Herstein Topics in Algebra Exercise Solutions

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1 Set Theory

1.1 Question 1

1. If A is a subset of B and B is a subset of C, prove that A is a subset of C.
2. If $B \subset A$, prove that $A \cup B = A$, and conversely.
3. If $B \subset A$, prove that for any set C both $B \cup C \subset A \cup C$ and $B \cap C \subset A \cap C$.

1.2 Question 9

Let S be a set and let S^* be the set whose elements are the various subsets of S. In S^* we define an addition and multiplication as follows: If $A, B \in S^*$ (remember, this means that they are substes of S):

1. $A + B = (A - B) \cup (B - A)$.
2. $A \cdot B = A \cap B$.

Prove the following laws that governs these operations:

1. $(A + B) + C = A + (B + C)$.
2. $A \cdot (B + C) = A \cdot B + A \cdot C$.
3. $A \cdot A = A$.
4. $A + A = \varnothing$.
5. If $A + B = A + C$ then $B = C$.

(The system described is an example of a boolean algebra.)

Solution. We first show the following lemma,

$$
\begin{aligned}
(A + B) + C &= ((A - B) \cup (B - A)) + C \\
&= (((A - B) \cup (B - A)) - C) \cup (C - ((A - B) \cup (B - A)))\,.
\end{aligned}
$$

Now we expand the RHS,

$$
\begin{aligned}
A + (B + C) &= A + ((B - C) \cup (C - B)) \\
&= (A - ((B - C) \cup (C - B))) \cup (((B - C) \cup (C - B)) - A)\,..
\end{aligned}
$$

Using lemma 1 we get,

now we using bi directional containment relation technique.