MATH2701 Problem Set Solutions

Gaurish Sharma

October 2, 2023

Contents

1 Problem set 1

1.1 Problem 1

Let $A(a_1, a_2)$, $B(b_1, b_2)$ be two points in $\mathbb{R}^2$. Find the equation for the line $\ell(A, B)$ through A, B.

1. in Cartesian form.
2. in parametric vector form.

Solution. Cartesian form: For all $(x, y) \in l$

$$\begin{aligned}
\frac{b_2 - a_2}{b_1 - a_1} &= \frac{y - a_2}{x - a_1} \\
(x - a_1)(b_2 - a_2) &= (y - a_2)(b_1 - a_1) \\
xb_2 - xa_2 - a_1 b_1 + a_1 a_2 &= yb_1 - ya_1 - a_2 b_2 + a_1 a_2 \\
x(b_2 - a_2) - a_1 b_2 + a_2 b_1 &= y(b_1 - a_1) \\
y &= \frac{x(b_2 - a_1)}{b_1 - a_1} + \frac{-a_1 b_2 + b_2 b_1}{b_1 - b_1}.
\end{aligned}$$

So, above is the cartesian equation of ℓ.

Parametric form: $A = \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. We treat A and B as position veectors. Then,

$$\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}.$$

is parallel to ℓ. Hence the parametric form is, for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \ell$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_1 \end{pmatrix} + \lambda \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}.$$

for some $\lambda \in \mathbb{R}$. (Ask lect best way to write eqn)

1.2 Problem 2

For any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and scalar $\lambda \in \mathbb{R}$,

1. $\mathbf{a} \cdot \mathbf{b} \in \mathbb{R}$ (Scalar).
2. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$, $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.
3. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$, (commutativity)
4. $\mathbf{a} \cdot (\lambda \mathbf{b}) = \lambda (\mathbf{a} \cdot \mathbf{b})$
5. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$, (distributive)
6. If $\mathbf{a}, \mathbf{a} \in \mathbb{R}^n$ then $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ (cauchy-schwarz Inequality)
7. Hence, the angle θ between $\mathbf{a}, \mathbf{b}$ via $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ is well defined.

8. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}| + \mathbf{b}$ (Triangle inequality).
9. Use the dot product to prove that a real $n \times n$ matrix Q is an orthogonal matrix if and only if $Q\mathbf{x} \cdot Q\mathbf{x} = \mathbf{x} \cdot \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Solution. We first prove $Q\mathbf{x} \cdot Q\mathbf{x} = \mathbf{x} \cdot \mathbf{x}$, $\forall \mathbf{x} \in \mathbb{R}^n \Rightarrow Q$ is orthogonal.

$$\begin{aligned}
Q\mathbf{x} \cdot Q\mathbf{x} &= \mathbf{x} \cdot \mathbf{x} \\
(Q\mathbf{x} \cdot Q\mathbf{x}) &= (\mathbf{x} \cdot \mathbf{x}) \\
(Q\mathbf{x})^T Q\mathbf{x} &= \mathbf{x}^T \mathbf{x} \\
\mathbf{x}^T Q^T Q\mathbf{x} &= \mathbf{x}^T \mathbf{x}.
\end{aligned}$$

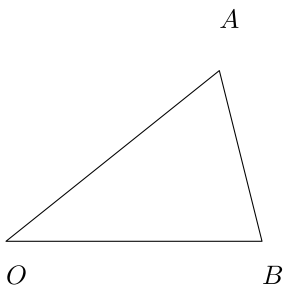
left multiplying $\left(\mathbf{x}^T\right)^{-1}$ on both sides and right multiplying $(\mathbf{x})^{-1}$ on both sides we get,

$$Q^T Q = I.$$

Note that the algebra above is reversible therefore the converse is also true.

1.3 Problem 3

Consider the points $A(\mathbf{a}), B\,(\mathbf{b})$ and the origina O and $\triangle OAB$ and let $\theta = \angle AOB$. Use the Cosine Law to deduce $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.



Solution. Using the cosine rule we get,

$$\begin{aligned}
|\mathbf{a}-\mathbf{b}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta \\
2|\mathbf{a}||\mathbf{b}|\cos\theta &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2 \\
\cos\theta &= \frac{|\mathbf{a}|^2 = |\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2}{2|\mathbf{a}||\mathbf{b}|} \\
\cos\theta &= \frac{\mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - (\mathbf{a}-\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b})}{2|\mathbf{a}||\mathbf{b}|} \\
&= \frac{2\mathbf{b} \cdot \mathbf{a}}{2|\mathbf{a}||\mathbf{b}|} \\
&= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.
\end{aligned}$$

1.4 Problem 4

Show that a collineation determines a 1-1 correspondence from the set of all lines to itself.

Solution. Clarify meaning of question.

1.5 Problem 5

Show that the lines with equations $aX + bY + c = 0$ and $dX + eY + f = 0$ are parallel iff $ae - bd = 0$ and are perpendicular iff $ad + be = 0$.

Solution. We first rewrite the equations in the form $Y = mX + b$. We get $Y = -\frac{a}{b}X - \frac{c}{b}$ and $Y = -\frac{d}{c} - \frac{c}{e}$ We know these lines are parellel if,

$$\begin{aligned} -\frac{a}{b} &= -\frac{d}{e} \\ -ae &= -db \\ ae - db &= 0. \end{aligned}$$

Therefore, backwards implication proved. Forward implication is just obtained with the same algebra above since we can assume that the lines are parallel and gradients are equal once again giving us $ae - db = 0$.

Now we prove the condition for perpendicular lines. We know these lines are perpendicular if,

$$\begin{aligned} \frac{b}{a} &= -\frac{d}{e} \\ ad + be &= 0. \end{aligned}$$

Hence, backward implication proved. Forward implication once again proved by same algebra above through similar reasoning with forward implication of parallel lines.

1.6 Problem 6

A multiplication for finite gorup is often called Caley table.