous and can be differentiated as many times you like.

C^∞ maps are called smooth

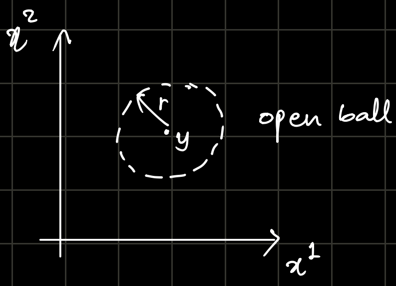
The sets M, N will be called diffeomorphic if there exist a C^∞ map $\phi : M \to N$ with a C^∞ inverse $\phi^{-1} : N \to M$; the map is called a diffeomorphism.

And here we are done with our preliminary definition. Lets move on to rigorous discussion.



→ Everything inside the white region is called "Manifold"
→ Yellow box is called a "chart".
→ Collection of the yellow boxes tha can cover the whole manifold is called "Atlas".

To define these things, we first have to define much more abstract thing "an open ball".

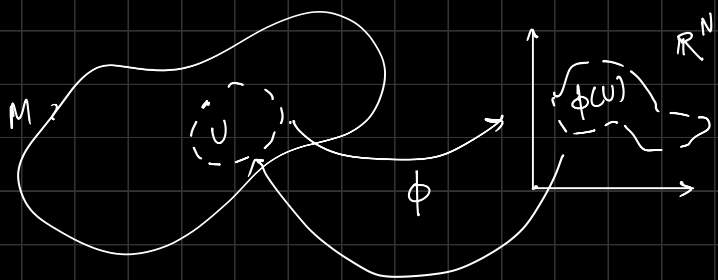


It is a set of all points x in $\mathbb{R}^N$ such that $|x-y| < r$ for some for fixed $y \in \mathbb{R}^N$ and $r \in \mathbb{R}$.

$$|x-y| = \left|\sum_i (x^i - y^i)\right|^{1/2}$$

The union of such open-balls will form an open set. or in other words $V \subset \mathbb{R}^N$ is open if for any $y \in V$ there is an open ball centered at y that completely inside V.

A chart or coordinate system consist of a subset U of a set M along with a one-to-one map $\phi : U \to \mathbb{R}^N$, such that $\phi(U)$ is open in $\mathbb{R}^N$.



∴ we then can say U is an open set in M.

ϕ^{-1} exist if it is one-one.

A C^∞ atlas is an indexed collection of charts $\{(U_\alpha, \phi_\alpha)\}$ that satisfies two conditions

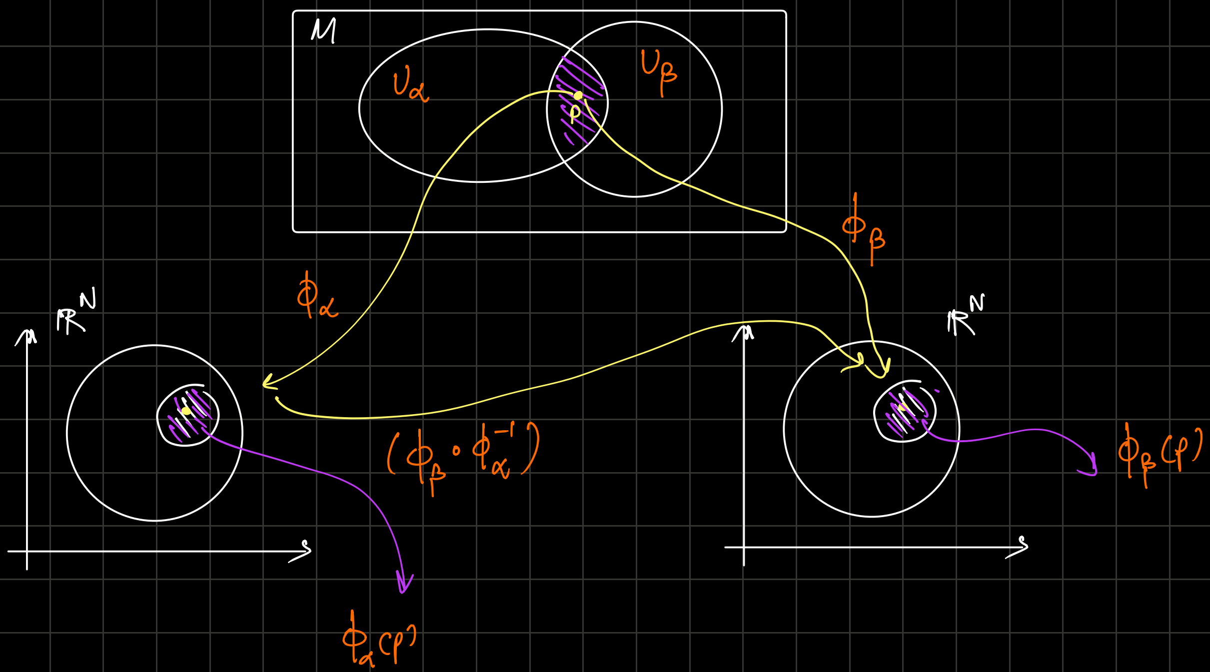
$$\bigcup_{\alpha=1}^{N} U_\alpha = M$$ (U_α will cover the manifold).

There also a term called "Maximal Atlas", that atlas has "all the possible compatible" manifolds.

• Overlapping of coordinate charts:

Now since atlas has all the charts that can cover the manifold then

$U_\alpha \cap U_\beta \neq \phi$!!



Imagine a point $p \in U_\alpha \cap U_\beta$, $\phi_\beta : p \to \phi_\beta(p)$ $\phi_\alpha : p \to \phi_\alpha(p)$

$(\phi_\beta \circ \phi_\alpha^{-1})$

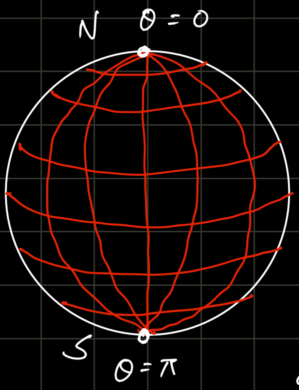
$\therefore (\phi_\beta \circ \phi_\alpha^{-1})(\phi_\alpha(p)) = \phi_\beta(p$

let's take sphere S^2 as an example.

Why can't we cover sphere S^2 with a single chart (map) $x : S^2 \to \mathbb{R}^2$

$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$

you want to use coordinates (a chart) to map points on the sphere to $\mathbb{R}^2$. One simple way to do that is to use spherical coordinates



at $\theta = 0$ $\sin\theta = 0$; now no matter what ϕ is $(x, y) = (0, 0)$

why it is a problem ?

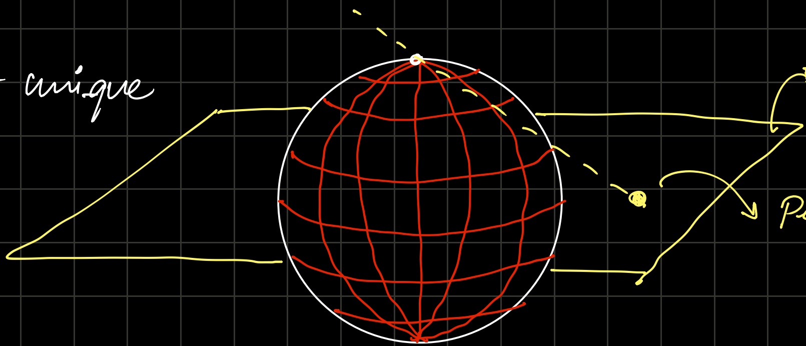
$\Rightarrow$ It violates injectivity (one to one)

different $(0, \phi)$ at $\theta = 0$ map to the same point on the sphere, map isn't invertible at the pole.

$\therefore$ Mapping is not a homeomorphism at the poles

$(0,0,1)$
$\downarrow$
$(0, \phi)$
$\hookrightarrow$ not unique ϕ.

To fix this problem, we use stereographic projection.



$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \qquad N = (0,0,1) \; ; \; S = (0,0,-1)$$

each projection (not including N pole) maps the sphere to the equatorial plane $z = 0$, we identify as $\mathbb{R}^2$.

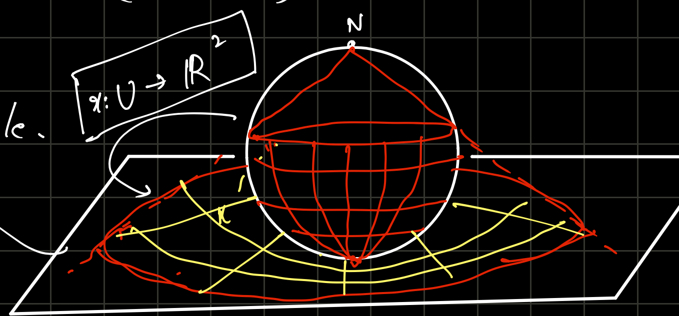
Projection from north pole : $\left(\phi_S : U_S \to \mathbb{R}^2\right)$

$$\phi_S(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$ $\Rightarrow$ this is bijective, continuous map $U_S \to \mathbb{R}^2$

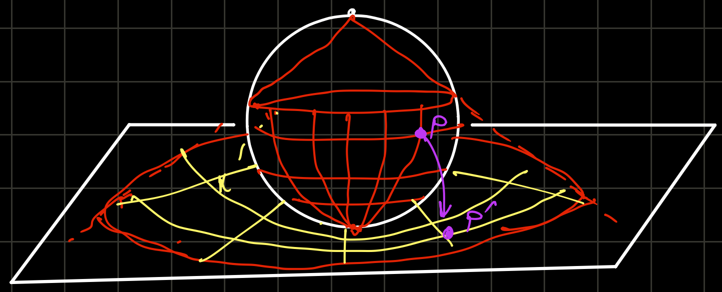
blows at $z = 1$. (Not -1).

$$\phi_S^{-1}(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right) \; ; \; (u, v \in \mathbb{R}^2)$$

we can do the same thing with south pole.



this is a flat circle my drawing is bad!



P' being the project point P on the sphere, the coordinates P' are non-physical, it just tells us where it is existing in that particular chart.

let's take one more example, M be a wire loop. We know that it exist in our 3D world that means, in is embedded in $\mathbb{R}^3$ but actually M is a 1D topological manifold, similarly torus is 2D topological manifold.

How? (Hint : Imagine you are an ant walking on these shapes & you will have your answer.

The metric and local flatness :

The metric provides the a mapping b/w vectors and one-forms at every point.

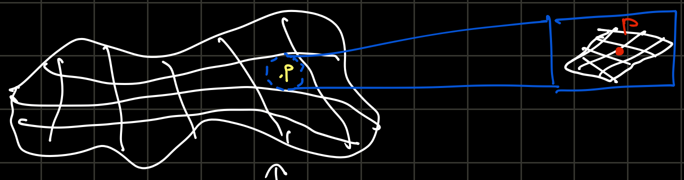
A vector field at point P is $\vec{V}(p)$, a unique one-form field $\tilde{V}(p) = g(\vec{V}(p), \ )$

The components of g are called $g_{\alpha\beta}$

$g_{\alpha\beta}$ permits raising and lowering of indices, e.g $V_\alpha = g_{\alpha\beta} V^\beta$.

Now let's talk about local flatness.

For a general curved metric, we cannot define a global Lorentz frame and that why will try to built a local lorentz frame.



← locally we can see that the space is "nearly" flat.

this is a general curved metric.

$$\therefore\ g_{\mu\nu} = \eta_{\mu\nu} + O((x^\mu)^2) \rightarrow \quad g_{\alpha\beta}(p) = \eta_{\alpha\beta} \quad \forall\, \alpha,\beta \quad - (1)$$

$$\partial_\gamma g_{\alpha\beta}(p) = 0 \quad \forall\, \alpha,\beta,\gamma \quad - (2)$$

$$\left[\frac{\partial^2}{\partial x^\gamma \cdot \partial x^\mu} g_{\alpha\beta} \neq 0 \right] \quad - (3)$$

<u>proof</u>:

let's take any arbitary coordinate system x^α at a certain fix point P.

$$x^\alpha = x^\alpha(x^{\mu'}) \quad ; \quad \Lambda^\alpha_{\mu'} = \frac{\partial x^\alpha}{\partial x^{\mu'}}$$

$$\Lambda^\alpha_{\mu'}(\vec{x}') = \Lambda^\alpha_{\mu'}(p) + (x^{\gamma'} - x_0^{\gamma'}) \frac{\partial \Lambda^\alpha_{\mu'}}{\partial x^\gamma} + \frac{1}{2}(x^{\gamma'} - x_0^{\gamma'})(x^{\lambda'} - x_0^{\lambda'}) \left(\frac{\partial^2 \Lambda^\alpha_{\mu'}}{\partial x^{\lambda'} \partial x^{\gamma'}} \right)$$

– Taylor expansion about P.

Expanding the metric in the same way gives

$$g_{\alpha\beta}(\vec{x}') = g_{\alpha\beta}\Big|_p + (x^{\gamma'} - x_0^{\gamma'}) \frac{\partial g_{\alpha\beta}}{\partial x^{\gamma'}}\Big|_p + \frac{1}{2}(x^{\gamma'} - x_0^{\gamma'})(x^{\lambda'} - x_0^{\lambda'}) \frac{\partial^2 g_{\alpha\beta}}{\partial x^{\lambda'} \cdot \partial x^{\gamma'}}\Big|_p + \cdots$$

$$\left\{ g_{\mu'\nu'} = \Lambda^\alpha_{\mu'} \Lambda^\beta_{\nu'} g_{\alpha\beta} \right\}$$

Now, we don't know what are these transformations, then how can we say $g_{\alpha\beta,\gamma,\mu} \neq 0$

let's calculate the number of free variables.

$\Lambda^{\alpha}_{\mu'}\big|_p$ has 16 free variable. (4 from α × 4 from μ')

then we have $\left.\dfrac{\partial^2 x^\alpha}{\partial x^{\gamma'}\partial x^{\mu'}}\right|_p$, it has $4\times4\times4$ available options but $\partial x^{\gamma'}\partial x^{\mu'} = \partial x^{\mu'}\partial x^{\gamma'}$ ∴,

	μ				
$\gamma - 1$	4	11	12	13	14
$\gamma - 2$	3	21	22	23	24
$\gamma - 3$	2	31	32	33	34
$\gamma \to 4$	1	41	42	43	44

10 free variables for $g_{\gamma\mu}$

for $\partial_{\gamma'}\partial_{\mu'}\partial x^\alpha = 4\times10$ free variables $= 40$

for $\partial_{\lambda'}\partial_{\gamma'}\partial_{\mu'}\partial x^\alpha = 4\times20$ free variables $= 80$

Well for $g_{\alpha\beta}$ we have,

In case of $\partial_\gamma g_{\alpha\beta} = 4\times10 = 40$

γ can take 4 values

In case of $\partial_\gamma\partial_\mu g_{\alpha\beta} = 10\times10 = 100$ free variables

$$g_{\mu\nu}\big|_p = \eta_{\mu\nu} \Rightarrow \eta_{\mu\nu} = \Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu}\cdot g_{\alpha\beta}\big|_p$$

For $g_{\alpha\beta}$ we have 10 eq^n but Λ^α_μ have 16 free variables, ∴ 10 eq^n's can be satisfied leaving six elements of Λ^α_μ unspecified. These unspecified six elements are six degrees of freedom in Lorentz transformation, 3 - boost and 3 - rotation. ∴ eq 1 is satisfied in Local flatness

Now eq 2

$\partial_\mu g_{\alpha\beta} = 0$ It will requiere 40 equations and yes when we transformation of $g_{\alpha\beta} = \Lambda^\mu_\alpha \Lambda^\nu_\beta g_{\mu\nu}$ and taylor expand $g_{\alpha\beta}$ we get $\left.\dfrac{\partial\Lambda^\mu_\alpha}{\partial x^\gamma}\right|_p$

$\partial_\gamma \Lambda^\mu_\alpha = 4\times10 = 40$ (It has exactly 40 numbers) ∴ eq 2 is also satisfied. (No extra D.O.F)

Now coming to eq(3).

From taylor expansion we have $\Lambda^{\mu}_{\alpha,\gamma\lambda} = 80$ and $g_{\alpha\beta,\gamma\lambda} = 100$ } $100 - 80 = 20 \leftarrow$ Degrees of freedom.

And these D.O.F will be responsible for representing the curvature of the metric.

Causality:

We can cast many physical problems an "initial value problem", given the state of a system at some t, can we say what will be the state at some time later.

eg) $\partial_t^2 \psi = \partial_x^2 \psi$ here we know how will ψ evolve in time but in general can we do it for all physical systems?

Answer to this comes from Causality, the idea that future events can be understood as consequence of initial conditions plus the laws of physics.

Since initial value formalism is very common in G.R, we are going briefly introduce some of the concepts used in understanding how causality works in G.R.

We will look at the problem of evolving fields (matter fields) on a fixed background spacetime.

Our guiding principle is that nothing can travel faster than speed of light $\therefore$ signals would travel on null or time like trajectories.

These trajectories won't be necessarily geodesics.

Alright is non-spacelike = time like ? or time-like = non-space like ?

Correct answer is 2nd option. non-spacelike ≤ 0, time-like < 0 } ds^2

Tensor densities:

lets have a discussion on levi-Civita symbol.

$$\varepsilon_{\mu_1\mu_2\cdots\mu_n} = \begin{cases} +1 & \mu_1\mu_2\cdots\mu_n \text{ is an even permutation of } 01\cdots(n-1) \\ -1 & \mu_1\mu_2\cdots\mu_n \text{ is an odd permutation of } 01\cdots(n-1) \\ 0 & \text{otherwise.} \end{cases}$$

e.g)

$\varepsilon_{123} = 1 \qquad \varepsilon_{213} = -1 \qquad \varepsilon_{113} = 0$

It's behaviour can be related to that of an ordinary tensor by noting that given any $n\times n$ matrix $M^{\mu}_{\mu'}$, the determinant $|M|$ obeys.

$$\tilde{\varepsilon}_{\mu_1'\mu_2'\cdots\mu_n'} = |M|^{-1}\,\tilde{\varepsilon}_{\mu_1\mu_2\cdots\mu_n}\, M^{\mu_1}_{\mu_1'} M^{\mu_2}_{\mu_2'} \cdots M^{\mu_n}_{\mu_n'} \quad ; \quad |M^{-1}| = |M|^{-1}$$

$$= \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right| \tilde{\varepsilon}_{\mu_1\mu_2\cdots\mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdot \frac{\partial x^{\mu_2}}{\partial x^{\mu_2'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}$$

levi-Civita symbol transforms in a way very close to the tensor transformation except the determinant out front.

Objects transforming in this way are know as tensor densities.

A quantity is called a tensor density of weight ω if it transforms as

$$\left[A'_{i_1\cdots i_n} = \left|\det\left(\frac{dx'}{dx}\right)\right|^{\omega} \left(\frac{\partial x^{j_1}}{\partial x'^{j_1}}\cdots\right) A_{j_1\cdots j_n} \right]$$

$\therefore$ Levi-Civita is weight 1 tensor density.

For someone who is confused about the transformation of the levi-Civita symbol wait for me to introduce differential forms and everything would be clear for now just take my word.

Now in order to make tensor density a true tensor, it does not take much work, we have to multiply a tensor density of weight ω with $|g|^{\omega/2}$ and what we will get is a true tensor.

Try to prove it why we are multiplying $|g|^{\omega/2}$.

$$\therefore \boxed{\varepsilon_{\mu_1\mu_2\cdots\mu_n} = \sqrt{|g|}\;\varepsilon_{\mu_1'\mu_2'\cdots\mu_n'}} \quad - \text{ Levi-Civita tensor.}$$

$$\left\{ \varepsilon^{\mu_1 \mu_2 \cdots \mu_n} = \frac{1}{\sqrt{|g|}} \varepsilon^{\mu_1' \mu_2' \cdots \mu_n'} \right\}$$

Differential forms :

let us now talk about a very special class of tensors, known as "Differential forms".

A differential p-form is simply a $(0,p)$ tensor that is completely anti-symmetric.

$\therefore$ A scalar is 0-form
Dual vectors are 1-forms.

The space of all p-forms is denoted by Λ^p and the space of all p-form fields over a manifold M is denoted by $\Lambda^p(M)$

Number of independent p-forms in n-dimensional space is $\binom{n}{p}$.

Wedge product :

let A be a p-form and q form B, we can form a $(p+q)$ form know as wedge product $A \wedge B$ by taking the antisymmetrized tensor product.

$$(A \wedge B)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!\, q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]}$$

$$\text{e.g)}\ (A \wedge B)_{\mu\nu} = 2 A_{[\mu} B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu$$

$$\left[A \wedge B = (-1)^{pq} B \wedge A \right]$$

Exterior derivative :

It allows us to differentiate p-form fields to obtain $(p+1)$-form fields. It is defined as an appropriately normalised, antisymmetrized partial derivative.

$$(dA)_{\mu_1 \cdots \mu_{p+1}} = (p+1)\, \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]}$$

Simplest example is 0-form $\Rightarrow (\partial\phi)_\mu = \partial_\mu \phi$

p-form ω and q-form η follows this : $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$

The reason why exterior derivative is necessary is that it is a tensor unlike the partial derivative. Another interesting fact about exterior differentiation is that, for any forms A.

$$d(dA) = 0 \quad - \text{ Try to prove it on your own.}$$

We define a p-form A to be closed if $dA = 0$, and exact if $A = dB$ for some $(p-1)$-form B. (obviously all exact are forms are closed)

On a manifold M, closed p-forms comprise a vector space $Z^p(M)$ and exact forms comprise a vector space $B^p(M)$.

The de-Rham Cohomology group : $H^p(M)$

It is the quotient space of the closed p-forms by the exact p-forms.

This means two closed forms ω_1 & ω_2, to be equivalent if they differ by an exact form. ω_1, ω_2 basically belong to the same cohomology class if

$$\omega_1 - \omega_2 = d\eta \quad \text{for some } (p-1)\text{-form } \eta.$$

e.g) $H^p(M) = \{0\}$ it means that every closed p-form on the manifold M is also exact.

If $H^p(M)$ is non-trivial, it means there are closed p-forms that are not exact.

The failure is directly related to the topology of the manifold. Loosely speaking, the dimentions of the cohomology called the "Betti number" counts the number of "p-dimentional holes" in a manifold.

Examples: 0^{th} cohomology, 0-form of a function f is $df = 0 \therefore$ function is constant locally.

If manifold is connected (continuous) then f is globally constant.

Example: for a circle (S^1)

$$H^0(S^1) = \frac{Z^0(S^1)}{B^0(S^1)} \qquad df = \frac{\partial f}{\partial \theta} . d\theta$$ – one-form of a function f.

0-form is closed if it's $df = 0$ (Exterior derivative)

$$df = 0 \Rightarrow \frac{\partial f}{\partial \theta} = 0 \qquad \therefore f(\theta) = C$$

$\therefore$ The space of closed 0-forms $Z^0(S^1)$ is the space of all constant of all constant functions on the circle. Any such function is uniquely determine by a single real number C. This means the space is a one-dim. vector space isomorphic to the real numbers.

$$Z^0(S^1) \simeq \mathbb{R}$$

$B^0(S^1)$

A 0-form is exact if it is the exterior derivative of (-1) form.
By the convention, the space of p-forms for a -ive value $(p<0)$ is defined to be the trivial vector space.

$$\therefore B^0(S^1) = \{0\} \qquad \therefore \left\{ H^0(S^1) = \frac{\mathbb{R}}{\{0\}} = \mathbb{R} \right\}$$

Hodge duality:

We define the Hodge star operator on a n-dimentional manifold as a map from p-forms to $(n-p)$-forms.

$$(* A)_{\mu_1 \mu_2 \cdots \mu_{n-p}} = \frac{1}{p!} \epsilon^{\nu_1 \nu_2 \cdots \nu_p}{}_{\mu_1 \mu_2 \cdots \mu_{n-p}} A_{\nu_1 \cdots \nu_p}$$

Mapping A to "A dual"

Applying Hodge star twice either gives original or negative of original form.

$$\left\{ (* * A) = (-1)^{s + p(n-p)} A \right\}$$ where s is the number of minus sign in the eigenvalues of the metric.