Boolean Expressons

CS 2LC3

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Truth Values

- **Boolean constants/values**: *false, true*
- **The type of Boolean values**: $\mathbb{B}$
  - This is the type of propositions, for example: $(x = 1) : \mathbb{B}$
  - For any type t, equality $\_ = \_$ can be used on expressions of that type: $\_ = \_ t \times t \to \mathbb{B}$
- **Boolean operators**:
  - $\neg\_ : \mathbb{B} \to \mathbb{B}$ - negation, complement, "logical not"
  - $\_\wedge\_ : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ - conjunction, "logical and"
  - $\_\vee\_ : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ - disjunction, "logical or"
  - $\_\Rightarrow\_ : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ - implication, "implies", "if ... then ... "
  - $\_\Leftarrow\_ : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ - consequence, "is a consequence of.."
  - $\_\Leftrightarrow\_ : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ - equivalence, "if and only if", "iff"
  - $\_\not\Leftrightarrow\_ : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ - inequivalence, "exclusive or"

$\Leftrightarrow$ is also denoted as $\equiv$ and $\not\Leftrightarrow$ as $\not\equiv$.

Boolean Binary Operators: Conjunction and Disjunction

- Conjunction:

Args.		$\wedge$	
F	F	F	The moon is green, and $2 + 2 = 7$.
F	T	F	The moon is green, and $1 + 1 = 2$.
T	F	F	$1 + 1 = 2$, and the moon is green.
T	T	T	$1 + 1 = 2$, and the sun is a star.

- Disjunction ("inclusive or"):

Args.		$\vee$	
F	F	F	The moon is green, or $2 + 2 = 7$.
F	T	T	The moon is green, or $1 + 1 = 2$.
T	F	T	$1 + 1 = 2$, or the moon is green.
T	T	T	$1 + 1 = 2$, or the sun is a star.

Boolean Binary Operators: Implication

Args.		$\Rightarrow$	
F	F	T	If the moon is green, then $2 + 2 = 7$.
F	T	T	If the moon is green, then $1 + 1 = 2$.
T	F	F	If $1 + 1 = 2$, then the moon is green.
T	T	T	If $1 + 1 = 2$, then the sun is a star.

$$\begin{aligned} p \Rightarrow q &\equiv \neg p \vee q \\ \neg p \Rightarrow q &\equiv \neg\neg p \vee q \\ \neg p \Rightarrow q &\equiv p \vee q \end{aligned}$$

If you don't eat your spinach, I'll spank you.	$\equiv$	You eat your spinach, or I'll spank you.

Boolean Binary Operators: Consequence

Args.		$\Leftarrow$	
F	F	T	The moon is green **if** $2+2=7$.
F	T	F	The moon is green **if** $1+1=2$.
T	F	T	$1+1=2$ **if** the moon is green.
T	T	T	$1+1=2$ **if** the sun is a star.

$$p \Leftarrow q \qquad \equiv \qquad p \vee \neg q$$

Equality of Boolean values is also called **equivalence** and written $\equiv$
(In some other places: $\Leftrightarrow$)

$p \equiv q$	can be read as:	p is equivalent to q
	or:	p exactly when q
	or:	p if-and-only-if q
	or:	p iff q

p	q	$p \equiv q$	
false	*false*	*true*	The moon is green **iff** $2+2=7$.
false	*true*	*false*	The moon is green **iff** $1+1=2$.
true	*false*	*false*	$1+1=2$ **iff** the moon is green.
true	*true*	*true*	$1+1=2$ **iff** the sun is a star.

The boolean expression $b \equiv c$ is evaluated exactly as $b = c$, except that $\equiv$ (or $\Leftrightarrow$) can be used only when b and c are *boolean expressions*.

Args.		$\not\equiv$	
F	F	F	Either the moon is green, or $2 + 2 = 7$.
F	T	T	Either the moon is green, or $1 + 1 = 2$.
T	F	T	Either $1 + 1 = 2$, or the moon is green.
T	T	F	Either $1 + 1 = 2$, or the sun is a star.

Evaluation of Boolean Expressions Using Truth Tables 1

p	q	$\neg p$	$q \wedge \neg p$	$p \vee (q \wedge \neg p)$
F	F	T	F	F
F	T	T	T	T
T	F	F	F	T
T	T	F	F	T

- Identify variables
- Identify subexpressions
- Enumerate possible states (of the variables)
- Evaluate (sub-)expressions in all states

p	q	r	$\neg r$	$q \wedge \neg r$	$p \vee (q \wedge \neg r)$
F	F	F	T	F	F
F	F	T	F	F	F
F	T	F	T	T	T
F	T	T	F	F	F
T	F	F	T	F	T
T	F	T	F	F	T
T	T	F	T	T	T
T	T	T	F	F	T

			$\wedge$					$\not\equiv$ $\neq$	$\vee$	nor	$\equiv$ $=$		$\Leftarrow$		$\Rightarrow$	nand	
F	F	F	F	F	F	F	F	F	F	T	T	T	T	T	T	T	T
F	T	F	F	F	F	T	T	T	T	F	F	F	F	T	T	T	T
T	F	F	F	T	T	F	F	T	T	F	F	T	T	F	F	T	T
T	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T

- A boolean expression is **satisfied** in state s iff it evaluates to *true* in state s.
- A boolean expression is **valid** iff it is satisfied in every state.
- A valid boolean expression is called a **tautology**.
- A boolean expression is **satisfiable** iff there is a state in which it is satisfied.
- A boolean expression is called a **contradiction** iff it evaluates to *false* in every state.
- Two boolean expressions are called **logically equivalent** iff they evaluate to the same truth value in every state.

These definitions rely on states / truth tables: **Semantic concepts**

- Henry VIII had one son and Cleopatra had two.

  Henry VIII had one son and Cleopatra had two sons.

  Declarations:

  $$h \;:\equiv\; \text{Henry VIII had one son}$$
  $$c \;:\equiv\; \text{Cleopatra had two sons}$$

  Formalisation:

  $$h \wedge c$$

Modeling English Propositions - Recipe

- Transform into shape with clear subpropositions
- Introduce Boolean variables to denote subpropositions
- Replace these subpropositions by their corresponding Boolean variables
- Translate the result into a Boolean expression, using (no perfect translation rules are possible!) **for example**:

and, but	becomes	$\wedge$
or	becomes	$\vee$
not	becomes	$\neg$
it is not the case that	becomes	$\neg$
if p then q	becomes	$p \Rightarrow q$

Raymond Smullyan provides, in **The Lady or the Tiger?**, the following context for a number of puzzles to follow:

> [...] the king explained to the prisoner that each of the two rooms contained either a lady or a tiger, but it *could* be that there were tigers in both rooms, or ladies in both rooms, or then again, maybe one room contained a lady and the other room a tiger.

In the first case, the following signs are on the doors of the rooms:

1	2
In this room there is a lady, and in the other room there is a tiger.	In one of these rooms there is a lady, and in one of these rooms there is a tiger.

We are told that one of the signs is true, and the other one is false.

"Which door would you open (assuming, of course, that you preferred the lady to the tiger)?"

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$$R1L \;:=\; \text{There is a lady in room 1}$$
$$R1T \;:=\; \text{There is a tiger in room 1}$$
$$R2L \;:=\; \text{There is a lady in room 2}$$
$$R2T \;:=\; \text{There is a tiger in room 2}$$

> [...] We are told that one of the signs is true, and the other one is false.

$$S_1 \;:=\; \text{Sign 1 is true}$$
$$S_2 \;:=\; \text{Sign 2 is true}$$