MA 202

Maths Project: Numerical Analysis

Numerical Analysis of Three-Body Problem Restricted to 2 Dimensions

Viramgami Gaurav 19110106

Yash More 19110123

Pahuni Jain 19110021

Kritika Kumawat 19110185

Maddela Siddarth 18110097

Contents

Problem Statement

We have learnt about Newton's law of gravitation in our high school syllabus. It involves the Gravitational Force between two objects (generally celestial bodies because of their considerable large masses). In view of the above law, we can use it to simulate the motion of two celestial bodies under Gravitational Force of each other. Using the above method, we can simulate the orbit of our planet earth under the Gravitational Force of the parent star, sun. This is the two-body problem.

The two-body problem is relatively easy to simulate analytically and numerically as it only involves mutual forces among two bodies. But when we add a third body in the system it becomes difficult to understand their motion analytically and can only be solved using computational methods. In this project we will try to simulate the motion of three-bodies (in 2-Dimension) numerically by using the Euler-Cromer method and Newton's law of gravitation.

In our solar system we generally observe the Sun-Earth system. But this is the simplest system that can be taken into consideration. In order to get a more accurate trajectory of the orbits, we also need to consider the Gravitational Force due to other celestial objects. We will show how the planetary motion of two planets changes when we add a third planet like Jupiter that can bring considerable changes in the existing system. The results will help us to understand how adding the third object (by experimenting with its intrinsic properties like mass and density) will affect the stable orbits of the two-body system. In general the three-body problem can be extended to n-body problems.[1]

Plan

1. Develop a physical model of the two-body Sun-Earth system using Free Body diagrams.
2. Derive the governing equations using Newton's Law of Gravitation.
3. Make appropriate assumptions for initial positions and velocities of the celestial objects.
4. Write a computer program to simulate the two-body system.
5. Introduce a third body to the two-body system and change the equations accordingly.
6. Simulate the equations for the three-body problem by making appropriate assumptions.
7. Experiment with the intrinsic properties of the third body (i.e., mass and density).
8. Verify the result with test cases such as the Sun-Earth-Moon system, etc and discuss the n-body problem.

1 Physical Model

1.1 2-Body Model

Consider two bodies, in a 2-dimensional cartesian plane, experiencing only Gravitational Force exerted on each other. The motion of these bodies are governed by the net effective force on each body. Figure below shows the free body diagram of two bodies having only Gravitational Force in between them.

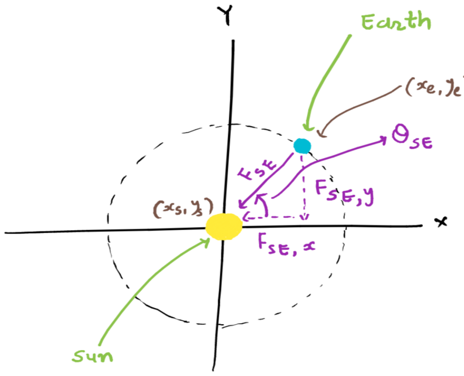


Figure 1: Sun-Earth System Free Body Diagram (2-Body)

In the figure: 1, the force acting on Earth is only the Gravitational Force due to Sun.

1.2 3-Body Model

Let us consider 3-bodies, exerting only Gravitational Force on each other, in a 2-dimensional cartesian plane, and their motion would be dependent on the net effective force on each body. The force would be dependent on the masses of the bodies.

Figure below shows the free body diagram of three bodies having only Gravitational Force in between them.

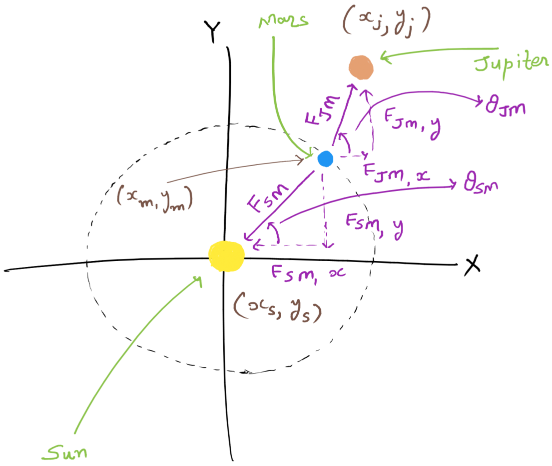


Figure 2: Sun-Mars-Jupiter System Free Body Diagram (3-Body)

In figure: 2, the forces acting on Mars are the Gravitational Forces of the Sun and Jupiter. Similarly, the force acting on Jupiter are the Gravitational Forces of Sun and Mars.

1.3 N-Body Model

Now, let us consider N-bodies, exerting only Gravitational Force on each other, in a 2-dimensional cartesian plane, and their motion would be dependent on the net effective force on each body. The force would be dependent on the masses of the bodies.

Figure below shows the free body diagram of N bodies having only Gravitational Force in between them.

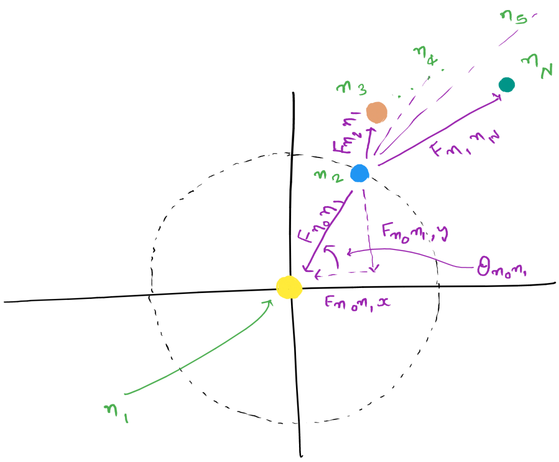


Figure 3: N Body System Free Body Diagram

In figure: 3, the forces acting on object n_1 are the Gravitational Forces of the objects n_0, n_2, n_3, ..., n_N. Similarly, the force acting on object n_2 are the Gravitational Forces of the objects n_0, n_1, n_3, ..., n_N.

Notations

Following are the notations that we have used throughout the report to be consistent:

- $F_{A,B}$: Gravitational Force on the celestial object B due to celestial object A (i.e. $F_{S,E}$: Gravitational Force on Earth due to Sun)
- $F_{AB,x}$: Component of the Gravitational Force on the celestial object B due to celestial object A in x - direction
- $F_{AB,y}$: Component of the Gravitational Force on the celestial object B due to celestial object A in y - direction
- F_A, x : Net Gravitational Force on celestial object A in x - direction
- F_A, y : Net Gravitational Force on celestial object A in y - direction
- $r_{A,B}$: The distance of the celestial object B with respect to celestial object A (i.e. $r_{S,E}$: Distance of Earth with respect to Sun)
- G : Gravitational Constant
- M_A : Mass of some celestial object A (i.e. M_S : Mass of Sun, M_E : Mass of Earth)
- x_a : x - coordinate of the celestial object A (i.e. x_e : x - coordinate of the Earth)
- y_a : y - coordinate of the celestial object A (i.e. y_e : y - coordinate of the Earth)
- $\theta_{A,B}$: Angle of the Gravitational Force between celestial objects A and B with horizontal as defined in the figure 1 and figure 2
- v_A : Velocity of the celestial object A (i.e. v_E : Velocity of Earth)
- $v_{a,x}$: Velocity of the celestial object A in x - direction (i.e. $v_{e,x}$: Velocity of Earth in x - direction)
- $v_{a,y}$: Velocity of the celestial object A in y - direction (i.e. $v_{e,y}$: Velocity of Earth in y - direction)
- t : To represent time

2 Assumptions

1. All the masses are assumed to be constant.
2. We take into account the coplanar movement of the celestial objects.
3. Analysis is done for the two dimensional motion of the bodies.

4. Celestial objects are considered to be spherical in shape.

5. We don't take into account the effect of celestial objects outside the considered system.

6. The body forces whose effect is considered on the system include Gravitational Force only.

7. Bodies are considered to be of uniform density.

8. The distance between any two planets is considered to be much larger than the radius of the planets.

3 Governing Equations

We will use Newton's Law for deriving equations for the 2 - Body Model as well as 3 - Body Model.
Newton's Gravitational law states that "The Gravitational Force between two point masses is proportional to the product of the two masses, and inversely proportional to the square of the distance between them".
Newton's second law of motion states that "The rate of change of momentum of an object is proportional to the force applied and is in the same direction of the applied force".

3.1 2 - Body Model

For the Sun-Earth system shown in the figure 1, we can write the Gravitational Force equation for the force acting on Earth due to Sun using the Newton's Gravitation Law as follows:
(The Sun is at the origin and the force $F_{S,E}$ is directed towards the Sun)

$$F_{S,E} = \frac{GM_S M_E}{r_{S,E}^2} \tag{1}$$

Now, by using the Newton's second law of motion we can write the following two second-order differential equations:

$$\begin{aligned} \frac{d^2x_e}{dt^2} &= \frac{F_{SE,x}}{M_E} \\ \frac{d^2y_e}{dt^2} &= \frac{F_{SE,y}}{M_E} \end{aligned} \tag{2}$$

Now, from the figure 1, we can use trigonometry to get the x and y component for $F_{S,E}$:

$$\begin{aligned}
\cos\theta_{S,E} &= \frac{x_e - x_s}{r_{S,E}} \\
\sin\theta_{S,E} &= \frac{y_e - y_s}{r_{S,E}} \\
F_{SE,x} &= -\frac{GM_S M_E}{r_{S,E}^2}\cos\theta_{S,E} = -\frac{GM_S M_E(x_e - x_s)}{r_{S,E}^3} \\
F_{SE,y} &= -\frac{GM_S M_E}{r_{S,E}^2}\sin\theta_{S,E} = -\frac{GM_S M_E(y_e - x_s)}{r_{S,E}^3}
\end{aligned} \tag{3}$$

3.2 3 - Body Model

For the Sun-Mars-Jupiter system shown in the figure 2, we can write the Gravitational Force equation for the force acting on Mars and Jupiter using Newton's Gravitational Law as follows:

(The Sun is at the origin)

The force $F_{S,M}$ and $F_{S,J}$ can be calculated as: (Both forces are directed towards Sun)

$$F_{S,M} = \frac{GM_S M_M}{r_{S,M}^2} \tag{4}$$

$$F_{S,J} = \frac{GM_S M_J}{r_{S,J}^2} \tag{5}$$

Now, from the figure 2, we can use trigonometry to get the x and y component for $F_{S,M}$ and $F_{S,J}$:

$$\begin{aligned}
\cos\theta_{S,M} &= \frac{x_m - x_s}{r_{S,M}} \\
\sin\theta_{S,M} &= \frac{y_m - y_s}{r_{S,M}} \\
F_{SM,x} &= -\frac{GM_S M_M}{r_{S,M}^2}\cos\theta_{S,M} = -\frac{GM_S M_M(x_m - x_s)}{r_{S,M}^3} \\
F_{SM,y} &= -\frac{GM_S M_M}{r_{S,M}^2}\sin\theta_{S,M} = -\frac{GM_S M_M(y_m - y_s)}{r_{S,M}^3}
\end{aligned} \tag{6}$$

And

$$\begin{aligned}
\cos\theta_{S,J} &= \frac{x_j - x_s}{r_{S,J}} \\
\sin\theta_{S,J} &= \frac{y_j - y_s}{r_{S,J}} \\
F_{SJ,x} &= -\frac{GM_S M_J}{r_{S,J}^2}\cos\theta_{S,J} = -\frac{GM_S M_J(x_j - x_s)}{r_{S,J}^3} \\
F_{SJ,y} &= -\frac{GM_S M_J}{r_{S,J}^2}\sin\theta_{S,J} = -\frac{GM_S M_J(y_j - y_s)}{r_{S,J}^3}
\end{aligned} \tag{7}$$

The force $F_{J,M}$ and $F_{M,J}$ can be calculated as: (force $F_{J,M}$ is directed towards Jupiter and the force $F_{M,J}$ is directed towards Mars)

$$F_{J,M} = \frac{GM_J M_M}{r_{J,M}^2} \tag{8}$$

$$F_{M,J} = \frac{GM_M M_J}{r_{M,J}^2} \tag{9}$$

Now, from the figure 2, we can use trigonometry to get the x and y component for $F_{J,M}$ and $F_{M,J}$:

$$\begin{aligned}
\cos\theta_{J,M} &= \frac{x_m - x_j}{r_{J,M}} \\
\sin\theta_{J,M} &= \frac{y_m - y_j}{r_{J,M}} \\
F_{JM,x} &= -\frac{GM_J M_M}{r_{J,M}^2}\cos\theta_{J,M} = -\frac{GM_J M_M(x_m - x_j)}{r_{J,M}^3} \\
F_{JM,y} &= -\frac{GM_J M_M}{r_{J,M}^2}\sin\theta_{J,M} = -\frac{GM_J M_M(y_m - y_j)}{r_{J,M}^3}
\end{aligned} \tag{10}$$

And

$$\begin{aligned}
\cos\theta_{M,J} &= \frac{x_j - x_m}{r_{M,J}} \\
\sin\theta_{M,J} &= \frac{y_j - y_m}{r_{M,J}} \\
F_{MJ,x} &= -\frac{GM_M M_J}{r_{M,J}^2}\cos\theta_{M,J} = -\frac{GM_M M_J(x_j - x_m)}{r_{M,J}^3} \\
F_{MJ,y} &= -\frac{GM_M M_J}{r_{M,J}^2}\sin\theta_{M,J} = -\frac{GM_M M_J(y_j - y_m)}{r_{M,J}^3}
\end{aligned} \tag{11}$$

So, the net Gravitational Force on Mars in x and y direction is (from equation (6) and (10)):

$$\begin{aligned}
F_{M,x} &= -\frac{GM_S M_M(x_m - x_s)}{r_{S,M}^3} - \frac{GM_J M_M(x_m - x_j)}{r_{J,M}^3} \\
F_{M,y} &= -\frac{GM_S M_M(y_m - y_s)}{r_{S,M}^3} - \frac{GM_J M_M(y_m - y_j)}{r_{J,M}^3}
\end{aligned} \tag{12}$$

Ans similarly the net Gravitational Force on Jupiter in x and y direction is (from equation (7) and (11)):

$$\begin{aligned}
F_{J,x} &= -\frac{GM_S M_J(x_j - x_s)}{r_{S,J}^3} - \frac{GM_M M_J(x_j - x_m)}{r_{M,J}^3} \\
F_{J,y} &= -\frac{GM_S M_J(y_j - y_s)}{r_{S,J}^3} - \frac{GM_M M_J(y_j - y_m)}{r_{M,J}^3}
\end{aligned} \tag{13}$$

Now, we can represent Force as mass times acceleration as follows:

$$\begin{aligned}
\frac{d^2 x_j}{dt^2} &= \frac{F_{J,x}}{M_J} \\
\frac{d^2 y_j}{dt^2} &= \frac{F_{J,y}}{M_J} \\
\frac{d^2 x_m}{dt^2} &= \frac{F_{M,x}}{M_M} \\
\frac{d^2 y_m}{dt^2} &= \frac{F_{M,y}}{M_M}
\end{aligned} \tag{14}$$

3.3 N - Body Model

For the N Body system shown in the figure 3, we can write the Gravitational Force equation for the force acting on each celestial object using Newton's Gravitational Law (The Sun is at the origin) and derive the net Gravitational Force on each celestial object. By generalizing the equations (12) and (13):
For force on some celestial object P_i having index i (1 indexing from Sun i.e. Sun - index 1)

$$\begin{aligned} F_{P_i,x} &= \sum_{j=1,\ j\neq i}^{j=N} -\frac{GM_{P_j}M_{P_i}(x_i - x_j)}{r^3_{P_j,P_i}} \\ F_{P_i,y} &= \sum_{j=1,\ j\neq i}^{j=N} -\frac{GM_{P_j}M_{P_i}(y_i - y_j)}{r^3_{P_j,P_i}} \end{aligned} \tag{15}$$

Now, we can represent Force as mass times acceleration as follows:

$$\begin{aligned} \frac{d^2x_i}{dt^2} &= \frac{F_{P_i,x}}{M_{P_i}} \\ \frac{d^2y_i}{dt^2} &= \frac{F_{P_i,y}}{M_{P_i}} \end{aligned} \tag{16}$$

4 Initial Conditions

Since the governing equations (2) (for 2 Body System), (14) (for 3 Body System) and (16) (for N Body System) are second order differential equations we will need two initial conditions to solve those second order differential equations.
These two initial conditions include the initial position ((x, y) at t = 0) and the velocity $((\frac{dx}{dt}, \frac{dy}{dt}) =$ (v_x, v_y) at t = 0) of the celestial objects.

4.1 Initial Position of the Celestial Objects

Reference celestial object:
For Reference celestial object, we have assumed that initially at t = 0, it is situated at the origin (i.e. (0, 0)).
Orbiting celestial objects:
For all other Orbiting Celestial Objects, we have assumed that initially at t = 0, the are situated on the positive x - axis with respect to the reference celestial object.
For example, if we consider Sun as the reference celestial object, then the initial position of Earth will be:
(x, y) = (Orbiting radius of Earth with respect to sun, 0) = (1.00 AU, 0)
Similarly for Mars the initial position with respect to Sun will be:
(x, y) = (Sun-Mars distance, 0) = (1.52 AU, 0)

The values of the orbiting radius of the planets of solar system are given in the Parameters section.

4.2 Initial Velocity of the Celestial Objects

Reference celestial object:

Since, we are calculating positions and velocities of other orbiting celestial objects with respect to the reference celestial object, we have assumed the reference celestial object to be at rest and hence its initial velocity will be:

$(\frac{dx}{dt}, \frac{dy}{dt}) = (\mathrm{v}_x, \mathrm{v}_y) = (0, 0)$ (in AU/yr)

Orbiting celestial objects:

Since we have assumed the initial position of the celestial objects on x - axis, the initial velocity of the celestial object will be only in the y - direction. And that velocity can be approximated using its orbiting radius and the orbiting time period.

For example, if we consider Sun as the reference celestial object, then the initial velocity of Earth will be:

$(\frac{dx}{dt}, \frac{dy}{dt}) = (\mathrm{v}_x, \mathrm{v}_y) = (0, 2\pi(\text{Orbiting Radius})/(\text{Orbiting Time Period})) = (0, 2\pi(1.00\text{ AU})/(1.0000\text{ yr}))$ (in AU/yr)

Similarly for Mars the initial velocity with respect to Sun will be:

$(\frac{dx}{dt}, \frac{dy}{dt}) = (\mathrm{v}_x, \mathrm{v}_y) = (0, 2\pi(\text{Orbiting Radius})/(\text{Orbiting Time Period})) = (0, 2\pi(1.52\text{ AU})/(1.8808\text{ yr}))$ (in AU/yr)

The values of the orbiting radius and orbiting time period of the planets of solar system are given in the Parameters section.

5 Parameters

5.1 Global Constants:

Serial No.	Planets	Mass (in kg)	Distance from the Sun (in AU)	Orbital Time Period (in yr)
1	Mercury	3.3×10^{23}	0.39	0.2410
2	Venus	4.9×10^{24}	0.72	0.6164
3	Earth	6.0×10^{24}	1.00	1.0000
4	Mars	6.4×10^{23}	1.52	1.8808
5	Jupiter	1.9×10^{27}	5.20	11.8600
6	Saturn	5.7×10^{26}	9.54	29.4571
7	Uranus	8.8×10^{25}	19.19	84.0205
8	Neptune	1.0×10^{26}	30.06	164.8000
9	Pluto	1.5×10^{22}	39.53	247.9400

5.2 Other constants:

1. Gravitational Constant = $6.674 \times 10^{-11}\ Nm^2/Kg^2$
2. Mass of Sun = 1.989×10^{30} Kg
3. One Astronomical Unit (1 AU) = 1.5×10^{11} m

   1 AU is defined as the average distance between the Sun and the Earth.
4. π(PI) = 3.141592
5. Conversion Constant = $(6.68455 \times 10^{-12})^3/(3.17098 \times 10^{-8})^2$

   It converts units from $\frac{m^3}{sec^2}$ to $\frac{AU^3}{yr^2}$.

6 Solution Methodology

We know that in symplectic problems, we deal with scenarios where energy conservation is applicable. We only consider the sum of kinetic energy and potential energy, therefore we had to use an algorithm where energy conservation is ensured. For the same reason, we have used the Euler Cromer and Velocity Verlet integration method to solve 2-body, 3-body and n-body problems. In a simple Euler method algorithm which is Non Energy Preserving, we are actually calculating the velocity (hence momentum) and position at the same time. Therefore we can see that it does not follow the Heisenberg uncertainty principle.

So, to get stable orbits of the celestial objects, we need to use Energy Preserving numerical methods.

Euler Cromer is first order Energy Preserving and Velocity Verlet method is second order Energy Preserving method.

Since, Euler Cromer is a first order method, we get the local truncation error in the order of O(h^2) (By using Taylor expansion). To minimize the error we have used the Velocity Verlet integration in which the local truncation error order is of order O(h^3) (By using Taylor expansion).

7 Numerical Solution

In case of 2 Body system, we can represent both the second-order differential equations (2) as two first-order differential equations with the help of these relationship:

$$\begin{aligned} \frac{d^2x_e}{dt^2} &= \frac{dv_{e,x}}{dt} \\ \frac{d^2y_e}{dt^2} &= \frac{dv_{e,y}}{dt} \end{aligned} \tag{17}$$

So, from the equation (3) and (4) the first-order differential order equations will be:

$$\begin{aligned} \frac{dv_{e,x}}{dt} &= -\frac{GM_S x_e}{r_{S,E}^3} \\ \frac{dx_e}{dt} &= v_{e,x} \\ \frac{dv_{e,y}}{dt} &= -\frac{GM_S y_e}{r_{S,E}^3} \\ \frac{dy_e}{dt} &= v_{e,y} \end{aligned} \tag{18}$$

Similarly, we can get system of first order equations for 3 Body and N Body systems.

7.1 Euler Cromer Algorithm

By using the Taylor series expansion of the velocity, we can write:

$$v(t+h) = v(t) + h\dot{v}(t) + O(h^2) \tag{19}$$

$$v(t+h) = v(t) + a(t)h + O(h^2) \tag{20}$$

and similarly for position,

$$x(t+h) = x(t) + h\dot{x}(t) + O(h^2) \tag{21}$$

Now, at this step the Euler Cromer method differs from the Normal Euler method. As we need to preserve energy of the system the two terms space (position) and momentum (velocity) should not be changed at the same time. And hence we need to first update velocity assuming the previous value of position and then update the value of position using new value of velocity. So,

$$\boxed{x(t+h) = x(t) + v(t+h)h + O(h^2)} \tag{22}$$

7.2 Velocity Verlet Algorithm

The second order differential equation of motion i.e. $\ddot{x}(t) = \frac{F(x)}{m}$ is split into two first order equations:

$$\begin{aligned}\dot{x}(t) &= v(t)\\ \dot{v}(t) &= F(x(t))/m\\ \dot{v}(t) &= a(t)\end{aligned} \tag{23}$$

Using the Taylor Series Expansion, a finite difference algorithm is derived below, starting with x(t+h):

$$x(t+h) = x(t) + h\dot{x}(t) + \frac{h^2}{2}\ddot{x}(t) + O(h^3) \tag{24}$$

Replacing $\dot{x}$ with v(t) and $\ddot{x}$ with a(t);

$$\boxed{x(t+h) = x(t) + hv(t) + \frac{h^2}{2}a(t) + O(h^3)} \tag{25}$$

Again, using the Taylor Series Expansion, a finite difference algorithm is developed for v(t+h):

$$v(t+h) = v(t) + h\dot{v}(t) + \frac{h^2}{2}\ddot{v}(t) + O(h^3) \tag{26}$$

Now, as we require $\ddot{v}$, we expand $\dot{v}(t+h)$;

$$\dot{v}(t+h) = \dot{v}(t) + h\ddot{v}(t) + O(h^2) \tag{27}$$

In order to go up to the order h^2, we multiply the above equation by h/2 and after rearranging, we get:

$$\frac{h^2}{2}\ddot{v}(t) = \frac{h}{2}(\dot{v}(t+h) - \dot{v}(t)) + O(h^3) \tag{28}$$

Above step was done to get a good approximation to order O(h^3). Now, substituting the above equation in the equation for v(t+h);

$$v(t+h) = v(t) + h\dot{v}(t) + \frac{h}{2}(\dot{v}(t+h) - \dot{v}(t)) + O(h^3) \tag{29}$$

Finally, the equation of motion involving velocity, position and Force acting between the celestial objects comes out to be:

$$v(t+h) = v(t) + \frac{h}{2}(a(t+h) + a(t)) + O(h^3) \tag{30}$$

8 Algorithm Used

8.1 Euler- Cromer Method

At every time step i we compute the position (x,y) and the velocity along x and y axis (v_x, v_y).

1. First we calculate the distance r_i from the reference celestial object: $r_i^2 = (x_i^2 + y_i^2)$
2. Calculate the values of $v_{x,i+1}$ and $v_{y,i+1}$ as $v_{x,i+1} = v_{x,i} + a_{x,i}\Delta t$ and $v_{y,i+1} = v_{y,i} + a_{y,i}\Delta t$.
3. Now compute x_{i+1} and y_{i+1} using $v_{x,i+1}$ and $v_{y,i+1}$ as $x_{i+1} = x_i + v_{x,i+1}\Delta t$ and $y_{i+1} = y_i + v_{y,i+1}\Delta t$.
4. Now compute $a_{x,i+1}$ and $a_{y,i+1}$ using the new position values x_{i+1} and y_{i+1}.
5. Memoize these positions x_{i+1} and y_{i+1} and accelerations $a_{x,i+1}$ and $a_{y,i+1}$ for the next iteration.

8.2 Velocity-Verlet Algorithm

1. Calculate x_{i+1} and y_{i+1} using $x_{i+1} = x_i + v_{x,i}\Delta t + a_{x,i}\frac{\Delta t^2}{2}$ and $y_{i+1} = y_i + v_{y,i}\Delta t + a_{y,i}\frac{\Delta t^2}{2}t^2$.
2. Now we can calculate the updated distance $r_{i+1}^2 = (x_{i+1}^2 + y_{i+1}^2)$ and hence update the acceleration.
3. We also have to compute the new $a_{x,i+1}$ and $a_{y,i+1}$ from new positions x_{i+1} and y_{i+1}.
4. Using the update acceleration we update the velocity as $v_{x,i+1} = v_{x,i} + (a_{x,i} + a_{x,i+1})\frac{\Delta t}{2}$ and $v_{y,i+1} = v_{y,i} + (a_{y,i} + a_{y,i+1})\frac{\Delta t}{2}$
5. Memoize current velocity, position and acceleration for the next iteration.

9 Higher Order Numerical Algorithms for celestial object trajectories

To further minimize the local truncation error, we can increase our order of equation. Here, we have extended our approach to Ronald Ruth algorithm.

Using the splitting method for separable Hamiltonian, we have equations below:

$$v_{i+1} = v_i + d_i a(x_i)\Delta t \tag{31}$$

$$x_{i+1} = x_i + c_i v_{i+1} \Delta t \tag{32}$$

For third order Ronald Ruth method which has local truncation error of the order $O(h^4)$, we have the coefficient $c_1 = 1$, $c_2 = -\frac{2}{3}$, $c_3 = \frac{2}{3}$ and $d_1 = -\frac{1}{24}$, $d_2 = \frac{3}{4}$, $d_3 = \frac{7}{24}$.

1. As we have three coefficients we find $v_{x,i+1,j+1}$ using the relation $v_{x,i+1,j+1} = v_{x,i+1,j} + (d_i a_{x,i+1,j} \Delta t)$ and similarly $v_{y,i+1,j+1} = v_{y,i+1,j} + (d_i a_{y,i+1,j} \Delta t)$
2. Now we update the position of the celestial objects $x_{i+1,j+1}$ as $x_{i+1,j+1} = x_{i+1,j} - (c_i v_{x,i+1,j+1} \Delta t)$ and similarly $y_{i+1,j+1} = y_{i+1,j} - (c_i v_{y,i+1,j+1} \Delta t)$
3. Now using the newly calculated position the acceleration component is updated in step 1.
4. After three iterations of the above steps (as there are three coefficients), we get $x_{i+1,4}$ which will be used as $x_{i+2,1}$.

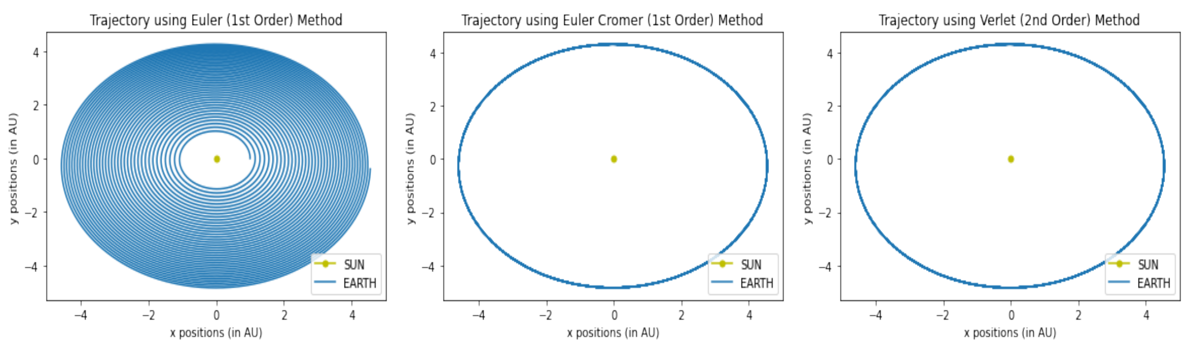
The local truncation error can further minimised to the order of $O(h^5)$ by using 4th Order Ronald Ruth method.

10 Results and Discussion

10.1 2-Body Trajectory

Implementation of the first-order Euler method showed that the Earth is orbiting around the Sun in a spiral path. It implies that its energy is not being conserved during its motion and hence, it will end up colliding with the Sun at the centre.

To solve this issue, symplectic methods which are energy conserving in Nature are used here for numerical analysis. Using Euler Cromer Method and Velocity Verlet Method, we got a stable orbiting path for Earth around the Sun.



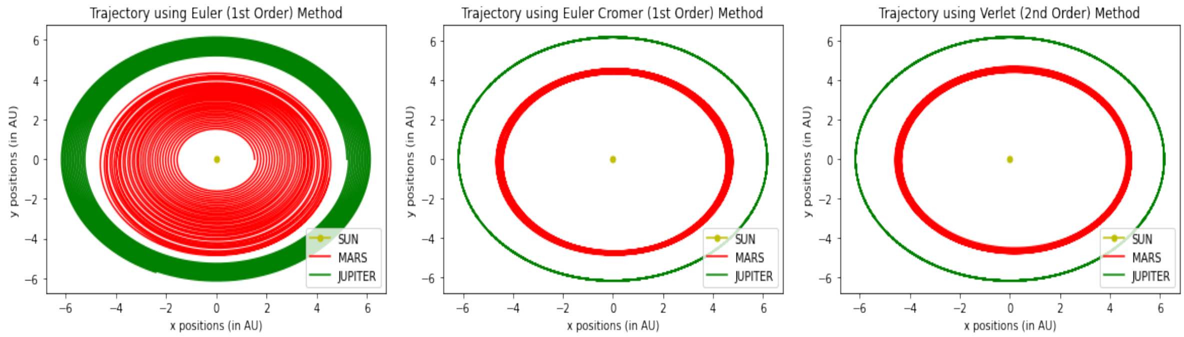
(a) Sun-Earth system (b) Sun-Earth system (c) Sun-Earth system

Figure 4: 2-Body system simulations using code

10.2 3-Body Trajectory

10.2.1 Standard Mass 3-Body system

Similar to 2-body system, orbiting trajectory for Mars and Jupiter around Sun were obtained. First-Order Euler method being non-conservative in nature gave a spiral trajectory for both Mars and Jupiter. The other two methods, Euler Cromer and Velocity Verlet, gave a perfect orbital trajectory for the given 3-Body system.

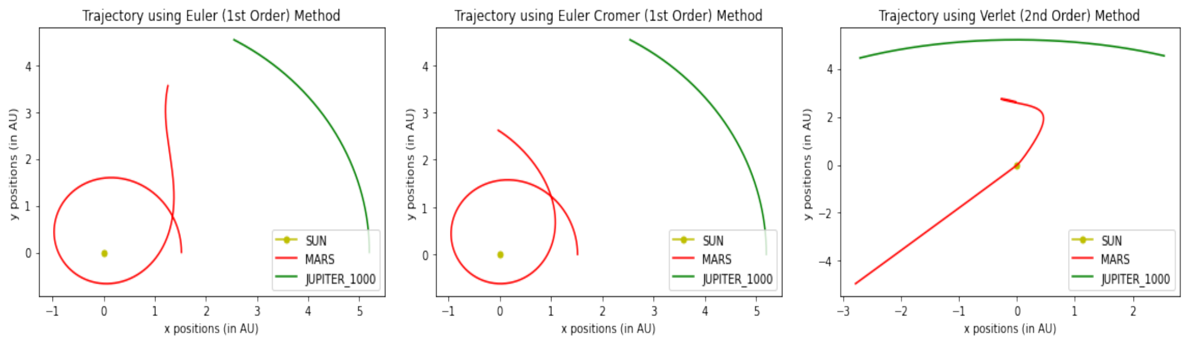


(a) Sun-Mars-Jupiter (b) Sun-Mars-Jupiter (c) Sun-Mars-Jupiter

Figure 5: 3-Body system simulations using code

10.2.2 3-Body system with change in Standard Mass

For the 3 body system, involving Sun, Mars and Jupiter (Mass equal to 1000 times the mass of actual Jupiter), we observed the trajectories of the two planets around Sun. Same three methods as used for above mentioned cases, showed a distorted orbit for Mars implying the influence of increased mass of Jupiter.



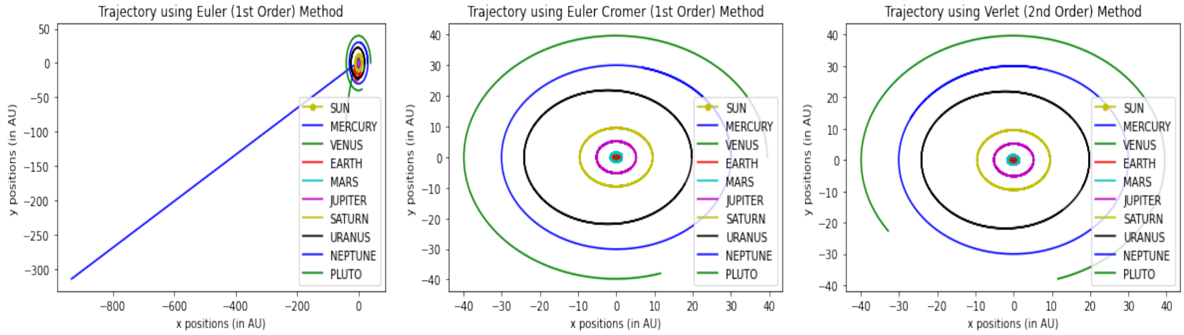
(a) Sun-Mars-Jupiter (b) Sun-Mars-Jupiter (c) Sun-Mars-Jupiter

Figure 6: 3-Body system simulations using code

10.3 N-Body Trajectory

Numerical Analysis methods were further done for N-body system. The trajectories for all the planets of the Solar system were observed firstly through Euler (1st order) method which gave unexpected results being a non- energy conserving method.

Perfect, stable orbits for the revolving planets around the Sun were obtained through Euler Cromer and Velocity Verlet Methods.



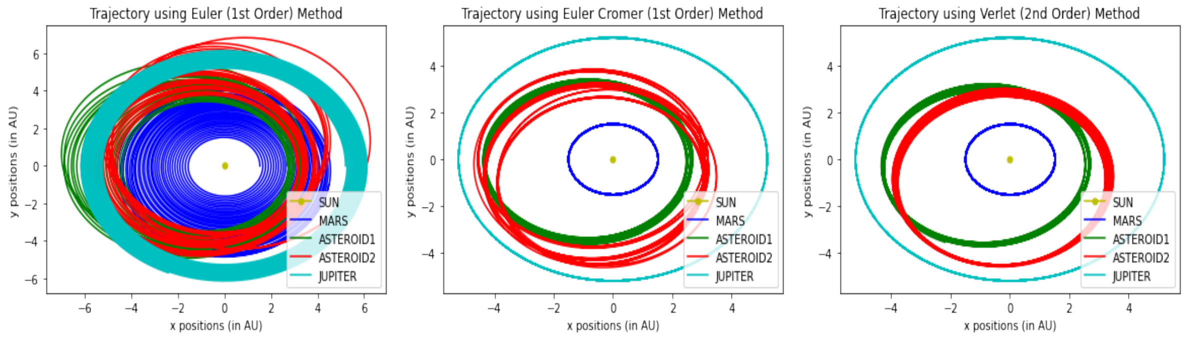
(a) Solar system (b) Solar system (c) Solar system

Figure 7: N-Body system (here solar system) simulations using code

Miscellaneous Case:

Here, the system in consideration involved Sun, Mars, Asteroid Belt(Between Mars and Jupiter) and Jupiter. First order Euler method showed spiral orbits, implying the loss of energy.

The same system considering two particluar asteroids along with Mars and Jupiter, was solved using Euler Cromer and Velocity Verlet method. These methods gave stable orbits for revolving celestial objects. It was also observed that as asteroid approaches near the Jupiter, its orbit get distorted due to effect of strong gravitational forces from the Jupiter.



(a) Sun-Mars-Asteroid Belt-Jupiter (b) Sun-Mars-Asteroid Belt-Jupiter (c) Sun-Mars-Asteroid Belt-Jupiter

Figure 8: N-Body system (Here asteroid belt between mars and jupiter)

GitHub Repository Link

We have open sourced the source code filed on GitHub and it can be accessed from the following link: `https://github.com/GauravViramgami/-MA-202-Maths-Project`

Notes and References

1. Giordano, Nicholas; Nakanishi, Hisao. *Computational Physics*. 2nd ed., Chapter 4, Pearson Education Ltd., 2006.

2. Jonsson, Hannes; *Classical Dynamics*, Lecture Notes, University of Washington., 2000. Link: Verlet Derivation

3. Forest, Etienne; Ruth, Ronald D. *Fourth-Order Symplectic Integration*, 1989. Link: Symplectic Method

4. Williams, David. (2019, October 21). *Planetary Fact Sheet- Metric*. NASA. Link: Global Constants