Topological manifolds:

There's no such set of topological notions known such that we can work out weather two spaces are homeomorphic by simply 'ticking' whether two spaces have these notions or not.

For classical physics, we may focus on topological spaces (M, O_M)

M is a set of all points, O_M is a topology on M

Together it gives us the topological space which allows us to talk about continuity, neighborhoods, limits without needing a full coordinate system.

Def: Let U_p denote an open neighbourhood containing the point p in some topological space. A topological space (M, O) is called a d-dimensional topological manifold if

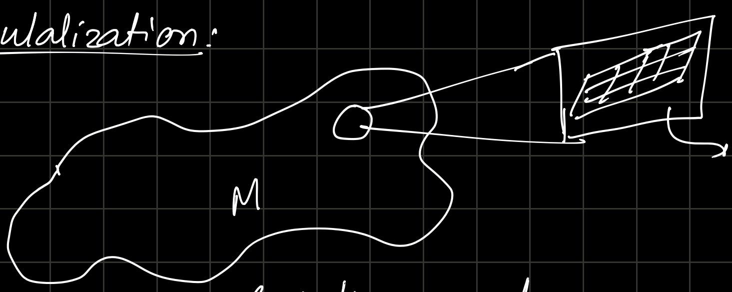
$$\forall p \in M \; \exists \, U_p \in O : \exists x : U_p \to x(U_p) \subseteq \mathbb{R}^d$$

for every point in the set M.

There exist an open set U_p in the topology O.

there exist a function x that maps the open set U_p into $\mathbb{R}^d$ (standard d-dim Euclidean space).

Visulalization:



locally it behaves as flat Euclidean space.

Globally curved

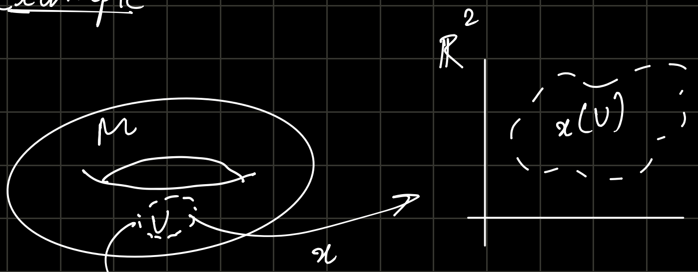
such that

i) x is invertible : $x^{-1} : x(U_p) \to U_p$ (Each point has a unique pre-img).
ii) x is continuous
iii) x^{-1} is continuous

These properties defines Homeomorphism, meaning you can smoothly translate between the abstract manifold & familiar coordinate space. Which is foundational in G.R and diff. geo.

<u>Example</u>

Let M be the surface of a torus. This is a subset of $\mathbb{R}^3$



map to a flat space.

boundariless closed shape on the surface

Now a common question here can be, "SINCE IT IS AN OPEN BOUNDARY WHOLE MANIFOLD IS HOMEOMORPHIC TO $\mathbb{R}^d$"

The answer is NOOO!!

$\Rightarrow$ A manifold is locally like $\mathbb{R}^d$, not globally

Each chart $x : U_p \to \mathbb{R}^d$ maps just a small open neighbourhood $U_p \subset M$ to an open set of $\mathbb{R}^d$, not whole $\mathbb{R}^d$; and in most of the manifolds no single chart covers the whole manifold.

let's take sphere S^2 as an example.

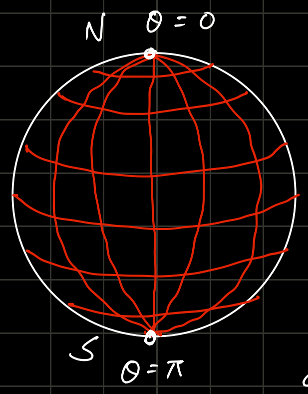
Why can't we cover sphere S^2 with a single chart (map) $x : S^2 \to \mathbb{R}^2$

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

you want to use coordinates (a chart) to map points on the sphere to $\mathbb{R}^2$. One simple way to do that is to use spherical coordinates

$$(\theta, \phi) \longmapsto (x,y,z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$\theta \in (0,\pi)$ is the polar angle ; $\phi \in (0, 2\pi)$ is the azimuthal angle.



at $\theta = 0$ $\sin\theta = 0$; now no matter what ϕ is $(x, y) = (0, 0)$

why it is a problem ?

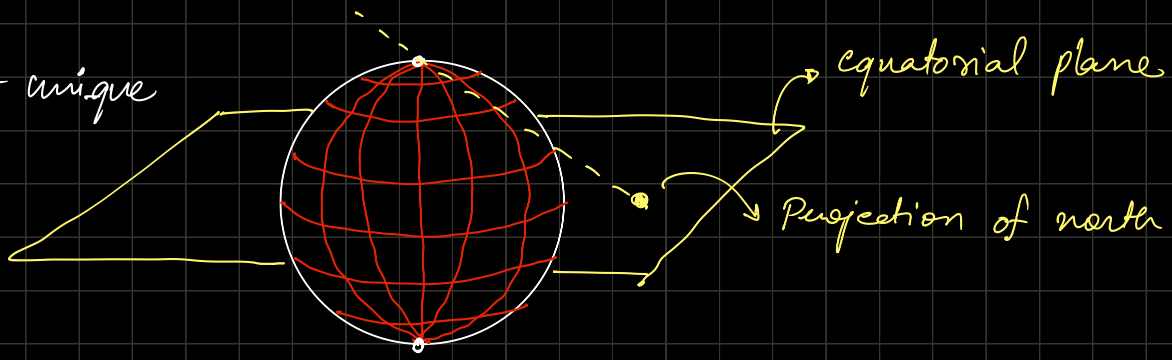
⇒ It violates injectivity (one to one)

different $(0, \phi)$ at $\theta = 0$ map to the same point on the sphere, map isn't invertible at the pole.

∴ Mapping is not a homeomorphism at the poles

$(0,0,1)$
$\downarrow$
$\underline{(0,\phi)}$
↳ not unique ϕ.

To fix this problem, we use stereographic projection.



$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ $\quad N = (0,0,1)$; $S = (0,0,-1)$

each projection (not including N pole) maps the sphere to the equatorial plane $z=0$, we identify as $\mathbb{R}^2$.

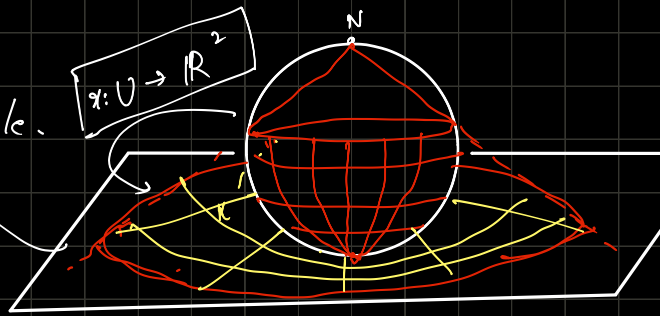
Projection from north pole : $\left(\phi_s : U_s \to \mathbb{R}^2\right)$

$$\phi_s(x,y,z) = \left(\frac{x}{\underline{1-z}}, \frac{y}{1-z}\right)$$ ⇒ this is bijective, continuous map $U_s \to \mathbb{R}^2$

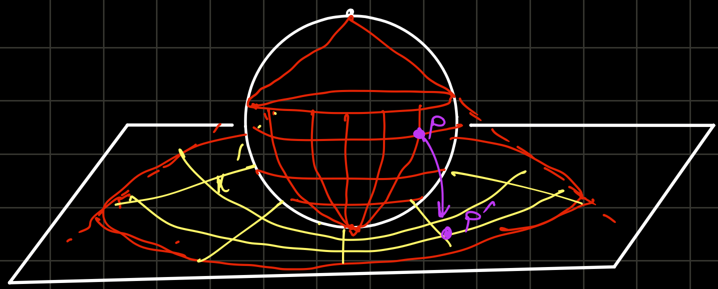
blows at $z = 1$. (Not -1).

$$\phi_s^{-1}(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right) ; \left(u, v \in \mathbb{R}^2\right)$$

we can do the same thing with south pole.



this is a flat circle
my drawing is bad!



P' being the project point P on the sphere, the coordinates P' are non-physical, it just tells us where it is existing in that particular chart.

Let's take one more example, M be a wire loop. We know that it exist in our 3D world that means, in is embedded in $\mathbb{R}^3$ but actually M is a 1D topological manifold, similarly torus is 2D topological manifold.

↳ How? (Hint: Imagine you are an ant walking on these shapes & you will have your answer.

Okay !! wait last example before moving forward!

is this a topological manifold?

So, we have mention charts in our previous discussion so let's look for it's terminology

- The pair (U, x) is called chart of (M, O)

→ The set $A = \{ (U_{(\alpha)}, x_{(\alpha)}) \mid \alpha \in A \}$ for some arbitary index set A, is called atlas of (M, O) is $\bigcup_{\alpha \in A} U_{(\alpha)} = M$.

→ $x : U \to x(U) \subseteq \mathbb{R}^d$ is called a chart map defined by $x(p) = (x^1(p), \ldots, x^d(p))$ where $x^i(p)$ is the i^{th} coordinate of p w.r.t the choosen chart (U, x).

→ $x^i : U \to \mathbb{R}$ are called the coordinate maps.

" An atlas that contains every possible cart for a topological manifold is called a maximal atlas."

Chart transition maps:

Imagine two charts (U, x) & (V, y) from the same topological space (M, O) with overlaping regions, i.e $U \cap V \neq \phi$, now a point p in this overlapped region can be mapped by both $x, y \rightarrow$ their respective $\mathbb{R}^d$ patches.