

MATH2701 Lecture Notes

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Contents

1	Recap	3
2	Hyperplanes and Reflections	3
2.1	Reflections	3
2.2	General Reflection Formula	4
2.3	Reflections in \mathbb{R}^2	5
2.4	Example reflection calculation	5
2.5	Points in a generic position	6
3	Translations and rotations in \mathbb{R}^n	7
3.1	Tanslations	7
3.2	Translations in \mathbb{R}^2	8
3.3	Rotations	8
3.4	Rotations in \mathbb{R}^2	8

1 Recap

A group is a set G equipped with a multiplication $*$: $G \times G \rightarrow G$ that satisfies Associativity, Existence of Identity and Inverse.

- The group $\mathcal{B}(\mathbb{R}^n)$ of all transformations in \mathbb{R}^n
- The group $\mathcal{T}(\mathbb{R}^n)$ of all translations in \mathbb{R}^n recall $T_{\mathbf{b}} : \mathbf{x} \rightarrow \mathbf{x} + \mathbf{b}$ This group is a subgroup of $\mathcal{B}(\mathbb{R}^n)$ is isomorphic to the group $(\mathbb{R}^n, +)$. A mapping $\tau : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called an isometry if it preserves distance between any two points. Any isometry is a transformation.
- The set $\mathcal{I}(\mathbb{R}^n)$ of all isometries in \mathbb{R}^n is a subgroup of $\mathcal{B}(\mathbb{R}^n)$. It contains $\mathcal{T}(\mathbb{R}^n)$ as a subgroup.
- If $\tau : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isometry, then there exist an orthogonal $n \times n$ matrix Q and a vector \mathbf{b} such that τ has the following equation.

$$\tau(\mathbf{x}) = Q\mathbf{x} + \mathbf{b}.$$

- Some special isometries: translations, reflections in \mathbb{R}^n , glide reflections in \mathbb{R}^n .

2 Hyperplanes and Reflections

Definition 1. A hyperplane in \mathbb{R}^n is the translation $\mathbb{H} = T_a(V)$ of an $n - 1$ dimensional subspace V . If V is the orthogonal complement of a line parallel to \mathbf{n} , i.e, $V = \langle \mathbf{n} \rangle^\perp$ then V has equation $\mathbf{n} \cdot \mathbf{x} = 0$. and, thus, \mathbb{H} has equation in point-normal form: $\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0$ or in cartesian form:

$$n_1X_1 + n_2X_2 + \dots + n_nX_n + c = 0.$$

where $\mathbf{n} = (a_1, a_2, \dots, a_n)$, $c = -\mathbf{n} \cdot \mathbf{a}$. In other words, every hyperplane \mathbb{H} is determined by its normal \mathbf{n} and a point $A(\mathbf{a})$ in \mathbb{H} :

$$\mathbb{H} = \mathbb{H}_{\mathbf{n}, \mathbf{a}} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0\}.$$

In parametric vector form: $\mathbf{x} = \mathbf{a} + \lambda_1 \mathbf{a}_1 + \dots + \lambda_{n-1} \mathbf{a}_{n-1}$, where $\mathbf{a}_1, \dots, \mathbf{a}_{n-1}$ form a basis for $\langle \mathbf{n} \rangle^\perp$

2.1 Reflections

Definition 2. Let \mathbb{H} . The reflection $\sigma_{\mathbb{H}}$ in \mathbb{H} is the mapping defined by:

$$\sigma_{\mathbb{H}} = \begin{cases} P & \text{if } P \in \mathbb{H}; \\ P' & \text{if } P \text{ is off } \mathbb{H} \text{ and } \mathbb{H} \text{ is perpendicular bisector of } \overline{PP'} \end{cases}$$

(in the sense that $d(P, X) = d(P', X)$ for all $X \in \mathbb{H}$)

Proposition 1. Let \mathbb{H} be a hyperplane

1. A reflection $\sigma_{\mathbb{H}}$ is an isometry satisfying $\sigma_{\mathbb{H}}^2 = 1$

-
2. $\sigma_{\mathbb{H}}$ fixes a line $m \not\subseteq \mathbb{H}$ if and only if $m \perp \mathbb{H}$.
 3. $\sigma_{\mathbb{H}}$ fixes a line pointwise if and only if $m \subseteq \mathbb{H}$.

Proof. Let $\mathbb{H} = \mathbb{H}_{\mathbf{n}, \mathbf{a}}$.

1. By definition, $\sigma_{\mathbb{H}}^2 = 1$. Hence, it is a bijection by existence of inverse I think (Ask Lect). For the proof of isometry, we may prove it geometrically mimicking the \mathbb{R}^2 case, or algebraically (to come.)
2. if $m \perp \mathbb{H}$, it is clear $\sigma_{\mathbb{H}}(m) = m$ (Ask lect). Conversely, $Q = \sigma_{\mathbb{H}}(P)$ for some $P \in m$, but off \mathbb{H} . Then $Q \in m$ and the line segment \overline{PQ} is perpendicular to \mathbb{H} (Ask Lect dont we get Q in m from knowing that PQ is perpendicular as reflection \rightarrow there is only one perpendicular from one pt so Q in m This deduction seems to go in the opposite direction.) Hence, $m \perp \mathbb{H}$.
3. is clear: $\sigma_{\mathbb{H}}(P) = P$ for all $P \in m \Leftrightarrow m \subseteq \mathbb{H}$ (Ask lect by definition ?)

□

2.2 General Reflection Formula

Theorem 1. If $\mathbb{H} = \mathbb{H}_{\mathbf{n}, \mathbf{a}}$, then there exist $Q = I - \frac{2}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} \mathbf{n}^T \in O_n(\mathbb{R})$ and $\mathbf{b} = 2 \frac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n}$

such that.

$$\sigma_{\mathbb{H}}(\mathbf{x}) = Q\mathbf{x} + \mathbf{b}.$$

Proof. By vector geometry (draw the expression below geometrically to see that it gives reflection),

$$\sigma_{\mathbb{H}}(\mathbf{x}) = \mathbf{a} + [(\mathbf{x} - \mathbf{a}) - 2\text{proj}_{\mathbf{n}}(\mathbf{x} - \mathbf{a})] = \mathbf{x} - 2\text{proj}_{\mathbf{n}}(\mathbf{x} - \mathbf{a}) \quad (1)$$

$$= \mathbf{x} - 2 \frac{\mathbf{x} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} + 2 \frac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n}. \quad (2)$$

□

Now the first assertion follows easily, noting

$$(\mathbf{x} \cdot \mathbf{n}) \mathbf{n} = (\mathbf{n}^T \mathbf{x}) \mathbf{n}.$$

Note that $(\mathbf{n}^T \mathbf{x}) \mathbf{n}$ is a scalar so using commutativity.

$$(\mathbf{n}^T \mathbf{x}) \mathbf{n} = \mathbf{n} (\mathbf{n}^T \mathbf{x}) = (\mathbf{n} \mathbf{n}^T) \mathbf{x}.$$

Now continuing from (2) by substituting,

$$\begin{aligned} &= \mathbf{x} - 2 \frac{(\mathbf{n} \mathbf{n}^T) \mathbf{x}}{\mathbf{n} \cdot \mathbf{n}} + \frac{2 \mathbf{a} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} \\ &= \mathbf{x} \left(I - 2 \frac{(\mathbf{n} \mathbf{n}^T)}{\mathbf{n} \cdot \mathbf{n}} \right) + \frac{2 \mathbf{a} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} \\ &= Q\mathbf{x} + \mathbf{b}. \end{aligned}$$

Since Q is symmetric (difference between two symmetric matrices) and $QQ = I$

2.3 Reflections in \mathbb{R}^2

Corollary 1. In \mathbb{R}^2 , if line ℓ has equation $aX + bY + c = 0$, then the reflection σ_ℓ has equation:

$$\begin{aligned}\sigma_\ell(\mathbf{x}) &= \frac{1}{a^2 + b^2} \begin{bmatrix} b^2 - a^2 & -2ab \\ -2ab & a^2 + b^2 \end{bmatrix} \mathbf{x} + \frac{1}{a^2 + b^2} \begin{bmatrix} -2ac \\ -2bc \end{bmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{a^2 + b^2} \begin{pmatrix} 2a(ax + by + c) \\ 2b(ax + by + c) \end{pmatrix}.\end{aligned}$$

Proof. Here, $\mathbf{n} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{a} \cdot \mathbf{n} = -c$. Substituting gives the required formula (substituting in the general reflection formula theorem before). Alternatively, let $\mathbf{x}' = \sigma_\ell(\mathbf{x})$. Then $\mathbf{x}' - \mathbf{x}$ is parallel to \mathbf{n} and $\frac{1}{2}(\mathbf{x}' + \mathbf{x}) \in \ell$. Thus, $b(x' - x) = a(y' - y)$ and $a(x' + x) + b(y' + y) + 2c = 0$. Hence simplifying both equations, $bx' - ay' = bx + ay$ and $ax' + by' = -(ax + by + 2c)$ which are two equations and two variables. Solving gives the second formula \square

2.4 Example reflection calculation

Problem. Find the equation of a reflection in the line tangent to the circle $x^2 + y^2 = 25$ at $(3, 4)$

Solution:

The normal of the tangent line ℓ is $\mathbf{n} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. If \mathbf{x} is a point on ℓ other than $(3, 4)$, then $\mathbf{n} \cdot (\mathbf{x} - \mathbf{n}) = 0$. So, ℓ has the equation $3x + 4y - 25 = 0$. The equation of σ_ℓ has the form

$$\begin{aligned}\sigma_\ell &= \begin{pmatrix} x \\ y \end{pmatrix} - \frac{2}{25} \begin{pmatrix} 3(3x + 4y - 25) \\ 4(3x + 4y - 25) \end{pmatrix} \\ &= \frac{1}{25} \begin{pmatrix} 4^2 - 3^2 & -2 \cdot 3 \cdot 4 \\ -2 \cdot 3 \cdot 4 & 3^2 - 4^2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 6 \\ 8 \end{pmatrix}.\end{aligned}$$

You can verify this answer by checking the reflection of something you intuitively know the answer for. Half of a point on the line expect reflection to be 3/2 of that point.

$$\sigma_\ell \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

and

$$\sigma_\ell \left(\frac{2}{3} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right) = -\frac{1}{2} \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ 8 \end{pmatrix} = \frac{3}{2} \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

2.5 Points in a generic position

Definition 3. We say that m points $P_1, P_2, \dots, P_m \in \mathbb{R}^n$ are in a generic position if the vectors $\mathbf{p}_i - \mathbf{p}_1$, for $i = 2, 3, \dots, m$, are linearly independent. In particular $n + 1$ points in \mathbb{R}^n are in generic position if every hyperplane contains at most n of the $n + 1$ points.

Theorem 2. Following are results about generic points:

1. An isometry on \mathbb{R}^n that fixes $n + 1$ points in generic position is the identity map
2. An isometry on \mathbb{R}^n that fixes n points in generic position is a reflection or the identity map.
3. Every isometry (in \mathbb{R}^n) is a product of at most $n + 1$ reflections.

Proof. We prove all these results:

1. If τ is an isometry, then by a theorem proved previously, there exists an orthogonal matrix $Q \in O_n(\mathbb{R})$, and a vector \mathbf{b} such that $\tau(\mathbf{x}) = Q\mathbf{x} + \mathbf{b}$. Now suppose points P_1, \dots, P_{n+1} are in a generic position with position vectors $\mathbf{p}_1, \dots, \mathbf{p}_{n+1}$ and $\tau(P_i) = P_i$ for every i . Then we have:

$$Q\mathbf{p}_i + \mathbf{b} = \mathbf{p}_i.$$

for all $1 \leq i \leq n + 1$. Thus subtracting $Q\mathbf{p}_1 + \mathbf{b} = \mathbf{p}_1$, from equation above we get

$$Q(\mathbf{p}_1 - \mathbf{p}_i) = \mathbf{p}_1 - \mathbf{p}_i.$$

for all $i = 2, \dots, n + 1$. Hence, Q fixes every element in the basis $\mathbf{p}_1 - \mathbf{p}_2, \dots, \mathbf{p}_1 - \mathbf{p}_{n+1}$ for \mathbb{R}^n so $Q = I_n$. Consequently, $\mathbf{b} = 0$ and $\tau = 1$, proving (1).

2. Suppose τ fixes points P_1, P_2, \dots, P_n which are in generic position. Then the hyperplane.

$$\mathbb{H} = \mathbf{x} = \mathbf{p}_1 + \lambda_1(\mathbf{p}_2 - \mathbf{p}_1) + \dots + \lambda_{n-1}(\mathbf{p}_n - \mathbf{p}_1).$$

where $\lambda_i \in \mathbb{R}$.

contains all n points. If $\tau \neq 1$, then there exists a point R off \mathbb{H} such that $\tau(R) = R' \neq R$. So we have $d(P_i, R) = d(\tau(P_i), \tau(R)) = d(P_i, R')$ for all $i = 1, 2, \dots, n$. Thus, for any point $A(\mathbf{a})$ in \mathbb{H} , since its position vector $\mathbf{a} = \mathbf{p}_1 + \lambda_1\mathbf{p}_2 - \mathbf{p}_1 + \dots + \lambda_{n-1}\mathbf{p}_n - \mathbf{p}_1$ for some λ_i , we see that

$$\begin{aligned} \tau(\mathbf{a}) &= Q(\mathbf{p}_1 + \lambda_1(\mathbf{p}_2 - \mathbf{p}_1) + \dots + \lambda_{n-1}(\mathbf{p}_n - \mathbf{p}_1)) + \mathbf{b} \\ &= [Q(\mathbf{p}_1) + \mathbf{b}] + \lambda_1(\mathbf{p}_1 - \mathbf{p}_1) + \dots + \lambda_{n-1}(\mathbf{p}_n - \mathbf{p}_1) = \mathbf{a}. \end{aligned}$$

The above simplification is because the \mathbf{b} 's will cancel out when you evaluate the $Q(\mathbf{p}_i)$ values upon distributing Q . So, $d(A, R) = d(\tau(A), \tau(R)) = d(A, R')$. This shows \mathbb{H} is the perpendicular bisector of $\overline{RR'}$. Hence $\tau = \sigma_{\mathbb{H}}$.

-
3. Suppose τ is an isometry, and fixes the points P_1, \dots, P_{n-1} in generic position. Choose a point $P = P_0$ so that P, P_1, \dots, P_{n-1} are in generic position. Then $P' = \tau(P) \neq P$. (Proof incomplete right now complete from slides)
 4. Suppose τ is an isometry and let m be the maximal number of points in generic positions which τ fixes. We apply a downward induction on m . If $m = n + 1$ or n , we are done by (1) and (2).

Suppose now that $m < n$ and the assertion is true for $m + 1$. Consider a point P such that $P' = \tau(P) \neq P$ and a hyperplane \mathbb{H} containing the m points in generic position and perpendicularly bisecting $\overline{PP'}$. Then $\sigma_{\mathbb{H}}\tau$ fixed $m + 1$ points in generic position. By induction, $\sigma_{\mathbb{H}}\tau$ is a product of $n - m$ reflections. Hence, τ is a product of $n - m + 1$ reflections. By induction, the assertion is true for all $m = n + 1, n, \dots, 1, 0$.

□

Corollary 2. Following results can be deduced from above.

1. Every plane isometry is a product of at most three reflections.
2. The group $\mathcal{I}(\mathbb{R}^n)$ of isometries on \mathbb{R}^n is generated by reflections $\mathbb{H}_{n,\mathbf{a}}$ for all $\mathbf{0} \neq \mathbf{n}, \mathbf{b} \in \mathbb{R}^n$.

Remark. We call $\mathcal{I}(\mathbb{R}^n)$ a reflection group. The study of finite reflection groups is an interesting research area in mathematics.

3 Translations and rotations in \mathbb{R}^n

3.1 Translations

Theorem 3. An isometry τ in \mathbb{R}^n is a translation if and only if τ is the product of two reflections in parallel hyperplanes.

Proof. Let $\mathbb{H} = \mathbb{H}_{n,\mathbf{a}}$ and $\mathbb{H}' = \mathbb{H}_{n,\mathbf{b}}$. Then, for all $\mathbf{x} \in \mathbb{R}^n$, we have

$$\begin{aligned}
 \sigma_{\mathbb{H}'}\sigma_{\mathbb{H}}(\mathbf{x}) &= \sigma_{\mathbb{H}'}\left(\mathbf{x} - 2\frac{\mathbf{x} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\mathbf{n} + 2\frac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\mathbf{n}\right) \\
 &= \left(\mathbf{x} - 2\frac{\mathbf{x} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\mathbf{n} + 2\frac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\mathbf{n}\right) - 2\frac{\left(\mathbf{x} - 2\frac{\mathbf{x} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\mathbf{n} + 2\frac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\mathbf{n}\right) \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\mathbf{n} + 2\frac{\mathbf{b} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\mathbf{n} \\
 &= \mathbf{x} + 2\left(\frac{\mathbf{b} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} - \frac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\right)\mathbf{n} \\
 &= \mathbf{x} + 2\text{proj}_{\mathbf{n}}(\mathbf{b} - \mathbf{a}) \\
 &= T_{2\text{proj}_{\mathbf{n}}(\mathbf{b}-\mathbf{a})}(\mathbf{x}).
 \end{aligned}$$

(Hence, $\sigma_{\mathbb{H}_{n,\mathbf{b}}}\sigma_{\mathbb{H}_{n,\mathbf{a}}} = T_{2\text{proj}_{\mathbf{n}}(\mathbf{b}-\mathbf{a})}$)

□

Conversely, suppose $\tau_{P,Q}$ be translation sending $P(\mathbf{p})$ to Q . Let \mathbf{n} be a vector parallel to \overrightarrow{PQ} and $M(\mathbf{m})$ the midpoint of \overrightarrow{PQ} . Then we have,

$$\sigma_{\mathbb{H}_{\mathbf{n},\mathbf{m}}} \sigma_{\mathbb{H}_{\mathbf{n},\mathbf{p}}} = T_{2proj_{\mathbf{n}}}(\mathbf{m} - \mathbf{p}) = T_{2(\mathbf{m}-\mathbf{p})} = T_{\mathbf{q}-\mathbf{p}} = \tau_{P,Q}.$$

We got $T_{2proj_{\mathbf{n}}}(\mathbf{m} - \mathbf{p})$ above from the same calculation as the proof.

3.2 Translations in \mathbb{R}^2

Corollary 3. A plane isometry is a translation if and only if it is a product of two reflections in parallel lines.

Example 1. For lines $\ell_1 : 2x + 3y = 0$, $\ell_2 : 2x + 3y = 5$ and $\ell_3 : 2x + 3y = 7$, find the vector and line ℓ such that $\sigma_{\ell_2} \sigma_{\ell_1} = \sigma_{\ell}$.

Solution. We have,

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

and points $A(1, 1) \in \ell_2$ and $B(1, 2) \in \ell_3$. Then

$$proj_{\mathbf{n}}(B - A) = proj_{\mathbf{n}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{2}{13} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

and

$$\sigma_{\ell_3} \sigma_{\ell_2} = T_{\frac{4}{13}} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

using the formula.

Thus,

$$\begin{aligned} \sigma_{\ell_3} \sigma_{\ell_2} \sigma_{\ell_1} &= T_{\frac{4}{13}} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \sigma_{\ell_1} : \mathbf{x} \mapsto \frac{1}{13} \begin{pmatrix} 5 & -12 \\ -12 & -5 \end{pmatrix} \mathbf{x} + \frac{4}{13} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{13} \begin{pmatrix} 2 \times 2(2x + 3y - 2) \\ 2 \times 3(2x + 3y - 2) \end{pmatrix}. \end{aligned}$$

Hence, $\ell : 2x + 3y - 2 = 0$

3.3 Rotations

3.4 Rotations in \mathbb{R}^2

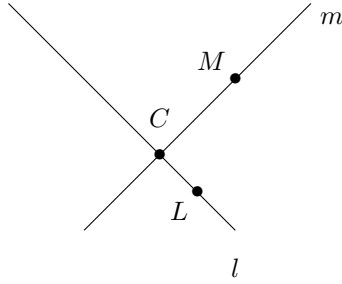
Definition 4. A rotation on \mathbb{R}^2 about a point C , through angle θ , is the transformation that fixes C and otherwise sends a point P to a point P' , where $d(C, P) = d(C, P')$, and the angle from \overrightarrow{CP} to $\overrightarrow{CP'}$ is θ (in anti-clockwise direction if $\theta > 0$, and clockwise if $\theta < 0$). We denote this transformation by $\rho_{C,\theta}$.

Theorem 4. An plane isometry is a rotation if and only if it is the product of two plane reflections in intersecting lines. More precisely,

1. if lines l, m intersect at C , and the directed angle from l to m is $\frac{\theta}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, then $\sigma_m \sigma_l = \rho_{C, \theta}$;
2. if lines p, q, r are concurrent, then there exists a line l such that

$$\sigma_r \sigma_q \sigma_p = \sigma_l.$$

Proof. (2) follows easily from (1). We now prove (1). Let $L \in l, L \neq C$, and let $M \in m, M \neq C$. By a geometrical argument, one checks easily that $\sigma_m \sigma_l(C) = C = \rho_{C, \theta}(C)$, $\sigma_m \sigma_l(L) = \sigma_m(M) = \rho_{C, \theta}(L)$.



Hence, $\sigma_m, \sigma_l = \rho_{C, \theta}$ (Ask lect why, not sure how the generic point argument explained in lecture works.) \square

Corollary 4. Some rotation results:

1. A non-identity rotation (on \mathbb{R}^2) fixes exactly one point.
2. A rotation with centre C fixes every circle with centre C .
3. The set of all rotations about a particular point (i.e. with centre at a particular point) is a subgroup of the group $\mathcal{I}(\mathbb{R}^2)$ of isometries; further still, it is a commutative subgroup. In other words,

$$\mathcal{R}_C := \{\rho_{C, \theta} : \theta \in \mathbb{R}\} \leq \mathcal{I}(\mathbb{R}^2).$$

and

$$\rho \rho' = \rho' \rho, \forall \rho, \rho' \in \mathcal{R}_C.$$

Theorem 5.

The rotation $\rho_{\mathbf{0}, \theta} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ about the origin $\mathbf{0}$ and through the angle θ is the linear isomorphism $T_{U, \mathbf{0}}(\mathbf{x}) = U\mathbf{x}$, where U is the following matrix:

$$U = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Reason for U being this you apply rotation by θ to the basis and that gives the matrix as $\rho_{\mathbf{0}, \theta}$ is a linear map (one of the properties of isometries).

If \mathbf{c} is the position vector of C , then $\rho_{C,\theta} = T_{\mathbf{c}}(\rho_{0,\theta})T_{-\mathbf{c}}$ (Ask lect, can you explain why this holds). Hence, $\rho_{C,\theta}$ has equation $\rho_{C,\theta}(\mathbf{x}) = U\mathbf{x} + \mathbf{b}$, where U defines $\rho_{0,\theta}$ as in (1) and $\mathbf{b} = (I - U)\mathbf{c}$. Moreover, at the group level, we have $\mathcal{R}_C = T_{\mathbf{c}}\mathcal{R}_0T_{-\mathbf{c}}$, or \mathcal{R}_C is conjugate to \mathcal{R}_0 .

Proof. By the theorem above, we may assume that $\rho_{0,\theta} = \sigma_m, \sigma_l$, where l is the x-axis, and m has equation:

$$\sin\left(\frac{\theta}{2}\right)X - \cos\left(\frac{\theta}{2}\right)Y = 0.$$

□

Hence, σ_m has the equations:

$$\begin{cases} x' = (1 - 2\sin^2(\frac{\theta}{2}))x + (2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2}))y = (\cos\theta)x + (\sin\theta)y \\ y' = (\sin\theta)x - (\cos\theta)y. \end{cases}$$

Also, σ_l has (more obvious) equation: $X' = X, Y' = -Y$. Hence, by multiplying matrices, we can see that:

$$\sigma_m\sigma_l = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{x} = U\mathbf{x}.$$

proving (1).

Maintain the notation in the proof of (1). For (2), we have

$$T_{\mathbf{c}}\rho_{0,\theta}T_{-\mathbf{c}} = T_{\mathbf{c}}\sigma_m\sigma_lT_{-\mathbf{c}} = (T_{\mathbf{c}}\sigma_mT_{-\mathbf{c}})(T_{\mathbf{c}}\sigma_lT_{-\mathbf{c}}) = \sigma_{m'}\sigma_{l'} = \rho_{C,\theta}.$$

(Ask lect about $(T_{\mathbf{c}}\sigma_mT_{-\mathbf{c}})(T_{\mathbf{c}}\sigma_lT_{-\mathbf{c}})\sigma_{m'}\sigma_{l'} = \rho_{C,\theta}$)