

# MATH2701 Problem Set Solutions

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October 2, 2023

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## 1 Problem set 1

### 1.1 Problem 1

Let  $A(a_1, a_2)$ ,  $B(b_1, b_2)$  be two points in  $\mathbb{R}^2$ . Find the equation for the line  $\ell(A, B)$  through  $A, B$ .

1. in Cartesian form.
2. in parametric vector form.

**Solution.** Cartesian form: For all  $(x, y) \in l$

$$\begin{aligned}\frac{b_2 - a_2}{b_1 - a_1} &= \frac{y - a_2}{x - a_1} \\ (x - a_1)(b_2 - a_2) &= (y - a_2)(b_1 - a_1) \\ xb_2 - xa_2 - a_1b_1 + a_1a_2 &= yb_1 - ya_1 - a_2b_2 + a_1a_2 \\ x(b_2 - a_2) - a_1b_2 + a_2b_1 &= y(b_1 - a_1) \\ y &= \frac{x(b_2 - a_1)}{b_1 - a_1} + \frac{-a_1b_2 + b_2b_1}{b_1 - b_1}.\end{aligned}$$

So, above is the cartesian equation of  $\ell$ .

Parametric form:  $A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ . We treat  $A$  and  $B$  as position vectors. Then,

$$\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}.$$

is parallel to  $\ell$ . Hence the parametric form is, for all  $\begin{pmatrix} x \\ y \end{pmatrix} \in \ell$ ,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \lambda \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}.$$

for some  $\lambda \in \mathbb{R}$ . (Ask lect best way to write eqn)

### 1.2 Problem 2

For any vectors  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$  and scalar  $\lambda \in \mathbb{R}$ ,

1.  $\mathbf{a} \cdot \mathbf{b} \in \mathbb{R}$  (Scalar).
2.  $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ ,  $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$ .
3.  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ , (commutativity)
4.  $\mathbf{a} \cdot (\lambda \mathbf{b}) = \lambda (\mathbf{a} \cdot \mathbf{b})$
5.  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ , (distributive)
6. If  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$  then  $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$  (cauchy-schwarz Inequality)
7. Hence, the angle  $\theta$  between  $\mathbf{a}, \mathbf{b}$  via  $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$  is well defined.

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8. For  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ ,  $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$  (Triangle inequality).
9. Use the dot product to prove that a real  $n \times n$  matrix  $Q$  is an orthogonal matrix if and only if  $Q\mathbf{x} \cdot Q\mathbf{x} = \mathbf{x} \cdot \mathbf{x}$  for all  $\mathbf{x} \in \mathbb{R}^n$ .

**Solution.** We first prove  $Q\mathbf{x} \cdot Q\mathbf{x} = \mathbf{x} \cdot \mathbf{x}$ ,  $\forall \mathbf{x} \in \mathbb{R}^n \Rightarrow Q$  is orthogonal.

$$\begin{aligned} Q\mathbf{x} \cdot Q\mathbf{x} &= \mathbf{x} \cdot \mathbf{x} \\ (Q\mathbf{x} \cdot Q\mathbf{x}) &= (\mathbf{x} \cdot \mathbf{x}) \\ (Q\mathbf{x})^T Q\mathbf{x} &= \mathbf{x}^T \mathbf{x} \\ \mathbf{x}^T Q^T Q\mathbf{x} &= \mathbf{x}^T \mathbf{x}. \end{aligned}$$

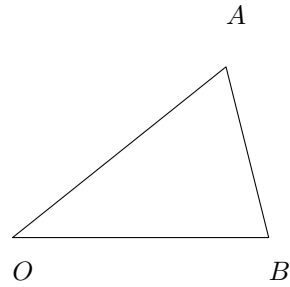
left multiplying  $(\mathbf{x}^T)^{-1}$  on both sides and right multiplying  $(\mathbf{x})^{-1}$  on both sides we get,

$$Q^T Q = I.$$

Note that the algebra above is reversible therefore the converse is also true.

### 1.3 Problem 3

Consider the points  $A(\mathbf{a})$ ,  $B(\mathbf{b})$  and the origina  $O$  and  $\triangle OAB$  and let  $\theta = \angle AOB$ . Use the Cosine Law to deduce  $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ .



**Solution.** Using the cosine rule we get,

$$\begin{aligned} |\mathbf{a} - \mathbf{b}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta \\ 2|\mathbf{a}||\mathbf{b}| \cos \theta &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2 \\ \cos \theta &= \frac{|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2}{2|\mathbf{a}||\mathbf{b}|} \\ \cos \theta &= \frac{\mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})}{2|\mathbf{a}||\mathbf{b}|} \\ &= \frac{2\mathbf{b} \cdot \mathbf{a}}{2|\mathbf{a}||\mathbf{b}|} \\ &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}. \end{aligned}$$

### 1.4 Problem 4

Show that a collineation determines a 1-1 correspondence from the set of all lines to itself.

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**Solution.** Clarify meaning of question.

### 1.5 Problem 5

Show that the lines with equations  $aX + bY + c = 0$  and  $dX + eY + f = 0$  are parallel iff  $ae - bd = 0$  and are perpendicular iff  $ad + be = 0$ .

**Solution.** We first rewrite the equations in the form  $Y = mX + b$ . We get  $Y = -\frac{a}{b}X - \frac{c}{b}$  and  $Y = -\frac{d}{e}X - \frac{f}{e}$ . We know these lines are parallel if,

$$\begin{aligned} -\frac{a}{b} &= -\frac{d}{e} \\ -ae &= -db \\ ae - db &= 0. \end{aligned}$$

Therefore, backwards implication proved. Forward implication is just obtained with the same algebra above since we can assume that the lines are parallel and gradients are equal once again giving us  $ae - db = 0$ .

Now we prove the condition for perpendicular lines. We know these lines are perpendicular if,

$$\begin{aligned} \frac{b}{a} &= -\frac{d}{e} \\ ad + be &= 0. \end{aligned}$$

Hence, backward implication proved. Forward implication once again proved by same algebra above through similar reasoning with forward implication of parallel lines.

### 1.6 Problem 6

A multiplication for finite group is often called Cayley table. Find the Cayley table for the group  $V_4 = \{1, \sigma, \alpha, \beta\}$  which is defined by the transformations

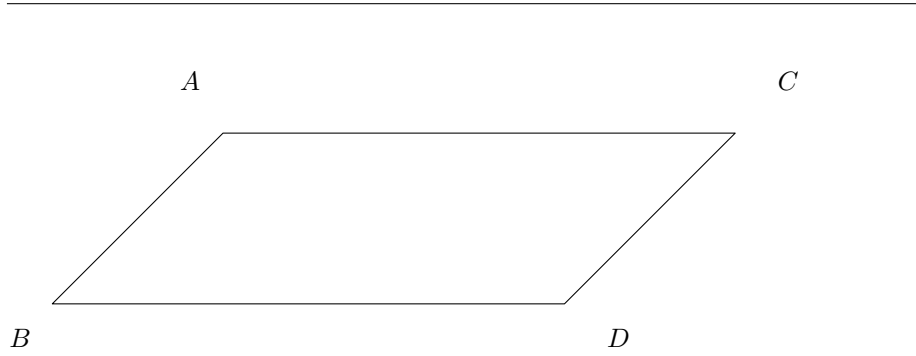
$$1(x, y) = (x, y), \sigma(x, y) = (-x, -y), \alpha(x, y) = (x, -y), \beta(x, y) = (-x, y).$$

For all  $(x, y) \in \mathbb{R}^2$ .

	$V_4$	1	$\sigma$	$\alpha$	$\beta$
	1	1	$\sigma$	$\alpha$	$\beta$
<b>Solution.</b>	$\sigma$	$\sigma$	1	$\beta$	$\alpha$
	$\alpha$	$\alpha$	$\beta$	1	$\sigma$
	$\beta$	$\beta$	$\alpha$	$\sigma$	1

### 1.7 Problem 7

Suppose  $A, B, C$  are non collinear points. Then  $\tau_{A,B} = \tau_{C,D} \Leftrightarrow \square ABCD$  is a parallelogram.



**Solution.** We show forward implication first.

$$\tau_{A,B} = \tau_{C,D} \Rightarrow \overrightarrow{AB} = \overrightarrow{CD}.$$

So,  $|\overrightarrow{AB}| = |\overrightarrow{CD}|$  and  $AB \parallel CD$ . Now, it remains to show  $AB$  and  $CD$  do not lie on the same line. We know that  $A, B, C$  are non collinear. We show that  $A, B, D$  are non collinear. Assume that  $A, B, D$  are collinear, then  $A, D, C$  are non collinear and form a triangle. Then  $CD$  intersects  $AD$  and hence  $\ell(AB)$  as  $B$  lies on  $\ell(AD)$ . If  $CD$  intersects  $\ell(AB)$  then  $AB$  and  $CD$  cannot be parallel which is a contradiction as we have already proven them to be parallel. Therefore our assumption that  $A, B, D$  are collinear is false. So,  $A, B, C$  are non collinear and  $A, B, D$  are non collinear. Hence  $CD$  does not lie  $\ell(AB)$ . Therefore  $AB$  and  $CD$  are congruent and parallel sides of a parallelogram  $CABD$ .

Backward implication follows immediately from the congruence of  $AB$  and  $CD$  and therefore their vectors being equal as they are two sides of a parallelogram. Giving is the equality of the isometry.