Differentiable manifolds :

In earlier chapters we spoke about the topological manifolds and curves on it now we are going to talk about their differentiability.

Curves, $\gamma : \mathbb{R} \to M$

function, $f : M \to \mathbb{R}$

Maps : $\varphi : M \to N$

1. Strategy

Choose a chart (U, x)

$$\begin{tikzcd} \gamma : \mathbb{R} \arrow[r, "\gamma"] \arrow[dr, "x\circ\gamma"'] & U \arrow[d, "x"] \\ & x(U) \subseteq \mathbb{R}^d \end{tikzcd}$$

Problem is, can this be well defined under the change of chart.

$$\begin{tikzcd} & y(U\cap V) \subseteq \mathbb{R}^d & \\ \gamma : \mathbb{R} \arrow[ur, "y\circ\gamma"] \arrow[r, "\gamma"'] \arrow[dr, "x\circ\gamma"'] & U \cap V \neq 0 \arrow[u, "y"'] \arrow[d, "x"] & \\ & x(U\cap V) \subseteq \mathbb{R}^d \arrow[uu, bend right=60, "y\circ x^{-1}"'] & \end{tikzcd}$$

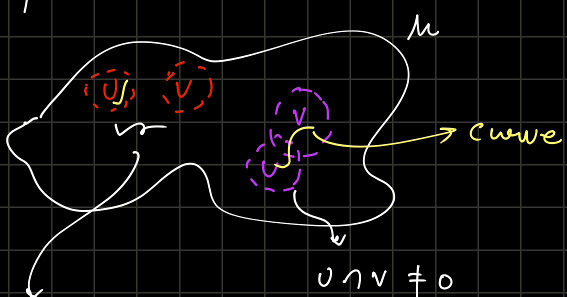
$\Rightarrow$ Here we need to make sure that in the overlapping region U, V should agree on the differentiability.

$\therefore \underbrace{y \circ x^{-1}}_{\mathbb{R}^d \to \mathbb{R}^d} \circ \underbrace{x \circ \gamma}_{\mathbb{R} \to \mathbb{R}^d} = y \circ \gamma$ $\rightarrow$ $\therefore$ It is continuous but not always differentiable.

continuous   differentiable

Now for sure we know that $y\circ x^{-1}$ is continuous but we dont know if it is differentiable.

2. Compatible charts



$U \cap V \neq 0$

$\therefore$ We need some rules for this curve to be differentiable.

$U \cap V = \phi$

$\therefore$ there no problem

. Def : Two chart (U, x) and (V, y) of a topological manifold $(M, \mathcal{O})$ are called $*$-compatible if either

a) $U \cap V = \phi$

b) $U \cap V \neq \emptyset$ and the chart transition maps $(x \circ y^{-1}) : y(U \cap V) \to x(U \cap V)$ and $(y \circ x^{-1}) : x(U \cap V) \to y(U \cap V)$ are differentiable.

Def : An atlas A is a $*$-compatible atlas if all of it's charts are $*$-compatible

Def: A $*$-manifold is a triple $(M, \mathcal{O}, A_*)$; where A_* is a $*$-compatible atlas.

Now, the reason for using '$*$' is that we can have various types of compatibility.

let's discuss what $*$ can be

$*$

C^0 — $C^0(\mathbb{R}^d \to \mathbb{R}^d)$, continuous maps w.r.t the standard topology on $\mathbb{R}^d$

C^1 — $C^1(\mathbb{R}^d \to \mathbb{R}^d)$, once differentiable with continuous result w.r.t $\mathcal{O}_{\text{standard}}$

C^k — $C^k(\mathbb{R}^d \to \mathbb{R}^d)$, k times continuously differentiable.

D^k — $D^k(\mathbb{R}^d \to \mathbb{R}^d)$, k times differentiable, don't need to be continuous

C^∞ — $C^\infty(\mathbb{R}^d \to \mathbb{R}^d)$, infinitely differentiable with continuous result known smooth.

C^ω — $C^\omega(\mathbb{R}^d \to \mathbb{R}^d)$, analytic function

$\mathbb{C}^\infty$ — $\mathbb{C}^\infty(\mathbb{R}^{2n} \to \mathbb{R}^{2n})$, $\dim M = 2n$ for integer n, they satisfy the Cauchy-Riemann equations pairwise. This gives us complex manifold

Theorem 4.2.1 (Whitney theorem) : Any C^k-atlas A_{C^k} for $k \geq 1$ for a topological manifold contains as a subatlas a C^∞ atlas.

4. Diffeomorphisms

$M \xrightarrow{\phi} N$

If M, N are naked set, the structure preserving maps are bijections (invertible maps)

$M \cong N$ (isomorphic) if $\exists$ bijection $\phi : M \to N$

Now $(M, \mathcal{O}_M) \cong_{Top} (N, \mathcal{O}_N)$ $\Rightarrow$ topologically isomorphic = "Homeomorphic"
$\exists$ bijection $\phi : M \to N$ such the ϕ, ϕ^{-1} are continuous.

Def: Let $(M, \mathcal{O}_M, \mathcal{A}_M)$ and $(N, \mathcal{O}_N, \mathcal{A}_N)$ be two smooth manifolds. They are isomorphic if there exist a bijection $\phi : M \to N$ such that ϕ & ϕ^{-1} are C^∞ maps. Such maps are known as diffeomorphism and manifolds are known to be diffeomorphic.

Theorem: The number of C^∞-manifolds one can make from a given C^0-manifolds up to diffeomorphism given by the following table.

dim M	No. of C^∞-manifolds	
1	1	Moise - Radon - theorem
2	1	
3	1	
4	uncountably infinitely many	Sod's law
5	finitely many	Surgery theory
6	finitely many	
7	finitely many	
⋮	⋮	