

# Herstein Topics in Algebra Second Edition Exercise Solutions

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# 1 Set Theory

## 1.1 Solution 1

1.  $A \subseteq B$  and  $B \subseteq C$ . Then let  $x \in A \Rightarrow x \in B \Rightarrow x \in C$ . Hence, for all  $x \in A, x \in C$ . Therefore  $A \subseteq C$ .
2.  $B \subseteq A$ . This means  $x \in B \Rightarrow x \in A$ . So, all elements of  $B$  are in  $A$ . Now, we show that  $A \cup B = A$ . Let  $x \in A \cup B$  then  $x \in A$  or  $x \in B$ .

If  $x \in A$  then we have nothing more to show. Otherwise if  $x \in B$ . Since  $B$  is a subset of  $A$ . This means that  $x \in A$  as well. Hence in either eventuality  $x \in A$ . So, for all  $x \in A \cup B, x \in A$ . Hence  $A \cup B \subseteq A$ .

It remains to show that  $A \subseteq A \cup B$ . Let  $x \in A$ . then by definition of union  $x \in A \cup B$ .

Hence, for all  $x \in A, x \in A \cup B$ . Therefore,  $A \subseteq A \cup B$ .

Hence we have shown that  $A \subseteq A \cup B$  and  $A \cup B \subseteq A$ . Therefore,  $A \cup B = A$ .

3.  $B \subseteq A$ , so  $x \in B \Rightarrow x \in A$ . Now, let  $x \in B \cup C$ . then  $x \in B$  or  $x \in C$ .

If  $x \in B$  then  $x \in A$ , hence  $x \in A \cup C$ .

Otherwise if  $x \in C$ . then  $x \in A \cup C$  by definition of union.

So, in either eventuality  $x \in A \cup C$ . Now, we show that  $B \cap C \subseteq A \cap C$ .

If  $x \in B \cap C, x \in B$  and  $x \in C$ . Since,  $x \in B, x \in A$ . So,  $x \in A$ . and  $x \in C$ . Therefore,  $x \in A$  and  $x \in C$ . Therefore,  $x \in A \cap C$ .

## 1.2 Solution 2

1. Let  $x \in A \cap B$  then  $x \in A$  and  $x \in B$ . Hence,  $x \in A \cap B$  by definition. Therefore,  $A \cap B \subseteq B \cap A$ .

Let  $x \in B \cap A$ , then  $x \in B$  and  $x \in A$ . Hence  $x \in A \cap B$ . Therefore,  $B \cap A \subseteq A \cap B$ .

Now we show that  $A \cup B = B \cup A$ . Similar to above, trivial.

2. Let  $x \in A \cap (B \cap C)$ . Then  $x \in A$  and  $x \in B \cap C \Rightarrow x \in B$  and  $x \in C$ . Hence  $x \in A$  and  $x \in (B \cap C)$ . Therefore by definition  $x \in A \cap (B \cap C)$ .

## 1.3 Solution 9

$$\begin{aligned}(A + B) + C &= ((A - B) \cup (B - A)) + C \\ &= (((A - B) \cup (B - A)) - C) \cup (C - ((A - B) \cup (B - A))).\end{aligned}$$

Now we expand the RHS,

$$\begin{aligned}A + (B + C) &= A + ((B - C) \cup (C - B)) \\ &= (A - ((B - C) \cup (C - B))) \cup (((B - C) \cup (C - B)) - A).\end{aligned}$$

Now, let  $x \in (A + B) + C$ , then there are two cases

- 
- $x$  is an element of  $((A - B) \cup (B - A)) - C$ . If this is the case then we know that  $x \in (A - B)$  or  $x \in (B - A)$ . However, in either case  $x \notin C$ . We examine both cases,

- $x \in (A - B)$  and  $x \notin C$ : In this case we know that  $x \notin B \cup C$ . So it is definitely not in a reduced version of this union which is  $(B - C) \cup (C - B)$  as this is just removing further elements from  $B$  and  $C$  before doing a union. Hence  $x \notin (B - C) \cup (C - B)$  but  $x \in A$ . Therefore,  $x \in A - ((B - C) \cup (C - B))$  and hence  $x \in (A - ((B - C) \cup (C - B))) \cup (((B - C) \cup (C - B)) - A)$ . Hence, in this case the subset relation  $(A + B) + C \subset A + (B + C)$  holds.
- $x \in (B - A)$  and  $x \notin C$ . Hence,  $x \notin B \cap C$ . So, it is in  $(B - C) \cup (C - B)$  as this is the same as  $(B \cup C) - (B \cap C)$  by definition of  $B + C$ . However,  $x \notin A$ . Therefore,  $x \in ((B - C) \cup (C - B)) - A$  and hence  $x \in (A - ((B - C) \cup (C - B))) \cup (((B - C) \cup (C - B)) - A)$ . Hence, in this case the subset relation  $(A + B) + C \subset A + (B + C)$  holds.
- not bothered rn come back to this later too tedious

#### 1.4 Solution 10

1. Nope, transitivity is not guaranteed.
2. Nope, transitivity is not satisfied.
3. Yep, all three conditions satisfied, transitivity since uniqueness of father, the other two are trivial.
4. Yep, all three conditions satisfied.

#### 1.5 Solution 11

Not sure about this one.

#### 1.6 Solution 12

The relation of concern is defined as follows. The set  $S$  of all integers and let  $n > 1$  be a fixed integer. Define for  $a, b \in S$ ,  $a \sim b$  if  $a - b$  is a multiple of  $n$ .

We first prove that this relation is an equivalence relation.

Let  $a \in S$  then,  $a - a = 0$ . Hence,  $a \sim a$ . Therefore, this relation is reflexive.

Let  $a, b \in S$  and  $a \sim b$ . Then  $a - b = k$  where  $k$  is a multiple of  $n$ . Hence,  $b - a = -k$  and  $-k$  is also a multiple of  $n$ . Therefore,  $b \sim a$ . Hence, this relation is symmetric.

Let  $a, b$  and  $c \in S$  and  $a \sim b$  and  $b \sim c$ . Then we know that  $a - b$  and  $b - c$  are multiples of  $n$ . So,

$$a - b = pn \tag{1}$$

$$b - c = qn. \tag{2}$$

For some  $p, q \in \mathbb{Z}$ . Now we consider  $(1) + (2)$ ,

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$$\begin{aligned}
a - b + b - c &= pn - qn \\
a - c &= (p - q)n.
\end{aligned}$$

Since,  $p - q \in \mathbb{Z}$  as  $p, q \in \mathbb{Z}$ . Therefore,  $a - c$  is a multiple of  $n$ . Hence,  $a \sim c$ . Therefore, this relation is transitive as well.

So, this relation is, reflexive, symmetric and transitive. Hence this is an equivalence relation.

Now we show that there are only  $n$ , equivalence classes for this relation. Note that,

$$\begin{aligned}
cl(0) &= \{m \times n \mid m \in \mathbb{Z}\} \\
cl(1) &= \{m \times n + 1 \mid m \in \mathbb{Z}\} \\
cl(2) &= \{m \times n + 2 \mid m \in \mathbb{Z}\} \\
&\vdots \\
cl(n) &= \{m \times n \mid m \in \mathbb{Z}\} = cl(0).
\end{aligned}$$

Hence, by inspection we notice that the equivalence classes start cycling every  $n$  terms  $n$ . It is also clear that,  $\{cl(0), cl(1), \dots, cl(n-1)\}$ , are distinct equivalence classes. Now to show that only  $n$  equivalence classes exist we show that  $cl(k) \in \{cl(0), \dots, cl(n-1)\}$  for all  $k \in \mathbb{Z}$ .

Now, let  $k \in \mathbb{Z}$ , then we know that,  $mod(k, n) = a$  where  $m \in \{0, \dots, n-1\}$ .

Therefore,  $cl(k) = \{m \times n + a \mid m \in \mathbb{Z}\} \in \{cl(0), cl(1), \dots, cl(n-1)\}$ .

Hence, there are  $n$  equivalence classes for this equivalence relation which are  $\{cl(0), \dots, cl(n-1)\}$ .

## 1.7 Solution 13

We state theorem 1.1.1 first below,

**Theorem 1.** The distinct equivalence classes of an equivalence relation on  $A$ , provide us with a decomposition of  $A$  as a union of mutually disjoint subsets. Conversely, given a decomposition of  $A$  as a union of mutually disjoint, non empty subsets, we can define an equivalence relation on  $A$  for which these subsets are distinct equivalence classes.