# Towards Sample-Optimal Methods for Solving Random Quadratic Equations with Structure

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**Abstract**

We consider the problem of estimating a structured high-dimensional parameter vector using random Gaussian quadratic samples. This problem is a generalization of the classical problem of phase retrieval and impacts numerous problems in computational imaging. We provide a generic algorithm based on alternating minimization that, if properly initialized, achieves information-theoretically optimal sample complexity. In essence, we show that solving a system of random quadratic equations with structural constraints is (nearly) as easy as solving the corresponding linear system with the same constraints, if a proper initial guess of the solution is available. As an immediate consequence, our approach improves upon the best known existing sample complexity results for phase retrieval (structured or otherwise). We support our theory via several numerical experiments.

## 1 Introduction

**Motivation:** Our focus in this paper is the following constrained estimation problem. An unknown vector of parameters, $\mathbf{x}^* \in \mathbb{R}^n$, is observed (or measured) to yield observations $\mathbf{y} \in \mathbb{R}^m$ of the form:

$$y_i = |\langle \mathbf{a}_i, \mathbf{x}^* \rangle|^p, \quad i = [m], \quad \text{s.t.} \quad \mathbf{x}^* \in \mathcal{M}_s \tag{1}$$

where $\mathcal{M}_s \subset \mathbb{R}^n$ is a *model* set that reflects the structural constraints on $\mathbf{x}^*$. We adopt the familiar setting of under-determined Gaussian observations, $\mathbf{A} = [\mathbf{a}_1 \dots \mathbf{a}_i \dots \mathbf{a}_m]^\top \in \mathbb{R}^{m \times n}$ with $m < n$. The task is to recover an estimate of $\mathbf{x}^*$ from either absolute-value ($p = 1$) or quadratic ($p = 2$) measurements $\mathbf{y}$.

An important application of the aforementioned setup is the classical signal processing problem of *phase retrieval*. Here, the measurements correspond to the magnitudes of complex 2D Fourier (or Short Time Fourier) transform coefficients. The sensing apparatus is incapable of detecting phase of the complex light-field reflected or transmitted from the illuminated object source. This necessitates a phase recovery strategy, and proposed solution approaches been explored dating back to the 1970s via several works[1, 2]. Recent, renewed interest by the statistical learning community in this problem has focused on the case of Gaussian observations, and have spawned several algorithms which are efficient as well as asymptotically (near) optimal [3, 4, 5, 6]. However, even in the best case, one requires $m > 2n - 1$ [6] measurements, and in the case of high dimension $n$, the sample complexity $m$ can be very large.

Similarly, polynomial neural networks have piqued attention in applications such as classification, where activation functions are *quadratic* mappings [7]. The problem in (1) then corresponds to *learning* weights of a single neuron, $\mathbf{x}^*$, for $m$ Gaussian distributed training examples $\mathbf{a}_i$, and $y_i$ being the corresponding output labels. The task is to design efficient algorithms for *learning* weights using fewer training samples.

To reduce sample complexity of such problems, several works introduce *sparsity* assumptions. Sparsity has been used to great advantage in compressive sensing and streaming algorithms [8, 9], and establish an information theoretically optimal [10] requirement of $\mathcal{O}\left(s \log \frac{n}{s}\right)$ samples for stable recovery of $\mathbf{x}^*$ from linear measurements. Sparsity assumptions for inverting quadratic (or magnitude-only) equations of the form (1) has similarly helped lower computational and memory requirements [11, 12, 13, 5]. Specifically, several

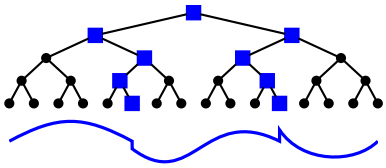


Figure 1: *A binary wavelet tree for a one-dimensional signal. The squares denote the large wavelet coefficients that arise from the discontinuities in the piecewise smooth signal drawn below. Figure taken from [23].*

papers consider the problem in (1), where $\mathcal{M}_s$ represents all $s$-sparse vectors $\mathbf{x}^*$, with the assumption of a Gaussian observation framework [14, 15, 16]. In previous work [16, 17], we have proposed a sparse phase retrieval algorithm called CoPRAM, which is linearly convergent and improves upon all other algorithms, obtaining a Gaussian sample complexity of $\mathcal{O}\left(s^2 \log n\right)$ in general ( $\mathcal{O}\left(s \log n\right)$ if power-law decaying sparse signals are considered [17]). Moreover, given a good initialization, one requires only $\mathcal{O}\left(s \log \frac{n}{s}\right)$ samples for sparse phase retrieval [18].

A natural extension of sparsity is the notion of *structured sparsity*. Several works in compressed sensing and statistical learning have utilized various structures such as blocks, clusters, and trees [19, 20]. Block structures in the context of sparse phase retrieval have been studied in [16]. Tree structures are popularly found in applications where sparsity is considered in wavelet basis [21]. While the impact of structured sparsity has been studied thoroughly for the case of linear measurements [22], recovery from quadratic or magnitude-only measurements is relatively less understood.

**Our contributions:** In this paper, we propose a new algorithmic framework called *Model-based CoPRAM* to solve the problem of phase retrieval of signals with underlying structured sparsity. Our framework is fairly generic and succeeds for parameters belonging to any structured sparsity model (defined formally below). Moreover, we provably show that if the algorithm is properly initialized, then its sample complexity is information-theoretically *optimal*. In particular, we analyze a special instantiation of our framework, called *Tree CoPRAM*, which is applicable in the case of rooted $s$-sparse tree structures for $\mathbf{x}^*$, and demonstrate the superior performance of our method both in theory and numerical simulations. In essence, our contributions imply that solving a system of under-determined quadratic equations under structural constraints is (essentially) as easy as solving the corresponding linear system under the same constraints, provided a good initial guess is available.

**Techniques:** The algorithmic techniques used in this paper are a combination of two focal points: (i) alternating minimization based sparse signal recovery from phaseless measurements (via our previous work on CoPRAM [16]), (ii) using a structured-sparsity promoting subroutine called ModelApprox (e.g. TreeApprox [23] in the context of tree-structured sparsity), which replaces the standard $s$-sparse projection rule used to enforce sparsity. Additionally, we also design a novel initialization heuristic, which yields an initial estimate $\mathbf{x^0}$ that is very close to $\mathbf{x}^*$ in practice. Our main theoretical contribution is a generalization of a recent result of [24] to the case where $\mathbf{x}^*$ belongs to a known model set $\mathcal{M}_s$.

# 2 Background

## 2.1 Structured sparsity

We provide some background for the problem formulation in (1). A vector $\mathbf{x}^* \in \mathbb{R}^n$ is said to be $s$-sparse if it has no more than $s$ non-zero entries. We use $S := \{j | x_j^* \neq 0\}$ to indicate the true support of $\mathbf{x}^*$, such that $|S| \leq s$. The model notation $\mathcal{M}_s$ is introduced as an indicator set comprising of all vectors which follow a given structural constraint, underscored with parameter $s$. Let $\mathbb{M}_s$ denote the set of all allowable supports $\{S_1 \dots S_i \dots S_N\}$, such that $S_i \subseteq [n]$ and $|S_i| \leq s$, then $\mathcal{M}_s = \{\mathbf{x} \in \mathbb{R}^n | \operatorname{supp}(\mathbf{x}) \in \mathbb{M}_s\}$. As a special case, $\mathcal{M}_s$ can be a model representing all $s$-sparse rooted tree supports, as illustrated in Figure 1.

## 2.2 Model-based CoSaMP

Model-based CoSaMP (Compressive Sensing using Matching Pursuit) [20] is a popular technique to recover structured (for example tree) $s$-sparse vectors or signals $\tilde{\mathbf{x}}^* \in \mathbb{R}^n$ from linear observations $\tilde{\mathbf{y}} \in \mathbb{R}^m$ of the form:

$$\tilde{\mathbf{y}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}}^*,$$

where $\tilde{\mathbf{A}} \in \mathbb{R}^{m \times n}$ and $m < n$. The sensing matrix $\tilde{\mathbf{A}}$ is required to satisfy model-RIP [20], with constant $\delta_{M_s}$, such that for all $\tilde{\mathbf{x}} \in \mathcal{M}_s$, the following holds:

$$(1 - \delta_{M_s}) \left\|\tilde{\mathbf{x}}\right\|_2^2 \leq \left\|\tilde{\mathbf{A}}\tilde{\mathbf{x}}\right\|_2^2 \leq (1 + \delta_{M_s}) \left\|\tilde{\mathbf{x}}\right\|_2^2.$$

This holds trivially, if the entries of $\tilde{\mathbf{A}}$, $\tilde{a}_{ij}$ are distributed according to normal distribution $\mathcal{N}(0, 1/\sqrt{m})$. Model CoSaMP solves the following minimization approximately:

$$\min_{\tilde{\mathbf{x}} \in \mathcal{M}_s} \left\|\tilde{\mathbf{y}} - \tilde{\mathbf{A}}\tilde{\mathbf{x}}\right\|_2^2. \tag{2}$$

Model-based CoSaMP (also referred as ModelCoSaMP), utilizes a model-based approximation stage (referred to as ModelApprox). A specific instantiation of Model-based CoSaMP, Tree CoSaMP employs an exact or approximate *tree projection* subroutine called TreeApprox [23, 25], to ensure that the output of the minimization in (2), $\tilde{\mathbf{x}}^+$, belongs to the model $\mathcal{M}_s$. This approach is largely parameter free and only requires knowledge of signal sparsity $s$ and assumption of tree structure.

## 2.3 Phaseless signal recovery

The recovery problem can be expressed by constructing a loss function of the form:

$$\min_{\mathbf{x} \in \mathcal{M}_s} \sum_{i=1}^{m} \left(y_i - |\langle \mathbf{a}_i, \mathbf{x} \rangle|^p\right)^2. \tag{3}$$

where $p = 1$ or $p = 2$. Gradient descent based approaches popularly use the Wirtinger Flow (which solves the quadratic variant, $p = 2$) [4, 26, 27, 14, 24] and Amplitude Flow (which solves the magnitude-only variant, $p = 1$) [28, 15, 6] approaches, to calculate the explicit gradient of the objective function in (3) composed of either squared or magnitude-only measurements. In this paper, we use the alternating minimization approach [2], with magnitude-only measurements ($p = 1$), by introducing a new variable to represent the missing phase information, hence *linearizing* the problem. We then update the phase variable and signal variable in an *alternating* fashion. In the signal estimation stage, we employ the Model-based CoSaMP algorithm to obtain a structurally sparse vector estimate. To evaluate the distance of the $\mathbf{x}$-estimate from $\mathbf{x}^*$, we introduce the expression $\text{dist}(\mathbf{x}_1, \mathbf{x}_2) := \min(\|\mathbf{x}_1 - \mathbf{x}_2\|_2, \|\mathbf{x}_1 + \mathbf{x}_2\|_2)$ for every $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$. This method is discussed in further detail in Section 3.1.

## 2.4 Spectral initialization

Non-convex approaches for phase retrieval [5, 4] rely on a spectral initialization technique to ensure that the initial estimate $\mathbf{x^0}$ is within a $\delta_0$-ball radius of the true solution $\mathbf{x}^*$. This is required to establish subsequent convergence of descent based algorithms. For this, one can construct an estimator matrix $\mathbf{M} = \frac{1}{m}\sum_{i=1}^{m} y_i^2 \mathbf{a}_i \mathbf{a}_i^\top$, and use the top left-singular vector of this matrix as an appropriate initialization. Sparse modifications of this strategy involve detecting (partial) support information from the diagonal of the estimator matrix $\mathbf{M}$, by using an approximate projection onto model $\mathcal{M}_s$. This method is discussed in further detail in Section 3.2.

**Algorithm 1** Model-based CoPRAM

**Input:** $\mathbf{A}, \mathbf{y}, s, t_0$

1: Compute signal power: $\phi^2 = \frac{1}{m}\sum_{i=1}^{m} y_i^2$.
2: Compute: $\text{diag}(\mathbf{M}) := M_{jj} = \frac{1}{m}\sum_{i=1}^{m} y_i^2 a_{ij}^2$ for $j = [n]$.
3: Set: $\hat{S} \leftarrow \text{ModelApprox}(\text{diag}(\mathbf{M}))$.
4: Set: $\mathbf{v}_1 \leftarrow$ top s.v. of $\mathbf{M}_{\hat{S}} = \frac{1}{m}\sum_{i=1}^{m} y_i^2 \mathbf{a}_{i\,\hat{S}} \mathbf{a}_{i\,\hat{S}}^T$.
5: Compute: $\mathbf{v} \in \mathbb{R}^n \leftarrow \mathbf{v}_1$ for $\hat{S}$, and $\mathbf{0}$ for $\hat{S}^c$.
6: Compute: $\mathbf{x^0} \leftarrow \phi\mathbf{v}$ .

7: **for** $t = 0, \cdots, t_0 - 1$ **do**
8: $\quad \mathbf{p}^{t+1} \leftarrow \text{sign}\left(\mathbf{A}\mathbf{x}^t\right)$,
9: $\quad \mathbf{x}^{t+1} = \text{ModelCoSaMP}(\frac{1}{\sqrt{m}}\mathbf{A}, \frac{1}{\sqrt{m}}\mathbf{p}^{t+1} \circ \mathbf{y}, s, \mathbf{x}^t)$.
10: **end for**

**Output:** $\mathbf{x}^{t_0} \leftarrow \mathbf{x}^t$.

# 3 Algorithm

In this section, we propose a new algorithm for solving the tree sparse phase retrieval problem and analyze its performance. We use a *spectral initialization*, followed by an alternating minimization based *descent* approach, similar to our previous work in [16]. Our algorithm is largely parameter-free except for knowledge of the underlying sparsity $s$. Moreover, the theoretical analysis requires no extra assumptions on the parameter vector, except that its support belongs to a structured sparsity model. We call our algorithm *Model-based CoPRAM*, which generalizes our previous algorithm called CoPRAM (or Compressive Phase Retrieval with Alternating Minimization) [16, 17]. The algorithm can be broken down into three types of update stages: (i) initialization; (ii) phase estimation; and (iii) signal estimation. The full algorithm is presented in pseudo-code form as Algorithm 1. The phase and signal estimation stages are described in detail in the Section 3.1. Due to the simplicity of our algorithm, it can easily be extended to a general class of signals defined by any model $\mathcal{M}_s$ In this paper, we focus on the special case where the model $\mathcal{M}_s$ corresponds to tree-sparse vectors in $\mathbb{R}^n$.

## 3.1 Convergence of Model-based CoPRAM

This part of the algorithm is described in Lines 7-10 of Algorithm 1. Once we obtain a good enough initial estimate $\mathbf{x^0} \in \mathcal{M}_s$ such that $\text{dist}\left(\mathbf{x^0}, \mathbf{x}^*\right) \leq \delta_0 \left\|\mathbf{x}^*\right\|_2$, we construct a method to accurately estimate the true $\mathbf{x}^*$. To achieve this, we adapt the alternating minimization approach from [5]. The observation model in (1) can be restated as follows:

$$\text{sign}\left(\langle \mathbf{a}_i, \mathbf{x}^* \rangle\right) \circ y_i = \langle \mathbf{a}_i, \mathbf{x}^* \rangle ,$$

for all $i = \{1, 2, \ldots m\}$. We denote the *phase vector* $\mathbf{p} \in \mathbb{R}^m$ as a vector that contains the unknown signs of the measurements, i.e., $p_i = \text{sign}\left(\langle \mathbf{a}_i, \mathbf{x} \rangle\right)$ for all $i = \{1, 2, \ldots m\}$. Let $\mathbf{p}^*$ denote the true phase vector and let $\mathcal{P}$ denote the set of all phase vectors, i.e. $\mathcal{P} = \{\mathbf{p} : p_i = \pm 1, \forall i\}$. Then our measurement model gets modified as:

$$\mathbf{p}^* \circ \mathbf{y} = \mathbf{A}\mathbf{x}^*.$$

The loss function in (3) gets modified and is composed of two variables $\mathbf{x}$ and $\mathbf{p}$,

$$\min_{\mathbf{x} \in \mathcal{M}_s, \mathbf{p} \in \mathcal{P}} \left\|\mathbf{A}\mathbf{x} - \mathbf{p} \circ \mathbf{y}\right\|_2 \tag{4}$$

Note that the problem above is *not convex*, because $\mathbf{p} \in \mathcal{P}$ is a set of all vectors with entries constrained to be in $\{-1, 1\}$. Instead, we alternate between estimating $\mathbf{p}$ and $\mathbf{x}$. We perform two estimation steps: (i) if we fix the signal estimate $\mathbf{x}$, then the minimizer $\mathbf{p} \in \mathcal{P}$ is given in closed form as:

$$\mathbf{p} = \text{sign}\left(\mathbf{A}\mathbf{x}\right), \tag{5}$$

(*phase estimation*, Line 8 of Algorithm 1) (ii) if we fix the phase vector $\mathbf{p}$, the signal vector $\mathbf{x} \in \mathcal{M}_s$ can be obtained by solving a sparse recovery problem,

$$\min_{\mathbf{x} \in \mathcal{M}_s} \frac{1}{\sqrt{m}} \|\mathbf{A}\mathbf{x} - \mathbf{p} \circ \mathbf{y}\|_2, \tag{6}$$

if $m < n$ and $\frac{\mathbf{A}}{\sqrt{m}}$ satisfies the *restricted isometry property* (*signal estimation*, Line 9 of Algorithm 1).

Here, we employ the Model-based CoSaMP [20] algorithm to (approximately) solve (6). Note that since (6) itself is a non-convex problem, exact minimization can be hard. Thus in each signal estimation step, we do not need to explicitly obtain the minimizer. However we still show a sufficient descent criterion, by analyzing the Model-based CoSaMP algorithm. For analysis reasons, we require that the entries of the input sensing matrix are distributed according to $\mathcal{N}\left(0, 1/\sqrt{m}\right)$. This can be achieved by scaling down the inputs to Model-based CoSaMP: $\mathbf{A}^t, \mathbf{p}^{t+1} \circ \mathbf{y}$ by a factor of $\sqrt{m}$. We also use a "warm start" Model-based CoSaMP routine for the $(t+1)^{th}$ update of $\mathbf{x}$, $\mathbf{x}^{t+1}$, for each iteration where the initial guess of the solution to (6) is given by the current signal estimate $\mathbf{x}^t$.

We now analyze our proposed descent scheme. We obtain the following theoretical result:

**Theorem 3.1.** *Given an initialization* $\mathbf{x^0} \in \mathcal{M}_s$ *satisfying* $\text{dist}\left(\mathbf{x}^0, \mathbf{x}^*\right) \leq \delta_0 \left\|\mathbf{x}^*\right\|_2$, *for* $0 < \delta_0 < 1$, *if we have number of Gaussian measurements,*

$$m > C\left(s + \text{card}(\mathbb{M}_{4s})\right),$$

*then the iterates* $\mathbf{x}^{t+1}$ *of Algorithm 1, satisfy:*

$$\text{dist}\left(\mathbf{x}^{t+1}, \mathbf{x}^*\right) \leq \rho_0 \text{dist}\left(\mathbf{x}^t, \mathbf{x}^*\right), \tag{7}$$

*where* $\mathbf{x}^t, \mathbf{x}^{t+1}, \mathbf{x}^* \in \mathcal{M}_s$, *and* $0 < \rho_0 < 1$ *is a constant, with probability greater than* $1 - e^{-\gamma m}$, *for positive constant* $\gamma$.

**Proof sketch:** The per-iteration error for the $t^{th}$ iteration of Model-based CoPRAM, with $L$ iterations of Model-based CoSaMP, can be derived as:

$$\left\|\mathbf{x}^{t+1} - \mathbf{x}^*\right\|_2 \leq (\rho_1\rho_3)^L \left\|\mathbf{x}^* - \mathbf{x}^t\right\|_2 + \frac{(\rho_1\rho_4 + \rho_2)}{(1 - \rho_1\rho_3)} \left\|E_{ph}\right\|_2, \tag{8}$$

where $\rho_1, \rho_2, \rho_3, \rho_4$ are appropriate constants, and $E_{ph}$ is the error in estimating phase in the $t^{th}$ run of Model-based CoPRAM. The second part of this proof requires a bound on the phase error term $\left\|E_{ph}\right\|_2$:

$$\|E_{ph}\|_2^2 = \frac{4}{m} \sum_{i=1}^{m} \left(\mathbf{a}_i^\top \mathbf{x}^*\right)^2 \cdot \mathbf{1}_{\{\text{sign}(\mathbf{a}_i\mathbf{x}^t)\,\text{sign}(\mathbf{a}_i\mathbf{x}^*) = -1\}}.$$

We do this through Lemma 3.2.

**Lemma 3.2.** *As long as the initial estimate is a small distance away from the true signal* $\mathbf{x}^* \in \mathcal{M}_s$, $\text{dist}\left(\mathbf{x^0}, \mathbf{x}^*\right) \leq \delta_0 \left\|\mathbf{x}^*\right\|_2$, *and subsequently,* $\text{dist}\left(\mathbf{x}^t, \mathbf{x}^*\right) \leq \delta_0 \left\|\mathbf{x}^*\right\|_2$, *where* $\mathbf{x}^t$ *is the* $t^{th}$ *update of Algorithm 1, then the following bound holds,*

$$\frac{4}{m} \sum_{i=1}^{m} \left(\mathbf{a}_i^\top \mathbf{x}^*\right)^2 \cdot \mathbf{1}_{\left\{(\mathbf{a}_i^\top \mathbf{x}^t)(\mathbf{a}_i^\top \mathbf{x}^*) \leq 0\right\}} \leq \rho_5^2 \left\|\mathbf{x}^t - \mathbf{x}^*\right\|_2^2,$$

*with probability greater than* $1 - e^{-\gamma_2 m}$, *where* $\gamma_2$ *is a positive constant, as long as* $m > C\left(s + \text{card}(\mathbb{M}_{4s})\right)$ *and* $\rho_5^2 = 0.0256$.

We therefore achieve a per-step error reduction scheme of the form:

$$\left\|\mathbf{x}^{t+1}-\mathbf{x}^{*}\right\|_{2} \leq \rho_{0}\left\|\mathbf{x}^{t}-\mathbf{x}^{*}\right\|_{2},$$

if the initial estimate $\mathbf{x^0}$ satisfies $\left\|\mathbf{x^0}-\mathbf{x}^{*}\right\|_{2} \leq \delta_{0}\left\|\mathbf{x}^{*}\right\|_{2}$, and this result can be trivially extended to the case where the initial estimate $\mathbf{x^0}$ satisfies $\left\|\mathbf{x^0}+\mathbf{x}^{*}\right\|_{2} \leq \delta_{0}\left\|\mathbf{x}^{*}\right\|_{2}$, hence giving the convergence criterion of the form (for $\rho_0 < 1$):

$$\operatorname{dist}\left(\mathbf{x}^{t+1}, \mathbf{x}^{*}\right) \leq \rho_{0} \operatorname{dist}\left(\mathbf{x}^{t}, \mathbf{x}^{*}\right).$$

The complete proof of Theorem 3.1 and Lemma 3.2 can be found in Appendix A. We present a corollary of Theorem 3.1 for *tree* sparse signals.

**Corollary 3.3.** *As a consequence of Theorem 3.1, if $\mathcal{M}_s$ is a model representing rooted tree sparse signals with sparsity $s$, then Algorithm 1 is linearly convergent and requires a Gaussian sample complexity of $m > Cs$, as long as the initialization $\mathbf{x^0}$ satisfies* $\operatorname{dist}\left(\mathbf{x}^{0}, \mathbf{x}^{*}\right) \leq \delta_{0}\left\|\mathbf{x}^{*}\right\|_{2}$.

The proof of this corollary can be found in Appendix A.

### 3.2 Initialization of Model-based CoPRAM

The first stage (Lines 1-6 of Algorithm 1) of Model-based CoPRAM uses a spectral initialization approach, similar to that provided in previous sparse phase retrieval methods [5, 27, 14, 15, 16]. We construct a *biased* estimator of the squared true signal coefficients, which we call the signal *marginal* matrix:

$$\mathbf{M}=\frac{1}{m} \sum_{i=1}^{m} y_{i}^{2} \mathbf{a}_{i} \mathbf{a}_{i}^{\top}.$$

The $j^{\text{th}}$ signal coefficient can be estimated from the the diagonal term $M_{jj} = \frac{1}{m}\sum_{i=1}^{m} y_i^2 a_{ij}^2$, and the set of all $M_{jj}$'s can be calculated in $\mathcal{O}(mn)$ time. The approximate support estimate $\hat{S}$ can be extracted by performing an exact or approximate tree projection algorithm [23] on the $n$-dimensional diagonal of the marginal matrix $\mathbf{M}$. From this we obtain the sub-matrix $\mathbf{M}_{\hat{S}}$, whose rows and columns are projected onto $\hat{S}$. This is followed by a spectral technique ([16, 5, 15, 14]), which extracts the top left singular vector (s.v.) of $\mathbf{M}$ to construct a good initial estimate $\mathbf{x^0}$ (Lines 4-6 of Algorithm 1).

To provide the intuition behind this strategy, we leverage the fact that the diagonal elements of the expectation matrix $\mathbb{E}[\mathbf{M}]$ are given by $\mathbb{E}[M_{jj}] = \|\mathbf{x}^*\|^2 + 2x_j^{*2}$. The signal marginals $M_{jj}$ corresponding to $j \in S$ are larger, in *expectation*, than those corresponding to $j \in S^c$. Therefore the signal marginals $M_{jj}$ serve as a good indicator to extract an approximate support $\hat{S}$ of $\mathbf{x}^*$. We additionally impose structure $\mathcal{M}_s$ to this sparse initial vector, by utilizing an approximate model projection algorithm (such as tree projection [23]) (Line 3 of Algorithm 1). We demonstrate experimentally that this initialization strategy produces a good estimate of $\mathbf{x}^*$. We do not have a full theoretical characterization of the initialization stage, and intend to pursue this in future work.

## 4 Experiments

In this section, we present some experimental results to demonstrate the empirical advantages of a special instantiation of the Model-based CoPRAM algorithm, called Tree CoPRAM, over a sparse phase retrieval algorithm such as CoPRAM. We consider two different sizes ($32 \times 32$ and $64 \times 64$) of an image of the Lovett Hall as shown in Figure 2. This image is considered to be sparse in the Haar wavelet basis. The number of levels of decomposition are chosen to be $\log_2 n$ where $n$ is the number of pixels in each image: $n$=1024 and $n$=4096.

Figure 2: *Image considered for simulations, resized to* $32 \times 32$ *and* $64 \times 64$ *pixels, considered to be sparse in Haar basis.*

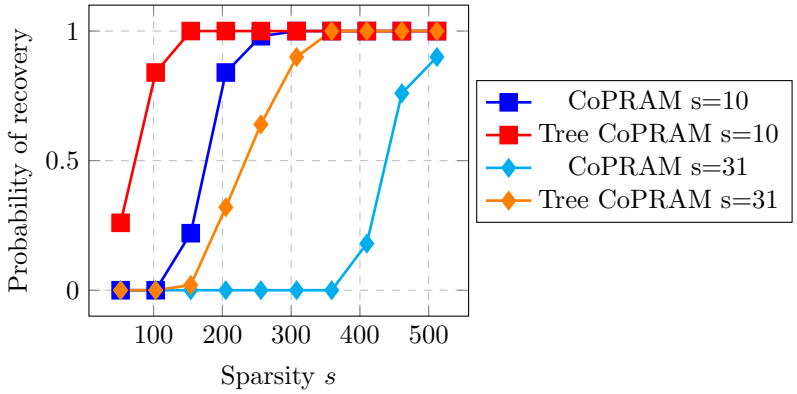



Figure 3: *Phase transitions for CoPRAM and Tree CoPRAM for sparsities s=10 and s=31 on an n=1024 dimensional signal.*

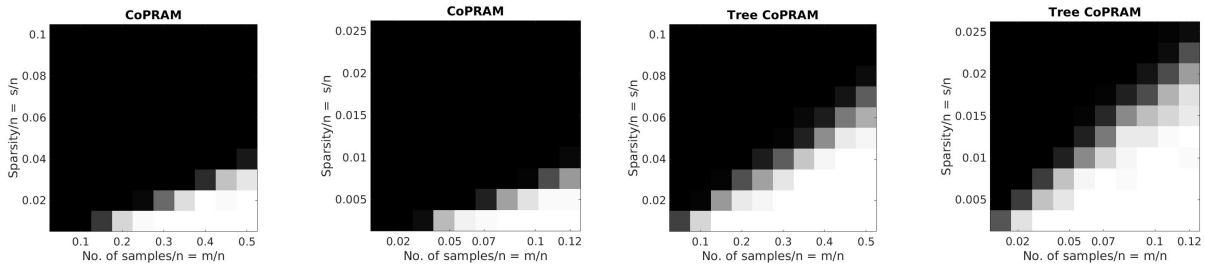
The Tree CoPRAM and CoPRAM algorithms were run for the following experimental settings: $n$=1024 and $n$=4096. The original image $\hat{\mathbf{x}}$ was *sparsified* by fixing $s$ and picking the top $s$- wavelet coefficients of $\hat{\mathbf{x}}$. This sparsified image is considered to be the $s$-sparse tree structured ground truth $\mathbf{x}^*$.

Different sparsities starting from $s \approx \log_2 n$ till $s \approx 10 \log_2 n$ are considered and number measurements are varied as $m$=52,103,154,205,256,308,359,410,461,512. These values are kept the same for generating the phase transitions for Tree CoPRAM and CoPRAM. Each experiment (fixed $n, s, m$) is run for a total of 50 trials.

**Phase transitions:** We demonstrate the superior performance of the Tree CoPRAM algorithm in comparison to CoPRAM, through a series of phase transition graphs and diagrams. In Figure 3, we illustrate two different settings of sparsities for $n = 1024$ dimensional $\mathbf{x}^*$: $s = 10$ and $s = 31$ and compare the performances of CoPRAM and Tree CoPRAM by plotting the variation in the number of measurements $m$ on the horizontal axis and the probability of successful recovery (fraction of trials in which $\|\mathbf{x}^{t_0} - \mathbf{x}^*\|_2 / \|\mathbf{x}^*\|_2 \leq 0.05$). It is clear that far fewer samples are required for successful recovery, when Tree CoPRAM is used instead of CoPRAM.

Figure 4 shows phase transitions for two different sizes of image $n$=1024 and $n$=4096, at different sparsities ($s$=10,20,31,41,51,61,72,82,92,102) and number of measurements ranging uniformly between $m$=52 and $m$=512. It is clear that the phase transition plot of Tree CoPRAM demonstrates better sample complexity w.r.t. CoPRAM.

**Running time performance:** In our final set of results, the running time performance of Tree CoPRAM w.r.t CoPRAM is tabulated in Table 1. We have only considered experiments in which all 50 trials gave




(a) n=1024 (b) n=4096 (c) n=1024 (d) n=4096

Figure 4: *Phase transition diagrams for CoPRAM on signal of dimension (a) $n = 1024$, (b) $n = 4096$; Tree CoPRAM on signal of dimension (c) $n = 1024$, (d) $n = 4096$.*

successful recovery. The simulations were run using MATLAB R2017b on a desktop computer with Intel Xeon E5-2620 processor with 12 CPUs at 2.4GHz and 64GB RAM.

It may be noted that the comparative performance of CoPRAM w.r.t. other sparse phase retrieval algorithms has been discussed in Section 5 of [16].

Table 1: Average running time in seconds of Tree CoPRAM v/s CoPRAM for $n$=1024.

| Algorithm | $s$=10,$m$=308 | $s$=20,$m$=410 | $s$=20,$m$=512 |
|---|---|---|---|
| CoPRAM | 0.0241 | 0.0455 | 0.0433 |
| Tree CoPRAM | 0.0119 | 0.0336 | 0.0302 |

# 5 Conclusions

Through our algorithmic framework called Model-based CoPRAM, we were able to demonstrate lower sample complexity, and lower running time for recovering structured signals from random quadratic equations, as compared to standard sparse phase retrieval algorithms.

# A Appendix

In this section, we provide the proof for our main theoretical result in Theorem 3.1.

**Note:** For evaluation of the distance measure $\text{dist}(\cdot,\cdot)$, we only consider $\text{dist}(\mathbf{x}^t, \mathbf{x}^*) = \|\mathbf{x}^t - \mathbf{x}^*\|_2$, assuming that $\text{dist}(\mathbf{x^0}, \mathbf{x}^*) = \|\mathbf{x^0} - \mathbf{x}^*\|_2$ at the end of initialization stage. We claim that wlog, the same results would hold, if $\text{dist}(\mathbf{x^0}, \mathbf{x}^*) = \|\mathbf{x^0} + \mathbf{x}^*\|_2$.

Small constants are denoted uniformly by $c$ and large constants by $C$, for simplicity of notation.

**Theorem 3.1.** *Given an initialization $\mathbf{x^0} \in \mathcal{M}_s$ satisfying $\text{dist}(\mathbf{x}^0, \mathbf{x}^*) \leq \delta_0 \|\mathbf{x}^*\|_2$, for $0 < \delta_0 < 1$, if we have number of Gaussian measurements,*

$$m > C\left(s + \text{card}(\mathbb{M}_{4s})\right),$$

*then the iterates $\mathbf{x}^{t+1}$ of Algorithm 1, satisfy:*

$$\text{dist}\left(\mathbf{x}^{t+1}, \mathbf{x}^*\right) \leq \rho_0 \text{dist}\left(\mathbf{x}^t, \mathbf{x}^*\right), \tag{7}$$

*where $\mathbf{x}^t, \mathbf{x}^{t+1}, \mathbf{x}^* \in \mathcal{M}_s$, and $0 < \rho_0 < 1$ is a constant, with probability greater than $1 - e^{-\gamma m}$, for positive constant $\gamma$.*

**Note:** Please refer to [17] for complete theoretical analysis of the CoPRAM algorithm.

To show the descent of our alternating minimization algorithm using Tree (or model-based) CoSaMP, we need to analyze the reduction in error, per step of Tree (or model-based) CoSaMP [29, 20], (refer Algorithm 5 in [17]) first. We analyze the inputs to Tree (or model-based) CoSaMP, in step 9 of Algorithm 1,

$$\begin{aligned}\frac{\mathbf{p}^t \circ \mathbf{y}}{\sqrt{m}} &= \operatorname{sign}\left(\mathbf{A}\mathbf{x}^t\right) \circ \frac{|\mathbf{A}\mathbf{x}^*|}{\sqrt{m}},\\ &= \operatorname{sign}\left(\mathbf{A}\mathbf{x}^t\right) \circ \left\{\left(\mathbf{A}\mathbf{x}^*\right) \circ \operatorname{sign}\left(\mathbf{A}\mathbf{x}^*\right)\right\},\\ &= \mathbf{A}\mathbf{x}^* + \left(\operatorname{sign}\left(\mathbf{A}\mathbf{x}^t\right) \circ \operatorname{sign}\left(\mathbf{A}\mathbf{x}^*\right) - \mathbf{1}\right) \circ \mathbf{A}\mathbf{x}^*.\\ \implies \mathbf{p}^t \circ \mathbf{y} - \mathbf{A}\mathbf{x}^* &= -2\mathbf{A}\mathbf{x}^* \circ \mathbb{1}, \qquad (9)\\ &:= \sqrt{m}E_{ph},\end{aligned}$$

where $E_{ph}$, is error due to failure in estimating the correct phase, $\mathbf{1} \in \mathbb{R}^m$ is a vector of ones, and $\mathbb{1}$ is an indicator vector such that:

$$\mathbb{1}_i = \begin{cases} 1, & \text{if} \quad \operatorname{sign}\left(\mathbf{a}_i\mathbf{x}^t\right)\operatorname{sign}\left(\mathbf{a}_i\mathbf{x}^*\right) = -1,\\ 0, & \text{if} \quad \operatorname{sign}\left(\mathbf{a}_i\mathbf{x}^t\right)\operatorname{sign}\left(\mathbf{a}_i\mathbf{x}^*\right) = 1.\end{cases}$$

Using equation (9), the per-step reduction in error for the $l^{th}$ run of Tree (or model-based) CoSaMP inside the $t^{th}$ iteration of Tree (or model-based) CoPRAM is (refer equation 35 and proof of Theorem IV.2 of [17], results naturally extend to Tree CoPRAM):

$$\left\|\mathbf{x}^{t+1,l+1} - \mathbf{x}^*\right\|_2 \leq \rho_1 \left\|\left(\mathbf{x}^* - \mathbf{x}^{t+1,l}\right)_{\Gamma^c}\right\|_2 + \rho_2 \left\|E_{ph}\right\|_2.$$

where constants $\rho_1 = 2\sqrt{\frac{1+\delta_{M_s}}{1-\delta_{M_s}}}$ and $\rho_2 = \frac{4}{\sqrt{1-\delta_{M_s}}}$ and $\delta_{M_s}$ is the model-RIP constant with parameter $s$.

Finally, the first term in the previous inequality can be bounded using (Lemmas 3 and 4 of Proof of Theorem 4 of model-based CoSaMP [20]),

$$\begin{aligned}&\left\|\mathbf{x}^{t+1,l+1} - \mathbf{x}^*\right\|_2\\ &\qquad \leq \rho_1\rho_3 \left\|\mathbf{x}^* - \mathbf{x}^{t+1,l}\right\|_2 + \left(\rho_1\rho_4 + \rho_2\right)\left\|E_{ph}\right\|_2, \qquad (10)\end{aligned}$$

where $\rho_3 = \frac{\delta_{M_{2s}} + \delta_{M_{4s}}}{1-\delta_{M_{2s}}}$ and $\rho_4 = \frac{2\sqrt{1+\delta_{M_{2s}}}}{1-\delta_{M_{2s}}}$ are constants. Assuming that Tree (or model-based) CoSaMP is let to run a maximum of $L$ iterations,

$$\left\|\mathbf{x}^{t+1} - \mathbf{x}^*\right\|_2 \leq \left(\rho_1\rho_3\right)^L \left\|\mathbf{x}^* - \mathbf{x}^t\right\|_2 + \frac{\left(\rho_1\rho_4 + \rho_2\right)}{\left(1 - \rho_1\rho_3\right)}\left\|E_{ph}\right\|_2. \qquad (11)$$

The second part of this proof requires a bound on the phase error term $\left\|E_{ph}\right\|_2$:

$$\begin{aligned}\sqrt{m}E_{ph} &= -2\mathbf{A}\mathbf{x}^* \circ \mathbb{1},\\ \left\|E_{ph}\right\|_2^2 &= \frac{4}{m}\sum_{i=1}^{m}\left(\mathbf{a}_i^\top \mathbf{x}^*\right)^2 \cdot \mathbf{1}_{\{\operatorname{sign}(\mathbf{a}_i\mathbf{x}^t)\operatorname{sign}(\mathbf{a}_i\mathbf{x}^*)=-1\}}.\end{aligned}$$

We proceed to finish this proof by invoking Lemma 3.2.

**Lemma 3.2.** *As long as the initial estimate is a small distance away from the true signal* $\mathbf{x}^* \in \mathcal{M}_s$, $\operatorname{dist}\left(\mathbf{x}^\mathbf{0}, \mathbf{x}^*\right) \leq \delta_0 \left\|\mathbf{x}^*\right\|_2$, *and subsequently,* $\operatorname{dist}\left(\mathbf{x}^t, \mathbf{x}^*\right) \leq \delta_0 \left\|\mathbf{x}^*\right\|_2$, *where* $\mathbf{x}^t$ *is the* $t^{th}$ *update of Algorithm 1, then the following bound holds,*

$$\frac{4}{m}\sum_{i=1}^{m}\left(\mathbf{a}_i^\top \mathbf{x}^*\right)^2 \cdot \mathbf{1}_{\left\{\left(\mathbf{a}_i^\top \mathbf{x}^t\right)\left(\mathbf{a}_i^\top \mathbf{x}^*\right) \leq 0\right\}} \leq \rho_5^2 \left\|\mathbf{x}^t - \mathbf{x}^*\right\|_2^2,$$

*with probability greater than* $1 - e^{-\gamma_2 m}$, *where* $\gamma_2$ *is a positive constant, as long as* $m > C\left(s + \operatorname{card}(\mathbb{M}_{4s})\right)$ *and* $\rho_5^2 = 0.0256$.

*Proof.* This proof has been adapted from Lemma 3 of [24] and uses the generic chaining techniques of [30, 31].

We are required to bound the following term:

$$\begin{aligned}
\|E_{ph}\|_2^2 &\leq \frac{4}{m}\sum_{i=1}^{m}\left(\mathbf{a}_i^\top \mathbf{x}^*\right)^2 \cdot \mathbf{1}_{\left\{|\mathbf{a}_i^\top \mathbf{x}^*|<|\mathbf{a}_i^\top \mathbf{h}|\right\}} \\
&\leq \frac{4}{m}\sum_{i=1}^{m}\left(\mathbf{a}_i^\top \mathbf{h}\right)^2 \cdot \mathbf{1}_{\left\{|\mathbf{a}_i^\top \mathbf{x}^*|<|\mathbf{a}_i^\top \mathbf{h}|\right\}}, \\
&\leq \frac{4}{m}\sum_{i=1}^{m}\chi_i\left(\left(\mathbf{a}_i^\top \mathbf{h}\right)^2\right) \qquad (12) \\
&\leq \frac{4}{m}\sum_{i=1}^{m}\left(\mathbf{a}_i^\top \mathbf{h}\right)^2 \cdot \mathbf{1}_{\left\{(1-\delta)|\mathbf{a}_i^\top \mathbf{x}^*|<|\mathbf{a}_i^\top \mathbf{h}|\right\}}, \qquad (13) \\
&:= \frac{4\|\mathbf{h}\|_2^2}{m}\sum_{i=1}^{m}\gamma_i
\end{aligned}$$

where we have a fixed $\mathbf{h}$ defined as $\mathbf{h} = \mathbf{x}^t \pm \mathbf{x}^*$ ($\pm$ corresponds to sign of minimum $\ell_2$ norm) and satisfying $\|\mathbf{h}\|_2 \leq \delta_0 \|\mathbf{x}^*\|_2$, $\delta$ is a small constant, and the pre-final steps in Eqs. (12) and (13) can be obtained via auxiliary random Lipschitz approximations $\chi_i\left(\left(\mathbf{a}_i^\top \mathbf{h}\right)^2\right)$, as in Eq. 52 of Section C.1 (refer Proof of Lemma 3) of [24].

Here we invoke Lemma 3 of [24], which we modify to suit our problem formulation. Firstly, we relax the constraint for the initial separation $\delta_0$. Secondly, we calculate the expectation of random variable $\gamma_i := \frac{(\mathbf{a}_i^\top \mathbf{h})^2}{\|\mathbf{h}\|_2^2}\mathbf{1}_{\left\{(1-\delta)|\mathbf{a}_i^\top \mathbf{x}^*|<|\mathbf{a}_i^\top \mathbf{h}|\right\}}$, by setting $\delta_0 = 0.0035$ and $\delta = 0.01$. We therefore evaluate the integral expansion of $\mathbb{E}[\gamma_i]$, (Section C.1, proof of Lemma 3 of [24]) and this expression can be bounded as:

$$\begin{aligned}
&\mathbb{E}[\gamma_i] \leq 0.0063 \quad \text{for} \quad \delta_0 < 0.0035 \quad \text{and} \quad \delta = 0.01, \\
&\implies \mathbb{E}\left[\chi_i\left(\left(\mathbf{a}_i^\top \mathbf{h}\right)^2\right)\right] \leq 0.0063\|\mathbf{h}\|_2^2, \\
&(\text{for} \quad \delta_0 < 0.0035 \quad \text{and} \quad \delta = 0.01).
\end{aligned}$$

Using Bernstein type inequality [32] on sub-exponential variable $\chi_i\left(\left(\mathbf{a}_i^\top \mathbf{h}\right)^2\right)$,

$$\mathbb{P}\left[\frac{1}{m}\sum_{i=1}^{m}\frac{\chi_i\left(\left(\mathbf{a}_i^\top \mathbf{h}\right)^2\right)}{\|\mathbf{h}\|_2^2} > (0.0063+\epsilon)\right] < \exp\left(-cm\epsilon^2\right).$$

At this point, we leverage the sparsity of the problem and consider a union bound over all $2s$-sparse $\mathbf{h}$'s (such that $\mathbf{x}^t$ and $\mathbf{x}^\mathbf{0}$ are tree sparse, contained in $\mathcal{M}_s$) lying in an $\epsilon'$- net $\mathcal{N}_{\epsilon'}$ sphere of radius $\delta_0\|\mathbf{x}^*\|_2$ and $\epsilon' = \epsilon\delta_0\|\mathbf{x}^*\|_2$. The $\epsilon'$-net has cardinality $\text{card}(\mathcal{N}_{\epsilon'}) \leq \text{card}(\mathbb{M}_{2s})\left(1+\frac{2}{\epsilon}\right)^{2s}$. For example, $\text{card}(\mathcal{N}_{\epsilon'}) \leq \binom{n}{2s}\left(1+\frac{2}{\epsilon}\right)^{2s}$ for general $2s$-sparse signals $\left(\text{card}(\mathbb{M}_{2s}) = \binom{n}{2s}\right)$. For the case of tree sparse signals, we use Proposition 1 of [20], which states that the cardinality of $\mathcal{M}_s$ is:

$$\text{card}(\mathbb{M}_s) \leq \frac{2e^s}{s+1} \quad \text{for} \quad s > \log_2 n.$$

Using this fact, for $\mathbf{x}^t, \mathbf{x}^* \in \mathcal{M}_s$ (tree sparse model), the $\epsilon'$-net for $\mathbf{h}$ has cardinality $\text{card}(\mathcal{N}_{\epsilon'}) \leq \frac{6^{2s}}{s}\left(1+\frac{2}{\epsilon}\right)^{2s}$.

Now the union bound over all such $\mathbf{h}_0 \in \mathcal{N}_{\epsilon'}$, such that $\|\mathbf{h} - \mathbf{h}_0\|_2 \leq \epsilon \|\mathbf{h}\|_2$ is:

$$\mathbb{P}\left[\frac{1}{m}\sum_{i=1}^{m}\frac{\chi_i\left(\left(\mathbf{a}_i^\top \mathbf{h}_0\right)^2\right)}{\|\mathbf{h}_0\|_2^2} \leq (0.0063+\epsilon)\right] \tag{14}$$

$$> 1 - \text{card}(\mathbb{M}_{2s})\left(1+\frac{2}{\epsilon}\right)^{2s}\exp\left(-cm\epsilon^2\right), \tag{15}$$

$$\forall \mathbf{h}_0 \in \mathcal{N}_{\epsilon'}.$$

Now, we bound the RHS of Eq.(12) as follows:

$$\begin{aligned}
&\frac{4}{m}\sum_{i=1}^{m}\chi_i\left(\left(\mathbf{a}_i^\top \mathbf{h}\right)^2\right) - \frac{4}{m}\sum_{i=1}^{m}\chi_i\left(\left(\mathbf{a}_i^\top \mathbf{h}_0\right)^2\right)\\
&\leq \frac{4}{m}\left|\sum_{i=1}^{m}\chi_i\left(\left(\mathbf{a}_i^\top \mathbf{h}\right)^2\right) - \frac{4}{m}\sum_{i=1}^{m}\chi_i\left(\left(\mathbf{a}_i^\top \mathbf{h}_0\right)^2\right)\right|\\
&\leq \frac{4}{m}\sum_{i=1}^{m}\left|\chi_i\left(\left(\mathbf{a}_i^\top \mathbf{h}\right)^2\right) - \chi_i\left(\left(\mathbf{a}_i^\top \mathbf{h}_0\right)^2\right)\right|\\
&\leq \frac{4}{m}\cdot\frac{1}{\delta}\sum_{i=1}^{m}\left|\left(\mathbf{a}_i^\top \mathbf{h}\right)^2 - \left(\mathbf{a}_i^\top \mathbf{h}_0\right)^2\right| &(16)\\
&\leq 4\cdot\frac{c}{\delta}\left\|\mathbf{h}\mathbf{h}^\top - \mathbf{h}_0\mathbf{h}_0^\top\right\|_F &(17)\\
&\leq 4\cdot\frac{3c}{\delta}\|\mathbf{h}-\mathbf{h}_0\|_2\cdot\|\mathbf{h}\|_2 \leq \frac{12c\epsilon}{\delta}\|\mathbf{h}\|_2^2 &(18)
\end{aligned}$$

where (16) is due to the $\chi_i$'s being Lipschitz functions with constant $\frac{1}{\delta}$ and (17) and (18) are through Lemma A.1 and Lemma 2 of [27] respectively, with probability $1 - c\,\text{card}(\mathbb{M}_{4s})\exp(-Cm)$.

**Lemma A.1.** *For all symmetric rank-2 matrices $\mathbf{H} \in \mathbb{R}^{4s\times 4s}$, if $m > Cs$, then with probability $1 - c\exp(-Cm)$,*

$$\frac{1}{m}\sum_{i=1}^{m}\left|\mathbf{a}_{i_\Omega}\mathbf{H}\mathbf{a}_{i_\Omega}^\top\right| \leq c\|\mathbf{H}\|_F, \tag{19}$$

*where $\Omega$ is a $4s$-dimensional support vector and $\mathbf{a}_{i_\Omega} \in \mathbb{R}^{4s}$ is a sub-vector of $\mathbf{a}_i$ (adapted from Lemma 1 of [27]).*

*Consequently, taking a union bound over all $4s$-dimensional subspaces in lying in $n$-dimension, the bound in (19) holds with probability at least $1 - c\,\text{card}(\mathbb{M}_{4s})\exp(-Cm)$, where $\mathbf{H} := \left(\mathbf{h}_\Omega\mathbf{h}_\Omega^\top - \mathbf{h}_{0_\Omega}\mathbf{h}_{0_\Omega}^\top\right)$ and $\mathbf{h}_{0_\Omega} \in \mathbb{R}^{4s}$ and $\mathbf{h}_\Omega \in \mathbb{R}^{4s}$ are sub-vectors of $\mathbf{h}$ and $\mathbf{h}_0$, such that $\Omega := supp(\mathbf{h}) \cup supp(\mathbf{h}_0)$.*

Effectively, we evaluate the sample complexity, by considering the probability with which the final expression in Equation 18 holds,

$$\begin{aligned}
&\text{card}(\mathbb{M}_{4s})\exp\left(-cm\epsilon^2\right) < \delta,\\
&\implies m > C\left(s + \text{card}(\mathbb{M}_{4s})\right).
\end{aligned}$$

Specifically, for tree structures,

$$\begin{aligned}
&\frac{6^{2s}}{s}\exp\left(-cm\epsilon^2\right) < \delta,\\
&\implies m > Cs(\log 6 - \log s) > Cs.
\end{aligned}$$

Using the result at the end of (18), and combining with (14) we have,

$$\frac{4}{m}\sum_{i=1}^{m}\chi_i\left(\left(\mathbf{a}_i^\top\mathbf{h}\right)^2\right) \leq 4\left(0.0063+\epsilon+\frac{3c\epsilon}{\delta}\right)\|\mathbf{h}\|_2^2$$
$$< 0.0256\,\|\mathbf{h}\|_2^2\,.$$

since $\epsilon$ can be chosen to be as small as required, hence concluding the proof of Lemma 3.2. □

Using this in addition to equation (11), we have our final per-step error reduction for a single run of Tree CoPRAM (Algorithm 1), as:

$$\begin{aligned}\left\|\mathbf{x}^{t+1}-\mathbf{x}^*\right\|_2 &\leq \left((\rho_1\rho_3)^L+\rho_5\frac{(\rho_1\rho_4+\rho_2)}{(1-\rho_1\rho_3)}\right)\left\|\mathbf{x}^t-\mathbf{x}^*\right\|_2\,,\\ &\leq \rho_0\left\|\mathbf{x}^t-\mathbf{x}^*\right\|_2\,,\end{aligned} \tag{20}$$

where $\rho_0 < 1$.

**Evaluating convergence parameter $\rho_0$:**. To obtain per-step reduction in error, we require $\rho_0 < 1$ in (20). For sake of numerical analysis, $\delta_{M_s}, \delta_{M_{2s}}, \delta_{M_{4s}} \leq 0.0001$, then $\rho_1 \approx 1, \rho_3 \approx 0.0002$. Let $\delta_0 = 0.0035$, then $\rho_5 \approx 0.16$. Similarly, $\rho_2 \approx 4$ and $\rho_4 \approx 2$. Suppose Tree CoSaMP is allowed to run for $L = 5$ iterations then, $\rho_0 \approx 0.96 < 1$.

The inequalities used for analysis of Tree CoSaMP, particularly (10) can be made tighter, which would give less stringent restrictions on the factor $\delta_0$, which controls how close the initial estimate is to the true signal $\mathbf{x}^*$.

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