LECTURE 0

Assumed Knowledge

This is a review of basic facts about complex numbers that ought to be familiar:

- the definition of complex numbers,
- their arithmetic,
- Cartesian and polar representations,
- the Argand diagram,
- de Moivre's theorem, and
- extracting nth roots of complex numbers.

Students who do not feel confident about this material need to do lots of exercises about these, such as those in the MATH1141 notes.

1. Complex numbers

Definition 0.1. A *complex number* is an expression of the form $x + iy$, where x and y are real numbers. The *real part* of $x + iy$ is x and the *imaginary part* of $x + iy$ is y. We denote this by $\mathrm{Re}(x + iy) = x$ and $\mathrm{Im}(x + iy) = y$. The set of all complex numbers is denoted $\mathbb{C}$.

We often write $w = u + iv$ and $z = x + iy$, and work with w and z rather than $u + iv$ and $x + iy$. In this case, we write, for instance, $\mathrm{Re}(z) = x$ and $\mathrm{Im}(w) = v$. We abbreviate $x + i0$ and $0 + iy$ to x and iy, and $0 + i1$ to i.

Definition 0.2. Suppose that $w = u + iv$ and $z = x + iy$, where $u, v, x, y \in \mathbb{R}$. Then we define the complex numbers $w + z$, $-z$, wz and, if $z \neq 0$, z^{-1} and w/z, as follows:

$$\begin{aligned} w + z &= (u + x) + i(v + y) \\ -z &= -x + i(-y) \\ wz &= (ux - vy) + i(uy + vx) \\ z^{-1} &= (x^2 + y^2)^{-1}(x - iy) \\ w/z = wz^{-1} &= (x^2 + y^2)^{-1}[(ux + vy) + i(vx - uy)]. \end{aligned}$$

Then $i^2 = -1$ and $-z = (-1)z$.

We may combine these operations to make sense of more complicated expressions such as $w^m - z^n$, where m and n are integers.

PROPOSITION 0.3. *Complex numbers have the following properties:*

$$\begin{aligned}
z_1 + z_2 &= z_2 + z_1 && \forall z_1, z_2 \in \mathbb{C} \\
(z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3) && \forall z_1, z_2, z_3 \in \mathbb{C} \\
0 + z &= z && \forall z \in \mathbb{C} \\
(-z) + z &= 0 && \forall z \in \mathbb{C} \\
z_1 z_2 &= z_2 z_1 && \forall z_1, z_2 \in \mathbb{C} \\
(z_1 z_2) z_3 &= z_1 (z_2 z_3) && \forall z_1, z_2, z_3 \in \mathbb{C} \\
1z &= z && \forall z \in \mathbb{C} \\
z z^{-1} &= 1 && \forall z \in \mathbb{C} \setminus \{0\} \\
z_1(z_2 + z_3) &= z_1 z_2 + z_1 z_3 && \forall z_1, z_2, z_3 \in \mathbb{C}
\end{aligned}$$

The symbol $\forall$ is read "for all". This proposition shows that the complex numbers form a *field*.

DEFINITION 0.4. If $z = x + iy$, then $\overline{z}$, the *(complex) conjugate* of z, and $|z|$, the *modulus* of z, are defined to be $x - iy$ and $(x^2 + y^2)^{1/2}$.

Some further properties of complex numbers relate conjugates and moduli.

PROPOSITION 0.5. *For all $z, z_1, z_2 \in \mathbb{C}$,*

(a)	$\operatorname{Re}(z) = \dfrac{z + \overline{z}}{2}$	(b)	$\operatorname{Im}(z) = \dfrac{z - \overline{z}}{2i}$
(c)	$\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$	(d)	$\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$
(e)	$\overline{-z} = -\overline{z}$	(f)	$\overline{z^{-1}} = (\overline{z})^{-1}$
(g)	$\lvert z_1 + z_2 \rvert \leq \lvert z_1 \rvert + \lvert z_2 \rvert$	(h)	$\lvert z_1 z_2 \rvert = \lvert z_1 \rvert \, \lvert z_2 \rvert$
(i)	$z\overline{z} = \lvert z \rvert^2$	(j)	$z^{-1} = \lvert z \rvert^{-2} \overline{z}$.

Inequality (g) is called the *triangle inequality*. If $|z| = 1$, then $z^{-1} = \overline{z}$, from (j).

PROOF. We only prove the triangle inequality, because this is hardest.

First, here is an algebraic proof. Recall that $2ab \leq a^2 + b^2$ for real numbers a and b. Taking a and b to be $x_1 y_2$ and $x_2 y_1$, we deduce that

$$2x_1 x_2 y_1 y_2 \leq x_1^2 y_2^2 + x_2^2 y_1^2,$$

and hence

$$x_1^2 x_2^2 + y_1^2 y_2^2 + 2x_1 x_2 y_1 y_2 \leq x_1^2 x_2^2 + y_1^2 y_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2 = (x_1^2 + y_1^2)(x_2^2 + y_2^2),$$

so, taking square roots,

$$x_1 x_2 + y_1 y_2 \leq |z_1| \, |z_2| \, .$$

Finally, $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$, and so

$$\begin{aligned}
|z_1 + z_2|^2 &= (x_1 + x_2)^2 + (y_1 + y_2)^2 \\
&= x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2(x_1 x_2 + y_1 y_2) \\
&\leq |z_1|^2 + |z_2|^2 + 2|z_1| \, |z_2| \\
&= \big(|z_1| + |z_2|\big)^2;
\end{aligned}$$

the triangle inequality follows by taking square roots.

Alternatively, here is a geometric version. Consider the triangle whose vertices are w, z, and $w+z$, and the parallelogram with vertices 0, w, z, and $w+z$. The side of the parallelogram that joins w to $w+z$ is congruent to the side joining 0 to z, and the side of the parallelogram that joins z to $w+z$ is congruent to the side joining 0 to w. So we are just asserting the obvious fact that the length of one side of a triangle is less than the sum of the other two sides. $\square$

2. Euler's formula

The usual exponential function has a Taylor series:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

so at least formally, for a real number θ,

$$\begin{aligned} e^{i\theta} &= 1 + i\frac{\theta}{1!} - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots \\ &= \Big(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\Big) + i\Big(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\Big) \\ &= \cos\theta + i\sin\theta, \end{aligned}$$

from the Taylor series for the cosine and sine functions.

Later we will make this rigorous and use power series very effectively. This observation leads us to make the following definition.

Definition 0.6. We define $e^{i\theta}$ to be $\cos\theta + i\sin\theta$, for any real number θ.

Suppose that $z = x + iy \neq 0$. Write r instead of $|z|$. Then $(x/r, y/r)$ lies on the unit circle in the Cartesian plane, so $(x/r, y/r) = (\cos\theta, \sin\theta)$ for an appropriate choice of θ, and hence

$$z = r\Big(\frac{x}{r} + i\frac{y}{r}\Big) = r\big(\cos\theta + i\sin\theta\big) = re^{i\theta}.$$

Definition 0.7. The *Cartesian form* of a complex number z is its representation in the form $x+iy$, where x and y are real. The *polar form* of a complex number z is its representation in the form $re^{i\theta}$, where $r \geq 0$ and $\theta \in \mathbb{R}$. The number θ is called the *argument* of z, and is written $\arg(z)$.

Lemma 0.8. *Suppose that $r, s \in \mathbb{R}^+$ and $\theta, \phi \in \mathbb{R}$. Then*

$$r(\cos\theta + i\sin\theta) = s(\cos\phi + i\sin\phi)$$

if and only if $r = s$ and $\theta - \phi = 2k\pi$ for some $k \in \mathbb{Z}$.

This lemma follows immediately from trigonometry. It tells us that the argument of a nonzero complex number z is ambiguous. The next definition is to avoid this ambiguity.

Definition 0.9. The *principal value* of the argument of a nonzero complex number z, written $\mathrm{Arg}(z)$, is the unique number θ such that $z = |z|\, e^{i\theta}$ and $-\pi < \theta \leq \pi$.

We do not define the argument of 0.

3. The Argand Diagram

Suppose that $w = u + iv$. Then to the complex number w we associate the point in the Cartesian plane with Cartesian coordinates (u, v). When we do this, we call the axes the real axis and the imaginary axis. See Figure 0.1.

Geometrically, $|w|$ is the length of the line joining w to O, and $\arg(w)$ is the angle between this line and the positive real axis (taking the anticlockwise direction to be positive). Further, $|w - z|$ is the length of the line joining w and z.

Adding two complex numbers corresponds to vector addition in the plane. Multiplying by r in $\mathbb{R}^+$ dilates by a factor of r, and multiplying by the complex number $e^{i\theta}$ (where $\theta \in \mathbb{R}$) rotates (anticlockwise) through the angle θ.

4. De Moivre's formula

THEOREM 0.10. *If $\theta, \phi \in \mathbb{R}$, then*

$$(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos(\theta+\phi) + i\sin(\theta+\phi).f$$

PROOF. For all $\theta, \phi \in \mathbb{R}$,

$$\begin{aligned}
&(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\\
&= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)\\
&= \cos(\theta+\phi) + i\sin(\theta+\phi),
\end{aligned}$$

as required. □

COROLLARY 0.11 (de Moivre's formula). *If $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$, then*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{0.1}$$

PROOF. This is obviously true if $n = 0$ or 1. The result may be proved for $n \in \mathbb{Z}^+$ by induction. Suppose that $k \in \mathbb{Z}^+$ and formula (0.1) holds when $n = k$, i.e.,

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta.$$

Then by Theorem 0.10,

$$\begin{aligned}
(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^k\\
&= (\cos\theta + i\sin\theta)(\cos k\theta + i\sin k\theta)\\
&= \cos(\theta + k\theta) + i\sin(\theta + k\theta)\\
&= \cos(k+1)\theta + i\sin(k+1)\theta,
\end{aligned}$$

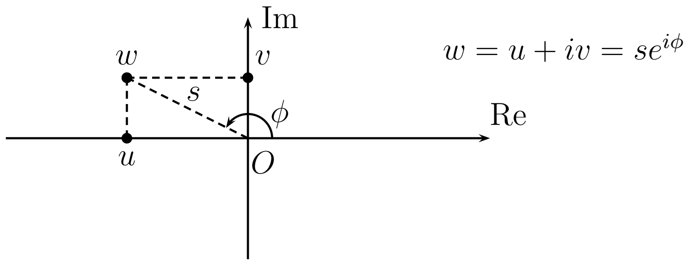

FIGURE 0.1. The complex plane

so the result holds when $n = k + 1$. By induction, the result holds for all $n \in \mathbb{Z}^+$.

To prove the result when $n \in \mathbb{Z}^-$, we use the fact that if $|z| = 1$, then $z\overline{z} = 1$, so $\overline{z} = z^{-1}$. That is,

$$(\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta),$$

as required. □

In polar notation, Theorem 0.10 and Corollary 0.11 become

$$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi} \qquad \text{and} \qquad (e^{i\theta})^n = e^{in\theta}.$$

These are more "obvious" and easier to remember than the trigonometric formulae.

5. Roots of complex numbers

We use complex exponentials to find roots of complex numbers. Fix $w \in \mathbb{C}$ and $n \in \mathbb{Z}^+$, and suppose that $w = z^n$ for some $z \in \mathbb{C}$. Then z is called *an nth root of w*. Write z as $re^{i\theta}$ and w as $se^{i\phi}$. Then

$$se^{i\phi} = w = z^n = r^n e^{in\theta}.$$

From Lemma 0.8, $r = s^{1/n}$ and $n\theta = \phi + 2k\pi$ for some $k \in \mathbb{Z}$. Therefore $z = s^{1/n}e^{i\theta}$, where

$$\theta = \frac{\phi}{n} + \frac{2k\pi}{n}$$

for some $k \in \mathbb{Z}$. If $k = nl + r$, where $l \in \mathbb{Z}$ and $r = 0, 1, 2, \dots, n-1$, then

$$e^{(\phi/n + 2k\pi/n)} = e^{(\phi/n + 2r\pi/n + 2l\pi)} = e^{(\phi/n + 2r\pi/n)},$$

so we get the same value of z by taking $k = nl + r$ as by taking $k = r$. Thus, to get all n possible values of z, it suffices to take the first n values of k, or any n consecutive values, or indeed any n values of k which give the n possible different remainders when divided by n.

The nth roots of any nonzero complex number w are uniformly spaced around the circle with centre 0 and radius $|w|^{1/n}$. For example, Figure 0.2 shows the seventh roots of unity. A symmetry argument shows that the sum of all these numbers is 0.

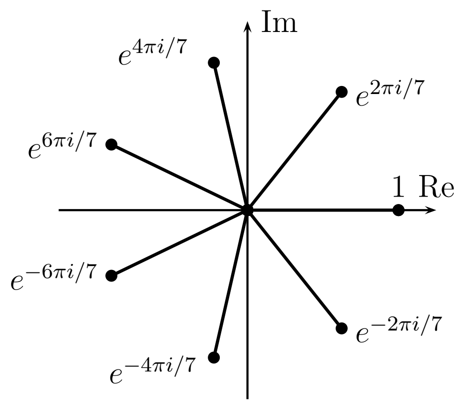

FIGURE 0.2. The seventh roots of unity

6. History†

Leonhard Euler found the formula that bears his name in 1748; he was one of the most prolific mathematicians ever. Some mathematicians did not believe that the geometric representation afforded by the Argand diagram was legitimate. The story of de Moivre's death is very curious. For more information, see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/`
and then find Euler, Argand, and de Moivre.

LECTURE 1

Inequalities and Sets of Complex Numbers

In complex analysis, we consider functions whose domains or ranges or both are regions in the complex plane. So to be able to discuss functions, we need to be able to describe regions. Curves and regions in the complex plane are often described by equalities and inequalities involving $|\cdot|$, Arg, Re, Im, ….

In the first part of this lecture, we review some equalities and inequalities. Then we discuss different types of regions. Finally, we consider some examples.

1. Equalities and inequalities

We begin with a lemma that is related to the cosine rule. It implies that $\operatorname{Re}(w\bar{z})$ is the inner product of the vectors represented by the complex numbers w and z.

Lemma 1.1. *For all complex numbers w and z,*

$$|w+z|^2 = |w|^2 + 2\operatorname{Re}(w\bar{z}) + |z|^2.$$

Proof. Observe that

$$\begin{aligned} |w+z|^2 &= (w+z)(\bar{w}+\bar{z}) = w\bar{w} + w\bar{z} + \bar{w}z + z\bar{z} \\ &= |w|^2 + w\bar{z} + (w\bar{z})^- + |z|^2 = |w|^2 + 2\operatorname{Re}(w\bar{z}) + |z|^2, \end{aligned}$$

as required. □

The triangle inequality is one of the most useful results about complex numbers. It states:

$$|w+z| \le |w| + |z| \qquad \forall w, z \in \mathbb{C}.$$

Here are a variation on the triangle inequality, sometimes called the circle inequality.

Lemma 1.2. *For all complex numbers w and z,*

$$\big||w| - |z|\big| \le |w-z|.$$

Proof. Observe that $w = (w-z) + z$, so $|w| \le |w-z| + |z|$ by the triangle inequality, and hence

$$|w| - |z| \le |w-z|. \tag{1.1}$$

Interchanging the roles of w and z in (1.1),

$$|z| - |w| \le |z-w|.$$

Combining these inequalities and recalling that $|w-z| = |z-w|$, we see that

$$\big||w| - |z|\big| = \max\{|w|-|z|, |z|-|w|\} \le |w-z|,$$

as required.

Alternatively, consider points on circles of radii $|z|$ and $|w|$, and compare the distance between the points with the difference of the radii. □

Recall that if $z = x + iy$, then e^z is defined to be $e^x(\cos y + i \sin y)$. Then $e^w e^z = e^{w+z}$ for all complex numbers w and z. Here is another very useful result.

LEMMA 1.3. *If $z \in \mathbb{C}$, then*

$$|e^z| = e^{\mathrm{Re}(z)}.$$

PROOF. See Problem Sheet 1. □

LEMMA 1.4. *For all real numbers θ,*

$$\left|e^{i\theta} - 1\right| \le |\theta|.$$

PROOF. See Problem Sheet 1. □

2. Properties of sets

DEFINITION 1.5. The *open ball* with centre z_0 and radius ε, written $\mathrm{B}(z_0, \varepsilon)$, is the set $\{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$.

The *punctured open ball* with centre z_0 and radius ε, written $\mathrm{B}^\circ(z_0, \varepsilon)$, is the set $\{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$.

Sometimes these sets are called discs rather than balls.

DEFINITION 1.6. Suppose that $S \subseteq \mathbb{C}$. For any point z_0 in $\mathbb{C}$, there are three mutually exclusive and exhaustive possibilities:

(1) When the positive real number ε is sufficiently small, $\mathrm{B}(z_0, \varepsilon)$ is a subset of S, that is, $\mathrm{B}(z_0, \varepsilon) \cap S = \mathrm{B}(z_0, \varepsilon)$. In this case, z_0 is an *interior point* of S.
(2) When the positive real number ε is sufficiently small, $\mathrm{B}(z_0, \varepsilon)$ does not meet S, that is, $\mathrm{B}(z_0, \varepsilon) \cap S = \emptyset$. In this case, z_0 is an *exterior point* of S.
(3) No matter how small the positive real number ε is, neither of the above holds, that is, $\emptyset \subset \mathrm{B}(z_0, \varepsilon) \cap S \subset \mathrm{B}(z_0, \varepsilon)$. In this case, z_0 is a *boundary point* of S.

These definitions are illustrated in Figure 1.1. We consider points z_1, z_2 and z_3.

If the radius of the ball centred at z_1 is small enough, then the ball lies inside the set S, and $\mathrm{B}(z_1, \varepsilon) \cap S = \mathrm{B}(z_1, \varepsilon)$. Thus z_1 is an interior point.

If the radius of the ball centred at z_2 is small enough, then the ball lies outside the set S, and $\mathrm{B}(z_2, \varepsilon) \cap S$ is empty. Thus z_2 is an exterior point.

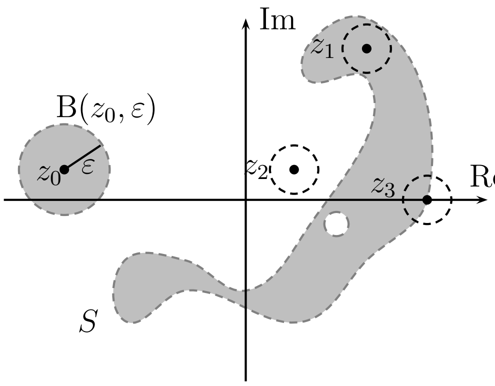

FIGURE 1.1. The ball with centre z_0 and radius ε, and interior, exterior and boundary points z_1, z_2 and z_3 of the set S

No matter how small the radius of the ball centred at z_3 is, part of the ball lies inside S, and part lies outside S, and $\mathrm{B}(z_3, \varepsilon) \cap S$ is neither empty nor all of $\mathrm{B}(z_3, \varepsilon)$. Thus z_3 is a boundary point.

DEFINITION 1.7. Suppose that $S \subseteq \mathbb{C}$.

(1) The set S is *open* if all its points are interior points.
(2) The set S is *closed* if it contains all of its boundary points, or equivalently, if its complement $\mathbb{C} \setminus S$ is open.
(3) The *closure* of S, written $\bar{S}$, is the set consisting of all the points of S together with all its boundary points.
(4) The set S is *bounded* if $S \subseteq \mathrm{B}(0, R)$ for some positive real number R.
(5) The set S is *compact* if it is both closed and bounded.
(6) The set S is a *region* if it is an open set together with none, some, or all of its boundary points.

For example, the dashed boundary lines of the set S in Figure 1.1 indicate that it does not contain any boundary points. Consequently, this set is open.

Note that open and closed are not exclusive nor exhaustive. There are sets that are open and closed, such as the whole plane, and sets that are neither open nor closed, such as $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) > 0\}$. In complex analysis, we focus on open sets. We often write Ω for an open set.

We now present connectedness.

DEFINITION 1.8. A *polygonal path* is a finite sequence of finite line segments, where the end point of one line segment is the initial point of the next one. A *simple closed polygonal path* is a polygonal path that does not cross itself, but the final point of the last segment is the initial point of the first segment. The complement of a simple closed polygonal path is made up of two pieces: one, the *interior of the path*, is bounded, and the other, the *exterior*, is not.

DEFINITION 1.9. Let $X \subseteq \mathbb{C}$ be a subset of the complex plane.

(1) The set X is *polygonally path-connected* if any two points of X can be joined by a polygonal arc lying inside X.
(2) The set X is *simply polygonally connected* if it is polygonally path-connected and if the interior of every simple closed polygonal arc in X lies in X, that is, if "X has no holes".
(3) The set X is a *domain* if it is open and polygonally path-connected.

The set Ω in Figure 1.2 is polygonally path-connected, because any two points in Ω (such as z_1 and z_4) can be joined by a polygonal path. However, Ω is not simply polygonally connected, because part of the interior of the closed path shown going through z_5 is not in Ω.

3. Describing sets in the complex plane

EXERCISE 1.10. Suppose that $a, b, c, d \in \mathbb{C}$. Show that the set

$$\{z \in \mathbb{C} : |az + b| = |cz + d|\}$$

may be empty, a point, a line, a circle, or the whole complex plane, and all these possibilities occur for suitable values of the parameters a, b, c, d.

ANSWER.

EXERCISE 1.11. Sketch the set $\{z \in \mathbb{C} : |z - 3 - 2i| < 4,\ \mathrm{Re}(z) > 0\}$ in the complex plane. Is it open, closed, bounded, compact, polygonally path-connected, simply polygonally connected, a region, or a domain?

ANSWER.

$\triangle$

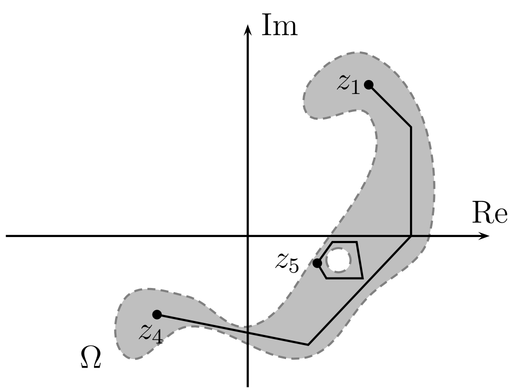

FIGURE 1.2. A polygonally path-connected, but not simply polygonally connected, set

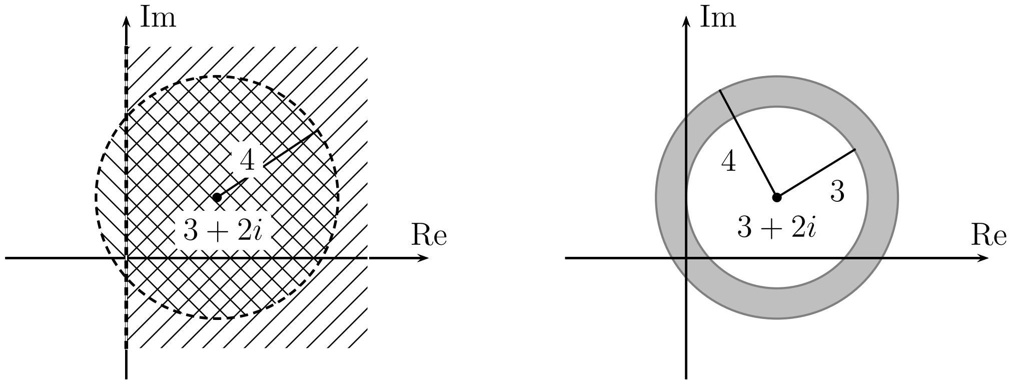

FIGURE 1.3. Two regions defined by inequalities

EXERCISE 1.12. Sketch the set $\{z \in \mathbb{C} : 4 \leq |z - 3 - 2i| \leq 5\}$ in the complex plane. Is it open, closed, bounded, compact, polygonally path-connected, simply polygonally connected, a region or a domain?

ANSWER.

Here is a more complicated example, related to conic sections.

EXERCISE 1.13. Sketch the set $\{z \in \mathbb{C} : |z + i| + |z - i| = 4\}$ in $\mathbb{C}$. Is it open or bounded? Describe the set $\{z \in \mathbb{C} : |z+i|+|z-i| < 4\}$. Is it open, closed, bounded, compact, polygonally path-connected, simply polygonally connected, a region or a domain?

ANSWER.

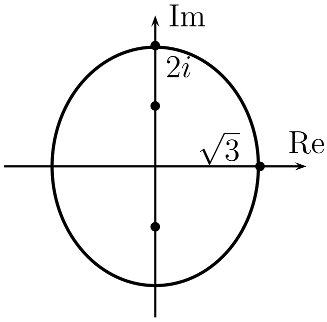

FIGURE 1.4. An ellipse

LECTURE 2

Functions of a complex variable

In this lecture, we introduce functions of a complex variable, and recall concepts such as domain and range. We examine some examples and consider the problem of estimating the size of a complex function.

1. Functions

In Mathematics, we often think of functions as machines: you give the machine a number, x say, press a button, and out comes $f(x)$. We sometimes write $x \mapsto f(x)$ to indicate that x is the input and $f(x)$ is the output.

- The *domain* of a function f, written Domain(f), is the set of all the numbers you are allowed to put in. Sometimes this is restricted in some way. If there is no explicit restriction, you should consider the *natural domain*, that is, the largest domain possible.
- A *codomain* is a set of numbers that includes all the numbers that you can get out, and perhaps more.
- The *range* or *image* of a function f, written Range(f), is the set of the numbers that you can get out, and no others.
- The *image* of a subset S of the domain of a function f, sometimes written $f(S)$, is the set of all possible values $f(s)$ as s varies over S.
- The *preimage* of a subset T of the codomain of a function f, sometimes written $f^{-1}(T)$, is the set of all x in Domain(f) such that $f(x) \in T$.

DEFINITION 2.1. A *complex function* is one whose domain, or whose range, or both, is a subset of the complex plane $\mathbb{C}$ that is not a subset of the real line $\mathbb{R}$. To emphasize that the domain is complex, not real, the expression *function of a complex variable* may be used. To emphasize that the range is complex, not real, the expression *complex-valued function* may be used.

REMARK 2.2. The word "domain" is then used in two different ways in this course. We say that a subset of $\mathbb{C}$ is a domain if it is open and connected which has nothing to do with a function. We just saw that the domain of a function f is the set of points where f is defined. Be careful to not confuse them and note that the domain of a function is not necessarily a domain in the sense of open and connected: for instance consider $f : \{0, 1\} \to \mathbb{C}, 0 \mapsto 0, 1 \mapsto 1$ and not that its domain is equal to two points which forms a nonconnected and nonopen subset of $\mathbb{C}$.

2. Examples of functions

Examples of functions of a complex variable include the real part function Re, the imaginary part function Im, the modulus function $z \mapsto |z|$, and the principal value of the argument Arg; these are all real-valued. Complex conjugation $z \mapsto \bar{z}$ is an example of a complex-valued function of a complex variable.

In this course, we are going to learn about a number of useful complex functions. Shortly we will define complex polynomials and then rational functions. In future lectures, we will define $\log z$, $\sin z$, and $\cosh z$ for a complex number z, and there are many other functions in the menagerie of complex functions.

EXERCISE 2.3. Suppose that $f(z) = 1/z$ for all $z \in \mathbb{C} \setminus \{0\}$, and that $g(z) = z$ for all $z \in \mathbb{C}$. Show that $f \circ f(z) = z$ for all $z \in \mathbb{C} \setminus \{0\}$. Is $f \circ f = g$?

ANSWER.

DEFINITION 2.4. A *complex polynomial* is a function $p : \mathbb{C} \to \mathbb{C}$ of the form

$$p(z) = a_d z^d + \cdots + a_1 z + a_0,$$

where $a_d, \ldots, a_1, a_0 \in \mathbb{C}$. If $a_d \neq 0$, we say that p is of *degree* d. A *rational function* is a quotient of polynomials.

Sums, differences, products and compositions of polynomials are polynomials.

THEOREM 2.5 (The fundamental theorem of algebra). *Every nonconstant complex polynomial p of degree d factorizes uniquely: there exist $\alpha_1, \alpha_2, \ldots, \alpha_d$ and c in $\mathbb{C}$ such that*

$$p(z) = c \prod_{j=1}^{d} (z - \alpha_j) \qquad \forall z \in \mathbb{C}.$$

Equivalently, every nonconstant complex polynomial has at least one root.

In the factorisation above, the roots α_j may occur more than once. Thus we could also write

$$p(z) = c \prod_{j=1}^{e} (z - \alpha_j)^{m_j},$$

where the α_j are all distinct, and the *multiplicities* m_j add to give the degree of p.

THEOREM 2.6 (Polynomial division and partial fractions). *Suppose that p and q are polynomials of degrees m and n. Then the rational function p/q may be written as a sum*

$$\frac{p(z)}{q(z)} = s(z) + \frac{r(z)}{q(z)},$$

where r and s are polynomials, and the degree of r is strictly less than the degree of q. Further, if

$$q(z) = c \prod_{j=1}^{e} (z - \beta_j)^{m_j},$$

then we may decompose the term r/q into partial fractions:

$$\frac{r(z)}{q(z)} = \sum_{j=1}^{e} \sum_{k=1}^{m_j} \frac{a_{jk}}{(z - \beta_j)^k}.$$

At this stage, we do not prove these results, which should be familiar, though perhaps not in this generality; we will give proofs later.

The natural domain of any complex polynomial is $\mathbb{C}$. Sometimes we cannot determine the range of a real polynomial exactly, because we cannot find maxima or minima exactly. However, for complex polynomials, things are easier.

EXERCISE 2.7. Suppose that p is a nonconstant complex polynomial. Show that the range of p is $\mathbb{C}$.

ANSWER.

3. Real and imaginary parts

To a function $f : S \to \mathbb{C}$, where $S \subseteq \mathbb{C}$, we associate two *real-valued* functions u and v of *two real variables*:

$$f(x + iy) = u(x, y) + iv(x, y).$$

Then $u(x, y) = \operatorname{Re} f(x + iy)$ and $v(x, y) = \operatorname{Im} f(x + iy)$. It is very useful and very important to be able to view a complex-valued function of a complex variable in this way.

EXERCISE 2.8. Suppose that $f(z) = z$ and that $g(z) = z^2$. Find the real and imaginary parts of f and g.

ANSWER.

EXERCISE 2.9. Suppose that $f(z) = z^3 + \overline{z} - 2$. Write the real and imaginary parts of this function as functions u and v of (x, y), where $z = x + iy$.

ANSWER.

EXERCISE 2.10. Suppose that $f(z) = 1/z$. Write the real and imaginary parts of this function as functions of x and y, where $z = x + iy$.

ANSWER.

Lecture 1

Sat 23 Sep 2023 16:28

LECTURE 0

Assumed Knowledge

This is a review of basic facts about complex numbers that ought to be familiar:

- the definition of complex numbers,
- their arithmetic,
- Cartesian and polar representations,
- the Argand diagram,
- de Moivre's theorem, and
- extracting nth roots of complex numbers.

Students who do not feel confident about this material need to do lots of exercises about these, such as those in the MATH1141 notes.

1. Complex numbers

DEFINITION 0.1. A *complex number* is an expression of the form $x + iy$, where x and y are real numbers. The *real part* of $x + iy$ is x and the *imaginary part* of $x + iy$ is y. We denote this by $\mathrm{Re}(x + iy) = x$ and $\mathrm{Im}(x + iy) = y$. The set of all complex numbers is denoted $\mathbb{C}$.

We often write $w = u + iv$ and $z = x + iy$, and work with w and z rather than $u + iv$ and $x + iy$. In this case, we write, for instance, $\mathrm{Re}(z) = x$ and $\mathrm{Im}(w) = v$. We abbreviate $x + i0$ and $0 + iy$ to x and iy, and $0 + i1$ to i.

DEFINITION 0.2. Suppose that $w = u + iv$ and $z = x + iy$, where $u, v, x, y \in \mathbb{R}$. Then we define the complex numbers $w + z$, $-z$, wz and, if $z \neq 0$, z^{-1} and w/z, as follows:

$$\begin{aligned}
w + z &= (u + x) + i(v + y) \\
-z &= -x + i(-y) \\
wz &= (ux - vy) + i(uy + vx) \\
z^{-1} &= (x^2 + y^2)^{-1}(x - iy) \\
w/z = wz^{-1} &= (x^2 + y^2)^{-1}[(ux + vy) + i(vx - uy)].
\end{aligned}$$

Then $i^2 = -1$ and $-z = (-1)z$.

We may combine these operations to make sense of more complicated expressions such as $w^m - z^n$, where m and n are integers.

PROPOSITION 0.3. *Complex numbers have the following properties:*

$$\begin{aligned}
z_1 + z_2 &= z_2 + z_1 && \forall z_1, z_2 \in \mathbb{C} \\
(z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3) && \forall z_1, z_2, z_3 \in \mathbb{C} \\
0 + z &= z && \forall z \in \mathbb{C} \\
(-z) + z &= 0 && \forall z \in \mathbb{C} \\
z_1 z_2 &= z_2 z_1 && \forall z_1, z_2 \in \mathbb{C} \\
(z_1 z_2) z_3 &= z_1 (z_2 z_3) && \forall z_1, z_2, z_3 \in \mathbb{C} \\
1z &= z && \forall z \in \mathbb{C} \\
z z^{-1} &= 1 && \forall z \in \mathbb{C} \setminus \{0\} \\
z_1(z_2 + z_3) &= z_1 z_2 + z_1 z_3 && \forall z_1, z_2, z_3 \in \mathbb{C}
\end{aligned}$$

The symbol $\forall$ is read "for all". This proposition shows that the complex numbers form a *field*.

DEFINITION 0.4. If $z = x + iy$, then $\overline{z}$, the *(complex) conjugate* of z, and $|z|$, the *modulus* of z, are defined to be $x - iy$ and $(x^2 + y^2)^{1/2}$.

Some further properties of complex numbers relate conjugates and moduli.

PROPOSITION 0.5. *For all* $z, z_1, z_2 \in \mathbb{C}$,

$$\begin{aligned}
&\text{(a)} \quad \operatorname{Re}(z) = \frac{z + \overline{z}}{2} && \text{(b)} \quad \operatorname{Im}(z) = \frac{z - \overline{z}}{2i} \\
&\text{(c)} \quad \overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2 && \text{(d)} \quad \overline{z_1 z_2} = \overline{z}_1 \overline{z}_2 \\
&\text{(e)} \quad \overline{-z} = -\overline{z} && \text{(f)} \quad \overline{z^{-1}} = (\overline{z})^{-1} \\
&\text{(g)} \quad |z_1 + z_2| \leq |z_1| + |z_2| && \text{(h)} \quad |z_1 z_2| = |z_1|\,|z_2| \\
&\text{(i)} \quad z\overline{z} = |z|^2 && \text{(j)} \quad z^{-1} = |z|^{-2}\overline{z}.
\end{aligned}$$

Inequality (g) is called the *triangle inequality*. If $|z| = 1$, then $z^{-1} = \overline{z}$, from (j).

PROOF. We only prove the triangle inequality, because this is hardest.

First, here is an algebraic proof. Recall that $2ab \leq a^2 + b^2$ for real numbers a and b. Taking a and b to be $x_1 y_2$ and $x_2 y_1$, we deduce that

$$2x_1 x_2 y_1 y_2 \leq x_1^2 y_2^2 + x_2^2 y_1^2,$$

and hence

$$x_1^2 x_2^2 + y_1^2 y_2^2 + 2x_1 x_2 y_1 y_2 \leq x_1^2 x_2^2 + y_1^2 y_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2 = (x_1^2 + y_1^2)(x_2^2 + y_2^2),$$

so, taking square roots,

$$x_1 x_2 + y_1 y_2 \leq |z_1|\,|z_2|\,.$$

Finally, $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$, and so

$$\begin{aligned}
|z_1 + z_2|^2 &= (x_1 + x_2)^2 + (y_1 + y_2)^2 \\
&= x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2(x_1 x_2 + y_1 y_2) \\
&\leq |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2| \\
&= \big(|z_1| + |z_2|\big)^2;
\end{aligned}$$

the triangle inequality follows by taking square roots.

Alternatively, here is a geometric version. Consider the triangle whose vertices are w, z, and $w+z$, and the parallelogram with vertices 0, w, z, and $w+z$. The side of the parallelogram that joins w to $w+z$ is congruent to the side joining 0 to z, and the side of the parallelogram that joins z to $w+z$ is congruent to the side joining 0 to w. So we are just asserting the obvious fact that the length of one side of a triangle is less than the sum of the other two sides. $\square$

2. Euler's formula

The usual exponential function has a Taylor series:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

so at least formally, for a real number θ,

$$\begin{aligned} e^{i\theta} &= 1 + i\frac{\theta}{1!} - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots \\ &= \Big(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\Big) + i\Big(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\Big) \\ &= \cos\theta + i\sin\theta, \end{aligned}$$

from the Taylor series for the cosine and sine functions.

Later we will make this rigorous and use power series very effectively. This observation leads us to make the following definition.

Definition 0.6. We define $e^{i\theta}$ to be $\cos\theta + i\sin\theta$, for any real number θ.

Suppose that $z = x + iy \neq 0$. Write r instead of $|z|$. Then $(x/r, y/r)$ lies on the unit circle in the Cartesian plane, so $(x/r, y/r) = (\cos\theta, \sin\theta)$ for an appropriate choice of θ, and hence

$$z = r\Big(\frac{x}{r} + i\frac{y}{r}\Big) = r\big(\cos\theta + i\sin\theta\big) = re^{i\theta}.$$

Definition 0.7. The *Cartesian form* of a complex number z is its representation in the form $x + iy$, where x and y are real. The *polar form* of a complex number z is its representation in the form $re^{i\theta}$, where $r \geq 0$ and $\theta \in \mathbb{R}$. The number θ is called the *argument* of z, and is written $\arg(z)$.

Lemma 0.8. *Suppose that* $r, s \in \mathbb{R}^+$ *and* $\theta, \phi \in \mathbb{R}$. *Then*

$$r(\cos\theta + i\sin\theta) = s(\cos\phi + i\sin\phi)$$

if and only if $r = s$ *and* $\theta - \phi = 2k\pi$ *for some* $k \in \mathbb{Z}$.

This lemma follows immediately from trigonometry. It tells us that the argument of a nonzero complex number z is ambiguous. The next definition is to avoid this ambiguity.

Definition 0.9. The *principal value* of the argument of a nonzero complex number z, written $\operatorname{Arg}(z)$, is the unique number θ such that $z = |z|\, e^{i\theta}$ and $-\pi < \theta \leq \pi$.

We do not define the argument of 0.

3. The Argand Diagram

Suppose that $w = u + iv$. Then to the complex number w we associate the point in the Cartesian plane with Cartesian coordinates (u, v). When we do this, we call the axes the real axis and the imaginary axis. See Figure 0.1.

Geometrically, $|w|$ is the length of the line joining w to O, and $\arg(w)$ is the angle between this line and the positive real axis (taking the anticlockwise direction to be positive). Further, $|w - z|$ is the length of the line joining w and z.

Adding two complex numbers corresponds to vector addition in the plane. Multiplying by r in $\mathbb{R}^+$ dilates by a factor of r, and multiplying by the complex number $e^{i\theta}$ (where $\theta \in \mathbb{R}$) rotates (anticlockwise) through the angle θ.

4. De Moivre's formula

THEOREM 0.10. *If $\theta, \phi \in \mathbb{R}$, then*

$$(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos(\theta+\phi) + i\sin(\theta+\phi).f$$

PROOF. For all $\theta, \phi \in \mathbb{R}$,

$$\begin{aligned}&(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\\ &= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)\\ &= \cos(\theta+\phi) + i\sin(\theta+\phi),\end{aligned}$$

as required. □

COROLLARY 0.11 (de Moivre's formula). *If $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$, then*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{0.1}$$

PROOF. This is obviously true if $n = 0$ or 1. The result may be proved for $n \in \mathbb{Z}^+$ by induction. Suppose that $k \in \mathbb{Z}^+$ and formula (0.1) holds when $n = k$, i.e.,

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta.$$

Then by Theorem 0.10,

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^k\\ &= (\cos\theta + i\sin\theta)(\cos k\theta + i\sin k\theta)\\ &= \cos(\theta + k\theta) + i\sin(\theta + k\theta)\\ &= \cos(k+1)\theta + i\sin(k+1)\theta,\end{aligned}$$

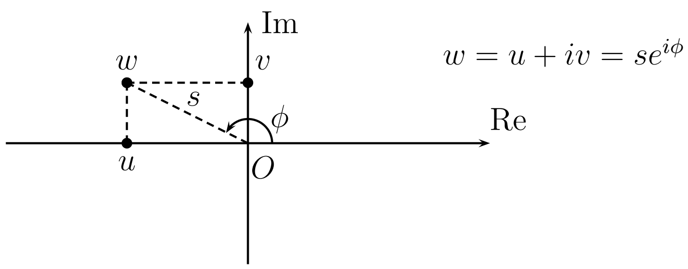

FIGURE 0.1. The complex plane

so the result holds when $n = k + 1$. By induction, the result holds for all $n \in \mathbb{Z}^+$.

To prove the result when $n \in \mathbb{Z}^-$, we use the fact that if $|z| = 1$, then $z\overline{z} = 1$, so $\overline{z} = z^{-1}$. That is,

$$(\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta),$$

as required. $\square$

In polar notation, Theorem 0.10 and Corollary 0.11 become

$$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi} \qquad \text{and} \qquad (e^{i\theta})^n = e^{in\theta}.$$

These are more "obvious" and easier to remember than the trigonometric formulae.

5. Roots of complex numbers

We use complex exponentials to find roots of complex numbers. Fix $w \in \mathbb{C}$ and $n \in \mathbb{Z}^+$, and suppose that $w = z^n$ for some $z \in \mathbb{C}$. Then z is called *an nth root of w*. Write z as $re^{i\theta}$ and w as $se^{i\phi}$. Then

$$se^{i\phi} = w = z^n = r^n e^{in\theta}.$$

From Lemma 0.8, $r = s^{1/n}$ and $n\theta = \phi + 2k\pi$ for some $k \in \mathbb{Z}$. Therefore $z = s^{1/n}e^{i\theta}$, where

$$\theta = \frac{\phi}{n} + \frac{2k\pi}{n}$$

for some $k \in \mathbb{Z}$. If $k = nl + r$, where $l \in \mathbb{Z}$ and $r = 0, 1, 2, \ldots, n-1$, then

$$e^{(\phi/n + 2k\pi/n)} = e^{(\phi/n + 2r\pi/n + 2l\pi)} = e^{(\phi/n + 2r\pi/n)},$$

so we get the same value of z by taking $k = nl + r$ as by taking $k = r$. Thus, to get all n possible values of z, it suffices to take the first n values of k, or any n consecutive values, or indeed any n values of k which give the n possible different remainders when divided by n.

The nth roots of any nonzero complex number w are uniformly spaced around the circle with centre 0 and radius $|w|^{1/n}$. For example, Figure 0.2 shows the seventh roots of unity. A symmetry argument shows that the sum of all these numbers is 0.

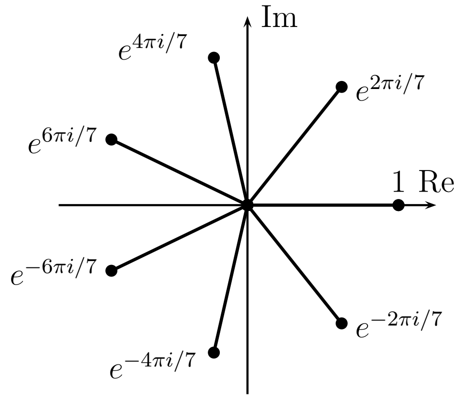

FIGURE 0.2. The seventh roots of unity

6. History†

Leonhard Euler found the formula that bears his name in 1748; he was one of the most prolific mathematicians ever. Some mathematicians did not believe that the geometric representation afforded by the Argand diagram was legitimate. The story of de Moivre's death is very curious. For more information, see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/`
and then find Euler, Argand, and de Moivre.

LECTURE 1

Inequalities and Sets of Complex Numbers

In complex analysis, we consider functions whose domains or ranges or both are regions in the complex plane. So to be able to discuss functions, we need to be able to describe regions. Curves and regions in the complex plane are often described by equalities and inequalities involving $|\cdot|$, Arg, Re, Im, ….

In the first part of this lecture, we review some equalities and inequalities. Then we discuss different types of regions. Finally, we consider some examples.

1. Equalities and inequalities

We begin with a lemma that is related to the cosine rule. It implies that $\operatorname{Re}(w\bar{z})$ is the inner product of the vectors represented by the complex numbers w and z.

LEMMA 1.1. *For all complex numbers w and z,*

$$|w+z|^2 = |w|^2 + 2\operatorname{Re}(w\bar{z}) + |z|^2.$$

PROOF. Observe that

$$\begin{aligned}|w+z|^2 &= (w+z)(\bar{w}+\bar{z}) = w\bar{w} + w\bar{z} + \bar{w}z + z\bar{z}\\ &= |w|^2 + w\bar{z} + (w\bar{z})^{-} + |z|^2 = |w|^2 + 2\operatorname{Re}(w\bar{z}) + |z|^2,\end{aligned}$$

as required. □

The triangle inequality is one of the most useful results about complex numbers. It states:

$$|w+z| \le |w| + |z| \qquad \forall w, z \in \mathbb{C}.$$

Here are a variation on the triangle inequality, sometimes called the circle inequality.

LEMMA 1.2. *For all complex numbers w and z,*

$$\big||w| - |z|\big| \le |w-z|.$$

PROOF. Observe that $w = (w-z)+z$, so $|w| \le |w-z| + |z|$ by the triangle inequality, and hence

$$|w| - |z| \le |w-z|. \tag{1.1}$$

Interchanging the roles of w and z in (1.1),

$$|z| - |w| \le |z-w|.$$

Combining these inequalities and recalling that $|w-z| = |z-w|$, we see that

$$\big||w| - |z|\big| = \max\{|w|-|z|, |z|-|w|\} \le |w-z|,$$

as required.

Alternatively, consider points on circles of radii $|z|$ and $|w|$, and compare the distance between the points with the difference of the radii. □

Recall that if $z = x + iy$, then e^z is defined to be $e^x(\cos y + i \sin y)$. Then $e^w e^z = e^{w+z}$ for all complex numbers w and z. Here is another very useful result.

LEMMA 1.3. *If $z \in \mathbb{C}$, then*

$$|e^z| = e^{\operatorname{Re}(z)}.$$

PROOF. See Problem Sheet 1. □

LEMMA 1.4. *For all real numbers θ,*

$$\left|e^{i\theta} - 1\right| \le |\theta|.$$

PROOF. See Problem Sheet 1. □

2. Properties of sets

DEFINITION 1.5. The *open ball* with centre z_0 and radius ε, written $\mathrm{B}(z_0, \varepsilon)$, is the set $\{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$.

The *punctured open ball* with centre z_0 and radius ε, written $\mathrm{B}^\circ(z_0, \varepsilon)$, is the set $\{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$.

Sometimes these sets are called discs rather than balls.

DEFINITION 1.6. Suppose that $S \subseteq \mathbb{C}$. For any point z_0 in $\mathbb{C}$, there are three mutually exclusive and exhaustive possibilities:

(1) When the positive real number ε is sufficiently small, $\mathrm{B}(z_0, \varepsilon)$ is a subset of S, that is, $\mathrm{B}(z_0, \varepsilon) \cap S = \mathrm{B}(z_0, \varepsilon)$. In this case, z_0 is an *interior point* of S.
(2) When the positive real number ε is sufficiently small, $\mathrm{B}(z_0, \varepsilon)$ does not meet S, that is, $\mathrm{B}(z_0, \varepsilon) \cap S = \emptyset$. In this case, z_0 is an *exterior point* of S.
(3) No matter how small the positive real number ε is, neither of the above holds, that is, $\emptyset \subset \mathrm{B}(z_0, \varepsilon) \cap S \subset \mathrm{B}(z_0, \varepsilon)$. In this case, z_0 is a *boundary point* of S.

These definitions are illustrated in Figure 1.1. We consider points z_1, z_2 and z_3.

If the radius of the ball centred at z_1 is small enough, then the ball lies inside the set S, and $\mathrm{B}(z_1, \varepsilon) \cap S = \mathrm{B}(z_1, \varepsilon)$. Thus z_1 is an interior point.

If the radius of the ball centred at z_2 is small enough, then the ball lies outside the set S, and $\mathrm{B}(z_2, \varepsilon) \cap S$ is empty. Thus z_2 is an exterior point.

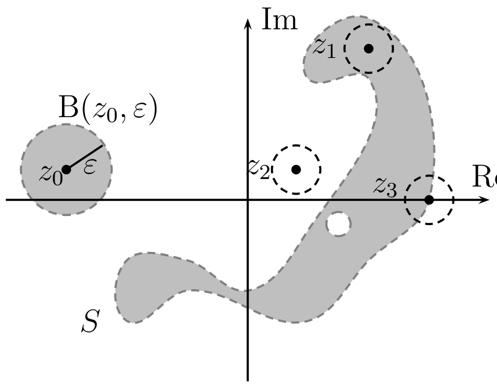

FIGURE 1.1. The ball with centre z_0 and radius ε, and interior, exterior and boundary points z_1, z_2 and z_3 of the set S

No matter how small the radius of the ball centred at z_3 is, part of the ball lies inside S, and part lies outside S, and $\mathrm{B}(z_3, \varepsilon) \cap S$ is neither empty nor all of $\mathrm{B}(z_3, \varepsilon)$. Thus z_3 is a boundary point.

Definition 1.7. Suppose that $S \subseteq \mathbb{C}$.

(1) The set S is *open* if all its points are interior points.
(2) The set S is *closed* if it contains all of its boundary points, or equivalently, if its complement $\mathbb{C} \setminus S$ is open.
(3) The *closure* of S, written $\bar{S}$, is the set consisting of all the points of S together with all its boundary points.
(4) The set S is *bounded* if $S \subseteq \mathrm{B}(0, R)$ for some positive real number R.
(5) The set S is *compact* if it is both closed and bounded.
(6) The set S is a *region* if it is an open set together with none, some, or all of its boundary points.

For example, the dashed boundary lines of the set S in Figure 1.1 indicate that it does not contain any boundary points. Consequently, this set is open.

Note that open and closed are not exclusive nor exhaustive. There are sets that are open and closed, such as the whole plane, and sets that are neither open nor closed, such as $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) > 0\}$. In complex analysis, we focus on open sets. We often write Ω for an open set.

We now present connectedness.

Definition 1.8. A *polygonal path* is a finite sequence of finite line segments, where the end point of one line segment is the initial point of the next one. A *simple closed polygonal path* is a polygonal path that does not cross itself, but the final point of the last segment is the initial point of the first segment. The complement of a simple closed polygonal path is made up of two pieces: one, the *interior of the path*, is bounded, and the other, the *exterior*, is not.

Definition 1.9. Let $X \subseteq \mathbb{C}$ be a subset of the complex plane.

(1) The set X is *polygonally path-connected* if any two points of X can be joined by a polygonal arc lying inside X.
(2) The set X is *simply polygonally connected* if it is polygonally path-connected and if the interior of every simple closed polygonal arc in X lies in X, that is, if "X has no holes".
(3) The set X is a *domain* if it is open and polygonally path-connected.

The set Ω in Figure 1.2 is polygonally path-connected, because any two points in Ω (such as z_1 and z_4) can be joined by a polygonal path. However, Ω is not simply polygonally connected, because part of the interior of the closed path shown going through z_5 is not in Ω.

3. Describing sets in the complex plane

Exercise 1.10. Suppose that $a, b, c, d \in \mathbb{C}$. Show that the set

$$\{z \in \mathbb{C} : |az + b| = |cz + d|\}$$

may be empty, a point, a line, a circle, or the whole complex plane, and all these possibilities occur for suitable values of the parameters a, b, c, d.

ANSWER.

EXERCISE 1.11. Sketch the set $\{z \in \mathbb{C} : |z - 3 - 2i| < 4,\ \mathrm{Re}(z) > 0\}$ in the complex plane. Is it open, closed, bounded, compact, polygonally path-connected, simply polygonally connected, a region, or a domain?

ANSWER.

$\triangle$

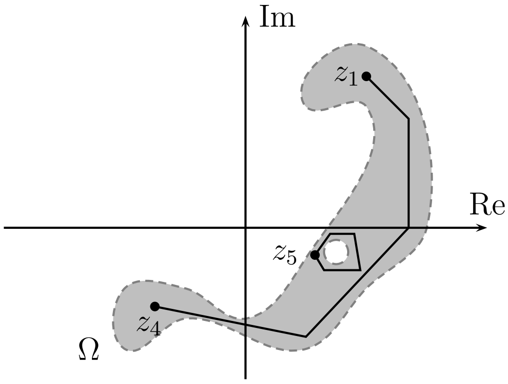

FIGURE 1.2. A polygonally path-connected, but not simply polygonally connected, set

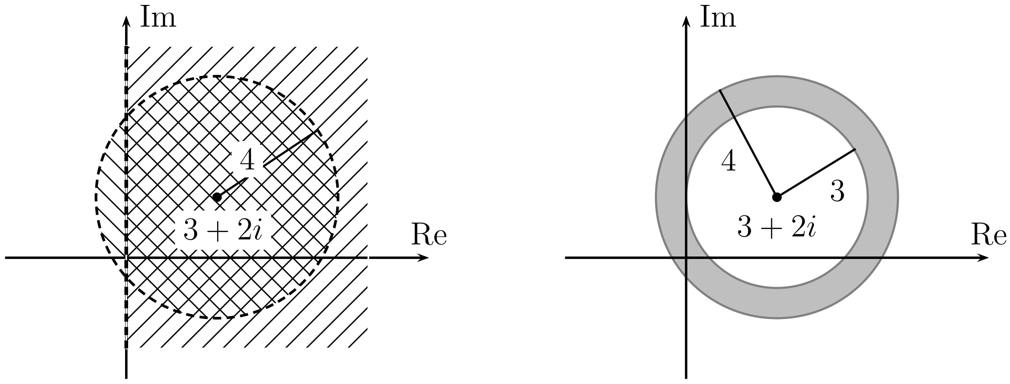

FIGURE 1.3. Two regions defined by inequalities

EXERCISE 1.12. Sketch the set $\{z \in \mathbb{C} : 4 \leq |z - 3 - 2i| \leq 5\}$ in the complex plane. Is it open, closed, bounded, compact, polygonally path-connected, simply polygonally connected, a region or a domain?

ANSWER.

Here is a more complicated example, related to conic sections.

EXERCISE 1.13. Sketch the set $\{z \in \mathbb{C} : |z + i| + |z - i| = 4\}$ in $\mathbb{C}$. Is it open or bounded? Describe the set $\{z \in \mathbb{C} : |z+i|+|z-i| < 4\}$. Is it open, closed, bounded, compact, polygonally path-connected, simply polygonally connected, a region or a domain?

ANSWER.

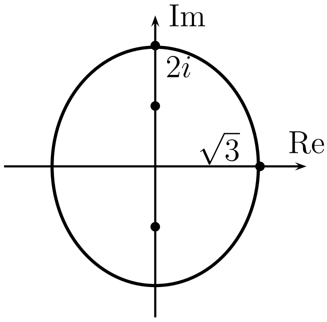

FIGURE 1.4. An ellipse

LECTURE 2

Functions of a complex variable

In this lecture, we introduce functions of a complex variable, and recall concepts such as domain and range. We examine some examples and consider the problem of estimating the size of a complex function.

1. Functions

In Mathematics, we often think of functions as machines: you give the machine a number, x say, press a button, and out comes $f(x)$. We sometimes write $x \mapsto f(x)$ to indicate that x is the input and $f(x)$ is the output.

- The *domain* of a function f, written $\mathrm{Domain}(f)$, is the set of all the numbers you are allowed to put in. Sometimes this is restricted in some way. If there is no explicit restriction, you should consider the *natural domain*, that is, the largest domain possible.
- A *codomain* is a set of numbers that includes all the numbers that you can get out, and perhaps more.
- The *range* or *image* of a function f, written $\mathrm{Range}(f)$, is the set of the numbers that you can get out, and no others.
- The *image* of a subset S of the domain of a function f, sometimes written $f(S)$, is the set of all possible values $f(s)$ as s varies over S.
- The *preimage* of a subset T of the codomain of a function f, sometimes written $f^{-1}(T)$, is the set of all x in $\mathrm{Domain}(f)$ such that $f(x) \in T$.

Definition 2.1. A *complex function* is one whose domain, or whose range, or both, is a subset of the complex plane $\mathbb{C}$ that is not a subset of the real line $\mathbb{R}$. To emphasize that the domain is complex, not real, the expression *function of a complex variable* may be used. To emphasize that the range is complex, not real, the expression *complex-valued function* may be used.

Remark 2.2. The word "domain" is then used in two different ways in this course. We say that a subset of $\mathbb{C}$ is a domain if it is open and connected which has nothing to do with a function. We just saw that the domain of a function f is the set of points where f is defined. Be careful to not confuse them and note that the domain of a function is not necessarily a domain in the sense of open and connected: for instance consider $f : \{0, 1\} \to \mathbb{C}, 0 \mapsto 0, 1 \mapsto 1$ and not that its domain is equal to two points which forms a nonconnected and nonopen subset of $\mathbb{C}$.

2. Examples of functions

Examples of functions of a complex variable include the real part function Re, the imaginary part function Im, the modulus function $z \mapsto |z|$, and the principal value of the argument Arg; these are all real-valued. Complex conjugation $z \mapsto \bar{z}$ is an example of a complex-valued function of a complex variable.

In this course, we are going to learn about a number of useful complex functions. Shortly we will define complex polynomials and then rational functions. In future lectures, we will define $\log z$, $\sin z$, and $\cosh z$ for a complex number z, and there are many other functions in the menagerie of complex functions.

EXERCISE 2.3. Suppose that $f(z) = 1/z$ for all $z \in \mathbb{C} \setminus \{0\}$, and that $g(z) = z$ for all $z \in \mathbb{C}$. Show that $f \circ f(z) = z$ for all $z \in \mathbb{C} \setminus \{0\}$. Is $f \circ f = g$?

ANSWER.

DEFINITION 2.4. A *complex polynomial* is a function $p : \mathbb{C} \to \mathbb{C}$ of the form

$$p(z) = a_d z^d + \cdots + a_1 z + a_0,$$

where $a_d, \ldots, a_1, a_0 \in \mathbb{C}$. If $a_d \neq 0$, we say that p is of *degree d*. A *rational function* is a quotient of polynomials.

Sums, differences, products and compositions of polynomials are polynomials.

THEOREM 2.5 (The fundamental theorem of algebra). *Every nonconstant complex polynomial p of degree d factorizes uniquely: there exist $\alpha_1, \alpha_2, \ldots, \alpha_d$ and c in $\mathbb{C}$ such that*

$$p(z) = c \prod_{j=1}^{d} (z - \alpha_j) \qquad \forall z \in \mathbb{C}.$$

Equivalently, every nonconstant complex polynomial has at least one root.

In the factorisation above, the roots α_j may occur more than once. Thus we could also write

$$p(z) = c \prod_{j=1}^{e} (z - \alpha_j)^{m_j},$$

where the α_j are all distinct, and the *multiplicities* m_j add to give the degree of p.

THEOREM 2.6 (Polynomial division and partial fractions). *Suppose that p and q are polynomials of degrees m and n. Then the rational function p/q may be written as a sum*

$$\frac{p(z)}{q(z)} = s(z) + \frac{r(z)}{q(z)},$$

where r and s are polynomials, and the degree of r is strictly less than the degree of q. Further, if

$$q(z) = c \prod_{j=1}^{e} (z - \beta_j)^{m_j},$$

then we may decompose the term r/q into partial fractions:

$$\frac{r(z)}{q(z)} = \sum_{j=1}^{e} \sum_{k=1}^{m_j} \frac{a_{jk}}{(z - \beta_j)^k}.$$

At this stage, we do not prove these results, which should be familiar, though perhaps not in this generality; we will give proofs later.

The natural domain of any complex polynomial is $\mathbb{C}$. Sometimes we cannot determine the range of a real polynomial exactly, because we cannot find maxima or minima exactly. However, for complex polynomials, things are easier.

EXERCISE 2.7. Suppose that p is a nonconstant complex polynomial. Show that the range of p is $\mathbb{C}$.

ANSWER.

3. Real and imaginary parts

To a function $f : S \to \mathbb{C}$, where $S \subseteq \mathbb{C}$, we associate two *real-valued* functions u and v of *two real variables*:

$$f(x + iy) = u(x, y) + iv(x, y).$$

Then $u(x, y) = \operatorname{Re} f(x + iy)$ and $v(x, y) = \operatorname{Im} f(x + iy)$. It is very useful and very important to be able to view a complex-valued function of a complex variable in this way.

EXERCISE 2.8. Suppose that $f(z) = z$ and that $g(z) = z^2$. Find the real and imaginary parts of f and g.

ANSWER.

EXERCISE 2.9. Suppose that $f(z) = z^3 + \overline{z} - 2$. Write the real and imaginary parts of this function as functions u and v of (x, y), where $z = x + iy$.

ANSWER.

EXERCISE 2.10. Suppose that $f(z) = 1/z$. Write the real and imaginary parts of this function as functions of x and y, where $z = x + iy$.

ANSWER.

EXERCISE 2.11. Write e^z in the form $u(x,y) + iv(x,y)$, where $z = x + iy$.

ANSWER.

Sometimes we view the complex number z in polar coordinates, that is, we write $z = re^{i\theta}$. In this case, we consider

$$f(z) = u(r,\theta) + iv(r,\theta).$$

EXERCISE 2.12. Write e^z in the form $u(r,\theta) + iv(r,\theta)$, where $z = re^{i\theta}$.

ANSWER.

4. The function $z \mapsto 1/z$

It is obvious that if $w = 1/z$, then $z = 1/w$, and the function $z \mapsto 1/z$ is one-to-one (injective). Further, the domain and the range of the function are both equal to $\mathbb{C} \setminus \{0\}$.

EXERCISE 2.13. Suppose that z varies on the line $x = 1$, and let $w = 1/z$. Show that w varies on the circle $|w - \frac{1}{2}| = \frac{1}{2}$.

ANSWER.

We can reverse the argument and show that every point on the circle except 0 arises in this way. Thus the image of the line is the circle with the point 0 removed.

5. Fractional linear transformations

The fractional linear transformations form an important family of complex functions. These are the functions of the form

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. We will study these functions in more detail later, but at the moment we just point out that if f is a fractional linear transformation and z varies on a line, then $f(z)$ varies on a line or on a circle. The same holds if z varies along a circle. Note that when $z \to -d/c$, then $cz + d \to 0$ and $f(z) \to \infty$ (we will define limits formally later). We can tell whether $f(z)$ varies on a line or on a circle as follows: if the points where z varies include $-d/c$, then

Exercise 2.13: We can set $w = u + iv$. If z is on the line $x = 1$, then $Re(z) = 1$, whence $Re(\frac{1}{w}) = 1$ (and $w \neq 0$). Now

$$1 = Re(\frac{\overline{w}}{\mid w \mid^2}) = \frac{u}{v^2 + u^2}.$$

It follows that $u^2 + v^2 = u$, whence $(u - \frac{1}{2})^2 + v_2 = \frac{1}{4}$, and the result follows we can reverse this mapping and show that every point on the circle except 0 arises this way. Thus the image of the line is the circle with the point 0 removed.