ASSIGNMENT - 1

MODULE - 1

1) Derive the cauchy-Riemann equation in cartesian form.

Sol Statement: if $w = f(z) = u(x,y) + iv(x,y)$ is an analytic function at any point $z = x+iy$ then there exist 4 continuous first order partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}$ & $\frac{\partial v}{\partial y}$ & satisfies the equation

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. These equations are known as C-R equations.

Proof: given $w = f(z) = u(x,y) + iv(x,y)$ is analytic

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \text{ exists} \quad -①$$

WKT $z = x+iy$

$$\delta z = \delta x + i\delta y$$

$$f(z) = u(x,y) + iv(x,y)$$

$$f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$$

Eq ①

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)] - [u(\delta x, y) + iv(x,y)]}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) - u(x,y)]}{\delta z} + \lim_{\delta z \to 0} i\frac{[v(x+\delta x, y+\delta y) - v(x,y)]}{\delta z} \quad ②$$

Since; $\delta z \to 0$ it has 2 possibilities

case i): $\delta z \to 0$ so then $\delta z = \delta x$

as $\delta z \Rightarrow 0 \Rightarrow \delta x \to 0$

Eq ② becomes.

$$f'(z) = \lim_{\delta x \to 0} \frac{[u(x+\delta x, y) - u(x-y)]}{\delta x}$$

$$+ \lim_{\delta z \to 0} \frac{i\,[v(x+\delta x, y) - v[x-y)]}{\delta x}$$

The above limits from the basic definition of partial derivation.

$$f'(x) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad —③$$

case (ii): $\delta x = 0$, so that $\delta z = i\delta y$

as $\delta z \to 0 \Rightarrow i\delta y \to 0 \Rightarrow \delta y \to 0$

Eq ② becomes

$$f'(z) = \lim_{\delta y \to 0} \frac{[u(x, y+\delta y) - u(x,y)]}{i\delta y} + \lim_{\delta y \to 0} \frac{[v(x, y+\delta y) - v}{i\delta y}$$

The above limits from the basic definition of partial derivation

$$f'(x) = \frac{\partial u}{\partial x} + \frac{i\partial v}{\partial x} \quad —③$$

case ii): $\delta x = 0$, so that $\delta x = i\delta y$

as $\delta z \to 0 \Rightarrow i\delta y \to 0 \Rightarrow \delta y \to 0$

Eq ② becomes

$$f'(z) = \lim_{\delta y \to 0} [$$

Equation ③ & ④ RHS

$$\frac{\partial u}{\partial x} + \frac{i\partial v}{\partial x} = -\frac{i\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

comparing

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}}$$

$$\boxed{u_x = v_y \; ; \; v_x = -u_y}$$

2) Derive the cauchy-Riemann equations in polar form.

Sol Statement: If $f(z) = u(r,\theta) + iv(r,\theta)$ is an analytical at a point $z = r.e^{i\theta}$ then there exists a continuous 1st order partial derivatives. $\frac{\partial u}{\partial r}, \frac{\partial v}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial \theta}$ and satisfies the equation $\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\cdot\frac{\partial u}{\partial \theta}$ the equation are known as CR eqns in polar form.

Proof: let $f(z)$ be analytic

$$f'(z) = \underset{\delta z \to 0}{lt} \frac{f(z+\delta z) - f(z)}{\delta z} \quad — ①$$

exists & is unique

WKT

$$f(z) = u(r,\theta) + iv(r,\theta)$$

$$\therefore f(z+\delta z) = u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)$$

$$f'(z) = \underset{\delta z \to 0}{lt} \frac{u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta) - (u(r,\theta) + iv(r,\theta))}{\delta z}$$

$$= \underset{\delta z \to 0}{lt} \frac{u(r+\delta r, \theta+\delta\theta) - u(r,\theta)}{\delta z} + i\underset{\delta z \to 0}{lt} \frac{v(r+\delta r, \theta+\delta\theta) - v(r,\theta)}{\delta z} \quad — ②$$

Since $z = r.e^{i\theta}$, z is a function of 2 variables r, θ

$$\delta z = \frac{\partial z}{\partial r}\cdot\delta r + \frac{\partial z}{\partial \theta}\cdot\delta\theta$$

$$= \frac{\partial}{\partial r}(r.e^{i\theta})\delta r + \frac{\partial}{\partial \theta}(r.e^{i\theta})\delta\theta$$

$$\delta z = e^{i\theta}\cdot\delta r + r\cdot ie^{i\theta}\delta\theta \quad — ③$$

case (i) let $\delta\theta = 0 \Rightarrow \delta z = e^{i\theta}\delta r$

as $\delta z \to 0 \Rightarrow e^{i\theta}\delta r$

$$f'(z) = \underset{\delta z \to 0}{lt} \frac{u(r+\delta r, \theta) - u(r,\theta)}{e^{i\theta}\delta r} + i\underset{\delta r \to 0}{lt} \frac{v(r+\delta r, \theta) - v(r,\theta)}{e^{i\theta}\cdot\delta r}$$

$$= e^{-i\theta}\left[\underset{\delta r \to 0}{lt} \frac{u(r+\delta r, \theta) - u(r,\theta)}{\delta r} + i\underset{\delta r \to 0}{lt} \frac{v(r+\delta r, \theta) - v(r,\theta)}{\delta r}\right]$$

$$f'(z) = e^{i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \quad \text{—④}$$

case (ii): Let $\delta r = 0 \rightarrow \delta z = i \cdot r \cdot e^{i\theta}\delta\theta$

as $\delta z \rightarrow 0 \rightarrow \delta\theta \rightarrow 0$

$$f'(z) = \lim_{\delta\theta \to 0} \frac{u(r,\theta+\delta\theta) - u(r,\theta)}{i \cdot r \cdot e^{i\theta}\delta\theta} + i \lim_{\delta\theta \to 0} \frac{v(r,\theta+\delta\theta) - v(r,\theta)}{i r \cdot e^{i\theta}\delta\theta}$$

$$= \frac{e^{-i\theta}}{r}\left[\lim_{\delta\theta \to 0} (-i)\frac{u(r,\theta+\delta\theta) - u(r,\theta)}{\delta\theta} - \lim_{\delta\theta \to 0} \frac{v(r,\theta+\delta\theta) - v(r,\theta)}{\delta\theta}\right]$$

$$= f'(z) = \frac{e^{-i\theta}}{r} \cdot \left[-i\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right] \quad \text{—⑤}$$

comparing ④ & ⑤

$$e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = \frac{e^{-i\theta}}{r} \cdot \left[-i\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right]$$

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \; ; \; \frac{\partial v}{\partial r} = \frac{-1}{r} \cdot \frac{\partial u}{\partial \theta}}$$

$$u_r = \frac{1}{r} \cdot v_\theta \, , \; v_r = \frac{-1}{r} \cdot u_\theta$$

$$\boxed{r u_r = v_\theta \; ; \; r \cdot v_r = -u_\theta}$$

3) Define Harmonic function. Prove that the real and imaginary parts of an analytic functions are harmonic.

Sol Definition - A function ϕ is said to be harmonic if it satisfies the Laplace equation $\nabla^2\phi = 0$.

Statement - ~~If~~ $f(z) = u + iv$ be analytic then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \; \& \; \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Proof:- $f(z) = u + iv$ is analytic

$$u_x = v_y \, , \; v_x = -u_y$$

$$\text{By } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \, , \; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{—②}$$

Diff ① partially wrt x &

② partially wrt y

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \cdot \partial y} , \frac{\partial^2 v}{\partial x \partial y} = \frac{-\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-\partial^2 u}{\partial y^2}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

$\therefore$ This shows that the real part is harmonic

Diff ① wrt y & ② wrt x

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} , \frac{\partial^2 v}{\partial x^2} = \frac{-\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{-\partial^2 v}{\partial x^2}$$

$$\boxed{\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

This shows that the imaginary part is also harmonic.

Thus u & v are hormonic in cartesian form.

b) If $f(z) = u + iv$ is analytic in r & θ then u & v are harmonic (or) if $f(z) = u + iv$ is analytic in r & θ then $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \theta} = 0$ & $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial v}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 v}{\partial \theta^2} = 0$

proof: Since $f(z) = u + iv$ is analytic

By CR eq $r \cdot \frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}$ — ①

$$r \cdot \frac{\partial v}{\partial r} = \frac{-\partial u}{\partial \theta} \text{ — ②}$$

Diff partially ① wrt r & ② wrt θ

$$r \cdot \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = \frac{\partial^2 v}{\partial r \cdot \partial \theta} , \frac{r \cdot \partial^2 v}{\partial r \cdot \partial \theta} = \frac{-\partial^2 u}{\partial \theta^2} ,$$

$$\frac{\partial^2 v}{\partial r \cdot \partial \theta} = \frac{-1}{r} \frac{\partial^2 u}{\partial \theta^2}$$

$$r\cdot\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{\partial r}=\frac{-1}{r}\frac{\partial^2 u}{\partial\theta^2}$$

$$r\cdot\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{\partial r}+\frac{1}{r}\cdot\frac{\partial^2 u}{\partial\theta^2}=0$$

$\div$ by r

$$\boxed{\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0}$$

Hence u is harmonic

Now Diff partially ① wrt θ & wrt r

$$r\cdot\frac{\partial^2 u}{\partial r\cdot\partial\theta}=\frac{\partial^2 v}{\partial\theta^2},\ r\cdot\frac{\partial^2 v}{\partial r^2}+\frac{\partial v}{\partial r}=\frac{-\partial^2 u}{\partial\theta\partial r}$$

$$\frac{\partial^2 u}{\partial r\cdot\partial\theta}=\frac{1}{r}\cdot\frac{\partial v^2}{\partial\theta^2};\ r\cdot\frac{\partial^2 v}{\partial r^2}+\frac{\partial v}{\partial r}=\frac{-\partial^2 u}{\partial r\cdot\partial\theta}$$

$$r\cdot\frac{\partial^2 v}{\partial r^2}+\frac{\partial v}{\partial r}=-\frac{1}{r}\cdot\frac{\partial v^2}{\partial\theta^2}$$

$$\frac{r\cdot\partial^2 v}{\partial r^2}+\frac{\partial v}{\partial r}+\frac{1}{r}\cdot\frac{\partial v^2}{\partial\theta}=0$$

$\div$ by r

$$\boxed{\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial v^2}{\partial\theta^2}=0}$$

Thus v is harmonic

u & v are harmonic function.

4) Show that $f(z)=\sin z$ is analytic and hence find $f'(z)$.

Sol: $w=\sin z$

let $z=x+iy$

$w=\sin(x+iy)$ [$\sin(A+B)=\sin A\cos B+\cos A\cdot\sin B$]

$w=\sin x\cdot\cos iy+\cos x\cdot\sin iy$

$u+iv=\sin x\cdot\cosh y+i\cos x\cdot\sinh y$

$u=\sin x\cdot\cosh y,\ v=\cos x\cdot\sinh y$

$u_x=\cos x\cdot\cosh y,\ v_x=-\sin x\cdot\sinh y$

classmate
Date
Page

$u_y = \sin x \sinh y$, $v_y = \cos x \cdot \cosh y$

$u_x = v_y$, $v_x = -u_y$

$\therefore$ w $f(z)$ is analytic

$\frac{dw}{dz}$ or $f'(z) = u_x + i v_x$

$= \cos x \cdot \cosh y + i(-\sin x \cdot \sinh y)$

Sub $z = x$ & $y = 0$

$\underline{\underline{f'(z) = \cos z}}$

(5) Show that $w = \log z, z \neq 0$ is analytic and hence find $\frac{dw}{dz}$

Sol: $w = \log z$

$z = r \cdot e^{i\theta}$

$w = \log(r \cdot e^{i\theta})$

$= \log r + \log e^{i\theta}$

$u + iv = \log r + i\theta \log e$

$u = \log r$, $v = \theta \log e$ $\quad \log e = 1$

$u = \log r$, $v = \theta$

$u_r = \frac{1}{r}$, $v_r = 0$

$u_\theta = 0$, $v_\theta = 1$

$\boxed{r \cdot u_r = v_\theta}$, $\boxed{r v_r = -u_\theta}$

$f'(z) = e^{-i\theta}[u_r + i v_r]$

$= e^{-i\theta}[1/r + i(0)]$

$= e^{-i\theta}[1/r]$

$\theta = 0$, $r = z$

$= e^{0}\left(\frac{1}{z}\right)$

$\boxed{f'(z) = 1/z}$

⑥ Find the analytic function $f(z)$ whose real part is $x^2-y^2+\frac{x}{x^2+y^2}$

Sol: $u = x^2-y^2+\frac{x}{x^2+y^2}$

$u_x = 2x + \frac{(x^2+y^2)1 - x\,2x}{(x^2+y^2)^2} = 2x + \frac{y^2-2x^2}{(x^2+y^2)^2}$

$u_y = -2y + \frac{(x^2+y^2)0 - x\cdot 2y}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2}$

consider $f'(z) = u_x + iv_x$

By CR eqn's $u_x = v_y$ & $v_x = -u_y$

$f'(z) = u_x - iu_y$

$f'(z) = 2x + \frac{y^2-2x^2}{(x^2+y^2)^2} + i\left(2y + \frac{2xy}{(x^2+y^2)^2}\right)$

Put $x = z$ & $y = 0$

$= 2z + \frac{0-2z^2}{(z^2+0)^2} + i(0+0)$

$= 2z - \frac{2z^2}{(z)^4}$

$f'(z) = 2z - \frac{1}{z^2}$

Integrate wrt dz

$f(z) = \frac{2z^2}{2} + \frac{1}{z}$

$\boxed{f(z) = z^2 + \frac{1}{z}} + c$

⑦ Find the analytic function $f(z) = u+iv$ given $v = e^{-x}(x\cos y + y\sin y)$

Sol: $v = e^{-x}(x\cos y + y\sin y)$

$v = e^{-x}x\cdot\cos y + e^{-x}y\sin y$

$v_x = e^{-x}\cdot(1)\cdot\cos y - xe^{-x}\cdot\cos y - e^{-x}\cdot y\sin y$

$V_y = -e^{-x} \cdot x \sin y + e^{-x} \cdot y \cdot \cos y + e^{-x} \sin y$

wkt $f'(z) = u_x + i v_x$

$u_x = v_y$.

$f'(z) = v_y + i v_x$

$= e^{-x} \cdot x \sin y + e^{-x} y \cdot \cos y + e^{-x} \sin y +$
$i (e^{-x} \cdot \cos y - x e^{-x} \cos y - e^{-x} y \sin y)$

$x = z, \; y = 0$

$f'(z) = 0 + 0 + e^{-z}(0) + i(e^{-z} - z e^{-z} - 0)$

$f'(z) = i (e^{-z} - z e^{-z})$

Integrate wrt z

$f(z) = i \int e^{-z} - z e^{-z} \, dz$

$= i\left(\frac{e^{-z}}{-1} - \left(\frac{z e^{-z}}{-1} - \int \frac{e^{-z}}{-1} \cdot (1) \cdot dz + c\right)\right)$

$= i(-e^{-z} + z e^{-z} + e^{-z} + c)$

$$\boxed{f(z) = i z e^{-z} + c}$$

(8) Find the analytic function whose real part is $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ also find its imaginary part.

Sol $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$u_x = 3x^2 - 3y^2 + 6x$

$u_{xx} = 6x + 6$

$u_y = -6xy - 6y$

$u_{yy} = -6x - 6$

$u_{xx} + u_{yy} = 0$

$\therefore$ It is harmonic

By CR equation

$u_x = v_y, \; v_x = -u_y$

$v_y = 3x^2 - 3y^2 + 6x, \; v_x = 6xy + 6y$

integrate wrt y , integrate wrt x

$V = 3x^2 y - y^3 + 6xy + f(x), \; V = 3x^2 y + 6xy + g(y)$

To find a common Expression for V

choose $f(x)=0$ & $g(y)=-y^3$

$\therefore V = 3x^2y - y^3 + 6xy$

WKT $f(z) = u + iv$

$f(z) = (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(3x^2y - y^3 + 6xy)$

$x = z,\ y = 0$

$f(z) = \underline{z^3 + 3z^2 + 1}$

(9) Find the analytic function whose real part is $\dfrac{x^4 - y^4 - 2x}{x^2 + y^2}$ hence determine v.

Sol $u = \dfrac{x^4 - y^4 - 2x}{x^2 + y^2}$

$u_x = \dfrac{(x^2+y^2)(4x^3 - 2) - (x^4 - y^4 - 2x)2x}{(x^2+y^2)^2}$

$u_y = \dfrac{(x^2+y^2)(-4y^3) - (x^4 - y^4 - 2x)2y}{(x^2+y^2)^2}$

WKT $f'(z) = u_x + iv_x$ But $v_x = -u_y$ by CR equation

$f'(z) = u_x - iu_y$

$f'(z) = \dfrac{(x^2+y^2)(4x^3 - 2) - (x^4 - y^4 - 2x)2x}{(x^2+y^2)^2} - i\left(\dfrac{(x^2+y^2)(-4y^3) - (x^4 - y^4 - 2x)2y}{(x^2+y^2)^2}\right)$

$x = z$ & $y = 0$

$f'(z) = \dfrac{(z^2)(4z^3 - 2) - (z^4 - 2z)2z}{z^4} - i\left(\dfrac{(z^2)(0) - (z^4)0}{(z^2)^2}\right)$

$f'(z) = \dfrac{4z^5 - 2z^2 - 2z^5 + 4z^2}{z^4} = \dfrac{2z^5 + 2z^2}{z^4}$

~~$f'(z) =$ integrate wrt z~~

$f'(z) = \dfrac{2z^5}{z^4} + \dfrac{2z^2}{z^4}$

$$f'(z) = 2z + \frac{2}{z^2}$$

Integrate wrt z

$$f(z) = \int 2z + \frac{2}{z^2}\, dz$$

$$f(z) = z^2 - \frac{2}{z} + c$$

$$\boxed{f(z) = z^2 - \frac{2}{z} + c}$$

By CR eq

$$u_x = v_y,\ v_x = -u_y$$

WKT $z = x + iy$

$$f(z) = (x+iy)^2 - \frac{2}{(x+iy)} + c$$

$$u + iv = (x^2 + i^2y^2 + 2xiy) - \frac{2(x-iy)}{(x+iy)(x-iy)} + c$$

$$= (x^2 - y^2) + 2xiy - \frac{2(x-iy)}{x^2+y^2} + c$$

$$= \left(x^2 - y^2 - \frac{2x}{x^2+y^2}\right) + i\left(2xy + \frac{2y}{x^2+y^2}\right) + c$$

$$u + iv = \left[\frac{x^4 - y^4 - 2x}{x^2+y^2}\right] + i\left[\frac{2x^3y + 2xy^3 + 2y}{x^2+y^2}\right] + c$$

$$\therefore \boxed{v = \frac{2x^3y + 2xy^3 + 2y}{x^2+y^2}}$$

(10) find the analytic function $f(z)$ whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$.

Sol: Let $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$u_x = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - (\sin 2x)(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$u_y = \frac{-2\sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$f'(z) = u_x + i v_x = u_x - i u_y$ by C-R eqn

$$f'(z) = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - (\sin 2x)(2\sin 2x)}{(\cosh 2y - \cos 2x)^2} + i\left(\frac{\sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}\right)$$

put $x = z$ & $y = 0$

$$f'(z) = \frac{(1-\cos 2z)(2\cos 2z) - (\sin 2z)(2\sin 2z)}{(1-\cos 2z)^2} + i(0)$$

$$f'(z) = \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1-\cos 2z)^2}$$

$$f'(z) = \frac{2\cos 2z - 2}{(1-\cos 2z)^2}$$

$$= \frac{-2\,\cancel{(1-\cos 2z)}}{(1-\cos 2z)^{\cancel{2}}}$$

$$= \frac{-2}{(1-\cos 2z)}$$

$$= \frac{-2}{2\sin^2 z}$$

$$\boxed{f'(z) = -\operatorname{cosec}^2 z}$$

integrate wrt z

$$\boxed{f(z) = \cot z + c}$$

(11) Determine the analytic function whose imaginary part is $v = \left(r - \frac{1}{r}\right)\sin\theta$

Sol: $v = \left(r - \frac{k^2}{r}\right)\sin\theta$

$$v_r = \sin\theta + \frac{k^2}{r^2}\sin\theta$$

$$= \sin\theta\left(1 + \frac{k^2}{r^2}\right)$$

classmate

Date
Page

$$v_\theta = \left(r - \frac{k^2}{r}\right)\cos\theta$$

$$f'(z) = e^{-i\theta}(u_r + i v_r) \quad u_r = \frac{1}{r} v_\theta$$

$$= e^{-i\theta}\left(\frac{1}{r} v_\theta + i v_r\right)$$

$$= e^{-i\theta}\left[\frac{1}{r}\cos\theta\left(r - \frac{k^2}{r}\right) + i\left(1 - \frac{k^2}{r^2}\right)\sin\theta\right]$$

$r = z,\ \theta = 0$

$$= 1\left[\frac{1}{z}\left(z - \frac{k^2}{z}\right) + i(0)\right]$$

$$= \frac{1}{z}\left(z - \frac{k^2}{z}\right)$$

$$f'(z) = 1 - \frac{k^2}{z^2}$$

$$f(z) = \int\left(1 - \frac{k^2}{z^2}\right)dz + c$$

$$f(z) = z - k^2\left(\frac{-1}{z}\right) + c$$

$$f(z) = z + \frac{k^2}{z} + c$$

(12) Show that the function $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ is harmonic.

Sol $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$

$u_x = \cos x \cosh y + (-2\sin x)\sinh y + 2x + 4y$

$u_x = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$

$u_y = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$

$u_{xx} = -\sin x \cosh y - 2\cos x \sinh y + 2$

$u_{yy} = \sin x \cosh y + 2\cos x \sinh y - 2$

$u_{xx} + u_{yy} = 0$

$\therefore$ u is harmonic.

(18) Show that $u = e^x(x\cos y - y\sin y)$ is harmonic & find its harmonic conjugate. Also determine the corresponding analytic function.

Sol $u = e^x(x\cos y - y\sin y)$

$u_x = e^x\cdot\cos y + (x\cos y - y\sin y)e^x$

$u_x = e^x(\cos y + x\cos y - y\sin y)$

$u_{xx} = e^x\cdot\cos y + (\cos y + x\cos y - y\sin y)e^x$

$u_{xx} = e^x(2\cos y + x\cos y - y\sin y)$

$u_y = e^x(-x\sin y - [y\cos y + \sin y])$

$u_y = -e^x(x\sin y + y\cos y + \sin y)$

$u_{yy} = -e^x(x\cos y + [-y\sin y + \cos y) + \cos y)$

$u_{yy} = -e^x(2\cos y + x\cos y - y\sin y)$

$u_{xx} + u_{yy} = 0$ Thus u is harmonic

By CR equations

$u_x = v_y$, $v_x = -u_y$

$v_y = e^x(\cos y + x\cos y - y\sin y)$ — ①

$v_x = e^x(x\sin y + y\cos y + \sin y)$ — ②

Integrate ① wrt y

$v = e^x[\int \cos y\,dy + x\int \cos y\,dy - \int y\sin y\,dy] + f(x)$

$v = e^x[\sin y + x\sin y - (y - \cos y - 1 - \sin y)] + f(x)$

$v = e^x[\cancel{\sin y} + x\sin y + y\cos y - \cancel{\sin y}] + f(x)$

$v = xe^x\sin y + e^x y\cos y + f(x)$ — ③

Integrate ② wrt x

$v = \sin y\int xe^x\,dx + y\cos y\int e^x\,dx + \sin y\int e^x\,dx + g(y)$

$v = \sin y(xe^x - e^x) + y\cos y\, e^x + \sin y\, e^x + g(y)$

$v = e^x x\sin y + e^x y\cos y + g(y)$ — ④

By comparing ③ & ④ $f(x) = 0$ & $g(y) = 0$

$v = xe^x\sin y + e^x y\cos y = e^x(x\sin y + y\cos y)$

$f(z) = u + iv$

$= e^x(x\cos y - y\sin y) + ie^x(x\sin y + y\cos y)$

$x = z$ & $y = 0$

$f(z) = ze^z$

classmate
Date
Page

(14) Find the analytic function given that
$u+v = x^3-y^3+3x^2y-3xy^2$

Sol $u+v = x^3-y^3+3x^2y-3xy^2$

$u_x+v_x = 3x^2+6xy-3y^2$ — ①

$u_y+v_y = -3y^2+3x^2-6xy$

By CR equation

$u_x = v_y$ & $v_x = -u_y$

$u_x-u_y = 3x^2+6xy-3y^2$ — ①

$u_y+u_x = -3y^2+3x^2-6xy$ — ②

add ① & ②

$2u_x = 6x^2-6y^2$

$\cancel{2}u_x = \cancel{6}3(\cancel{3}x^2-y^2)$

$u_x = 3(x^2-y^2)$

Sub ② & ②

$-\cancel{2}u_y = 12xy$

$u_y = -6xy$

$v_x = -u_y$

$v_x = 6xy$

$f'(z) = u_x+iv_x$

$= 3(x^2-y^2)+i6xy$

$x = z$ & $y = 0$

$= 3z^2$

$f'(z) = 3z^2$

integrate on bs wrt z

$f(z) = \int 3z^2+c$

$\boxed{f(z) = z^3+c}$

(15) if $f(z)$ is analytic, show that $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 4|f'(z)|^2$.

Sol $f(z) = u + iv$

$|f(z)| = \sqrt{u^2+v^2}$

$|f(z)|^2 = u^2+v^2 = \phi$

$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 4|f'(z)|^2$

$\phi_{xx} + \phi_{yy} = 4|f'(z)|^2$

let $\phi = u^2+v^2$

diff ϕ part wrt x

$\phi_x = 2uu_x + 2vv_x$

again diff ϕ part wrt x

$\phi_{xx} = 2[uu_{xx} + u_x^2 + vv_{xx} + v_x^2]$

Hly $\phi_{yy} = 2[uu_{yy} + u_y^2 + vv_{yy} + v_y^2]$

$\phi_{xx}+\phi_{yy} = 2[u(u_{xx}+u_{yy}) + v(v_{xx}+v_{yy}) +$
$u_x^2 + v_x^2 + u_y^2 + v_y^2]$

Since $f(z)$ is analytic u & v are harmonic

$u_{xx}+u_{yy} = 0$, $v_{xx}+v_{yy} = 0$

$\phi_{xx}+\phi_{yy} = 2[u_x^2 + v_x^2 + u_y^2 + v_y^2]$

By CR eqn $u_x = v_y$, $v_x = -u_y$

$\phi_{xx}+\phi_{yy} = 2[u_x^2 + v_x^2 + v_x^2 + u_x^2]$

$= 2[2u_x^2 + 2v_x^2]$

$= 4[u_x^2 + v_x^2]$

$\phi_{xx}+\phi_{yy} = 4|f'(z)|^2$

$f(z) = u+iv$

$f'(z) = u_x + iv_x$

$|f'(z)| = \sqrt{u_x^2+v_x^2}$

$|f'(z)|^2 = u_x^2+v_x^2$

(16) If $f(z)$ is a regular function of z. Show that $\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$

Sol wkt $f(z) = u+iv$

$|f(z)| = \sqrt{u^2+v^2} = \phi$

$|f(z)|^2 = u^2+v^2 = \phi^2$

$\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = |f'(z)|^2$

$\phi^2 x + \phi^2 y = |f'(z)|^2$

Let $\phi^2 = u^2 + v^2$

Diff. wrt x & y

$2\phi\phi x = 2uux + 2vvx$

$\phi\phi x = uux + vvx$ — ②

$\phi\phi y = uuy + vvy$ — ③

Squaring & adding ② & ③

$\phi^2\phi^2 x + \phi^2\phi^2 y = (uux + vvx)^2 + (uuy + vvy)^2$

$\phi(\phi x^2 + \phi^2 y) = u^2u^2x + v^2vx^2 + 2uv\,vx\,vx + u^2u^2y$
$+ v^2v^2y + 2uv\,vy\,uy$

$\phi[\phi x^2 + \phi^2 y] = u^2[ux^2 + u^2y] + v^2[v^2x + v^2y] +$
$2uv\,ux\,vx + 2uv(-vx)(ux)$

$ux = vy,\ vx = -uy$

$\phi^2[\phi^2 x + \phi^2 y] = u^2[ux^2 + (-vx)^2] + v^2[v^2x + ux^2]$
$= u^2[ux^2 + vx^2] + v^2\cdot[u^2x + vx^2]$
$= [ux^2 + vx^2](u^2 + v^2)$

$\phi^2(\phi^2x + \phi^2y) = (u^2x + v^2x)\phi^2$

$\phi x^2 + \phi y^2 = ux^2 + vx^2$

$\phi x^2 + \phi^2 y = |f'(z)|^2$

18) If $\phi + i\psi$ represents the complex potential of an electrostatic field where $\psi = (x^2 - y^2) + \frac{x}{x^2 + y^2}$. find the complex potential as a function of the complex variable z. and also determine ϕ.

Sol $\psi x = 2x + \frac{(x^2+y^2)1 - x\cdot 2x}{(x^2+y^2)^2} = 2x + \frac{y^2 - x^2}{(x^2+y^2)^2}$

$\psi y = -2y + \frac{(x^2+y^2)0 - x\cdot 2y}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2}$

$f'(z) = \phi x + i\psi x$ But $\phi x = \psi_1 x$ CR eq

$f'(z) = \psi_1 y + i\psi x$

$f'(z) = 2x + \frac{y^2 - x^2}{(x^2+y^2)^2} + i\left(-2y - \frac{2xy}{(x^2+y^2)^2}\right)$

$x = z, y = 0$

$f'(z) = 0 + i\left(2z - \frac{z^2}{(z^2)^2}\right) = i\left(2z - \frac{1}{z^2}\right)$

integrate wrt z

$f(z) = i\int\left(2z - \frac{1}{z^2}\right)dz + c = i\left(z^2 + \frac{1}{z}\right) + c$

$$\boxed{f(z) = i\left(z^2 + \frac{1}{z}\right) + c}$$

$\phi + i\psi = i\left\{(x+iy)^2 + \frac{1}{x+iy}\right\} + c$

$= i\left\{(x^2 + i^2y^2 + 2xiy) + \frac{x-iy}{(x+iy)(x-iy)}\right\} + c$

$\phi + i\psi = i\{(x^2-y^2) + 2xiy\} + i\left\{\frac{x-iy}{x^2+y^2}\right\} + c$

$\phi + i\psi = i(x^2-y^2) - 2xy + \frac{ix}{x^2+y^2} + \frac{y}{x^2+y^2} + c$

$\therefore \phi + i\psi = \left(-2xy + \frac{y}{x^2+y^2}\right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + c$

$$\boxed{\phi = -2xy + \frac{y}{x^2+y^2}}$$

17) If $f(u) = u + iv$ is analytic find $f(z)$ if $u - v = (x-y)(x^2 + 4xy + y^2)$

Sol $u - v = x^3 + 3x^2y - 3xy^2 - y^3$

$u_x - v_x = 3x^2 + 6xy - 3y^2$ — ①

$u_y - v_y = 3x^2 - 6xy - 3y^2$

$u_y = -v_x$ & $v_y = u_x$ by C-R equations

$-v_x - u_x = 3x^2 - 6xy - 3y^2$ — ②

Let us solve for u_x & v_x from ① & ②.

①+② : $-2v_x = 6(x^2 - y^2)$

$v_x = 3(y^2 - x^2)$

①−② : $2u_x = 12xy$

$u_x = 6xy$

$f'(z) = u_x + i v_x$

$f'(z) = 6xy + i\cdot 3(y^2 - x^2)$

$x = z$ & $y = 0$

$f'(z) = 0 + i\cdot 3(0 - z^2)$

$f'(z) = -3iz^2$

integrate wrt z

$f(z) = i\int -3z^2\, dz + c$

$\boxed{f(z) = -iz^3 + c}$

(19) If $f(z) = u + iv$ is analytic, find $f(z)$ if $u + v = (x+y) + e^x(\cos y + \sin y)$

Sol: $u + v = (x+y) + e^x(\cos y + \sin y)$

$u_x + v_x = 1 + e^x(\cos y + \sin y)$

$u_y + v_y = 1 + e^x(-\sin y + \cos y)$

By CR eq

$u_x = v_y$ & $v_x = -u_y$

$u_x - u_y = 1 + e^x(\cos y + \sin y)$ — ①

$u_y + u_x = 1 + e^x(-\sin y + \cos y)$ — ②

add ① & ②

$2u_x = 2 + 2e^x\cos y$

$\cancel{2}u_x = (1 + e^x\cos y)\cancel{2}$

$u_x = 1 + e^x\cos y$

sub ① & ②

$-2u_y = 2e^x \sin y$

$u_y = -e^x \sin y$

$v_x = -u_y$

$v_x = e^x \sin y$

WKT $f'(z) = u_x + i v_x$

$f'(z) = 1 + e^x \cos y + i(e^x \sin y)$

Sub $x = z$ & $y = 0$

$f'(z) = 1 + e^z$

Integrate on b s wrt z

$f(z) = \int 1 + e^z \, dz + c$

$\boxed{f(z) = z + e^z + c}$

20) Show that the analytic function with constant is constant.

Sol: if $f(z) = u + iv$ is analytic, then

$|f(z)| = \sqrt{u^2 + v^2}$ is constant $= c$ or $u^2 + v^2 = c^2$ — ①

Diff ① partially wrt x & y we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0; \quad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \text{ — ②}; \quad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \text{ — ③}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ by CR eqn}$$

③ becomes $-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0$ — ④

Squaring & adding ② & ④

$$u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2 = 0$$

$(u^2 + v^2)\left[u_x^2 + v_x^2\right] = 0$

(or)

$u_x^2 + v_x^2 = 0$

$f'(z) = u_x + i v_x$

$|f'(z)|^2 = u_x^2 + v_x^2 = 0$