Special

Relativity

Theory of special relativity:

Inertial frames and the principle of relativity

$\Rightarrow$ Frames are one of the most essential concept in the theory of relativity, we are free to mark any steady frame S and a moving frame w.r.t S as S'.

There exist a class of preferred reference systems called inertial frames in which a free particle is either at rest of moving with a constant velocity.

This specifically means that it is not changing directions

$\therefore$ In cartesian coordinates it means $\left\{ \frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0 \right\}$

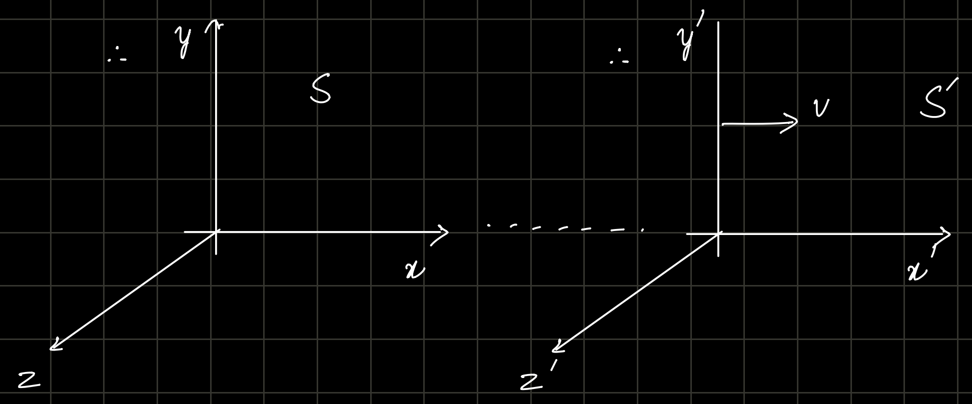
$\therefore$ In absence of gravity, if we have two inertial frames then S' can differ from S only by

i) Translation

ii) Rotation

iii) A motion of one frame w.r.t the other frame at a constant velocity.

These inertial frames are the fundamental concept of the special relativity. In relativity a point in a 4-dimentional space is described by (ct, x, y, z).



$\therefore$ Transformations for S & S' are

$$\left.\begin{aligned} t' &= At + Bx \\ x' &= Ct + Dx \\ y' &= y \\ z &= z \end{aligned}\right\}$$ Don't worry, this will be proven later.

$x' = 0 \Rightarrow x = vt \quad \therefore \quad x = 0 \Rightarrow x' = -vt'$

$\therefore \; t' = At$ at $x = 0 \quad \therefore \; x' = 0 \quad Ct + vDt = 0$

$x = 0 \Rightarrow x' = Ct = \frac{C}{A}t' \quad \Rightarrow \boxed{C/A = -v}$

$\therefore \; C + vD = 0 \Rightarrow -Av + vD = 0 \Rightarrow \boxed{A = D}$

$\therefore$ Finally the transformations looks like.

$$\begin{aligned} t' &= At + Bx \\ x' &= A(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

I am assuming how does classical relativity works (i.e $v << c$) !!

So, let's move to the topic of geometry of space-time.

In SR, Einstein removed the concept of absolute time and replaced it by saying that the speed of light c is the same in all inertial frames

$\therefore$ By applying this postulate together with the principle of relativity we will be able of connect coordinates of an event P in two different cartesian inertial frames, S and S'.

- Consider a photon being emitted at $t = 0$ & $t' = 0$ and travelling in arbitrary dx^n.

It will satisfy the following eq^n

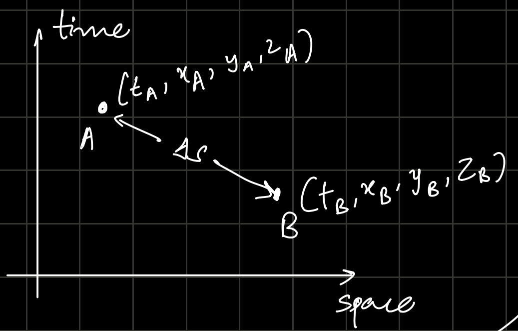
$$\left\{ c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 = 0 \right\}$$

Plugging the above transformation here will give us A and B.

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \\ x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z \end{aligned} \qquad \text{where } \left\{\beta = V/c\right\} \text{ and } \left\{\gamma = \frac{1}{\sqrt{1 - V^2/c^2}}\right\}$$

This is called Lorentz transformation and it is particularly boost in x-direction.

for $\beta << 1 \Rightarrow$ It becomes Galilean transformation.



$$\left[\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2\right]$$

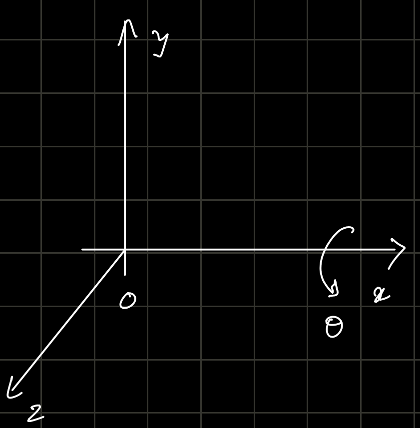
this quantity is invariant under any Lorentz transformation.

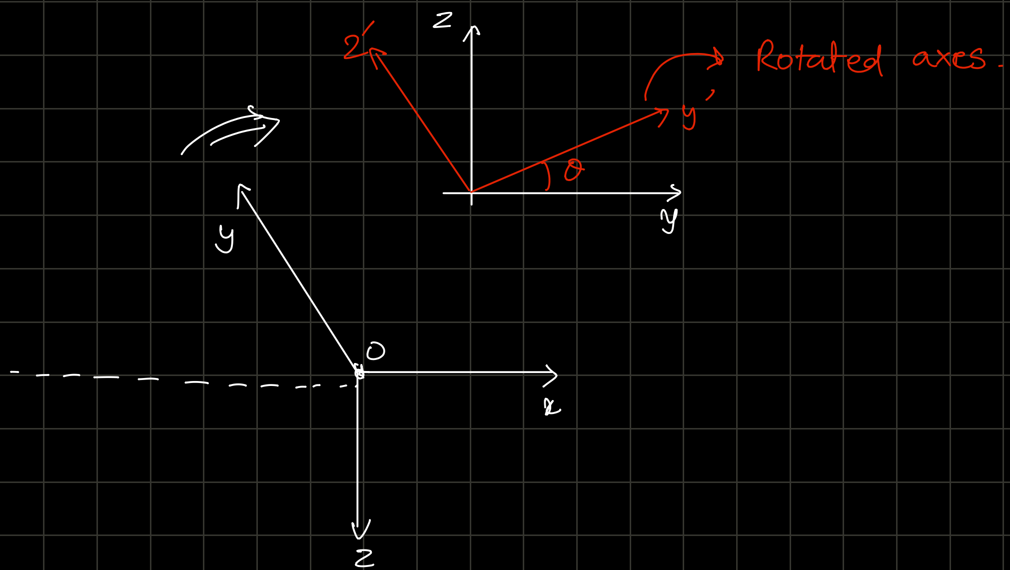
And that's the reason why we call this 4D continuum as space-time.

this is often called as 'Minkowski Geometry'.

* For fixed t, it becomes Euclidean.

"Rotations"



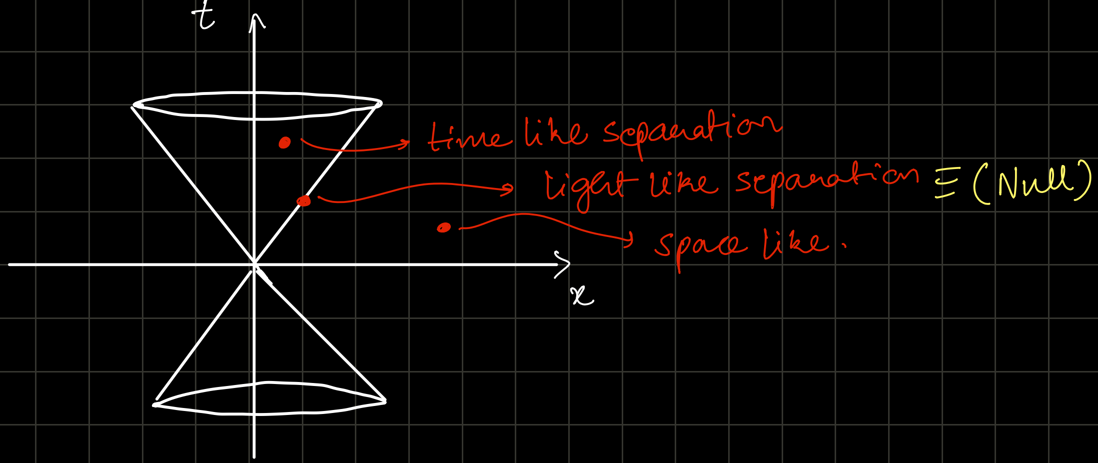


$$\begin{aligned} ct' &= ct \\ x' &= x \\ y' &= y\cos\theta - z\sin\theta \\ z' &= y\sin\theta + z\cos\theta. \end{aligned}$$

"Light cone"

Internal

$$\Delta s^2 = c\Delta t^2 - \Delta x' - \Delta y' - \Delta x' \implies \text{For this equation} \quad \Delta s = \begin{cases} > 0 & \text{Space-like} \\ = 0 & \text{light-like} \\ < 0 & \text{time-like.} \end{cases}$$



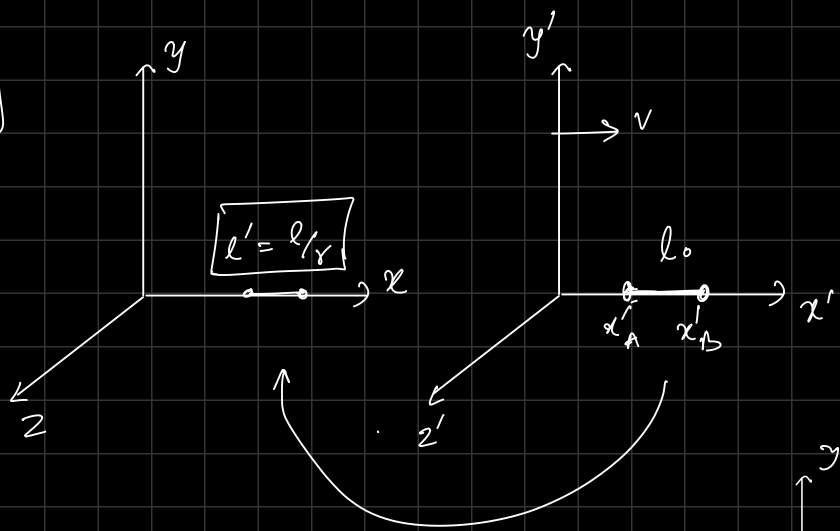
Length contraction and time dilation:

length contraction:

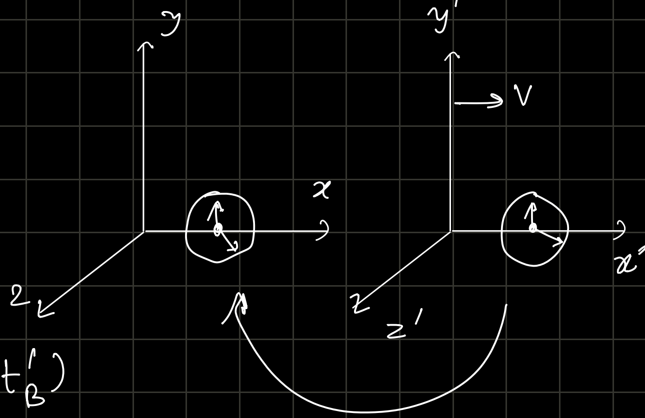
Assume a rod of length l_0, where $l_0 = x'_B - x'_A$ s' is at rest

$$\left.\begin{aligned} x'_A &= \gamma(x_A - vt_A) \\ x'_B &= \gamma(x_B - vt_B) \end{aligned}\right\} \quad l'_B = \gamma(l_B) \quad \text{since } t_A = t_B = t$$

$$\therefore \boxed{\frac{l_0}{\gamma} = l}$$



Time-dilation:



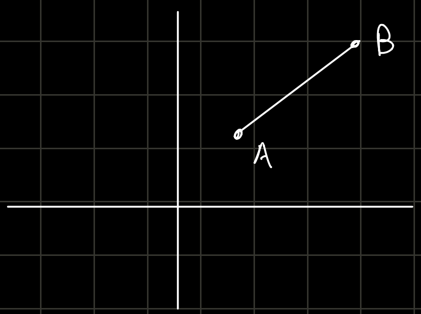
$$\left.\begin{aligned} t_A &= \gamma\left(t'_A + \frac{vx'_a}{c^2}\right) \\ t_B &= \gamma\left(t'_B + \frac{vx'_B}{c^2}\right) \end{aligned}\right\} \quad t_B - t_A = \gamma(t'_A - t'_B)$$

$x'_A = x'_B$ (clocks are fixed)

$$\therefore \boxed{\Delta t = \gamma T_0}$$

The faster you move, slower your time click observed by a person a rest.

"Minkowski line-element".



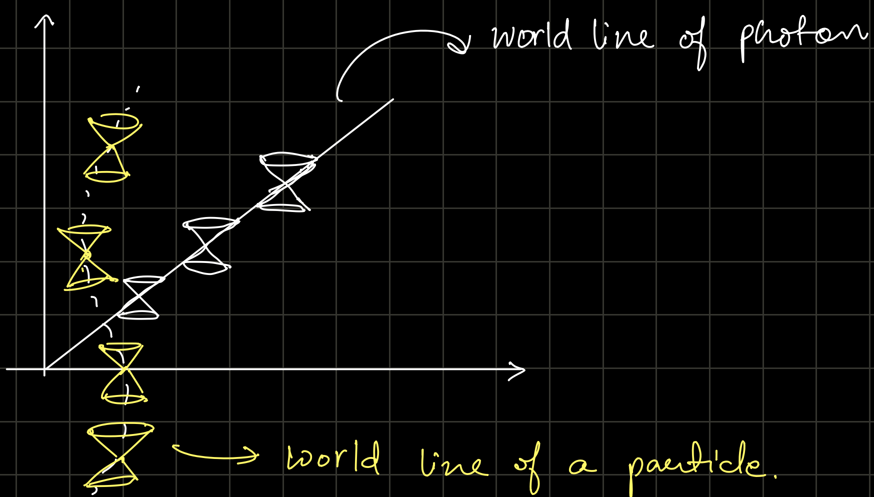
$$\Delta s = \int_A^B ds$$

$$\therefore ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$\therefore ds^2 > 0$ Time like

$ds^2 = 0$ Light like

$ds^2 < 0$ space-like $\Rightarrow$ we cannot have $v > c$.



A four dimentional way of describing a world line is to give the four coordinates (t, x, y, z) as functions of parameter λ, that varies monotonically along the world line.

For a massive particle, such parameter is called it's proper time.

$$c^2 d\tau^2 = ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

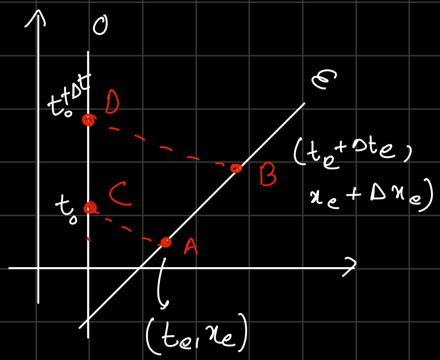
we define this for infinitesimally seperated events.

$$\therefore d\tau^2 = \left(1 - \frac{v^2}{c^2}\right) dt^2 \Rightarrow \left\{ d\tau = \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt = \frac{dt}{\gamma} \right\}$$

$$\left\{ \Delta\tau = \int_a^b d\tau = \int_a^b \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt \right\}$$

proper time τ is just the coordinate time t measured by clocks at rest in S.

"Doppler effect"



$$\Delta\tau_{AB} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \Delta t_e$$

$$\Delta\tau_{CD} = \Delta t_o$$

worldline representing photons : $ds^2 = 0 \Rightarrow c^2dt^2 - dx^2 - \underbrace{dy^2 - dz^2}_{=0} = 0$

$\therefore$ World line connecting A & C $\Rightarrow \displaystyle\int_{t_e}^{t_o} c\,dt = -\int_{x_e}^{x_o} dx$

because photon is travelling from $A \rightarrow C$ (−ive X direction)

$$\therefore c(t_o - t_e) = -(x_o - x_e)$$

similarly along the world line B, D

$$\int_{t_e+\Delta t_e}^{t_o+\Delta t_o} c\,dt = -\int_{x_e+\Delta x_e}^{x_o} dx \Rightarrow \boxed{c\Delta t_o - c\Delta t_e = \Delta x_e}$$

(add both the integrals)

$$\therefore \Delta t_o = \left(1 + \frac{1}{c}\frac{\Delta x_e}{\Delta t_e}\right)\Delta t_e = \left(1 + \frac{v}{c}\right)\Delta t_e$$

$$\therefore \Delta t_{CD} = (1+\beta)\Delta t_e \quad ; \quad \Delta t_{AB} = (1-\beta^2)^{+1/2}\Delta t_e$$

$$\therefore \boxed{\frac{\Delta t_{CD}}{\Delta t_{AB}} = \left(\frac{1+\beta}{1-\beta}\right)^{0.5}} \Rightarrow \frac{\nu_{AB}}{\nu_{CD}} = \frac{\nu_e}{\nu_o}$$

"Addition of velocities in SR"

$$u_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \qquad u_z = \frac{dz}{dt}$$

$$dt' = \gamma_v \left(dt - \frac{v\,dx}{c^2} \right) \;;\; dx' = \gamma_v \left(dx - v\,dt \right) \;;\; dy' = dy \;;\; dz' = dz$$

$$\therefore \left[u'_x = \frac{u_x - v}{\left(1 - \frac{u_x v}{c^2}\right)} \right] \qquad \left[\begin{array}{l} u'_i = \dfrac{u_i}{\gamma_v \left(1 - \frac{v_x v}{c^2}\right)} \\ i = y, z \end{array} \right]$$

And similarly we can find acceleration. $\left[\begin{array}{l} a_i = \dfrac{dU_i}{dt} \\ i = x, y, z \end{array} \right]$