MATH3711 2024 probelm set solutions

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1 Problem set 1

1.1 Problem 1

1. Given the following equation in a group

$$x^{-1}yxz^2 = 1.$$

solve for y.

Solution. Let the group that these three elements x, y, z belong to be G.

$$\begin{aligned}
x^{-1}yxz^2 &= 1 \\
xx^{-1}yxz^2 &= x \\
1_G yxz^2 &= x \\
1_G yxz^2 z^{-2} &= xz^{-2} \\
1yxz^2 z^{-2} &= xz^{-2} \\
yx1_G &= xz^{-2} \\
yxx^{-1} &= xz^{-2}x^{-1} \\
yxx^{-1} &= xz^{-2}x^{-1} \\
y1_G &= xz^{-2}x^{-1} \\
y &= xz^{-2}x^{-1}.
\end{aligned}$$

Note: ask if need to explicitly state all these algebraic manipulations. Also clarify if need to be explicit with identity with group as subscript.

1.2 Problem 2

In any group G, show that $(g^{-1})^{-1} = g$ for any $g \in G$. Show that for any $m, n \in \mathbb{Z}$ that $g^m g^n = g^{m+n}$ and $(g^m)^n = g^{mn}$.

Solution. Let $m, n \in \mathbb{Z}$ then,

$$\begin{aligned}
g^m g^n &= (g^m)(g^n) \\
&= \underbrace{(gg \ldots g)}_{m \text{ terms}} \underbrace{(gg \ldots g)}_{n \text{ terms}} \\
&= \underbrace{gg \ldots g \, gg \ldots g}_{m+n \text{ terms}} \\
&= g^{m+n}.
\end{aligned}$$

Hence, this is true for all $m, n \in \mathbb{Z}$.

Now again let $m, n \in \mathbb{Z}$, then,

$$\begin{aligned}
(g^m)^n &= \underbrace{g^m \dots g^m}_{n \text{ terms}} \\
&= \underbrace{\underbrace{(g \dots g)}_{m \text{ terms}} \dots \underbrace{(g \dots g)}_{m \text{ terms}}}_{n \text{ times}} \\
&= \underbrace{g \dots g}_{nm \text{ terms}} \\
&= g^{nm} \\
&= g^{mn}
\end{aligned}$$

.

Now let $g \in G$ then,

$$g^{-1}g = 1.$$

So, this is true for all g. Hence, $(g^{-1})^{-1} = g$ for all $g \in G$.

1.3 Problem 3

Prove disprove or salvage if possible the following statement. Given subgroups $J, H \leq G$. The union $H \cup J$ is a subgroup of G.

Solution. The union $H \cup J$ is not necessarily a subgroup of G. We give a counter example to disprove this statement.

Consider the groups $\mathbb{Z}/3$ and $\mathbb{Z}/4 \leq \mathbb{Z}$ equipped with integer addition as the group binary operation. Now, $2 \in \mathbb{Z}$ and $3 \in 4$. $2 \times 3 = 6 \notin \frac{\mathbb{Z}}{3} \cup \frac{\mathbb{Z}}{4}$. Therefore this statement is false. However, we can salvage this statement by considering the intersection $H \cap J$ instead of the union. This is indeed a subgroup of G. Following is the proof.

Proof. Since, $1_G \in H$ and $1_G \in J$. $1_G \in H \cap J$. Therefore, the identity element is in $H \cap J$.

Now, let $x, y \in H \cap J$. This means $x, y \in J \Rightarrow xy \in J$ by closure under multiplication and $x, y \in H \Rightarrow xy \in H$ by closure under multiplication. Hence $xy \in H \cap J$. This is true for all $xy \in H \cap J$. Hence $H \cap J$ is closed under group multiplication.

Remains to prove closure under group inverse. Let $x \in H \cap J$. Then $x \in H \Rightarrow x^{-1} \in H$ and $x \in J \Rightarrow x^{-1} \in J$ due to closure under group inverse of H and J. Hence $x^{-1} \in H \cap J$. Hence, this is true for all $x \in H \cap J$.

We have proven closure under group multiplication and group inverse and also existence of identity. Hence by subgroup theorem $H \cap J \leq G$. $\square$

1.4 Quesiton 4

Let G be a group and $H \subseteq G$. Show that H is a subgroup iff it is non empty and for every $h, j \in H$ we have $hj^{-1} \in H$. This gives an alternate characterization for subgroups. (there is an analogue here for subspaces do you know it?).

Solution. Lets prove the forwards implication i.e $H \leq G \Rightarrow H$ is non empty and $hj^{-1} \in H$.

Since H is a subgroup we know it contains an identity element so it must be nonempty.

Now, let $h, j \in H$. Since H is closed under inverses we know that $j^{-1} \in H$. Also, H is closed under group multiplication. Therefore, $hj^{-1} \in H$.

Therefore, this is true for all $h, j \in H$. Hence for all $h, j \in H$ we have $hj^{-1} \in H$.

Now, we prove the reverse implication i.e H is non empty and for all $h, j \in H$ $hj^{-1} \in H \Rightarrow H \leq G$.

Since, H is non empty we know that there exists an element $h \in H$. We also know that $h, j^{-1} \in H$ for all $h, j \in H$. Hence, $hh^{-1} \in H$. Therefore $1_G \in H$. Therefore, H contains the identity element. Note: Ask about this kind of variable naming. Is this too confusing perhaps ?.

Now, we show existence of inverse. Let $h \in H$ and we know $1_G \in H$, $1_G h^{-1} \in H$. Hence $h^{-1} \in H$.

Finally we show closure under group multiplication. Now, let $h, j \in H$. Therefore, by closure of inverse proven above $j^{-1} \in H$. Hence, $h(j^{-1})^{-1} \in H \Rightarrow hj \in H$. Therefore for all $h, j \in H$ we have $hj \in H$. Hence we have proven closure under group multiplication, group inverse and existence of identity.

Therefore, by subgroup theorem we have $H \leq G$.

Hence we have proven both implications and therefore the statement.

The vector space analogue is that $V \subseteq W$ is a vector subspace of W with scalar field F equipped with vector addition $(+)$ and scalar multiplication $(*)$. iff V is non empty and $\forall \mathbf{x}, \mathbf{y} \in V$ and $\lambda, \mu \in F$ we have $\lambda \mathbf{x} + \mu \mathbf{y} \in V$. Note: ask if you were allowed to assume subgroup theorem here since I have.

1.5 Question 5

Let G be a group with group multiplication $\mu : G \times G \to G$. We define a new group multiplication by $\nu : G \times G \to G : (g, g') \mapsto \mu(g', g)$. We let G_{op} be the set G equipped with this map. Show that G^{op} is a group. (It is called the opposite group to G). Remark: when there are two group structures on a set, then a product expression like gg' can mean two different things depending on which multiplication you use. A simple remedy is to introduce more complicated notation like $g * g' \coloneqq v(g, g'), gg' \coloneqq (g, g')$. Then the relation between the two group structures is $g * g' = g'g$.

Solution. We show associatovity of the binary operation. Let $g, h, k \in G_{op}$.

Then

$$(g * h) * k = (hg) * k = k(hg) = khg = (kh)\, g = g * (kh) = g * (h * k)\,.$$

Since, this is true for all $g, h, k \in G_{op}$ we have associativity.

Now, we show existence of identity. We know $1_G \in G_{op}$. Let $g \in G_{op}$, then $1_G * g = g1_G = g$. Similarly $g * 1_G = 1_G g = g$. This is true for all $g \in G_{op}$ Therefore, 1_G is also the identity of G_{op} and G_{opp} contains an identity element.

Now, show closure under inverse. Let $g \in G_{op}$ then $g \in G$ and $g^{-1} \in G$ as G is closed under inverse being a group. So $g^{-1} \in G_{op}$ (Note: kinda confusing notation beacuse this already kinda implies it being an inverse do need something to denote its an inverse specificllay in G). Now we need to show that this is an inverse in G_{op} as well.

$$g * g^{-1} = g^{-1} g = 1_G.$$

Similarly

$$g^{-1} * g = g g^{-1} = 1_G.$$

Therefore, g^{-1} is an inverse of g in G_{op} as well. This is true for all $g \in G$. Hence, G_{op} is closed under inverse.

Therefore, we have proven associativity, closure under group multiplication and group inverse for G_{op}. Hence, G_{op} is a group by definition.

1.6 Question 6