LECTURE 0

Assumed Knowledge

This is a review of basic facts about complex numbers that ought to be familiar:

- the definition of complex numbers,
- their arithmetic,
- Cartesian and polar representations,
- the Argand diagram,
- de Moivre's theorem, and
- extracting nth roots of complex numbers.

Students who do not feel confident about this material need to do lots of exercises about these, such as those in the MATH1141 notes.

1. Complex numbers

DEFINITION 0.1. A *complex number* is an expression of the form $x + iy$, where x and y are real numbers. The *real part* of $x + iy$ is x and the *imaginary part* of $x + iy$ is y. We denote this by $\mathrm{Re}(x + iy) = x$ and $\mathrm{Im}(x + iy) = y$. The set of all complex numbers is denoted $\mathbb{C}$.

We often write $w = u + iv$ and $z = x + iy$, and work with w and z rather than $u + iv$ and $x + iy$. In this case, we write, for instance, $\mathrm{Re}(z) = x$ and $\mathrm{Im}(w) = v$. We abbreviate $x + i0$ and $0 + iy$ to x and iy, and $0 + i1$ to i.

DEFINITION 0.2. Suppose that $w = u + iv$ and $z = x + iy$, where $u, v, x, y \in \mathbb{R}$. Then we define the complex numbers $w + z$, $-z$, wz and, if $z \neq 0$, z^{-1} and w/z, as follows:

$$\begin{aligned}
w + z &= (u + x) + i(v + y) \\
-z &= -x + i(-y) \\
wz &= (ux - vy) + i(uy + vx) \\
z^{-1} &= (x^2 + y^2)^{-1}(x - iy) \\
w/z = wz^{-1} &= (x^2 + y^2)^{-1}[(ux + vy) + i(vx - uy)].
\end{aligned}$$

Then $i^2 = -1$ and $-z = (-1)z$.

We may combine these operations to make sense of more complicated expressions such as $w^m - z^n$, where m and n are integers.

PROPOSITION 0.3. *Complex numbers have the following properties:*

$$\begin{aligned}
z_1 + z_2 &= z_2 + z_1 && \forall z_1, z_2 \in \mathbb{C} \\
(z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3) && \forall z_1, z_2, z_3 \in \mathbb{C} \\
0 + z &= z && \forall z \in \mathbb{C} \\
(-z) + z &= 0 && \forall z \in \mathbb{C} \\
z_1 z_2 &= z_2 z_1 && \forall z_1, z_2 \in \mathbb{C} \\
(z_1 z_2) z_3 &= z_1 (z_2 z_3) && \forall z_1, z_2, z_3 \in \mathbb{C} \\
1z &= z && \forall z \in \mathbb{C} \\
z z^{-1} &= 1 && \forall z \in \mathbb{C} \setminus \{0\} \\
z_1(z_2 + z_3) &= z_1 z_2 + z_1 z_3 && \forall z_1, z_2, z_3 \in \mathbb{C}
\end{aligned}$$

The symbol $\forall$ is read “for all”. This proposition shows that the complex numbers form a *field*.

DEFINITION 0.4. If $z = x + iy$, then $\overline{z}$, the *(complex) conjugate* of z, and $|z|$, the *modulus* of z, are defined to be $x - iy$ and $(x^2 + y^2)^{1/2}$.

Some further properties of complex numbers relate conjugates and moduli.

PROPOSITION 0.5. *For all* $z, z_1, z_2 \in \mathbb{C}$,

(a)	$\operatorname{Re}(z) = \dfrac{z + \overline{z}}{2}$	(b)	$\operatorname{Im}(z) = \dfrac{z - \overline{z}}{2i}$
(c)	$\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$	(d)	$\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$
(e)	$\overline{-z} = -\overline{z}$	(f)	$\overline{z^{-1}} = (\overline{z})^{-1}$
(g)	$\lvert z_1 + z_2\rvert \leq \lvert z_1\rvert + \lvert z_2\rvert$	(h)	$\lvert z_1 z_2\rvert = \lvert z_1\rvert\,\lvert z_2\rvert$
(i)	$z\overline{z} = \lvert z\rvert^2$	(j)	$z^{-1} = \lvert z\rvert^{-2}\overline{z}.$

Inequality (g) is called the *triangle inequality*. If $|z| = 1$, then $z^{-1} = \overline{z}$, from (j).

PROOF. We only prove the triangle inequality, because this is hardest.

First, here is an algebraic proof. Recall that $2ab \leq a^2 + b^2$ for real numbers a and b. Taking a and b to be $x_1 y_2$ and $x_2 y_1$, we deduce that

$$2x_1 x_2 y_1 y_2 \leq x_1^2 y_2^2 + x_2^2 y_1^2,$$

and hence

$$x_1^2 x_2^2 + y_1^2 y_2^2 + 2x_1 x_2 y_1 y_2 \leq x_1^2 x_2^2 + y_1^2 y_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2 = (x_1^2 + y_1^2)(x_2^2 + y_2^2),$$

so, taking square roots,

$$x_1 x_2 + y_1 y_2 \leq |z_1|\,|z_2|\,.$$

Finally, $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$, and so

$$\begin{aligned}
|z_1 + z_2|^2 &= (x_1 + x_2)^2 + (y_1 + y_2)^2 \\
&= x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2(x_1 x_2 + y_1 y_2) \\
&\leq |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2| \\
&= \big(|z_1| + |z_2|\big)^2;
\end{aligned}$$

the triangle inequality follows by taking square roots.

Alternatively, here is a geometric version. Consider the triangle whose vertices are w, z, and $w+z$, and the parallelogram with vertices 0, w, z, and $w+z$. The side of the parallelogram that joins w to $w+z$ is congruent to the side joining 0 to z, and the side of the parallelogram that joins z to $w+z$ is congruent to the side joining 0 to w. So we are just asserting the obvious fact that the length of one side of a triangle is less than the sum of the other two sides. $\square$

2. Euler's formula

The usual exponential function has a Taylor series:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

so at least formally, for a real number θ,

$$\begin{aligned} e^{i\theta} &= 1 + i\frac{\theta}{1!} - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots \\ &= \Big(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\Big) + i\Big(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\Big) \\ &= \cos\theta + i\sin\theta, \end{aligned}$$

from the Taylor series for the cosine and sine functions.

Later we will make this rigorous and use power series very effectively. This observation leads us to make the following definition.

Definition 0.6. We define $e^{i\theta}$ to be $\cos\theta + i\sin\theta$, for any real number θ.

Suppose that $z = x + iy \neq 0$. Write r instead of $|z|$. Then $(x/r, y/r)$ lies on the unit circle in the Cartesian plane, so $(x/r, y/r) = (\cos\theta, \sin\theta)$ for an appropriate choice of θ, and hence

$$z = r\Big(\frac{x}{r} + i\frac{y}{r}\Big) = r\big(\cos\theta + i\sin\theta\big) = re^{i\theta}.$$

Definition 0.7. The *Cartesian form* of a complex number z is its representation in the form $x+iy$, where x and y are real. The *polar form* of a complex number z is its representation in the form $re^{i\theta}$, where $r \geq 0$ and $\theta \in \mathbb{R}$. The number θ is called the *argument* of z, and is written $\arg(z)$.

Lemma 0.8. *Suppose that $r, s \in \mathbb{R}^+$ and $\theta, \phi \in \mathbb{R}$. Then*

$$r(\cos\theta + i\sin\theta) = s(\cos\phi + i\sin\phi)$$

if and only if $r = s$ and $\theta - \phi = 2k\pi$ for some $k \in \mathbb{Z}$.

This lemma follows immediately from trigonometry. It tells us that the argument of a nonzero complex number z is ambiguous. The next definition is to avoid this ambiguity.

Definition 0.9. The *principal value* of the argument of a nonzero complex number z, written $\mathrm{Arg}(z)$, is the unique number θ such that $z = |z|\, e^{i\theta}$ and $-\pi < \theta \leq \pi$.

We do not define the argument of 0.

3. The Argand Diagram

Suppose that $w = u + iv$. Then to the complex number w we associate the point in the Cartesian plane with Cartesian coordinates (u, v). When we do this, we call the axes the real axis and the imaginary axis. See Figure 0.1.

Geometrically, $|w|$ is the length of the line joining w to O, and $\arg(w)$ is the angle between this line and the positive real axis (taking the anticlockwise direction to be positive). Further, $|w - z|$ is the length of the line joining w and z.

Adding two complex numbers corresponds to vector addition in the plane. Multiplying by r in $\mathbb{R}^+$ dilates by a factor of r, and multiplying by the complex number $e^{i\theta}$ (where $\theta \in \mathbb{R}$) rotates (anticlockwise) through the angle θ.

4. De Moivre's formula

THEOREM 0.10. *If $\theta, \phi \in \mathbb{R}$, then*

$$(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos(\theta + \phi) + i\sin(\theta + \phi).f$$

PROOF. For all $\theta, \phi \in \mathbb{R}$,

$$\begin{aligned} &(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi) \\ &= \cos(\theta + \phi) + i\sin(\theta + \phi), \end{aligned}$$

as required. □

COROLLARY 0.11 (de Moivre's formula). *If $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$, then*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{0.1}$$

PROOF. This is obviously true if $n = 0$ or 1. The result may be proved for $n \in \mathbb{Z}^+$ by induction. Suppose that $k \in \mathbb{Z}^+$ and formula (0.1) holds when $n = k$, i.e.,

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta.$$

Then by Theorem 0.10,

$$\begin{aligned} (\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^k \\ &= (\cos\theta + i\sin\theta)(\cos k\theta + i\sin k\theta) \\ &= \cos(\theta + k\theta) + i\sin(\theta + k\theta) \\ &= \cos(k+1)\theta + i\sin(k+1)\theta, \end{aligned}$$

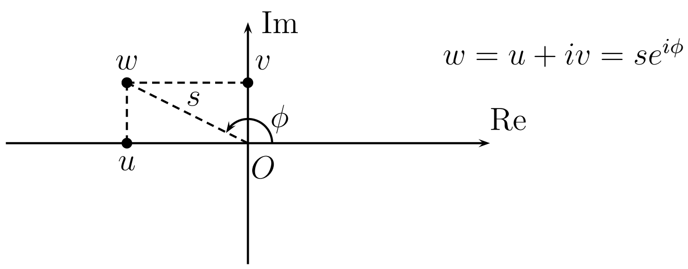

FIGURE 0.1. The complex plane

so the result holds when $n = k + 1$. By induction, the result holds for all $n \in \mathbb{Z}^+$.

To prove the result when $n \in \mathbb{Z}^-$, we use the fact that if $|z| = 1$, then $z\overline{z} = 1$, so $\overline{z} = z^{-1}$. That is,

$$(\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta),$$

as required. □

In polar notation, Theorem 0.10 and Corollary 0.11 become

$$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi} \qquad \text{and} \qquad (e^{i\theta})^n = e^{in\theta}.$$

These are more "obvious" and easier to remember than the trigonometric formulae.

5. Roots of complex numbers

We use complex exponentials to find roots of complex numbers. Fix $w \in \mathbb{C}$ and $n \in \mathbb{Z}^+$, and suppose that $w = z^n$ for some $z \in \mathbb{C}$. Then z is called *an nth root of w*. Write z as $re^{i\theta}$ and w as $se^{i\phi}$. Then

$$se^{i\phi} = w = z^n = r^n e^{in\theta}.$$

From Lemma 0.8, $r = s^{1/n}$ and $n\theta = \phi + 2k\pi$ for some $k \in \mathbb{Z}$. Therefore $z = s^{1/n}e^{i\theta}$, where

$$\theta = \frac{\phi}{n} + \frac{2k\pi}{n}$$

for some $k \in \mathbb{Z}$. If $k = nl + r$, where $l \in \mathbb{Z}$ and $r = 0, 1, 2, \dots, n-1$, then

$$e^{(\phi/n+2k\pi/n)} = e^{(\phi/n+2r\pi/n+2l\pi)} = e^{(\phi/n+2r\pi/n)},$$

so we get the same value of z by taking $k = nl + r$ as by taking $k = r$. Thus, to get all n possible values of z, it suffices to take the first n values of k, or any n consecutive values, or indeed any n values of k which give the n possible different remainders when divided by n.

The nth roots of any nonzero complex number w are uniformly spaced around the circle with centre 0 and radius $|w|^{1/n}$. For example, Figure 0.2 shows the seventh roots of unity. A symmetry argument shows that the sum of all these numbers is 0.

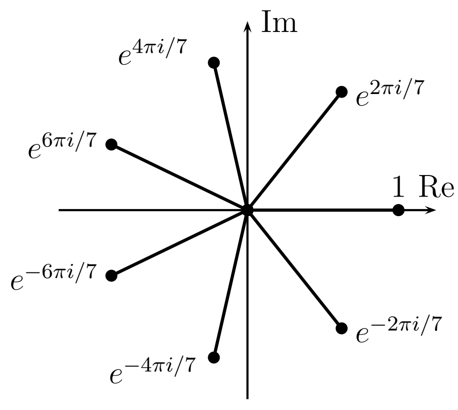

Figure 0.2. The seventh roots of unity

6. History†

Leonhard Euler found the formula that bears his name in 1748; he was one of the most prolific mathematicians ever. Some mathematicians did not believe that the geometric representation afforded by the Argand diagram was legitimate. The story of de Moivre's death is very curious. For more information, see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/`
and then find Euler, Argand, and de Moivre.

LECTURE 1

Inequalities and Sets of Complex Numbers

In complex analysis, we consider functions whose domains or ranges or both are regions in the complex plane. So to be able to discuss functions, we need to be able to describe regions. Curves and regions in the complex plane are often described by equalities and inequalities involving $|\cdot|$, Arg, Re, Im, ...

In the first part of this lecture, we review some equalities and inequalities. Then we discuss different types of regions. Finally, we consider some examples.

1. Equalities and inequalities

We begin with a lemma that is related to the cosine rule. It implies that $\operatorname{Re}(w\bar{z})$ is the inner product of the vectors represented by the complex numbers w and z.

Lemma 1.1. *For all complex numbers w and z,*

$$|w+z|^2 = |w|^2 + 2\operatorname{Re}(w\bar{z}) + |z|^2.$$

Proof. Observe that

$$\begin{aligned} |w+z|^2 &= (w+z)(\bar{w}+\bar{z}) = w\bar{w} + w\bar{z} + \bar{w}z + z\bar{z} \\ &= |w|^2 + w\bar{z} + (w\bar{z})^{-} + |z|^2 = |w|^2 + 2\operatorname{Re}(w\bar{z}) + |z|^2, \end{aligned}$$

as required. □

The triangle inequality is one of the most useful results about complex numbers. It states:

$$|w+z| \leq |w| + |z| \qquad \forall w, z \in \mathbb{C}.$$

Here are a variation on the triangle inequality, sometimes called the circle inequality.

Lemma 1.2. *For all complex numbers w and z,*

$$\big||w| - |z|\big| \leq |w-z|.$$

Proof. Observe that $w = (w-z)+z$, so $|w| \leq |w-z| + |z|$ by the triangle inequality, and hence

$$|w| - |z| \leq |w-z|. \tag{1.1}$$

Interchanging the roles of w and z in (1.1),

$$|z| - |w| \leq |z-w|.$$

Combining these inequalities and recalling that $|w-z| = |z-w|$, we see that

$$\big||w| - |z|\big| = \max\{|w|-|z|, |z|-|w|\} \leq |w-z|,$$

as required.

Alternatively, consider points on circles of radii $|z|$ and $|w|$, and compare the distance between the points with the difference of the radii. □

Recall that if $z = x + iy$, then e^z is defined to be $e^x(\cos y + i \sin y)$. Then $e^w e^z = e^{w+z}$ for all complex numbers w and z. Here is another very useful result.

LEMMA 1.3. *If $z \in \mathbb{C}$, then*

$$|e^z| = e^{\mathrm{Re}(z)}.$$

PROOF. See Problem Sheet 1. ☐

LEMMA 1.4. *For all real numbers θ,*

$$\left|e^{i\theta} - 1\right| \le |\theta|.$$

PROOF. See Problem Sheet 1. ☐

2. Properties of sets

DEFINITION 1.5. The *open ball* with centre z_0 and radius ε, written $\mathrm{B}(z_0, \varepsilon)$, is the set $\{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$.

The *punctured open ball* with centre z_0 and radius ε, written $\mathrm{B}^\circ(z_0, \varepsilon)$, is the set $\{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$.

Sometimes these sets are called discs rather than balls.

DEFINITION 1.6. Suppose that $S \subseteq \mathbb{C}$. For any point z_0 in $\mathbb{C}$, there are three mutually exclusive and exhaustive possibilities:

(1) When the positive real number ε is sufficiently small, $\mathrm{B}(z_0, \varepsilon)$ is a subset of S, that is, $\mathrm{B}(z_0, \varepsilon) \cap S = \mathrm{B}(z_0, \varepsilon)$. In this case, z_0 is an *interior point* of S.
(2) When the positive real number ε is sufficiently small, $\mathrm{B}(z_0, \varepsilon)$ does not meet S, that is, $\mathrm{B}(z_0, \varepsilon) \cap S = \emptyset$. In this case, z_0 is an *exterior point* of S.
(3) No matter how small the positive real number ε is, neither of the above holds, that is, $\emptyset \subset \mathrm{B}(z_0, \varepsilon) \cap S \subset \mathrm{B}(z_0, \varepsilon)$. In this case, z_0 is a *boundary point* of S.

These definitions are illustrated in Figure 1.1. We consider points z_1, z_2 and z_3.

If the radius of the ball centred at z_1 is small enough, then the ball lies inside the set S, and $\mathrm{B}(z_1, \varepsilon) \cap S = \mathrm{B}(z_1, \varepsilon)$. Thus z_1 is an interior point.

If the radius of the ball centred at z_2 is small enough, then the ball lies outside the set S, and $\mathrm{B}(z_2, \varepsilon) \cap S$ is empty. Thus z_2 is an exterior point.

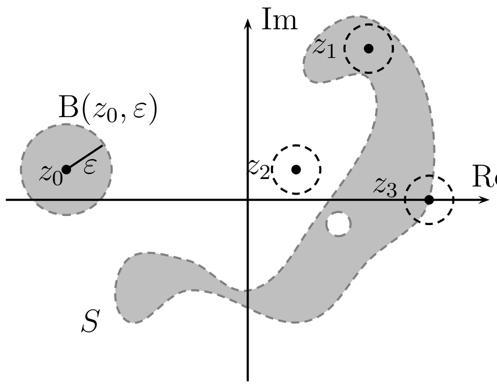

FIGURE 1.1. The ball with centre z_0 and radius ε, and interior, exterior and boundary points z_1, z_2 and z_3 of the set S

No matter how small the radius of the ball centred at z_3 is, part of the ball lies inside S, and part lies outside S, and $\mathrm{B}(z_3, \varepsilon) \cap S$ is neither empty nor all of $\mathrm{B}(z_3, \varepsilon)$. Thus z_3 is a boundary point.

DEFINITION 1.7. Suppose that $S \subseteq \mathbb{C}$.

(1) The set S is *open* if all its points are interior points.
(2) The set S is *closed* if it contains all of its boundary points, or equivalently, if its complement $\mathbb{C} \setminus S$ is open.
(3) The *closure* of S, written $\bar{S}$, is the set consisting of all the points of S together with all its boundary points.
(4) The set S is *bounded* if $S \subseteq \mathrm{B}(0, R)$ for some positive real number R.
(5) The set S is *compact* if it is both closed and bounded.
(6) The set S is a *region* if it is an open set together with none, some, or all of its boundary points.

For example, the dashed boundary lines of the set S in Figure 1.1 indicate that it does not contain any boundary points. Consequently, this set is open.

Note that open and closed are not exclusive nor exhaustive. There are sets that are open and closed, such as the whole plane, and sets that are neither open nor closed, such as $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) > 0\}$. In complex analysis, we focus on open sets. We often write Ω for an open set.

We now present connectedness.

DEFINITION 1.8. A *polygonal path* is a finite sequence of finite line segments, where the end point of one line segment is the initial point of the next one. A *simple closed polygonal path* is a polygonal path that does not cross itself, but the final point of the last segment is the initial point of the first segment. The complement of a simple closed polygonal path is made up of two pieces: one, the *interior of the path*, is bounded, and the other, the *exterior*, is not.

DEFINITION 1.9. Let $X \subseteq \mathbb{C}$ be a subset of the complex plane.

(1) The set X is *polygonally path-connected* if any two points of X can be joined by a polygonal arc lying inside X.
(2) The set X is *simply polygonally connected* if it is polygonally path-connected and if the interior of every simple closed polygonal arc in X lies in X, that is, if "X has no holes".
(3) The set X is a *domain* if it is open and polygonally path-connected.

The set Ω in Figure 1.2 is polygonally path-connected, because any two points in Ω (such as z_1 and z_4) can be joined by a polygonal path. However, Ω is not simply polygonally connected, because part of the interior of the closed path shown going through z_5 is not in Ω.

3. Describing sets in the complex plane

EXERCISE 1.10. Suppose that $a, b, c, d \in \mathbb{C}$. Show that the set

$$\{z \in \mathbb{C} : |az + b| = |cz + d|\}$$

may be empty, a point, a line, a circle, or the whole complex plane, and all these possibilities occur for suitable values of the parameters a, b, c, d.

ANSWER.

EXERCISE 1.11. Sketch the set $\{z \in \mathbb{C} : |z - 3 - 2i| < 4,\ \mathrm{Re}(z) > 0\}$ in the complex plane. Is it open, closed, bounded, compact, polygonally path-connected, simply polygonally connected, a region, or a domain?

ANSWER.

$\triangle$

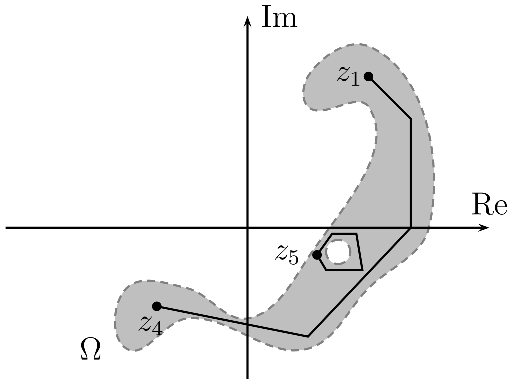

FIGURE 1.2. A polygonally path-connected, but not simply polygonally connected, set

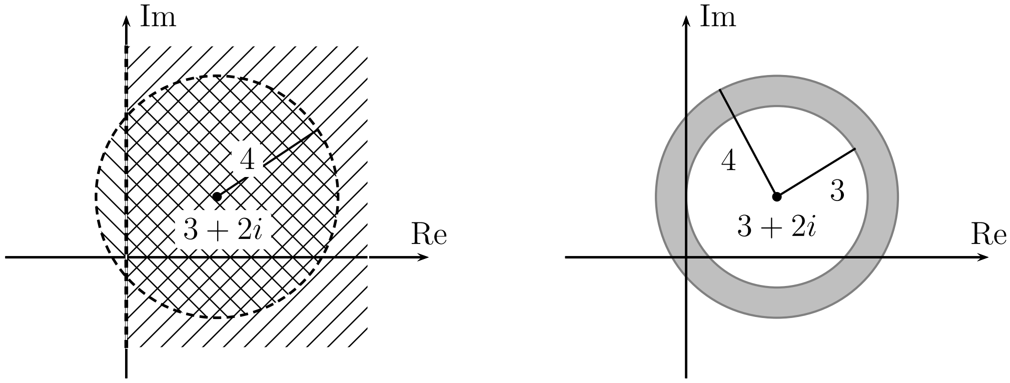

FIGURE 1.3. Two regions defined by inequalities

EXERCISE 1.12. Sketch the set $\{z \in \mathbb{C} : 4 \leq |z - 3 - 2i| \leq 5\}$ in the complex plane. Is it open, closed, bounded, compact, polygonally path-connected, simply polygonally connected, a region or a domain?

ANSWER.

Here is a more complicated example, related to conic sections.

EXERCISE 1.13. Sketch the set $\{z \in \mathbb{C} : |z + i| + |z - i| = 4\}$ in $\mathbb{C}$. Is it open or bounded? Describe the set $\{z \in \mathbb{C} : |z+i|+|z-i| < 4\}$. Is it open, closed, bounded, compact, polygonally path-connected, simply polygonally connected, a region or a domain?

ANSWER.

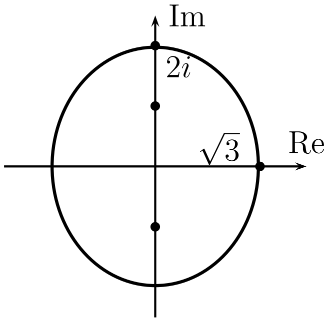

FIGURE 1.4. An ellipse

LECTURE 2

Functions of a complex variable

In this lecture, we introduce functions of a complex variable, and recall concepts such as domain and range. We examine some examples and consider the problem of estimating the size of a complex function.

1. Functions

In Mathematics, we often think of functions as machines: you give the machine a number, x say, press a button, and out comes $f(x)$. We sometimes write $x \mapsto f(x)$ to indicate that x is the input and $f(x)$ is the output.

- The *domain* of a function f, written $\mathrm{Domain}(f)$, is the set of all the numbers you are allowed to put in. Sometimes this is restricted in some way. If there is no explicit restriction, you should consider the *natural domain*, that is, the largest domain possible.
- A *codomain* is a set of numbers that includes all the numbers that you can get out, and perhaps more.
- The *range* or *image* of a function f, written $\mathrm{Range}(f)$, is the set of the numbers that you can get out, and no others.
- The *image* of a subset S of the domain of a function f, sometimes written $f(S)$, is the set of all possible values $f(s)$ as s varies over S.
- The *preimage* of a subset T of the codomain of a function f, sometimes written $f^{-1}(T)$, is the set of all x in $\mathrm{Domain}(f)$ such that $f(x) \in T$.

Definition 2.1. A *complex function* is one whose domain, or whose range, or both, is a subset of the complex plane $\mathbb{C}$ that is not a subset of the real line $\mathbb{R}$. To emphasize that the domain is complex, not real, the expression *function of a complex variable* may be used. To emphasize that the range is complex, not real, the expression *complex-valued function* may be used.

Remark 2.2. The word "domain" is then used in two different ways in this course. We say that a subset of $\mathbb{C}$ is a domain if it is open and connected which has nothing to do with a function. We just saw that the domain of a function f is the set of points where f is defined. Be careful to not confuse them and note that the domain of a function is not necessarily a domain in the sense of open and connected: for instance consider $f : \{0, 1\} \to \mathbb{C}, 0 \mapsto 0, 1 \mapsto 1$ and not that its domain is equal to two points which forms a nonconnected and nonopen subset of $\mathbb{C}$.

2. Examples of functions

Examples of functions of a complex variable include the real part function Re, the imaginary part function Im, the modulus function $z \mapsto |z|$, and the principal value of the argument Arg; these are all real-valued. Complex conjugation $z \mapsto \bar{z}$ is an example of a complex-valued function of a complex variable.

In this course, we are going to learn about a number of useful complex functions. Shortly we will define complex polynomials and then rational functions. In future lectures, we will define $\log z$, $\sin z$, and $\cosh z$ for a complex number z, and there are many other functions in the menagerie of complex functions.

EXERCISE 2.3. Suppose that $f(z) = 1/z$ for all $z \in \mathbb{C} \setminus \{0\}$, and that $g(z) = z$ for all $z \in \mathbb{C}$. Show that $f \circ f(z) = z$ for all $z \in \mathbb{C} \setminus \{0\}$. Is $f \circ f = g$?

ANSWER.

DEFINITION 2.4. A *complex polynomial* is a function $p : \mathbb{C} \to \mathbb{C}$ of the form

$$p(z) = a_d z^d + \cdots + a_1 z + a_0,$$

where $a_d, \ldots, a_1, a_0 \in \mathbb{C}$. If $a_d \neq 0$, we say that p is of *degree* d. A *rational function* is a quotient of polynomials.

Sums, differences, products and compositions of polynomials are polynomials.

THEOREM 2.5 (The fundamental theorem of algebra). *Every nonconstant complex polynomial p of degree d factorizes uniquely: there exist $\alpha_1, \alpha_2, \ldots, \alpha_d$ and c in $\mathbb{C}$ such that*

$$p(z) = c \prod_{j=1}^{d} (z - \alpha_j) \qquad \forall z \in \mathbb{C}.$$

Equivalently, every nonconstant complex polynomial has at least one root.

In the factorisation above, the roots α_j may occur more than once. Thus we could also write

$$p(z) = c \prod_{j=1}^{e} (z - \alpha_j)^{m_j},$$

where the α_j are all distinct, and the *multiplicities* m_j add to give the degree of p.

THEOREM 2.6 (Polynomial division and partial fractions). *Suppose that p and q are polynomials of degrees m and n. Then the rational function p/q may be written as a sum*

$$\frac{p(z)}{q(z)} = s(z) + \frac{r(z)}{q(z)},$$

where r and s are polynomials, and the degree of r is strictly less than the degree of q. Further, if

$$q(z) = c \prod_{j=1}^{e} (z - \beta_j)^{m_j},$$

then we may decompose the term r/q into partial fractions:

$$\frac{r(z)}{q(z)} = \sum_{j=1}^{e} \sum_{k=1}^{m_j} \frac{a_{jk}}{(z - \beta_j)^k}\,.$$

At this stage, we do not prove these results, which should be familiar, though perhaps not in this generality; we will give proofs later.

The natural domain of any complex polynomial is $\mathbb{C}$. Sometimes we cannot determine the range of a real polynomial exactly, because we cannot find maxima or minima exactly. However, for complex polynomials, things are easier.

EXERCISE 2.7. Suppose that p is a nonconstant complex polynomial. Show that the range of p is $\mathbb{C}$.

ANSWER.

3. Real and imaginary parts

To a function $f : S \to \mathbb{C}$, where $S \subseteq \mathbb{C}$, we associate two *real-valued* functions u and v of *two real variables*:

$$f(x+iy) = u(x,y) + iv(x,y).$$

Then $u(x,y) = \operatorname{Re} f(x+iy)$ and $v(x,y) = \operatorname{Im} f(x+iy)$. It is very useful and very important to be able to view a complex-valued function of a complex variable in this way.

EXERCISE 2.8. Suppose that $f(z) = z$ and that $g(z) = z^2$. Find the real and imaginary parts of f and g.

ANSWER.

EXERCISE 2.9. Suppose that $f(z) = z^3 + \overline{z} - 2$. Write the real and imaginary parts of this function as functions u and v of (x,y), where $z = x+iy$.

ANSWER.

EXERCISE 2.10. Suppose that $f(z) = 1/z$. Write the real and imaginary parts of this function as functions of x and y, where $z = x+iy$.

ANSWER.

EXERCISE 2.11. Write e^z in the form $u(x,y)+iv(x,y)$, where $z=x+iy$.

ANSWER.

Sometimes we view the complex number z in polar coordinates, that is, we write $z=re^{i\theta}$. In this case, we consider

$$f(z)=u(r,\theta)+iv(r,\theta).$$

EXERCISE 2.12. Write e^z in the form $u(r,\theta)+iv(r,\theta)$, where $z=re^{i\theta}$.

ANSWER.

4. The function $z \mapsto 1/z$

It is obvious that if $w=1/z$, then $z=1/w$, and the function $z\mapsto 1/z$ is one-to-one (injective). Further, the domain and the range of the function are both equal to $\mathbb{C}\setminus\{0\}$.

EXERCISE 2.13. Suppose that z varies on the line $x=1$, and let $w=1/z$. Show that w varies on the circle $|w-\frac{1}{2}|=\frac{1}{2}$.

ANSWER.

We can reverse the argument and show that every point on the circle except 0 arises in this way. Thus the image of the line is the circle with the point 0 removed.

5. Fractional linear transformations

The fractional linear transformations form an important family of complex functions. These are the functions of the form

$$f(z)=\frac{az+b}{cz+d},$$

where $a,b,c,d\in\mathbb{C}$ and $ad-bc\neq 0$. We will study these functions in more detail later, but at the moment we just point out that if f is a fractional linear transformation and z varies on a line, then $f(z)$ varies on a line or on a circle. The same holds if z varies along a circle. Note that when $z\to -d/c$, then $cz+d\to 0$ and $f(z)\to\infty$ (we will define limits formally later). We can tell whether $f(z)$ varies on a line or on a circle as follows: if the points where z varies include $-d/c$, then the points where $f(z)$ varies will include ∞, and this means that $f(z)$ must vary on a line. Conversely, if the points where z varies do not include $-d/c$, then $f(z)$ will stay bounded, and this means that $f(z)$ must vary on a circle. Once we know whether $f(z)$ varies on a line or on a circle, we may find the equation of the line or the circle quite easily by finding a few values of $f(z)$.

EXAMPLE 2.14. Let $f(z) = 1/z$. As z varies on the line $x = 1$, its image $f(z)$ varies on a circle, because z stays away from 0 and so $1/z$ stays away from ∞. This circle passes through the points 1 and 0 (since $f(z) \to 0$ as $z \to \infty$), and is symmetric about the real axis, since $1/(1 - it) = (1/(1 + it))^{-}$. This must be the circle that we found above.

6. Estimating the size of the values of a function

We will need to use what we know about inequalities to estimate how large the values of a complex function are.

EXERCISE 2.15. Suppose that $f(z) = \dfrac{1}{z^4 - 1}$ for all $z \in \mathbb{C} \setminus \{\pm 1, \pm i\}$. Show that

$$|f(z)| \leq \frac{1}{15}$$

if $|z| \geq 2$ (that is, if z lies on or outside the circle with centre 0 and radius 2).

ANSWER.

EXERCISE 2.16. Suppose that $p(z) = 10z^4 - 3z^3 + z - 10$. Show that when $|z|$ is large enough, $|p(z)| \leq 11|z|^4$.

ANSWER.

LECTURE 3

Sketching complex functions

We often use graphical methods to gain useful intuition about complex functions, and we spend some time investigating these. It is hard to represent complex functions, because there are up to four variables involved. Just as we often write $y = f(x)$ for a real function, it is common to consider a function in the form $w = f(z)$, and to use real variables x and y to describe the domain and u and v to describe the range. Typically, we draw "elementary" curves in the z plane, such as lines parallel to the axes, or concentric circles around and rays exiting from the origin, and then examine their images in the w plane, or we draw similar elementary curves in the w plane and then examine their preimages.

1. Linear and affine mappings

Suppose that $a \neq 0$, and consider the bijective linear map $z \mapsto az$. We write $a = c + id$ and $z = x + iy$; then

$$az = (c + id)(x + iy) = (cx - dy) + i(cy + dx).$$

In Cartesian coordinates, the map may be written

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that

$$\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $r = \sqrt{c^2 + d^2}$, while $\cos\theta = c/r$ and $\sin\theta = d/r$. Hence multiplying by the matrix

$$\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

is the same as rotating through the angle θ and then dilating by the real number r. This corresponds to the representation of a in the form $re^{i\theta}$, where $r = |a|$ and $\theta = \operatorname{Arg}(a)$.

An affine map of the complex plane is a map of the form $z \mapsto az + b$; such mappings are also bijective (as long as $a \neq 0$). We may also represent this as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. We write $a = c + id$ and $b = e + if$. Then in Cartesian coordinates, we have

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}.$$

It is an exercise in algebra to see that the image of a line under an affine mapping is a line, and the image of a circle under an affine mapping is a circle.

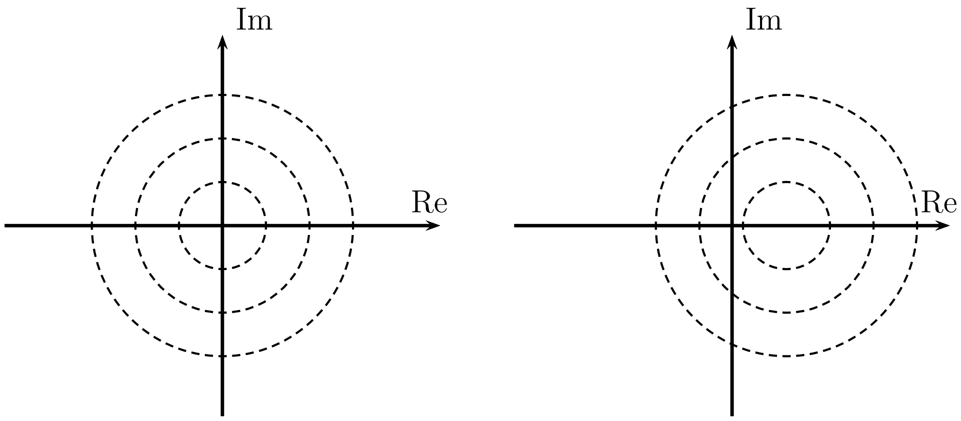

FIGURE 3.1. The translation $z \mapsto z + 1$

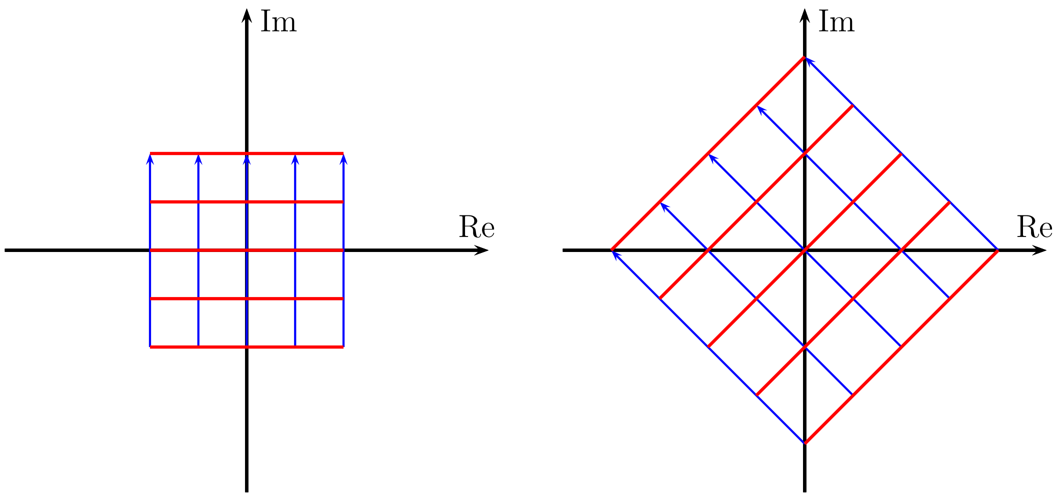

FIGURE 3.2. The multiplication $z \mapsto (1 + i)z$

The inverse of an affine mapping is an affine mapping. It follows that the preimages of lines are lines and preimages of circles are circles; the preimage of a grid parallel to the axes is a rectangular grid, but not necessarily parallel to the axes.

In Figures 3.1 and 3.2, we illustrate the images of lines parallel to the axes and circles around the origin under affine mappings.

This is a good way to show how the function behaves, although a lot of space is used and care is needed to choose the points that are moved in a way that is not ambiguous. Indeed, it might be better to draw a very asymmetrical figure in the z plane and then its image in the w plane.

Sometimes we just draw the right hand figure of the two drawn above, labelling the curves with the corresponding curve in the domain of the function. In Figure 3.1, these are the circles $r = 1$, $r = 2$, and $r = 3$; in Figure 3.2, they are the horizontal lines $x = 0$, $x = \pm 1$, $x = \pm 2$, and the vertical lines $y = 0$, $y = \pm 1$, $y = \pm 2$.

On the other hand, we may look for curves in the xy plane whose images in the uv plane are the lines $u = c$ and $v = d$. So we are finding the level curves of the real and imaginary parts of the function. People who are used to map reading can build a picture in their mind of terrain, just knowing the contours that represent different heights. They imagine a surface above the page at the height indicated by the contour, and then fill in the gaps.

2. Quadratic functions

Now we consider the function $z \mapsto z^2$. Notice that this function is two-to-one in $\mathrm{B}^\circ(0,\infty)$.

On the one hand, we may represent the images in the uv plane of the curves in the xy plane given by $x = a$ and $y = b$, or by $r = a$ and $\theta = b$. For instance, if $x = a$ and $y = t$, where a is fixed and t varies, then

$$w = z^2 = (a + it)^2 = a^2 - t^2 + 2iat.$$

That is, $u = a^2 - t^2$ and $v = 2at$. We eliminate t to show that

$$u = a^2 - \frac{v^2}{4a^2}\,.$$

Alternatively, if $y = b$ and $x = t$, where b is fixed and t varies, then

$$w = z^2 = (t + ib)^2 = t^2 - b^2 + 2ibt.$$

That is, $u = t^2 - b^2$ and $v = 2bt$. We eliminate t to show that

$$u = \frac{v^2}{4b^2} - b^2.$$

See, for example, Figure 3.3.

Exercise† 3.1. Find the focus and the directrix of the parabola $u = v^2/4b^2 - b^2$ in the uv plane.

On the other hand, we may look for the values of x and y so that $\mathrm{Re}(z^2)$ or $\mathrm{Im}(z^2)$ takes a fixed value. For instance, if $\mathrm{Re}(z^2) = a$, then

$$x^2 - y^2 = a,$$

and this is a hyperbola opening to the left and right, or up and down, depending on the sign of a. Similarly, if $\mathrm{Im}(z^2) = b$, then

$$2xy = b,$$

and this is a right hyperbola in the first and third quadrants, or in the second and fourth quadrants, depending on the sign of b. See, for example, Figure 3.3.

Exercise 3.2. How would you “sketch the graph” of $w = (z - 1)^2 - 1$?

3. The function $w = 1/z$

As we saw in the last lecture, the function $w = 1/z$ is one-to-one. Further, it sends lines through the origin to lines through the origin. Actually, this is easiest to see using polar coordinates: as r varies in $\mathbb{R}^+$, the point $z = re^{i\theta}$ varies along a ray from the origin. Now $w = (1/r)e^{-i\theta}$, and $1/r$ also varies in $\mathbb{R}^+$ and this point too varies along a ray. However the new ray is the reflection of the old ray in the real axis. Since moreover diametrically opposed rays are mapped to diametrically opposed rays, lines through the origin do indeed go to lines through the origin.

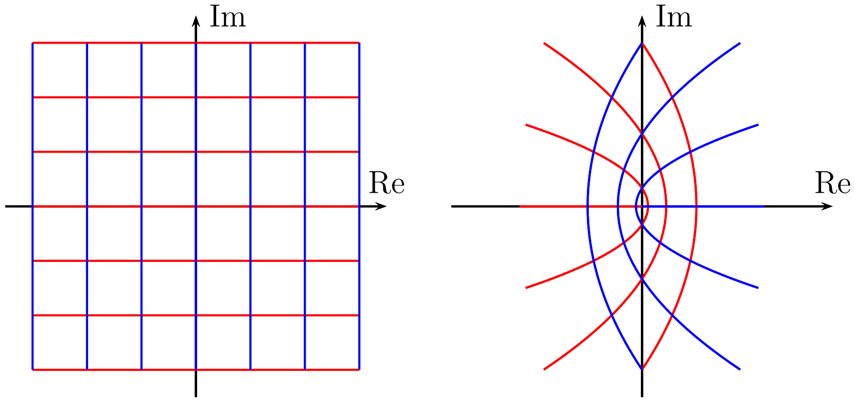

FIGURE 3.3. Images of lines $x = c$ and $y = d$ for $w = z^2$

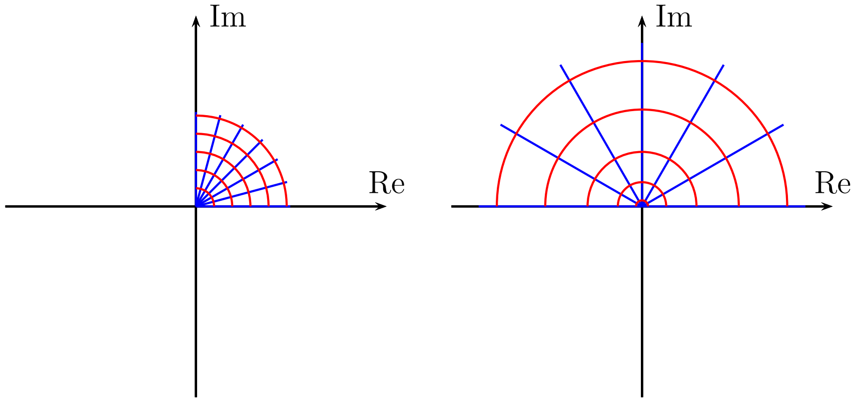

FIGURE 3.4. Images of curves $r = c$ and $\theta = d$ for $w = z^2$

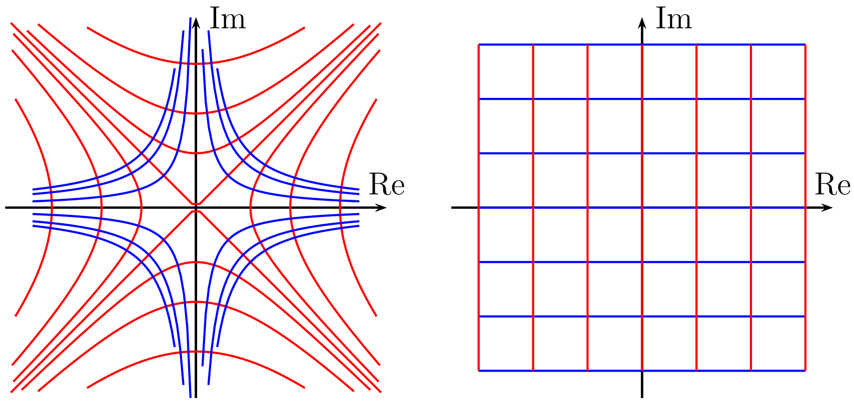

FIGURE 3.5. The level curves for $\mathrm{Re}(z^2)$ and $\mathrm{Im}(z^2)$

Similarly, as θ varies, the point $re^{i\theta}$ moves around a circle centred at the origin. Now $w = (1/r)e^{-i\theta}$, and this point moves around the same circle, but in the opposite direction.

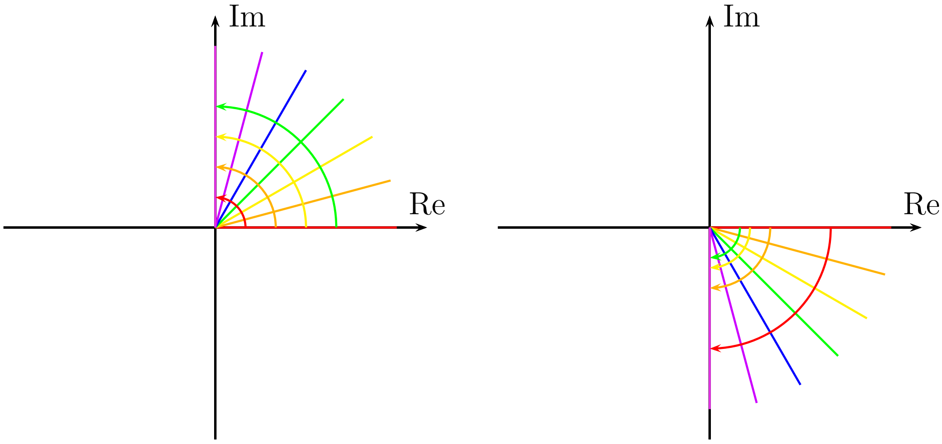

FIGURE 3.6. Images of curves $r = c$ and $\theta = d$ for $w = 1/z$

We represent this graphically in Figure 3.6.

For completeness, we now describe the images of lines and circles under the map $w = 1/z$ in more detail.

LEMMA 3.3. *Consider the mapping $w = 1/z$.*

(1) The image of a line through 0 (not including 0) is a line through 0 (not including 0).
(2) The image of a line that does not pass through 0 is a circle through 0, with 0 removed. If p is the closest point on the line to 0, then the line segment between 0 and $1/p$ is a diameter of the circle.
(3) The image of a circle that passes through 0 is a line. If q is the furthest point on the circle from 0, then the closest point on the line to 0 is $1/q$.
(4) The image of a circle that does not pass through 0 is a circle. If p and q are the points on the circle closest to and furthest from 0, then the points on the image circle closest to and furthest from 0 are $1/q$ and $1/p$.

PROOF. See the exercise sheet. $\square$

EXERCISE 3.4. Suppose that z varies on the line $ax + by = c$, where $a, b, c \in \mathbb{R}$, and let $w = 1/z$. Show that w varies on a line when $c = 0$ and on a circle otherwise.

ANSWER.

4. The exponential function

The exponential function $w = e^z$ is ∞-to-1; that is, infinitely many different points in the xy plane are sent to the same point in the uv plane. See Figure 3.7 for a graphical representation.

5. More on graphical representations of complex functions

There are many good web-sites that explore different ways to represent complex functions.

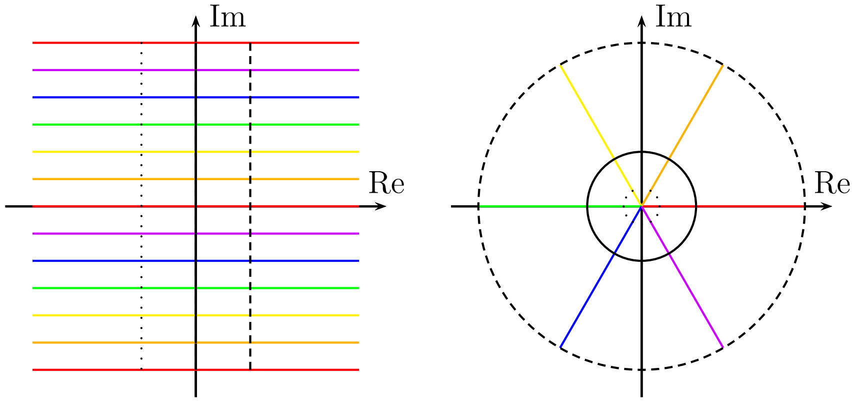

FIGURE 3.7. Images of curves $x = c$ and $y = d$ for $w = e^z$

LECTURE 4

Fractional linear transformations

In this lecture, we study a particular type of function: fractional linear transformations. These are easy to handle because we can use linear algebra to simplify computations.

1. Domains and ranges of fractional linear transformations

Let M be a 2×2 complex matrix:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We write T_M for the associated *fractional linear transformation*:

$$T_M(z) = \frac{az+b}{cz+d}. \tag{4.1}$$

First, we need to assume, and will always do so, that $(c,d) \neq (0,0)$, otherwise the denominator is always 0. If $\det M = 0$, then $(a,b) = \lambda(c,d)$ for some $\lambda \in \mathbb{C}$, whence $az+b = \lambda(cz+d)$ and so $f(z) = \lambda$ for all $z \in \mathrm{Domain}(f)$. Thus fractional linear transformations associated to matrices with determinant 0 are essentially just constant functions, and we do not consider them further.

Observe now that $T_{\lambda M} = T_M$ if $\lambda \neq 0$, and recall that $\det(\lambda M) = \lambda^2 \det(M)$. This means that when $\det M \neq 0$, if we set $M' = (\det M)^{-1/2} M$, then $\det M' = 1$ and $T_M = T_{M'}$. Thus we may and shall henceforth restrict our attention to fractional linear transformations associated to matrices with determinant 1.

First we find the domain of T_M. If $c = 0$ and $d \neq 0$, then the denominator is never 0, so $T_M(z)$ is defined for all a, that is, $\mathrm{Domain}(T_M) = \mathbb{C}$; otherwise, the denominator is nonzero when $z \neq -d/c$, whence $\mathrm{Domain}(T_M) = \mathbb{C} \setminus \{-d/c\}$.

Now suppose that $w \in \mathrm{Range}(T_M)$. Then

$$w = \frac{az+b}{cz+d} \quad \Longrightarrow \quad wcz + wd = az + b \quad \Longrightarrow \quad z(wc-a) = b - wd,$$

and as long as $wc \neq a$, this in turn implies that

$$z = \frac{b-wd}{wc-a} = \frac{(-d)w+b}{cw+(-a)} = \frac{dw-b}{(-c)w+a} = T_{M'}(w), \tag{4.2}$$

where

$$M' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then M' is exactly M^{-1}, since $\det M = 1$. We conclude that

$$\mathrm{Range}(T_M) = \begin{cases} \mathbb{C} \setminus \{a/c\} & \text{if } c \neq 0 \\ \mathbb{C} & \text{if } c = 0. \end{cases}$$

Our discussion of the domain and range had several cases; we can simplify the statements by enlarging the set of complex numbers by adding ∞. Indeed, we define

$$f(-d/c) = \lim_{z \to -d/c} \frac{az+b}{cz+d} = \infty \quad \text{and} \quad f(\infty) = \lim_{z \to \infty} \frac{az+b}{cz+d} = \frac{a}{c}$$

(we will define limits formally in the next lecture) and now we can just write

$$T_M : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}.$$

We often call $\mathbb{C} \cup \{\infty\}$ the Riemann sphere, and write it S: we imagine a unit sphere in three dimensions with centre at 0. We can define a function $\sigma : \mathbb{C} \to S$ geometrically by joining a point p in the plane to the north pole n of the sphere by a straight line. The line will cut the sphere at n and at one other point, which we call $\sigma(p)$. Then we may think of n as being $\sigma(\infty)$. The function σ is called *stereographic projection*.

2. Matrix products and composition of mappings

Exercise 4.1. Suppose that $M, N \in M_{2,2}(\mathbb{C})$. Show that $T_M T_N = T_{MN}$ (on the Riemann sphere). Deduce that the transformation T_M is bijective and that $(T_M)^{-1} = T_{M^{-1}}$.

Answer.

3. Factorisations of fractional linear transformations

Matrices may be factorised, and hence fractional linear transformations may be factorised too.

Theorem 4.2. *Every 2×2 complex matrix with determinant 1 may be written as a product of at most three matrices of the following special types:*

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

PROOF. Consider the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We consider three cases, according to whether any of a or c is 0. since the determinant is 1, both cannot be 0.

If $c = 0$, the matrix itself is of the desired form.

Now suppose that $a = 0$. Then

$$\begin{aligned}
\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c & -d \\ 0 & b \end{pmatrix},
\end{aligned}$$

which is of the desired form.

Finally, if neither a nor c is 0, then we take $x = a/c$, and write

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - cx & b - dx \\ c & d \end{pmatrix};
\end{aligned}$$

since $a - cx = 0$, combining with the previous case shows that the factorisation holds in this case too. □

This factorisation simplifies a number of arguments; the next result is an example.

THEOREM 4.3. *Let T_M be a fractional linear transformation. Then the image of a line under T_M is a line or a circle, and the image of a circle under T_M is also a line or a circle.*

PROOF. If M is of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, then T_M is an affine transformation, and the theorem holds in this case. Otherwise, T_M is composed of some affine transformations and the inversion map $z \mapsto -1/z$; it therefore suffices to treat the inversion map.

We may write the equation of the circle with centre c and radius r in the form $|z - c|^2 = r^2$. Note that this is the same as $|z|^2 - 2\operatorname{Re}(z\bar{c}) + |c|^2 - |r|^2 = 0$.

If we set $w = -1/z$, then we find that

$$\left| \frac{-1}{w} - c \right| = r^2,$$

whence

$$|wc + 1| = r^2 |w|^2,$$

and

$$\left(|c|^2 - r^2\right) |w|^2 + 2\operatorname{Re}(wc) + 1 = 0,$$

which is the equation of a circle, unless $|c|^2 = r^2$, in which case we get

$$2\,\mathrm{Re}\,(wc) = -1,$$

which is the equation of a straight line.

The argument to show that the inversion mapping sends straight lines to straight lines or circles is similar, and we omit it; the starting point is that the equation of a straight line may be written as $|z-p| = |z-q|$. □

REMARK 4.4. We are slightly imprecise here. Given a fractional linear transformation T_M (e.g. $T_M(z) = -1/z$) and X a circle or a line (e.g. $X = C(1,1)$ the circle centre 1 radius 1), it might happen that T_M is defined on $X \setminus \{x\}$ where x is precisely the point that will make vanish the denominator of T_M (e.g. $x = 0$). We are then looking at $T_M(X \setminus \{x\})$ rather than $T_M(X)$. The image is then alway a line as we will see in the next corollary.

COROLLARY 4.5. *Let T_M be a fractional linear transformation with $M = \begin{matrix} a & b \\ c & d \end{matrix}$ satisfying that $c \neq 0$. If $X \subset \mathbb{C}$ is a line or a circle containing the point $-d/c$, then $T_M(X \setminus \{-d/c\})$ is a line.*

PROOF. By the previous theorem we have that $T_M(X \setminus \{-d/c\})$ is either a line or a circle. Note that $\lim_{z \to -d/c} T_M(z) = \infty$ (see the next section on limits) implying that $T_M(X \setminus \{-d/c\})$ is unbounded. Therefore, $T_M(X \setminus \{-d/c\})$ must be a line. □

EXERCISE 4.6. Show that the fractional linear transformation $T : z \mapsto \dfrac{z-i}{z+i}$ sends the upper half plane onto the unit disc $\{w \in \mathbb{C} : |w| < 1\}$.

Find the image under T of the sector $\{z \in \mathbb{C} : \varphi < \mathrm{Arg}(z) < \pi - \varphi\}$.

ANSWER.

4. Special classes of fractional linear transformations

EXERCISE 4.7. Suppose that $M \in M_{2,2}(\mathbb{R})$ and $\det(M) = 1$. Show that if $\mathrm{Im}(z) > 0$, then $\mathrm{Im}(T_M z) > 0$. Deduce from this and Exercise 4.1 that T_M maps the upper half plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ onto itself bijectively.

ANSWER.

LECTURE 5

Limits and continuity

In this lecture, we outline the key ideas and facts about limits and continuity, as a preliminary to defining differentiability.

1. Limits

We define limits for complex functions much as for real functions.

Recall that, given a set S, we define its *closure* $\bar{S}$ or S^{-} to be the set consisting of all points of S together with all boundary points.

DEFINITION 5.1. Suppose that f is a complex function, $\ell \in \mathbb{C}$, and z_0 is in $\mathrm{Domain}(f)^{-}$. We say that $f(z)$ tends to ℓ as z tends to z_0, or that ℓ is the limit of $f(z)$ as z tends to z_0, and we write $f(z) \to \ell$ as $z \to z_0$, or

$$\lim_{z \to z_0} f(z) = \ell,$$

if, for every $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $|f(z) - \ell| < \varepsilon$ provided that z is in $\mathrm{Domain}(f)$ and $0 < |z - z_0| < \delta$.

Suppose also S is a subset of $\mathrm{Domain}(f)$ and that $z_0 \in \bar{S}$. We say that $f(z)$ tends to ℓ as z tends to z_0 in S, or that ℓ is the limit of $f(z)$ as z tends to z_0 in S, and we write $f(z) \to \ell$ as $z \to z_0$ in S, or

$$\lim_{\substack{z \to z_0 \\ z \in S}} f(z) = \ell,$$

if, for every $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $|f(z) - \ell| < \varepsilon$ provided that $z \in S$ and $0 < |z - z_0| < \delta$.

Informally, $f(z)$ tends to ℓ if we can make $f(z)$ arbitrarily close to ℓ by taking z close to, but not equal to, z_0.

Most of what follows about limits of the form $\lim_{z \to z_0} f(z)$ also applies to restricted limits, that is, limits of the form $\lim_{\substack{z \to z_0 \\ z \in S}} f(z)$.

We may rewrite the conditions $0 < |z - z_0| < \delta$ and $|f(z) - \ell| < \varepsilon$ as $z \in \mathrm{B}^{\circ}(z_0, \delta)$ and $f(z) \in \mathrm{B}(\ell, \varepsilon)$. We define limits involving infinity in a similar way by defining balls centred at infinity, and extending our previous definition slightly.

DEFINITION 5.2. Suppose that $\varepsilon > 0$. We define both $\mathrm{B}(\infty, \varepsilon)$ and $\mathrm{B}^{\circ}(\infty, \varepsilon)$ to be the set $\{z \in \mathbb{C} : |z| > 1/\varepsilon\}$.

DEFINITION 5.3. Suppose that f is a complex function, that $\ell \in \mathbb{C} \cup \{\infty\}$, and that either $z_0 \in \mathrm{Domain}(f)^{-}$ or $\mathrm{Domain}(f)$ is unbounded and $z_0 = \infty$. We say that $f(z)$ tends to ℓ as z tends to z_0, or that ℓ is the limit of $f(z)$ as z tends to z_0, and we write $f(z) \to \ell$ as $z \to z_0$, or

$$\lim_{z \to z_0} f(z) = \ell,$$

if for all $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $f(z) \in \mathrm{B}(\ell, \varepsilon)$ provided that $z \in \mathrm{Domain}(f) \cap \mathrm{B}^\circ(z_0, \delta)$.

With this definition, the following lemma holds; we omit the proof.

LEMMA 5.4 (Standard limits). *Suppose that $\alpha, c \in \mathbb{C}$. Then*

$$\lim_{z\to\alpha} c = c \qquad\qquad \lim_{z\to\infty} c = c$$
$$\lim_{z\to\alpha} z - c = \alpha - c \qquad\qquad \lim_{z\to\infty} z - \alpha = \infty$$
$$\lim_{z\to\alpha} \frac{1}{z-\alpha} = \infty \qquad\qquad \lim_{z\to\infty} \frac{1}{z-\alpha} = 0.$$

As the statement of these limits indicates, we are sometimes allowed to consider ∞ as a limit in this course.

The next results follows from the definition; we omit the proofs, which generalise arguments from calculus.

LEMMA 5.5. *Suppose that f is a complex function, that $T \subseteq S \subseteq \mathrm{Domain}(f)$, and that $z_0 \in \bar{T}$. If $\lim_{\substack{z\to z_0\\ z\in S}} f(z)$ exists, then so does $\lim_{\substack{z\to z_0\\ z\in T}} f(z)$, and these limits are equal.*

LEMMA 5.6. *Suppose that f is a complex function, and that $z_0 \in \mathrm{Domain}(f)^{-}$. If $\lim_{z\to z_0} f(z)$ exists, then it is unique.*

We may break complicated limits up into sums, products, and so on, of simpler limits.

THEOREM 5.7. *Suppose that f and g are complex functions and that $c \in \mathbb{C}$. Then*

$$\begin{aligned}
\lim_{z\to z_0} cf(z) &= c \lim_{z\to z_0} f(z)\\
\lim_{z\to z_0} f(z) + g(z) &= \lim_{z\to z_0} f(z) + \lim_{z\to z_0} g(z)\\
\lim_{z\to z_0} f(z)\, g(z) &= \lim_{z\to z_0} f(z) \lim_{z\to z_0} g(z)\\
\lim_{z\to z_0} \frac{f(z)}{g(z)} &= \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)},
\end{aligned}$$

in the sense that if the right hand side exists, then so does the left hand side, and they are equal. In particular, for the quotient, we require that $\lim_{z\to z_0} g(z) \neq 0$.

We also omit the proof of this theorem, which is very similar to that of the corresponding theorem for limits of functions of a real variable.

Limits respect complex conjugation and related operations.

THEOREM 5.8. *Suppose that f is a complex function and that either* $\mathrm{Domain}(f)$ *is unbounded and $z_0 = \infty$ or $z_0 \in \mathrm{Domain}(f)^{-}$. Then*

$$\begin{aligned}
\lim_{z\to z_0} \overline{f(z)} &= \overline{\lim_{z\to z_0} f(z)}\\
\lim_{z\to z_0} \mathrm{Re}(f(z)) &= \mathrm{Re} \lim_{z\to z_0} f(z)\\
\lim_{z\to z_0} \mathrm{Im}(f(z)) &= \mathrm{Im} \lim_{z\to z_0} f(z)\\
\lim_{z\to z_0} f(z) &= \lim_{z\to z_0} \mathrm{Re}(f(z)) + i \lim_{z\to z_0} \mathrm{Im}(f(z)),
\end{aligned}$$

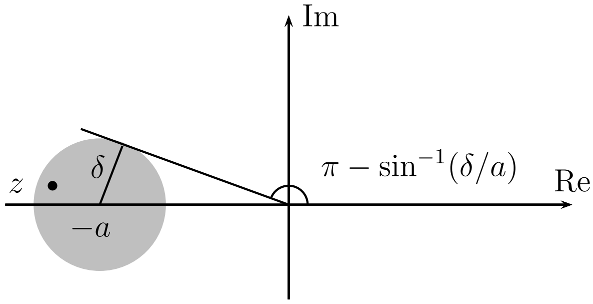

FIGURE 5.1. The trigonometry of the argument function

in the sense that if the right hand side exists, then so does the left hand side, and they are equal. In particular, $f(z)$ tends to ℓ as z tends to z_0 if and only if $\mathrm{Re}(f(z))$ *tends to* $\mathrm{Re}(\ell)$ *and* $\mathrm{Im}(f(z))$ *tends to* $\mathrm{Im}(\ell)$ *as z tends to z_0.*

PROOF. The proof of part (1) uses that fact that

$$\left|\overline{f(z)} - \bar{\ell}\right| = |f(z) - \ell| .$$

The rest follows from the first part and the first two parts of Theorem 5.7. □

2. Examples of limits

EXERCISE 5.9. Show from first principles that $\lim_{z\to z_0} z = z_0$.

ANSWER.

EXERCISE 5.10. Suppose that $f(z) = z^2 - \bar{z} + i$. Does $\lim_{z\to 2i} f(z)$ exist: if so find it, and if not, explain why not.

ANSWER.

We now consider an important example.

EXERCISE 5.11. Suppose that $a > 0$. Show that

$$\lim_{\substack{z\to -a\\ \mathrm{Im}(z)\geq 0}} \mathrm{Arg}(z) = \pi \qquad \text{and} \qquad \lim_{\substack{z\to -a\\ \mathrm{Im}(z)<0}} \mathrm{Arg}(z) = -\pi.$$

Does $\lim_{z\to -a} \mathrm{Arg}(z)$ exist, and if so, what is it?

ANSWER.

We may also show that

$$\lim_{\substack{z\to 0\\ \operatorname{Arg}(z)=\theta}} \operatorname{Arg}(z) = \theta.$$

Thus Arg is not continuous at any point of $(-\infty, 0]$. The function Arg is one of many important discontinuous complex functions.

EXERCISE 5.12. Suppose that $f(z) = \bar{z}/z$ and $g(z) = z^2/\bar{z}$. Does $\lim_{z\to 0} f(z)$ or $\lim_{z\to 0} g(z)$ exist? If so, find the limit; otherwise, explain why it does not exist.

ANSWER.

3. Stereographic projection and the Riemann sphere

To explain the role of ∞, we imagine a unit sphere S in xyz space, with center at 0, and we identity the xy plane with the complex plane. We may define a function $\sigma : \mathbb{C} \to S$ geometrically by joining a point p in the plane to the north pole n of the sphere by a straight line. The line will cut the sphere at n and at one other point, which we call $\sigma(p)$. Then we may think of n as being $\sigma(\infty)$.

The function σ is called *stereographic projection*, and the sphere is called the Riemann sphere. Then balls $\mathrm{B}(0, \varepsilon)$ correspond to spherical caps in the Riemann sphere, centred at $\sigma(0)$, while balls $\mathrm{B}(z_0, \varepsilon)$ correspond to spherical caps in the Riemann sphere containing the point $\sigma(z_0)$, and the "punctured balls" $\mathrm{B}^\circ(\infty, \varepsilon)$ correspond to "punctured" spherical caps in the Riemann sphere centred at $\sigma(\infty)$. These spherical caps shrink down towards the points $\sigma(0)$, to $\sigma(z)$ and to $\sigma(\infty)$ as ε tends to 0.

4. Continuity

DEFINITION 5.13. Suppose that f is a complex function. We say that f is continuous at a point z_0 if $f(z_0)$ is defined and $\lim_{z\to z_0} f(z) = f(z_0)$. We say that f is continuous in a set S if it is continuous at all points of S. We say that f is continuous if it is continuous at all points of its domain.

The functions $z \mapsto z$, $z \mapsto \overline{z}$, $z \mapsto |z|$, $z \mapsto \operatorname{Re}(z)$, and $z \mapsto \operatorname{Im}(z)$ are all continuous. The function Arg is continuous in the set $\{z \in \mathbb{C} \setminus \{0\} : \operatorname{Arg}(z) \neq \pi\}$.

Properties of limits lead to similar properties of continuous functions.

THEOREM 5.14. *Suppose that $c \in \mathbb{C}$, and that $f : S \to \mathbb{C}$ and $g : S \to \mathbb{C}$ are continuous complex functions in $S \subseteq \mathbb{C}$. Then cf, $f + g$, $|f|$, $\overline{f}$, $\operatorname{Re} f$, $\operatorname{Im} f$ and fg are continuous in S, as is f/g provided that $g(z) \neq 0$ for any z in S.*

THEOREM 5.15. *Suppose that $f : S \to \mathbb{C}$ and $g : T \to \mathbb{C}$ are continuous complex functions in $S \subseteq \mathbb{C}$ and $T \subseteq \mathbb{C}$. Then $f \circ g$ is continuous where it is defined, that is, in $\{z \in T : g(z) \in S\}$.*

By using the theorems above, it follows that functions that are composed of the standard functions (except Arg), such as

$$z \mapsto \frac{\mathrm{Re}(z^2) + i\,\mathrm{Im}(z^3)}{|z| + 1 + \overline{z}}\,,$$

are also continuous where they are defined (with this example, the tricky bit is finding the domain of definition; the natural domain is actually $\mathbb{C}$).

Generally speaking, any function that can be written down using the standard functions, and without choices in the definition, is continuous in its domain of definition, except when Arg is involved. Where there are choices in the definition, the difficulties usually lie where the different definitions match up.

Continuity is useful for two reasons. First, when functions are continuous, we do not have to worry about limits much. Next, continuous functions have some important properties.

THEOREM 5.16. *Suppose that the set $S \subseteq \mathbb{C}$ is compact (i.e., closed and bounded) and that f is a continuous complex function defined on S. Then there exists a point z_0 in S such that*

$$|f(z_0)| = \max\{|f(z)| : z \in S\}.$$

One says that the modulus of a continuous function *attains its maximum* in a compact set. As a consequence, if f is a continuous complex function defined in a compact set $S \subseteq \mathbb{C}$, then there is a number R such that

$$|f(z)| \leq R \qquad \forall z \in S.$$

Thus f is *bounded in S*.

Last but not least, continuous functions in compact sets are *uniformly continuous.* We will not explain this now.

5. Examples of continuous functions

EXERCISE 5.17. Show from first principles that $z \mapsto |z|$ is a continuous function in $\mathbb{C}$.

ANSWER.

EXERCISE 5.18. Show that $\mathrm{Arg}(z)$ is continuous in $\mathbb{C} \setminus (-\infty, 0]$.

ANSWER.

DEFINITION 5.19. The function $\mathrm{Log} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is defined by

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z).$$

EXERCISE 5.20. Show that $\mathrm{Log}(z)$ is continuous in $\mathbb{C} \setminus (-\infty, 0]$, and is not continuous at any point on $(-\infty, 0]$.

ANSWER.

LECTURE 6

Complex Differentiability

In this lecture, we investigate the differentiability of a function of a complex variable, defined much as for functions of a real variable. Many calculations are similar to the real-variable case; however, some functions that we might expect to be differentiable are not.

1. Definition

DEFINITION 6.1. Suppose that $S \subseteq \mathbb{C}$ and that $f : S \to \mathbb{C}$ is a complex function. Then we say that f is *differentiable at the point* z_0 in S if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{6.1}$$

exists and is finite. If it does, it is called the *derivative* of f at z_0, and is written $\dfrac{df(z_0)}{dz}$ or $f'(z_0)$.

We say that f is differentiable in a set S if it is differentiable at all points of S, and that f is differentiable if it is differentiable at all points of its domain.

REMARK 6.2. The limit (6.1) may also be written as

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}, \tag{6.2}$$

and the definition may be given with this limit instead.

2. Examples

EXERCISE 6.3. Suppose that $f_1(z) = z^2 + iz + 2$. Is f_1 differentiable at z_0 in $\mathbb{C}$? If so, find $f_1'(z_0)$?

ANSWER.

The computation above is almost identical to that to find the derivative of the real function $x^2 + x + 2$. Indeed, many formulae from the real case also hold in the complex case when x is replaced by z. So do many theorems.

3. More examples of differentiation of complex functions

The following examples show that there is a twist to the story.

EXERCISE 6.4. Suppose that $f_2(z) = \overline{z}$. Is f_2 differentiable at z_0 in $\mathbb{C}$? If so, find $f_2'(z_0)$?

ANSWER.

EXERCISE 6.5. Suppose that $f_3(z) = |z|^2$. Is f_3 differentiable at z_0 in $\mathbb{C}$? If so, find $f_3'(z_0)$?

ANSWER.

These examples show that a function may be differentiable everywhere, or nowhere, or at some points and not others. The nondifferentiable examples involved the complex conjugate, explicitly or implicitly.

4. The Cauchy–Riemann equations

We will now investigate differentiability using theoretical tools. If $\lim_{w\to 0} q(w)$ exists, then

$$\lim_{\substack{w\to 0\\ w\in\mathbb{R}}} q(w) = \lim_{\substack{w\to 0\\ w\in i\mathbb{R}}} q(w) = \lim_{w\to 0} q(w),$$

in the sense that the first two limits also exist, and are equal to the third. This allows us to relate the complex derivative to partial derivatives.

THEOREM 6.6. *Suppose that Ω is an open subset of $\mathbb{C}$, that f is a complex function defined in Ω, that $f(x+iy) = u(x,y) + iv(x,y)$, where u and v are real-valued functions of two real variables, and that f is differentiable at $z_0 \in \Omega$. Then the partial derivatives*

$$\frac{\partial u}{\partial x}(x_0,y_0), \quad \frac{\partial u}{\partial y}(x_0,y_0), \quad \frac{\partial v}{\partial x}(x_0,y_0) \quad \textit{and} \quad \frac{\partial v}{\partial y}(x_0,y_0)$$

all exist, and

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \qquad \textit{and} \qquad \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0)\,. \tag{6.3}$$

Further,

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) = -i\Big(\frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0)\Big). \tag{6.4}$$

REMARK 6.7. The pair of equations (6.3), which relate the partial derivatives of u and v, are known as the Cauchy–Riemann equations.

PROOF. If $f'(z_0)$ exists, then

$$\begin{aligned}
f'(z_0) &= \lim_{\substack{w\to 0\\ w\in\mathbb{R}}} \frac{f(z_0+w) - f(z_0)}{w} \\
&= \lim_{\substack{w\to 0\\ w\in\mathbb{R}}} \frac{u(x_0+w,y_0) + iv(x_0+w,y_0) - u(x_0,y_0) - iv(x_0,y_0)}{w} \\
&= \lim_{\substack{w\to 0\\ w\in\mathbb{R}}} \frac{u(x_0+w,y_0) - u(x_0,y_0)}{w} + i\lim_{\substack{w\to 0\\ w\in\mathbb{R}}} \frac{v(x_0+w,y_0) - v(x_0,y_0)}{w} \\
&= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0),
\end{aligned}$$

because a limit exists if and only if its real and imaginary parts do. Thus

$$\frac{\partial u}{\partial x}(x_0,y_0) = \operatorname{Re}(f'(z_0)) \qquad \text{and} \qquad \frac{\partial v}{\partial x}(x_0,y_0) = \operatorname{Im}(f'(z_0)).$$

Since $f'(z_0) = \operatorname{Re}(f'(z_0)) + i\operatorname{Im}(f'(z_0))$, part of (6.4) follows.

Similarly, if $f'(z_0)$ exists, then

$$\begin{aligned}
f'(z_0) &= \lim_{\substack{w\to 0\\ w\in i\mathbb{R}}} \frac{f(z_0+w) - f(z_0)}{w} \\
&= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(x_0,y_0+h) + iv(x_0,y_0+h) - u(x_0,y_0) - iv(x_0,y_0)}{ih} \\
&= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{v(x_0,y_0+h) - v(x_0,y_0)}{h} - i\lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(x_0,y_0+h) - u(x_0,y_0)}{h} \\
&= \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0) = -i\Big(\frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0)\Big).
\end{aligned}$$

Thus

$$\frac{\partial v}{\partial y}(x_0,y_0) = \operatorname{Re}(f'(z_0)) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x_0,y_0) = -\operatorname{Im}(f'(z_0)).$$

The Cauchy–Riemann equations follow by equating the two expressions for the real part of $f'(z_0)$ and the two expressions for the imaginary part of $f'(z_0)$, and the remaining part of (6.4) also follows.. □

One consequence of the previous theorem is that if f is differentiable at every point of an open set Ω in $\mathbb{C}$, then the Cauchy–Riemann equations hold at every point of Ω. Later on, we will see that in addition the four partial derivatives $\partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial y$ and $\partial v/\partial y$ are all continuous. For open sets Ω, the converse is true.

THEOREM 6.8. *If the four partial derivatives $\partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial y$ and $\partial v/\partial y$ are all continuous in an open set Ω, then f is complex differentiable at $z_0 \in \Omega$ if and only if the Cauchy–Riemann equations hold at z_0, and if so, then*

$$f'(x_0 + iy_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

We will justify this result later. When the partial derivatives are continuous in a set that is not open, the function might be differentiable, or it might not be.

5. Examples

We revisit our previous examples, and add another, using the Cauchy–Riemann equations.

EXAMPLES 6.9. (1) Suppose that $f_1(z) = z^2 + iz + 2$. Then

$$u(x,y) = x^2 - y^2 - y + 2 \quad \text{and} \quad v(x,y) = 2xy + x,$$

so

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial u}{\partial y} = -2y - 1, \qquad \frac{\partial v}{\partial x} = 2y + 1 \quad \text{and} \quad \frac{\partial v}{\partial y} = 2x,$$

and hence the Cauchy–Riemann equations hold for all (x, y) in $\mathbb{R}^2$. Since the partial derivatives are continuous and $\mathbb{C}$ is open, f_1 is differentiable in $\mathbb{C}$, and

$$f_1'(z) = 2x + i(2y + 1) = 2z + i.$$

(2) Suppose that $f_2(z) = \overline{z}$. Then

$$u(x,y) = x \quad \text{and} \quad v(x,y) = -y,$$

so

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = -1,$$

and hence the Cauchy–Riemann equations do not hold for any (x, y) in $\mathbb{R}^2$.

Hence f_2 is not differentiable at any point in $\mathbb{C}$.

(3) Suppose that $f_3(z) = |z|^2$. Then

$$u(x,y) = x^2 + y^2 \quad \text{and} \quad v(x,y) = 0,$$

so

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial u}{\partial y} = 2y, \qquad \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = 0,$$

and hence the Cauchy–Riemann equations hold if and only if $x = y = 0$. The partial derivatives are continuous in $\mathbb{C}$, which is open, and hence f_3 is differentiable at 0. Finally, f is not differentiable at any other point than 0, since the Cauchy–Riemann equations do not hold at any other point.

(4) Suppose that $f_4(z) = e^z$. Then

$$u(x,y) = e^x \cos y \quad \text{and} \quad v(x,y) = e^x \sin y,$$

so

$$\frac{\partial u}{\partial x} = e^x \cos y, \qquad \frac{\partial u}{\partial y} = -e^x \sin y, \qquad \frac{\partial v}{\partial x} = e^x \sin y \quad \text{and} \quad \frac{\partial v}{\partial y} = e^x \cos y,$$

and hence the Cauchy–Riemann equations hold for all (x, y) in $\mathbb{R}^2$. Since the partial derivatives are continuous and $\mathbb{C}$ is open, f_4 is differentiable in $\mathbb{C}$, and

$$f_4'(z) = \frac{\partial u}{\partial x}(x,y) + \frac{\partial u}{\partial y}(x,y) = e^x(\cos y + i \sin y) = e^z.$$

Remark 6.10. Doing (4) using limits would be rather messy!

6. Properties of the derivative

Theorem 6.11. *Suppose that $z_0 \in \mathbb{C}$, that the complex functions f and g are differentiable at z_0, and that $c \in \mathbb{C}$. Then the functions cf, $f + g$ and $f\,g$ are differentiable at z_0, and*

$$\begin{aligned} (cf)'(z_0) &= c\, f'(z_0), \\ (f+g)'(z_0) &= f'(z_0) + g'(z_0), \\ (f\,g)'(z_0) &= f'(z_0)\, g(z_0) + f(z_0)\, g'(z_0). \end{aligned}$$

Further, if $g(z_0) \neq 0$, then the function f/g is differentiable at z_0, and

$$(f/g)'(z_0) = \frac{f'(z_0)\, g(z_0) - f(z_0)\, g'(z_0)}{g(z_0)^2}\,.$$

Theorem 6.12. *Suppose that $z_0 \in \mathbb{C}$, that the complex function f is differentiable at $g(z_0)$, and that the complex function g is differentiable at z_0. Then the function $f \circ g$ is differentiable at z_0, and*

$$(f \circ g)'(z_0) = f'(g(z_0))\, g'(z_0).$$

Theorem 6.13. *Suppose that f is a complex function and that $z_0 \in \mathrm{Domain}(f)$. If f is differentiable at z_0, then f is continuous at z_0.*

Proof. The proof of these results are very similar to those of the corresponding results for real functions, and we omit them. □

Theorem 6.14 (l'Hôpital's rule). *Suppose that $z_0 \in \mathbb{C} \cup \{\infty\}$ and that the complex functions f and g are differentiable at z_0. If $\lim_{z \to z_0} f(z)/g(z)$ is indeterminate, that is, of the form $0/0$ or ∞/∞, and if $\lim_{z \to z_0} f'(z)/g'(z)$ exists, then*

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}\,. \tag{6.5}$$

We do not prove l'Hôpital's rule at this time. It follows from results that we prove later about Taylor series and Laurent series.

LECTURE 7

Connections with calculus in the plane† (Not examinable)

In this lecture, we link limits, continuity and differentiability of complex functions with their analogues for functions of two real variables.

1. Limits and continuity

The definitions and properties of limits and continuity for functions of two real variables are essentially the same as limits for functions of one complex variable. To save space, we represent vectors in $\mathbb{R}^2$ as row vectors rather than column vectors.

DEFINITION 7.1. Suppose that u is a real-valued function of two real variables, and that $(x_0, y_0) \in \mathrm{Domain}(u)^{-}$. We say that $u(x, y)$ tends to ℓ as (x, y) tends to (x_0, y_0), or that ℓ is the limit of $u(x, y)$ as (x, y) tends to (x_0, y_0), and we write $u(x, y) \to \ell$ as $(x, y) \to (x_0, y_0)$, or

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = \ell,$$

if, for every $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $|u(x, y) - \ell| < \varepsilon$ provided that $(x, y) \in \mathrm{Domain}(f)$ and $0 < |(x, y) - (x_0, y_0)| < \delta$. The same definition applies to a vector-valued function F, provided that we interpret $|F(x, y) - \ell|$ as a vector length.

Given a complex function f, we associate real-valued functions u and v and an $\mathbb{R}^2$-valued function F of two real variables by the formulae

$$\begin{aligned} f(x + iy) &= u(x, y) + iv(x, y) \\ F(x, y) &= (u(x, y), v(x, y)). \end{aligned}$$

The definitions of limit imply the following link between the real and complex functions.

CONNECTION 1. *Let functions f, u, v and F be related as above. Then the following are equivalent:*

(1) $f(z) \to \ell$ as $z \to z_0$;
(2) $u(x, y) \to \operatorname{Re} \ell$ and $v(x, y) \to \operatorname{Im} \ell$ as $(x, y) \to (x_0, y_0)$;
(3) $F(x, y) \to (\operatorname{Re} \ell, \operatorname{Im} \ell)$ as $(x, y) \to (x_0, y_0)$.

This means that to most theorems about limits of real functions such as u and v, there is a corresponding theorem about complex-valued functions f, and vice versa. For example, a theorem about vector-valued functions states that a vector-valued function tends to a limit ℓ if and only if each component of the function tends to the corresponding component of ℓ. This is the analogue of the theorem that a complex-valued function tends to a complex limit if and only if the real and imaginary parts of the function tend to the real and imaginary parts of the limit.

Continuity for functions of two real variables is defined much as for functions of a complex variable.

DEFINITION 7.2. Suppose that u is a real-valued function of two real variables, and that $(x_0, y_0) \in \mathrm{Domain}(u)^{-}$. We say that u is continuous at (x_0, y_0) if $\lim_{(x,y)\to(x_0,y_0)} u(x, y)$ and $u(x_0, y_0)$ both exist and are equal. We say that u is continuous if it is continuous at all points of $\mathrm{Domain}(u)$.

The same definition applies to vector-valued functions.

CONNECTION 2. *Let functions f, u, v and F be related as in* (1). *Then the following are equivalent:*

(1) f is continuous;
(2) u and v are continuous;
(3) F is continuous.

2. Differentiability

For functions of two real variables, we cannot define the derivative as for functions of a real variable, because this would involve dividing by a vector. But we note that an equivalent definition of the derivative of a function of one (real or complex) variable is that f is differentiable at z_0 and $f'(z_0) = D$ if

$$\lim_{h\to 0} \frac{|f(z_0 + h) - f(z_0) - Dh|}{|h|} = 0.$$

To see this, note that

$$\begin{aligned} \lim_{h\to 0} \frac{|f(z_0 + h) - f(z_0) - Dh|}{|h|} &= \lim_{h\to 0} \left| \frac{f(z_0 + h) - f(z_0) - Dh}{h} \right| \\ &= \lim_{h\to 0} \left| \frac{f(z_0 + h) - f(z_0)}{h} - D \right|. \end{aligned}$$

We define the derivative by extending this modified definition.

DEFINITION 7.3. A real- or vector-valued function u of two real variables is differentiable at (x_0, y_0), and its derivative is the linear transformation D, if

$$\lim_{(h,k)\to(0,0)} \frac{|u((x_0, y_0) + (h, k)) - u(x_0, y_0) - D(h, k)|}{|(h, k)|} = 0,$$

or equivalently, if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|u(x, y) - u(x_0, y_0) - D(x - x_0, y - y_0)|}{|(x - x_0, y - y_0)|} = 0.$$

The linear transformation D sends $\mathbb{R}^2$ to $\mathbb{R}$ if u is real-valued, and sends $\mathbb{R}^2$ to $\mathbb{R}^2$ if u is $\mathbb{R}^2$-valued.

It turns out that a vector-valued function is differentiable if and only if each of its components is differentiable, so that much of the theory may be developed for real-valued functions and then extended to vector-valued functions component by component.

One of the first steps in the development of calculus in several real variables is the identification of the linear transformation D. If u is real-valued and differentiable, then the partial derivatives exist and

$$\begin{aligned} D(h,k) &= \frac{\partial u}{\partial x}(x_0,y_0)h + \frac{\partial u}{\partial y}(x_0,y_0)k \\ &= \left(\frac{\partial u}{\partial x}(x_0,y_0), \frac{\partial u}{\partial y}(x_0,y_0) \right) \cdot (h,k) \\ &= \left(\frac{\partial u}{\partial x}(x_0,y_0) \quad \frac{\partial u}{\partial y}(x_0,y_0)) \right) \begin{pmatrix} h \\ k \end{pmatrix}, \end{aligned}$$

where the second line is a dot product of vectors, and the third is a product of two matrices. The plane $z = u(x_0, y_0) + D(x - x_0, y - y_0)$ in $\mathbb{R}^3$ is the tangent plane to the surface $z = u(x, y)$ in $\mathbb{R}^3$ at the point $(x_0, y_0, u(x_0, y_0))$.

To avoid having to deal with limits all the time, most treatments of multivariable calculus prove the next result as soon as possible. For ease of notation, we state it for a real-valued function, but it also holds for vector-valued functions.

THEOREM 7.4. *Suppose that u is a real-valued function of two real variables, and that the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ exist and are continuous in an open set Ω. Then u is differentiable in Ω.*

PROOF. Take $(x_0, y_0) \in \Omega$. To show that u is differentiable at (x_0, y_0), we need to make the quotient

$$\frac{|u(x,y) - u(x_0,y_0) - D(x-x_0, y-y_0)|}{|(x-x_0, y-y_0)|}$$

small, by taking (x, y) close to (x_0, y_0). To make this quantitative, we take $\varepsilon \in \mathbb{R}^+$, and make the quotient less than ε. Recall that

$$D(x-x_0, y-y_0) = \frac{\partial u}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial u}{\partial y}(x_0,y_0)(y-y_0).$$

Choose δ such that $\mathrm{B}((x_0, y_0), \delta) \subset \Omega$ and

$$\left| \frac{\partial u}{\partial x}(x_1,y_1) - \frac{\partial u}{\partial x}(x_0,y_0) \right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left| \frac{\partial u}{\partial y}(x_1,y_1) - \frac{\partial u}{\partial y}(x_0,y_0) \right| < \frac{\varepsilon}{2}$$

whenever $(x_1, y_1) \in \mathrm{B}((x_0, y_0), \delta)$. This is possible because Ω is open and because both the partial derivatives are continuous at (x_0, y_0). If $(x, y) \in \mathrm{B}((x_0, y_0), \delta)$, then the line segments joining (x_0, y_0) to (x, y_0) and joining (x, y_0) to (x, y) lie in $\mathrm{B}((x_0, y_0, \delta)$, by the geometry of balls in $\mathbb{R}^2$.

The fundamental theorem of calculus and the mean value theorem for integrals implies that there exist y_1 between y and y_0 and x_1 between x and x_0 such that

$$\begin{aligned} u(x,y) - u(x_0,y_0) &= u(x,y) - u(x,y_0) + u(x,y_0) - u(x_0,y_0) \\ &= \int_{y_0}^{y} \frac{\partial u}{\partial y}(x,t)\,dt + \int_{x_0}^{x} \frac{\partial u}{\partial x}(s,y_0)\,ds \\ &= \frac{\partial u}{\partial y}(x,y_1)(y-y_0) + \frac{\partial u}{\partial x}(x_1,y_0)(x-x_0), \end{aligned}$$

and this in turn implies that

$$
\begin{aligned}
&|u(x,y) - u(x_0,y_0) - D(x-x_0, y-y_0)| \\
&\quad = \Big| \frac{\partial u}{\partial y}(x,y_1)(y-y_0) - \frac{\partial u}{\partial y}(x_0,y_0)(y-y_0) \\
&\qquad\quad + \frac{\partial u}{\partial x}(x_1,y_0)(x-x_0) - \frac{\partial u}{\partial x}(x_0,y_0)(x-x_0) \Big| \\
&\quad \le \left| \frac{\partial u}{\partial y}(x,y_1) - \frac{\partial u}{\partial y}(x_0,y_0) \right| |y-y_0| + \left| \frac{\partial u}{\partial x}(x_1,y_0) - \frac{\partial u}{\partial x}(x_0,y_0) \right| |x-x_0| \\
&\quad < \frac{\varepsilon}{2} |y-y_0| + \frac{\varepsilon}{2} |x-x_0| ,
\end{aligned}
$$

whence

$$
\frac{|u(x,y) - u(x_0,y_0) - D(x-x_0,y-y_0)|}{|(x-x_0,y-y_0)|} < \frac{\frac{\varepsilon}{2}|y-y_0| + \frac{\varepsilon}{2}|x-x_0|}{|(x-x_0,y-y_0)|} \le \varepsilon,
$$

as required. □

Here is another theorem about differentiation of functions of two real variables; we omit the proof.

THEOREM 7.5. *Suppose that u is a twice continuously differentiable real-valued function of two real variables. Then*

$$
\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x} .
$$

A consequence of this theorem is that the derivative of u, which is the vector-valued function $(\partial u/\partial x, \partial u/\partial y)$, satisfies the condition that the derivative with respect to y of the first component is equal to the derivative with respect to x of the second component. In general, a vector-valued function (p,q) is said to be *closed* or *conservative* if $\partial p/\partial y = \partial q/\partial x$; and we have just seen that a derivative is conservative. An important question in multivariable calculus is whether every conservative vector-valued function (p,q) is a derivative of a real-valued function, called the *potential*; whether this is always true or not depends on whether the domain of definition of the function (p,q) is simply connected or not. We will return to this later.

THEOREM 7.6. *Suppose that Ω is a simply connected subset of $\mathbb{R}^2$, that p and q are differentiable functions from Ω to $\mathbb{R}$, and that $\partial p/\partial y = \partial q/\partial x$. Then there exists a function $f : \Omega \to \mathbb{R}$ such that $\partial f/\partial x = p$ and $\partial f/\partial y = q$.*

We omit the proof of this result, as we will prove an equivalent result for complex functions later.

EXAMPLE 7.7. Consider the vector-valued function on $\mathbb{R}^2 \setminus \{(0,0)\}$ given by

$$
F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) = (p(x,y), q(x,y)),
$$

say. It is easy to check that

$$
\frac{\partial p}{\partial y}(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \text{and} \qquad \frac{\partial p}{\partial y}(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2} ,
$$

and so $\partial p/\partial y = \partial q/\partial x$. However there is no function on $\mathbb{R}^2 \setminus \{(0,0)\}$ whose gradient is F. When we integrate, we get the argument function (plus a constant), and this cannot be defined continuously on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Observe that $\mathbb{R}^2 \setminus \{(0,0)\}$ is not simply connected.

3. Differentiability in $\mathbb{R}^2$ and complex differentiability

Suppose that f is a complex function. As before, we associate to f the functions u, v and F:

$$\begin{aligned} f(x+iy) &= u(x,y) + iv(x,y) \\ F(x,y) &= (u(x,y), v(x,y)). \end{aligned}$$

Then f is differentiable at z_0 with derivative $f'(z_0)$ if and only if

$$\lim_{h+ik\to 0} \frac{|f(z_0+h+ik) - f(z_0) - f'(z_0)(h+ik)|}{|h+ik|} = 0.$$

And F is differentiable at (x_0, y_0) with derivative $F'(x_0, y_0)$ if and only if

$$\lim_{(h,k)\to 0} \frac{|F((x_0,y_0)+(h,k)) - F(x_0,y_0) - F'(x_0,y_0)(h,k)|}{|(h,k)|} = 0.$$

Now $F'(x_0, y_0)$ is a linear transformation that may be identified with multiplication by the matrix

$$\begin{pmatrix} \dfrac{\partial u}{\partial x}(x_0,y_0) & \dfrac{\partial u}{\partial y}(x_0,y_0) \\[2ex] \dfrac{\partial v}{\partial x}(x_0,y_0) & \dfrac{\partial v}{\partial y}(x_0,y_0) \end{pmatrix}.$$

Multiplication by $f'(z_0)$ may be identified with multiplication by the matrix

$$\begin{pmatrix} b & -c \\ c & b \end{pmatrix},$$

where $b = \operatorname{Re} f'(z_0)$ and $c = \operatorname{Im} f'(z_0)$.

These two operations correspond provided that

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0),$$

and then

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial u}{\partial x}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0) + \dots.$$

These conditions are exactly the Cauchy–Riemann equations.

In conclusion, we have established the following link between complex functions and $\mathbb{R}^2$-valued functions of two real variables.

Connection 3. *A complex function f is complex differentiable if and only if the associated vector-valued function of two real variables F is differentiable* and *the derivative of F corresponds to multiplication by a complex number.*

A consequence of this connection and Theorem 7.4 is that a complex function f is differentiable in an open set if the Cauchy–Riemann equations hold for the associated functions u and v and the partial derivatives of u and v are continuous.

Here is another. As we have seen, multiplication by the nonzero complex number c corresponds to scalar multiplication by the modulus $|c|$, which preserves angles, and then rotating through $\operatorname{Arg} c$, which also preserves angles.

It can be shown that each linear transformation of $\mathbb{R}^2$ that preserve angles is the composition of a nonzero scalar multiplication and a rotation. Thus differentiable complex functions whose derivatives do not vanish in some open set correspond to differentiable real functions on $\mathbb{R}^2$ whose derivatives preserve angles at every point in the same open set. Such functions will be called *conformal*.

Later we will see the noteworthy result that functions that are complex differentiable in open sets in $\mathbb{C}$ are infinitely differentiable. This contrasts with what happens in the theory of real functions, where a function can be differentiable k times but not $k + 1$ times. Thus a small difference in definition can lead to very large differences in behaviour.

LECTURE 8

Properties of differentiable functions

We explore some of the consequences of the Cauchy–Riemann equations.

1. Examples

The Cauchy Riemann equations enable us to define new functions and show that they are complex differentiable. For instance, recall the definition of the hyperbolic functions:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \forall x \in \mathbb{R}.$$

Recall also that the derivative of cosh is sinh and the derivative of sinh is cosh, and that

$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

and

$$\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y).$$

We define two new functions of a complex variable as follows:

$$\mathrm{ch}(x + iy) = \cosh(x)\cos(y) + i\sinh(x)\sin(y)$$

and

$$\mathrm{sh}(x + iy) = \sinh(x)\cos(y) + i\cosh(x)\sin(y).$$

Exercise 8.1. Is ch differentiable? What is its derivative? What about sh?

Answer.

Recall that we defined $\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$ for all $z \in \mathbb{C} \setminus \{0\}$.

Exercise 8.2. Where is Log differentiable? What is its derivative?

ANSWER.

the Cauchy–Riemann equations hold in the open set and so is differentiable in this set

2. Geometry of the Cauchy–Riemann equations (Not examinable)

The Cauchy–Riemann equations are important in physical applications because they imply that the functions u and v are related in a very significant way.

COROLLARY 8.3. *Suppose that f is a differentiable function in an open set Ω, and that $f(x+iy) = u(x,y) + iv(x,y)$ for all $x+iy \in \Omega$. Then the gradients ∇u and ∇v satisfy*

$$\|\nabla u\| = \|\nabla v\| \qquad \textit{and} \qquad \nabla u \cdot \nabla v = 0,$$

that is, the two gradient vectors are of equal length and perpendicular to each other. Consequently when the derivative is nonzero, the contour lines for u are perpendicular to the contour lines for v.

PROOF. The gradient vectors ∇u and ∇v are given by

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \qquad \text{and} \qquad \nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right).$$

The Cauchy–Riemann equations imply that the second vector may be obtained from the first by rotating through 90 degrees, and the result follows. $\square$

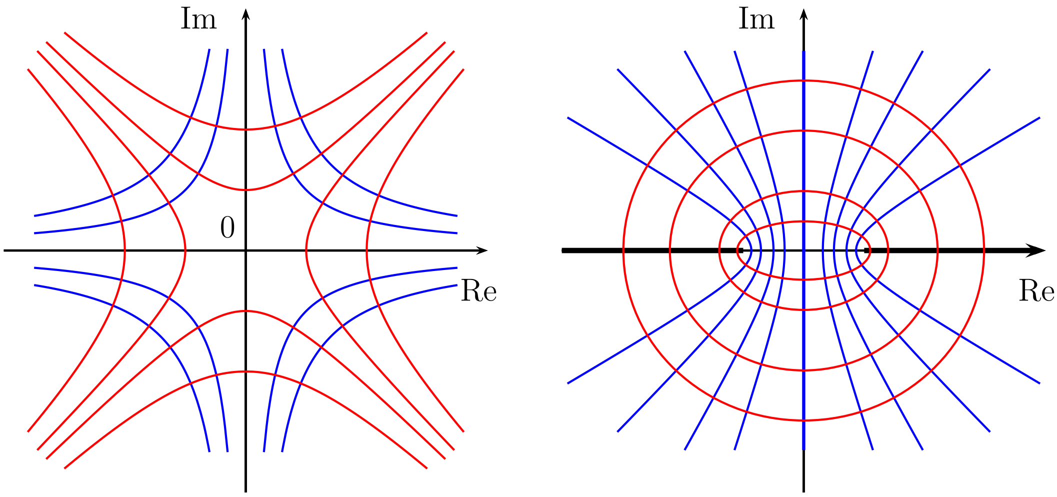

FIGURE 8.1. Sketches of level curves for two functions

When we sketched complex functions, we drew level curves for the functions u and v. The sketches suggested that the level curves for u and for v are perpendicular to each other; the corollary explains why this is so. It is also true that the images of the lines $x = c$ and $y = d$ are perpendicular, at least when $f'(z) \neq 0$.

There are examples in physics where u is a potential (electrical or gravitational), so the contour lines for u are lines of equipotential; the contour lines for v are then the curves along which particles (with charge or mass) move under the influence of the field.

EXAMPLE 8.4. Suppose that $f(z) = z^2$, and that $f(x + iy) = u(x, y) + iv(x, y)$. Then $u(x, y) = x^2-y^2$ and $v(x, y) = 2xy$. The contours for u, that is, the level curves given by an equation of the form $u(x, y) = c$, for some constant c, are hyperbolae asymptotic to the lines $y = \pm x$. The contours for v, that is, the level curves given by an equation of the form $v(x, y) = d$, for some constant d, are hyperbolae asymptotic to the x and y axes. See the left hand sketch in Figure 8.1.

EXAMPLE 8.5. The next example is related to ch, and we are going to study this in detail later. It corresponds to the electric field between two "infinite" charged plates, represented by the infinite intervals $(-\infty, -1]$ and $[1, \infty)$: the equipotential lines are hyperbolae, while the paths followed by charged particles are elliptical. See the right hand sketch in Figure 8.1.

3. The Cauchy–Riemann equations and complex differentiability

Before we look at other implications of complex differentiability, we consider a "topological" question: if f is defined in an open set Ω, and f is constant along all horizontal and vertical line segments contained in Ω, must f be constant? It is easy to see that if Ω is not connected, then f need not be constant. But what if Ω is connected? Recall that a connected open subset of $\mathbb{C}$ is called a *domain*.

PROPOSITION 8.6. *Suppose that f is a function defined on a domain Ω in $\mathbb{C}$, and f is constant along all horizontal and vertical line segments contained in Ω. Then f is constant in Ω.*

PROOF.

□

THEOREM 8.7. *Suppose that f is differentiable in a* domain Ω *in* $\mathbb{C}$. *Then*

(a) if $f' = 0$ in Ω, then f is constant on Ω;
(b) if $|f|$ is constant, then f is constant on Ω;
(c) if $\operatorname{Re}(f)$ *or* $\operatorname{Im}(f)$ *is constant, then f is constant on Ω.*

PROOF. As usual, write $f(x + iy) = u(x, y) + iv(x, y)$.

First, suppose that $f' = 0$. Then $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ are all 0. By Proposition 8.6, f is constant.

Next, suppose that $|f|$ is a constant, C say, in Ω. If $C = 0$, then $f = 0$, a constant, so without loss of generality we may suppose that $C \neq 0$. Then

$$u^2 + v^2 = C^2 > 0. \tag{8.1}$$

Differentiating (8.1) with respect to x and with respect to y gives

$$2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 0 \tag{8.2}$$

$$2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} = 0. \tag{8.3}$$

Using the Cauchy–Riemann equations with (8.3), we get

$$2v \frac{\partial u}{\partial x} - 2u \frac{\partial v}{\partial x} = 0. \tag{8.4}$$

Eliminating $\partial v/\partial x$ from (8.2) and (8.4) shows that $2(u^2 + v^2)\,\partial u/\partial x = 0$. From (8.1), $\partial u/\partial x = 0$. Similarly, $\partial v/\partial x = 0$, so $f' = 0$, and f is constant.

Finally, suppose that $\operatorname{Re}(f)$ is constant. Then $\partial u/\partial x = \partial u/\partial y = 0$, so from the Cauchy–Riemann equations, $\partial v/\partial x = \partial v/\partial y = 0$, whence $f' = 0$ and f is constant. The argument if $\operatorname{Im}(f)$ is constant is similar. □

In fact, it is possible to show that if f is complex differentiable in a connected open set Ω and nonconstant, then the range of f is open, which means that f cannot satisfy any equations that restrict its range to lie in a one-dimensional set, such as a curve.

4. Polar coordinates

The Cauchy–Riemann equations are a very powerful tool. Consequently it is worth stating them in polar coordinates as well.

THEOREM 8.8. *Suppose that the complex function f is differentiable at a point z_0 in $\mathbb{C} \setminus \{0\}$, and that $z_0 = r_0 e^{i\theta_0}$. Then*

$$\frac{\partial u}{\partial \theta}(r_0, \theta_0) = -r_0 \frac{\partial v}{\partial r}(r_0, \theta_0) \qquad \textit{and} \qquad \frac{\partial v}{\partial \theta}(r_0, \theta_0) = r_0 \frac{\partial u}{\partial r}(r_0, \theta_0).$$

Further,

$$f'(z_0) = e^{-i\theta_0} \left(\frac{\partial u}{\partial r}(r_0, \theta_0) + i \frac{\partial v}{\partial r}(r_0, \theta_0) \right) = \frac{-ie^{-i\theta_0}}{r} \left(\frac{\partial u}{\partial \theta}(r_0, \theta_0) + i \frac{\partial v}{\partial \theta}(r_0, \theta_0) \right).$$

PROOF. Write z in polar coordinates as $re^{i\theta}$. Much as argued to prove the Cauchy–Riemann equations, since f is differentiable at z_0,

$$\begin{aligned}
\frac{\partial f}{\partial r}(r_0 e^{i\theta_0}) &= \lim_{s \to 0} \frac{f((r_0 + s)e^{i\theta_0}) - f(r_0 e^{i\theta_0})}{s} \\
&= e^{i\theta_0} \lim_{s \to 0} \frac{f(r_0 e^{i\theta_0} + s e^{i\theta_0}) - f(r_0 e^{i\theta_0})}{s e^{i\theta_0}} \\
&= e^{i\theta_0} \lim_{s \to 0} \frac{f(z_0 + s e^{i\theta_0}) - f(z_0)}{s e^{i\theta_0}} \\
&= e^{i\theta_0} f'(z_0),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial f}{\partial \theta}(r_0 e^{i\theta_0}) &= \lim_{\varphi \to 0} \frac{f(r_0 e^{i(\theta_0 + \varphi)}) - f(r_0 e^{i\theta_0})}{\varphi} \\
&= \lim_{\varphi \to 0} \frac{f(r_0 e^{i(\theta_0 + \varphi)}) - f(r_0 e^{i\theta_0})}{r_0 e^{i(\theta_0 + \varphi)} - r_0 e^{i\theta_0}} \frac{r_0 e^{i(\theta_0 + \varphi)} - r_0 e^{i\theta_0}}{\varphi} \\
&= \lim_{\varphi \to 0} \frac{f(r_0 e^{i(\theta_0 + \varphi)}) - f(r_0 e^{i\theta_0})}{r_0 e^{i(\theta_0 + \varphi)} - r_0 e^{i\theta_0}} \lim_{\varphi \to 0} \frac{r_0 e^{i(\theta_0 + \varphi)} - r_0 e^{i\theta_0}}{\varphi} \\
&= i r_0 \, e^{i\theta_0} \, f'(z_0),
\end{aligned}$$

where we used trigonometric limits to evaluate the last limit.

We now have two formulae, both involving $e^{i\theta_0} \, f'(z_0)$; we equate these to get

$$\frac{\partial f}{\partial \theta}(r_0 e^{i\theta_0}) = i r_0 \frac{\partial f}{\partial r}(r_0 e^{i\theta_0}).$$

Now we write f in $u + iv$ form, and see that

$$\frac{\partial u}{\partial \theta}(r_0, \theta_0) + i \frac{\partial v}{\partial \theta}(r_0, \theta_0) = i r_0 \left(\frac{\partial u}{\partial r}(r_0, \theta_0) + i \frac{\partial v}{\partial r}(r_0, \theta_0) \right).$$

Equating the real and imaginary parts gives the Cauchy–Riemann equations.

The formula for $f'(z_0)$ is found similarly. □

REMARK 8.9. It is easiest to remember this version of the Cauchy–Riemann equations in the form

$$\frac{\partial f(re^{i\theta})}{\partial \theta} = ir \frac{\partial f(re^{i\theta})}{\partial r}.$$

COROLLARY 8.10. *The function* Log *is differentiable in* $\mathbb{C} \setminus (-\infty, 0]$.

PROOF. Write $\mathrm{Log}(re^{i\theta})$ in the form $u(r,\theta) + iv(r,\theta)$. Then $u(r,\theta) = \ln(r)$ and $v(r,\theta) = \theta$; v is discontinuous when $\theta = \pi$. Hence

$$\frac{\partial u}{\partial r}(r_0,\theta_0) = \frac{1}{r_0} \qquad \frac{\partial u}{\partial \theta}(r_0,\theta_0) = 0$$
$$\frac{\partial v}{\partial r}(r_0,\theta_0) = 0 \qquad \frac{\partial v}{\partial \theta}(r_0,\theta_0) = 1,$$

unless $\theta = \pi$. Hence the Cauchy–Riemann equations in polar form are satisfied in $\mathbb{C} \setminus (-\infty, 0]$. Further, the partial derivatives are continuous in the open set $\mathbb{C} \setminus (-\infty, 0]$, so Log is differentiable in this set. $\square$

5. Inverse functions

Suppose that Ω and Υ are open subsets of $\mathbb{C}$, and that f is one-to-one from Ω onto Υ. Then f has an inverse function, usually written f^{-1}, from Υ to Ω: we define $f^{-1}(w) = z$ if $f(z) = w$.

THEOREM 8.11. *Suppose that Ω and Υ are open subsets of $\mathbb{C}$, that $f : \Omega \to \Upsilon$ is one-to-one, and that $f(z_0) = w_0$. If f is differentiable at z_0 and f^{-1} is differentiable at w_0, then $(f^{-1})'(w_0) = 1/f'(z_0)$.*

PROOF. By definition, $z = f^{-1}(f(z))$; the chain rule implies that

$$1 = \frac{df^{-1}}{dw}(w_0)\, f'(z_0). \qquad \square$$

Later, we will investigate whether the inverse function is differentiable.

6. Definition

The examples above show that functions that are complex differentiable in an open set have special properties; we are going to study them in much greater detail. This justifies giving them a name.

DEFINITION 8.12. Suppose that Ω is an open subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is a function. If f is differentiable in Ω, that is, if it is differentiable at every point of Ω, then we say that f is *holomorphic* or *complex analytic* or *analytic* or *regular* in Ω, and we write $f \in H(\Omega)$.

If $\Omega = \mathbb{C}$ and f is differentiable in Ω, then we say that f is *entire*.

LECTURE 9

Harmonic functions

In this lecture, we introduce harmonic functions, and we discuss their significance. We see how to find them using holomorphic functions, and conversely, how to find holomorphic functions using harmonic functions. One of the reasons why complex variable theory is an important area of mathematics is that it enables us to find harmonic functions.

1. Harmonic functions

We begin with a definition.

Definition 9.1. Suppose that $u : \Omega \to \mathbb{R}$ is a function, where Ω is an open subset of $\mathbb{R}^2$, and that u is twice continuously differentiable, that is, all the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x\,\partial y$, $\partial^2 u/\partial y\,\partial x$ and $\partial^2 u/\partial y^2$ exist and are continuous. Then we say that u is *harmonic* in Ω if it satisfies Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Remark 9.2. In three dimensions, where the three real variables are labelled x, y and z, we say that a real-valued twice continuously differentiable function u is harmonic if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

If a function u on $\mathbb{R}^3$ is independent of the variable z, then we may consider it as a function of x and y only, and then it is harmonic in the two-dimensional sense above.

How do we find harmonic functions? The next theorem tells us that it is enough to find holomorphic functions.

Theorem 9.3. *Suppose that $f \in H(\Omega)$, where Ω is an open subset of $\mathbb{C}$, that f is twice continuously differentiable, and that*

$$f(x + iy) = u(x, y) + iv(x, y)$$

for all $x + iy$ in Ω, where u and v are real-valued. Then u and v are harmonic functions.

Remark 9.4. Later we will see that $f \in H(\Omega)$ implies that f is actually infinitely differentiable, so the "twice continuously differentiable" hypothesis is not really necessary.

Proof. Suppose that $f \in H(\Omega)$, where Ω is an open subset of $\mathbb{C}$, and that

$$f(x + iy) = u(x, y) + iv(x, y)$$

in Ω, where u and v are real-valued. Since f is twice continuously differentiable, the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x\,\partial y$, $\partial^2 u/\partial y\,\partial x$ and $\partial^2 u/\partial y^2$ and $\partial v/\partial x$, $\partial v/\partial y$, $\partial^2 v/\partial x^2$, $\partial^2 v/\partial x\,\partial y$, $\partial^2 v/\partial y\,\partial x$ and $\partial^2 v/\partial y^2$ all exist and are continuous.

From vector calculus, this implies that

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} \qquad \text{and} \qquad \frac{\partial^2 v}{\partial x\,\partial y} = \frac{\partial^2 v}{\partial y\,\partial x}\,.$$

Now, from the Cauchy–Riemann equations,

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \\ &= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \\ &= \frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial^2 v}{\partial y\,\partial x} = 0,\end{aligned}$$

so u is harmonic. Similarly, v is harmonic. □

Now we ask whether we get all harmonic functions in this way. Let us first consider some examples.

2. Examples

EXERCISE 9.5. Suppose that $u(x, y) = x^3 - 3xy^2$. Show that u is harmonic in $\mathbb{C}$, and find a function v such that the function f, given by $f(x+iy) = u(x, y)+iv(x, y)$, is holomorphic in $\mathbb{C}$.

ANSWER.

EXERCISE 9.6. Suppose that $u(x, y) = x^2 - y^2$. Show that u is harmonic in $\mathbb{C}$. Find a function v such that the function f, given by $f(x + iy) = u(x, y) + iv(x, y)$, is holomorphic in $\mathbb{C}$.

ANSWER.

These examples suggest that it might always be possible to find a function v such that $u + iv$ is holomorphic. In fact we have the following theorem.

THEOREM 9.7. *If Ω is a simply connected domain, and $u : \Omega \to \mathbb{R}$ is harmonic, then there exists a harmonic function $v : \Omega \to \mathbb{R}$ such that f, given by*

$$f(x + iy) = u(x, y) + iv(x, y)$$

in Ω, is holomorphic. Any two such functions v differ by an additive constant.

REMARK 9.8. The function v is called a harmonic conjugate of u. The function f may often be determined using the fact that

$$f'(x + iy) = u_x(x, y) + iv_x(x, y) = u_x(x, y) - iu_y(x, y);$$

in fact, often when we write out $u_x(x, y) - iu_y(x, y)$, we can express this in terms of z and hence find f.

We give a partial proof of this theorem in Section 4.

To see why it might be necessary to require the simple connectedness of Ω in the theorem, consider the following exercise.

EXERCISE 9.9. Suppose that $u(x, y) = \ln(x^2 + y^2)^{1/2}$ in $\mathbb{R}^2 \setminus \{(0, 0)\}$. Show that u is harmonic. Can you find a function v such that the function f in $\mathbb{C} \setminus \{0\}$, defined by $f(x + iy) = u(x, y) + iv(x, y)$, is holomorphic.

EXERCISE 9.10. Suppose that $u(x, y) = x^2 - y^2$. Show that u is harmonic in $\mathbb{C}$. Find a function v such that the function f, given by $f(x + iy) = u(x, y) + iv(x, y)$, is holomorphic in $\mathbb{C}$.

ANSWER.

The trouble with this method is that it may be hard to express $u_x - iu_y$ as a function of z, or hard to find the integral of this.

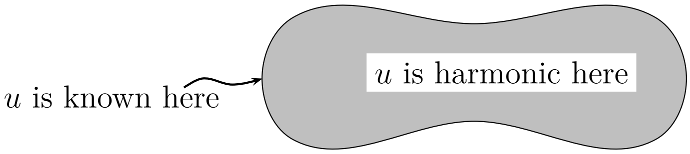

FIGURE 9.1. The Dirichlet problem: find u inside the body

3. Applications of harmonic functions† (Not examinable)

Harmonic functions are important because they appear in many physical applications. For instance, electrical and gravitational potentials are harmonic, and harmonic functions appear in fluid flow too. Similarly, components of electrical and gravitational fields in fixed directions are harmonic. Physicists often talk of infinite plates, or infinite wires, so that the associated potentials and fields do not depend on one of the variables, and may be modelled in two dimensions.

Here is a basic problem. We can measure a potential on the boundary of a body. How do we determine the potential inside the body? Mathematically, the problem is to find a harmonic function inside which, on the boundary, is equal to a given function. This is called the *Dirichlet problem.*

4. Proof of the existence theorem† (Not examinable)

PROOF. We give the proof for the case where Ω is a rectangle with sides parallel to the axes. In general, we need to be more sophisticated.

It is enough to find v such that the Cauchy–Riemann equations hold, for if we can do this, then $u + iv$ will be holomorphic in Ω.. Choose (x_0, y_0) in Ω. For any differentiable function v in Ω,

$$v(x,y) = \int_{y_0}^{y} \frac{\partial v}{\partial y}(x_0, t)\, dt + \int_{x_0}^{x} \frac{\partial v}{\partial x}(s, y)\, ds + v(x_0, y_0).$$

But if the Cauchy–Riemann equations hold, then

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x_0, t)\, dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(s, y)\, ds + v(x_0, y_0).$$

So we simply define v by the formula

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x_0, t)\, dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(s, y)\, ds + C.$$

Now

$$\frac{\partial v}{\partial x}(x,y) = \frac{\partial}{\partial x}\left(\int_{y_0}^{y} \frac{\partial u}{\partial x}(x_0, t)\, dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(s, y)\, ds + C \right) = -\frac{\partial u}{\partial x}(x,y)$$

as needed, and, since u is harmonic,

$$\begin{aligned}
\frac{\partial v}{\partial y}(x,y) &= \frac{\partial}{\partial y}\left(\int_{y_0}^{y}\frac{\partial u}{\partial x}(x_0,t)\,dt - \int_{x_0}^{x}\frac{\partial u}{\partial y}(s,y)\,ds + C\right)\\
&= \frac{\partial u}{\partial x}(x_0,y) - \int_{x_0}^{x}\frac{\partial^2 u}{\partial y^2}(s,y)\,ds\\
&= \frac{\partial u}{\partial x}(x_0,y) + \int_{x_0}^{x}\frac{\partial^2 u}{\partial x^2}(s,y)\,ds\\
&= \frac{\partial u}{\partial x}(x_0,y) + \frac{\partial u}{\partial x}(s,y)\Big|_{s=x_0}^{s=x}\\
&= \frac{\partial u}{\partial x}(x_0,y) + \frac{\partial u}{\partial x}(x,y) - \frac{\partial u}{\partial x}(x_0,y)\\
&= \frac{\partial u}{\partial x}(x,y),
\end{aligned}$$

as required.

In summary, the function v, given by an integral, has partial derivatives so that the Cauchy–Riemann equations hold in Ω, and hence f is holomorphic. □

In general, we may prove this theorem by appealing to Theorem 7.6.

LECTURE 10

Power series

In this lecture, we define and study complex power series. The proofs of many theorems about complex power series are almost identical to the proof of the corresponding theorem for real power series, and so we omit most proofs. Power series are important for several reasons.

(a) There are formulae for manipulating them, so they may be used for calculations, for instance, in MAPLE.

(b) Holomorphic functions may be expressed in power series, and, as we will see later, *vice versa*.

1. Definition and convergence

A (complex) power series is an expression of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{10.1}$$

where the *centre* z_0 and the *coefficients* a_n are all fixed complex numbers, and the *variable* z is complex. We take $(z - z_0)^0$ to be 1 for all z, even when $z = z_0$.

The first problem is whether the sum (10.1) makes sense. If $z = z_0$, then the sum converges trivially; for other values of z, the series may or may not converge.

THEOREM 10.1. *Every power series* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *has a* radius of convergence ρ, *given by the formulae*

$$\rho = \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1} = \left(\lim_{k\to\infty} \sup_{n\geq k} |a_n|^{1/n}\right)^{-1}.$$

The radius of convergence $\rho \in [0, +\infty]$ *has the following properties:*

(a) $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges *if* $|z - z_0| < \rho$

(b) $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ does not converge *if* $|z - z_0| > \rho$

(c) $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ *may converge for no, some or all* z *such that* $|z - z_0| = \rho$.

If $\rho = 0$, then the series converges only when $z = z_0$, while if $\rho = +\infty$, then the series converges for all $z \in \mathbb{C}$. We often write $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in $\mathrm{B}(a, r)$ to mean that the sum converges for all z in $\mathrm{B}(a, r)$. When $a = z_0$, this means that $r \leq \rho$.

There are several tests for convergence that carry over to complex power series from the theory of real power series.

LEMMA 10.2 (The ratio test). *The radius of convergence is given by*

$$\rho = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|},$$

as long as the limit exists or is $+\infty$.

LEMMA 10.3 (The root test). *The radius of convergence is given by*

$$\rho = \lim_{n\to\infty} \frac{1}{|a_n|^{1/n}}\,,$$

as long as the limit exists or is $+\infty$.

In order to use the root test when the coefficients involve factorials, the following facts, collectively known as *Stirling's formula*, may sometimes be useful:

$$\lim_{n\to\infty} \frac{\ln(n!)}{\ln((2\pi)^{1/2}\, e^{-n}\, n^{n+1/2})} = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{n!}{(2\pi)^{1/2}\, e^{-n}\, n^{n+1/2}} = 1.$$

There are variations on this that may also be useful.

2. Examples

EXERCISE 10.4. Find the centre and radius of convergence of $\displaystyle\sum_{n=0}^{\infty} 3^{n-1}(z+1)^n$.

ANSWER.

This is a *geometric series* whose sum is $\big[3(1-3(z+1))\big]^{-1}$.

EXERCISE 10.5. Find the centre and radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(z-2)^n}{n!}$.

ANSWER.

This is an *exponential series* whose sum is e^{z-2}.

EXERCISE 10.6. Fix $\alpha \in \mathbb{C}$. Find the centre and radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} a_n(z-1)^n$, where $a_0 = 1$ and, for all $n \in \mathbb{Z}^+$,

$$a_n = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}.$$

ANSWER.

This is a *binomial series*, which converges to $(1+(z-1))^\alpha$, that is, z^α.

EXERCISE 10.7. Find the centre and radius of convergence of the series

$$\sum_{k=1}^{\infty} \frac{k(k+1)(k^2+2)}{2^k}(z+3)^k.$$

ANSWER.

EXERCISE 10.8. Find the centre and radius of convergence of the series

$$\sum_{j=1}^{\infty} \frac{(-j)^j}{j!}(z-5)^j.$$

ANSWER.

EXERCISE 10.9. Find the centre and radius of convergence of the series

$$\sum_{\substack{m \in \mathbb{N} \\ m \text{ even}}} \frac{z^m}{2^m}.$$

ANSWER.

Note that $|a_m|^{-1/m}$ is equal to 0 if m is odd and to 2 if m is even, and

$$\limsup_{m \to \infty} |a_m|^{-1/m} = 2.$$

3. The algebra and calculus of power series

THEOREM 10.10. *Suppose that the complex power series* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *and* $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ *both converge in* $\mathrm{B}(z_0,\rho)$*, and that* $c \in \mathbb{C}$*. Then the following series also converge in* $\mathrm{B}(z_0,\rho)$*:*

(a) $\displaystyle\sum_{n=0}^{\infty} c\, a_n(z-z_0)^n$*, and its sum is* $\displaystyle c\sum_{n=0}^{\infty} a_n(z-z_0)^n$*;*

(b) $\displaystyle\sum_{n=0}^{\infty}(a_n+b_n)(z-z_0)^n$*, and its sum is* $\displaystyle\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=0}^{\infty} b_n(z-z_0)^n$*;*

(c) $\displaystyle\sum_{n=0}^{\infty} c_n(z-z_0)^n$*, where* $c_0 = a_0\, b_0$*,* $c_1 = a_0\, b_1 + a_1\, b_0$*, ..., and* $\displaystyle c_n = \sum_{j=0}^{n} a_j\, b_{n-j}$*,*

and its sum is $\displaystyle\sum_{n=0}^{\infty} a_n(z-z_0)^n \times \sum_{n=0}^{\infty} b_n(z-z_0)^n$*.*

PROOF. We omit the proof. □

THEOREM 10.11. *Suppose that* $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ *in* $\mathrm{B}(z_0,\rho)$*, and that* $\rho > 0$*. Then* $f'(z)$ *exists in* $\mathrm{B}(z_0,\rho)$*, and*

$$f'(z) = \sum_{n=0}^{\infty} a_n n(z-z_0)^{n-1} = \sum_{m=0}^{\infty} a_{m+1}(m+1)(z-z_0)^m$$

in $\mathrm{B}(z_0,\rho)$*.*

PROOF. We omit the proof. □

This theorem allows us to differentiate power series term by term.

COROLLARY 10.12. *Suppose that* $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ *in* $\mathrm{B}(z_0,\rho)$*. Then* f *may be differentiated as many times as desired, and*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)a_n(z-z_0)^{n-k}.$$

In particular,

$$f^{(k)}(z_0) = k!\, a_k.$$

Further, the real-valued functions u *and* v*, such that* $f(x+iy) = u(x,y)+iv(x,y)$*, may be differentiated as many times as desired, and all their partial derivatives are continuous.*

PROOF. We omit the proof, but point out that induction may be used. □

COROLLARY 10.13. *Suppose that* $g(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ *in* $\mathrm{B}(z_0,\rho)$*, and that* $\varepsilon > 0$*. If* $g(z_0+t) = 0$ *for all real* t *in* $(-\varepsilon,\varepsilon)$*, then* $g(z) = 0$ *for all* z *in* $\mathrm{B}(z_0,\rho)$*.*

PROOF. First, we prove by induction that $g^{(k)}(z_0+t) = 0$ for all real t in $(-\varepsilon,\varepsilon)$ and all natural numbers n. This is true when $k=0$, by hypothesis. Suppose now that $g^{(k)}(z_0+t) = 0$ for all real t in $(-\varepsilon,\varepsilon)$ for some natural number k, and write $f(z) = g^{(k)}(z)$ for all $z \in \mathrm{B}(z_0,\rho)$, so that $f(z_0+t) = 0$ for all real t in $(-\varepsilon,\varepsilon)$. By Theorem 10.11, applied as many times as necessary, f is holomorphic in $\mathrm{B}(z_0,\rho)$.

As usual, we write $z_0 = x_0 + iy_0$, and $f(x + iy) = u(x, y) + iv(x, y)$. Then $u(x_0+t, y_0) = 0$ and $u(x_0+t, y_0) = 0$ for all $t \in (-\varepsilon, \varepsilon)$. Hence $\partial u/\partial x(x_0+t, y_0) = 0$ and $\partial v/\partial x(x_0 + t, y_0) = 0$ for all $t \in (-\varepsilon, \varepsilon)$. Since

$$f'(z_0 + it) = \frac{\partial u}{\partial x}(x_0 + t, y_0) + i\frac{\partial v}{\partial x}(x_0 + t, y_0),$$

we see that $f'(z_0 + it) = 0$ for all $t \in (-\varepsilon, \varepsilon)$. This implies that $g^{(k+1)}(z_0 + t) - 0$ for all $t \in (-\varepsilon, \varepsilon)$, and the inductive step is established.

To conclude, we recall that the power series for $g^{(k)}$ is of the form

$$\sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)a_n(z-z_0)^{n-k},$$

whence $g^{(k)}(z_0) = k!\,a_k$, and so $a_k = 0$. This is true for all k, and hence $g(z) = \sum_{n=0}^{\infty} 0(z - z_0)^n = 0$ for all $z \in \mathrm{B}(z_0, \rho)$. □

Corollary 10.14. *Suppose that* $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ *and moreover that* $g(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$ *in* $\mathrm{B}(z_0, \rho)$. *If* $f(z_0 + t) = g(z_0 + t)$ *for all* $t \in (-\varepsilon, \varepsilon)$, *then* $f(z) = g(z)$ *for all* $z \in \mathrm{B}(z_0, \rho)$.

Proof. We apply the previous corollary to $f - g$. □

There is a stronger version of Corollary 10.13 that says that if f is holomorphic in a domain Ω, and $f(z_n) = 0$ for distinct points $z_n \in \Omega$ such that $z_n \to z_0 \in \Omega$ as $n \to \infty$, then $f = 0$. This leads to a stronger version of the last corollary.

These last corollaries will later lead us to the concept of *analytic continuation*: if a function is defined in a domain Ω, then it is determined by its values in a small set. In particular, if f is an entire function, then it is determined by its values on $\mathbb{R}$. This explains why, in finding an analytic function with certain properties, it suffices to find it on $\mathbb{R}$; this fact is useful in dealing with harmonic functions.

LECTURE 11

Exponential, Hyperbolic and Trigonometric functions

1. The exponential function

DEFINITION 11.1. We define the exponential series by the formula

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \forall z \in \mathbb{C}.$$

This is the only possible power series extension of the real exponential into the whole complex plane. Indeed, we saw in the previous lecture that a complex power series $\sum_{n=0}^{\infty} a_n z^n$ is determined by its values on any interval $(-\varepsilon, \varepsilon)$.

THEOREM 11.2. *The exponential series has the following properties:*

(1) $\exp(0) = 1$*;*
(2) $\exp(z + w) = \exp(z)\exp(w)$ *for all* $z, w \in \mathbb{C}$*;*
(3) $\exp(-z) = \exp(z)^{-1}$ *for all* $z \in \mathbb{C}$*;*
(4) $\exp(z) \neq 0$ *for all* $z \in \mathbb{C}$*;*
(5) $\exp'(z) = \exp(z)$ *for all* $z \in \mathbb{C}$*;*
(6) if a function $f : \mathbb{C} \to \mathbb{C}$ *satisfies* $f(0) = 1$ *and* $f'(z) = f(z)$ *for all* $z \in \mathbb{C}$*, then* $f(z) = \exp(z)$ *for all* $z \in \mathbb{C}$*;*
(7) $\exp(x + iy) = e^x(\cos(y) + i\sin(y))$ *for all* $x, y \in \mathbb{R}$*.*

PROOF. Item (1) is obvious.

Item (2) follows from the binomial theorem and manipulation of sums:

$$\sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} z^j w^{n-j} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{z^j}{j!} \frac{w^{n-j}}{(n-j)!} = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{z^j}{j!} \frac{w^m}{m!}.$$

Item (3) and (4) follows from the first two: $\exp(z)\exp(-z) = \exp(z + (-z)) = 1$, and so $\exp(z)$ cannot be 0 and moreover $\exp(z)^{-1} = \exp(-z)$.

Item (5) follows from the calculus of power series.

To prove (6), we consider the derivative of the quotient function $f/\exp$:

$$\Big(\frac{f}{\exp}\Big)'(z) = \frac{f'(z)\exp(z) - f(z)\exp'(z)}{\exp^2(z)} = \frac{f(z)\exp(z) - f(z)\exp(z)}{\exp^2(z)} = 0.$$

Thus $f/\exp$ is constant; when $z = 0$ its value is 1, so it is identically 1, and $f(z) = \exp(z)$ for all $z \in \mathbb{C}$.

Finally, we prove (7). From part (2), $\exp(z) = \exp(x)\exp(iy)$; using results for real power series, we see that $\exp(x) = e^x$ and $\exp(iy) = \cos(y) + i\sin(y)$. $\square$

COROLLARY 11.3. *The exponential* $\exp$ *maps* $\mathbb{C}$ *onto* $\mathbb{C} \setminus \{0\}$*, and* $\exp(z_1) = \exp(z_2)$ *if and only if* $z_1 - z_2 \in 2\pi i\mathbb{Z}$*.*

PROOF. From the theorem, $\exp(z) \neq 0$ for all $z \in \mathbb{C}$. If $w \neq 0$, then take $z = \ln(|w|) + i\,\mathrm{Arg}(w)$; we may check that $\exp(z) = w$. Hence exp maps $\mathbb{C}$ *onto* $\mathbb{C} \setminus \{0\}$. If $\exp(z_1) = \exp(z_2)$, then $\exp(z_1 - z_2) = 1$, and so $\mathrm{Re}(z_1 - z_2) = 0$ and $\mathrm{Im}(z_1 - z_2) \in 2\pi\mathbb{Z}$, by trigonometry. $\square$

The fact that $\exp(z) = \exp(z + 2\pi i k)$ for all $k \in \mathbb{Z}$ is called the *periodicity* of exp. Often we write e^z rather than $\exp(z)$.

2. The hyperbolic functions

DEFINITION 11.4. We define the hyperbolic cosine and sine by the formulae

$$\cosh(z) = \frac{e^z + e^{-z}}{2} = \sum_{n\in\mathbb{N}} \frac{z^{2n}}{(2n)!} \qquad \text{and} \qquad \sinh(z) = \frac{e^z - e^{-z}}{2} = \sum_{n\in\mathbb{N}} \frac{z^{2n+1}}{(2n+1)!}$$

for all $z \in \mathbb{C}$.

It follows from the definitions that these are the only power series extensions of the real functions cosh and sinh into the whole complex plane.

THEOREM 11.5. *The hyperbolic sine and cosine have the following properties:*

$$\begin{array}{llll} (i) & \cosh(-z) = \cosh(z) & (ii) & \sinh(-z) = -\sinh(z) \\ (iii) & \cosh'(z) = \sinh(z) & (iv) & \sinh'(z) = \cosh(z) \\ (v) & \cosh(z + 2\pi i k) = \cosh(z) & (vi) & \sinh(z + 2\pi i k) = \sinh(z) \end{array}$$

for all $z \in \mathbb{C}$ and all $k \in \mathbb{Z}$. Further, for all $w, z \in \mathbb{C}$,

$$\begin{array}{ll} (vii) & \cosh(z + w) = \cosh(z)\cosh(w) + \sinh(z)\sinh(w) \\ (viii) & \sinh(z + w) = \sinh(z)\cosh(w) + \cosh(z)\sinh(w) \end{array}$$

$$(ix) \quad \cosh^2(z) - \sinh^2(z) = 1.$$

Finally, for all $x, y \in \mathbb{R}$,

$$\begin{array}{ll} (x) & \cosh(x + iy) = \cosh(x)\cos(y) + i\sinh(x)\sin(y) \\ (xi) & \sinh(x + iy) = \sinh(x)\cos(y) + i\cosh(x)\sin(y). \end{array}$$

PROOF. Items (i) and (ii) follow from the definition. Items (iii) and (iv) follow from the definition and the calculus of power series. Items (v) and (vi), often called the "periodicity of cosh and sinh", follow from the periodicity of exp.

To prove item (vii), observe that

$$\begin{aligned} &4(\cosh(z)\cosh(w) + \sinh(z)\sinh(w)) \\ &= (e^z + e^{-z})(e^w + e^{-w}) + (e^z - e^{-z})(e^w - e^{-w}) \\ &= e^{z+w} + e^{z-w} + e^{-z+w} + e^{-z-w} + e^{z+w} - e^{z-w} - e^{-z+w} + e^{-z-w} \\ &= 2(e^{z+w} + e^{-z-w}) = 4\cosh(z + w), \end{aligned}$$

and divide by 4. Items (viii) and (ix) may be proved similarly. Items (x) and (xi) follow from the identifications $\sinh(iy) = i\sin(y)$ and $\cosh(iy) = \cos(y)$, which may be seen using power series. $\square$

EXERCISE 11.6. Expand $\cosh(z + i\pi/2)$ and $\sinh(z + i\pi/2)$ in terms of $\cosh(z)$ and $\sinh(z)$. Hence find $\cosh(z + i\pi)$.

ANSWER.

3. The trigonometric functions

DEFINITION 11.7. We define the cosine and sine by the formulae

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = \sum_{n\in\mathbb{N}} \frac{(-1)^n \, z^{2n}}{(2n)!}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = \sum_{n\in\mathbb{N}} \frac{(-1)^n \, z^{2n+1}}{(2n+1)!}$$

for all $z \in \mathbb{C}$.

It follows from the definitions that these are the only power series extensions of the real functions cos and sin into the whole complex plane.

EXERCISE 11.8. Show that the complex sine and cosine have the following properties:

$$\begin{array}{llll} \text{(i)} & \cos(-z) = \cos(z) & \text{(ii)} & \sin(-z) = -\sin(z) \\ \text{(iii)} & \cos'(z) = -\sin(z) & \text{(iv)} & \sin'(z) = \cos(z) \\ \text{(v)} & \cos(z + 2\pi k) = \cos(z) & \text{(vi)} & \sin(z + 2\pi k) = \sin(z) \end{array}$$

for all $z \in \mathbb{C}$ and all $k \in \mathbb{Z}$. Show also that

$$\text{(vii)} \quad \cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$$

$$\text{(viii)} \quad \sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$$

$$\text{(ix)} \quad \cos^2(z) + \sin^2(z) = 1$$

for all $w, z \in \mathbb{C}$.

EXERCISE 11.9. Express the following as functions of z:

(a) $\sinh(iz)$, (b) $\cosh(iz)$, (c) $\sin(iz)$, (d) $\cos(iz)$.

ANSWER.

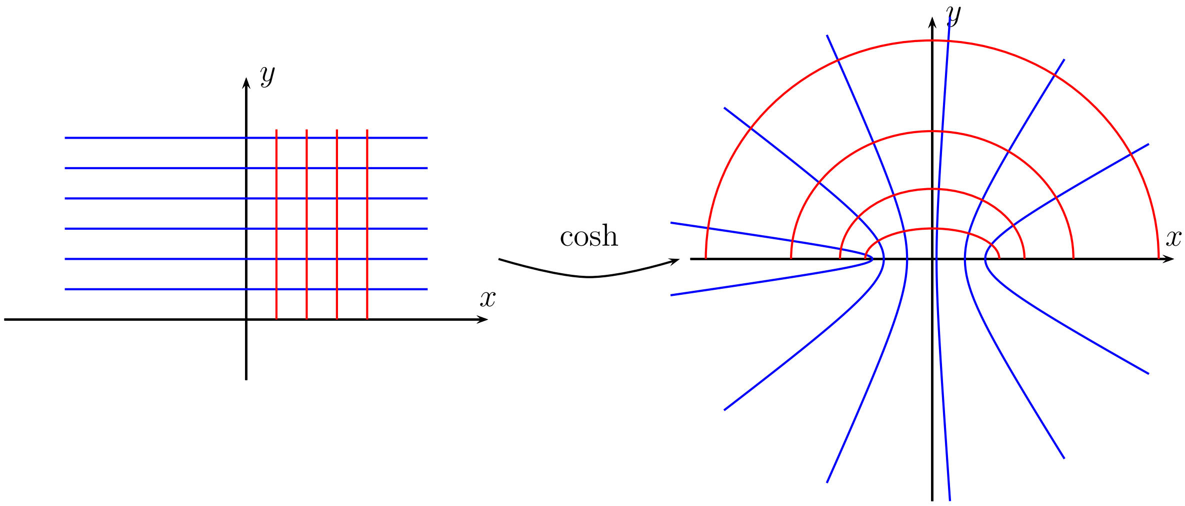

FIGURE 11.1. The map cosh

4. Graph sketching

EXAMPLE 11.10. Find the images of the lines $\mathrm{Re}(z) = c$ and $\mathrm{Im}(z) = d$ under the function cosh, and sketch these for various values of c and d.

ANSWER.

EXERCISE 11.11. Find the images of the lines $\text{Re}(z) = c$ and $\text{Im}(z) = d$ under the functions sinh, sin and cos, and sketch these for various values of c and d.

5. Harmonic functions

EXERCISE 11.12. Suppose that $u(x, y) = \cos(x)\sinh(y)$ for all $x, y \in \mathbb{R}$. Show that u is harmonic and find its harmonic conjugate.

ANSWER.

LECTURE 12

Logarithms and roots

In this lecture, we investigate some inverse functions, in particular, roots and logarithms.

This part of the theory can become quite complicated, and we start by just considering a few examples.

1. Some algebra

Many complex functions are not one-to-one. This means that inverse functions are not well-defined. One solution is to deal with "multi-valued functions", and another is to restrict the domain of the original function. The archetypical example is the argument function, and we usually write $\arg(z)$ to indicate the multi-valued function and $\operatorname{Arg}(z)$ to indicate the particular choice where the values lie in the range $(-\pi, \pi]$.

EXAMPLE 12.1. Suppose that $w = z^2$. Then we may write $z = \sqrt{w}$ or $w^{1/2}$; the question is what this means. Unfortunately, different writers mean different things. Some writers mean that z may be any of the two possible values; others mean that a particular choice has been made. We will try to use the expressions above for the multi-valued function, and add the words "principal value" (symbolically, PV) or "principal branch" to indicate that a particular choice is being made. In particular, we define

$$\operatorname{PV} w^{1/2} = \begin{cases} |w|^{1/2} e^{i \operatorname{Arg}(w)/2} & \text{if } w \neq 0 \\ 0 & \text{if } w = 0. \end{cases}$$

In words, we might say "the principal branch of the square root".

We may think of the graph of the multi-valued function $w^{1/2}$ as being like two copies of the plane, slit along the branch cut, and then joined together cleverly. It is not possible to do this in three dimensions, but it is possible in four dimensions.

EXAMPLE 12.2. Suppose that $w = e^z$ and $z = x + iy$. Then $w = e^x e^{iy}$, so

$$|w| = e^x \qquad \text{and} \qquad \operatorname{Arg} w = \operatorname{Arg} e^{iy}.$$

Then $x = \ln |w|$, and x is single-valued, but $y = \operatorname{Arg}(w) + 2\pi k$, where $k \in \mathbb{Z}$; and y is multiple-valued. When $w \neq 0$, we write $z = \log(w)$ to indicate that z may be any one of the infinitely many complex numbers such that $e^z = w$ and we write $z = \operatorname{Log}(w)$ to indicate the choice that $z = \ln |w| + i \operatorname{Arg}(w)$.

It is important to be very clear in what you write about multi-valued functions; always include the words "multi-valued" or "principal value" or the symbol PV.

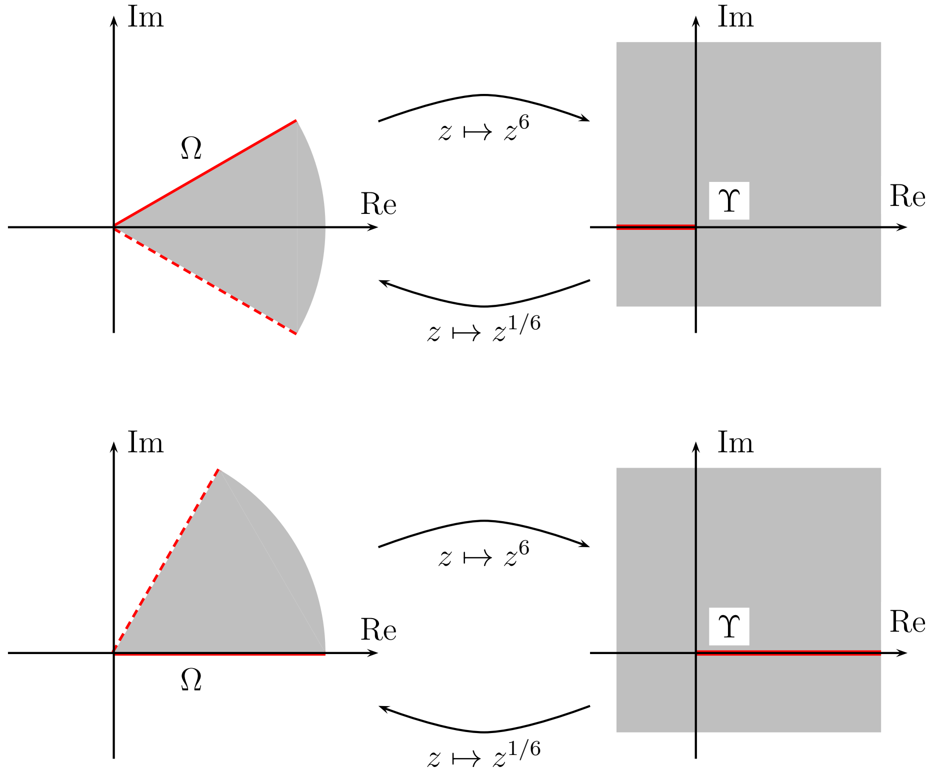

FIGURE 12.1. Two branches of the 6th root function

2. Defining nth roots

Suppose that $f(z) = z^n$, where $n \geq 2$. Then f is not one-to-one. However, if we restrict f to a region Ω such as $\{z \in \mathbb{C} : |\operatorname{Arg}(z)| < \pi/n\}$, then f becomes one-to-one, and we may define an inverse function, $z \mapsto z^{1/n}$. If we choose a different region, we get a different inverse function. The *principal value* of the nth root is given by

$$\operatorname{PV} z^{1/n} = \exp\left(\frac{\operatorname{Log}(z)}{n}\right) = |z|^{1/n} e^{i \operatorname{Arg}(z)/n}.$$

PROPOSITION 12.3. *The function* $\operatorname{PV} z^{1/n}$ *is differentiable in* $\mathbb{C} \setminus (-\infty, 0]$.

PROOF. This function is the composition of Log, followed by division by n, followed by the exponential. Compositions of differentiable functions are differentiable. □

Note that where this function is differentiable and where it is continuous are nearly the same; the only difference is that $\operatorname{PV} z^{1/n}$ is continuous at 0 but not differentiable.

All the possible inverse functions are constant multiples of each other on connected sets where both are defined, where the multiplying factors are nth roots of unity. The different possible inverse functions are called *branches* of the nth root function.

Figure 12.1 illustrates some possibilities in the case where $n = 6$. In the upper figure, we usually choose the inverse function on the negative real axis to map to the upper side of the cone. It is given by $f^{-1}(w) = |w|^{1/6}e^{i\,\mathrm{Arg}(w)/6}$.

In the lower figure, we usually choose the inverse function to map the positive real axis to the lower side of the cone. It is given by $f^{-1}(w) = |w|^{1/6}e^{i\,\mathrm{Arg}^0(w)/6}$, where Arg^0 denotes the choice of argument in the range $[0, 2\pi)$.

The first inverse function is the "standard" choice; it is called the *principal value* of the 6th root; the notation $\mathrm{PV}\, z^{1/6}$ should be used.

The regions Ω that we chose are relatively simple, but we could have chosen more complicated versions, such as a region between two curves coming out of the origin. This would have led to a curved branch cut.

In general, when we restrict a function f to a smaller domain in order to define an inverse function for f, we try to make the smaller domain Ω as large as possible, so that the domain Υ of the inverse function in as large as possible. The boundary of Υ is called the *branch cut* (or cuts, as it may have a number of connected pieces). By choosing Ω and Υ carefully, we may usually avoid having an inverse function with discontinuities where we want to work. Although we may vary these sets by varying the branch cut, there are some points, called *branch points*, which must appear in any branch cut. For the case of the function $f(z) = z^n$, the only branch point is 0.

3. Square roots of polynomials

The next level of difficulty with inverse functions is with defining square roots of polynomials.

First, we consider the problem of defining $(z \pm 1)^{1/2}$. The principal branch of $(z-1)^{1/2}$ is given by

$$\mathrm{PV}(z-1)^{1/2} = \begin{cases} |z-1|^{1/2}e^{i\,\mathrm{Arg}(z-1)/2} & \text{if } z \neq 1 \\ 0 & \text{if } z = 1. \end{cases}$$

This function is continuous in $\mathbb{C} \setminus (-\infty, 1)$; the branch cut is the infinite interval $(-\infty, 1]$ and the branch point is 1. In much the same way, the principal branch of $(z+1)^{1/2}$ is given by

$$\mathrm{PV}(z+1)^{1/2} = \begin{cases} |z+1|^{1/2}e^{i\,\mathrm{Arg}(z+1)/2} & \text{if } z \neq -1 \\ 0 & \text{if } z = -1. \end{cases}$$

Now we consider the problem of defining $(z^2-1)^{1/2}$. We could define

$$(z^2-1)^{1/2} = \mathrm{PV}(z^2-1)^{1/2} = \begin{cases} |z^2-1|^{1/2}e^{i\,\mathrm{Arg}(z^2-1)/2} & \text{when } z \neq \pm 1 \\ 0 & \text{when } z = \pm 1. \end{cases} \tag{12.1}$$

Alternatively, since $z^2 - 1 = (z-1)(z+1)$, we could define

$$\begin{aligned}(z^2-1)^{1/2} &= \mathrm{PV}(z-1)^{1/2}\,\mathrm{PV}(z+1)^{1/2} \\ &= \begin{cases} |z^2-1|^{1/2}e^{i\,\mathrm{Arg}(z-1)/2 + i\,\mathrm{Arg}(z+1)/2} & \text{when } z \neq \pm 1 \\ 0 & \text{when } z = \pm 1. \end{cases} \end{aligned} \tag{12.2}$$

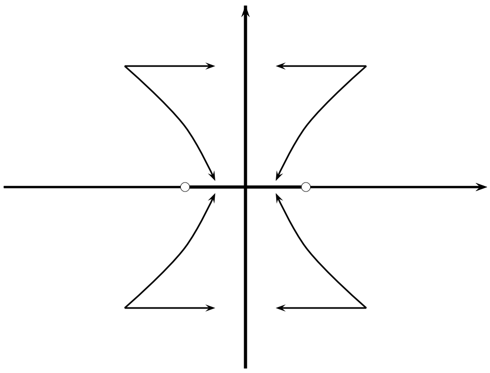

FIGURE 12.2. The discontinuities of $\operatorname{Arg}(z^2-1)$

Let us now determine whether these possible definitions coincide and where they are continuous and where they are differentiable.

Exercise 12.4. Find an explicit formula for the "branch of $(1 - z^2)^{1/2}$ that is continuous except on the real intervals $(-\infty, -1)$ and $(1, \infty)$ and that takes the value 1 at 0". Where is this function differentiable?

Answer.

The next step up in complication is the square root of a cubic polynomial. For instance, we might take the multi-valued function $w = (z(z^2 - 1))^{1/2}$. "Graphs" of functions like this are called *elliptic curves*. They are important in number theory—they are central to Andrew Wiles' proof of Fermat's last theorem—and they are important in cryptography—they give rise to "better" codes.

Exercise 12.5. Show that there is a choice of definition of $(z(z^2 - 1))^{1/2}$ that is continuous except on the intervals $(-\infty, -1)$ and $(0, 1)$.

The graph of the multi-valued function is like two copies of the plane, with slits along the branch cuts, joined together appropriately.

4. More examples

The logarithm log is multi-valued because the argument arg is multi-valued. There are two common choices of argument, one between $-\pi$ and π, and the other between 0 and 2π. Both of these give logarithms with the property that $\log(i) = \pi/2$, but the first has its branch cut along the negative real axis and the second has its branch cut along the positive real axis. If we want to deal with a logarithm that is continuous around 1, the first choice is better, but if we want to arrange continuity around -1, then the second is better.

Soon we will need to look at expressions such as $\log(w + (w^2 - 1)^{1/2})$. We try to choose a branch of log and a branch of the square root to make the function continuous in as large a domain as possible. Usually this function will be differentiable where it is continuous, with some possible exceptions.

LECTURE 13

Inverses of exponential and related functions

In this lecture, we discuss the inverse functions for the exponential, hyperbolic and trigonometric functions introduced in the last lecture.

1. Inverse functions

The following exercise is a warm-up.

EXERCISE 13.1. Fix w in $\mathbb{C}$. Find all z in $\mathbb{C}$ such that
(a) $\exp(z) = w$, (b) $\cosh(z) = w$, (c) $\sinh(z) = w$.

ANSWER.

Note that cosh is even, so if $\cosh(z) = w$, then $\cosh(-z) = w$. Ideally, the solution set should be chosen so that this fact is evident.

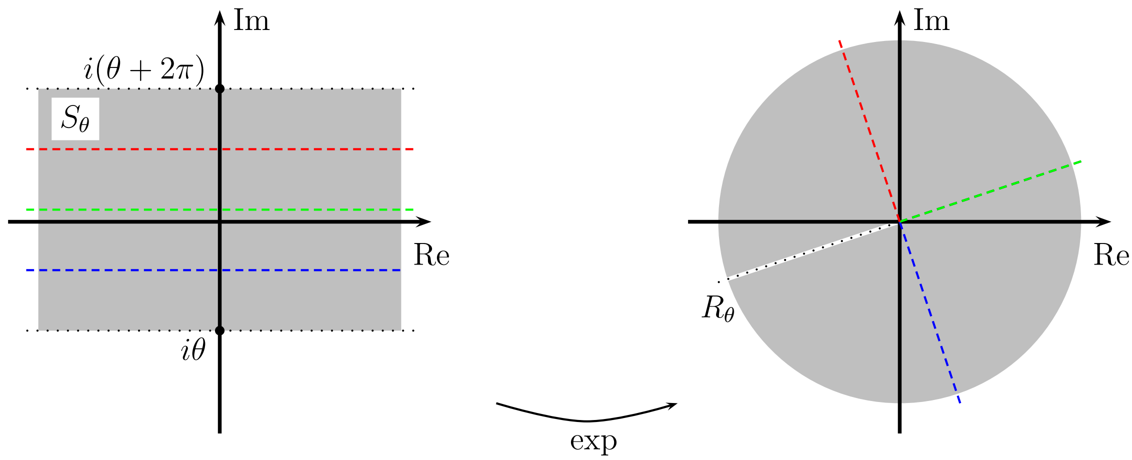

FIGURE 13.1. The exponential map

2. The exponential function

Now we consider inverse functions for the exponential function. Here is one candidate.

DEFINITION 13.2. The *principal branch* of the complex logarithm is the function Log from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C}$, given by

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z),$$

where $\mathrm{Arg}(z)$ takes values in the range $(-\pi, \pi]$.

By properties of the exponential, if $z = re^{i\theta}$, then

$$e^{\mathrm{Log}(z)} = e^{\ln(r)+i\theta} = re^{i\theta} = z;$$

however, if $z = x + iy$, then

$$\mathrm{Log}(e^z) = \ln|e^z| + i\,\mathrm{Arg}(e^z) = \ln(e^x) + i\,\mathrm{Arg}(e^{iy}) = x + iy + 2\pi ik,$$

where the integer k is such that $-\pi < y + 2\pi k \leq \pi$, and it is possible that $\mathrm{Log}(e^z) \neq z$.

Other candidates for inverse function are sometimes more useful than Log. We define the ray R_θ and the horizontal strip S_θ (where $\theta \in \mathbb{R}$) in the complex plane by

$$R_\theta = \{w \in \mathbb{C} : \mathrm{Arg}(w) - \theta \in 2\pi\mathbb{Z}\}$$
$$S_\theta = \{z \in \mathbb{C} : \theta < \mathrm{Im}(z) < \theta + 2\pi\}.$$

The exponential map exp takes horizontal lines to rays, and is one-to-one and onto from the open horizontal strip S_θ to $\mathbb{C} \setminus R_\theta$.

We may define an inverse function $\log_\theta$ from $\mathbb{C} \setminus R_\theta$ to S_θ:

$$\log_\theta(w) = \ln|w| + i\arg_\theta(w),$$

where $\arg_\theta(w)$ is the argument in the range $(\theta, \theta + 2\pi)$.

As θ varies, we get different inverse functions. These inverse functions are called *branches* of the complex logarithm, and the rays R_θ are called *branch cuts*. Different branches of the logarithm differ by a constant in connected open sets where they are both defined, The point 0, which is common to all the branch cuts, is called a *branch point*.

LEMMA 13.3. *For any branch* $\log_\theta$ *of the complex logarithm,*

$$\log_\theta'(w) = \frac{1}{w}$$

for all $w \in \mathbb{C} \setminus R_\theta$.

PROOF. We apply the Cauchy–Riemann equations. □

The notation $\log_\theta$ is not standard, and we will not use it any more. Rather, we use the expression "the branch of the logarithm with imaginary part in $(\theta, \theta + 2\pi)$".

A different sort of "inverse function" of the exponential function is the "multi-function" log from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C}$ given by

$$\log(z) = \ln|z| + i\arg(z).$$

This is a "multifunction" in the sense that it takes multiple values, because $\arg(z)$ takes multiple values.

3. Complex powers

We define complex powers of complex numbers using exponentials and logarithms.

DEFINITION 13.4. Given $z \in \mathbb{C} \setminus \{0\}$ and $\alpha \in \mathbb{C}$, we define

$$z^\alpha = \exp(\alpha \log(z)).$$

The *principal branch of* z^α is found by using Log, the principal branch of the logarithm. That is, $\mathrm{PV}\, z^\alpha = \exp(\alpha \operatorname{Log}(z))$.

The possible values of z^α are $\exp(\alpha \operatorname{Log}(z) + 2\pi i k \alpha)$ where $k \in \mathbb{Z}$. Different values of k may give very different values of z^α.

LEMMA 13.5. *The function* $z \mapsto \mathrm{PV}\, z^\alpha$ *is differentiable in* $\mathbb{C} \setminus (-\infty, 0]$, *with derivative* $\alpha \, \mathrm{PV}\, z^\alpha / z$.

PROOF. This follows because Log is differentiable in $\mathbb{C} \setminus (-\infty, 0]$. □

EXERCISE 13.6. Compute the possible values of i^i; which is the principal value?

ANSWER.

Often we write e^z rather than $\exp(z)$. Note that this is ambiguous, since complex powers are multi-valued! So arguably $\exp(z)$ is better notation.

4. The inverse hyperbolic sine

Last lecture, we defined the hyperbolic sine and cosine, and established some of their properties. Now we consider the inverse function(s) of sinh.

EXERCISE 13.7. Find all z in $\mathbb{C}$ such that $\sinh(z) = w$.

ANSWER.

The principal branch of the inverse hyperbolic sine function is given by

$$\operatorname{PV}\sinh^{-1} w = \operatorname{Log}(w + \operatorname{PV}(w^2 + 1)^{1/2}).$$

It is easy to show that for any complex number w,

$$\sinh \operatorname{PV}\sinh^{-1} w = w;$$

however, it is not necessarily true that $\operatorname{PV}\sinh^{-1} \sinh w = w$.

EXERCISE 13.8. Find the possible values of $\operatorname{PV}\sinh^{-1} \sinh w - w$, where $w \in \mathbb{C}$.

ANSWER.

Both the logarithm and the square root are possible causes of discontinuity. The function $\operatorname{PV}(w^2+1)^{1/2}$ is continuous as long as w^2+1 is not in the interval $(-\infty, 0]$, and the logarithm is continuous as long as $w + \operatorname{PV}(w^2 + 1)^{1/2}$ is not in the interval $(-\infty, 0]$.

On the one hand, if $w^2 + 1$ is not in $(-\infty, 0]$, then w^2 is not in $(-\infty, -1]$. So one possible discontinuity is when $w = iv$, where $|v| \geq 1$.

On the other hand, we may try to solve the equation $w + \mathrm{PV}(w^2+1)^{1/2} = -t$ for $t \in [0, \infty)$; we get

$$\begin{aligned} \mathrm{PV}(w^2+1)^{1/2} &= -t - w \\ w^2 + 1 &= t^2 + w^2 + 2tw \\ w &= \frac{1-t^2}{2t}, \end{aligned}$$

and so w is real. But if w is real, then

$$w + \mathrm{PV}(w^2+1)^{1/2} = w + (w^2+1)^{1/2} > 0,$$

and so $w + \mathrm{PV}(w^2+1)^{1/2}$ is not in the interval $(-\infty, 0]$. Thus the only possible discontinuities are when $w = iv$, where v is real and $|v| \geq 1$.

LEMMA 13.9. *The principal branch of the inverse hyperbolic sine function is differentiable in* $\mathbb{C} \setminus ([i, +i\infty) \cup (-i\infty, -i])$. *Further,*

$$\big(\mathrm{PV}\sinh^{-1}\big)'(w) = \frac{1}{\mathrm{PV}\sqrt{w^2+1}}.$$

PROOF. We compute the derivative:

$$\begin{aligned} \frac{d\sinh^{-1}(w)}{dw} &= \frac{d\,\mathrm{Log}(w + \mathrm{PV}(w^2+1)^{1/2})}{dw} \\ &= \frac{1 + w/\mathrm{PV}(w^2+1)^{1/2}}{w + \mathrm{PV}\sqrt{w^2+1}} \\ &= \frac{1}{\mathrm{PV}(w^2+1)^{1/2}}, \end{aligned}$$

as required. □

The inverse hyperbolic cosine may be treated in a similar way. We define

$$\mathrm{PV}\cosh^{-1}(w) = \mathrm{Log}(w + \mathrm{PV}(w+1)^{1/2}\,\mathrm{PV}(w-1)^{1/2}).$$

EXERCISE 13.10. Show that

$$\frac{d\,\mathrm{PV}\cosh^{-1}(w)}{dw} = \frac{1}{\mathrm{PV}(w+1)^{1/2}\,\mathrm{PV}(w-1)^{1/2}}$$

for most $w \in \mathbb{C}$. Where is $\cosh^{-1}$ not differentiable?

ANSWER.

5. The inverse trigonometric functions

We may define the inverse trigonometric functions using the formulae $\cos(iz) = \cosh(z)$ and $\sin(iz) = i\sinh(z)$. For example, if $z = \cos^{-1}(w)$, then $w = \cos(z) = \cosh(iz)$, and so $iz = \cosh^{-1}(w)$.

EXERCISE 13.11. What are the ranges of $\sinh^{-1}$, $\cos^{-1}$, and $\sin^{-1}$? Where are the branch cuts for $\cos^{-1}$ and $\sin^{-1}$?

LECTURE 14

Paths and path integrals† (Not examinable)

In this lecture, we define paths and path integrals, and see a key theorem about these. This material should be familiar to students who have studied multi-variable calculus.

The main question underlying this lecture is the relation between integration and differentiation. As before, we represent points in $\mathbb{R}^2$ by row vectors rather than by column vectors.

1. Curves

DEFINITION 14.1. A *curve* $\boldsymbol{\gamma}$ in $\mathbb{R}^2$ is a continuous function from an interval $[a, b]$ of real numbers into $\mathbb{R}^2$. We may write $\boldsymbol{\gamma}(t) = (\gamma_1(t), \gamma_2(t))$, where γ_1 and γ_2 are real-valued; then the continuity of $\boldsymbol{\gamma}$ is equivalent to the continuity of both γ_1 and γ_2.

The *initial point* of the curve is $\boldsymbol{\gamma}(a)$ and the *final point* of the curve is $\boldsymbol{\gamma}(b)$. The *range* of the curve is the set of points $\{\boldsymbol{\gamma}(t) : t \in [a, b]\}$.

A curve $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^2$ is said to be *closed* if $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$, and *simple* if $\boldsymbol{\gamma}(s) \neq \boldsymbol{\gamma}(t)$ when $s < t$, except perhaps if $s = a$ and $t = b$.

EXAMPLES 14.2. In Figure 14.1, the curve $\boldsymbol{\alpha}$ moves from $\boldsymbol{p}$ to $\boldsymbol{q}$, the curve $\boldsymbol{\beta}$ moves up the parabolic arc, the curve $\boldsymbol{\gamma}$ moves from the right to the left of the line segment and then back again, the curve $\boldsymbol{\delta}$ moves once around the circle in the anticlockwise direction, starting at the right-most point, the curve $\boldsymbol{\varepsilon}$ moves twice around the "infinity-shaped" figure, starting at the right-most point and moving upwards, and the curve $\boldsymbol{\zeta}$ moves along the "alpha-shaped" figure.

The curves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are simple but not closed; $\boldsymbol{\gamma}$ and $\boldsymbol{\varepsilon}$ are closed but not simple; $\boldsymbol{\delta}$ is both simple and closed; and $\boldsymbol{\zeta}$ is neither closed not simple. The curve $\boldsymbol{\gamma}$ in the figure "goes back on itself", while the curves $\boldsymbol{\varepsilon}$ and $\boldsymbol{\zeta}$ intersect themselves. The curve $\boldsymbol{\varepsilon}$ repeats itself.

DEFINITION 14.3. Suppose that $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^2$ and $\boldsymbol{\beta} : [c, d] \to \mathbb{R}^2$ are curves such that the final point of $[a, b]$ is the initial point of $[c, d]$ and the final point of $\boldsymbol{\alpha}$ is the initial point of $\boldsymbol{\beta}$, that is, $\boldsymbol{\alpha}(b) = \boldsymbol{\beta}(c)$. The *join* $\boldsymbol{\alpha} \sqcup \boldsymbol{\beta}$ of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is the curve

$$\boldsymbol{\alpha} \sqcup \boldsymbol{\beta}(t) = \begin{cases} \boldsymbol{\alpha}(t) & \text{when } a \leq t \leq b \\ \boldsymbol{\beta}(t) & \text{when } c \leq t \leq d. \end{cases}$$

Suppose that $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^2$ is a curve. The *reverse curve* $\boldsymbol{\gamma}^* : [-b, -a] \to \mathbb{R}^2$ of $\boldsymbol{\gamma}$ is given by

$$\boldsymbol{\gamma}^*(t) = \boldsymbol{\gamma}(-t) \qquad \text{where } -b \leq t \leq -a.$$

$$\begin{aligned}\boldsymbol{\alpha}(t) &= \boldsymbol{p} + (\boldsymbol{q} - \boldsymbol{p})t \\ &= \boldsymbol{p}(1-t) + \boldsymbol{q}t \quad \forall t \in [0,1]\end{aligned}$$

$$\boldsymbol{\beta}(t) = (\tfrac{3}{2} + t, t^2) \quad \forall t \in [0,1]$$

$$\boldsymbol{\gamma}(t) = (1 + 2t^2, -3) \quad \forall t \in [-1,1]$$

$$\boldsymbol{\delta}(t) = (1 + \tfrac{22}{10}\cos(t), \tfrac{22}{10}\sin(t)) \quad \forall t \in [0, 2\pi]$$

$$\boldsymbol{\varepsilon}(\theta) = (\cos(\theta), \sin(\theta)\cos(\theta)) \quad \forall \theta \in [0, 4\pi]$$

$$\begin{aligned}\boldsymbol{\zeta}(\theta) &= (\cos(\theta) - 2, \sin(\theta)\cos(\theta) - 3) \\ &\qquad \forall \theta \in [\pi/4, 7\pi/4]\end{aligned}$$

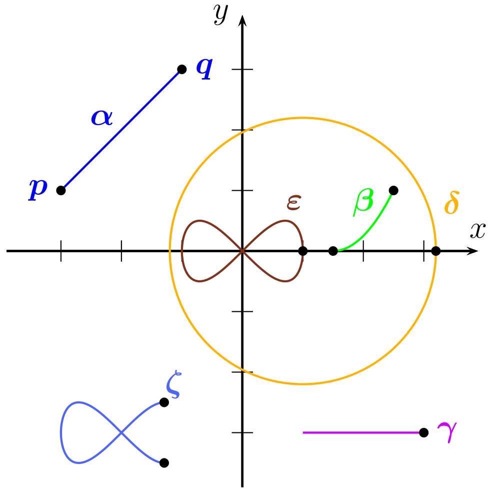

FIGURE 14.1. Examples of curves

$$\begin{aligned}\boldsymbol{\alpha}(t) &= (t, 0) && \forall t \in [0,1] \\ \boldsymbol{\beta}(t) &= (1, t-1) && \forall t \in [1,2] \\ \boldsymbol{\gamma}(t) &= (3-t, 1) && \forall t \in [2,3] \\ \boldsymbol{\delta}(t) &= (0, 4-t) && \forall t \in [3,4]\end{aligned}$$

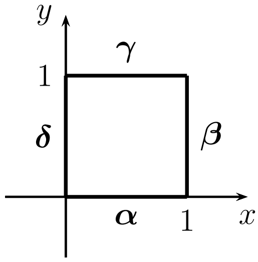

FIGURE 14.2. Example of a join of curves

EXAMPLE 14.4. In Figure 14.2, the join $\boldsymbol{\alpha} \sqcup \boldsymbol{\beta} \sqcup \boldsymbol{\gamma} \sqcup \boldsymbol{\delta}$ moves anticlockwise around the perimeter of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, starting and ending at $(0,0)$, as t varies from 0 to 4. The curve $\boldsymbol{\alpha}^*$ moves from 1 to 0 as t varies between 0 and 1. What does $(\boldsymbol{\alpha} \sqcup \boldsymbol{\beta} \sqcup \boldsymbol{\gamma} \sqcup \boldsymbol{\delta})^*$ do?

DEFINITION 14.5. Suppose that $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^2$ is a curve, and that h is a continuous bijection from $[c,d]$ to $[a,b]$ such that $h(c) = a$ and $h(d) = b$. Then $\boldsymbol{\gamma} \circ h : [c,d] \to \mathbb{R}^2$ is also a curve, called a reparametrisation of $\boldsymbol{\gamma}$.

A reparametrised curve $\boldsymbol{\gamma} \circ h$ has the same initial and final point as the original curve, and the same range; further it is traversed in the same direction and it goes through each point the same number of times. What may change is the interval of definition, and the speed.

There are some very strange curves, such as space-filling curves and snowflake curves that violate some of our intuitions about curves. It is helpful to restrict to certain "nice" curves that are more tractable.

DEFINITION 14.6. Suppose that $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^2$ is a curve and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$, where $\gamma_1, \gamma_2 : [a,b] \to \mathbb{R}$. Then we define

$$\boldsymbol{\gamma}'(t) = (\gamma_1'(t), \gamma_2'(t)),$$

provided that both $\gamma_1'(t)$ and $\gamma_2'(t)$ exist. We say that $\boldsymbol{\gamma}$ is *continuously differentiable* if the derivative $\boldsymbol{\gamma}'$ exists and is continuous in $[a,b]$, and $\boldsymbol{\gamma}$ is *smooth* if it is continuously differentiable and moreover $\boldsymbol{\gamma}'(t) \neq 0$ for all $t \in [a,b]$.

$$\boldsymbol{\alpha}(t) = (\cos(t), \sin(t)) \quad \forall t \in [0, 2\pi]$$
$$\boldsymbol{\alpha}'(t) = (-\sin(t), \cos(t)) \quad \forall t \in [0, 2\pi]$$
$$\boldsymbol{\beta}(t) = (\cos^3(t), \sin^3(t)) \quad \forall t \in [0, 2\pi]$$
$$\boldsymbol{\beta}'(t) = (-3\sin(t)\cos^2(t), 3\sin^2(t)\cos(t)) \quad \forall t \in [0, 2\pi]$$

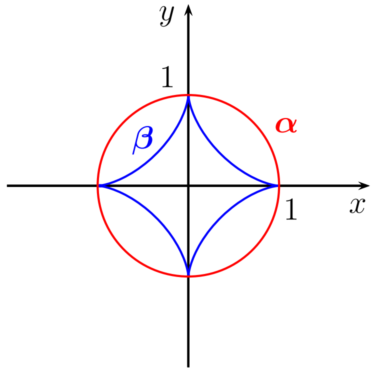

FIGURE 14.3. Examples of differentiable curves

EXAMPLES 14.7. In Figure 14.3, both curves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are continuously differentiable. Only $\boldsymbol{\alpha}$ is smooth.

DEFINITION 14.8. A curve is *piecewise smooth* if it is a join of finitely many smooth curves. The length of a piecewise smooth curve $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^2$ is given by

$$\operatorname{Length}(\boldsymbol{\gamma}) = \int_a^b |\boldsymbol{\gamma}'(t)| \, dt.$$

Note that $\boldsymbol{\gamma}'(t)$ may not be defined at finitely many points, where different smooth curves are joined. This is just a formal difficulty; we take $\boldsymbol{\gamma}'(t)$ to be 0 where it is not defined, and then the integral exists as a Riemann integral. For this it is important that $\boldsymbol{\gamma}'$ is bounded.

2. Orientation

If $\boldsymbol{\gamma}$ is a simple curve that is not closed, then it has an initial point and an endpoint and a "direction of motion", and even if we reparametrise it, these do not change. Similarly, simple closed curves have a "direction of motion". We call this *orientation*. For simple closed curves, the next theorem lets us find an orientation that does not depend on the parametrisation.

THEOREM 14.9 (The Jordan curve theorem). *If $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^2$ is a simple closed curve, then the complement of the range of $\boldsymbol{\gamma}$ is the union of two disjoint domains. One of these is bounded and the other is not. The bounded component is called the* interior *of $\boldsymbol{\gamma}$ and written $\operatorname{Int}(\boldsymbol{\gamma})$, and the unbounded component is called the* exterior *of $\boldsymbol{\gamma}$ and written $\operatorname{Ext}(\boldsymbol{\gamma})$.*

PROOF. We do not prove this, but remark that it relies on approximating a simple curve by a simple polygonal curve. □

This theorem seems obvious but is actually quite difficult, particularly when curves such as snowflakes are concerned. We will consider only curves for which it is easy to identify the interior and exterior.

If $\boldsymbol{\gamma}$ is a simple closed curve, then as we travel around $\boldsymbol{\gamma}$, the interior of $\boldsymbol{\gamma}$ will always lie on our left, or always on our right. The *standard orientation* of $\boldsymbol{\gamma}$ is the direction of motion such that the interior is always on our left. With the standard orientation, we move around the perimeter of $\operatorname{Int}(\boldsymbol{\gamma})$ in an anti-clockwise direction.

EXERCISE 14.10. What is the standard orientation of each of the simple closed curves described above? Can you define orientation for closed curves that are not simple?

We will use the expression *oriented range* of a curve to describe the image of a simple curve in $\mathbb{R}^2$ and a "direction of motion" along the curve.

3. Vector fields and line integrals

A *vector field* V on an open subset Ω of $\mathbb{R}^2$ is a function from Ω to $\mathbb{R}^2$. A vector field may be viewed as a pair of functions (v, w), where $v, w : \Omega \to \mathbb{R}$.

DEFINITION 14.11. Given a piecewise smooth curve $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^2$ and a continuous vector field V defined on $\operatorname{Range}(\boldsymbol{\gamma})$, we define the *line integral* $\int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\, d\boldsymbol{s}$ by

$$\int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\, d\boldsymbol{s} = \int_a^b V(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt.$$

Again, $\boldsymbol{\gamma}'(t)$ may not be defined for finitely many points, where different continuously differentiable curves are joined. We take $\boldsymbol{\gamma}'(t)$ to be 0 wherever it is not defined, and then the integral exists as a Riemann integral.

We now consider the approximation of piecewise smooth curves by polygonal curves. If $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^2$ is a piecewise smooth curve, we may define "approximating curves" $\boldsymbol{\gamma}_N$ (where $N \in \mathbb{Z}^+$) as follows. Fix N, and subdivide the interval $[a, b]$ into N equal subintervals of equal length, $[a_{n-1}, a_n]$ say, where $n = 1, \dots, N$ (we do this by choosing $a_n = (N - n)a/N + nb/N$). Now let $\boldsymbol{\gamma}_N$ be the polygonal curve composed of the N line segments from $\boldsymbol{\gamma}(a_{n-1})$ to $\boldsymbol{\gamma}(a_n)$, in order.

LEMMA 14.12. *Suppose also V is a continuous vector field in a domain Ω in $\mathbb{R}^2$. Suppose that $\boldsymbol{\gamma}$ is a piecewise smooth curve in Ω, and that the polygonal curves $\boldsymbol{\gamma}_N$ defined above also lie in Ω. Then the curves $\boldsymbol{\gamma}_N$ approximate $\boldsymbol{\gamma}$, in the sense that $\boldsymbol{\gamma}_N(t) \to \boldsymbol{\gamma}(t)$ for all $t \in [a, b]$ and $\operatorname{Length}(\boldsymbol{\gamma}_N) \to \operatorname{Length}(\boldsymbol{\gamma})$ as $N \to \infty$; further,*

$$\int_{\boldsymbol{\gamma}_N} V(\boldsymbol{s})\, d\boldsymbol{s} \to \int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\, d\boldsymbol{s}.$$

PROOF. Omitted. □

The point of this lemma is that we may prove many results for integration along general curves by proving easier results for integration along polygonal curves.

The line integral has the usual properties of integration.

THEOREM 14.13. *Suppose that $\lambda, \mu \in \mathbb{R}$, that $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^2$ is a piecewise smooth curve, and that V and W are vector fields defined on $\operatorname{Range}(\boldsymbol{\gamma})$. Then the following hold.*

(a) The integral is linear:

$$\int_{\boldsymbol{\gamma}} \lambda V(\boldsymbol{s}) + \mu W(\boldsymbol{s})\, d\boldsymbol{s} = \lambda \int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\, d\boldsymbol{s} + \mu \int_{\boldsymbol{\gamma}} W(\boldsymbol{s})\, d\boldsymbol{s}.$$

(b) *The integral is independent of parametrisation: if $\boldsymbol{\delta}$ is a reparametrisation of $\boldsymbol{\gamma}$ that is also a piecewise smooth curve, then*

$$\int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\, d\boldsymbol{s} = \int_{\boldsymbol{\delta}} V(\boldsymbol{s})\, d\boldsymbol{s}.$$

(c) *The integral is additive for joins: if $\boldsymbol{\gamma} = \boldsymbol{\alpha} \sqcup \boldsymbol{\beta}$, then*

$$\int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\, d\boldsymbol{s} = \int_{\boldsymbol{\alpha}} V(\boldsymbol{s})\, d\boldsymbol{s} + \int_{\boldsymbol{\beta}} V(\boldsymbol{s})\, d\boldsymbol{s}.$$

(d) *The integral depends on the orientation:*

$$\int_{\boldsymbol{\gamma}^*} V(\boldsymbol{s})\, d\boldsymbol{s} = -\int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\, d\boldsymbol{s}.$$

(e) *The size of the integral is controlled by the size of the vector field V and the length of the curve $\boldsymbol{\gamma}$:*

$$\left| \int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\, d\boldsymbol{s} \right| \le ML,$$

where L is the length of $\boldsymbol{\gamma}$ and M is a number such that $|V(\boldsymbol{s})| \le M$ for all $\boldsymbol{s} \in \mathrm{Range}(\boldsymbol{\gamma})$.

Proof. We omit this proof. □

Definition 14.14. We define a path $\boldsymbol{\Gamma}$ to be the oriented range of a piecewise smooth curve $\boldsymbol{\gamma}$. The theorem above implies that the line integral depends only on $\boldsymbol{\Gamma}$, and hence we define

$$\int_{\boldsymbol{\Gamma}} V(\boldsymbol{s})\, d\boldsymbol{s} = \int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\, d\boldsymbol{s},$$

where $\boldsymbol{\gamma}$ is any parametrisation of $\boldsymbol{\Gamma}$.

We may extend the notation for joins and reverse curves to paths.

Note that for paths that are not closed, different parametrisations differ by a change of variable, and part (d) of the theorem applies. But for closed paths, parametrisations might have different initial points and endpoints. To show that the integral around a closed path does not depend on where we start, we break up the closed path $\boldsymbol{\Gamma}$ into two parts: $\boldsymbol{\Gamma} = \mathbf{A} \sqcup \mathbf{B}$, and observe that

$$\begin{aligned}\int_{\mathbf{A}\sqcup\mathbf{B}} V(\boldsymbol{s})\, d\boldsymbol{s} &= \int_{\mathbf{A}} V(\boldsymbol{s})\, d\boldsymbol{s} + \int_{\mathbf{B}} V(\boldsymbol{s})\, d\boldsymbol{s} \\ &= \int_{\mathbf{B}} V(\boldsymbol{s})\, d\boldsymbol{s} + \int_{\mathbf{A}} V(\boldsymbol{s})\, d\boldsymbol{s} = \int_{\mathbf{B}\sqcup\mathbf{A}} V(\boldsymbol{s})\, d\boldsymbol{s}.\end{aligned}$$

This implies that the integral around $\boldsymbol{\Gamma}$ starting at the initial point of $\mathbf{A}$ is the same as the integral around $\boldsymbol{\Gamma}$ starting at the initial point of $\mathbf{B}$.

4. Closed and exact vector fields

Suppose that the vector field $V : \Omega \to \mathbb{R}^2$ is given in coordinates by (v, w). Then V is said to be *curl-free* or *closed* if $\partial w/\partial x = \partial v/\partial y$. Further, V is said to be *conservative* or *exact* if there is a function $u : \Omega \to \mathbb{R}$ such that $\partial u/\partial x = v$ and $\partial u/\partial y = w$; this is often written as $V = \nabla u$. If the function u is twice continuously differentiable, then

$$\frac{\partial w}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial v}{\partial y}.$$

This leads to the conclusion that a continuously differentiable exact vector field is closed. The converse of this is very useful.

THEOREM 14.15. *Suppose that Ω is a simply connected domain, and that V is a closed continuously differentiable vector field in Ω. Then*

$$\int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\,d\boldsymbol{s} = 0$$

for all closed piecewise smooth curves $\boldsymbol{\gamma}$ whose range lies in Ω.

PROOF. We omit this proof, but mention that Green's theorem is often used when $\boldsymbol{\gamma}$ is simple. We will prove a more general result about complex integrals, whose proof may be modified to establish this result.

Then the result is extended to closed polygonal curves, and finally an approximation argument is used to extend to general closed piecewise smooth curves. □

THEOREM 14.16. *Suppose that Ω is a simply connected domain, and that V is a closed continuously differentiable vector field in Ω. Then V is exact.*

PROOF. We omit this proof, but mention that the previous theorem implies that $\int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\,d\boldsymbol{s}$ depends only on the initial and final points of the curve $\boldsymbol{\gamma}$, which allows us to define

$$u(\boldsymbol{q}) = \int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\,d\boldsymbol{s}$$

for any curve γ from a fixed point $\boldsymbol{p}$ to $\boldsymbol{q}$. □

This theorem has variants; in particular, it is equivalent to the statement that a line integral depends only on the initial point and the end point.

LECTURE 15

Contour integrals

In this lecture, we define contour integrals, and compute some examples.

1. Curves and contours

We define a curve in $\mathbb{C}$ much as in $\mathbb{R}^2$: a curve $t \mapsto (\gamma_1(t), \gamma_2(t))$ is replaced by a curve $t \mapsto \gamma_1(t) + i\gamma_2(t)$. The definitions of simple and closed curves are almost identical, as are joins, reverse curves, reparametrisations, and orientations.

DEFINITION 15.1. Suppose that $\gamma : [a, b] \to \mathbb{C}$ is a curve and $\gamma(t) = \gamma_1(t)+i\gamma_2(t)$, where $\gamma_1, \gamma_2 : [a, b] \to \mathbb{R}$, that is, $\gamma_1(t) = \operatorname{Re}(\gamma(t))$ and $\gamma_2(t) = \operatorname{Im}(\gamma(t))$ for all $t \in [a, b]$. Then we define

$$\gamma'(t) = \gamma_1'(t) + i\gamma_2'(t),$$

provided that both $\gamma_1'(t)$ and $\gamma_2'(t)$ exist. We say that $\gamma : [a, b] \to \mathbb{C}$ is continuously differentiable if γ' is defined and continuous on $[a, b]$, smooth if γ is continuously differentiable and moreover $\gamma'(t) \neq 0$ for all $t \in [a, b]$, and piecewise smooth if it is the join of finitely many smooth curves.

Note that

$$\gamma'(t) = \lim_{h\to 0} \frac{1}{h} \left(\gamma(t + h) - \gamma(t)\right).$$

We may differentiate many complex-valued functions of a real variable as if they were real-valued. For instance, if $\alpha(t) = e^{it}$, then $\alpha'(t) = ie^{it}$. The usual rules of differentiation hold for complex curves and functions defined on complex curves, and we do not always need to break things up into their real and imaginary parts in order to differentiate. In particular, if f is a holomorphic function and α is a curve in the complex plane, then

$$\frac{d}{dt} f(\alpha(t)) = f'(\alpha(t))\, \alpha'(t).$$

We can prove this by breaking everything up into real and imaginary parts, and applying the chain rule from multi-variable calculus, or by copying the proof from calculus of one real variable.

DEFINITION 15.2. A *contour* is an oriented range of a piecewise smooth curve in the complex plane.

A contour is the analogue in the complex plane of a path in $\mathbb{R}^2$; usually we deal with simple contours. We may extend the definition of joins and reverse curves in $\mathbb{R}^2$ to curves in $\mathbb{C}$ and hence to contours.

EXAMPLES 15.3. Figure 15.1 illustrates some contours in the complex plane. In the absence of other information, we assume that closed contours have the standard orientation.

$$\alpha(t) = \cos(t) + i\sin(t) = e^{it} \quad \forall t \in [0, 2\pi]$$

$$\beta(t) = (1-t) + it \quad \forall t \in [0, 1]$$

$$\gamma(t) = -t + i(1-t) \quad \forall t \in [0, 1]$$

$$\delta(t) = (t-1) - it \quad \forall t \in [0, 1]$$

$$\varepsilon(t) = t + i(t-1) \quad \forall t \in [0, 1]$$

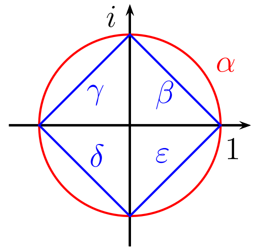

FIGURE 15.1. Examples of parametrised contours

2. Integrals with complex integrands

DEFINITION 15.4. Suppose that $u, v : [a, b] \to \mathbb{R}$ are real-valued functions, and that $f : [a, b] \to \mathbb{C}$ is given by $f = u + iv$. We define

$$\int_a^b f(t)\, dt = \int_a^b \big(u(t) + iv(t)\big)\, dt = \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt,$$

provided that the two real integrals on the right hand side exist. In other words,

$$\operatorname{Re}\left(\int_a^b f(t)\, dt\right) = \int_a^b \operatorname{Re}\left(f(t)\right)\, dt \quad \text{and} \quad \operatorname{Im}\left(\int_a^b f(t)\, dt\right) = \int_a^b \operatorname{Im}\left(f(t)\right)\, dt.$$

Integration of complex-valued functions on an interval has similar properties to standard integration. For instance, it is linear, we may integrate by substitution and by parts, and exponentials integrate as we would expect: for $a, b, c, d \in \mathbb{R}$, $\lambda, \mu \in \mathbb{C}$, a real-valued differentiable function $h : [c, d] \to [a, b]$ such that $h(c) = a$ and $h(d) = b$, and complex-valued functions f and g,

$$\int_a^b \lambda f(t) + \mu g(t)\, dt = \lambda \int_a^b f(t)\, dt + \mu \int_a^b g(t)\, dt$$

$$\int_c^d f(h(t))\, h'(t)\, dt = \int_a^b f(s)\, ds$$

$$\int_a^b f'(t)\, g(t)\, dt = \big(f(b)\, g(b) - f(a)\, g(a)\big) - \int_a^b f(t)\, g'(t)\, dt$$

$$\int_a^b e^{\lambda t}\, dt = \left[\frac{e^{\lambda t}}{\lambda}\right]_{t=a}^{t=b} = \frac{e^{\lambda b} - e^{\lambda a}}{\lambda}$$

$$\left|\int_a^b f(t)\, dt\right| \leq \int_a^b |f(t)|\, dt\,.$$

We prove only the last part: if the left hand side is 0, there is nothing to prove, otherwise we take θ to be the argument of the left hand side. Then

$$\left|\int_a^b f(t)\, dt\right| = e^{-i\theta} \int_a^b f(t)\, dt = \int_a^b e^{-i\theta} f(t)\, dt.$$

Since the left-hand side is real, so is the right-hand side, and so

$$\begin{aligned}\left|\int_a^b f(t)\,dt\right| &= \operatorname{Re}\left(\int_a^b e^{-i\theta} f(t)\,dt\right)\\ &= \int_a^b \operatorname{Re}\left(e^{-i\theta} f(t)\right)\,dt\\ &\le \int_a^b \left|e^{-i\theta} f(t)\right|\,dt\\ &= \int_a^b |f(t)|\,dt.\end{aligned}$$

EXERCISE 15.5. Evaluate $\int_0^\pi t\,e^{it}\,dt$.

ANSWER.

We take real and imaginary parts and deduce that $\int_0^\pi t\cos(t)\,dt = -2$ and $\int_0^\pi t\sin(t)\,dt = \pi$. It is much easier to find an integral like $\int e^{(1+i)t}\,dt$ by using complex powers than to evaluate $\int e^t\cos(t)\,dt$.

3. Contour integrals

DEFINITION 15.6. Given a piecewise smooth curve $\gamma : [a,b] \to \mathbb{C}$ and a continuous (not necessarily differentiable) function f defined on the range of γ, we define the *complex line integral* $\int_\gamma f(z)\,dz$ by

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt,$$

provided that the integral on the right hand side exists.

To remember this formula, write $z(t)$ rather than $\gamma(t)$: then $dz = \dfrac{dz}{dt}\,dt$, and

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,\frac{dz}{dt}\,dt.$$

If we now convert back to writing $\gamma(t)$, which is a good idea, because we use z to mean too many different things, then we get the formula in the definition.

Note that in general,

$$\operatorname{Re}\left|\int_\gamma f(z)\,dz\right| \ne \int_\gamma \operatorname{Re}(f(z))\,dz,$$

and similarly for the imaginary part.

THEOREM 15.7. *Suppose that $\lambda, \mu \in \mathbb{C}$, that $\gamma : [a,b] \to \mathbb{C}$ is a piecewise smooth curve, and that f and g are complex functions defined on* $\mathrm{Range}(\gamma)$. *Then the following hold.*

(a) *The integral is linear:*
$$\int_\gamma \lambda f(z) + \mu g(z)\, dz = \lambda \int_\gamma f(z)\, dz + \mu \int_\gamma g(z)\, dz.$$

(b) *The integral is independent of parametrisation: if δ is a reparametrisation of γ that is also a piecewise smooth curve, then*
$$\int_\gamma f(z)\, dz = \int_\delta f(z)\, dz.$$

(c) *The integral is additive for joins: if $\gamma = \alpha \sqcup \beta$, then*
$$\int_\gamma f(z)\, dz = \int_\alpha f(z)\, dz + \int_\beta f(z)\, dz.$$

(d) *The integral depends on the orientation:*
$$\int_{\gamma^*} f(z)\, dz = -\int_\gamma f(z)\, dz.$$

(e) *The size of the integral is controlled by the size of the function f and the length of the curve γ:*
$$\left|\int_\gamma f(z)\, dz\right| \leq ML,$$
where L is the length of γ and M is a maximiser *for $|f|$ on the curve, that is, a number such that $|f(z)| \leq M$ for all $z \in \mathrm{Range}(\gamma)$.*

PROOF. We omit the proof of parts (a) to (d).

For part (e), observe that
$$\left|\int_\gamma f(z)\, dz\right| = \left|\int_a^b f(\gamma(t))\, \gamma'(t)\, dt\right| \leq \int_a^b \left|f(\gamma(t))\right| \left|\gamma'(t)\right| dt$$
$$\leq \int_a^b M \left|\gamma'(t)\right| dt = ML,$$
by the formula for the length of a curve. □

Part (e) of the theorem is often called the ML Lemma. Note that M is not necessarily the *maximum* of $|f|$ on the curve.

Recall that a contour Γ is the oriented range of a piecewise smooth curve γ. The theorem above implies that the complex line integral depends only on Γ, and not on the parametrisation γ.

DEFINITION 15.8. We define
$$\int_\Gamma f(z)\, dz = \int_\gamma f(z)\, dz,$$
where γ is any parametrisation of Γ.

4. Examples of contour integrals

We consider some examples of contour integration.

EXERCISE 15.9. Suppose that $p, q \in \mathbb{C}$, that Γ is the line segment from p to q, and that $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$. Find $\displaystyle\int_\Gamma \lambda_1 + \lambda_2 z + \lambda_3 \overline{z} + \lambda_4 e^z \, dz$.

ANSWER.

EXERCISE 15.10. Suppose that Γ is the circle $\{z \in \mathbb{C} : |z| = r\}$ with the standard orientation, traversed k times, and $n \in \mathbb{Z}$. Compute $\displaystyle\int_\Gamma z^n \, dz$.

ANSWER.

LECTURE 16

The Cauchy–Goursat theorem

In this lecture, we begin with an exercise, and then state and discuss one of the key theorems of complex analysis.

1. An exercise

EXERCISE 16.1. Suppose that Γ is a simple closed contour in $\mathbb{C}$ and $c_0, c_1 \in \mathbb{C}$. Show that $\int_\Gamma (c_1 z + c_0)\, dz = 0$. Is it always true that $\int_\Gamma \operatorname{Re}(z)\, dz = 0$?

ANSWER.

2. Simply connected domains

First, we consider the case of a simply connected domain.

THEOREM 16.2 (Cauchy–Goursat). *Suppose that Ω is a simply connected domain, that $f \in H(\Omega)$, and that Γ is a closed contour in Ω. Then*

$$\int_\Gamma f(z)\, dz = 0.$$

PROOF. We prove a more general version of this result later for the case in which Γ is simple. To extend from the simple case to the general case, we argue as we did for path integrals in Lecture 13. □

COROLLARY 16.3 (Independence of contour). *Suppose that Ω is a simply connected domain in $\mathbb{C}$, that $f \in H(\Omega)$, and that Γ and Δ are contours with the same initial point p and the same final point q. Then*

$$\int_\Gamma f(z)\,dz = \int_\Delta f(z)\,dz.$$

PROOF. The contour $\Gamma \sqcup \Delta^*$ is closed, and so

$$0 = \int_{\Gamma \sqcup \Delta^*} f(z)\,dz = \int_\Gamma f(z)\,dz - \int_\Delta f(z)\,dz,$$

and the desired equality follows. □

COROLLARY 16.4 (Existence of primitives). *Suppose that Ω is a simply connected domain in $\mathbb{C}$, and that $f \in H(\Omega)$. Then there exists a function F on Ω such that*

$$\int_\Gamma f(z)\,dz = F(q) - F(p)$$

for all contours Γ in Ω from p to q. Further, F is differentiable, and $F' = f$. Finally, if F_1 is any other function such that $F_1' = f$, then $F_1 - F$ is a constant and

$$\int_\Gamma f(z)\,dz = F_1(q) - F_1(p),$$

where p and q are the initial and final points of Γ.

PROOF. Fix a "base point" b in Ω, and for $p \in \Omega$, define $F(p)$ to be $\int_\Gamma f(z)\,dz$, where Γ is any contour in Ω with initial point b and final point p. This definition makes sense in light of the previous corollary.

Given p and q in Ω, and a contour Γ from p to q, take contours Γ_p from b to p and Γ_q from b to q. Then $\Gamma \sqcup (\Gamma_q)^* \sqcup \Gamma_p$ is a closed contour for which

$$\int_{\Gamma \sqcup (\Gamma_q)^* \sqcup \Gamma_p} f(z)\,dz = 0.$$

Writing this as a combination of integrals, we see that

$$\int_\Gamma f(z)\,dz = F(q) - F(p).$$

Now we take $p \in \Omega$, and show that $F'(p) = f(p)$. To do so, we need to make

$$\left| \frac{F(q) - F(p)}{q - p} - f(p) \right|$$

small by taking q sufficiently close to p.

Take $q \in \Omega$ close to p, and let Δ be the line segment from p to q. On the one hand,

$$\frac{F(q) - F(p)}{q - p} = \frac{1}{q - p} \int_\Delta f(z)\,dz;$$

this is correct provided that Δ is contained in Ω. On the other hand, by direct calculation, $\int_\Delta dz = q - p$, whence

$$f(p) = \frac{1}{q-p} \int_\Delta f(p)\, dz.$$

Thus

$$\begin{aligned}\left|\frac{F(q)-F(p)}{q-p} - f(p)\right| &= \left|\frac{1}{q-p}\int_\Delta f(z)\, dz - \frac{1}{q-p}\int_\Delta f(p)\, dz\right| \\ &= \left|\frac{1}{q-p}\int_\Delta \big(f(z)-f(p)\big)\, dz\right|.\end{aligned}$$

We can make this small by making Δ small enough that Δ is contained in Ω and f does not vary much on Δ(f is differentiable and hence continuous), and then using the ML Lemma.

More precisely, take any small positive ε. Since Ω is open and f is continuous at p, there exists δ such that $B(p,\delta) \subset \Omega$ and $|f(z)-f(p)| < \varepsilon$ when $z \in B(p,\delta)$. Take $q \in B^\circ(p,\delta)$ and let Δ be the straight line segment from p to q. Then $\Delta \subset B(p,\delta)$ and so $|f(z) - f(p)| < \varepsilon$ for all $z \in \Delta$. Thus

$$\begin{aligned}\left|\frac{F(q)-F(p)}{q-p} - f(p)\right| &= \left|\frac{1}{q-p}\int_\Delta \big(f(z)-f(p)\big)\, dz\right| \\ &\le \frac{1}{|q-p|}\max\{|f(z)-f(p)| : z \in \Delta\}\, |q-p| \\ &= \max\{|f(z)-f(p)| : z \in \Delta\} < \varepsilon,\end{aligned}$$

by the ML lemma and the fact that $\max\{|f(z)-f(p)| : z \in \Delta\} = f(z^*)$ for some $z^* \in \Delta$ (the maximum is attained). It follows that F is differentiable at p, with derivative $f(p)$, as required.

If F_1 is another function such that $F_1' = f$, then $(F_1 - F)' = 0$, so $F_1 - F$ is a constant, C say. This means that

$$F_1(q) - F_1(p) = (F(q)+C) - (F(p)+C) = F(q) - F(p),$$

so that F_1 can also be used to compute $\int_\Gamma f(z)\, dz$. □

We call a function F such that $F' = f$ a *primitive* or an *anti-derivative* of f. In some of our earlier computations, there are hints that it might be possible to compute contour integrals using primitives; now we have the proof of this, at least when f is holomorphic.

3. Multiply connected domains

Perhaps unfortunately, many domains are not simply connected. For these domains, the discussion is more complicated. Suppose that Ω is a bounded domain whose boundary $\partial\Omega$ consists of finitely many disjoint contours. Then one of the contours, Γ_0 say, is outside Ω while the others, Γ_1, ..., Γ_n say, are inside Ω. We orient these contours in the standard way. This is illustrated in Figure 16.1.

Exercise 16.5. Determine the orientations of the contours in Figure 16.1.

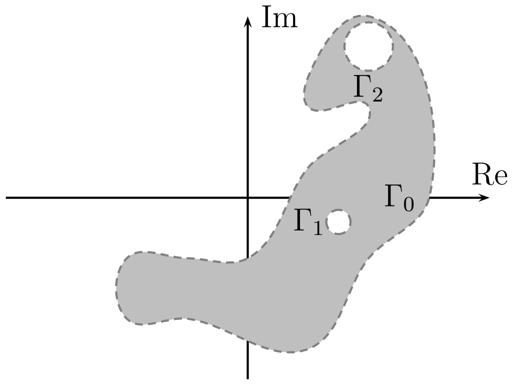

FIGURE 16.1. A multiply connected domain

THEOREM 16.6 (Cauchy–Goursat). *Suppose that Ω is a bounded domain whose boundary $\partial\Omega$ consists of finitely many contours Γ_0, Γ_1, ..., Γ_n. Suppose also that $f \in H(\Upsilon)$, where $\overline{\Omega} \subset \Upsilon$. Then*

$$\int_{\partial\Omega} f(z)\, dz = \sum_{j=0}^{n} \int_{\Gamma_j} f(z)\, dz = 0.$$

PROOF. We will see this next lecture. □

COROLLARY 16.7. *Suppose that Υ is a simply connected domain, that Γ is a simple closed contour in Υ, and that f is a differentiable function in Υ. Then*

$$\int_{\Gamma} f(z)\, dz = 0.$$

PROOF. We let Ω be the interior of Γ and apply the previous result. □

COROLLARY 16.8. *Suppose that Ω is a bounded domain whose boundary $\partial\Omega$ consists of finitely many contours Γ_0, Γ_1, ..., Γ_n, that $\overline{\Omega} \subset \Upsilon$, and that f is a differentiable function in Υ. If $\int_{\Gamma_j} f(z)\, dz = 0$ when $j = 1, \dots, n$, then $\int_{\Gamma} f(z)\, dz = 0$ when Γ is a closed curve in Ω, and there is a differentiable function F in Ω such that $F' = f$, and*

$$\int_{\Delta} f(z)\, dz = F(q) - F(p)$$

whenever Δ is a contour in Ω from p to q.

PROOF. We omit this proof. □

4. History† (Not examinable)

The Cauchy–Goursat theorem is mathematically important because it will lead to the Cauchy integral formula, one of the most useful formulae in complex analysis. One of the uses of this formula is to compute integrals. Another is to show that a holomorphic function in a domain Ω has continuous partial derivatives, and indeed is infinitely differentiable. We stated several theorems earlier about holomorphic functions which include the hypotheses that f is holomorphic and that f' is continuous, and it is useful to know that the continuity hypothesis is automatically true. At least in principle, we should check that the hypotheses of a theorem are satisfied before we apply the theorem, and so it is good to make these hypotheses unnecessary.

Augustin Cauchy was one of the finest French mathematicians of the first half of the 1800s, and he developed much of what is in a course on complex analysis today, as well as making precise the idea of limit that had been worrying mathematicians and philosophers of mathematics since the time of Newton and Leibnitz. The Cauchy–Goursat theorem is Goursat's 1884 improvement of a theorem of Cauchy from 1829—it says something of Goursat's ability that he could improve the work of Cauchy.

Some of Cauchy's ideas were being developed simultaneously by George Green, an "uneducated miller" from Nottingham, who gave us Green's theorem in 1828.

5. Connection with vector fields† (Not examinable)

Take a simple piecewise smooth curve $\gamma : [a, b] \to \mathbb{C}$, and write $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ and $f(z) = u(x, y) + iv(x, y)$. Let $\boldsymbol{\gamma}(t)$ be the analogue of γ in $\mathbb{R}^2$, that is, $\boldsymbol{\gamma}(t) = (\gamma_1(t), \gamma_2(t))$. Then

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_a^b (u(\boldsymbol{\gamma}(t)) + iv(\boldsymbol{\gamma}(t)))\,(\gamma_1'(t) + i\gamma_2'(t))\,dt \\
&= \int_a^b (u(\boldsymbol{\gamma}(t))\,\gamma_1'(t) - v(\boldsymbol{\gamma}(t)))\,\gamma_2'(t))\,dt \\
&\qquad + i\int_a^b (u(\boldsymbol{\gamma}(t))\,\gamma_2'(t) + v(\boldsymbol{\gamma}(t)\,\gamma_1'(t))\,dt \\
&= \int_a^b U(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,dt + i\int_a^b V(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,dt \\
&= \int_{\boldsymbol{\gamma}} U(\boldsymbol{s})\,d\boldsymbol{s} + i\int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\,d\boldsymbol{s},
\end{aligned}$$

where the vector fields U and V are given by

$$U(x, y) = (u(x, y), -v(x, y)) \qquad \text{and} \qquad V(x, y) = (v(x, y), u(x, y)).$$

We know that, in a simply connected domain, the line integrals depend only on the initial point and the final point when the vector fields are closed. The condition that U and V be closed is the condition

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{and} \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.$$

These are exactly the Cauchy–Riemann equations. So the complex line integrals depend only on the initial point and the final point when the Cauchy–Riemann equations hold.

LECTURE 17

Cauchy's integral formula

In this lecture, we begin with a sketch proof of the Cauchy–Goursat Theorem. Then we state and prove Cauchy's integral formula. Finally we see some applications.

1. Proof of the Cauchy–Goursat Theorem

First we restate the Cauchy–Goursat Theorem.

THEOREM. *Suppose that Ω is a bounded domain whose boundary $\partial\Omega$ consists of finitely many contours Γ_0, Γ_1, ..., Γ_n. Suppose also that $f \in H(\Upsilon)$, where $\overline{\Omega} \subset \Upsilon$. Then*

$$\sum_j \int_{\Gamma_j} f(z)\, dz = 0.$$

PROOF. The proof of the theorem involves three steps. First, we prove it in the case where Ω is a triangle. Second, we consider the case where Ω is a domain whose boundary is made up of finitely many closed polygonal contours. Third, we treat the general case.

Step one. Suppose that Ω is a triangle in the complex plane. We write T_0 for Ω and $\partial\mathrm{T}_0$ for its boundary. Suppose that $f \in H(\Upsilon)$, where $\overline{\mathrm{T}}_0 \subset \Upsilon$, and let

$$\int_{\partial \mathrm{T}_0} f(z)\, dz = I.$$

We have to show that $I = 0$, and we suppose towards a contradiction that $I \neq 0$. We may subdivide T_0 into four congruent sub-triangles, T', T'', T''' and T'''' say, by taking the midpoint of each side, and joining these midpoints; see Figure 17.1.

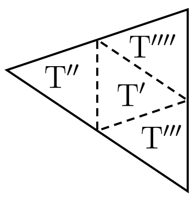

FIGURE 17.1. Subdividing a triangle

Now

$$\begin{aligned} I &= \int_{\partial \mathrm{T}_0} f(z)\,dz \\ &= \int_{\partial \mathrm{T}'} f(z)\,dz + \int_{\partial \mathrm{T}''} f(z)\,dz + \int_{\partial \mathrm{T}'''} f(z)\,dz + \int_{\partial \mathrm{T}''''} f(z)\,dz. \end{aligned}$$

At least one of the triangles T', T'', T''' and T'''', which we call T_1, must satisfy

$$\Big|\int_{\partial \mathrm{T}_1} f(z)\,dz\Big| \geq \frac{1}{4}\Big|\int_{\partial \mathrm{T}_0} f(z)\,dz\Big| = \frac{|I|}{4}\,.$$

We now subdivide T_1 into 4 congruent triangles, and argue in the same way that there must be one of these, T_2 say, with the property that

$$\Big|\int_{\partial \mathrm{T}_2} f(z)\,dz\Big| \geq \frac{1}{4}\Big|\int_{\partial \mathrm{T}_1} f(z)\,dz\Big| \geq \frac{|I|}{16}\,.$$

Continuing inductively in this way, we find a sequence $(\mathrm{T}_n)_{n\in\mathbb{N}}$ of nested triangles, such that

$$\Big|\int_{\partial \mathrm{T}_n} f(z)\,dz\Big| \geq \frac{|I|}{4^n}\,. \tag{17.1}$$

Write $\mathrm{Length}(\partial \mathrm{T}_n)$ for the perimeter of T_n. Then

$$\mathrm{Length}(\partial \mathrm{T}_n) = 2^{-n}\,\mathrm{Length}(\partial \mathrm{T}_0).$$

By compactness, there is a point z_0 that lies in each of the closed triangles $\overline{\mathrm{T}}_n$, and by hypothesis, f is differentiable at z_0. If $z \in \partial \mathrm{T}_n$, then $|z - z_0|$ is less than half the perimeter of T_n, that is,

$$|z - z_0| \leq \tfrac{1}{2}\,\mathrm{Length}(\partial \mathrm{T}_n) = 2^{-n-1}\,\mathrm{Length}(\partial \mathrm{T}_0),$$

and this tends to 0 as $n \to \infty$.

Since f is differentiable at z_0, we may write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z),$$

where the error term $E(z)$ satisfies

$$\frac{|E(z)|}{|z - z_0|} \to 0 \quad \text{as } z \to z_0.$$

In particular, we can ensure that

$$\frac{|E(z)|}{|z - z_0|} \leq \frac{|I|}{\mathrm{Length}(\partial \mathrm{T}_0)^2} \qquad \forall z \in \partial \mathrm{T}_n \tag{17.2}$$

by taking n large enough. In what follows, we take such an n.

Recall that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z).$$

This means that

$$\int_{\partial \mathrm{T}_n} f(z)\,dz = \int_{\partial \mathrm{T}_n} f(z_0)\,dz + \int_{\partial \mathrm{T}_n} f'(z_0)(z - z_0)\,dz + \int_{\partial \mathrm{T}_n} E(z)\,dz.$$

The first two integrals on the right hand side are 0, by calculation, and hence by (17.1), the last equation, the ML Lemma, properties of maxima, and (17.2),

$$\begin{aligned}
|I| &\leq 4^n \Big| \int_{\partial \mathrm{T}_n} f(z)\, dz \Big| \\
&= 4^n \Big| \int_{\partial \mathrm{T}_n} E(z)\, dz \Big| \\
&\leq 4^n \max\{|E(z)| : z \in \partial \mathrm{T}_n\} \operatorname{Length}(\partial \mathrm{T}_n) \\
&= 4^n \max \left\{ \frac{|E(z)|}{|z - z_0|} |z - z_0| : z \in \partial \mathrm{T}_n \right\} \operatorname{Length}(\partial \mathrm{T}_n) \\
&\leq 4^n \max \left\{ \frac{|E(z)|}{|z - z_0|} : z \in \partial \mathrm{T}_n \right\} \max\{|z - z_0| : z \in \partial \mathrm{T}_n\} \operatorname{Length}(\partial \mathrm{T}_n) \\
&\leq 4^n \frac{|I|}{\operatorname{Length}(\partial \mathrm{T}_0)^2} \frac{\operatorname{Length}(\partial \mathrm{T}_n)^2}{2} \\
&= \frac{|I|}{2},
\end{aligned}$$

which is absurd. Hence $I = 0$.

Step 2. The next step is to deal with a domain Ω with a polygonal boundary. Any such domain may be subdivided into triangles T_n (although this may seem obvious, it is hard to prove), in such a way that

$$\int_{\partial \Omega} f(z)\, dz = \sum_n \int_{\partial \mathrm{T}_n} f(z)\, dz;$$

by the result of the previous step,

$$\int_{\partial \Omega} f(z)\, dz = 0.$$

Step 3. Finally, we have to deal with a domain whose boundary is the union of finitely many disjoint closed contours. This can be done by approximating unions of closed contours by unions of polygonal contours; the integral is 0 for all the unions of approximating polygonal contours, and so the integral around the union of general contours that we want is also 0. □

2. Cauchy's integral formula

THEOREM 17.1 (Cauchy's integral formula). *Suppose that Ω is a simply connected domain in $\mathbb{C}$, that $f \in H(\Omega)$, that Γ is a simple closed contour in Ω and that $w \in \operatorname{Int}(\Gamma)$. Then*

$$f(w) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - w}\, dz. \tag{17.3}$$

PROOF. Let Γ_ε be the circle with centre w and radius ε, *traversed clockwise*, and take ε small enough that $\Gamma_\varepsilon \subset \operatorname{Int}(\Gamma)$. We consider the domain Υ consisting of $\operatorname{Int}(\Gamma) \cap \operatorname{Ext}(\Gamma_\varepsilon)$, the domain between Γ and Γ_ε, whose boundary consists of Γ, traversed anti-clockwise, and Γ_ε, traversed clockwise. The quotient function $z \mapsto f(z)/(z - w)$ is holomorphic in $\Omega \setminus \{w\}$, a domain that contains $\Upsilon \cup \partial \Upsilon$.

By the Cauchy–Goursat theorem,

$$\int_{\partial\Upsilon} \frac{f(z)}{z-w}\,dz = \int_{\Gamma} \frac{f(z)}{z-w}\,dz + \int_{\Gamma_\varepsilon} \frac{f(z)}{z-w}\,dz = 0;$$

that is,

$$\int_{\Gamma} \frac{f(z)}{z-w}\,dz = \int_{\Gamma_\varepsilon^*} \frac{f(z)}{z-w}\,dz.$$

The left hand side of this equality does not depend on ε, so the limit as ε tends to 0 of the right hand side exists.

To compute the limit, we parametrise Γ_ε^*. Define $\gamma_\varepsilon^*(\theta) = w + \varepsilon e^{i\theta}$, where $0 \le \theta \le 2\pi$, and observe that

$$\begin{aligned}
\int_{\Gamma_\varepsilon^*} \frac{f(z)}{z-w}\,dz &= \lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon^*} \frac{f(z)}{z-w}\,dz \\
&= \lim_{\varepsilon\to 0} \int_0^{2\pi} \frac{f(w+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}\,d\theta \\
&= \lim_{\varepsilon\to 0} \int_0^{2\pi} f(w+\varepsilon e^{i\theta})\, i\,d\theta \\
&= i\int_0^{2\pi} \lim_{\varepsilon\to 0} f(w+\varepsilon e^{i\theta})\,d\theta \\
&= i\int_0^{2\pi} f(w)\,d\theta \\
&= 2\pi i f(w).
\end{aligned}$$

We can move the limit inside the integral because $\lim_{\varepsilon\to 0} f(w+\varepsilon e^{i\theta}) = f(w)$ uniformly in θ, since $\lim_{z\to w} f(z) = f(w)$. Formula (17.3) follows. □

3. Two corollaries of Cauchy's integral formula

COROLLARY 17.2 (independence of contour). *Suppose that w lies in a simply connected domain Ω, and that $f \in H(\Omega)$. If Γ and Δ are simple closed contours such that $w \in \operatorname{Int}(\Gamma)$ and $w \in \operatorname{Int}(\Delta)$, then*

$$\int_{\Gamma} \frac{f(z)}{z-w}\,dz = \int_{\Delta} \frac{f(z)}{z-w}\,dz.$$

PROOF. By Cauchy's integral formula, both sides are equal to $2\pi i f(w)$. □

This means that if we need to compute the integral $\displaystyle\int_{\Gamma} \frac{f(z)}{z-w}\,dz$, we may change the contour to make the calculation easier.

COROLLARY 17.3 (mean value formula). *Suppose that Ω is a simply connected domain in $\mathbb{C}$, that $f \in H(\Omega)$, and that $w \in \Omega$. If $\overline{B}(w,r) \subset \Omega$, then*

$$f(w) = \frac{1}{2\pi} \int_0^{2\pi} f(w+re^{i\theta})\,d\theta. \tag{17.4}$$

PROOF. This formula is virtually proved in the course of the proof of the Cauchy integral formula; let $\gamma(\theta) = w + re^{i\theta}$, where $0 \le \theta \le 2\pi$. Then

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w}\, dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(w+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(w+re^{i\theta})\, d\theta, \end{aligned}$$

as required. □

The Cauchy integral formula may be considered as a way to write $f(w)$ as a weighted average of the values of $f(z)$ around any contour surrounding w.

4. A computation

Cauchy's integral formula enables us to compute some integrals without integrating!

EXERCISE 17.4. Compute $\displaystyle\int_\Gamma \frac{\sin z}{z}\, dz$, where Γ is the circle with centre 0 and radius R.

ANSWER.

LECTURE 18

Cauchy's generalised integral formula

Cauchy's integral formula,

$$f(w) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - w} \, dz, \tag{18.1}$$

where w lies inside a simple closed contour Γ in a simply connected domain Ω, and $f \in H(\Omega)$, is perhaps the most important formula in complex analysis.

In this lecture, we establish various consequences of Cauchy's integral formula. These include both theoretical results and explicit computations.

1. Cauchy's generalised integral formula

The most important theoretical application of Cauchy's integral formula is the following extension.

COROLLARY 18.1. *Suppose that $f \in H(B(z_0, R))$, and that Γ is a simple closed contour in $B(z_0, R)$ such that $z_0 \in \operatorname{Int}(\Gamma)$. Then*

$$f(w) = \sum_{n=0}^{\infty} c_n (w - z_0)^n \qquad \forall w \in B(z_0, R),$$

where

$$c_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}} \, dz.$$

The radius of convergence of the power series is at least R.

REMARK 18.2. This corollary, combined with the fact that $f^{(n)}(z_0) = n! \, c_n$, implies that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}} \, dz.$$

This is often called Cauchy's generalised integral formula.

Notice that we just assumed that f is differentiable once; Corollary 18.1 implies that f is actually infinitely differentiable.

PROOF. Write Γ_r for the circle with centre z_0 and radius r, where $r < R$. By independence of contour,

$$\int_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}} \, dz = \int_{\Gamma_r} \frac{f(z)}{(z - z_0)^{n+1}} \, dz,$$

so we may assume that $\Gamma = \Gamma_r$. By the Cauchy integral formula, if $w \in B(z_0, r)$, then

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(z)}{z-w}\, dz \\ &= \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(z)}{(z-z_0)-(w-z_0)}\, dz \\ &= \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(z)}{(z-z_0)}\, \frac{1}{(1-(w-z_0)/(z-z_0))}\, dz. \end{aligned}$$

Observe that $|w - z_0| < |z - z_0| = r$ for all $z \in \Gamma_r$, so

$$\frac{1}{1-(w-z_0)/(z-z_0)} = \sum_{n=0}^{\infty} \frac{(w-z_0)^n}{(z-z_0)^n},$$

and, for fixed w, this series converges uniformly for $z \in \Gamma_r$. This means that

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(z)}{(z-z_0)} \sum_{n=0}^{\infty} \frac{(w-z_0)^n}{(z-z_0)^n}\, dz \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\Gamma_r} \frac{f(z)}{(z-z_0)}\, \frac{(w-z_0)^n}{(z-z_0)^n}\, dz \\ &= \sum_{n=0}^{\infty} c_n (w-z_0)^n, \end{aligned}$$

where

$$c_n = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z-z_0)^{n+1}}\, dz,$$

by independence of contour. (Here we have exchanged the order of summation and integration.) Once we know that f has this power series representation, it follows that $c_n = \frac{f^{(n)}(z_0)}{n!}$ from results on power series in Lecture 9.

We chose r and w such that $|w - z_0| < r < R$. If we take any $w \in B(z_0, R)$, then we may choose r such that these inequalities hold, so the series converges at w. Since w is an arbitrary element of $B(z_0, R)$, the series converges in $B(z_0, R)$, and the radius of convergence is at least R. □

Here is an example of the use of Cauchy's generalised integral formula.

Exercise 18.3. Compute $\displaystyle\int_{\Gamma} \frac{e^z}{z^{n+1}}\, dz$, where Γ is the circle with centre 0 and radius 1.

Answer.

2. Liouville's theorem and the fundamental theorem of algebra

Cauchy's generalised integral formula has some very surprising consequences. Here is one.

THEOREM 18.4 (Liouville's Theorem). *Suppose that f is a bounded entire function. Then f is constant.*

PROOF. Since f is bounded, we may choose a positive constant C such that $|f(z)| \leq C$ for all $z \in \mathbb{C}$.

Since f is entire, we may take Γ_R to be the circle centre 0 and radius R, and use Cauchy's generalised integral formula to find the power series for f inside Γ_R:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \tag{18.2}$$

where

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_{\Gamma_R} \frac{f(z)}{z^{n+1}} \, dz.$$

The power series (18.2) converges inside Γ_R, so its radius of convergence ρ is at least R; as R may be made arbitrarily large, $\rho = \infty$.

Further, when $|z| = R$,

$$\left| \frac{f(z)}{z^{n+1}} \right| \leq \frac{C}{R^{n+1}},$$

and so, by the ML Lemma,

$$\left| f^{(n)}(0) \right| = \frac{n!}{2\pi} \left| \int_{\Gamma_R} \frac{f(z)}{z^{n+1}} \, dz \right| \leq \frac{n!}{2\pi} \frac{C}{R^{n+1}} 2\pi R = \frac{n! \, C}{R^n}.$$

If $n \geq 1$, then the left hand side of the formula above is 0, since R may be made arbitrarily large, and so $f^{(n)}(0) = 0$. Thus the power series (18.2) simplifies to show that $f(z) = f(0)$. □

COROLLARY 18.5 (Fundamental theorem of algebra). *Suppose that f is a nonconstant complex polynomial. Then f has at least one root, and hence f may be factorised as a product of a constant and finitely many linear factors.*

SKETCH PROOF. Suppose that f has no root. First, $f(z) \to \infty$ as $z \to \infty$, and so there exists R such that $|f(z)| \geq 1$ when $|z| \geq R$. Next, in the compact set $\overline{B}(0, R)$, the function $|f|$ is continuous and takes positive values, so it has a minimum value, m say, which cannot be 0 as f has no root. Thus $|f(z)| \geq m$ when $|z| \leq R$. It follows that the function $1/f$ is bounded and entire, so $1/f$ is constant, and f is constant. Since f is not constant by hypothesis, f must have a root.

The complete factorisation of a polynomial follows by dividing out a factor of $z - r$ for each root r. If the quotient is a nonconstant polynomial, we can find another root, and keep on dividing out and finding more roots until the quotient is constant and we have a complete factorisation. □

3. Behaviour of a holomorphic function near a zero

A zero of a function f that is holomorphic in an open set Ω is a point $w \in \Omega$ such that $f(w) = 0$. For such a point w, Cauchy's generalised integral formula implies that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-w)^n \qquad \forall z \in B(w,r),$$

for some open ball $B(w,r)$. Note that $a_0 = f(w) = 0$. If all a_n are 0, then $f(z) = 0$ for all $z \in B(w,r)$. Otherwise, we define $N = \min\{n \in \mathbb{N} : a_n \neq 0\}$; then $a_n = 0$ when $n < N$ and $a_n \neq 0$.

One idea that we will use in several ways in this course is that $f(z)$ behaves like $a_N(z-w)^N$ near to w. Here is one way in which this is true.

Proposition 18.6. *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n(z-w)^n$ for all $z \in B(w,r)$, and that $a_n \neq 0$ for some $n \in \mathbb{N}$. Let $N = \min\{n \in \mathbb{N} : a_n \neq 0\}$. Then*

$$\lim_{z \to w} \frac{f(z)}{a_N(z-w)^N} = 1.$$

Proof†. We shall show that, given $\varepsilon \in \mathbb{R}^+$, there exists $r_\varepsilon \in \mathbb{R}^+$ such that

$$\left| \sum_{n=N+1}^{\infty} a_n(z-w)^n \right| < \varepsilon \left| a_N(z-w)^N \right| \qquad \forall z \in B(w, r_\varepsilon). \tag{18.3}$$

Since $a_n = 0$ when $n < N$, it follows that

$$\frac{f(z)}{a_N(z-w)^N} = \frac{\sum_{n=0}^{\infty} a_n(z-w)^n}{a_N(z-w)^N} = 1 + \frac{\sum_{n=N+1}^{\infty} a_n(z-w)^n}{a_N(z-w)^N},$$

and the lemma follows.

Now the argument involves power series. Take $z_0 \in B^\circ(w,r)$ such that $|z_0 - r|$ is close to r. Since the series $\sum_{n=0}^{\infty} a_n(z_0-w)^n$ converges, there is a constant C such that $|a_n(z_0-w)^n| \leq C$ for all $n \in \mathbb{N}$, so $|a_n| \leq C\,|z_0-w|^{-n}$. It follows that, when $|z-w| < |z_0-w|$,

$$\sum_{n=N+1}^{\infty} |a_n|\,|z-w|^n \leq \sum_{n=N+1} C\frac{|z-w|^n}{|z_0-w|^n} = C\,\frac{|z-w|^N}{|z_0-w|^N}\,\frac{|z-w|}{|z_0-w|-|z-w|}.$$

By taking r_ε small enough, we can ensure that

$$C\,\frac{1}{|z_0-w|^N}\,\frac{r_\varepsilon}{|z_0-w|-r_\varepsilon} < \varepsilon\,|a_N|\,,$$

and then when $z \in B(w, r_\varepsilon)$, it follows that

$$C\,\frac{|z-w|^N}{|z_0-w|^N}\,\frac{|z-w|}{|z_0-w|-|z-w|} < \varepsilon\left|a_N(z-w)^N\right|,$$

and then (18.3) holds. □

Later we will use this fact to prove l'Hôpital's rule for analytic functions.

Corollary 18.7. *Suppose that Ω is an open set, that $f \in H(\Omega)$, and that $f(w) = 0$ for some $w \in \Omega$. Then there exists $r \in \mathbb{R}^+$ such that either $f(z) = 0$ for all $z \in B(w,r)$ or $f(z) \neq 0$ for all $z \in B^\circ(w,r)$.*

PROOF†. Write $f(z) = \sum_{n=0}^{\infty} a_n(z-w)^n$ for all $z \in B(w,r)$, and suppose that $a_n \neq 0$ for some $n \in \mathbb{N}$. Then $f^{(n)}(w) \neq 0$, and so f is not identically equal to 0 near w. Take N as in Proposition 18.6. Then there exists $r \in \mathbb{R}^+$ such that

$$\left| \frac{f(z)}{a_N(z-w)^N} - 1 \right| < \frac{1}{2} \qquad \forall z \in B^\circ(w,r),$$

and for these z it follows that $f(z) \neq 0$. □

In summary, if a holomorphic function f is not constant, then the zeroes of f are *isolated*.

4. Examples

EXERCISE 18.8. Compute $\displaystyle\int_0^{2\pi} \frac{4}{5+3\cos(\theta)}\,d\theta$.

ANSWER.

This integral may also be computed using the substitution $t = \tan(\theta/2)$.

LECTURE 19

Morera's Theorem and analytic continuation

In this lecture, we prove Morera's theorem, which completes a logical circle relating differentiability and independence of contour for integrals. Then we discuss *analytic continuation* for holomorphic functions. Finally, we compute more examples of contour integrals.

1. Morera's theorem

THEOREM 19.1 (Morera's theorem). *Suppose that Ω is a domain, that the function $f : \Omega \to \mathbb{C}$ is continuous, and that*

$$\int_\Gamma f(z)\,dz = 0,$$

whenever Γ is a closed contour in Ω. Then f is holomorphic in Ω.

PROOF. This proof is in two steps.

Step 1. We fix a base point b in Ω, and for all $w \in \Omega$, define $F(w)$ to be $\int_\Gamma f(z)\,dz$, where Γ is a contour from b to w. Then $F'(w) = f(w)$ for all $w \in \Omega$, and hence F is holomorphic. The details are in the proof of the corollary of the Cauchy–Goursat theorem on existence of primitives.

Step 2. We deduce that f is holomorphic from the fact that F is holomorphic. To do this, we appeal to the corollary to Cauchy's integral formula from the preceding lecture. We take an arbitrary point $z_0 \in \Omega$, and then $B(z_0, r) \subseteq \Omega$ for some $r \in \mathbb{R}^+$ because Ω is open. Now $F \in H(B(z_0, r))$, and so by the corollary,

$$F(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

in $B(z_0, r)$. We may differentiate a power series term by term in any ball in which it converges, so

$$f(z) = F'(z) = \sum_{n=0}^{\infty} c_n n (z - z_0)^{n-1} = \sum_{m=0}^{\infty} c_{m+1}(m + 1)(z - z_0)^m$$

in $B(z_0, r)$, and hence f is holomorphic in $B(z_0, r)$. Now f is holomorphic in Ω because z_0 was an arbitrary point in Ω. $\square$

COROLLARY 19.2. *Suppose that Λ is a (possibly infinite) line segment in an open set Ω and $\Omega \setminus \Lambda$ is open. If function $f : \Omega \to \mathbb{C}$ is continuous in Ω and is holomorphic in $\Omega \setminus \Lambda$, then f is holomorphic in Ω.*

SKETCH PROOF. By Morera's theorem, and an approximation argument, it suffices to show that $\int_\Gamma f(z)\,dz = 0$ for all closed polygonal contours Γ in Ω. We can break such an integral into a sum of integrals over closed contours in $\Omega \setminus \Lambda$ together with an error term that may be made arbitrarily small. $\square$

2. The logic of the theorems on contour integration

We have seen a number of results about contour integration, involving various hypotheses and conclusions, and we now summarise these results. We suppose that f is a continuous function defined in a simply connected domain Ω, and Γ denotes a contour in Ω. Consider the following conditions:

(a) f is holomorphic in Ω

(b) $\displaystyle\int_\Gamma f(z)\,dz = 0$ for every closed Γ

(b′) $\displaystyle\int_\Gamma f(z)\,dz$ depends only on the start and end of Γ

(b″) there is a function F in Ω such that $\displaystyle\int_\Gamma f(z)\,dz = F(q) - F(p)$, where Γ goes from p to q, and $F' = f$

(c) $\displaystyle f(w) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-w}\,dz$ for any simple closed Γ such that $w \in \operatorname{Int}(\Gamma)$

(d) $\displaystyle f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ in any open ball $B(z_0, r)$ contained in Ω.

We have seen that:

(a) $\implies$ (b) (the Cauchy–Goursat theorem)
(b) $\implies$ (b′) (independence of contour)
(b′) $\implies$ (b″) (existence of primitives)
(a) $\implies$ (c) (Cauchy's integral formula)
(c) $\implies$ (d) (corollary to Cauchy's integral formula)
(d) $\implies$ (a) (power series)
(b′) $\implies$ (a) (Morera's theorem)
(b″) $\implies$ (a) (proof of Morera's theorem)

What is important in all this is that:

(a) for holomorphic functions in simply connected domains, integrals depend only on the initial and final points of contours,
(b) to integrate a holomorphic function f along a contour from p to q, we may find a primitive F and compute $F(q) - F(p)$
(c) integrals of functions around closed contours where the function is not holomorphic at finitely many points inside the contour may be calculated in terms of the values of the function or its derivatives at these points.

3. Analytic continuation

We have already seen that the values of a holomorphic function f on an interval determine f in an open ball. The key to this is showing that if f is 0 on an interval, then f is 0 on the ball. Now we see that the values of a holomorphic function in an open subset of a domain determine the values in the whole domain.

LEMMA 19.3. *Suppose that $B(z_1, r_1) \subset B(w, R)$, that $f \in H(B(w, R))$, and that $f(z) = 0$ for all $z \in B(z_1, r_1)$. Then $f(z) = 0$ for all $z \in B(w, R)$.*

PROOF. We can find a finite sequence of open balls, $B(z_1, r_1)$, $B(z_2, r_2)$, ..., $B(z_J, r_J)$, say, with the properties that

$$B(z_1, r_1) \subset B(z_2, r_2) \subset \cdots \subset B(z_J, r_J) = B(w, R),$$

and the centre of $B(z_{j+1}, r_{j+1})$ is contained in $B(z_j, r_j)$. We will show that $f(z) = 0$ for all $z \in B(z_J, r_J)$ by induction.

By hypothesis, $f(z) = 0$ for all $z \in B(z_1, r_1)$.

Suppose that $f(z) = 0$ for all $z \in B(z_j, r_j)$. Then $f(z) = 0$ for all z near the centre of $B(z_{j+1}, r_{j+1})$. Hence $f^{(n)}(z_{j+1}) = 0$ for all $n \in \mathbb{N}$. We may expand f in a power series in $B(z_{j+1}, r_{j+1})$, and the coefficients in the power series are multiples of the derivatives $f^{(n)}(z_{j+1})$. It follows that $f(z) = 0$ for all $z \in B(z_{j+1}, r_{j+1})$.

By induction, $f(z) = 0$ for all $z \in B(z_J, r_J)$. □

THEOREM 19.4. *Suppose that Υ is a nonempty open subset of a domain Ω in $\mathbb{C}$, and that $f \in H(\Omega)$. If $f(z) = 0$ for all z in Υ, then $f(z) = 0$ for all z in Ω.*

PROOF†. Take a base point $b \in \Upsilon$. Since Ω is connected, any point $w \in \Omega$ may be joined to b by a polygonal contour Γ in Ω. Each point z on Γ is in the open set Ω, and so there is a ball $B(z, r)$ with centre z that is contained in Ω. Because Γ is compact, there are finitely many of these balls, $B(z_j, r_j)$ say, such that $B(z_j, r_j) \cap B(z_{j+1}, r_{j+1})$ is not empty, $b \in B(z_1, r_1)$ and $w \in B(z_J, r_J)$. Since $f(z) = 0$ for all z near b, the lemma implies that $f(z) = 0$ for all $z \in B(z_1, r_1)$. Now $f(z) = 0$ for all z in an open subset of $B(z_2, r_2)$, and the lemma implies that $f(z) = 0$ for all $z \in B(z_2, r_2)$. Continuing inductively, $f(z) = 0$ for all $z \in B(z_J, r_J)$ and hence $f(w) = 0$. □

COROLLARY 19.5. *Suppose that Υ is a nonempty open subset of a domain Ω in $\mathbb{C}$, and that $f, g \in H(\Omega)$. If $f(z) = g(z)$ for all z in Υ, then $f(z) = g(z)$ for all z in Ω.*

PROOF. Apply the previous result to $f - g$. □

4. Further examples of contour integrals

EXERCISE 19.6. Compute $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{5 + 3\cos\theta}\, d\theta$.

ANSWER.

EXERCISE 19.7. Compute $\displaystyle\int_0^{2\pi} \frac{\sin\theta}{\cos\theta}\, d\theta$.

ANSWER.

In all the following exercises, Γ is the circle of radius 3 and centre 0, traversed in the usual anticlockwise direction, and $f \in H(B(0,\pi))$.

EXERCISE 19.8. Compute $\displaystyle\int_\Gamma \frac{f(z)}{z^2-1}\, dz$.

ANSWER.

Exercise 19.9. Compute $\displaystyle\int_\Gamma \frac{f(z)}{(z-1)^2}\,dz$.

Answer.

Exercise 19.10. Compute $\displaystyle\int_\Gamma \frac{f(z)}{(z^2-1)^2}\,dz$.

Answer.

5. Remarks

The exercises that we have done suggest that we will be able to compute "arbitrary" integrals around closed contours of functions of the form $f(z)/p(z)$, where p is a polynomial. To do this, we will need to be able to factorise p, and then expand $1/p$ into partial fractions, that is, a sum of terms of the form $1/(z-\alpha)^k$. We will need to be able to compute with partial fractions!

LECTURE 20

Taylor series

We recall some facts about power series, Cauchy's integral formula, and Taylor series. We discuss computation with and manipulation of Taylor series.

1. Power series, Taylor series, and Maclaurin series

DEFINITION 20.1. A *power series* (with centre z_0) is an expression of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the *coefficients* a_n, the *centre* z_0, and the *variable* z are complex. A *Taylor series* (with centre z_0) for a function f is a series of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

A *Maclaurin series* for a function f is a Taylor series for f with centre 0, that is, a series of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

Maclaurin series and Taylor series are particular examples of power series.

Recall from Lecture 10 that a power series has a *radius of convergence*, ρ, which can often be found using the ratio test or the root test. The power series converges inside $B(z_0, \rho)$ and fails to converge outside $\overline{B}(z_0, \rho)$; if a power series converges in $B(z_0, r)$, then $r \leq \rho$, but it is possible that $r < \rho$.

We say that (or write)

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{ in } B(z_0, r)$$

to mean that the domain of f includes $B(z_0, r)$, the power series converges in $B(z_0, r)$ and that $f(z)$ is the sum of the power series for each $z \in B(z_0, r)$. If this holds, then f is holomorphic in $B(z_0, r)$, and moreover

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \tag{20.1}$$

that is, this power series is the Taylor series for f with centre z_0. Conversely, from the lecture on Cauchy's generalised integral formula, if a function f is holomorphic in $B(z_0, r)$, then it can be represented as a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ in the same ball.

When we ask how large a ball with centre z_0 in which a function f is represented by a power series can be, we often find that the maximum value of the radius r

of the ball is equal to the distance of z_0 from the set of points where f fails to be holomorphic.

EXERCISE 20.2. Show that the function f defined by $f(z) = 1/z$ can be represented as a power series in a ball $B(z_0, r)$, where $z_0 \neq 0$. Find the radius of convergence of this power series.

ANSWER.

The function f fails to be holomorphic at 0, which is on the edge of the ball $B(z_0, |z_0|)$. We say that "the *singularity* of f at 0 is a barrier to the convergence of the power series in any larger ball with centre z_0".

2. The algebra and calculus of power series

To determine a Taylor series, we need to be able to find *all* the derivatives of a function. There are only a few basic examples, and variations on these, for which finding all the derivatives is possible. Geometric series and exponential series are particularly important. Putting together these basic series to find more complicated series is our next topic.

First, we can add and multiply power series.

THEOREM 20.3. *Suppose that* $c \in \mathbb{C}$, *and that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n \quad \text{in } B(z_0, r).$$

Then

(a) $c\,f(z) = \displaystyle\sum_{n=0}^{\infty} c\,a_n (z - z_0)^n$ *in* $B(z_0, r)$;

(b) $(f + g)(z) = \displaystyle\sum_{n=0}^{\infty} (a_n + b_n)(z - z_0)^n$ *in* $B(z_0, r)$;

(c) $(fg)(z) = \displaystyle\sum_{n=0}^{\infty} c_n (z - z_0)^n$ *in* $B(z_0, r)$, *where* $c_n = \sum_{j=0}^{n} a_j\, b_{n-j}$.

PROOF. First, $f, g \in H(B(z_0, r))$, and hence $c\,f, f + g, fg \in H(B(z_0, r))$ from properties of holomorphic functions. Thus these new functions may also be represented by power series in $B(z_0, r)$. To determine the coefficients, we observe that

$$(c\,f)^{(n)}(z_0) = (c\,f)^{(n)}(z_0), \qquad (f + g)^{(n)}(z_0) = f^{(n)}(z_0) + g^{(n)}(z_0),$$

and

$$(fg)^{(n)}(z_0) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z_0)\, g^{(n-k)}(z_0),$$

and the theorem follows from this and (20.1). □

We can differentiate and integrate power series term by term.

THEOREM 20.4. *Suppose that* $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ *in* $B(z_0, r)$*, where* $r > 0$*. Then* f *is differentiable in* $B(z_0, r)$*, and*

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z-z_0)^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} (z-z_0)^m \qquad \text{in } B(z_0, r).$$

Further, let

$$b_m = \begin{cases} c & \text{when } m = 0 \\ \dfrac{a_{m-1}}{m} & \text{when } m \geq 1. \end{cases}$$

Then $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ *converges in* $B(z_0, r)$*, to a function* F *say, such that* $F' = f$ *and* $F(z_0) = c$*.*

PROOF. We omit this proof. □

It is sometimes possible to divide power series, since the quotient of holomorphic functions is homomorphic, at least when the denominator does not vanish, and holomorphic functions may be represented as power series. It is also sometimes possible to compose power series, since the composition of holomorphic functions is homomorphic. Rather than try to state theorems about these operations, we will look at some examples.

3. Examples

EXERCISE 20.5. Consider the function Log. Determine its Taylor series with centre $i-1$. What is the radius of convergence ρ of this series. Does the series represent Log in the ball $B(i-1, \rho)$?

ANSWER.

EXERCISE 20.6. Define the function f by $f(z) = \dfrac{\sin(z)}{z}$ if $z \neq 0$ and $f(0) = 1$. Does this function have a Maclaurin series? If so, then what is its radius of convergence?

ANSWER.

We note that it follows that $f^{(2n)}(0) = (-1)^n/(2n+1)$ and $f^{(2n+1)}(0) = 0$ for all $n \in \mathbb{N}$. It would also be possible to compute the derivatives of the function $z \mapsto \sin(z)/z$ "by hand", and to find a formula for these using induction, but this would be much longer.

When we discuss singularities in more detail, we will say that "the function $z \mapsto \sin(z)/z$ has a removable singularity at 0".

EXERCISE 20.7. Define the function $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \dfrac{z}{\sin(z)}$ when $z \neq 0$ and $g(0) = 1$. Does this function have a Maclaurin series? If so, then what is its radius of convergence?

ANSWER.

4. Compositions of power series

We begin with an easy exercise.

Exercise 20.8. Find the Maclaurin series for e^{z^2}.

Answer.

More generally, if $f(z) = \sum_{m=0}^{\infty} a_m(z-z_0)^m$ and $g(w) = \sum_{n=0}^{\infty} b_n(w-w_0)^n$, then

$$g(f(z)) = \sum_{n=0}^{\infty} b_n \left(\sum_{m=0}^{\infty} a_m(z-z_0)^m - w_0 \right)^n;$$

at least in principle we can expand the inner power and then gather terms. Usually this is only done if $a_0 = w_0$; then the first nonzero term of the expanded series is $[a_1(z-z_0)]^n$, and only finitely many terms are involved in the coefficient of each of the powers $(z-z_0)^k$. Using this, it is possible to determine the power series corresponding to the inverse function of a holomorphic function.

Theorem† 20.9 (Lagrange inversion theorem). *Suppose that $f \in H(\Omega)$, and that $f(a) = b$ and $f'(a) \neq 0$. Then there is a holomorphic function g, defined in an open set Υ that contains b, such that $g \circ f(z) = z$ for all z near to a, and $f \circ g(w) = w$ for all w near to b. Further, $g(w) = a + \sum_{n=1}^{\infty} c_n(w-b)^n$, where*

$$c_n = \frac{1}{n!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{z-a}{f(z)-f(a)} \right)^n$$

when $n \geq 1$.

Proof. Omitted. □

Exercise 20.10. Define the function f by $f(z) = e^{1/z}$ when $z \neq 0$. Find a series that represents f in $B(1,1)$.

Answer.

Laurent series

In this lecture, we prove Laurent's theorem about representing holomorphic functions in an annulus by a series, which we now call a Laurent series, and we present some examples of these series.

An annulus is a set of the form $\{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$; we allow R_1 to be 0 or R_2 to be ∞. We also call the set $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ a punctured ball, and we denote it by $B^\circ(z_0, R)$.

1. Laurent's theorem

THEOREM 21.1. *Suppose that* A *is the annulus* $\{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$ *and* $R_1 < r < R_2$. *If* $f \in H(\mathrm{A})$, *then*

$$f(w) = \sum_{n=-\infty}^{\infty} c_n (w - z_0)^n \qquad \forall w \in \mathrm{A},$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(z)}{(z - z_0)^{n+1}} \, dz.$$

PROOF. First, take r_1 and r_2 such that $R_1 < r_1 < r_2 < R_2$, and let Ω_1 be the domain $B(z_0, r_2) \setminus \overline{B}(z_0, r_1)$, as shown in Figure 21.1.

Suppose that $g \in H(\mathrm{A})$. By the Cauchy–Goursat theorem, $\int_{\partial\Omega_1} g(z)\, dz = 0$. It follows that

$$\int_{\partial B(z_0,r_1)} g(z)\, dz = \int_{\partial B(z_0,r_2)} g(z)\, dz.$$

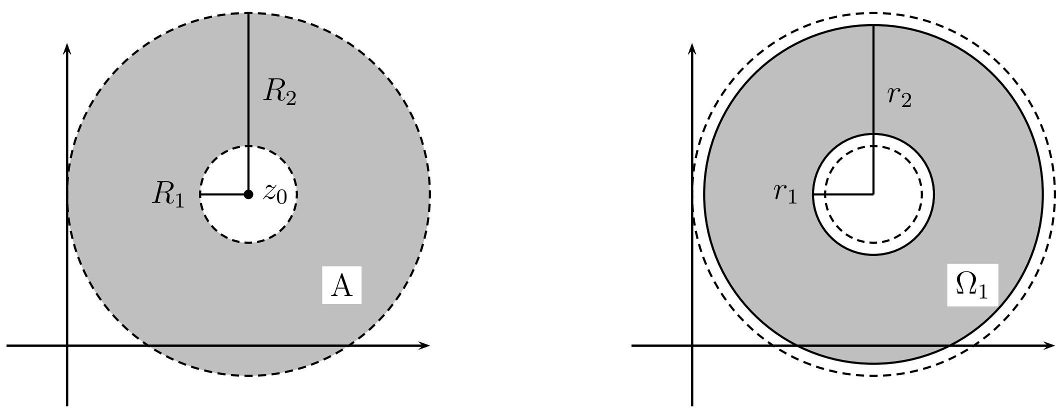

FIGURE 21.1. The annulus A and the first region Ω_1

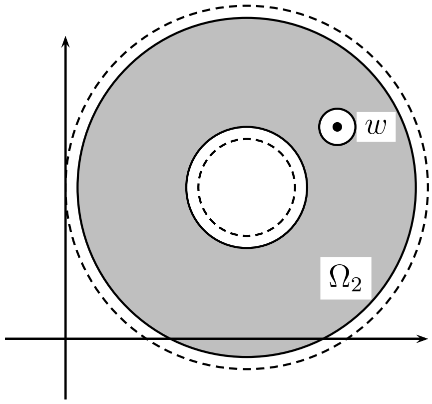

FIGURE 21.2. The second region Ω_2

In particular, by taking $g(z) = f(z)/(z - z_0)^n$, where $f \in H(\mathrm{A})$, we see that the expression defining c_n does not depend on r.

Now suppose that $f \in H(\mathrm{A})$ and $w \in \mathrm{A}$. Choose r_1 and r_2 such that

$$R_1 < r_1 < |w - z_0| < r_2 < R_2,$$

and take a small ball $B(w, r)$ whose closure lies inside $B(z_0, r_2) \setminus \overline{B}(z_0, r_1)$, as shown in Figure 21.2. Let $\Omega_2 = \Omega_1 \setminus \overline{B}(w, r)$.

By the Cauchy–Goursat theorem,

$$\frac{1}{2\pi i} \int_{\partial \Omega_2} \frac{f(z)}{z - w}\, dz = 0.$$

Breaking the integral over the boundary of Ω_2 up into its three constituent integrals, we see that

$$\frac{1}{2\pi i} \int_{\partial B(w,r)} \frac{f(z)}{z - w}\, dz = \frac{1}{2\pi i} \int_{\partial B(z_0,r_2)} \frac{f(z)}{z - w}\, dz - \frac{1}{2\pi i} \int_{\partial B(z_0,r_1)} \frac{f(z)}{z - w}\, dz.$$

From the Cauchy integral formula,

$$f(w) = \frac{1}{2\pi i} \int_{\partial B(z_0,r_2)} \frac{f(z)}{z - w}\, dz - \frac{1}{2\pi i} \int_{\partial B(z_0,r_1)} \frac{f(z)}{z - w}\, dz.$$

If z lies on $\partial B(z_0, r_2)$, then $|z - z_0| = r_2$ and $|w - z_0| < r_2$, so

$$\begin{aligned}
\int_{\partial B(z_0,r_2)} \frac{f(z)}{z - w}\, dz &= \int_{\partial B(z_0,r_2)} \frac{f(z)}{(z - z_0) - (w - z_0)}\, dz \\
&= \int_{\partial B(z_0,r_2)} \frac{f(z)}{(z - z_0)(1 - (w - z_0)/(z - z_0))}\, dz \\
&= \int_{\partial B(z_0,r_2)} \frac{f(z)}{(z - z_0)} \sum_{n=0}^{\infty} \frac{(w - z_0)^n}{(z - z_0)^n}\, dz \\
&= \sum_{n=0}^{\infty} (w - z_0)^n \int_{\partial B(z_0,r_2)} \frac{f(z)}{(z - z_0)^{n+1}}\, dz.
\end{aligned}$$

Similarly, if z lies on $\partial B(z_0, r_1)$, then $|z - z_0| = r_1$ and $|w - z_0| < r_1$, so

$$\begin{aligned}
-\int_{\partial B(z_0,r_1)} \frac{f(z)}{z-w}\,dz &= \int_{\partial B(z_0,r_1)} \frac{f(z)}{(w-z_0)-(z-z_0)}\,dz \\
&= \int_{\partial B(z_0,r_1)} \frac{f(z)}{(w-z_0)(1-(z-z_0)/(w-z_0))}\,dz \\
&= \int_{\partial B(z_0,r_1)} \frac{f(z)}{(w-z_0)} \sum_{m=0}^{\infty} \frac{(z-z_0)^m}{(w-z_0)^m}\,dz \\
&= \sum_{m=0}^{\infty} \frac{1}{(w-z_0)^{m+1}} \int_{\partial B(z_0,r_1)} f(z)\,(z-z_0)^m\,dz \\
&= \sum_{n=-\infty}^{-1} (w-z_0)^n \int_{\partial B(z_0,r_1)} \frac{f(z)}{(z-z_0)^{n+1}}\,dz
\end{aligned}$$

(at the last step, we substituted $n = -m - 1$). It follows that

$$f(w) = \sum_{n=-\infty}^{\infty} c_n (w - z_0)^n \qquad \forall w \in \mathrm{A},$$

where

$$c_n = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\partial B(z_0,r_2)} \frac{f(z)}{(z-z_0)^{n+1}}\,dz & \text{if } n \geq 0 \\ \dfrac{1}{2\pi i} \displaystyle\int_{\partial B(z_0,r_1)} \frac{f(z)}{(z-z_0)^{n+1}}\,dz & \text{if } n \leq -1. \end{cases}$$

To conclude, we recall that the integrals defining c_n around $\partial B(z_0, r)$ do not depend on r in the range (R_1, R_2). □

A series in powers of $(z - z_0)$ that converges in an annulus with centre z_0 is called a *Laurent series*. We do not always find Laurent series by computing integrals; rather, often we compute integrals by finding Laurent series.

2. Finding Laurent series

EXERCISE 21.2. Suppose that

$$f(z) = \frac{1}{z(z^2-1)} = \frac{1}{z(z-1)(z+1)} = \frac{-1}{z} + \frac{1/2}{z-1} + \frac{1/2}{z+1}\,.$$

Find Laurent series for f in the largest annuli with centre 0 in which f is holomorphic.

ANSWER.

REMARKS. This form of presentation is legitimate as long as there are sufficiently many terms that the pattern is clear.

Note that both series have only odd powers, reflecting the symmetry of f; indeed, z^n is an odd or even function as n is an odd or even integer. Note also that the radius of convergence of the first power series is 1; this corresponds to the fact that f is holomorphic in $B^\circ(0,1)$ but not in any larger punctured ball with centre 0. The second series converges when $|z| > 1$; indeed, it is a geometric series with ratio $|1/z^2|$.

Note also that the first Laurent series has only one negative power of z, while the second has many, starting with z^{-3}. This is because $f(z)$ behaves like z^{-1} when z is small and like z^{-3} when z is large.

EXERCISE 21.3. Suppose that

$$f(z) = \frac{1}{z(z^2-1)} = \frac{1}{z(z-1)(z+1)} = \frac{-1}{z} + \frac{1/2}{z-1} + \frac{1/2}{z+1}\,.$$

Find Laurent series for f in the largest annuli with centre 1 in which f is holomorphic.

ANSWER.

Remarks. These series involve both odd and even powers, reflecting the fact that f is neither even nor odd about 1.

None of these series converges in any larger annulus, since otherwise f would be holomorphic in a larger domain.

Note also that the last Laurent series starts with z^{-3}. This corresponds to the fact that $f(z)$ behaves like z^{-3} at infinity, and that $(z-1)^{-3}$ and z^{-3} behave similarly there. Thus we should have expected cancellations in the terms $(z-1)^{-1}$ and $(z-1)^{-2}$.

LECTURE 22

Singularities

In this lecture, we study holomorphic functions in a punctured ball, and illustrate this study with some examples. Recall that the punctured ball $B^\circ(z_0, R)$ is defined to be $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$; we allow R to be ∞.

1. Laurent series at an isolated singularity

DEFINITION 22.1. A function f has an *isolated singularity* at z_0 in $\mathbb{C}$ if f is holomorphic in the punctured ball $B^\circ(z_0, R)$ for some $R \in \mathbb{R}^+$, and f is not differentiable at z_0, perhaps because $f(z_0)$ is not defined or f is not continuous at z_0.

Suppose that $f \in H(B^\circ(z_0, R))$. Take r such that $0 < r < R$. By Laurent's theorem,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \qquad \forall z \in B^\circ(z_0, R),$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

If $c_n = 0$ for all $n \in \mathbb{Z}$, then f is identically zero in $B^\circ(0, R)$. In this case we can define $f(z_0)$ to be 0 and then $f \in H(B(z_0, R))$. Otherwise, are three mutually exclusive and exhaustive possibilities:

(i) There are infinitely many $n \in \mathbb{Z}^-$ such that $c_n \neq 0$. In this case, we say that f has an *(isolated) essential singularity* at z_0.

(ii) There are no $n \in \mathbb{Z}^-$ such that $c_n \neq 0$. In this case, we say that f has a *removable singularity* at z_0. If there exists $M \in \mathbb{Z}^+$ such that $c_M \neq 0$ and $c_n = 0$ for all $n < M$, then we say that f has a *zero of order (or multiplicity)* M at z_0. Zeros of order 1 are also known as *simple zeros*.

(iii) There are at least one and finitely many $n \in \mathbb{Z}^-$ such that $c_n \neq 0$. In this case, there exists $M \in \mathbb{Z}^-$ such that $c_M \neq 0$ and $c_n = 0$ for all $n < M$. In this case, we say that f has a *pole* at z_0 of order $-M$. Poles of order 1 are also known as *simple poles*.

EXAMPLES 22.2. The following examples illustrate different types of singularities.

1. Suppose that $f(z) = \dfrac{\sin(z)}{z}$. The natural domain for f is $\mathbb{C} \setminus \{0\}$, because $f(0)$ is not defined. As we have seen, it is possible to extend the definition of f to 0, by setting $f(0) = 1$, and then the extended function (still written f) is entire, and the singularity has been removed.

2. Suppose that $f(z) = \dfrac{(1 - \cos(z))^2}{z}$. The natural domain for f is $\mathbb{C} \setminus \{0\}$, because $f(0)$ is not defined. As we will see, it is possible to extend the definition of f to 0,

by setting $f(0) = 0$, and then f becomes an entire function, and the singularity has been removed. This extended function has a zero of order three at 0.

3. Suppose that $f(z) = \dfrac{1}{z^3 - z}$. The natural domain for f is $\mathbb{C} \setminus \{0, \pm 1\}$, because $f(0)$, $f(1)$ and $f(-1)$ are not defined. As we will see, $\lim_{z\to 0} f(z) = \infty$, so we will leave $f(0)$ undefined. For this function,

$$f(z) = \sum_{n=0}^{\infty} (-1)^{n+1} z^{2n-1}$$

in $B^\circ(0, 1)$, and there is a pole of order 1, that is, a simple pole, at 0.

4. Suppose that $f(z) = e^{1/z}$. The natural domain for f is $\mathbb{C} \setminus \{0\}$, because $f(0)$ is not defined. As we will see, $\lim_{z\to 0} f(z)$ does not exist, and f has an essential singularity at 0.

5. Suppose that $f(z) = \mathrm{Log}(z)$. The domain of f is $\mathbb{C} \setminus \{0\}$, and we say that f has a singularity at 0. However, f is not differentiable at all points on the negative real axis, due to the jump in Arg there, and so the singularity at 0 is not isolated.

6. Suppose that $f(z) = \dfrac{1}{\sin(\pi/z)}$. Then f is defined in $\mathbb{C}\setminus(\{0\}\cup\{1/n : n \in \mathbb{Z}\setminus\{0\}\})$. In this example, f has isolated singularities at the points $1/n$, where $n \in \mathbb{Z} \setminus \{0\}$, and a nonisolated singularity at 0.

THEOREM 22.3. *Suppose that $f \in H(B^\circ(z_0, R))$, and that*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \qquad \forall z \in B^\circ(z_0, R).$$

Then the following are equivalent:

(i) $c_n = 0$ *for all* $n < 0$
(ii) $\lim\limits_{z\to z_0} f(z)$ *exists in* $\mathbb{C}$, *and by defining* $f(z_0)$ *to be* $\lim\limits_{z\to z_0} f(z)$, *we may extend* f *to a holomorphic function on* $B(z_0, R)$
(iii) $\lim\limits_{z\to z_0} f(z)$ *exists in* $\mathbb{C}$
(iv) there exists $C \in \mathbb{R}^+$ *and* $r \in (0, R)$ *such that*

$$|f(z)| \le C \qquad \forall z \in B^\circ(z_0, r).$$

PROOF. Suppose that (i) holds. Then $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ in $B^\circ(z_0, R)$, and the power series converges in $B^\circ(z_0, R)$, so its radius of convergence is at least R. Hence $\lim_{z\to z_0} f(z) = c_0$, and by defining $f(z_0)$ to be c_0, we extend f to agree with the power series in $B(z_0, R)$, which is holomorphic there, that is, (ii) holds.

Suppose that (ii) holds. Then (iii) holds trivially.

Suppose that (iii) holds, i.e., that $\lim_{z\to z_0} f(z) = c$ for some $c \in \mathbb{C}$. Take $\varepsilon = 1$ in the definition of the limit. Then there exists r such that $0 < r < R$ and

$$|f(z) - c| < 1$$

when $z \in B^\circ(z_0, r)$. For such z,

$$|f(z)| = |f(z) - c + c| \le |f(z) - c| + |c| < 1 + |c|,$$

so taking C to be $1 + |c|$, we have $|f(z)| < C$, and (iv) holds.

Suppose that (iv) holds. We use an argument like that to prove Liouville's theorem. According to Laurent's theorem, if $0 < r < R$, then

$$c_n = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(z)}{(z - z_0)^{n+1}} \, dz \, .$$

Clearly when z lies on $\partial B(z_0, r)$,

$$\Big| \frac{f(z)}{(z - z_0)^{n+1}} \Big| = \frac{|f(z)|}{r^{n+1}} \le \frac{C}{r^{n+1}}.$$

The length of $\partial B(z_0, r)$ is $2\pi r$, and so, by the ML Lemma,

$$|c_n| \le \frac{1}{2\pi} \Big| \int_{\partial B(z_0,r)} \frac{f(z)}{(z - z_0)^{n+1}} \, dz \Big| \le \frac{1}{2\pi} \frac{C}{r^{n+1}} 2\pi r = 2\pi C r^{-n}.$$

If $n < 0$, then $-n > 0$, and $r^{-n} \to 0$ as $r \to 0$. In this case, we can make r^{-n} arbitrarily small, so $c_n = 0$, i.e., (i) holds. □

The reason for the terminology "removable singularity" should now be clear: by defining, or perhaps redefining, $f(z_0)$ appropriately, we may extend f to be holomorphic in $B(z_0, R)$, that is, we *remove the singularity* of f at z_0.

It is possible to generalize the proof of the theorem above to prove the following result.

THEOREM 22.4. *Suppose that $f \in H(B^\circ(z_0, R))$, and that*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \qquad \forall z \in B^\circ(z_0, R).$$

Then the following are equivalent:

(i) $c_n = 0$ *for all* $n < M$ *and* $c_M \ne 0$
(ii) *there exists* $F \in H(B(z_0, R))$ *such that* $f(z) = (z - z_0)^M F(z)$ *in* $B^\circ(z_0, R)$ *and* $F(z_0) \ne 0$
(iii) $\lim_{z \to z_0} (z - z_0)^{-M} f(z)$ *exists and is in* $\mathbb{C} \setminus \{0\}$
(iv) *there exists* $C \in \mathbb{R}^+$ *and* $r \in (0, R)$ *such that*

$$|f(z)| \le C |z - z_0|^M \qquad \forall z \in B^\circ(z_0, r),$$

but $\lim_{z \to z_0} (z - z_0)^{-M} f(z) \ne 0$ *(either because the limit does not exist or because it exists and is not* 0*).*

We omit the details of the proof, but observe that it involves defining the function F by $F(z) = (z - z_0)^{-M} f(z)$, applying the techniques of the proof of the previous theorem to F, and being careful about when limits are 0.

We write $f(z) \sim c(z - z_0)^N$ as $z \to z_0$, where $c \in \mathbb{C} \setminus \{0\}$ and $N \in \mathbb{Z}$, to mean that

$$\lim_{z \to z_0} \frac{f(z)}{c(z - z_0)^N} = 1.$$

Note that if $f(z) \sim c(z - z_0)^N$ as $z \to z_0$, then there is a punctured ball $B(z_0, r)$ in which f does not vanish.

Summary. If f has a zero of order N at z_0, then $f(z) \sim c_N(z-z_0)^N$ as $z \to z_0$, where $c_N \neq 0$, and vanishes at z_0; when the order of the zero is higher, $f(z)$ vanishes more rapidly.
If f has a pole of order N at z_0, then $f(z) \sim c_N(z-z_0)^{-N}$ as $z \to z_0$, where $c \neq 0$, and diverges to ∞; when the order of the pole is higher, the divergence is more rapid.

If f has a removable singularity at z_0, then f is bounded near z_0, and vice versa.

Corollary 22.5. *Suppose that $f, g, h \in H(B^\circ(z_0, R))$ and that f has a zero of order M at z_0 while g has a zero of order N at z_0, while h has a pole of order P at z_0. Then:*

(i) *$1/h$ has a zero of order P at z_0;*
(ii) *$1/f$ has a pole of order M at z_0;*
(iii) *fg has a zero of order $M+N$ at z_0;*
(iv) *if $M \geq N$, then f/g has a removable singularity at z_0, and if $M > N$, then f/g has a zero of order $M-N$;*
(v) *if $M < N$, then f/g has a pole of order $N-M$.*

Proof. We prove only (v). If f has a zero of order M at z_0 while g has a zero of order N at z_0, then $f(z) \sim c(z-z_0)^M$ and $g(z) \sim d(z-z_0)^N$ as $z \to z_0$, where $c, d \in \mathbb{C} \setminus \{0\}$. If follows that $(f/g)(z) \sim (c/d)(z-z_0)^{M-N}$ as $z \to z_0$, where $(c/d) \neq 0$. □

We may now prove l'Hôpital's rule.

Theorem 22.6. *Suppose that $f, g \in H(\Omega)$ and $z_0 \in \Omega$. Suppose also that $\lim_{z\to z_0} f(z)/g(z)$ is of the form $0/0$. If $\lim_{z\to z_0} f'(z)/g'(z)$ exists, then so does $\lim_{z\to z_0} f(z)/g(z)$, and the limits are equal.*

Proof. By hypothesis, f and g have Taylor series in some ball centred at z_0, and $f(z) \sim a(z-z_0)^M$ while $g(z) \sim b(z-z_0)^N$ as $z \to z_0$, where $a, b \in \mathbb{C} \setminus \{0\}$. Then $f'(z) \sim aM(z-z_0)^{M-1}$ and $g'(z) \sim bN(z-z_0)^{N-1}$ as $z \to z_0$. If $\lim_{z\to z_0} f'(z)/g'(z)$ exists, then

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{a(z-z_0)^M}{b(z-z_0)^N} = \lim_{z\to z_0} \frac{N}{M} \frac{aM(z-z_0)^{M-1}}{bN(z-z_0)^{N-1}} = \frac{N}{M} \lim_{z\to z_0} \frac{f'(z)}{g'(z)}.$$

Since the right hand side exists, so does the left hand side. Further, the right hand side may be 0, in which case the left hand side is too, and the desired equality holds, or a nonzero complex number, in which case $M = N$ and then the desired equality also holds. □

2. Examples

Exercise 22.7. Suppose that $f(z) = (1-\cos(z))^2/z$. What kind of singularity does f have at 0?

Answer.

EXERCISE 22.8. Suppose that $f(z) = e^{1/z}$. How does f behave near 0?

ANSWER.

3. More about singularities† (Not examinable)

We can say more about the behaviour of a function near a singularity.

THEOREM 22.9 (Picard's theorem). *If f has an essential singularity at z_0, then for all $\delta > 0$, the set $\mathbb{C} \setminus f(B^\circ(z_0, \delta))$ has at most one element.*

We shall not prove this result. But consider Exercise 22.8. Given any c in $\mathbb{C} \setminus \{0\}$, we can find w such that $e^w = c$. Let w_n be $w + 2\pi i n$, for all $n \in \mathbb{N}$. Then $e^{w_n} = c$. Now take $z_n = 1/w_n$; this sequence tends to 0. Thus there is a z_n as close to 0 as we like for which $e^{1/z_n} = c$. This illustrates Picard's theorem.

LECTURE 23

Residues and the residue theorem

In this lecture, we define residues, and prove Cauchy's residue theorem. We show how to find residues. This will enable us to find integrals over closed contours efficiently.

1. Residues

DEFINITION 23.1. Suppose that the function f has an isolated singularity at z_0. Then f is holomorphic in some punctured ball $B^\circ(z_0, r)$, and has a Laurent series $\sum_{n=-\infty}^{\infty} c_n(z - z_0)^n$ there. The *residue of f at z_0*, written $\operatorname{Res}(f, z_0)$ or $\operatorname{Res}(f(z), z = z_0)$, is defined to be c_{-1}, the coefficient of $(z - z_0)^{-1}$ in this series.

From Laurent's theorem, we see that if f is holomorphic in $B^\circ(z_0, r)$, and Γ is a simple closed contour in $B^\circ(z_0, r)$ around z_0, then

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi i} \int_\Gamma f(z)\, dz. \tag{23.1}$$

Observe that $\operatorname{Res}(g, z_0) = 0$ if g is holomorphic in $B(z_0, r)$; this can be seen from the definition above, or from the Cauchy–Goursat theorem and (23.1).

We are interested in residues because of the following generalization of Cauchy's integral formula.

THEOREM 23.2 (Cauchy's residue theorem). *Suppose that Γ is a simple closed contour with the standard orientation in a domain Ω and $f \in H(\Omega)$, and further that $\operatorname{Int}(\Gamma) \cap \Omega = \operatorname{Int}(\Gamma) \setminus \{z_1, z_2, \dots, z_K\}$. Then*

$$\int_\Gamma f(z)\, dz = 2\pi i \sum_{k=1}^{K} \operatorname{Res}(f, z_k).$$

We prove this theorem later. This theorem enables us to calculate integrals over closed curves as long as we can calculate residues.

EXERCISE 23.3. Suppose that

$$f(z) = \frac{\alpha_1}{z - a} + \frac{\beta_1}{z - b} + \frac{\beta_2}{(z - b)^2}\,.$$

Find the residues of f at a and b, and hence find $\int_\Gamma f(z)\, dz$, where Γ is a simple closed contour surrounding a and b.

ANSWER.

This example shows that, for rational functions, residues are coefficients of certain terms in the partial fraction expansions, and that not all the coefficients are needed to compute contour integrals (in the example above, β_2 does not matter at all). If we can compute residues efficiently, then we will not need to find partial fraction expansions.

2. Computing residues

Suppose that f has an isolated singularity at z_0. There are three ways to compute $\operatorname{Res}(f, z_0)$, according to the type of singularity,

First, if the singularity is removable, that is, if f is bounded near z_0, the residue is 0.

Second, if f has a pole of order N at z_0, then

$$\begin{aligned} f(z) = c_{-N}(z-z_0)^{-N} + c_{1-N}(z-z_0)^{1-N} + \dots \\ + c_{-1}(z-z_0)^{-1} + c_0(z-z_0)^0 + c_1(z-z_0)^1 + \dots \end{aligned}$$

so

$$\begin{aligned} (z-z_0)^N f(z) = c_{-N}(z-z_0)^0 + c_{1-N}(z-z_0)^1 + \dots \\ + c_{-1}(z-z_0)^{N-1} + c_0(z-z_0)^N + c_1(z-z_0)^{N+1} + \dots \end{aligned}$$

and

$$\begin{aligned} &\frac{d^{N-1}}{dz^{N-1}}\Big((z-z_0)^N f(z)\Big) \\ &\quad = \frac{d^{N-1}}{dz^{N-1}}\Big(c_{-N}(z-z_0)^0 + c_{1-N}(z-z_0)^1 + \dots \\ &\qquad\qquad + c_{-1}(z-z_0)^{N-1} + c_0(z-z_0)^N + c_1(z-z_0)^{N+1} + \dots\Big) \\ &\quad = (N-1)!\, c_{-1}(z-z_0)^0 + N!\, c_0(z-z_0)^1 + \frac{(N+1)!}{2!}\, c_1(z-z_0)^2 + \dots, \end{aligned}$$

whence

$$\lim_{z\to z_0} \frac{d^{N-1}}{dz^{N-1}}\,(z-z_0)^N f(z) = (N-1)!\, c_{-1}.$$

Thus

$$\operatorname{Res}(f, z_0) = \frac{1}{(N-1)!} \lim_{z\to z_0} \frac{d^{N-1}}{dz^{N-1}}(z-z_0)^N f(z). \tag{23.2}$$

Note that this formula is not the definition of a residue; that definition is at the start of this lecture. To use this formula, we have to know the order of the pole, and it does not apply at all singularities.

Third, if f has an essential singularity at z_0 then we need a different approach, usually involving integration or series.

If f has a singularity at z_0, we could proceed by looking at

$$\lim_{z \to z_0} (z - z_0)^n f(z)$$

for increasing values of n, starting at 0, until such time as we find a finite number; this is then the order of the pole, N. Once we know N, we use formula (23.2). This would be very inefficient if the order of the pole is large, and would not give a result at all for an essential singularity! *It is therefore important to know how to find the order of the pole.* The key information about the orders of zeros and poles is in the previous lecture.

EXERCISE 23.4. Suppose that $f(z) = \dfrac{z - \pi/2}{1 - \sin(z)}$. Find $\operatorname{Res}(f, \pi/2)$.

ANSWER.

We could have worked out that the pole was simple by working out that the zero of the numerator is of order 1 and that of the denominator is of order 2, and then doing the second calculation only.

The following result is often useful.

PROPOSITION 23.5 (The p/q' formula). *Suppose that $f(z) = p(z)/q(z)$ in Ω, and that $p(z_0) \neq 0$ while $q(z_0) = 0$. If z_0 is a simple zero of q, that is, a zero of order 1, then*

$$\operatorname{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}.$$

PROOF. Since the pole of f is simple,

$$\begin{aligned} \operatorname{Res}(f, z_0) &= \lim_{z \to z_0} (z - z_0) \frac{p(z)}{q(z)} = \lim_{z \to z_0} p(z) \frac{(z - z_0)}{q(z)} \\ &= p(z_0) \lim_{z \to z_0} \frac{(z - z_0)}{q(z)} = p(z_0) \frac{1}{q'(z_0)} = \frac{p(z_0)}{q'(z_0)}, \end{aligned}$$

by l'Hôpital's rule. □

EXERCISE 23.6. Suppose that

$$f(z) = \tan(z).$$

Find the residue of f at $(2k + 1)\pi/2$, where $k \in \mathbb{Z}$.

Answer.

Of course, we could also have done this calculation directly, without using the formula.

Exercise 23.7. Suppose that

$$f(z) = 2z \sin(z^{-2}).$$

Find the residue of f at 0.

Answer.

Exercise 23.8. Suppose that

$$f(z) = \sin(z)\, e^{1/z}.$$

Find the residue of f at 0.

Answer.

The expression for the residue in the last example does not simplify but may be computed numerically to any desired degree of accuracy. This behaviour is common with essential singularities.

Essential singularities usually appear because there is a "transcendental function" (such as exp or sinh or tan) which can be written as a power series, with something like $1/z$ in its argument. Residues involving transcendental functions

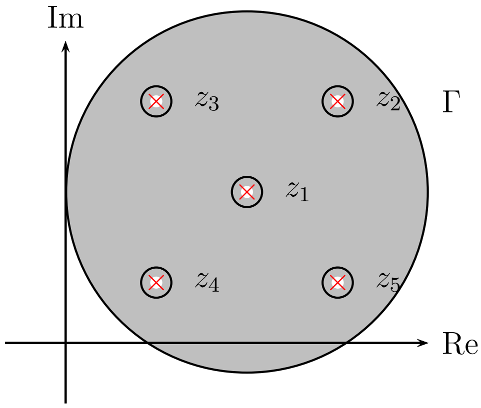

FIGURE 23.1. Curves around the singularities inside Γ

can often be expressed as series which are derived from the corresponding power series.

3. Proof of Cauchy's Residue Theorem

For convenience, we restate the theorem.

THEOREM (Cauchy's residue theorem). *Suppose that* Γ *is a simple closed contour with the standard orientation in a (multiply connected) domain* Ω*, that* $f \in H(\Omega)$*, and that* $\operatorname{Int}(\Gamma) \cap \Omega = \operatorname{Int}(\Gamma) \setminus \{z_1, z_2, \dots, z_K\}$*. Then*

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{k=1}^{K} \operatorname{Res}(f, z_k).$$

PROOF. We take balls $B(z_k, \varepsilon)$ centred at the singularities z_k, where the ε are chosen small enough that the closed balls $\bar{B}(z_i, \varepsilon)$ and $\bar{B}(z_j, \varepsilon)$ are disjoint if $i \neq j$, and such that each closed ball $\bar{B}(z_k, \varepsilon)$ is contained in $\operatorname{Int}(\Gamma)$. This ensures that the contours around the perimeters of the balls $B(z_k, \varepsilon)$ are well defined and do not meet each other or Γ. Define

$$\Upsilon = \operatorname{Int}(\Gamma) \setminus \Big(\bigcup_{k=1}^{K} \overline{B}(z_k, \varepsilon) \Big).$$

See Figure 23.1. Then f is holomorphic on the open set Ω which contains $\bar{\Upsilon}$, and $\partial\Upsilon$ is made up of Γ, traversed anti-clockwise, together with the boundaries $\partial B(z_k, \varepsilon)$ traversed clockwise.

The Cauchy–Goursat theorem implies that

$$0 = \int_{\partial\Upsilon} f(z)\,dz = \int_\Gamma f(z)\,dz + \sum_{k=1}^{K} \int_{\partial B(z_k,\varepsilon)^*} f(z)\,dz,$$

that is,

$$\int_\Gamma f(z)\,dz = \sum_{k=1}^{K} \int_{\partial B(z_k,\varepsilon)} f(z)\,dz.$$

From (23.1),

$$\operatorname{Res}(f, z_k) = \frac{1}{2\pi i} \int_{\partial B(z_k, \varepsilon)} f(z)\, dz,$$

so

$$\int_{\partial B(z_k, \varepsilon)} f(z)\, dz = 2\pi i \operatorname{Res}(f, z_k),$$

and hence

$$\int_{\Gamma} f(z)\, dz = \sum_{k=1}^{K} \int_{\partial B(z_k, \varepsilon)} f(z)\, dz = 2\pi i \sum_{k=1}^{K} \operatorname{Res}(f, z_k);$$

the theorem is proved. □

LECTURE 24

Computing integrals. 1

In this lecture, we give three examples of the use of Cauchy's residue theorem to calculate integrals.

1. Trigonometric integrals

Exercise 24.1. Evaluate $\displaystyle\int_{-\pi}^{\pi} \frac{\cos(\theta)}{5 - 4\cos(\theta)}\, d\theta.$

Answer.

Remarks on this calculation. If the integrand had contained a sin term, then we could have used the fact that

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}.$$

Similarly, expressions such as $\sin(2\theta)$ and $\tan(\theta)$ may be expressed in terms of $e^{i\theta}$ and hence of z. In summary, any integral that can be put in the form

$$\int_{-\pi}^{\pi} f(\sin(\theta), \cos(\theta))\, d\theta$$

where f is a rational function of two variables, can be tackled in this way, and becomes an integral of a different rational function around a closed contour, which can be evaluated, at least in principle, and often in practice.

There are some other integrals that may be converted to integrals of this form. For instance, since cos is an even function, so is $\theta \mapsto \cos(\theta)/(5 - 4\cos(\theta))$, and

$$\int_{0}^{\pi} \frac{\cos(\theta)}{5 - 4\cos(\theta)}\, d\theta = \int_{-\pi}^{0} \frac{\cos(\theta)}{5 - 4\cos(\theta)}\, d\theta$$

and so

$$\int_{0}^{\pi} \frac{\cos(\theta)}{5 - 4\cos(\theta)}\, d\theta = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\cos(\theta)}{5 - 4\cos(\theta)}\, d\theta.$$

If the integrand involves expressions like $\cos(\theta/2)$, this method does not work: square roots appear, and these mess up the holomorphy.

2. A rational function on the real line

Exercise 24.2. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4 + 1}\, dx$.

Answer.

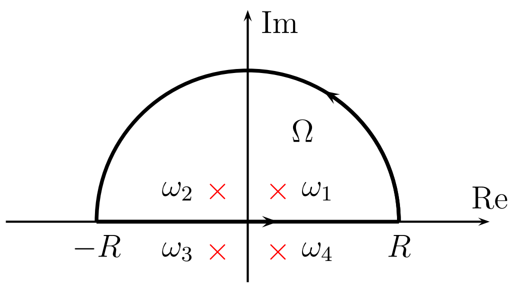

FIGURE 24.1. The region Ω

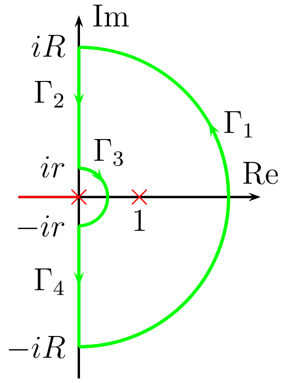

FIGURE 24.2. An indented contour

3. An integral involving a root

EXERCISE 24.3. Compute the contour integral

$$\int_\Gamma \frac{z^{1/2}}{z^2-1}\,dz,$$

where Γ is the join of the contours Γ_1, Γ_2, Γ_3 and Γ_4 shown in Figure 24.2, and $z^{1/2}$ denotes the principal branch of the square root. What happens to the component integrals when $r \to 0$ and $R \to \infty$? Find

$$\int_0^{+\infty} \frac{x^{1/2}}{x^2+1}\,dx.$$

ANSWER.

We can compute integrals involving functions with branches as long as the branch cut is outside the contour. The process of avoiding a singularity, such as 0 here, by adding a small circular arc to the contour, is called *indenting*.

LECTURE 25

Computing Integrals. 2

In this lecture, we compute more definite integrals, and introduce Jordan's lemma, which can be useful in this context.

1. An integral involving an exponential

EXERCISE 25.1. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2+1}\,dx$, where $\xi \in \mathbb{R}^+$.

ANSWER.

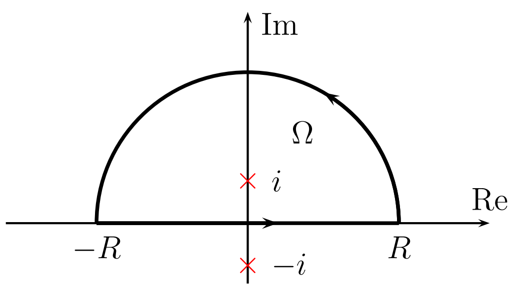

FIGURE 25.1. The region Ω

Remarks on this calculation. There are many integrals on the whole line $\mathbb{R}$ that may be evaluated in this way. Integrals between 0 and ∞ may also be tackled if the integrand is even. When the integrand involves $\cos(\xi x)$ or $\sin(\xi x)$, then we write the trigonometric function in terms of exponentials, and if necessary change variables to get an integral involving $e^{i\xi x}$, where $\xi \geq 0$. Then $|e^{i\xi z}| \leq 1$ when $z \in \text{Range}(\gamma)$. Sometimes this inequality is not enough and we have to use Jordan's lemma (see below).

For example, by the changes of variable $x' = -x$ and then $x = x'$,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos(\xi x)}{x^2+1}\,dx &= \frac{1}{2}\Big(\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2+1}\,dx + \int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{x^2+1}\,dx\Big)\\ &= \frac{1}{2}\Big(\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2+1}\,dx + \int_{-\infty}^{\infty} \frac{e^{i\xi x'}}{(x')^2+1}\,dx'\Big)\\ &= \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2+1}\,dx,\end{aligned}$$

which we have just computed. Since $\cos(x)$ and x^2 are even functions of x,

$$\int_0^{\infty} \frac{\cos(\xi x)}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(\xi x)}{x^2+1}\,dx = \frac{\pi e^{-\xi}}{2}\,.$$

2. Jordan's lemma

LEMMA 25.2. *Suppose that f is continuous in $\{z \in \mathbb{C} : \text{Im}(z) \geq 0,\ |z| \geq S\}$, for some positive S, that Γ_R is the upper half of the circle with centre 0 and radius R, and that $|f(z)| \leq M_R$ for all $z \in \Gamma_R$ where $\lim_{R\to\infty} M_R = 0$.*

Then for any $\xi > 0$,

$$\lim_{R\to\infty}\Big|\int_{\Gamma_R} e^{i\xi z}\, f(z)\,dz\Big| = 0.$$

PROOF. We give the proof at the end of this lecture. □

3. An example of the use of Jordan's lemma

EXERCISE 25.3. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x\,e^{i\xi x}}{x^2+1}\,dx$, where $\xi \in \mathbb{R}^+$.

ANSWER.

We may write the integral in the previous exercise as

$$\int_{\mathbb{R}} \frac{x\cos(\xi x)}{x^2+1}\,dx + i\int_{\mathbb{R}} \frac{x\sin(\xi x)}{x^2+1}\,dx.$$

The result implies that the first integral is 0, as it should be since the integrand is odd, while the second integral is $\pi e^{-\xi}$.

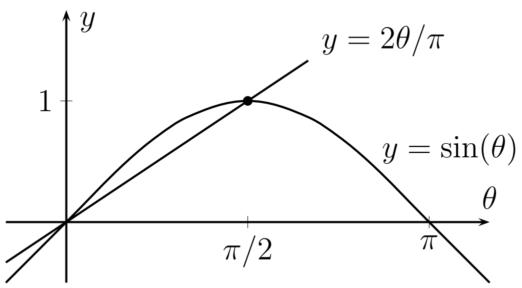

FIGURE 25.2. The graphs $y = \sin\theta$ and $y = \pi\theta/2$.

4. Proof of Jordan's lemma†

For convenience, we restate the result.

LEMMA. *Suppose that f is continuous in the set $\{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0,\ |z| \geq S\}$, for some positive S, that γ_R is the upper half of the circle with centre 0 and radius R, and that $|f(z)| \leq M_R$ for all $z \in \gamma_R$ where $\lim_{R\to\infty} M_R = 0$.*

Then for any $\xi > 0$,

$$\lim_{R\to\infty}\left|\int_{\gamma_R} e^{i\xi z}\, f(z)\, dz\right| = 0.$$

PROOF. We parametrise γ_R by setting $\gamma_R(\theta) = Re^{i\theta}$. Then

$$\begin{aligned}
\left|\int_{\gamma_R} e^{i\xi z}\, f(z)\, dz\right| &= \left|\int_0^\pi e^{i\xi Re^{i\theta}}\, f(Re^{i\theta})\, iRe^{i\theta}\, d\theta\right| \\
&\leq \int_0^\pi \left|e^{i\xi R(\cos(\theta)+i\sin(\theta))}\, f(Re^{i\theta})\, iRe^{i\theta}\right| d\theta \\
&= \int_0^\pi e^{-\xi R\sin(\theta)}\, \left|f(Re^{i\theta})\right|\, R\, d\theta \qquad (25.7)\\
&\leq \int_0^\pi e^{-\xi R\sin(\theta)}\, M_R\, R\, d\theta \\
&= 2\int_0^{\pi/2} e^{-\xi R\sin(\theta)}\, M_R\, R\, d\theta,
\end{aligned}$$

because sin is symmetric about $\pi/2$.

It is clear from the graph in Figure 25.2, and could be proved by using either the mean value theorem or convexity, that $\sin(\theta) \geq 2\theta/\pi$ when $0 \leq \theta \leq \pi/2$, so $-\xi R\sin(\theta) \leq -2\xi R\theta/\pi$ and

$$e^{-\xi R\sin(\theta)} \leq e^{-2\xi R\theta/\pi}.$$

Thus, substituting $t = 2\xi R\theta/\pi$, we see from (25.7) that

$$\begin{aligned}\Big|\int_{\gamma_R} e^{i\xi z}\, f(z)\, dz\Big| &\leq 2M_R \int_0^{\pi/2} e^{-2\xi R\theta/\pi}\ R\, d\theta \\ &= \frac{\pi}{\xi}\, M_R \int_0^{\xi R} e^{-t}\ dt \\ &\leq \frac{\pi}{\xi}\, M_R \int_0^{\infty} e^{-t}\ dt \\ &= \frac{\pi}{\xi}\, M_R \\ &\to 0\end{aligned}$$

as $R \to \infty$, as required. □

LECTURE 26

The Theory of Functions

In this lecture, we count the number of times a closed curve winds around a point, and use this to count the number of zeros and poles of a function. This leads to some surprising facts about holomorphic functions.

We start with several definitions that could have come earlier.

DEFINITION 26.1. A point z of a subset S of a set Ω is *isolated* if there exists $\varepsilon \in \mathbb{R}^+$ such that $B(z, \varepsilon) \cap S = \{z\}$. We say that S is *discrete* if every point of S is isolated.

DEFINITION 26.2. A function is said to be *meromorphic* in the open set Ω if it is holomorphic in $\Omega \setminus \Delta$, where Δ is a discrete subset of Ω, and the singularity at each point of Δ is a pole.

1. The winding number, zeros and poles

If $\gamma : [a, b] \to \mathbb{C}$ is a closed curve that does not pass through 0, then $\gamma(b) = \gamma(a)$ and so $\log \gamma(b) = \log \gamma(a)$. We may choose $\arg(\gamma(t))$ to vary continuously with t. When we do this, $\arg(\gamma(b)) - \arg(\gamma(a))$ is 2π multiplied by an integer, and the integer is the total number of times γ winds about 0 in the anti-clockwise sense. If we now define log using this version of arg, then

$$\begin{aligned}\log(\gamma(b)) - \log(\gamma(a)) &= \ln|\gamma(b)| + i\arg(\gamma(b)) - \ln|\gamma(a)| - i\arg(\gamma(a)) \\ &= i(\arg(\gamma(b)) - \arg(\gamma(a))),\end{aligned}$$

and so $(\log(\gamma(b)) - \log(\gamma(a)))/(2\pi i)$ is the number of times γ winds about 0.

Moreover, if $\gamma : [a, b] \to \mathbb{C}$ is a closed curve that does not pass through w, then the number of times that γ winds about w should be the same as the number of times that $\gamma - w$ winds about 0. This, together with the fact that $\log(z - w)$ is the integral of $1/(z - w)$, inspires the following definition.

DEFINITION 26.3. The *winding number* of a closed curve γ about a point w is defined to be

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z - w}\, dz.$$

Now we see how to count zeros and poles inside a simple closed curve.

THEOREM 26.4 (Cauchy's argument principle). *Suppose that f is meromorphic in the simply connected domain Ω, with zeros of order m_j at the points a_j and poles of order n_k at the points b_k. Suppose also that Γ is a simple closed contour in Ω that does not pass through any zero or pole of f. Then*

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\, dz = \sum_{j:\ a_j \in \mathrm{Int}(\Gamma)} m_j - \sum_{k:\ b_k \in \mathrm{Int}(\gamma)} n_k. \tag{26.1}$$

Proof. From Cauchy's residue theorem, the left hand side of the equality above is equal to the sum of the residues of f'/f at its singularities inside γ. The function f'/f has a singularity at w if and only if w is a zero or a pole of f, so it suffices to compute the residues at these points.

Suppose that f has a zero or a pole at w. Then, from the lecture on singularities, we know that there exist $\varepsilon \in \mathbb{R}^+$ and $m \in \mathbb{Z}$ such that $f(z) = (z-w)^m\, g(z)$, where $g, 1/g \in H(B(w,\varepsilon))$. If w is a zero, then the order of the zero is m, while if w is a pole, then the order of the pole is $-m$. Hence

$$\frac{f'(z)}{f(z)} = \frac{m(z-w)^{m-1}\, g(z) + (z-w)^m\, g'(z)}{(z-w)^m\, g(z)} = \frac{m\, g(z) + (z-w)\, g'(z)}{(z-w)\, g(z)}\,.$$

Then f'/f has a simple pole at w, and

$$\begin{aligned}\operatorname{Res}(f'/f, w) &= \lim_{z\to w} (z-w)\frac{m\, g(z) + (z-w)\, g'(z)}{(z-w)\, g(z)}\\ &= \lim_{z\to w} \frac{m\, g(z) + (z-w)\, g'(z)}{g(z)}\\ &= m.\end{aligned}$$

□

This result has been used to provide experimental confirmation of the Riemann hypothesis, which states that the zeros of the Riemann zeta function lie on the line $\{z \in \mathbb{C} : \operatorname{Re}(z) = \frac{1}{2}\}$. We will discuss this later.

Theorem 26.4 may also be used to show that functions have zeros in certain regions.

Corollary 26.5. *Suppose that Ω is a domain, that $f \in H(\Omega)$, that $\gamma : [a,b] \to \Omega$ is a simple closed curve, and that $f(\gamma(t)) \neq 0$ for all $t \in [a,b]$. Then the number of zeros of f in $\operatorname{Int}(\gamma)$ (counting multiplicities) is equal to the number of times $f \circ \gamma$ winds around 0.*

Proof. Let $\delta = f \circ \gamma$. Then $\delta : [a,b] \to \mathbb{C}$ is also a closed curve, though not necessarily simple. Further, $\delta'(t) = f'(\gamma(t))\,\gamma'(t)$, and so

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = \int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\,\gamma'(t)\, dt = \int_a^b \frac{\delta'(t)}{\delta(t)}\, dt = \int_\delta \frac{1}{z}\, dz.$$

□

2. Rouché's theorem

The next result gives us a useful tool for counting zeros.

Theorem 26.6 (Rouché's theorem). *Suppose that $\gamma : [a,b] \to \Omega$ is a closed curve in a simply connected domain Ω, that $f, g \in H(\Omega)$, and that $|f(z)| < |g(z)|$ for all $z \in \operatorname{Range}(\gamma)$. Then the number of zeros of $f+g$ inside γ is equal to the number of zeros of g inside γ.*

Proof. From the hypotheses, no zero of $f+g$ or g lies on γ; indeed, if $t \in [a,b]$, then

$$|(f+g)(\gamma(t))| = |f(\gamma(t)) + g(\gamma(t))| \geq |g(\gamma(t))| - |f(\gamma(t))| > 0$$

and

$$|g(\gamma(t))| > |f(\gamma(t))| \geq 0.$$

We need to show that the winding number of $(f+g)\circ\gamma$ is the same as the winding number of $g\circ\gamma$. We choose the argument $\arg(g(\gamma(t))$ so that it varies continuously with t. Since $|f(\gamma(t))| < |g(\gamma(t))|$, we may choose $\arg(f(\gamma(t)) + g(\gamma(t)))$ in such a way that

$$|\arg((f+g)(\gamma(t))) - \arg(g(\gamma(t)))| < \frac{\pi}{2}\,;$$

when we do this, $\arg((f+g)(\gamma(t)))$ varies continuously with t, and

$$|\arg((f+g)(\gamma(b))) - \arg((f+g)(\gamma(a))) - \arg(g(\gamma(b))) + \arg(g(\gamma(a)))| < \pi\,.$$

Since the left-hand side is equal to $2\pi k$ for some integer k, it must be 0, and so

$$\arg((f+g)(\gamma(b))) - \arg((f+g)(\gamma(a))) = \arg(g(\gamma(b))) - \arg(g(\gamma(a))),$$

which means that the winding numbers are equal. □

Here is a corollary.

Corollary 26.7. *Suppose that $p(z) = \sum_{j=0}^{n} c_j\, z^j$, and that $|c_m| > \sum_{j\neq m} |c_j|$. Then p has m zeros (counting multiplicities) inside the unit circle.*

Proof. Take $\gamma(t) = e^{it}$, where $0 \le t \le 2\pi$, and define $f(z) = \sum_{j\neq m} c_j\, z^j$ and $g(z) = c_m\, z^m$. The hypothesis on the coefficients of p means that $|f(e^{it})| < |g(e^{it})|$, and by Rouché's theorem, $(f+g)$ has the same number of zeros inside γ as g, that is, m zeros. □

Exercise 26.8. Suppose that $n \ge 3$, and let $p(z) = 2z^n + z^2 - 3z - 1$. Using Rouché's theorem, show that all the zeros of p lie inside the circle of radius 2.

Answer.

3. More consequences of Rouché's theorem

Rouché's theorem has many other corollaries that help us understand the behaviour of holomorphic functions.

Theorem 26.9. *Suppose that $R > 0$, and that*

$$h(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{in } B(z_0, R).$$

Let M be the smallest positive integer for which $a_m \neq 0$. Then there exist $\delta, \varepsilon \in \mathbb{R}^+$ such that for all $w \in B^\circ(a_0, \varepsilon)$, there are exactly M points z in $B^\circ(z_0, \delta)$ such that $f(z) = w$ (counting multiplicities).

The proof of this result may be found at the end of this lecture.

Remark 26.10. Suppose that $f(z) = \sum_{n=M}^{\infty} a_n (z-z_0)^n$ in $B(z_0, r)$, where $a_M \neq 0$. We have already remarked that $f(z)$ "behaves like $a_M(z - z_0)^M$ near z_0", in the sense that

$$\lim_{z\to z_0} \frac{f(z)}{a_M(z - z_0)^M} = 1.$$

Another way in which this assertion is true is that both f and the leading term of the Taylor series are both M-to-1 near z_0.

In the case where $a_1 \neq 0$, Theorem 26.9 implies that f is one-to-one near z_0. This is the hard part of the inverse function theorem.

COROLLARY 26.11 (Inverse function theorem). *Suppose that f is a holomorphic function in Ω, that $f(z_0) = w_0$, and that $f'(z_0) \neq 0$. Then there is an open subset Υ of Ω, containing z_0, and an open ball B containing w_0 such that the restriction of f to Υ is a bijection from Υ onto B, and the inverse function F^{-1} from B to Υ is holomorphic. Further,*

$$(f^{-1})'(w_0) = \frac{1}{f'(z_0)} \,.$$

PROOF. By Theorem 26.9 above, there exist $\delta, \varepsilon \in \mathbb{R}^+$ such that for every point $w \in B(w_0, \varepsilon)$, there exists exactly one point z in $B(z_0, \delta)$ such that $f(z) = w$. We define $f^{-1}(w) = z$.

It may be shown (but we do not do so here) that a continuous bijection on a compact set has a continuous inverse. From this result, it follows that f^{-1} is continuous, and so $f^{-1}(w) \to f^{-1}(w_0)$ as $w \to w_0$.

Now, taking $w = f(z)$ and $w_0 = f(z_0)$, we see that

$$\begin{aligned}\lim_{w\to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} &= \lim_{f^{-1}(w)\to f^{-1}(w_0)} \left(\frac{w - w_0}{f^{-1}(w) - f^{-1}(w_0)}\right)^{-1} \\ &= \lim_{z\to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0}\right)^{-1} = f'(z_0)^{-1} \,,\end{aligned}$$

which shows both that the derivative exists and finds it. □

COROLLARY 26.12. *Suppose that Ω and Υ are domains, and f is a holomorphic and bijective function from Ω to Υ. Then $f'(z) \neq 0$ for all $z \in \Omega$, and hence the inverse function f^{-1} is also holomorphic.*

PROOF. If $f'(z_0) = 0$ for some $z_0 \in \Omega$, then f is not one-to-one near z_0, by Theorem 26.9 above. This is a contradiction, and hence f' never vanishes in Ω.

Now the inverse function f^{-1} is holomorphic by the inverse function theorem. □

THEOREM 26.13 (The open mapping theorem). *Suppose that Ω is a domain in $\mathbb{C}$ and $f \in H(\Omega)$ is nonconstant. Then* $\mathrm{Range}(f)$ *is open in* $\mathbb{C}$.

PROOF. Take $w_0 \in \mathrm{Range}(f)$; then $w_0 = f(z_0)$ for some $z_0 \in \Omega$.

From the theorem above, there exists $\varepsilon \in \mathbb{R}^+$ such that every point in $B(w_0, \varepsilon)$ also lies in $\mathrm{Range}(f)$. Hence $\mathrm{Range}(f)$ is open. □

EXERCISE 26.14. Find an example of a holomorphic function defined on an open set Ω that is not constant, but whose range is not open.

ANSWER.

4. A proof†

In this section, we prove Theorem 26.9, whose statement we first recall.

Theorem. *Suppose that h is a nonconstant holomorphic function on $B(z_0, R)$, where $R > 0$, and that*

$$h(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \textit{in } B(z_0, R).$$

Let M be the smallest positive integer for which $a_m \neq 0$. Then there exist $\delta, \varepsilon \in \mathbb{R}^+$ such that for all $w \in B^\circ(a_0, \varepsilon)$, there are exactly M points z in $B^\circ(z_0, \delta)$ such that $f(z) = w$ (counting multiplicities).

Proof†. Since h is not constant, it is not possible that $a_n = 0$ for all positive n. Thus M exists. Note that a_0 need not be 0.

We may write

$$h(z) - a_0 = \sum_{n=M}^{\infty} a_n (z - z_0)^n \quad \text{and} \quad h'(z) = \sum_{n=M}^{\infty} n a_n (z - z_0)^{n-1}$$

in $B(z_0, R)$. In the lecture on Taylor series, we showed that

$$\lim_{z \to z_0} \frac{h'(z)}{M a_M (z - z_0)^{M-1}} = 1.$$

Thus there exists $\delta_1 \in \mathbb{R}^+$ such that

$$\left| \frac{h'(z)}{M a_M (z - z_0)^{M-1}} - 1 \right| < \frac{1}{2}$$

for all $z \in \overline{B}(z_0, \delta_1) \setminus \{z_0\}$; this implies that $h'(z) \neq 0$ for all such z. Similarly,

$$\lim_{z \to z_0} \frac{h(z) - a_0}{a_M (z - z_0)^M} = 1,$$

and there exists $\delta_0 \in \mathbb{R}^+$ such that $h(z) - a_0 \neq 0$ for all $z \in \overline{B}(z_0, \delta_0) \setminus \{z_0\}$. Take $\delta = \min\{\delta_0, \delta_1\}$.

The function $|h(z) - a_0|$ is continuous and nonvanishing on $\partial B(z_0, \delta)$. Define $\varepsilon = \min\{|h(z) - a_0| : z \in \partial B(z_0, \delta)\}$; then $\varepsilon > 0$, by compactness.

Suppose that $w \in B^\circ(a_0, \varepsilon)$. Take $f(z) = a_0 - w$, and $g(z) = h(z) - a_0$; then $|f(z)| < \varepsilon \leq |g(z)|$ for all $z \in \partial B(z_0, \delta)$. Rouché's Theorem implies that $f + g$ and g have the same number of zeros in $B(z_0, \delta)$, counting multiplicities. But z_0 is a zero of multiplicity M for g, and so $f + g$ has M zeros. The derivative of $f + g$ does not vanish in $B^\circ(z_0, \delta)$, and so these zeros all have order one. Since $(f+g)(z) = h(z) - w$, there are M distinct points in $B(z_0, \delta)$ such that $h(z) = w$. □

LECTURE 27

The Riemann zeta function

In this lecture, we show how residue calculus can be used to evaluate sums. We also introduce the Riemann ζ function, a focus of current mathematical research.

1. Computing sums

We begin with an exercise.

EXERCISE 27.1. Denote by Γ_N the perimeter of the square with vertices at the points $\pm(N+1/2) \pm i(N+1/2)$, where $N \in \mathbb{N}$, as in Figure 27.1, and define

$$f(z) = \frac{\cot(\pi z)}{z^2}.$$

(a) Find the singularities of f inside Γ_N, classify them, and compute the residues of f at the singularities.

(b) Show that

$$|\cot(\pm(N+1/2)\pi + iy)| \le 1 \qquad \forall y \in \mathbb{R},$$

and

$$|\cot(x \pm i(N+1/2)\pi)| \le \coth(\pi/2) \qquad \forall x \in \mathbb{R}.$$

(c) Deduce that if Λ is a vertical side and Δ is a horizontal side of Γ_N, then

$$\left|\int_\Lambda f(z)\,dz\right| \le \frac{4}{2N+1} \quad \text{and} \quad \left|\int_\Delta f(z)\,dz\right| \le \frac{4\coth(\pi/2)}{2N+1}.$$

(d) Hence find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

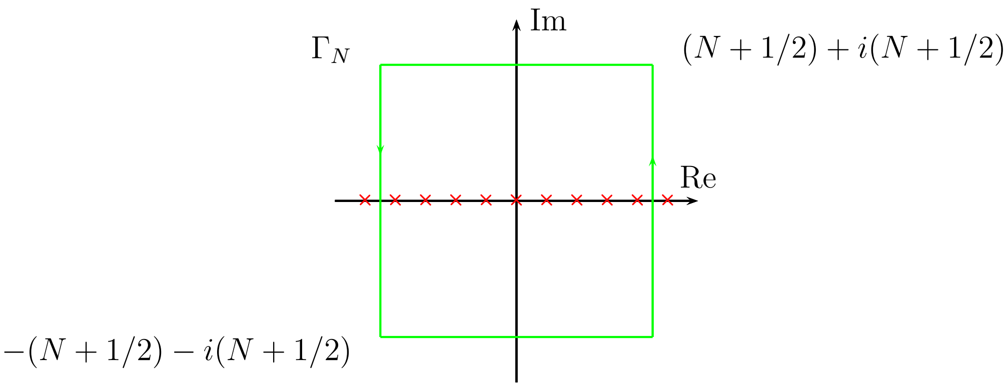

FIGURE 27.1. A contour

Answer.

EXERCISE 27.2. Compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$, where $k = 2, 3, 4, \ldots$.

2. The Riemann zeta function

EXERCISE 27.3. Suppose that $s \in \mathbb{C}$ and $s = \sigma + it$, where $\sigma > 1$. Show that

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^s}\right| < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} \left|\frac{\ln n}{n^s}\right| < \infty.$$

ANSWER.

The Riemann zeta function is defined by an infinite sum.

DEFINITION 27.4. The function ζ is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This sum converges absolutely when $\mathrm{Re}(s) > 1$, and continues meromorphically into $\mathbb{C}$.

It may be shown that ζ is holomorphic in the half plane $\{s \in \mathbb{C} : \operatorname{Re} s > 1\}$, and

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\ln n}{n^s} .$$

The argument is similar to that needed to show that we may differentiate inside an integral, which we will see shortly.

2.1. The Euler product formula. The zeta function is connected to prime numbers via the Euler product formula:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

Indeed, when $\operatorname{Re}(s) > 1$, we see that

$$(1 - p^{-s})^{-1} = 1 + p^{-s} + p^{-2s} + p^{-3s} + \dots,$$

and the summands in the expanded product are therefore all products over all primes of the form $p_1^{-k_1 s} p_2^{-k_2 s} p_3^{-k_3 s} \dots$. Each positive integer n has a unique factorisation of the form $p_1^{k_1} p_2^{k_2} p_3^{k_3} \dots$ (in which most of the powers k are 0), so the summands in the expanded product are therefore the powers n^{-s}, for all positive integer n.

2.2. The meromorphic continuation. Riemann showed that ζ extends holomorphically to $\mathbb{C} \setminus \{1\}$, and has a simple pole at 1. The first step in this meromorphic continuation is to show that

$$\zeta(s) = \frac{1}{s-1} \sum_{n=1}^{\infty} \left(\frac{n}{(n+1)^s} - \frac{n-s}{n^s} \right) \tag{27.1}$$

when $\operatorname{Re} s > 0$ (and $s \neq 1$). To do this, we observe first that when $\operatorname{Re}(s) > 2$, then

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left(\frac{n}{(n+1)^s} - \frac{n-s}{n^s} \right) \\
&\quad = \sum_{n=1}^{\infty} \frac{n+1}{(n+1)^s} - \sum_{n=1}^{\infty} \frac{1}{(n+1)^s} - \sum_{n=1}^{\infty} \frac{n}{n^s} + \sum_{n=1}^{\infty} \frac{s}{n^s} \\
&\quad = \zeta(s-1) - 1 - \zeta(s) + 1 - \zeta(s-1) + s\,\zeta(s) \\
&\quad = (s-1)\zeta(s).
\end{aligned}$$

However, it may be seen that the sum on the right hand side of (27.1) converges when $\operatorname{Re}(s) > 0$, and then analytic continuation (or more precisely meromorphic continuation) gives the desired result.

We may find other ways to rewrite $(s-1)\zeta(s)$ that converge when $\operatorname{Re}(s) > -1$, or when $\operatorname{Re}(s) > -2$, and so on.

2.3. More on the zeta function. It is possible to express the zeta function as an integral:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx.$$

To see this, assume that $\mathrm{Re}(s) > 1$, and write

$$\begin{aligned}\int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx &= \int_0^\infty \frac{x^{s-1}}{e^x(1-e^{-x})}\,dx = \int_0^\infty \sum_{n=0}^\infty \frac{x^{s-1}e^{-nx}}{e^x}\,dx \\ &= \sum_{n=0}^\infty \int_0^\infty x^{s-1}e^{-(n+1)x}\,dx = \Gamma(s)\sum_{n=0}^\infty \frac{1}{(n+1)^s}\,.\end{aligned}$$

Taking the infinite summation out of the integral requires exchanging the order of a limit and an integral.

Another interesting formula concerning the zeta function is the following *functional equation*

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,\zeta(1-s).$$

2.4. The Riemann hypothesis. Riemann hypothesised that if $\zeta(z) = 0$, then either $\mathrm{Re}(z) = 1/2$ or $z = -1, -3, -5, \dots$. The so-called Riemann hypothesis is one of the key problems in pure mathematics today. The first person to prove—or disprove—it will earn a prize of USD 1 million from the Clay Mathematics Institute. For more information, see

`http://www.claymath.org/millennium/`.

It is known that, apart from the so-called trivial zeros at -1, -3, -5, …, all the zeros lie inside the strip $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$, and that the zeta function has infinitely many zeros on the line $\{z \in \mathbb{C} : \mathrm{Re}(z) = 1/2\}$. It can be shown by numerical computation that all the nontrivial zeros z such that $|\,\mathrm{Im}(z)| \le 10^N$ lie on the line for some large N (in fact, finding zeros numerically is one of the test problems used by computer scientists to demonstrate their virtuosity).

To do this, one takes a rectangular contour which surrounds a segment of the vertical axis, above the horizontal axis; the vertical sides are where $\mathrm{Re}(z) = 0$ or $\mathrm{Re}(z) = 1$. Then there are no poles inside the contour; there are formulae which give the number of zeros on the axis. It is possible to compute the integral on the left hand side numerically, because the value must be an integer, and if the error is strictly less than $\frac{1}{2}$, then the real value must be the integer closest to the numerical value. If the number of zeros inside the contour is equal to the predicted number of zeros on the line $\{z \in \mathbb{C} : \mathrm{Re}(z) = \frac{1}{2}\}$, then there cannot be any zeros not on the line inside the contour.

In some sense, the Riemann hypothesis is equivalent to "randomness" of prime numbers. For instance it tells us that the number of primes in a large interval of integers can be given by a formula which amounts to saying that the probability of a large number n being prime is about $1/\log(n)$. If we could compute enough prime numbers, then we could test this computationally. However, it would require us to compute more prime numbers than there are atoms in the universe, so we would never be able to store all the necessary data.

The Riemann hypothesis is also of interest in theoretical physics, as the distribution of the zeros of a random matrix are related to those of the Riemann zeta function.

LECTURE 28

Interlude: Integrals[†]

In this lecture, we consider integrals in which the integrand contains a parameter, and consider the question of whether

$$\lim_{t\to t_0}\int_{-\infty}^{\infty} F(s,t)\,ds = \int_{-\infty}^{\infty}\lim_{t\to t_0} F(s,t)\,ds. \tag{28.1}$$

In our theorem, we suppose that F is defined on $\mathbb{R}\times P$, where the parameter space P is either $\mathbb{R}$ or $\mathbb{C}$. To cover both cases, let us agree that the notation $B(t,r)$ means the open interval $\{x\in\mathbb{R} : |x-t|<r\}$ when P is $\mathbb{R}$, and the open ball $\{z\in\mathbb{C} : |z-t|<r\}$ when P is $\mathbb{C}$.

We will often write

$$\int_{\mathbb{R}} |f(x)|\;dx \quad\text{rather than}\quad \int_{-\infty}^{\infty} |f(x)|\;dx,$$

$$\int_{[-R,R]} |f(x)|\;dx \quad\text{rather than}\quad \int_{-R}^{R} |f(x)|\;dx,$$

and

$$\int_{\mathbb{R}\setminus[-R,R]} |f(x)|\;dx \quad\text{rather than}\quad \int_{-\infty}^{-R} |f(x)|\;dx + \int_{R}^{\infty} |f(x)|\;dx.$$

So when the improper integral exists,

$$\int_{\mathbb{R}\setminus[-R,R]} |f(x)|\;dx \to 0 \quad\text{as } R\to\infty.$$

Other obvious variations of this notation will also be used.

1. An example

Just to be sure that there is a real issue to discuss, we consider an example.

Exercise 28.1. Define the function F on $\mathbb{R}\times(0,\infty)$ by

$$F(s,t) = t^{-3/2}s^2\exp(-s^2/t).$$

Find $\lim_{t\to 0+} F(s,t)$ and $\lim_{t\to+\infty} F(s,t)$, and decide whether or not (28.1) holds when $t_0 = 0$ and when $t = \infty$.

Answer.

It is a good idea to draw the graphs of the function $s \mapsto F(s,t)$ for different values of t to try to understand what is going on in the exercise above.

2. A theorem on limits

Now we prove a theorem giving sufficient conditions for the validity of exchanging the orders of limits and integrals. In this theorem P may be $\mathbb{R}$ or $\mathbb{C}$.

THEOREM 28.2. *Suppose that $F : \mathbb{R} \times P \to \mathbb{R}$ and $t_0 \in P$. Suppose also that there exists $\mu \in \mathbb{R}^+$ such that*

(a) F is continuous on $\mathbb{R} \times B(t_0, \mu)$, and
(b) there is a nonnegative function $M \in L^1(\mathbb{R})$ such that

$$|F(s,t)| \le M(s) \qquad \forall s \in \mathbb{R} \quad \forall t \in B(t_0, \mu).$$

Then

$$\lim_{t \to t_0} \int_{\mathbb{R}} F(s,t)\, ds = \int_{\mathbb{R}} \lim_{t \to t_0} F(s,t)\, ds = \int_{\mathbb{R}} F(s,t_0)\, ds.$$

PROOF. Suppose that $\varepsilon \in \mathbb{R}^+$. We need to find $\eta \in \mathbb{R}^+$ such that

$$\left| \int_{\mathbb{R}} F(s,t)\,ds - \int_{\mathbb{R}} F(s,t_0)\,ds \right| < \varepsilon$$

whenever $t \in B(t_0, \eta)$.

We will break the difference into two parts and make each part less than $\varepsilon/2$:

$$\begin{aligned}
&\left| \int_{\mathbb{R}} F(s,t)\,ds - \int_{\mathbb{R}} F(s,t_0)\,ds \right| \\
&= \left| \int_{\mathbb{R}} \left(F(s,t) - F(s,t_0)\right)\,ds \right| \\
&\le \int_{\mathbb{R}} \left|F(s,t) - F(s,t_0)\right|\,ds \\
&= \int_{\mathbb{R}\setminus[-R,R]} \left|F(s,t) - F(s,t_0)\right|\,ds + \int_{[-R,R]} \left|F(s,t) - F(s,t_0)\right|\,ds.
\end{aligned} \tag{28.2}$$

We make the first integral (which actually has two parts) small by making R big and using condition (b), and we make the second integral small by using condition (a).

To make the first integral small, recall that $|F(s,t)| \le M(s)$ when $t \in B(t_0, \mu)$. Hence, for such t,

$$\begin{aligned}
\int_{\mathbb{R}\setminus[-R,R]} \left|F(s,t) - F(s,t_0)\right|\,ds &\le \int_{\mathbb{R}\setminus[-R,R]} |F(s,t)| + |F(s,t_0)|\,ds \\
&\le 2\int_{\mathbb{R}\setminus[-R,R]} M(s)\,ds.
\end{aligned}$$

Since the improper integral $\int_{\mathbb{R}} M(s)\,ds$ converges, we may make $\int_{\mathbb{R}\setminus[-R,R]} M(s)\,ds$ small by taking R large enough. Take R such that

$$\int_{\mathbb{R}\setminus[-R,R]} M(s)\,ds < \frac{\varepsilon}{4}\,.$$

Then

$$\int_{\mathbb{R}\setminus[-R,R]} \left|F(s,t) - F(s,t_0)\right|\,ds < \frac{\varepsilon}{2}\,. \tag{28.3}$$

Further, from (b), F is uniformly continuous on compact subsets of $\mathbb{R} \times B(t_0, \mu)$. Hence there exists $\delta \in \mathbb{R}^+$ such that

$$\left|F(s,t) - F(s,t_0)\right| < \frac{\varepsilon}{4R} \qquad \forall s \in [-R,R]$$

provided that $t \in B(t_0, \delta)$. For such t,

$$\int_{[-R,R]} \left|F(s,t) - F(s,t_0)\right|\,ds \le \int_{[-R,R]} \frac{\varepsilon}{4R}\,ds = \frac{\varepsilon}{2}\,. \tag{28.4}$$

Set $\eta = \min\{\delta, \mu\}$. Putting everything together, (28.3) and (28.4) hold when $t \in B(t_0, \eta)$. Combining with (28.2), we see that

$$\left| \int_{\mathbb{R}} F(s,t)\,ds - \int_{\mathbb{R}} F(s,t_0)\,ds \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

when $t \in B(t_0, \eta)$, as required. $\square$

3. Remarks

Though we state our results for real-valued integrands F, they also hold for complex-valued integrands, by an identical argument.

The arguments also apply when the parameter t is constrained to lie in a subset of $\mathbb{R}$ or $\mathbb{C}$; we just have to add this constraint to the statement of the theorem, and to each step of the proof. By replacing t by $1/t$, we can also establish a version of the theorem for limits as t tends to ∞.

Our arguments also apply to integrals where one or both the limits of integration is finite, that is, to integrals such as

$$\int_{[a,b]} H(s,t)\,ds;$$

in this case, the proof of Theorem 28.2 simplifies, as we replace $[-R,R]$ by $[a,b]$ and do not need to consider the other component of the integral.

Finally, the result also extends to integration in $\mathbb{R}^n$; we replace the integrals over $[-R,R]$ and over $\mathbb{R}\setminus[-R,R]$ by integrals over $B(0,R)$ and over $\mathbb{R}\setminus B(0,R)$.

4. Exchanging integration and differentiation

Now we consider integrals in which the integrand contains a parameter, and consider the question of whether

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} F(s,t)\,ds\,\Big|_{t=t_0} = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}F(s,t)\,\Big|_{t=t_0}\,ds. \tag{28.5}$$

Just to be sure that there is a real issue to discuss, we consider an example.

Exercise 28.3. Define $F:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$F(s,t) = \begin{cases} \dfrac{s^3 t}{(s^2+t^2)^2} & \text{if } (s,t)\neq(0,0) \\ 0 & \text{if } (s,t)=(0,0). \end{cases}$$

Compute $\displaystyle\int_0^1 \frac{\partial}{\partial t}F(s,t)\,\Big|_{t=0}\,ds$ and $\displaystyle\frac{d}{dt}\int_0^1 F(s,t)\,ds\,\Big|_{t=0}$, and show that they are not equal.

Answer.

Finally, we state a theorem giving sufficient conditions for the validity of exchanging the order of differentiation and integration.

Theorem 28.4. *Suppose that $G:\mathbb{R}\times P\to\mathbb{R}$ and $t_0\in P$, where $P\subseteq\mathbb{R}$, and that the function $s\mapsto G(s,t_0)\in L^1(\mathbb{R})$. Suppose also that there exists $\mu\in\mathbb{R}^+$ such that*

(a) $(s,t)\mapsto \frac{\partial}{\partial t}G(s,t)$ is continuous on $\mathbb{R}\times B(t_0,\mu)$, and
(b) there is a nonnegative function $M\in L^1(\mathbb{R})$ such that

$$\left|\frac{\partial}{\partial t}G(s,t)\right| \le M(s) \qquad \forall s\in\mathbb{R} \quad \forall t\in B(t_0,\mu).$$

Then

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} G(s,t)\,ds \,\Big|_{t=t_0} = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} G(s,t)\,\Big|_{t=t_0}\, ds. \tag{28.6}$$

PROOF. The idea of the proof is to write

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} G(s,t)\,ds \,\Big|_{t=t_0} = \lim_{h\to 0}\int_{-\infty}^{\infty} F(s,h)\,ds,$$

where

$$F(s,h) = \begin{cases} \dfrac{1}{h}\displaystyle\int_{t_0}^{t_0+h} \frac{\partial}{\partial t} G(s,t)\,dt & \text{if } h \neq 0 \\ \dfrac{\partial}{\partial t} G(s,t)\,\Big|_{t=t_0} & \text{if } h = 0. \end{cases}$$

Then we show that F satisfies the hypotheses of the previous theorem (where we suppose that $h \to 0$). It follows that

$$\begin{aligned} \frac{\partial}{\partial t}\int_{-\infty}^{\infty} G(s,t)\,ds \,\Big|_{t=t_0} &= \lim_{h\to 0}\int_{-\infty}^{\infty} F(s,h)\,ds \\ &= \int_{-\infty}^{\infty} \lim_{h\to 0} F(s,h)\,ds \\ &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t} G(s,t_0)\,ds, \end{aligned}$$

as required. □

REMARK 28.5. This result also applies when t is a complex variable, and differentiation with respect to t is complex differentiation.

Differentiation with respect to a parameter may be used to compute integrals. For instance, to compute

$$\int_0^1 \frac{x^2-1}{\ln x}\,dx,$$

we may find

$$\frac{d}{dt}\int_0^1 \frac{x^t-1}{\ln x}\,dx \qquad \text{and} \qquad \int_0^1 \frac{x^t-1}{\ln x}\,dx\,\Big|_{t=0}\;.$$

5. Applications†

The gamma function Γ is defined by an integral.

DEFINITION 28.6. The function Γ is given by

$$\Gamma(z) = \int_0^{\infty} s^{z-1}\,e^{-s}\,ds.$$

This integral converges absolutely when $\operatorname{Re} z > 0$, and continues meromorphically into $\mathbb{C}$.

It follows from Theorem 28.4 that the gamma function is holomorphic, and

$$\Gamma'(z) = \int_0^{\infty} s^{z-1}\ln(s)\,e^{-s}\,ds.$$

It is easy to check that that $\Gamma(z) = (z-1)\Gamma(z-1)$, by integration by parts, and hence $\Gamma(n) = (n-1)!$ when $n \in \mathbb{Z}^+$. This formula allows to extend Γ meromorphically into $\mathbb{C}$.

Now we consider some integrals that may be evaluated in terms of this function.

EXERCISE 28.7. Find changes of variables or other transformations such that the following integrals may be converted into an integral of the form

$$\int_0^{\pi/2} \cos^a(\theta)\sin^b(\theta)\,d\theta.$$

(a) $\displaystyle\int_0^1 (1-x^2)^\alpha\, x^\beta\, dx$; (b) $\displaystyle\int_0^1 (1-y)^\alpha\, y^\beta\, dy$;

(c) $\displaystyle\int_0^{\pi/2} \frac{\tan^c(\varphi)}{\sec^d(\varphi)}\, d\varphi$;

(d) $\displaystyle\int_0^\infty \frac{t^\gamma}{(1+t^2)^\delta}\, dt$; (e) $\displaystyle\int_0^\infty \frac{s^\varepsilon}{(1+s)^\varsigma}\, ds$.

ANSWER.

Observe that

$$\int_0^\infty x^a e^{-x^2}\,dx = \tfrac{1}{2}\int_0^\infty (x')^{(a-1)/2} e^{-x'}\,dx' = \tfrac{1}{2}\Gamma(\tfrac{a+1}{2});$$

similarly,

$$\int_0^\infty y^b e^{-y^2}\,dy = \tfrac{1}{2}\Gamma(\tfrac{b+1}{2}),$$

and

$$\int_0^\infty r^{a+b+1} e^{-r^2}\,dr = \tfrac{1}{2}\Gamma(\tfrac{a+b+2}{2}).$$

By using polar coordinates, we see that

$$\begin{aligned}
\tfrac{1}{2}\Gamma(\tfrac{a+1}{2})\,\tfrac{1}{2}\Gamma(\tfrac{b+1}{2}) &= \int_0^\infty x^a e^{-x^2}\,dx \int_0^\infty y^b e^{-y^2}\,dy \\
&= \int_0^\infty \int_0^\infty x^a y^b e^{-x^2-y^2}\,dx\,dy \\
&= \int_0^{\pi/2} \int_0^\infty r^a \cos^a(\theta)\, r^b \sin^b(\theta)\, e^{-r^2} r\,dr\,d\theta \\
&= \int_0^\infty r^{a+b+1} e^{-r^2}\,dr \int_0^{\pi/2} \cos^a(\theta)\sin^b(\theta)\,d\theta \\
&= \tfrac{1}{2}\Gamma(\tfrac{a+b+2}{2}) \int_0^{\pi/2} \cos^a(\theta)\sin^b(\theta)\,d\theta.
\end{aligned}$$

Hence

$$\int_0^{\pi/2} \cos^a(\theta)\sin^b(\theta)\,d\theta = \frac{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}{2\Gamma(\frac{a+1}{2}+\frac{b+1}{2})}.$$

LECTURE 29

The Fourier transformation

In this lecture we introduce the Fourier transform $\widehat{f}$ of an integrable function f on $\mathbb{R}$; we then compute some examples. We will assume that it is legitimate to exchange limits and integrals; in a future lecture, we will examine whether this is really so.

1. Locally integrable and integrable functions on $\mathbb{R}$

We say that a function $f : \mathbb{R} \to \mathbb{C}$ is locally integrable if f is Riemann integrable on all finite intervals $[-R, R]$. For example, continuous functions are locally integrable.

We write $L^1(\mathbb{R})$ for the collection of all locally integrable functions $f : \mathbb{R} \to \mathbb{C}$ such that the improper integral

$$\int_{-\infty}^{\infty} |f(x)|\, dx = \lim_{R\to\infty} \int_{[-R,R]} |f(x)|\, dx$$

converges (and is finite). Note that $\int_{-R}^{R} |f(x)|\, dx$ increases as R increases, so the limit exists (and is finite) if and only if the integrals are bounded, uniformly in R.

Example 29.1. Define the Gaussian $\varphi : \mathbb{R} \to \mathbb{C}$ by $\varphi(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$ for all $x \in \mathbb{R}$. Then $\varphi \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} \varphi(x)\, dx = 1$.

2. The Fourier transform

Definition 29.2. Suppose that $f \in L^1(\mathbb{R})$. The Fourier transform of f is the function $\widehat{f} : \mathbb{R} \to \mathbb{C}$ given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-ix\xi}\, dx = \lim_{R\to\infty} \int_{[-R,R]} f(x)\, e^{-ix\xi}\, dx.$$

The convergence of the improper integral follows from that fact that $|\, e^{-ix\xi}| = 1$ for all $x, \xi \in \mathbb{R}$, and properties of the Riemann integral.

3. Examples

Exercise 29.3. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^{-|x|}$. Show that

$$\widehat{f}(\xi) = \frac{2}{1+\xi^2} \quad \forall \xi \in \mathbb{R}.$$

Answer.

EXERCISE 29.4. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1/(1+x^2)$. Show that

$$\widehat{f}(\xi) = \pi e^{-|\xi|} \quad \forall \xi \in \mathbb{R}.$$

ANSWER.

EXERCISE 29.5. The Gaussian φ is defined by $\varphi(x) = \dfrac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$. Show that

$$\widehat{\varphi}(\xi) = e^{-\xi^2/2}.$$

ANSWER.

EXERCISE 29.6. Let $f(x) = 1$ if $x \in [-1, 1]$ and $f(x) = 0$ otherwise. Show that $\widehat{f} \notin L^1(\mathbb{R})$.

ANSWER.

The unfortunate fact that the Fourier transform of an integrable function need not be integrable causes some problems. However, Fourier transforms of integrable functions do have some good properties.

LEMMA 29.7 (The Riemann–Lebesgue Lemma). *If $f \in L^1(\mathbb{R})$, then the function $\widehat{f}$ is bounded and continuous, and vanishes at infinity.*

PROOF. For the boundedness, observe that

$$\left|\widehat{f}(\xi)\right| = \left|\int_{\mathbb{R}} f(x)\, e^{-ix\xi}\, dx\right| \le \int_{\mathbb{R}} \left| f(x)\, e^{-ix\xi}\right|\, dx = \int_{\mathbb{R}} | f(x)|\, dx < \infty,$$

since $f \in L^1(\mathbb{R})$. Further, assuming that it is legitimate to exchange limits and integrals, we see that

$$\begin{aligned} \lim_{\xi\to\xi_0} \widehat{f}(\xi) &= \lim_{\xi\to\xi_0} \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx = \int_{-\infty}^{\infty} \lim_{\xi\to\xi_0} f(x)\, e^{-ix\xi}\, dx \\ &= \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi_0}\, dx = \widehat{f}(\xi_0), \end{aligned}$$

and so f is continuous.

To justify this last step, write $F(x,\xi) = f(x)\, e^{-ix\xi}$ for all $z, \xi \in \mathbb{R}$. Observe that

$$|F(x,\xi)| = |f(x)| \in L^1(\mathbb{R}),$$

by hypothesis, and so condition (b) of Theorem 28.2 holds. Further, if f is continuous, then F is continuous, and condition (a) of Theorem 28.2 holds.

We will not consider the vanishing at infinity. □

4. The inversion formula

Now we show how to recover the original function f from $\widehat{f}$. In other words, we find the inverse of the linear transformation $\mathcal{F}$ given by $\mathcal{F}(f) = \widehat{f}$.

To simplify the statement of the inversion theorem, we make an additional definition.

Definition 29.8. We define $\mathcal{M}(\mathbb{R})$ to be the set of "moderately nice" functions on $\mathbb{R}$, that is, of continuous bounded functions whose absolute value is integrable.

Theorem 29.9 (The inversion formula for the Fourier transform). *If $f \in \mathcal{M}(\mathbb{R})$ and $\widehat{f} \in \mathcal{M}(\mathbb{R})$, then*

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi)\, e^{ix\xi}\, d\xi.$$

Proof. We give the proof in the final section. □

The inversion formula implies that the Fourier transformation is bijective on the vector space of functions on $\mathcal{M}(\mathbb{R})$ whose Fourier transform lies in $\mathcal{M}(\mathbb{R})$.

The inversion theorem does not hold without a continuity hypothesis. For example, we have shown that $\widehat{\varphi} = \sqrt{2\pi}\varphi$, where φ is the Gaussian. Now define

$$f(x) = \begin{cases} \varphi(x) & \text{if } x \neq x_0 \\ 0 & \text{if } x = x_0. \end{cases}$$

Since changing the value of a function at a point does not change the integral of the function, it follows that $\widehat{f} = \widehat{\varphi}$, so

$$\int_{\mathbb{R}} \widehat{f}(\xi)\, e^{i\xi x_0}\, d\xi = \int_{\mathbb{R}} \widehat{\varphi}(\xi)\, e^{i\xi x_0}\, d\xi = \varphi(x_0) \neq f(x_0),$$

and the inversion formula for f fails at x_0.

However, counterexamples of this rather artificial type are the only problem. It may be shown that, if $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then f may be modified on a "small set" in order to become continuous without changing $\widehat{f}$, and then the inversion formula holds.

5. Proof of the inversion formula†

This proof requires a preliminary lemma.

Lemma 29.10. *Suppose that $g, \psi \in L^1(\mathbb{R})$. Then*

$$\int_{-\infty}^{\infty} g(x)\, \widehat{\psi}(x)\, dx = \int_{-\infty}^{\infty} \widehat{g}(\xi)\, \psi(\xi)\, d\xi.$$

PROOF. Observe that

$$\begin{aligned}\int_{\mathbb{R}} f(x)\,\widehat{g}(x)\,dx &= \int_{\mathbb{R}} f(x)\Big(\int_{\mathbb{R}} g(\xi)\,e^{-i\xi x}\,d\xi\Big)\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)\,g(\xi)\,e^{-i\xi x}\,d\xi\,dx\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)\,e^{-ix\xi}\,g(\xi)\,dx\,d\xi = \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(x)\,e^{-ix\xi}\,dx\Big)\,g(\xi)\,d\xi\\ &= \int_{\mathbb{R}} \widehat{f}(\xi)\,g(\xi)\,d\xi,\end{aligned}$$

as required. □

The change in the order of integration in the double integral is justified by the fact that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |g(x)|\,|\psi(\xi)|\;dx\,d\xi < \infty.$$

And now we give the proof of the inversion formula.

PROOF. Suppose that $\psi_\delta(x) = e^{-\delta^2 x^2/2}$, where $\delta \in \mathbb{R}^+$. Then by a variation of Exercise 29.5, we may show that

$$\widehat{\psi_\delta}(\xi) = \frac{\sqrt{2\pi}}{\delta}\,e^{-\xi^2/2\delta^2}.$$

By the change of variables $x = \delta y$ and Lemma 29.10, if $g, \widehat{g} \in \mathcal{M}(\mathbb{R})$, then

$$\begin{aligned}\sqrt{2\pi}\int_{-\infty}^{\infty} g(\delta y)\,e^{-y^2/2}\,dy &= \int_{-\infty}^{\infty} g(x)\,\frac{\sqrt{2\pi}}{\delta}\,e^{-x^2/2\delta^2}\,dx = \int_{-\infty}^{\infty} g(x)\,\widehat{\psi_\delta}(x)\,dx\\ &= \int_{-\infty}^{\infty} \widehat{g}(\xi)\,\psi_\delta(\xi)\,d\xi = \int_{-\infty}^{\infty} \widehat{g}(\xi)\,e^{-\delta^2\xi^2/2}\,d\xi.\end{aligned}$$

Now we claim that we may exchange limits and integrals to show that

$$\begin{aligned}\lim_{\delta\to 0}\sqrt{2\pi}\int_{-\infty}^{\infty} g(\delta y)\,e^{-y^2/2}\,dy &= \sqrt{2\pi}\int_{-\infty}^{\infty}\lim_{\delta\to 0} g(\delta y)\,e^{-y^2/2}\,dy\\ &= \sqrt{2\pi}\int_{-\infty}^{\infty} g(0)\,e^{-y^2/2}\,dy = 2\pi\,g(0),\end{aligned}$$

since g is continuous, and

$$\lim_{\delta\to 0}\int_{-\infty}^{\infty} \widehat{g}(\xi)\,e^{-\delta^2\xi^2/2}\,d\xi = \int_{-\infty}^{\infty}\lim_{\delta\to 0}\widehat{g}(\xi)\,e^{-\delta^2\xi^2/2}\,d\xi = \int_{-\infty}^{\infty}\widehat{g}(\xi)\,d\xi.$$

To justify the first of these, write $F(y,\delta) = g(\delta y)\,e^{-y^2/2}$. Since g is continuous, so is F, and condition (a) of Theorem 28.2 holds. Further, since g is bounded, that is, there is a constant C such that $|g(y)| \le C$ for all $y \in \mathbb{R}$, it follows that $|F(y,\delta)| \le C\,e^{-y^2/2}$ for all $y, \delta \in \mathbb{R}$, and condition (b) of Theorem 28.2 also holds. Thus we may exchange the limit and the integral. The second formula may be justified similarly.

Hence

$$g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{g}(\xi)\,d\xi. \tag{29.4}$$

Now suppose that $x \in \mathbb{R}$, $f \in \mathcal{M}(\mathbb{R})$ and $g(y) = f(y+x)$. Then, by the change of variable $u = y + x$, we see that

$$\begin{aligned}
\widehat{g}(\xi) &= \int_{\mathbb{R}} g(y)\, e^{-iy\xi}\, dy = \int_{-\infty}^{\infty} f(y+x)\, e^{-iy\xi}\, dy \\
&= \int_{-\infty}^{\infty} f(u)\, e^{-i(u-x)\xi}\, du = e^{ix\xi} \int_{-\infty}^{\infty} f(u)\, e^{-iu\xi}\, du = e^{ix\xi}\widehat{f}(\xi)
\end{aligned}$$

for all $\xi \in \mathbb{R}$, and by applying (29.4) to this g, we obtain

$$f(x) = g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{g}(\xi)\, d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{ix\xi}\, d\xi,$$

completing the proof of Theorem 29.9. $\square$

LECTURE 30

The Laplace Transform

In this lecture, we introduce the Laplace transformation. The first step is to define the class of functions on which it will act.

1. Functions of exponential type

DEFINITION 30.1 (Functions of exponential type). Suppose that $A \in \mathbb{R}$. A function $f : [0, \infty) \to \mathbb{C}$ is said to be of *exponential type* A if there exists a constant C such that

$$|f(t)| \le C\, e^{At} \qquad \forall t \in [0, \infty).$$

A function $f : [0, \infty) \to \mathbb{C}$ is of *exponential type* $A+$ if it is exponential type $A + \varepsilon$ for all $\varepsilon \in \mathbb{R}^+$.

The definition of type $A+$ is not standard, but is convenient.

The following examples illustrate these concepts.

EXERCISE 30.2. In this exercise, $a, b \in \mathbb{C}$ and p denotes a polynomial. Show that:

(a) $t \mapsto a\, e^{bt}$ is of exponential type $\mathrm{Re}(b)$;
(b) $t \mapsto p(t)\, e^{bt}$ is of exponential type $\mathrm{Re}(b)+$;
(c) $t \mapsto e^{t^2}$ is *not* of exponential type A for *any* $A \in \mathbb{R}$.

ANSWER.

Note that if f is of exponential type A, then it is of exponential type $A+$ and also of exponential type B for all $B \in [A, \infty)$. For computing exponential type, the following lemma may be useful.

LEMMA 30.3. *Suppose that the functions $f, g : [0, +\infty) \to \mathbb{C}$ are of exponential types A and B respectively. Then the functions $af + bg$ and afg are of exponential type* $\max\{A, B\}$ *and $A + B$ respectively, for all $a, b \in \mathbb{C}$. Similarly, if the functions f and g are of exponential types $A+$ and $B+$, then the functions $af + bg$ and afg are of exponential types* $\max\{A, B\}+$ *and* $(A + B)+$.

PROOF. We leave this as an exercise. □

We will also require a certain type of integrability of the functions to which we will apply the Laplace transformation, which we now define.

DEFINITION 30.4. A function $f : [0, +\infty) \to \mathbb{C}$ is said to be *locally integrable* if it is Riemann integrable on all intervals $[0, R]$ where $R \in \mathbb{R}^+$.

Finally, we will deal with many sets of the form $\{z \in \mathbb{C} : \mathrm{Re}(z) > A\}$; for brevity, we will denote this (right) half-plane by H_A.

2. The Laplace transform

DEFINITION 30.5. Suppose that $f : [0, \infty) \to \mathbb{C}$ is locally integrable and of exponential type A. The *Laplace transform* $\mathcal{L}f : H_A \to \mathbb{C}$ of f is the function given by

$$\mathcal{L}f(z) = \int_0^\infty f(t)\, e^{-zt}\, dt = \lim_{R\to\infty} \int_0^R f(t)\, e^{-zt}\, dt.$$

The integral in the definition converges by comparison with $\int_0^\infty C\, e^{(A-\mathrm{Re}(z))t}\, dt$.

Notice that $\mathcal{L}$ is linear. More precisely, if f and g are locally integrable and of exponential types A and B, and $a, b \in \mathbb{C}$, then $af + bg$ is also locally integrable and is of exponential type C, where $C = \max\{A, B\}$. Moreover, if $\mathrm{Re}(z) > C$, then

$$\mathcal{L}(af + bg)(z) = a\mathcal{L}f(z) + b\mathcal{L}g(z).$$

EXERCISE 30.6. Find the Laplace transforms of the functions
(a) $t \mapsto a$ (b) $t \mapsto p(t)$ (c) $t \mapsto e^{at}$ (d) $t \mapsto p(t)\, e^{at}$.

ANSWER.

THEOREM 30.7. *If $f : [0,\infty) \to \mathbb{C}$ is locally integrable and of exponential type $A+$, then $\mathcal{L}f$ is holomorphic on H_A. Further,*

$$\frac{d}{dz}\mathcal{L}f(z) = \mathcal{L}g(z) \qquad \forall z \in H_A,$$

where $g : [0,+\infty) \to \mathbb{C}$ is given by $g(t) = -t\,f(t)$.

PROOF. We give the proof at the end of this lecture. □

Next, here are a number of properties of the Laplace transformation.

PROPOSITION 30.8. *Suppose that $f : [0,\infty) \to \mathbb{C}$ is locally integrable and of exponential type $A+$. Then the following hold:*

(a) if $a \in \mathbb{C}$ and $g(t) = e^{-at}f(t)$ for all $t \in [0,+\infty)$, then

$$\mathcal{L}g(z) = \mathcal{L}f(z+a) \qquad \forall z \in H_{A-\mathrm{Re}(a)};$$

(b) if $a \in \mathbb{R}^+$ and $g(t) = f(t/a)$ for all $t \in [0,+\infty)$, then

$$\mathcal{L}g(z) = a\mathcal{L}f(az) \qquad \forall z \in H_{A/a};$$

(c) if $g(t) = tf(t)$ for all $t \in [0,+\infty)$, then

$$\mathcal{L}g(z) = -\frac{d}{dz}\mathcal{L}f(z) \qquad \forall z \in H_A;$$

(d) if f is differentiable and f' is also of exponential type $A+$, then

$$\mathcal{L}(f')(z) = z\mathcal{L}f(z) - f(0) \qquad \forall z \in H_A.$$

PROOF. To prove (a), observe that

To prove (b), we make the change of variables $s = at$:

$$\begin{aligned}\mathcal{L}g(z) &= \int_0^\infty g(s)\,e^{-zs}\,ds = \int_0^\infty f(s/a)\,e^{-zs}\,ds \\ &= \int_0^\infty f(t)\,e^{-azt}a\,dt = a\mathcal{L}f(az).\end{aligned}$$

We prove something very close to (c) in Theorem 30.7, and omit this.

To prove (d), we integrate by parts:

□

Remembering these properties can save time; for example, by part (a), since the Laplace transform of t^3 is $6/z^4$, the Laplace transform of $t^3\,e^{-2t}$ is just $6/(z+2)^4$.

3. The inversion formulae for the Laplace transform

THEOREM 30.9. *If $f : [0,\infty) \to \mathbb{C}$ is continuous and of exponential type $A+$, then*

$$f(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_\lambda \mathcal{L}f(z)\, e^{tz}\, dz \qquad \forall t \in \mathbb{R}^+, \tag{30.1}$$

where λ is the line segment from $\sigma - iR$ to $\sigma + iR$ and $\sigma \in (A, \infty)$.

Suppose further that $\mathcal{L}f$ extends to a holomorphic function on $\mathbb{C} \setminus \{a_1, \ldots, a_n\}$, and that there are positive constants M and k such that

$$|\mathcal{L}f(z)| \le M\, |z|^{-k}$$

whenever $|z|$ is sufficiently large. Then for any $t \in \mathbb{R}^+$,

$$f(t) = \sum_{j=1}^{n} \operatorname{Res}(\mathcal{L}f(z)\, e^{zt}; z = a_j) \qquad \forall t \in \mathbb{R}^+. \tag{30.2}$$

The integral on the right-hand-side of (30.1) is known as the *Bromwich Integral*.

PROOF. We give a sketch of the argument.

The first inversion formula may be deduced from the inversion formula for the *Fourier* transformation rather quickly. If we write $z = \sigma + iy$, then

$$\mathcal{L}f(z) = \int_0^\infty f(t)\, e^{-(\sigma+iy)t}\, dt = \int_{-\infty}^\infty g_\sigma(t)\, e^{-iyt}\, dt = (g_\sigma\hat{}\,(y),$$

where $g_\sigma(t) = f(t)\, e^{-\sigma t}$ if $t \ge 0$ and $g_\sigma(t) = 0$ if $t < 0$. By the inversion formula for the Fourier transform, if $t > 0$ then

$$f(t)\, e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^\infty \mathcal{L}f(\sigma + iy)\, e^{iyt}\, dy = e^{-\sigma t} \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\sigma - iR}^{\sigma + iR} \mathcal{L}f(z)\, e^{zt}\, dz.$$

Cancelling the factor of $e^{-\sigma t}$ now leads to formula (30.1).

For formula (30.2), we consider the semicircular contour $\gamma \sqcup \lambda$, where $\gamma(\theta) = \sigma + R\, e^{i\theta}$ and $\pi/2 \le \theta \le 3\pi/2$. The result will follow from the Cauchy Residue Theorem once we show that

$$\lim_{R\to\infty} \int_\gamma g(z)\, e^{tz}\, dz = 0.$$

This may be done using a Jordan's lemma-type argument. □

Strictly speaking, this is not a *proof*, because we proved the Fourier inversion formula for a more limited class of functions. However, with a little more work the Fourier inversion formula may be proved for more general functions, justifying the above argument.

The key fact is that the Laplace transformation is *invertible*. Thus to find the inverse Laplace transform of g, it suffices to find a continuous function f of exponential type such that $\mathcal{L}f = g$; then f is the desired inverse transform.

4. Examples

EXERCISE 30.10. Find the continuous function $f : [0, \infty) \to \mathbb{C}$ of exponential type 1+ such that $\mathcal{L}f(z) = \dfrac{1}{(z-1)^2}$.

ANSWER.

EXERCISE 30.11. Find the continuous function $f : [0, \infty) \to \mathbb{C}$ of exponential type 1+ for which $\mathcal{L}f(z) = \dfrac{1}{z^2-1}$.

ANSWER.

5. A proof†

For convenience, we restate the result that we are going to prove.

THEOREM. *If $f : [0, \infty) \to \mathbb{C}$ is locally integrable and of exponential type $A+$, then $\mathcal{L}f$ is holomorphic on H_A. Further,*

$$\frac{d}{dz}\mathcal{L}f(z) = \mathcal{L}g(z) \qquad \forall z \in H_A,$$

where $g : [0, +\infty) \to \mathbb{C}$ is given by $g(t) = -t\, f(t)$.

PROOF. We do this by applying Theorem 26.4, which states that we may exchange the order of differentiation with respect to a parameter and integration when three conditions are satisfied. By definition,

$$\mathcal{L}f(z) = \int_0^\infty f(s)\, e^{-zs}\, ds.$$

Consider the function $G : [0, +\infty) \times H_A \to \mathbb{C}$ given by $G(s, z) = f(s)\, e^{-zs}$. Observe that

$$\frac{dG}{dz}(s, z) = -s\, f(s)\, e^{-zs},$$

which is a continuous function on $[0, +\infty) \times \mathbb{C}$. Hence the continuity condition of Theorem 26.4 is satisfied.

Take $z \in H_A$, so that $\mathrm{Re}(z) > A$. Set $\varepsilon = (\mathrm{Re}(z) - A)/2$; then $\mathrm{Re}(z) - A = 2\varepsilon > \varepsilon$, that is, $z \in H_{A+\varepsilon}$. Since f is of exponential type $A+$, there exists C such that

$$|f(s)| \le C\, e^{(A+\varepsilon/2)s} \qquad \text{and} \qquad |s\, f(s)| \le C\, e^{(A+\varepsilon/2)s}$$

for all $s \in [0, \infty)$. If $z \in H_{A+\varepsilon}$, then

$$|G(s, z)| \le |f(s)|\, \left|e^{-zs}\right| = |f(s)|\; e^{-\mathrm{Re}(z)s} \le C\, e^{(A+\varepsilon/2)s}\, e^{-(A+\varepsilon)s} = C\, e^{-\varepsilon s/2}$$

and similarly

$$\left|\frac{dG}{dz}(s,z)\right| \leq C\,e^{-\varepsilon s/2},$$

and the integrability conditions of Theorem 26.4 are also satisfied. Thus we may differentiate under the integral and obtain the desired result. $\square$

LECTURE 31

Applications to differential and integral equations

We may use the Laplace transform to solve differential and integral equations on $[0, \infty)$.

1. Differential equations

The Laplace transform may be used to solve differential equations on $[0, +\infty)$.

EXERCISE 31.1. Use the Laplace transform to solve the ordinary differential equation

$$\frac{d^2u}{dt^2}(t) - 2\frac{du}{dt}(t) + u(t) = t \qquad \forall t \in [0, +\infty),$$

subject to the initial conditions $u(0) = \dfrac{du}{dt}(0) = 0$.

ANSWER.

Notice that this solution could have been found by more elementary methods, and the use of the Laplace transform here is using a sledgehammer to crack a nut. The same cannot be said so easily of the next examples.

EXAMPLE 31.2. Suppose that someone wiggles one end of a very long straight piece of string up and down for a certain time. How does the string move?

Let us suppose that the vertical displacement of the string at position x and time t is given by $u(x,t)$. The string satisfies the *wave equation*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},$$

and we may take the initial conditions $u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0$ when $x > 0$ and $u(0,t) = f(t)$, where $f(t) = 0$ unless $a < t < b$, where $b > a > 0$. In addition, we shall make other assumptions along the way to get a solution.

We write $\mathcal{L}u(x,z)$ for the Laplace transform of u in the t variable; this means that $x \neq \mathrm{Re}(z)$. Then (assuming that there are no problems with exchanging the order of differentiation and integration)

$$\frac{\partial^2 \mathcal{L}u}{\partial x^2}(x,z) = \mathcal{L}\frac{\partial^2 u}{\partial x^2}(x,z) = \mathcal{L}\frac{\partial^2 u}{\partial t^2}(x,z).$$

Further, from Proposition 30.8,

$$\mathcal{L}\frac{\partial^2 u}{\partial t^2}(x,z) = z\mathcal{L}\frac{\partial u}{\partial t}(x,z) - \frac{\partial u}{\partial t}(x,0) = z^2\mathcal{L}u(x,z) - u(x,0) - \frac{\partial u}{\partial t}(x,0).$$

The initial conditions imply that $u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0$, and so

$$\frac{\partial^2 \mathcal{L}u}{\partial x^2}(x,z) = z^2\mathcal{L}u(x,z).$$

The solution to this equation is

$$\mathcal{L}u(x,z) = A(z)e^{zx} + B(z)e^{-zx}.$$

It is physically sensible to suppose that $u(x,t) \to 0$ as $x \to +\infty$, so it seems reasonable to suppose that $A(z) = 0$. Thus, taking $x = 0$, we see that

$$B(z) = \mathcal{L}u(0,z) = \mathcal{L}f(z).$$

Finally,

$$\mathcal{L}u(x,z) = e^{-zx}\mathcal{L}f(z).$$

To unravel the Laplace transform, observe that if $g(t) = f(t-c)$, then $g(t) = 0$ if $t < c$, and so

$$\begin{aligned}
\mathcal{L}g(z) &= \int_0^\infty g(t)e^{-tz}\,dt = \int_c^\infty g(t)e^{-tz}\,dt \\
&= \int_c^\infty f(t-c)e^{-tz}\,dt = \int_0^\infty f(t)e^{-(t+c)z}\,dt = e^{-cz}\mathcal{L}f(z).
\end{aligned}$$

Hence

$$u(x,t) = f(t-x) \qquad \forall t \in \mathbb{R}^+.$$

EXAMPLE 31.3. Use the Laplace transform to find the solution $u : [0,\infty) \times [0,\infty) \to \mathbb{R}$ of the partial differential equation

$$\frac{\partial u}{\partial t}(s,t) = \frac{\partial^2 u}{\partial s^2}(s,t) \qquad \forall (s,t) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

subject to the initial conditions $u(s,0) = 0$ for all $s \in \mathbb{R}^+$ and $u(0,t) = 1$ for all $t \in \mathbb{R}^+$.

We omit the answer to this problem, which is known as the *heat equation on the half-line* $[0,\infty)$.

2. Convolution on $[0, \infty)$

Throughout this section, f and g will denote locally integrable functions on $[0, \infty)$.

DEFINITION 31.4 (Convolution on $[0, \infty)$). The convolution of f and g is the function $f * g : [0, \infty) \to \mathbb{C}$ given by

$$f * g(t) = \int_0^t f(s)\, g(t-s)\, ds.$$

Now we examine the properties of convolution.

PROPOSITION 31.5. *Suppose that $a, b \in \mathbb{C}$ and f, g, h are locally integrable functions of exponential type $A+$. Then $f * g$, $f * h$ and $f * (a\, g + b\, h)$ are also of exponential type $A+$. Further*

(a) $f * (a\, g + b\, h) = a\, f * g + b\, f * h$;
(b) $f * g = g * f$;
(c) $\mathcal{L}(f * g) = \mathcal{L} f \mathcal{L} g$.

PROOF. Fix $\varepsilon \in \mathbb{R}^+$. Since f and g are of exponential type $A + \varepsilon$, there exist constants C and D such that $|f(t)| \leq C\, e^{(A+\varepsilon)t}$ and $|g(t)| \leq D\, e^{(A+\varepsilon)t}$ for all $t \geq 0$. Hence

$$|f * g(t)| \leq \int_0^t C\, e^{(A+\varepsilon)s}\, D\, e^{(A+\varepsilon)(t-s)}\, ds = C D\, t\, e^{(A+\varepsilon)t} \leq \frac{CD}{\varepsilon}\, e^{(A+2\varepsilon)t}.$$

Since ε is arbitrary, $f * g$ is of exponential type $A+$. Similarly, $f * h$ and $f * (a\, g + b\, h)$ are of exponential type $A+$.

Item (a) follows from the definition of convolution as an integral, and linearity of that integral.

To prove item (b), take an arbitrary $t \in [0, +\infty)$.

Item (c) is a consequence of the fact that the exponential function is multiplicative (that is, $e^{a+b} = e^a\, e^b$).

$\square$

3. Integral equations

EXERCISE 31.6. Use the Laplace transform to find a function u satisfying the integral equation

$$u(t) = t^2 - \frac{2}{3}\int_0^t (t-s)^3\, u(s)\, ds.$$

This is an example of a *Volterra integral equation*; these arise in many areas of application, including finance and biology.

ANSWER.

Suppose that X is a positive random variable, described by a probability density function f; then $f(t) = 0$ when $t < 0$. Let F be the cumulative distribution function associated to f; that is,

$$F(t) = \int_{-\infty}^{t} f(s)\, ds.$$

The *renewal equation* associated to X is the Volterra integral equation

$$m(t) = F(t) + \int_{0}^{t} f(s)\, m(t-s)\, ds. \tag{31.1}$$

EXERCISE 31.7. Use the Laplace transform to find a function m satisfying the renewal equation (31.1), when $f(t) = e^{-t}$ for all $t \in \mathbb{R}^{+}$.

ANSWER.

LECTURE 32

The Dirichlet problem

In this lecture, we introduce the Dirichlet problem. We show that the solution to this problem in a domain is unique, if it exists, using the so-called maximum principle; later we consider the questions of existence and of finding explicit solutions.

Given a subset S of $\mathbb{C}$, we write $C(S)$ for the collection of all continuous functions on S. At the risk of some confusion, we will write Ω both for an open subset of $\mathbb{C}$ and for the corresponding subset of $\mathbb{R}^2$.

Recall that a real-valued function h of two real variables u and v is said to be harmonic in Υ its partial derivatives of order one and two are continuous in Υ, and $\Delta h(u,v) = 0$ for all $(u,v) \in \Upsilon$, where

$$\Delta h(u,v) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) h(u,v).$$

The *differential operator* Δ is known as the *Laplacian* or *Laplace operator.*

PROBLEM 32.1 (Dirichlet problem). Suppose that Ω is a bounded domain and that $\partial\Omega$ is a finite union of contours. Given a function $b \in C(\partial\Omega)$, find a harmonic function u on Ω such that

$$\begin{aligned} \Delta u(x,y) &= 0 && \text{for all } (x,y) \in \Omega \\ \lim_{(x,y)\to(x_0,y_0)} u(x,y) &= b(x_0,y_0) && \text{for all } (x_0,y_0) \in \partial\Omega. \end{aligned}$$

1. The maximum modulus principle

Recall from Lecture 26 that the open mapping theorem states that if Ω is a domain and $f \in H(\Omega)$ is nonconstant, then $\mathrm{Range}(f)$ is open.

THEOREM 32.2 (Maximum modulus principle). *Suppose that Ω is a bounded domain in $\mathbb{C}$, that $f \in H(\Omega)$, and that f extends continuously to the compact set $\overline{\Omega}$. If there is a point $z_0 \in \Omega$ such that*

$$|f(z_0)| = \max\{|f(z)| : z \in \overline{\Omega}\},$$

then f is constant.

PROOF. If f were not constant, then $\mathrm{Range}(f)$ would contain an open ball around $f(z_0)$, by the open mapping theorem, and $|f(z_0)|$ could not be the maximum value. $\square$

EXERCISE 32.3. Find an example of a nonconstant holomorphic function defined on a bounded open set Ω for which there is a point $z_0 \in \Omega$ such that

$$|f(z_0)| = \max\{|f(z)| : z \in \overline{\Omega}\}.$$

ANSWER.

EXERCISE 32.4. Find the maximum value of $|\sin(z)|$ as z varies over the closed unit disc $\overline{B}(0,1)$.

ANSWER.

2. The maximum principle for harmonic functions

THEOREM 32.5 (Maximum principle for harmonic functions). *Suppose that Ω is a bounded domain in $\mathbb{R}^2$, that $h : \Omega \to \mathbb{R}$ is harmonic and that h extends continuously to the compact set $\overline{\Omega}$. If there is a point $(x_0, y_0) \in \Omega$ such that*

$$h(x_0, y_0) = \max\{h(x, y) : (x, y) \in \overline{\Omega}\},$$

then h is constant.

PROOF. We will first prove this for a set Ω_1 that is simply connected. Suppose that there is a point $(x_0, y_0) \in \Omega_1$ such that $|h(x_0, y_0)| = \max\{|h(x, y)| : (x, y) \in \overline{\Omega}_1\}$. Since Ω_1 is simply connected, we may find a harmonic conjugate of h, and hence write $h(x, y) = \mathrm{Re}(f(x + iy))$, for some $f \in H(\Omega_1)$. Consider the holomorphic function $\exp \circ f$. Evidently

$$|\exp \circ f(x + iy)| = \exp(\mathrm{Re}(f(x + iy))) = \exp(h(x, y)),$$

and this attains its maximum at $x_0 + iy_0$, which lies in the interior of Ω_1. By the maximum modulus principle, $\exp \circ f$ is constant in Ω_1, that is, f is constant in Ω_1, and so h is constant in Ω_1.

To deal with the general case of a domain Ω that need not be simply connected, we take a arbitrary point (x_1, y_1) of Ω, and show that $h(x_1, y_1) = h(x_0, y_0)$, and conclude that h is constant. Since Ω is connected, there is a polygonal path joining (x_1, y_1) and (x_0, y_0); by omitting any loops, we may assume that this is simple. Now by "fattening up" this path a little, we may find a connected and simply connected open subset Ω_1 of Ω that contains both points. Observe that

$$h(x_0, y_0) \leq \max\{h(x, y) : (x, y) \in \Omega_1\} \leq \max\{h(x, y) : (x, y) \in \Omega\} \leq h(x_0, y_0),$$

since (x_0, y_0) is just one of the points considered in the first maximum, since $\Omega_1 \subseteq \Omega$, and by hypothesis. In particular, $h(x_0, y_0) = \max\{h(x, y) : (x, y) \in \overline{\Omega}_1\}$, and so we may apply the result for the simply connected set Ω_1, and conclude that $h(x_0, y_0) = h(x_1, y_1)$; hence h is constant. □

3. Uniqueness for the Dirichlet problem

Corollary 32.6 (Uniqueness for the Dirichlet problem). *Suppose that Ω is a bounded domain, that $b \in C(\partial\Omega)$, and that u_1 and u_2 are solutions of the Dirichlet problem*

$$\begin{aligned} \Delta u(x, y) &= 0 && \text{for all } (x, y) \in \Omega \\ \lim_{(x,y)\to(x_0,y_0)} u(x, y) &= b(x_0, y_0) && \text{for all } (x_0, y_0) \in \partial\Omega. \end{aligned}$$

Then $u_1 = u_2$.

Proof. By linearity, $u_1 - u_2$ is a harmonic function in Ω that tends to 0 as we approach the boundary of Ω; it suffices to show that $u_1 - u_2 = 0$. We write $u = u_1 - u_2$, and extend u to $\overline{\Omega}$ by continuity; then $u \big|_{\partial\Omega} = 0$.

If there were a point $(x, y) \in \Omega$ such that $u(x, y) > 0$, then there would be a point $(x_0, y_0) \in \overline{\Omega}$ such that

$$u(x_0, y_0) = \max\{u(x, y) : (x, y) \in \overline{\Omega}\} > 0.$$

By the maximum principle, $(x_0, y_0) \in \partial\Omega$. However, by definition, $u \big|_{\partial\Omega} = 0$, and so $u(x_0, y_0) = 0$, which is a contradiction. Thus $u \leq 0$ in Ω, that is $u_1 \leq u_2$.

Similarly, by taking $u = u_2 - u_1$, we may show that $u_2 \leq u_1$. Hence $u_2 = u_1$, as required. □

4. The Dirichlet problem and holomorphic mappings

We will spend some time looking at the Dirichlet problem later, but here is a very important idea, which is one of the reasons that complex analysis is useful.

Theorem 32.7. *Suppose that Ω is a domain, that $f : \Omega \to \Upsilon$ is holomorphic, and that $h : \Upsilon \to \mathbb{R}$ is harmonic. Write $f(x + iy) = u(x, y) + iv(x, y)$, where u and v*

are real-valued, and let $F(x,y) = (u(x,y), v(x,y))$ be the corresponding vector-valued function. Then $h \circ F$ is harmonic in Ω.

PROOF. It is evident that $h \circ F$ is defined in Ω, so it suffices to show that it is harmonic. To do this, it suffices to take an arbitrary point (x_0, y_0), and show that the function is harmonic in $B((x_0, y_0), r)$ for some small positive r. Let $z_0 = x_0 + iy_0$, and $(u_0, v_0) = (u(x_0, y_0), v(x_0, y_0))$.

Since h is harmonic, it has a conjugate harmonic function k, at least in some ball centred at (u_0, v_0) and lying in Ω. Let $g(u + iv) = h(u, v) + ik(u, v)$, for (u, v) in this ball.

From the definitions, we see that

$$\begin{aligned} h(u(x,y), v(x,y)) &= \operatorname{Re}(h(u(x,y), v(x,y)) + ik(u(x,y), v(x,y))) \\ &= \operatorname{Re}(g(f(x+iy))). \end{aligned}$$

Since g and g and holomorphic, so is $g \circ f$, and hence $(x,y) \mapsto h(u(x,y), v(x,y))$ is harmonic. □

The Dirichlet problem is important. If we can find holomorphic functions from a "harder" region to an "easier" one, we will be able to solve the Dirichlet problem in the "harder" case. Indeed, we have the following theorem.

THEOREM 32.8. *Suppose that Ω is a domain in $\mathbb{C}$, that B is a subset of $\partial\Omega$, and that $b : B \to \mathbb{R}$ is a continuous function. Suppose also that $f : \Omega \to \Upsilon$ is holomorphic, that f extends continuously to B, in the sense that if $z_0 \in B$ and $z \to z_0$, then $f(z) \to f(z_0) \in \partial\Upsilon$, and that $c : f(B) \to \mathbb{R}$ is given by $c(w) = b(f^{-1}(w))$ for all $w \in f(B)$. If $h : \Upsilon \to \mathbb{R}$ solves the Dirichlet problem*

$$\begin{aligned} \Delta h(u,v) &= 0 && \text{for all } (u,v) \in \Upsilon \\ \lim_{(u,v)\to(u_0,v_0)} h(u,v) &= b(f^{-1}(u_0,v_0)) && \text{for all } (u_0,v_0) \in f(B), \end{aligned}$$

then $(x,y) \mapsto h(u(x,y), v(x,y))$ solves the Dirichlet problem

$$\begin{aligned} \Delta u(x,y) &= 0 && \text{for all } (x,y) \in \Omega \\ \lim_{(x,y)\to(x_0,y_0)} u(x,y) &= b(x_0,y_0) && \text{for all } (x_0,y_0) \in B. \end{aligned}$$

If moreover $f : \Omega \to \Upsilon$ is bijective, and extends to a continuous bijection of the boundaries, then the solution to every Dirichlet problem in Ω arises by transferring a solution to the Dirichlet problem in Υ.

PROOF. Omitted. □

The next result is one of the most important theorems in complex analysis.

THEOREM 32.9 (Riemann mapping theorem). *Given any simply connected domain Ω in $\mathbb{C}$, other then $\mathbb{C}$ itself, there exists a bijective holomorphic mapping $f : \Omega \to B(0,1)$.*

The proof of this is too long and hard for this course. But it relies on being able to find a harmonic function on Ω such that $u(z) = -\ln|z - z_0|$ on the boundary of Ω, where z_0 is a point in Ω. It seems to be circular to use the Riemann mapping theorem to solve the Dirichlet problem and then use the Dirichlet problem to prove the Riemann mapping theorem, but it is not!

5. Examples

In the exercises below, we set $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, that is, H is the upper half plane. Boundary values should be interpreted as continuous limits at points of continuity on the boundary.

EXERCISE 32.10. Find bounded harmonic functions h in H such that

(a) $h(x, 0) = 1$ for all $x \in \mathbb{R}$;
(b) $h(x, 0) = 0$ if $x > 0$ and $h(x, 0) = \pi$ if $x < 0$;
(c) $h(x, 0) = \operatorname{sgn}(x)$ for all $x \in \mathbb{R}$;
(d) $h(x, 0) = 1$ if $0 < x < 1$ and $h(x, 0) = 0$ if $x < 0$ or $x > 1$.

ANSWER.

6. Solving the Dirichet problem in the upper half plane

The preceding exercise suggests that we can find harmonic functions in the upper half plane whose boundary values are "step functions". But actually we can do more. The key is the following: let H be the "Heaviside function": $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x \geq 0$. If $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and both g and g' tend to 0 at $\pm\infty$, then

$$\begin{aligned} g(x) &= \int_{-\infty}^{x} g'(s)\, ds = \int_{-\infty}^{\infty} H(x - s)\, g'(s)\, ds \\ &= \int_{-\infty}^{\infty} (1 - \frac{1}{\pi} \operatorname{Arg}(x - s))\, g'(s)\, ds \\ &= \int_{-\infty}^{\infty} g'(s)\, ds - \frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{Arg}(x - s)\, g'(s)\, ds = -\frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{Arg}(x - s)\, g'(s)\, ds. \end{aligned}$$

Now Arg extends into the upper half plane as a harmonic function, and so we examine the function h, defined on the upper half plane, by

$$\begin{aligned} h(z) &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{Arg}(z - s)\, g'(s)\, ds \\ &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{Arg}(x - s + iy)\, g'(s)\, ds \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\partial}{\partial s} \operatorname{Arg}(x - s + iy)\, g(s)\, ds. \end{aligned}$$

Finally,

$$\begin{aligned} \frac{\partial}{\partial s} \operatorname{Arg}(x - s + iy) &= \frac{\partial}{\partial s} \Big(\frac{\pi}{2} - \tan^{-1}((x - s)/y) \Big) \\ &= \frac{1}{((x - s)^2/y^2 + 1)y} \\ &= \frac{y}{(x - s)^2 + y^2}\,. \end{aligned}$$

THEOREM 32.11. *Suppose that g is a bounded continuous function on $\mathbb{R}$, and define h on the upper half-plane by the formula*

$$h(x + iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x - s)^2 + y^2}\, g(s)\, ds \qquad \forall x \in \mathbb{R} \quad \forall y \in \mathbb{R}^+.$$

Then the integral defining h converges, and h has the following properties:

(a) h is harmonic
(b) h is bounded
(c) $\lim_{x+iy\to x_0} h(x+iy) = g(x_0)$ for all $x_0 \in \mathbb{R}$.

Further, h is the only function with these properties.

PROOF. See next lecture. □

7. Solving the Dirichet problem in the unit ball

We may also compute an explicit solution to the Dirichlet problem in the unit disc.

THEOREM 32.12. *Suppose that g is a bounded continuous function on $\partial B(0,1)$, and define h inside $B(0,1)$ by the formula*

$$h(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\varphi)+r^2}\, g(e^{i(\theta-\varphi)})\, d\varphi$$

for all $re^{i\theta} \in B(0,1)$. Then the integral defining h converges, and moreover

(a) h is harmonic;
(b) h is bounded;
(c) $\lim_{re^{i\theta}\to e^{i\theta_0}} h(re^{i\theta}) = g(e^{i\theta_0})$ for all $\theta_0 \in [-\pi,\pi]$.

Further, h is the only function with these properties.

PROOF. Omitted. □

We can now solve the Dirichlet problem in several different regions.

LECTURE 33

Conformal mappings and harmonic functions

In this lecture, we first show that our proposed solution to the Dirichlet problem in the upper half plane has the desired properties. Next, we examine conformal mappings: mappings that preserve "form", at least locally. Finally we solve the Dirichlet problem in some other regions.

1. The Dirichlet problem in the upper half plane

Now we prove our theorem on harmonic functions in the upper half plane. First we recall the theorem for completeness.

THEOREM. *Suppose that g is a bounded continuous function on $\mathbb{R}$, and define h on the upper half-plane by the formula*

$$h(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-s)^2+y^2}\, g(s)\, ds \qquad \forall x \in \mathbb{R} \quad \forall y \in \mathbb{R}^+.$$

Then the integral defining h converges, and h has the following properties:

(a) h is harmonic
(b) h is bounded
(c) $\lim_{x+iy\to x_0} h(x+iy) = g(x_0)$ for all $x_0 \in \mathbb{R}$.

Further, h is the only function with these properties.

PROOF. To show that h is harmonic, we observe that

$$\operatorname{Arg}(z-s) = \operatorname{Re}(-i \operatorname{Log}(z-s)),$$

and so

$$\frac{\partial}{\partial s}\operatorname{Arg}(z-s) = \frac{\partial}{\partial s}\operatorname{Re}(-i\operatorname{Log}(z-s)) = \operatorname{Re}\frac{\partial}{\partial s}(-i\operatorname{Log}(z-s)) = \operatorname{Re}\left(i\frac{1}{z-s}\right),$$

which is harmonic, so if we can differentiate under the integral sign, the harmonicity will follow. It is possible to verify that it is legitimate to exchange the order of differentiation and integration; we omit the details.

To show that h is bounded, we observe that

$$\begin{aligned}
\left|\int_{\mathbb{R}} \frac{1}{\pi}\frac{y}{(x-s)^2+y^2}\, g(s)\, ds\right| &\le \int_{\mathbb{R}} \frac{1}{\pi}\frac{y}{(x-s)^2+y^2}\, |\, g(s)|\; ds \\
&\le \sup\{|g(s)| : s \in \mathbb{R}\} \int_{\mathbb{R}} \frac{1}{\pi}\frac{y}{(x-s)^2+y^2}\, ds \\
&= \sup\{|g(s)| : s \in \mathbb{R}\} \int_{\mathbb{R}} \frac{1}{\pi}\frac{1}{t^2+1}\, dt \\
&= \sup\{|g(s)| : s \in \mathbb{R}\},
\end{aligned}$$

by the change of variables $t = (x-s)/y$.

Finally, to establish the boundary limiting behaviour, change variables to see that

$$h(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{s^2+y^2}\, g(x-s)\, ds$$

whence, changing variable again ($s = yt$), we see that

$$\begin{aligned} h(x+iy) - g(x_0) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{s^2+y^2}\,(g(x-s) - g(x_0))\; ds \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{t^2+1}\,(g(x-yt) - g(x_0))\; dt \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{t^2+1}\,(g(x-yt) - g(x))\; dt \\ &\qquad + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{t^2+1}\,(g(x) - g(x_0))\; dt. \end{aligned}$$

Yet another "limit inside the integral" argument shows that this goes to 0 as $x+iy$ goes to x_0, that is, as $x \to x_0$ and $y \to 0+$. $\square$

2. Affine mappings and affine approximations

Consider the *translation* mapping $f : \mathbb{C} \to \mathbb{C}$, given by $z \mapsto z + b$, where $b \in \mathbb{C}$. This is a congruence, that is, it preserves distances, so that if T is a triangle in $\mathbb{C}$, then so is $f(T)$, and $f(T)$ is congruent to T, and the corresponding angles are equal. But more is true: the sides of $f(T)$ are parallel to those of T.

Now consider the *linear* mapping $f : \mathbb{C} \to \mathbb{C}$, given by $z \mapsto az$, where $a \in \mathbb{C}\setminus\{0\}$. This is composed of a rotation (through the angle $\operatorname{Arg}(a)$) and a dilation (by a factor of $|a|$); it is not a congruence unless $|a| = 1$, but it still preserves angles and *ratios* of lengths. More precisely, if T is a triangle in $\mathbb{C}$, then so is $f(T)$, and $f(T)$ is *similar* to T; corresponding angles are equal, and ratios of side lengths are preserved. Note that the *orientation* of triangles is also preserved: the equal angles of T and $f(T)$ appear in the same order as we go around the boundaries in the usual anti-clockwise direction.

We now combine these two observations, and consider the affine mapping $f : \mathbb{C} \to \mathbb{C}$, given by $z \mapsto az + b$, where $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$. This mapping also preserves angles, orientation, and ratios of lengths.

Consider now a differentiable function $f : \Omega \to \mathbb{C}$, defined in a domain Ω, and fix $z_0 \in \Omega$. The definition of differentiability tells us that

$$f(z) \approx f(z_0) + f'(z_0)(z - z_0),$$

at least when z is close to z_0. What does this tell us about the geometric properties of f? Because we are only dealing with an approximation, with errors that increase the further z is from z_0, it seems unlikely that f will preserve ratios of lengths. However, if might still be true that f preserves angles. But since $f(T)$ might have curved sides, even if T has straight sides, we need to be able to measure angles between curves.

3. Differentiability and angles between two curves

Suppose that $\gamma : I \to \mathbb{C}$ and $\delta : J \to \mathbb{C}$ are curves that both pass once through z_0, where I and J are open intervals in $\mathbb{R}$; we suppose that $\gamma(s_0) = z_0$ and $\delta(t_0) = z_0$. The tangent vectors to the curves at z_0 are $\dot\gamma(s_0)$ and $\dot\delta(t_0)$, where the dot denotes differentiation (with respect to a real variable). If these tangent vectors are both nonzero, then the angle between the tangent vectors is well-defined. Because we are working in the complex plane, we may define the angle between γ and δ to be $\operatorname{Arg}(\dot\gamma(s_0)/\dot\delta(t_0))$. This means that, if we align a pointer in the direction in which δ is moving at z_0, and turn the pointer anti-clockwise through the angle $\operatorname{Arg}(\dot\gamma(s_0)/\dot\delta(t_0))$, then the pointer indicates the direction in which γ is moving. Note that this is an *oriented angle*, and the angle between δ and γ is $\operatorname{Arg}(\dot\delta(t_0)/\dot\gamma(s_0))$, which has the opposite sign (unless the argument is 0 or π).

Now suppose also that $f : \Omega \to \mathbb{C}$ is differentiable at z_0. Then $f \circ \gamma(s_0) = f(z_0)$ and $f \circ \delta(t_0) = f(z_0)$, and the curves $f \circ \gamma$ and $f \circ \delta$ pass through $f(z_0)$ at s_0 and t_0. Further, the chain rule implies that $f \circ \gamma$ and $f \circ \delta$ are differentiable at s_0 and t_0, and $(f \circ \gamma)^{\cdot}(s_0) = f'(z_0)\,\dot\gamma(s_0)$ while $(f \circ \delta)^{\cdot}(t_0) = f'(z_0)\,\dot\delta(t_0)$. It follows immediately that

$$\frac{(f \circ \gamma)^{\cdot}(s_0)}{(f \circ \delta)^{\cdot}(t_0)} = \frac{f'(z_0)\,\dot\gamma(s_0)}{f'(z_0)\,\dot\delta(t_0)} = \frac{\dot\gamma(s_0)}{\dot\delta(t_0)}\,,$$

and so

$$\operatorname{Arg}\left(\frac{(f \circ \gamma)^{\cdot}(s_0)}{(f \circ \delta)^{\cdot}(t_0)}\right) = \operatorname{Arg}\left(\frac{\dot\gamma(s_0)}{\dot\delta(t_0)}\right)\,;$$

that is, f preserves the (oriented) angles between curves that intersect at z_0.

Note that if f is differentiable but $f'(z_0) = 0$, then f does not preserve angles. Indeed, if f is constant, then all curves map to the same point in $\mathbb{C}$, and there are no angles at all to measure; otherwise, by Taylor series, we may write $f(z) = f(z_0) + c_n(z - z_0)^n + \dots$, where $n > 1$ since $f'(z_0) = 0$; it is then an exercise to show that angles are not preserved.

Exercise 33.1. Suppose that $f(z) = z^n$ for all $z \in \mathbb{C}$, where $n > 1$. Let $I = (-1, 1)$, and consider the curves $\gamma_\theta : I \to \mathbb{C}$ given by $\gamma_\theta(t) = te^{i\theta}$. Show that the curves $f \circ \gamma_0$ and $f \circ \gamma_{2\pi/n}$ coincide and are horizontal, and deduce that f does not preserve the angles between curves at 0.

Exercise 33.2. Suppose that $f(z) = \overline{z}$ for all $z \in \mathbb{C}$. Show that f changes the sign of angles between curves.

Definition 33.3. We say that a mapping f from a domain Ω of $\mathbb{C}$ to $\mathbb{C}$ is *conformal* if it is differentiable and $f'(z_0) \neq 0$ for all $z_0 \in \Omega$. If the mapping is one-to-one, then the condition that $f'(z_0 \neq 0$ for all z_0 in Ω tells us that f is invertible, and its inverse is also differentiable.

Exercise 33.4. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear mapping. Show that T preserves the (oriented) angles between vectors if and only if T is composed of a rotation and a dilation. (Hint: write T as RU, where R is a rotation and U is upper triangular.)

Deduce that a mapping from an open subset of $\mathbb{R}^2$ into $\mathbb{R}^2$ that preserves (oriented) angles between curves arises as the "real version" of a holomorphic mapping.

Conformal mappings arose when mathematicians started to analyse the ways in which map-makers present a piece of the earth's curved surface on a plane. Map-makers' projections cannot preserve distance (or even scale), but some map projections are conformal.

Conformal mappings became important in applied mathematics because composing with conformal mappings preserves harmonic functions, and this allows us to solve many physical problems.

They are also used to explain morphology in biology.

4. Examples of conformal mappings

In the following exercises, we do not distinguish between points in the complex plane and points in $\mathbb{R}^2$.

EXERCISE 33.5. Find a conformal mapping f from the upper half plane H to the unit ball B.

ANSWER.

EXERCISE 33.6. Let Σ_α be the sector $\{z \in \mathbb{C} : |\operatorname{Arg}(z)| < \alpha\}$ in the complex plane. Find a conformal mapping g from the sector Σ_α (where $0 < \alpha < \pi$ to the upper half plane H. Hence find a harmonic function h in Σ_α such that $h(r\cos\alpha, -r\sin\alpha) = 0$ and $h(r\cos\alpha, r\sin\alpha) = 1$ for all positive r.

ANSWER.

Observe that the composed mapping

$$w = \frac{iz^{\pi/2\alpha} - i}{iz^{\pi/2\alpha} + i} = \frac{z^{\pi/2\alpha} - 1}{z^{\pi/2\alpha} + 1}$$

takes the sector to the unit ball in the complex plane.

EXERCISE 33.7. Let S_b be the infinite strip $\{z \in \mathbb{C} : 0 < \operatorname{Im} z < b\}$. Find a conformal mapping h from the infinite strip S_b to the upper half plane H.

ANSWER.

Other simply connected regions for which it is possible to find "explicit" conformal mappings from the region to the unit ball include polygons (the mappings that arise are called Schwarz–Christoffel mappings).

It is also possible to solve the Dirichlet problem in a doubly connected domain (a domain with one hole, which must not be just a missing point). First, a generalisation of the Riemann mapping theorem shows that any such domain may be mapped conformally onto an annulus $\{z \in \mathbb{C} : r < |z| < 1\}$. Next, it is possible to solve the problem in this kind of annulus, using separation of variables. More precisely, we may express a harmonic function in the annulus in a "generalised Laurent series":

$$h(z) = \sum_{n=-\infty}^{\infty} a_n z^n + \sum_{n=-\infty}^{\infty} \bar{a}_n \bar{z}^n,$$

and choose the coefficients a_n to get the desired boundary value functions.

Note that $a + b\ln(|\cdot|)$ is a radial harmonic function in the annulus, which may be given arbitrary boundary values that are constant on the inner and the outer circles. This kind of problem arises in electrostatics.

LECTURE 34

Loose ends

In this last lecture, we consider an application in aerodynamics.

1. An application to fluid mechanics

In the theory of steady flow of an ideal fluid, it is shown that the "streamlines" of the flow, that is, the lines that a particle in the flow will follow, are level curves of a harmonic function.

EXAMPLE 34.1. The streamlines for a 2-dimensional ideal flow around a solid obstacle in the shape of the unit ball are the parts of the level sets $\{(x, y) \in \mathbb{R}^2 : \psi(x, y) = d\}$ that lie outside the unit ball, where

$$\psi(x, y) = y - \frac{y}{x^2 + y^2} \, .$$

To understand the flow around a more complicated object, we may employ a Joukowsky transformation: this is the composition of an affine map close to the identity with the map $z \mapsto z + 1/z$.