Manifolds:

Topological space : $\exists$ (there are) or (there exist) so many that mathematicians cannot even classify them

For space-time physics, we may focus on topological spaces $(M, \mathcal{O})$ that can be charted, analogously to how the surface of the Earth is charted in an atlas.

1. Topological manifolds

Def. A top. space $(M, \mathcal{O})$ is called a d-dimensional topological manifold if

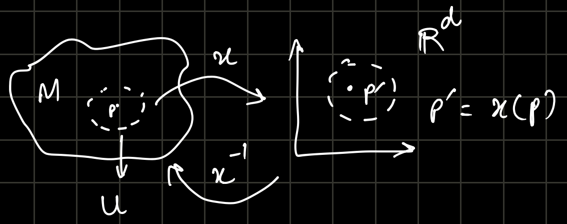
$$\forall p \in M : \exists \underset{\underset{p}{\cup}}{\mathcal{U}} \in \mathcal{O} : \exists x : \mathcal{U} \to x(\mathcal{U}) \subseteq \mathbb{R}^d$$

such that (i) x is invertible

$x^{-1} : x(\mathcal{U}) \to \mathcal{U}$

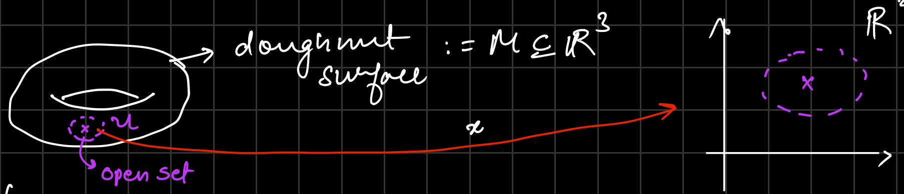
(ii) x is continuous

(iii) x^{-1} is continuous



Examples:

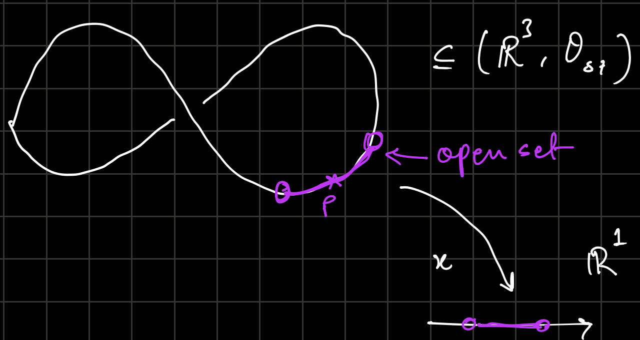
(i)



↳ This is a 2-D topological manifold

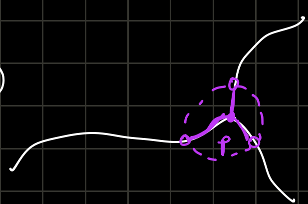
Terminology: $(\mathcal{U}, x)$ called a chart of $(M, \mathcal{O})$

(ii)



$\therefore$ This is a one-dimensional topological manifold.

(iii)



$\subseteq \mathbb{R}^2$ → It is a topological space.

→ There won't be a invertible map that's why this is a topological manifold

Terminology: $\mathcal{A} = \{ (\mathcal{U}_{(\alpha)}, x_{(\alpha)}) \mid \alpha \in A \}$ an atlas of $(M, \mathcal{O})$ if $M = \bigcup_{\alpha \in A} \mathcal{U}_\alpha$

$x : \mathcal{U} \to x(\mathcal{U}) \subseteq \mathbb{R}^d = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (d numbers in order)

↳ chart map $\quad \therefore \; x(p) = (x^1(p), \ldots, x^d(p))$

$\therefore \; x^i : \mathcal{U} \to \mathbb{R}$ (mapping each point) ≡ Component maps or the coordinate maps.

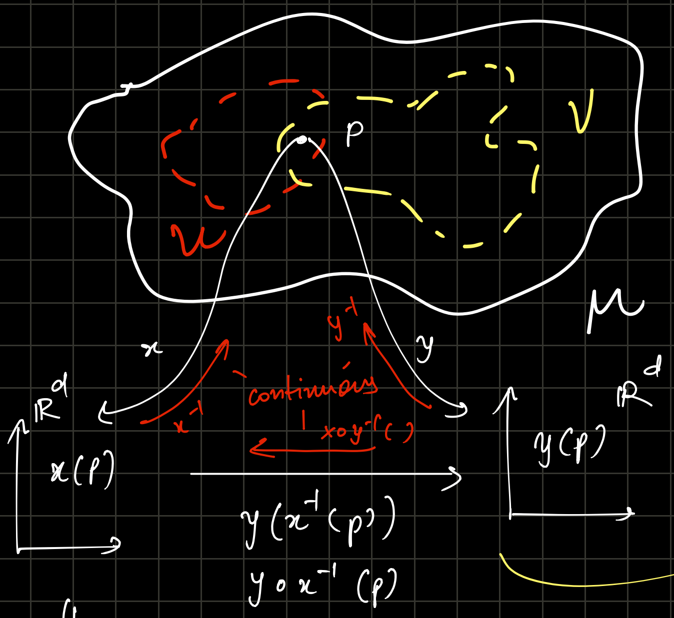
- $p \in \mathcal{U}$ then $x^1(p)$ is the first coordinate of point p w.r.t to chosen chart $(\mathcal{U}, x)$

  $x^2(p)$ is the 2nd coordinate of p.

3. Chart transition maps

Imagine two charts with overlapping regions

$U \cap V \neq \phi$

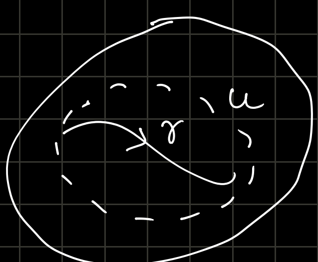


This is called the chart transition map.

↳ Obviously we can do that the full chart U & V.

4. Manifold Philosophy:

Often it is desired to define prospectives of real world object by judging suitable coordinates not on the real world object but on the chart representation of that object.



$$\begin{tikzcd} \mathbb{R} \arrow[r, "\gamma"] \arrow[rr, bend right, "x \circ f"'] & \mathcal{U} \arrow[r, "x"] & x(\mathcal{U}) \subseteq \mathbb{R}^d \end{tikzcd}$$

But it may be ill defined, so we need to make sure that properties doesn't change if we change the chart