MATH3711 2024 probelm set solutions

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1 Problem set 1

1.1 Problem 1

1. Given the following equation in a group

$$x^{-1}yxz^2 = 1.$$

solve for y.

Solution. Let the group that these three elements x, y, z belong to be G.

$$\begin{aligned}
x^{-1}yxz^2 &= 1 \\
xx^{-1}yxz^2 &= x \\
1_G yxz^2 &= x \\
1_G yxz^2 z^{-2} &= xz^{-2} \\
1yxz^2 z^{-2} &= xz^{-2} \\
yx1_G &= xz^{-2} \\
yxx^{-1} &= xz^{-2}x^{-1} \\
yxx^{-1} &= xz^{-2}x^{-1} \\
y1_G &= xz^{-2}x^{-1} \\
y &= xz^{-2}x^{-1}.
\end{aligned}$$

Note: ask if need to explicitly state all these algebraic manipulations. Also clarify if need to be explicit with identity with group as subscript.

1.2 Problem 2

In any group G, show that $(g^{-1})^{-1} = g$ for any $g \in G$. Show that for any $m, n \in \mathbb{Z}$ that $g^m g^n = g^{m+n}$ and $(g^m)^n = g^{mn}$.

Solution. Let $m, n \in \mathbb{Z}$ then,

$$\begin{aligned}
g^m g^n &= (g^m)(g^n) \\
&= \underbrace{(gg \dots g)}_{m \text{ terms}} \underbrace{(gg \dots g)}_{n \text{ terms}} \\
&= \underbrace{gg \dots g \, gg \dots g}_{m+n \text{ terms}} \\
&= g^{m+n}.
\end{aligned}$$

Hence, this is true for all $m, n \in \mathbb{Z}$.

Now again let $m, n \in \mathbb{Z}$, then,

$$\begin{aligned}
(g^m)^n &= \underbrace{g^m \dots g^m}_{n \text{ terms}} \\
&= \underbrace{\underbrace{(g \dots g)}_{m \text{ terms}} \dots \underbrace{(g \dots g)}_{m \text{ terms}}}_{n \text{ times}} \\
&= \underbrace{g \dots g}_{nm \text{ terms}} \\
&= g^{nm} \\
&= g^{mn}
\end{aligned}$$

.

Now let $g \in G$ then,

$$g^{-1}g = 1.$$

So, this is true for all g. Hence, $(g^{-1})^{-1} = g$ for all $g \in G$.

1.3 Problem 3

Prove disprove or salvage if possible the following statement. Given subgroups $J, H \leq G$. The union $H \cup J$ is a subgroup of G.

Solution. The union $H \cup J$ is not necessarily a subgroup of G. We give a counter example to disprove this statement.

Consider the groups $\mathbb{Z}/3$ and $\mathbb{Z}/4 \leq \mathbb{Z}$ equipped with integer addition as the group binary operation. Now, $2 \in \mathbb{Z}$ and $3 \in 4$. $2 \times 3 = 6 \notin \frac{\mathbb{Z}}{3} \cup \frac{\mathbb{Z}}{4}$. Therefore this statement is false. However, we can salvage this statement by considering the intersection $H \cap J$ instead of the union. This is indeed a subgroup of G. Following is the proof.

Proof. Since, $1_G \in H$ and $1_G \in J$. $1_G \in H \cap J$. Therefore, the identity element is in $H \cap J$.

Now, let $x, y \in H \cap J$. This means $x, y \in J \Rightarrow xy \in J$ by closure under multiplication and $x, y \in H \Rightarrow xy \in H$ by closure under multiplication. Hence $xy \in H \cap J$. This is true for all $xy \in H \cap J$. Hence $H \cap J$ is closed under group multiplication.

Remains to prove closure under group inverse. Let $x \in H \cap J$. Then $x \in H \Rightarrow x^{-1} \in H$ and $x \in J \Rightarrow x^{-1} \in J$ due to closure under group inverse of H and J. Hence $x^{-1} \in H \cap J$. Hence, this is true for all $x \in H \cap J$.

We have proven closure under group multiplication and group inverse and also existence of identity. Hence by subgroup theorem $H \cap J \leq G$. $\square$

1.4 Quesiton 4

Let G be a group and $H \subseteq G$. Show that H is a subgroup iff it is non empty and for every $h, j \in H$ we have $hj^{-1} \in H$. This gives an alternate characterization for subgroups. (there is an analogue here for subspaces do you know it?).

Solution. Lets prove the forwards implication i.e $H \leq G \Rightarrow H$ is non empty and $hj^{-1} \in H$.

Since H is a subgroup we know it contains an identity element so it must be nonempty.

Now, let $h, j \in H$. Since H is closed under inverses we know that $j^{-1} \in H$. Also, H is closed under group multiplication. Therefore, $hj^{-1} \in H$.

Therefore, this is true for all $h, j \in H$. Hence for all $h, j \in H$ we have $hj^{-1} \in H$.

Now, we prove the reverse implication i.e H is non empty and for all $h, j \in H$ $hj^{-1} \in H \Rightarrow H \leq G$.

Since, H is non empty we know that there exists an element $h \in H$. We also know that $h, j^{-1} \in H$ for all $h, j \in H$. Hence, $hh^{-1} \in H$. Therefore $1_G \in H$. Therefore, H contains the identity element. Note: Ask about this kind of variable naming. Is this too confusing perhaps ?.

Now, we show existence of inverse. Let $h \in H$ and we know $1_G \in H$, $1_G h^{-1} \in H$. Hence $h^{-1} \in H$.

Finally we show closure under group multiplication. Now, let $h, j \in H$. Therefore, by closure of inverse proven above $j^{-1} \in H$. Hence, $h(j^{-1})^{-1} \in H \Rightarrow hj \in H$. Therefore for all $h, j \in H$ we have $hj \in H$. Hence we have proven closure under group multiplication, group inverse and existence of identity.

Therefore, by subgroup theorem we have $H \leq G$.

Hence we have proven both implications and therefore the statement.

The vector space analogue is that $V \subseteq W$ is a vector subspace of W with scalar field F equipped with vector addition $(+)$ and scalar multiplication $(*)$. iff V is non empty and $\forall \mathbf{x}, \mathbf{y} \in V$ and $\lambda, \mu \in F$ we have $\lambda \mathbf{x} + \mu \mathbf{y} \in V$. Note: ask if you were allowed to assume subgroup theorem here since I have.

1.5 Question 5

Let G be a group with group multiplication $\mu : G \times G \to G$. We define a new group multiplication by