MATH2701 Lecture Notes

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Contents

1 Recap

A group is a set G equipped with a multiplication $*: G \times G \to G$ that satisfies Associativity, Existence of Identity and Inverse.

- The group $\mathscr{B}(\mathbb{R}^n)$ of all transformations in $\mathbb{R}^n$
- The group $\mathscr{T}(\mathbb{R}^n)$ of all translations in $\mathbb{R}^n$ recall $T_{\mathbf{b}}: \mathbf{x} \to \mathbf{x}+\mathbf{b}$ This group is a subgroup of $\mathscr{B}(\mathbb{R}^n)$ is isomorphic to the group $(\mathbb{R}^n, +)$. A mapping $\tau: \mathbb{R}^n \to \mathbb{R}^n$ is called an isometry if it preserves distance between any two points. Any isometry is a transformation.
- The set $\mathscr{I}(\mathbb{R}^n)$ of all isometries in $\mathbb{R}^n$ is a subgroup of $\mathscr{B}(\mathbb{R}^n)$. It contains $\mathscr{T}(\mathbb{R}^n)$ as a subgroup.
- If $\tau: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, then there exist and orthogonal $n \times n$ matrix Q and a vector $\mathbf{b}$ such that τ has the following equation.

$$\tau(\mathbf{x}) = Q\mathbf{x} + \mathbf{b}.$$

- Some special isometries: translations, reflections in R^n, glide reflections in $\mathbb{R}^n$.

2 Hyperplanes and Reflections

Definition 1. A hyperplance in $\mathbb{R}^n$ is the translation $\mathbb{H} = T_a(V)$ of an $n-1$ dimensional subspace V. If V is the orthogonal complement of a line parallel to $\mathbf{n}$, i,e, $V = \langle \mathbf{n} \rangle^{\perp}$ then V has equation $\mathbf{n} \cdot \mathbf{x} = 0$. and, thus, $\mathbb{H}$ has equation in point-normal form: $\mathbf{n} \cdot (\mathbf{x} - \mathbf{a})$ or in in cartesian form:

$$n_1 X_1 + n_2 X_2 + \ldots + n_n X_n + c = 0.$$

where $\mathbf{n}\left(a_1, a_2, \ldots, a_n\right), \mathbf{c} = -\mathbf{n} \cdot \mathbf{a}$. In other words, every hyperplance $\mathbb{H}$ is determined by its normal $\mathbf{n}$ and a point $A(\mathbf{a})$ in $\mathbb{H}$:

$$\mathbb{H} = \mathbb{H}_{\mathbf{n},\mathbf{a}} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0\}.$$

In parametric vector form: $\mathbf{x} = \mathbf{a} + \lambda_1 \mathbf{a}_1 + \ldots + \lambda_{n-1} \mathbf{a}_{n-1}$, where $\mathbf{a}_1, \ldots \mathbf{a}_n$ form a basis for $\langle \mathbf{n} \rangle^{\perp}$

2.1 Reflections

Definition 2. Let $\mathbb{H}$. The reflection $\sigma_{\mathbb{H}}$ in $\mathbb{H}$ is the mapping defined by:

$$\sigma_{\mathbb{H}} = \begin{cases} P & \text{if } P \in \mathbb{H}; \\ P' & \text{if if } P \text{ is off } \mathbb{H} \text{ and } \mathbb{H} \text{ is perpendicular bisector of } \overline{PP'} \end{cases}$$

(in the sense that $d(P, X) = d\left(P', X\right)$ for all $X \in \mathbb{H}$)

Proposition 1. Let $\mathbb{H}$ be a hyperplance

1. A reflection $\sigma_{\mathbb{H}}$ is an isometry satisfying $\sigma_{\mathbb{H}} = 1$

2. $\sigma_{\mathbb{H}}$ fixes a line $m \not\subseteq \mathbb{H}$ if and only if $m \perp \mathbb{H}$.

3. $\sigma_{\mathbb{H}}$ fixes a line pointwise if and only if $m \not\subseteq \mathbb{H}$.

Proof. Let $\mathbb{H} = \mathbb{H}_{\mathbf{n},\mathbf{a}}$.

1. By definition, $\sigma_{\mathbb{H}}^2 = 1$. Hence, it is a bijection by existence of inverse I think (Ask Lect). A For the proof of isometry, we may prove it geometrically mimicking the $\mathbb{R}^2$ case, or algebraically (to come.)

2. if $m \perp \mathbb{H}$, it is clear $\sigma_{\mathbb{H}}(m) = m$ (Ask lect) . Conversely, $Q = \sigma_{\mathbb{H}}(P)$ for some $P \in m$, but off $\mathbb{H}$. Then $Q \in m$ and the line segment $\overline{PQ}$ is perpendicular to $\mathbb{H}$ (Ask Lect dont we get Q in m from knowing that PQ is perpendicular as reflection $\rightarrow$ there is only on perpendicular form one pt so Q in m This deduction seems to go in the opposite direction.) Hence, $m \perp \mathbb{H}$.

3. is clear: $\sigma_{\mathbb{H}} = P$ for all $P \in m \Leftrightarrow m \subseteq \mathbb{H}$ (Ask lect by definition ?)

□

2.1.1 General Reflection Formula

Theorem 1. If $\mathbb{H} = \mathbb{H}_{\mathbf{n},\mathbf{a}}$, the there exist $Q = I - \frac{2}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}\mathbf{n}^T \in O_n(\mathbb{R})$ and $\mathbf{b} = 2\frac{\mathbf{a}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}$
such that.

$$\sigma_{\mathbb{H}}(\mathbf{x}) = Q\mathbf{x} + \mathbf{b}.$$

Proof. By vector geometry (draw the expression below geometrically to see that it gives reflection),

$$\begin{aligned}\sigma_{\mathbb{H}}(\mathbf{x}) &= \mathbf{a} + [(\mathbf{x}-\mathbf{a}) - 2\mathrm{proj}_{\mathbf{n}}(\mathbf{x}-\mathbf{a})] = \mathbf{x} - 2\mathrm{proj}_{\mathbf{n}}(\mathbf{x}-\mathbf{a}) &(1)\\ &= \mathbf{x} - 2\frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n} + 2\frac{\mathbf{a}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}. &(2)\end{aligned}$$

□

Now the first assertion follows easily, noting

$$(\mathbf{x}\cdot\mathbf{n})\mathbf{n} = \left(\mathbf{n}^T\mathbf{x}\right)\mathbf{n}.$$

Note that $(\mathbf{n}^T\mathbf{x})\mathbf{n}$ is a scalar so using commutativity.

$$\left(\mathbf{n}^T\mathbf{x}\right)\mathbf{n} = \mathbf{n}\left(\mathbf{n}^T\mathbf{x}\right) = \left(\mathbf{n}\mathbf{n}^T\right)\mathbf{x}.$$

Now continuing from (2) by substituting,

$$\begin{aligned}&= \mathbf{x} - 2\frac{(\mathbf{n}\mathbf{n}^T)\mathbf{x}}{\mathbf{n}\cdot\mathbf{n}} + \frac{2\mathbf{a}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}\\ &= \mathbf{x}\left(I - 2\frac{(\mathbf{n}\mathbf{n}^T)}{\mathbf{n}\cdot\mathbf{n}}\right) + \frac{2\mathbf{a}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}\\ &= Q\mathbf{x} + \mathbf{b}.\end{aligned}$$

Since Q is symmetric (difference between two symmetric matrices) and $QQ = I$

2.1.2 Reflections in $\mathbb{R}^2$

Corollary 1. In $\mathbb{R}^2$, if line ℓ has equation $aX + bX + c = 0$, then ther reflection σ_ℓ has equation:

$$\begin{aligned}\sigma_\ell(\mathbf{x}) &= \frac{1}{a^2+b^2}\begin{bmatrix} b^2-a^2 & -2ab \\ -2ab & a^2+b^2\end{bmatrix}\mathbf{x} + \frac{1}{a^2+b^2}\begin{bmatrix}-2ac \\ -2bc\end{bmatrix} \\ &= \begin{pmatrix} x \\ y\end{pmatrix} - \frac{1}{a^2+b_2}\begin{pmatrix} 2a(ax+by+c) \\ 2b(ax+by+c)\end{pmatrix}.\end{aligned}$$

Proof. Here, $\mathbf{n} = \begin{pmatrix} a \\ b\end{pmatrix}$ and $\mathbf{a}\cdot\mathbf{n} = -c$. Substituting gives the required formula (substituting in the general reflection formula theorem before). Alternatively, let $\mathbf{x}' = \sigma_\ell(\mathbf{x})$. Then $\mathbf{x}' - \mathbf{x}$ is parallel to $\mathbf{n}$ and $\frac{1}{2}(\mathbf{x}' + \mathbf{x}) \in \ell$. Thus, $b(x' - x) = a(y' - y)$ and $a(x' + x) + b(y' + y) + 2c = 0$. Hence simplifying both equations, $bx' - ay' = bx + ay$ and $ax' + by' = -(ax + by + 2c)$ which are two equations and two variables. Solving gives the second formula ☐

2.1.3 Example reflection calculation

Problem. Find the equation of a reflection in the line tangent to the circle $x^2 + y_2 = 25$ at $(3, 4)$

Solution:
The normal of the tangent line ℓ is $\mathbf{n} = \begin{pmatrix} 3 \\ 4\end{pmatrix}$. If $\mathbf{x}$ is a point on ℓ other than $(3, 4)$, then $\mathbf{n}\cdot(\mathbf{x} - \mathbf{n}) = 0$. So, ℓ has the equation $3x + 4y - 25 = 0$. The equation of σ_ℓ has the form

$$\begin{aligned}\sigma_\ell &= \begin{pmatrix} x \\ y\end{pmatrix} - \frac{2}{25}\begin{pmatrix} 3(3x+5y-25) \\ 4(3x+4y-25)\end{pmatrix} \\ &= \frac{1}{25}\begin{pmatrix} 4^2-3^3 & -2\cdot3\cdot4 \\ -2\cdot3\cdot4 & 3^3-4^2\end{pmatrix}\mathbf{x} + \begin{pmatrix} 6 \\ 8\end{pmatrix}.\end{aligned}$$

You can verify this answer by checking the reflection of something you intuitively know the answer for. Half of a point on the line expect reflection to be 3/2 of that point.

$$\sigma_\ell\begin{pmatrix} 3 \\ 4\end{pmatrix} = \begin{pmatrix} 3 \\ 4\end{pmatrix}$$

and

$$\sigma_\ell\left(\frac{2}{3}\begin{pmatrix} 3 \\ 4\end{pmatrix}\right) = -\frac{1}{2}\begin{pmatrix} 3 \\ 4\end{pmatrix} + \begin{pmatrix} 6 \\ 8\end{pmatrix} = \frac{3}{2}\begin{pmatrix} 3 \\ 4\end{pmatrix}.$$

2.2 Points in a generic position

Definition 3. We say that m points $P_1, P_2, \ldots P_m \in \mathbb{R}^n$ are in a generic position if the vectors $\mathbf{p}_i - \mathbf{p}_1$, for $i = 2, 3, \ldots, m$. are linearly independent. In particular $n+1$ points in $\mathbb{R}^n$ are in generic position if every hyperplance contains at most n of the $n+1$ points.

Theorem 2. Following are results about generic points:

1. An isometry on $\mathbb{R}^n$ that fixes $n+1$ points in generic position is the identity map
2. An isometry on $\mathbb{R}^n$ that fixes n points in generic position is a reflection or the identity map.
3. Every isometry (in $\mathbb{R}^n$) is a product of at most $n+1$ reflections.

Proof. We prove all these results:

1. If τ is an isometry, then by a theorem proved previously, there exists and orthogonal matrix $Q \in O_n(\mathbb{R})$, and a vector $\mathbf{b}$ such that $\tau(\mathbf{x}) = Q\mathbf{x} + \mathbf{b}$. Now suppose points $P_1, \ldots, P_{n+1}$ are in a generic position with position vectors $\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}$ and $\tau(P_i) = P_i$ for every i. Then we have:

$$Q\mathbf{p}_i + \mathbf{b} = \mathbf{p}_i.$$

for all $1 \leq i \leq n+1$. Thus subtracting $Q\mathbf{p}_1 + \mathbf{b} = \mathbf{p}_1$, from equation above we get

$$Q(\mathbf{p_1} - \mathbf{p_i}) = \mathbf{p}_1 - \mathbf{p}_i.$$

for all $i = 2, \ldots, n+1$. Hence, Q fixes every element in the basis $\mathbf{p}_1 - \mathbf{p}_2, \ldots, \mathbf{p}_1 - \mathbf{p}_{n+1}$ for $\mathbb{R}^n$ so $Q = I_n$. Consequently, $b = 0$ and $\tau = 1$, proving (1).

2. Suppose τ fixes points $P_1, P_2, \ldots, P_n$ which are in generic position. Then the hyperplane.

$$\mathbb{H} = \mathbf{x} = \mathbf{p}_1 + \lambda_1(\mathbf{p}_2 - \mathbf{p}_1) + \ldots + \lambda_{n-1}(\mathbf{p}_n - \mathbf{p}_1).$$

where $\lambda_i \in \mathbb{R}$.

contains all n points.If $\tau \neq 1$, then there exists a point R off $\mathbb{H}$ such that $\tau(R) = R' \neq R$. So we have $d(P_i, R) = d(\tau(P_i), \tau(R)) = d(P_i, R')$ for all $i = 1, 2, \ldots n$. Thus, for any point $A(\mathbf{a})$ in $\mathbb{H}$, since its position vector $\mathbf{a} = \mathbf{p}_1 + \lambda_1\mathbf{p}_2 - \mathbf{p}_1 + \ldots + \lambda_{n-1}\mathbf{p}_n - \mathbf{p}_1$ for some λ_i, we see that

$$\begin{aligned}\tau(\mathbf{a}) &= Q(\mathbf{p}_1 + \lambda_1(\mathbf{p}_2 - \mathbf{p}_1) + \ldots \lambda_{n-1}(\mathbf{p}_n - \mathbf{p}_q)) + \mathbf{b} \\ &= [Q(\mathbf{p}_1) + \mathbf{b}] + \lambda_1(\mathbf{p}_1 - \mathbf{p}_1) + \ldots \lambda_{n-1}(\mathbf{p}_n - \mathbf{p}_1) = \mathbf{a}.\end{aligned}$$

The above simplification is because the $\mathbf{b}$'s will cancel out when you evaluate the $Q(\mathbf{p}_i)$ values upon distributing Q. So, $d(A, R) = d(\tau(A), \tau(R)) = d(A, R')$. This shows $\mathbb{H}$ is the perpendicular bisector of $\overline{RR'}$. Hence $\tau = \sigma_{\mathbb{H}}$.

3. Suppose τ is an isometry, and fixes the points $P_1, \ldots, P_{n-1}$ in generic position. Choose a point $P = P_0$ so that $P, P_1, \ldots P_{n-1}$ are in generic position. Then $P' = \tau(P) \neq P$. (Proof incomplete right now complete from slides)
4. Suppose τ is an isometry and let m be the maximal number of points in generic positions which τ fixes. We apply a downward induction on m. If $m = n + 1$ or n. we are done by (1) and (2).

   Suppose now that $m < n$ and the assertion is true for $m + 1$. Consider a point P such that $P' = \tau(P) \neq P$ and a hyperplane $\mathbb{H}$ containing the m points in generic position and perpendicularly bisection $\overline{PP'}$. Then $\sigma_{\mathbb{H}}\tau$ fixed $m + 1$ points in generic position. By induction, $\sigma_{\mathbb{H}}\tau$ is a product of $n - m$ reflections. Hence, τ is a product of $n - m + 1$ reflections. By induction, the assertion is true for all $m = n + 1, n \ldots, 1, 0$.

□

Corollary 2. Following results can be deduced from above.

1. Every plane isometry is a product of at most three reflections.
2. The group $\mathscr{I}(\mathbb{R}^n)$ of isometries on $\mathbb{R}^n$ is generated by reflections $\mathbb{H}_{n,\mathbf{a}}$ for all $\mathbf{0} \neq \mathbf{n}, \mathbf{b} \in \mathbb{R}^n$.

Remark. We call $\mathscr{I}(\mathbb{R}^n)$ a reflection group. The study of finite ferflection groups is an interesting research area in mathemtics.

3 Translations and rotations in $\mathbb{R}^n$

3.1 Tanslations

Theorem 3. An isometry τ in $\mathbb{R}^n$ is a translation if and only if τ is the product of two reflections in parallel hyperplanes.

Proof. Let $\mathbb{H} = \mathbb{H}_{\mathbf{n},\mathbf{a}}$ and $\mathbb{H}' = \mathbb{H}_n, \mathbf{b}$. Then, for all $\mathbf{x} \in \mathbb{R}^n$, we have

$$\begin{aligned}
\sigma_{\mathbb{H}'}\sigma_{\mathbb{H}}(\mathbf{x}) &= \sigma_{\mathbb{H}'}\left(\mathbf{x} - 2\frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n} + 2\frac{\mathbf{a}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}\right) \\
&= \left(\mathbf{x} - 2\frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n} + 2\frac{\mathbf{a}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}\right) - 2\frac{\left(\mathbf{x} - 2\frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n} + 2\frac{\mathbf{a}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}\right)\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n} + 2\frac{\mathbf{b}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n} \\
&= \mathbf{x} + 2\left(\frac{\mathbf{b}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} - \frac{\mathbf{a}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} \\
&= \mathbf{x} + 2proj_n(\mathbf{b} - \mathbf{a}) \\
&= T_{2proj_{\mathbf{n}}(\mathbf{b}-\mathbf{a})}(\mathbf{x}).
\end{aligned}$$

(Hence, $\sigma_{\mathbb{H}_{\mathbf{n},\mathbf{b}}}\sigma_{\mathbb{H}_{\mathbf{n},\mathbf{a}}} = T_{2proj_{\mathbf{n}}(\mathbf{b}-\mathbf{a})}$) □

Conversely, suppose $\tau_{P,Q}$ be translation sneding $P(\mathbf{p})$ to Q. Let $\mathbf{n}$ be a vector parallel to $\overrightarrow{PQ}$ and $M(\mathbf{m})$ the midpoint of $\overrightarrow{PQ}$. Then we have,

$$\sigma_{\mathbb{H}_{\mathbf{n},\mathbf{m}}}\sigma_{\mathbb{H}_{\mathbf{n},\mathbf{p}}} = T_{2proj_n}(\mathbf{m}-\mathbf{p}) = T_{2(\mathbf{m}-\mathbf{p})} = T_{\mathbf{q}-\mathbf{p}} = \tau_{P,Q}.$$

We got $T_{2proj_n}(\mathbf{m}-\mathbf{p})$ above from the same calculation as the proof.

3.2 Translations in $\mathbb{R}^2$

Corollary 3. A plane isometry is a translation if and only if it is a produ t of two reflections in parallel lines.

Example 1. For lines $\ell_1 : 2x+3y=0, \ell_2 : 2x+3y=5$ and $\ell_3 : 2x+3y=7$, find the vector **b**and line ℓ such that $\sigma_{\ell_2}\sigma_{\ell_2}\sigma_{\ell_1} = \sigma_\ell$.

Solution. We have,

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

and points $A(1,1) \in \ell_2$ and $B(1,2) \in \ell_3$. Then

$$proj_{\mathbf{n}}(B-A) = proj_{\mathbf{n}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{2}{13}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

and

$$\sigma_{\ell_3}\sigma_{\ell_2} = T_{\frac{4}{13}\begin{pmatrix} 2 \\ 3 \end{pmatrix}}.$$

using the formula.

Thus,

$$\begin{aligned}\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1} &= T_{\frac{4}{13}\begin{pmatrix} 2 \\ 3 \end{pmatrix}}\sigma_{\ell_1} : \mathbf{x} \mapsto \frac{1}{13}\begin{pmatrix} 5 & -12 \\ -12 & -5 \end{pmatrix}\mathbf{x} + \frac{4}{13}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{13}\begin{pmatrix} 2\times 2(2x+3y-2) \\ 2\times 3(2x+3y-2) \end{pmatrix}.\end{aligned}$$

Hence, $\ell : 2x+3y-2=0$

3.3 Rotations

3.4 Rotations in $\mathbb{R}^2$

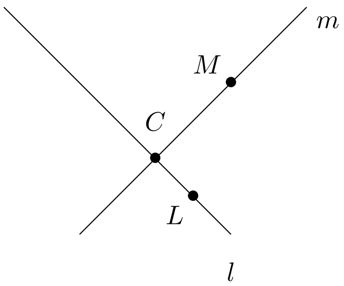
Definition 4. A rotation on $\mathbb{R}^2$ about a point C, through angle θ, is the transformation that fixes C and otherwise sends a point P to a point P', where $d(C,P) = d(C,P')$, and the angle from $\overrightarrow{CP}$ to $\overrightarrow{CP'}$ is θ (in anti-clockwise direction if $\theta > 0$, and clockwise if $\theta < 0$). We denote this transformation by $\rho_{C,\theta}$.

Theorem 4. An plane isometry is a rotation if and only if it is the product of two plane reflections in intersecting lines. More precisely,

1. if lines l, m intersect at C, and the directed angle from l to m is $\frac{\theta}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, then $\sigma_m \sigma_l = \rho_{C,\theta}$;
2. if lines p, q, r are concurrent, then there exists a line l such that

$$\sigma_r \sigma_q \sigma_p = \sigma_l.$$

Proof. (2) follows easily from (1). We now prove (1). Let $L \in l, L \neq C$, and let $M \in m$, $M \neq C$. By a geometrical argument, one checks easily that $\sigma_m \sigma_l (C) = C = \rho_{C,\theta} (C), \sigma_m \sigma_l (L) = \sigma_m (L) = \rho_{C,\theta} (L)$.



Hence, $\sigma_m, \sigma_l = \rho_{C,\theta}$ (Ask lect why, not sure how the generic point argument explained in lecture works.) $\square$

Corollary 4. Some rotation results:

1. A non-identity rotation (on $\mathbb{R}^2$) fixes exactly one point.
2. A rotation with centre C fixes every circle with centre C.
3. The set of all rotations about a particular point (i.e. with centre at a particular point) is a sugbroup of the group $\mathscr{I}\left(\mathbb{R}^2\right)$ of isometries; further still, it is a commutative subgroup. In other words,

$$\mathscr{R}_C \coloneqq \{\rho_{C,\theta} : \theta \in \mathbb{R}\} \leq \mathscr{I}\left(\mathbb{R}^2\right).$$

   and

$$\rho\rho' = \rho'\rho, \forall \rho, \rho' \in \mathscr{R}_C.$$

Theorem 5.

The rotation $\rho_{\mathbf{0},\theta} : \mathbb{R}^2 \to \mathbb{R}^2$ about the origin $\mathbf{0}$ and through the angle θ i the linear isomorphism $T_{U,\mathbf{0}} (\mathbf{x}) = U\mathbf{x}$, where U is the following matrix:

$$U = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Reason for this you apply rotation by θ to the basis and that gives the matrix.

If $\mathbf{c}$ is the position vector of C, then $\rho_{C,\theta} = T_{\mathbf{c}} (\rho_{0,\theta}) T_{-\mathbf{c}}$. Hence, $\rho_{C,\theta}$ has equation $\rho_{C,\theta} (\mathbf{x}) = U\mathbf{x} + \mathbf{b}$, where U defines $\rho_{\mathbf{0},\theta}$ as in (1) and $\mathbf{b} = (I - U)\mathbf{c}$. Moreover, at the group level, we have $\mathscr{R}_C = T_{\mathbf{c}} \mathscr{R}_{\mathbf{0}} T_{-\mathbf{c}}$, or $\mathscr{R}_C$ is conjugate to $\mathscr{R}_0$.

Proof. By the theorem above, we may assume that $\rho_{0,\theta} = \rho_m, \rho_l$, where l is the x-axis, and m has equation:

$$\sin\left(\frac{\theta}{2}\right) X - \cos\left(\frac{\theta}{2}\right) Y = 0.$$

$\square$

Hence, σ_m has the equations:

$$\begin{cases} x' = \left(1 - 2\sin^2(\frac{\theta}{2})\right) x + \left(2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right) y = (\cos\theta)\, x + (\sin\theta)\, y \\ y' = (\sin\theta)\, x - (\cos\theta)\, y. \end{cases}$$

Also, σ_l has (more obvious) equation: $X' = X, Y' = -Y$. Hence, by multiplying matrices, we can see that:

$$\sigma_m\sigma_l = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{x} = U\mathbf{x}.$$

proving (1).