Indian Institute of Technology, Delhi

1, 2, 3 - Conjecture and Chordal graphs

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Abstract

In this term paper, we first introduce and study the 1-2-3 Conjecture which states that for any connected graph G with atleast 3 vertices, we can weigh the edges of G from the set $\{1, 2, 3\}$ in such a way that adjacent vertices have distinct weighted degrees. We study some of the results that have been established so far in quest to prove this conjecture which still remains a big open problem. Along with this, we also present many variants of the 1-2-3 Conjecture that have been proposed as a part of the literature survey. Next we discuss about chordal graph and their properties. We also calculate the the total vertex irregularity strength for chordal graphs as an attempt for reaching towards the goal.

1 Introduction

Some definitions, unless otherwise stated, that we shall be using throughout this research study is represented as follows:

- G = (V,E) is a simple, finite, and undirected graph.
- **edge-weighting** w of a graph G is function that maps each edge of the graph G to a weight. $w : E(G) \rightarrow \{1, 2, \ldots k\}$.
- **vertex coloring** c of a graph G is a function that maps every vertex v of the graph to a colour value. For a vertex v, it computes the sum of the weights of its incident edges. $c : V(G) \rightarrow \{1, 2, \ldots r\}$, $c(v) = \sum_{e \sim v} w(e)$
- The coloring c is called **proper vertex coloring** if for every two adjacent vertices u and v, $c(u) \neq c(v)$.
- The weighting is w is proper if c is a proper coloring.
- $\chi^e_{\Sigma}(G)$ for a graph G denotes the least $k \geq 1$ (if any) such that G has proper k-weighting.
- A graph G is called **nice graph** if it has no component isomorphic to K_2.
- The chromatic number, $\chi(G)$ of a graph G is defined as the least number of colors in a proper coloring of G.

2 1-2-3 Conjecture

Despite the ongoing research and attempts during the last decade, the 1-2-3 Conjecture (Karonski, Luczak, Thomason) still remains an open problem and has not been proven yet. Some of the current research and investigations have been primarily focused on 1) proving the 1-2-3 Conjecture for the new class of graphs called nice graphs as defined above, 2) improving the constant upper bounds on

$\chi^e_{\Sigma}(G)$. We have done a regressive literature survey and summarised the important investigations and results in the following sections. Some of the different form of this conjecture as collected from various literature is as follows:

- *If G is a graph with no component isomorphic to K_2, then the edges of G may be assigned weights from the set 1, 2, 3 so that, for any adjacent vertices u, $v \in V(G)$, the sum of weights of edges incident to u differs from the sum of weights of edges incident to v.*
- *Every graph without isolated edges admits a vertex-colouring edge 3-weighting.*
- *For a nice graph G, we have $\chi^e_{\Sigma}(G) \leq 3$*

Many researchers and mathematicians have tried to prove 1-2-3 conjecture but the best result proved so far is $\chi^e_{\Sigma}(G) \leq 5$ for graph G that has no component isomorphic to K_2. However it has been proved that for every 3-colorable graph G, the above conjecture holds i.e. $\chi^e_{\Sigma}(G) \leq 3$. In general, this fact does not hold true for every 2-colorable graphs.

2.1 Main results established so far

Some of these following results have been proven so far to improve the bounds on the $\chi^e_{\Sigma}(G)$. The early approaches to solving the 1-2-3 Conjecture focused mainly on relating $\chi^e_{\Sigma}(G)$ to $\chi(G)$, the chromatic number of the graph. We shall not discuss the proofs of these theorems but these theorems help us understand about $\chi^e_{\Sigma}(G)$ in detail.

Theorem 2.1 (Kalkowski, Karoński, Pfender). For every nice graph G, $\chi^e_{\Sigma}(G) \leq 5$.

Theorem 2.2 For every 5-regular graph G, we have $\chi^e_{\Sigma}(G) \leq 4$.

Theorem 2.3 If G is 2-connected and $\chi(G) \geq 3$, then $\chi^e_{\Sigma}(G) \leq \chi(G)$. Particularly, for any integer $k \geq 3$ and a nice graph G, the following results hold:

- if G is k-colorable for k odd, then $\chi^e_{\Sigma}(G) \geq k$.
- if G is k-colorable for $k \equiv 0 \ (mod\ 4)$, then $\chi^e_{\Sigma}(G) \geq k$.
- if $\delta(G) \leq k - 2$, then then $\chi^e_{\Sigma}(G) \geq k$.
- if G is 2-connected, k-colorable, and has $\delta(G) \geq k + 1$ for $k \equiv 2(mod 4)$, then $\chi^e_{\Sigma}(G) \geq k$.

Theorem 2.4 For a connected graph G, if |V(G)| is odd or if there is a proper $\chi(G)$-coloring where one colour class has even size, then $\chi^e_{\Sigma}(G) \leq \chi(G)$.

2.2 Characterizations of graphs based on $\chi^e_\Sigma(G)$

We can see that there exist nice graphs G like K_3, C_6 for which two edge weights are not sufficient to colour the vertices of G by sums i.e. $\chi^e_\Sigma(G) \leq 2$ does not hold. And so the next best possible conjectured bound on all nice graphs $\chi^e_\Sigma(G)$ is equal to 3. However, it is has been already proven that for a constant $p \in (0,1)$, if G is a random graph chosen from $G_{n,p}$, then almost surely $\chi^e_\Sigma(G) \leq 3$ asymptotically. In this section, we try to characterise graph G based on $\chi^e_\Sigma(G)$ as follows:

- $\chi^e_\Sigma(G) = 1$ if and only if for every vertex $u, v \in V(G)$ such that u, v are adjacent, u and v should have different degrees.
- For $d \geq 3$, if G is bipartite and d-regular, then $\chi^e_\Sigma(G) \leq 2$.
- If a nice graph G, is either bipartite and 3-connected or has minimum degree $\delta(G) \geq 8\chi(G)$, then $\chi^e_\Sigma(G) \leq 2$.
- For a graph G, if $\chi^e_\Sigma(G) \leq 2$, then the length of all cycles of G is divisible by 4.

3 Chordal Graphs

A graph G is said to contain a chord-less cycle if and only if it has some induced subgraph isomorphic to a cycle C_t, for $t \geq 4$. If a graph does not contain any chord-less cycles, it is called chordal.

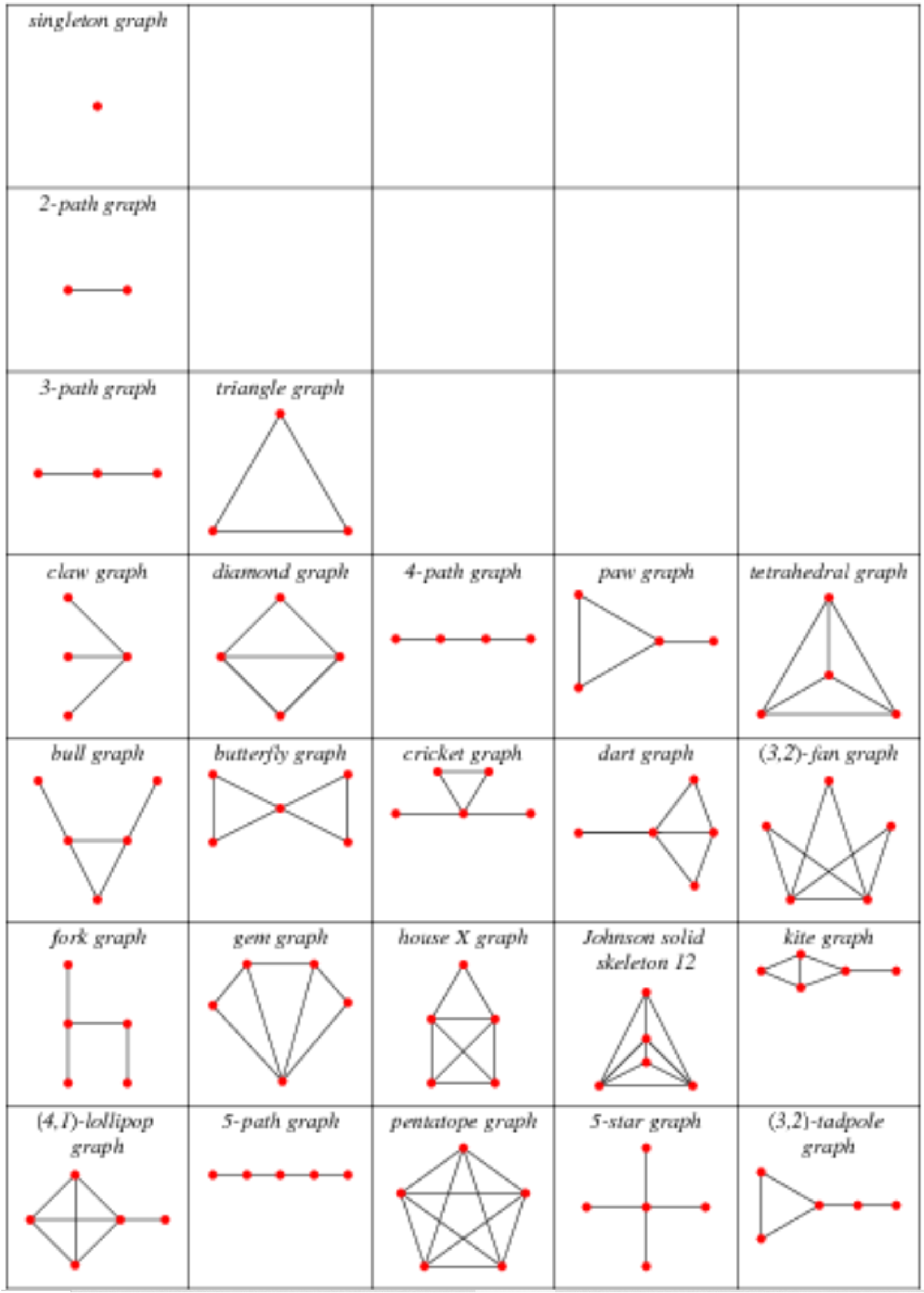
In other words, in a chordal graph, every cycle of four or more vertices has a chord in it, i.e. there is an edge between two non consecutive vertices of the cycle.

Clearly, any induced subgraph of a chordal graph is also chordal

Definition: In a graph G, a vertex v is called **simplicial** if and only if the subgraph of G induced by the vertex set $\{v\} \cup N(v)$ is a complete graph. A graph G on n vertices is said to have a **perfect elimination ordering** if and only if there is an ordering $\{v_1, \ldots v_n\}$ of G's vertices, such that each v_i is simplicial in the subgraph induced by the vertices $\{v_1, \ldots v_i\}$.

Definition: For any two vertices $x, y \in G$ such that $\{x, y\} \notin E(G)$, a $x - y$ **separator** is a set $S \subset V(G)$ such that the graph $G \backslash S$ has at least two disjoint connected components, one of which contains x and another of which contains y.

Here are some illustrations of chordal graphs, though many are trivially chordal since they possess no cycles of length $t \geq 4$.



Definition: In a graph G, if a and b are two vertices separated by a separator S then S is said to be an ab-separator. The set S is a **minimal separator** of G if S is a separator and no proper subset of S separates the graph.

Likewise S is a minimal ab-separator if S is an ab-separator and no proper subset of S separates a and b into distinct connected components. When the pair of vertices remains unspecified, we refer to S as a minimal vertex separator. It does not necessarily follow that a minimal vertex separator is also a minimal separator of the graph.

For instance, in Figure.1 the set S = {b, e} is a minimal dc-separator. Nevertheless, S is not a minimal separator of G since $e \in S$ is also a separator of G.

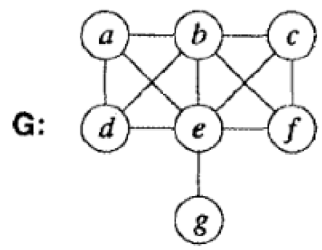


Figure 1: Minimal dc-separator {b,e} is not a minimal separator of G

Theorem 3.1 A graph G is chordal if and only if every minimal vertex separator of G is complete in G.

Theorem 3.2 For a graph G on n vertices, the following conditions are equivalent:

1) G is chordal.
2) G has a perfect elimination ordering.
3) If H is any induced subgraph of G and S is a vertex separator of H of minimal size, then S's vertices induce a clique.

4 Total Vertex Irregularity Strength

One motivation for the 1-2-3 Conjecture is the study of graph **irregularity strength**, which is is the smallest positive integer k for which there exists an edge weighting $w : E(G) \to [k]$ such that $\sum_{e \ni u} w(e) \neq \sum_{e \ni v} w(e)$ for all $u, v \in V(G)$. Irregularity strength and its variations have generated a significant amount of interest since the topic's inception in 1986.

Baˇca et al. [2] introduced the **total vertex irregularity strength** of a graph G on a labelling $w : V \cup E \to \{1, 2, \ldots k\}$, denoted $s^{t}_{\Sigma}(G)$, which is the smallest k such that G has a total k-weighting w such that $w(u) + \sum_{e \ni u} w(e) \neq w(v) + \sum_{e \ni v} w(e)$ for any distinct $u, v \in V(G)$. The best known bound on $s^{t}_{\Sigma}(G)$ was given by *Anholcer, Kalkowski, and Przyby lo* [3], who showed that $s^{t}_{\Sigma}(G) \leq 3\lceil \frac{n}{\delta} \rceil + 1$ for a graph on n vertices with minimum degree $\delta(G) = \delta$.

4.1 Result for Chordal Graphs:

Theorem: For chordal graphs, total vertex irregularity strength $(\underset{\Sigma}{s}^{t}(G)) = \lceil \frac{n+3}{4} \rceil$ for $n \geq 2$.

Proof: Let $CH(n)$ be the chordal graph with $V(CH(n)) = \{v_1, \ldots v_n\}$ and $E(CH(n)) = \{e_i, 1 \leq i \leq n\} \cup \{g_i, 1 \leq i \leq n/2\}$ where $e_i = (v_i, v_{i+1})$, $1 \leq i \leq n-1$, $e_n = (v_n, v_1)$, $g_i = (v_i, v_{n-i+1})$, $1 \leq i \leq n/2$. Let k denote the number of parallel edges. Then clearly $k = n/2$. To show that $\lceil \frac{n+3}{4} \rceil$ is an upper bound for $\underset{\Sigma}{s}^{t}(CH(n))$ we describe a total $\lceil \frac{n+3}{4} \rceil$ - labelling for $CH(n)$.

For $n \geq 4$, we construct the function ϕ as follows:

Case 1: k odd

$$\phi(v_i) = \begin{cases} \lceil \frac{i}{2} \rceil & \text{for } 1 \leq i \leq n/2 - 1 \\ \lceil \frac{n+3}{4} \rceil & \text{for } i = n/2 \\ \lceil \frac{n+3}{4} \rceil - 1 & \text{for } i = n/2 + 1 \end{cases}$$

For $n/2 + 2 \leq i \leq n$,

$$\phi(v_i) = \begin{cases} 2 + 2\lceil \frac{n-i}{4} \rceil & \text{for } n - i \equiv 2, 3 (mod 4) \\ 1 + 2\lceil \frac{n-i}{4} \rceil & \text{for } n - i \equiv 0, 1 (mod 4) \end{cases}$$

$$\phi(e_i) = \lceil \frac{i}{2} \rceil + 1 \text{ for } 1 \leq i \leq n/2 - 1$$

For $n/2 \leq i \leq n$,

$$\phi(e_i) = \begin{cases} 2 + 2\lceil \frac{n-i}{4} \rceil & \text{for } n - i \equiv 2, 3 (mod 4) \\ 1 + 2\lceil \frac{n-i}{4} \rceil & \text{for } n - i \equiv 0, 1 (mod 4) \end{cases}$$

$$\phi(g_i) = \lceil \frac{i+1}{2} \rceil \text{ for } 1 \leq i \leq n/2$$

Case 2: k even

$$\phi(v_i) = \lceil \frac{i}{2} \rceil \text{ for } 1 \leq i \leq n/2$$

For $n/2 + 1 \leq i \leq n$,

$$\phi(v_i) = \begin{cases} 2 + 2\lceil \frac{n-i}{4} \rceil & \text{for } n - i \equiv 2, 3 (mod 4) \\ 1 + 2\lceil \frac{n-i}{4} \rceil & \text{for } n - i \equiv 0, 1 (mod 4) \end{cases}$$

$$\phi(e_i) = \lceil \frac{i}{2} \rceil + 1 \text{ for } 1 \leq i \leq n/2$$

For $n/2 + 1 \leq i \leq n$,

$$\phi(e_i) = \begin{cases} 2 + 2\lceil \frac{n-i}{4} \rceil & \text{for } n - i \equiv 2, 3 (mod 4) \\ 1 + 2\lceil \frac{n-i}{4} \rceil & \text{for } n - i \equiv 0, 1 (mod 4) \end{cases}$$

$$\phi(g_i) = \lceil \frac{i+1}{2} \rceil \text{ for } 1 \leq i \leq n/2$$

We observe that
$wt(v_1) = \phi(v_1) + \phi(e_1) + \phi(e_n) + \phi(g_1)$,

For $2 \leq i \leq n/2$,
$wt(v_i) = \phi(v_i) + \phi(e_i) + \phi(e_{i-1}) + \phi(g_i)$,

For $n/2 + 1 \leq i \leq n$,
$wt(v_i) = \phi(v_i) + \phi(e_i) + \phi(e_{i-1}) + \phi(g_{n-i+1})$

Hence

$$wt(v_i) = \begin{cases} 2i + 3 & \text{for } 1 \leq i \leq n/2 \\ 2n + 4 - 2i & \text{for } n/2 \leq i \leq n \end{cases}$$

So the weights of the vertices of $CH(n)$ under the labelling ϕ constitute the set $\{4, 5, 6, ..., n + 3\}$ and the function ϕ is a mapping from $V(CH(n)) \cup E(CH(n))$ into $\{1, 2, 3, \ldots \lceil \frac{n+3}{4} \rceil\}$

For $CH(2)$, we give the following labelling:

$\phi(v_1) = 2, \phi(v_2) = 1, \phi(e_1) = 1, \phi(e_2) = 1, \phi(g_1) = 1.$

It is easy to see that this total labelling has the required properties.

This concludes the proof.

5 Summary:

We have stated various definitions and results for 1-2-3 conjecture, irregularity strength and for chordal graphs.

Although the ultimate goal of obtaining a bound for $\chi^e_\Sigma(G)$ (*Irregularity Strength*) for chordal graphs remains out of reach, we have obtained a result for *Total Vertex Irregularity Strength* for chordal graphs.

For chordal graphs, total vertex irregularity strength $(s^t_\Sigma(G)) = \lceil \frac{n+3}{4} \rceil$ for $n \geq 2$.

References

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