Multilinear algebra:

- **Vector spaces:**

In order to define a vector space, we first need to know what the field is.

Def: Let K be a set and let $\cdot : k \to k$. The double $(k, \cdot)$ is a Abelian group if the following axioms are satisfied.

(i) Commutative: $a \cdot b = b \cdot a$

(ii) Associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(iii) Neutral element; $\exists\, 0 \in k$ such that $a \cdot 0 = 0 \cdot a = a$

(iv) Inverse; $\exists\, a^{-1} \in K$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 0$.

$\mathbb{R}$ is not an abelian group; $a \times 0 = 1$ | there's no a such that $a \times 0 = 1$

Def: A field is a triple $(\mathbb{F}, +, \cdot)$ where

- $\mathbb{F}$ is a set, and
- $+, \cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ are maps.

They must satisfy the following axioms

(i) $(F, +)$ is an Abelian group

(ii) $(\mathbb{F}^*, \cdot)$ is an Abelian group, where $\mathbb{F}^* = F/\{0\}$

(iii) Distributive; $\forall\, a, b, c \in \mathbb{F}$ $a \cdot (b + c) = a \cdot b + a \cdot c$

- If we don't require cond^n (ii) but in its place just require associative cond^n $(a \cdot (b \cdot c) = (a \cdot b) \cdot c)$, then we get a weaker notion called a "RING". There are types of rings eg. unital ring, commutative ring.

Def: A $\mathbb{F}$-Vector space is the triple $(V, +, \cdot)$ where

- V is a set
- $+$ is a addition map, $+ : V \times V \to V$
- $\cdot$ is the s-multiplication map, $\cdot : \mathbb{F} \times V \to V$

Imp: We can build a $\mathbb{F}$ vector space over a field $(\mathbb{F}, +, \cdot)$, we can built a so called "R-module" over a ring $(R, +, \cdot)$. It's done in exactly same fashion

Okay, this statement for physics peep would a bit strange because we generally exploit maths but

"informally elements of vector space are referred as vectors".

Obvious no??

Ex:

$$P := \left\{ p : (-1,1) \to \mathbb{R} \;\middle|\; p(x) = \sum_{n=1}^{N} p_n x^n, \; p_n \in \mathbb{R} \right\}$$

Is this a vector? or $\square : (-1,1) \to \mathbb{R}$; $\square(x) = x^2$ is a vector?

Umm.... No!!

WHY??

⇒ Because first we need to built a vector "space".

We have to define add & s-multiplication.

$+ : (p, q) \mapsto p +_p q$; $(p +_p q)(x) := p(x) +_{\mathbb{R}} q(x)$

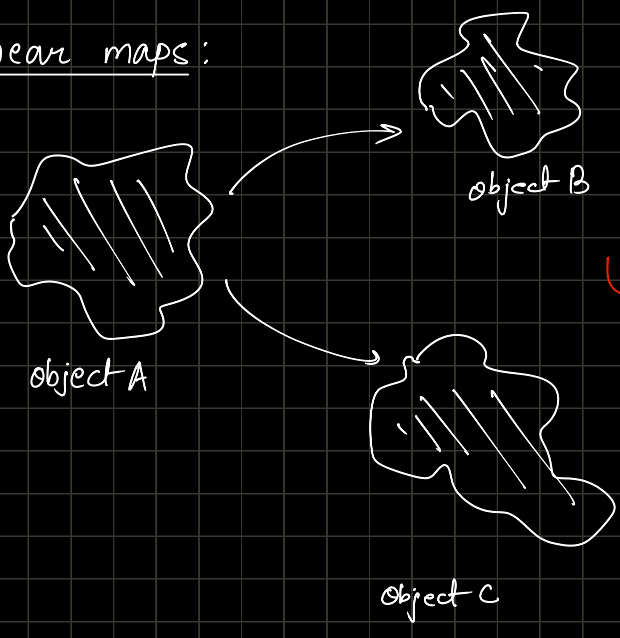
(This describes normal addition)

$\cdot_p : \mathbb{R} \times V \to \mathbb{R}$

⇒ Now since we have defined $(+, \cdot)$, now we can say that $\square$ is a vector.

⇒ The whole goal of above exercise to make you understand the abstractness

Linear maps:



All the 3 objects have similar structure and can be derived from the structure on the other.

For this we can use a linear map, such maps are called "Isomorphisms".

Def: Let $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ be vector spaces. Then a map $\varphi : V \to W$ is called linear if: for all $v, \tilde{v} \in V$ and $\lambda \in \mathbb{R}$

(i) $\varphi(v +_V \tilde{v}) = \varphi(v) +_W \varphi(\tilde{v})$

(ii) $\varphi(\lambda \cdot_V v) = \lambda \cdot_W \varphi(v)$

Ex: Consider a map $\delta : P \to P$, defined by $\delta(p) := p'$ (Differential operator).

$$\delta(p+q) = (p+q)' = p' + q' = \delta(p) + \delta(p)$$

$$\delta(\lambda \cdot p) = (\lambda \cdot p)' = \lambda \cdot p' = \lambda \cdot \delta(p)$$

Notation: $\varphi : V \to W \Rightarrow \varphi : V \xrightarrow{\sim} W$

linear map.

Theorem: Suppose we have the following linear maps $\varphi : V \xrightarrow{\sim} W$; $\psi : W \xrightarrow{\sim} U$. Then the map $(\psi \cdot \varphi) : V \xrightarrow{\sim} U$ is also linear.

Proof:

$$\varphi(v +_V \tilde{v}) = \varphi(v) +_W \varphi(\tilde{v})$$

$$\psi(w +_W \tilde{w}) = \psi(u) +_U \psi(\tilde{u})$$

$$\therefore (\psi \cdot \varphi)(v +_V \tilde{v}) = \psi\big(\varphi(v) +_W \varphi(\tilde{v})\big) = \psi(\varphi(v)) +_U \psi(\varphi(\tilde{v}))$$

$$\therefore \boxed{(\psi \cdot \varphi) : V \xrightarrow{\sim} U}$$

Vector space of Homomorphisms:

Def: Let $(V, +, \cdot)$ and $(W, +, \cdot)$ be vector spaces. Then we can define the set

$$\mathrm{Hom}(V, W) := \{\varphi : V \xrightarrow{\sim} W\}$$

We can can turn this into a vector space by defining

$$\oplus : \mathrm{Hom}(V, W) \times \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$$

$$\odot : \mathbb{R} \times \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W).$$

The triple $(\mathrm{Hom}(V,W), \oplus, \odot)$ is this vector space of homomorphisms.

A homomorphisms is a structure-preserving map b/w two algebraic objects like groups, rings or vector spaces.

Dual vector space: Let $(V,+,\cdot)$ be a vector space. We define the dual vector space (to V) $(V^*, \oplus, \odot)$, where

$$V^* := \mathrm{Hom}(V, \mathbb{R}) := \{\varphi : V \xrightarrow{\sim} \mathbb{R}\}$$

where $\oplus, \odot$ are necessarily defined.

Terminology: As a vector, an element $\varphi \in V^*$ is informally called a covector.

↳ But But But..... If you remember from above, we vaguely said that elements of vector space are called vector.

Umm... $\mathrm{Hom}(V, W)$ is a vector space then why we call the elements to be covectors.

↳ So, we also need to know that it is a dual to some other vector space, whose elements we have already called vectors.

Ex: $I : P \xrightarrow{\sim} \mathbb{R}$, which tells us $I \in P^*$

$$I(p) := \underbrace{\int_0^1 dx}_{\text{covector}} \; \underbrace{p(x)}_{\text{vector}} \quad ; \quad {}^{\#}\text{covector} : V \to \mathbb{R}$$

Theorem: Let $(V,+,\cdot)$ be a vector space. If it is finite dimensional then the double dual is the vector itself. that is

$$(V^*)^* = V$$

when $\dim(V) < \infty$

So now if we sum up till now, we now have the knowledge of vectors, fields, rings, maps, & covectors, so its the high time we should introduce tensors.

Tensors:

If we are considering finite dimensional vector spaces, then there is a very natural definition for tensors as multilinear maps.

Def: Let $(V,+,\cdot)$ be a vector space. A (r,s)-tensor T, over V is a multilinear map

$$T : \underbrace{V^* \times V^* \times \cdots \times V^*}_{r\text{-terms}} \times \underbrace{V \times V \times \cdots \times V}_{s\text{-terms}} \xrightarrow{\sim} \mathbb{R}$$

e.g.
Let T be a $(1,1)$-tensor. This means it takes in as its argument one covector and one vector. The multilinearity of T tells us that: for all $\varphi, \psi \in V^*$, $v, w \in V$ & $\lambda \in \mathbb{R}$

$$T(\varphi + \psi, v) = T(\varphi, v) + T(\psi, v)$$

$$T(\lambda \cdot \varphi, v) = \lambda \cdot T(\varphi, v)$$

$$T(\varphi, v+w) = T(\varphi, v) + T(\varphi, w)$$

$$T(\varphi, \lambda \cdot v) = \lambda T(\varphi, v)$$

eg We can give an example of the tensor using our polynomial space. The map $g : P \times P \xrightarrow{\sim} \mathbb{R}$ defined by.

$$g(p,q) = \int_{-1}^{1} dx \; p(x)\, q(x)$$

is a $(0,2)$-tensor over P, it is just the inner product on the real numbers.

Terminology: The number r is often known as the covariant order of T and s the contravariant order. $(r+s)$ is known as the rank of T.

Def: (Tensor (via tensor product)). Let $(V,+,\cdot)$ be a vector space. A (r,s)-tensor space is defined by

$$T = \underbrace{V \otimes V \cdots \otimes V}_{r\text{-terms}} \otimes \underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{s\text{-terms}} \equiv V^{\otimes r} \otimes (V^*)^{\otimes s}$$

$\otimes$ is the so called Tensor product.

we has interchanged r, s, $\therefore$ we have r V terms & s V^* terms. Now, because we're assuming our vector space to be finite dimentional

$V = (V^*)^*$, so we can think of V as the set of all linear maps from V^* to $\mathbb{R}$

$V: V^* \xrightarrow{\sim} \mathbb{R}$ $\qquad$ $V^*: V \xrightarrow{\sim} \mathbb{R}$

r- linear maps $\qquad$ s- linear maps.

Eg. Let $(V,+,\cdot)$ be a finite dimensional vector space and let T be a $(1,1)$ tensor, then T maps one covector and one vector to be a real number

A $(1,1)$ - tensor T is a bilinear map

$T: V^* \times V \to \mathbb{R}$

This means:
- It takes one covector $\phi \in V^*$
- And one vector $v \in V$
- And returns of scalar $T(\phi, v) \in \mathbb{R}$

Now let's say I fix one input $v \in V$, then $T(\cdot, v)$ is a function

$\phi \mapsto T(\phi, v)$ for all $\phi \in V^*$ so it defines $\phi: V \xrightarrow{\sim} \mathbb{R}$

" Feeding a vector v into the $(1,1)$ - tensor leaves us with a function of one covector - i.e., an element of the double dual $(V^*)^*$."

since V is finite dimensional, $(V^*)^* \cong V$.

$\therefore$ $T(\cdot, v) \in (V^*)^* \in V$. $\quad ; \; \varphi_T : V \to V$

<u>Summary:</u>

What we start with	Following output
$T \in \text{Tensor}^{(1,1)} = V \otimes V^*$	Bilinear map $T: V^* \times V \to \mathbb{R}$
Fix vector v	$T(\cdot, v) \in (V^*)^* \cong V$ gives $\varphi: V \to V$
Fix covector V^*	$T(\varphi, \cdot) \in V^*$ => gives map $V^* \to V^*$.

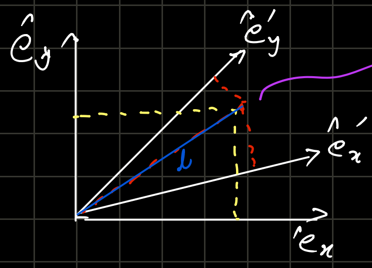
<u>Vectors & Co-vectors as Tensors:</u>

A covector $\varphi \in V^*$, i.e. $\varphi: V \xrightarrow{\sim} \mathbb{R}$ is a $(0,1)$ tensor

For a finite dimensional vector space, a vector $v \in V$ is a $(0,1)$ tensor.

Bases:

$\mathbb{R}^2$



there are infinetely many ways to describe this point in $\mathbb{R}^2$, the only thing which is independent of basis vectors is the length.

↳ there our main goal would be to describe results that are basis independent.

Def: Let $(V,+,\cdot)$ be a vector space. A subset $B \subseteq V$ is called a (Hamel-) basis if

$$\forall v \in V, \exists! \text{ finite } F = \{f_1, \ldots, f_n\} \subset B : \exists! \, v_1, \ldots, v_n \in \mathbb{R} : v = v^1 f_1 + v^2 f_2 + \ldots + v^n f_n$$

Def: Let $(V,+,\cdot)$ be a vector space. A subset $B = \{e_1, \ldots, e_d\} \subseteq V$ is called a basis if

i) The basis spans V; that is any $v \in V$ can be written as a linear combination of the basis elements, and

ii) the basis elements are linearly independent; that i

$$\sum_{i=1}^{d} \lambda^i e_i = 0 \Rightarrow \lambda^i = 0 \;\forall\, i = \{1, 2, \ldots, d\}$$

Def: If there exist a basis $B \subseteq V$ for a vector space $(V,+,\cdot)$ with finitely many elements, say d many, then we will call d the dimension of the vector space $[\dim V = d]$.