Herstein Topics in Algebra Exercise Solutions

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1 Set Theory

1.1 Question 1

1. If A is a subset of B and B is a subset of C, prove that A is a subset of C.
2. If $B \subset A$, prove that $A \cup B = A$, and conversely.
3. If $B \subset A$, prove that for any set C both $B \cup C \subset A \cup C$ and $B \cap C \subset A \cap C$.

1.2 Question 9

Let S be a set and let S^* be the set whose elements are the various subsets of S. In S^* we define an addition and multiplication as follows: If $A, B \in S^*$ (remember, this means that they are substes of S):

1. $A + B = (A - B) \cup (B - A)$.
2. $A \cdot B = A \cap B$.

Prove the following laws that governs these operations:

1. $(A + B) + C = A + (B + C)$.
2. $A \cdot (B + C) = A \cdot B + A \cdot C$.
3. $A \cdot A = A$.
4. $A + A = \varnothing$.
5. If $A + B = A + C$ then $B = C$.

(The system described is an example of a boolean algebra.)

Solution.

$$\begin{aligned}(A+B)+C &= ((A-B)\cup(B-A))+C \\ &= (((A-B)\cup(B-A))-C)\cup(C-((A-B)\cup(B-A)))\,.\end{aligned}$$

Now we expand the RHS,

$$\begin{aligned}A+(B+C) &= A+((B-C)\cup(C-B)) \\ &= (A-((B-C)\cup(C-B)))\cup(((B-C)\cup(C-B))-A)\,.\end{aligned}$$

Now, let $x \in (A + B) + C$, then there are two cases

- x is an element of $((A - B) \cup (B - A)) - C$. If this is the case then we know that $x \in (A - B)$ or $x \in (B - A)$. However, in either case $x \notin C$. We examine both cases,
  - $x \in (A - B)$ and $x \notin C$: In this case we know that $x \notin B \cup C$. So it is definitely not in a reduced version of this union which is $(B - C) \cup (C - B)$ as this is just removing further elements from B and C before doing a union. Hence $x \notin (B - C) \cup (C - B)$ but $x \in A$. Therefore, $x \in A - ((B - C) \cup (C - B))$ and hence

$x \in (A - ((B - C) \cup (C - B))) \cup (((B - C) \cup C - B) - A)$. Hence, in this case the subset relation $(A + B) + C \subset A + (B + C)$ holds.

- Fix this $x \in (B - A)$ and $x \notin C$. Hence, $x \notin B \cap C$. So, it is in $(B - C) \cup (C - B)$ as this is the same as $(B \cup C) - (B \cap C)$ by definition of $B + C$. However, $x \notin A$. Therefore, $x \in ((B - C) \cup (C - B)) - A$ and hence $x \in (A - ((B - C) \cup (C - B))) \cup (((B - C) \cup (C - B)) - A)$. Hence, in this case the subset relation $(A + B) + C \subset A + (B + C)$ holds.
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