PS 4

1) Example 3 (pg. 252)

2) Example 4 (pg. 253)

4) ① $\boxed{(1+x+x^2+x^3+x^4+x^5)^5}$

② $e_1 \in \{2, 4, 6\}$

$e_2 \in \{3, 5, 7\}$

$2 \le e_3, e_4 \le 7$

$$\boxed{(x^2+x^4+x^6)(x^3+x^5+x^7)(x^2+x^3+x^4+x^5+x^6+x^7)^2}$$

3) $e_M + e_A + e_T + e_H = 5$; $e_M, e_A \ge 0$

$0 \le e_T, e_H \le 1$

$$\boxed{(1+x+x^2+\cdots)^2 (1+x)^2}$$

5) ① $e_1 + e_2 + e_3 + e_4 = 25$; $0 \le e_i \le 25$

we want the coefficient of x^{25} in

$$\boxed{(1+x+x^2+\cdots+x^{25})^4}$$

② $e_1 + e_2 + e_3 + e_4 = 25$; $1 \le e_i \le 25$

coefficient of x^{25} in

$$\boxed{(x+x^2+x^3+\cdots+x^{25})^4}$$

③ $e_1 + e_2 + e_3 + e_4 = 25$; $0 \le e_i \le 12$

coefficient of x^{25} in

$$\boxed{(1+x+x^2+\cdots+x^{12})^4}$$

6) $g(x, y, z) = (xy + yz + zx)^8$

7) $x_1 + \cdots + x_{10} = 25 \quad ; \quad 1 \le x_i \le 6$

we want : coefficient of x^{25} in $(x + x^2 + \cdots + x^6)^{10}$

$= x^{10}(1 + x + x^2 + \cdots + x^5)^{10}$

i.e, coefficient of x^{15}

in $(1 + x + x^2 + \cdots + x^5)^{10} = \left[\dfrac{1-x^6}{1-x}\right]^{10} = (1-x^6)^{10}(1-x)^{-10}$

$= \left[1 - \binom{10}{1}x^6 + \binom{10}{2}x^{12} - \cdots\right] \cdot \left[\sum_{r=0}^{\infty} \binom{r+9}{r} x^r\right]$

So, required answer $= 1 \cdot \binom{15+9}{15} - \binom{10}{1}\binom{9+9}{9} + \binom{10}{2}\binom{3+9}{3}$

$= \binom{24}{15} - 10 \cdot \binom{18}{9} + \binom{10}{2} \cdot \binom{12}{3}$

$= 1,307,504 - 486,200 + 9,900$

$= 831,204$

8) To show : $\underbrace{\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}} = \binom{m+n}{r}$

coeff. of x^r in $(1+x)^m (1+x)^n$

= " " in $(1+x)^{m+n}$ = RHS ✓

9)

$$\frac{(1-x^2)^n}{(1-x)^n} = (1+x)^n$$

coeff. of x^m in $(1+x)^n = \binom{n}{m} = RHS$

Now, coeff. of x^m in $(1-x^2)^n (1-x)^{-n}$

$$= \left[1-\binom{n}{1}x^2+\binom{n}{2}x^4-\binom{n}{3}x^6+\dots+(-1)^{m/2}\binom{n}{m/2}x^m + \dots\right]$$

$$\times \left[\sum_{r=0}^{\infty}\binom{r+n-1}{r}x^r\right]$$

$$= \sum_{k=0}^{m/2}(-1)^k\binom{n}{k}\binom{m-2k+n-1}{m-2k}$$

$$= \sum_{k=0}^{m/2}(-1)^k\binom{n}{k}\binom{n+m-2k-1}{n-1} = LHS \checkmark$$

10)

$$g(x) = a_0+a_1x+a_2x^2+\dots+a_kx^k+\dots+a_nx^n$$

$$= \sum_{r=0}^{n} a_r x^r$$

Now, $g^{(1)}(x) = \sum_{r=1}^{n} r\, a_r x^{r-1}$

$$g^{(2)}(x) = \sum_{r=2}^{n}(r-1)\cdot r\cdot a_r \cdot x^{r-2}$$

$$g^{(3)}(x) = \sum_{r=3}^{n}(r-2)\cdot(r-1)\cdot r\cdot a_r\cdot x^{r-3}$$ and so on...

By induction, we can conclude that:

$$g^{(k)}(x) = \sum_{r=k}^{n}\frac{r!}{(r-k)!}a_r x^{r-k}$$

$$\Rightarrow g^{(k)}(0) = k!\, a_k \quad \Rightarrow \boxed{\frac{g^{(k)}(0)}{k!} = a_k}$$

11) $P_X(t) := \sum_r \mathbb{P}(X=r)\, t^r$

PGF of X

a) for $0 \le r \le n$, $\mathbb{P}(X=r) = \binom{n}{r}\left(\frac{1}{2}\right)^n$

$$P_X(t) = \sum_{r=0}^{n} \binom{n}{r}\left(\frac{1}{2}\right)^n t^r$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n} \binom{n}{r} t^r = \left(\frac{1}{2}\right)^n (1+t)^n \quad ✓$$

b) for $0 \le r \le n$, $\mathbb{P}(X=r) = \binom{n}{r} p^r q^{n-r}$

$$P_X(t) = \sum_{r=0}^{n} \binom{n}{r} p^r q^{n-r} t^r$$

$$= q^n \sum_{r=0}^{n} \binom{n}{r} \left(\frac{pt}{q}\right)^r = q^n \left(1 + \frac{pt}{q}\right)^n = (q+pt)^n \quad ✓$$

c) for $r \ge 5$, $\mathbb{P}(X=r) = \binom{r-1}{4}\left(\frac{1}{2}\right)^{r-1} \cdot \left(\frac{1}{2}\right)$

$$= \binom{r-1}{4}\left(\frac{1}{2}\right)^r$$

$$\Rightarrow P_X(t) = \sum_{r=5}^{\infty} \binom{r-1}{4}\left(\frac{1}{2}\right)^r t^r \quad ✓$$

d) for $r \ge m$, $\mathbb{P}(X=r) = \binom{r-1}{m-1} p^{m-1} q^{r-m} \cdot p$

$$= \binom{r-1}{m-1} p^m q^{r-m}$$

$$\Rightarrow P_X(t) = \sum_{r=m}^{\infty} \binom{r-1}{m-1} p^m q^{r-m} t^r \quad ✓$$

12) Example 4 (pg. 268)

13) 1 $10 = 10$

$$\left.\begin{aligned} &= 1+9 \\ &= 2+8 \\ &= 3+7 \\ &= 4+6 \end{aligned}\right\}$$

$$\left.\begin{aligned} &= 1+2+7 \\ &= 1+3+6 \\ &= 1+4+5 \\ &= 2+3+5 \end{aligned}\right\}$$

$$= 1+2+3+4$$

$$\left.\begin{aligned} 10 &= 1+9 \\ &= 3+7 \\ &= 5+5 \end{aligned}\right\}$$

$$\left.\begin{aligned} &= 3+3+3 \\ &= 1+1+1+3+3 \\ &= 1+1+1+1+1+1+3 \\ &= 1+1+1+1+1+1+1+1+1+1 \end{aligned}\right\}$$

$$\left.\begin{aligned} &= 1+1+1+7 \\ &= 1+1+3+5 \end{aligned}\right\} \quad = 1+1+1+1+1+5$$

(2) The generating function for partitions of r into distinct parts $= (1+x)(1+x^2)(1+x^3)\cdots$

$$= \frac{\cancel{1-x^2}}{1-x}\cdot\frac{\cancel{1-x^4}}{\cancel{1-x^2}}\cdot\frac{\cancel{1-x^6}}{1-x^3}\cdot\frac{\cancel{1-x^8}}{\cancel{1-x^4}}\cdot\frac{\cancel{1-x^{10}}}{1-x^5}\cdots$$

$$= \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdots$$

$$= (1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots$$

= the generating f^n for partition of r into odd parts

14) Idea : $283 = 2\times10^2 + 8\times10^1 + 3\times10^0$

a generating f^n term $\hookrightarrow$ $(x^{10^0})^2\cdot(x^{10^3})^5 = x^{\overbrace{2\times10^0+5\times10^3}^{= N \text{ (say)}}}$

Thus, coefficient of $x^N = 1$, $\forall N\in\mathbb{N}$ in generating f^n

$\iff$ unique decimal rep.

The generating f^n for a_r, the no. of decimal representations of $r\in\mathbb{N}$ is :

$$g(x) = \left[(x^{10^0})^0+(x^{10^0})^1+(x^{10^0})^2+\cdots+(x^{10^0})^9\right]$$
$$\times\left[(x^{10^1})^0+(x^{10^1})^1+(x^{10^1})^2+\cdots+(x^{10^1})^9\right]$$
$$\times\left[(x^{10^2})^0+(x^{10^2})^1+(x^{10^2})^2+\cdots+(x^{10^2})^9\right]$$
$$\times$$

PTO $\rightarrow$

$$= \frac{1-x^{10}}{1-x} \cdot \frac{\cancel{1-x^{100}}}{\cancel{1-x^{10}}} \cdot \frac{\cancel{1-x^{1000}}}{\cancel{1-x^{100}}} \cdots$$

$$= \frac{1}{1-x} = 1 + x + x^2 + \cdots$$

Since all coefficients of $g(x)$ are 1, every $r \in \mathbb{N}$ has a unique decimal representation

15) ①
$$\left.\begin{aligned} r &= 1 + \cdots \\ &= 1 + \cdots \\ &= 1 + \cdots \end{aligned}\right\} R(r-1, k-1)$$

no. of partitions containing 1

$$\left.\begin{aligned} &= \cdots \\ &= \cdots \end{aligned}\right\} R(r-k, k)$$

no. of partitions not containing 1

$y_1 + \cdots + y_k = r \; ; \; 2 \le y_i \le r$

$\Leftrightarrow (y_1 - 1) + (y_2 - 1) + \cdots + (y_k - 1) = r - k \; ; \; 1 \le y_i - 1 \le r-1$

$\Leftrightarrow z_1 + z_2 + \cdots + z_k = r - k \; ; \; 1 \le z_i \le r-k$

no. of partitions $= R(r-k, k)$

$\therefore R(r,k) = R(r-1, k-1) + R(r-k, k)$ $\square$

② Repeatedly applying a), we have :

$$R(n, r) = R(n-1, r-1) + R(n-r, r)$$

$$= R(n-2, r-2) + [R(n-r, r-1) + R(n-r, r)]$$

$$= R(n-3, r-3) + [R(n-r, r-2) + R(n-r, r-1) + R(n-r, r)]$$

$$\vdots$$

$$= \underbrace{R(n-r, 0)}_{= 0} + \sum_{k=1}^{r} R(n-r, k)$$

$$\therefore \sum_{k=1}^{r} R(n-r, k) = R(n, r) \qquad \square$$

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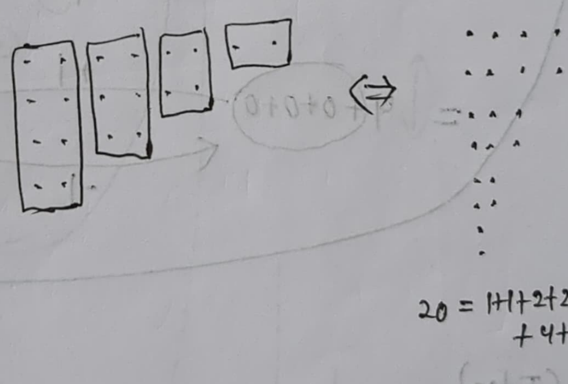
16) (Idea)

Example : $20 = 2+4+6+8$



$20 = 1+1+2+2+3+3+4+4$

HINT : Difference b/w no. of elements in two consecutive rows is even

17) (Idea)

we'll show a bijection b/w partitions of n and partitions of $2n$ into n parts :

Example :

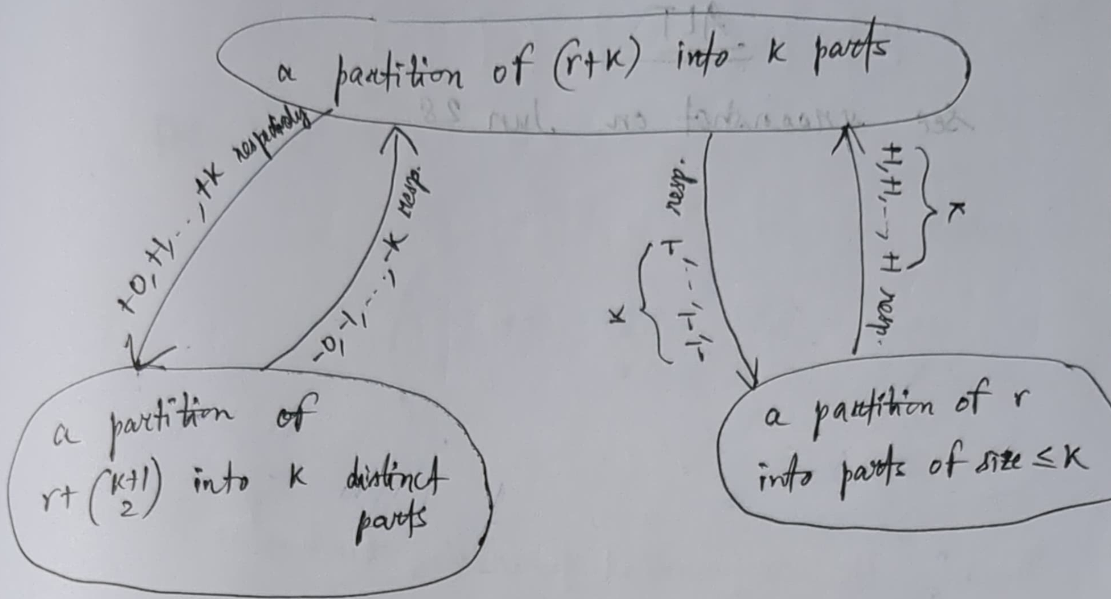
partitions of 4		partitions of 8 into 4 parts
		$8 =$
$4 = 1+1+1+1$	$\underset{-1}{\overset{+1}{\rightleftarrows}}$	$2+2+2+2$
$= 1+1+2+0$	$\underset{-1}{\overset{+1}{\rightleftarrows}}$	$1+2+2+3$
$= 1+3+0+0$	$\underset{-1}{\overset{+1}{\rightleftarrows}}$	$1+1+2+4$
$= 2+2+0+0$	$\underset{-1}{\overset{+1}{\rightleftarrows}}$	$1+1+3+3$
$= 4+0+0+0$	$\underset{-1}{\overset{+1}{\rightleftarrows}}$	$1+1+1+5$

18) (Idea)

WLOG assume that a partition is in non-decreasing order

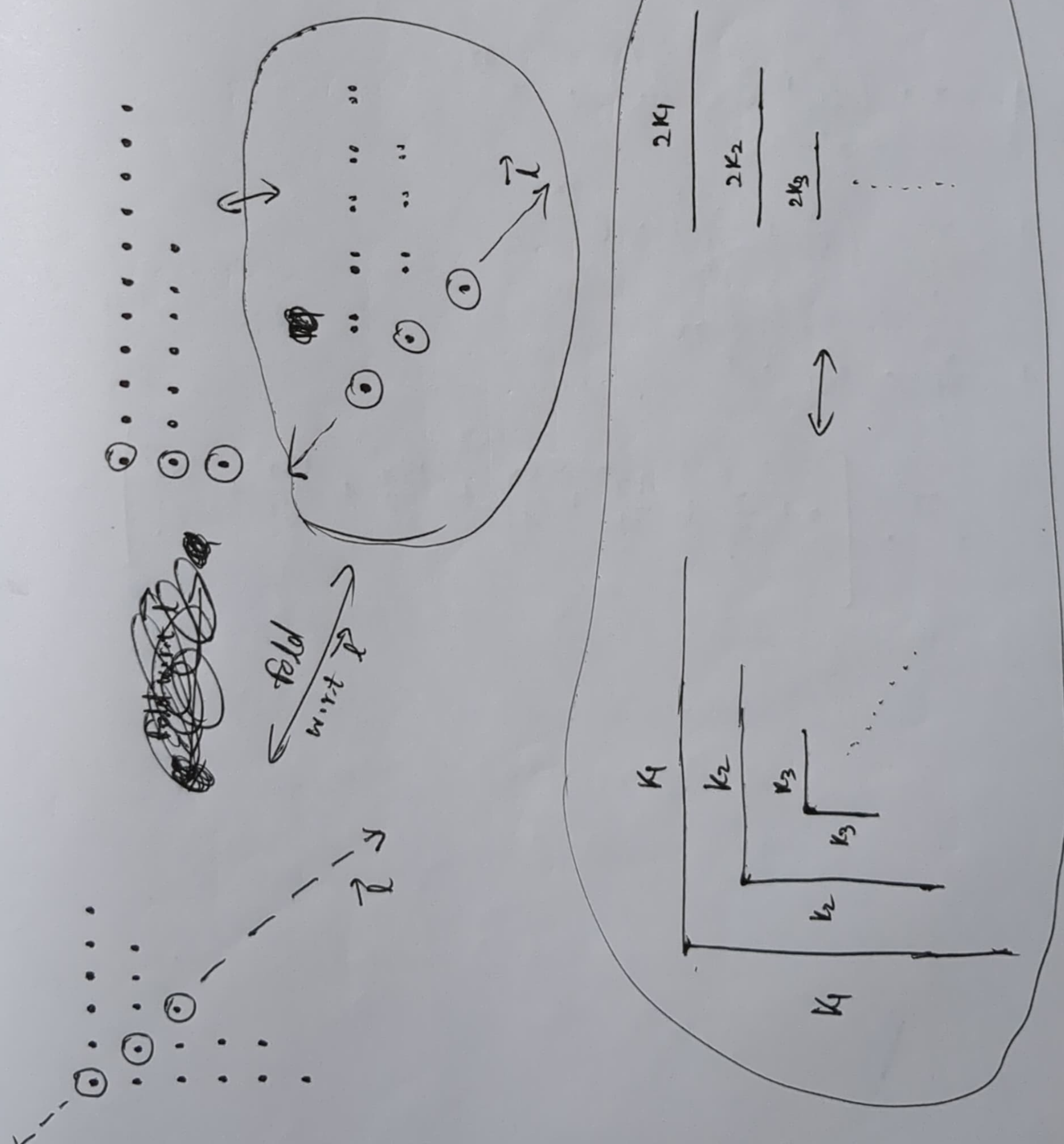
The following maps are bijective :

PTO →



19) ① [Idea : Stack exchange]

$19 = 1+2+2+3+5+6 \Longleftrightarrow 1+7+11$



20)

1	2	3	4	5
$\boxed{b_1}$	$\boxed{b_2}$	$\boxed{b_3}$	$\boxed{b_4}$	$\boxed{b_5}$

$$0 \le b_1 < b_2 \le 4$$

possible cases of (b_1, b_2):

(0, 1)	(1, 2)	(2, 3)	(3, 4)
(0, 2)	(1, 3)	(2, 4)	
(0, 3)	(1, 4)		
(0, 4)			

Thus, the exponential generating function is :

$$1 \cdot \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right) \cdot \underbrace{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)}_{= e^x}{}^{3}$$

$$+ \; x \cdot \left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right) \cdot e^{3x}$$

$$+ \; \frac{x^2}{2!} \left(\frac{x^3}{3!} + \frac{x^4}{4!}\right) \cdot e^{3x}$$

$$+ \; \frac{x^3}{3!} \cdot \frac{x^4}{4!} \cdot e^{3x}$$

$$= e^{3x} \cdot \left[x + \frac{x^2}{2!} + \frac{4x^3}{3!} + \frac{5x^4}{4!} + \frac{15x^5}{5!} + \frac{15x^6}{6!} + \frac{35x^7}{7!}\right]$$

22) $\boxed{h(x) = \sum_{r=1}^{\infty} r(r+2)x^r = (1\cdot3)x + (2\cdot4)x^2 + (3\cdot5)x^3 + (4\cdot6)x^4 + \cdots}$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\Rightarrow \underbrace{\frac{d}{dx}\left(\frac{1}{1-x}\right)}_{= \frac{1}{(1-x)^2}} = 1 + 2\cdot x + 3\cdot x^2 + 4\cdot x^3 + 5\cdot x^4 + \cdots$$

$$\Rightarrow \frac{x^3}{(1-x)^2} = x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + \cdots$$

$$\Rightarrow \underbrace{\frac{d}{dx}\left[\frac{x^3}{(1-x)^2}\right]}_{= \frac{x^2(3-x)}{(1-x)^3}} = (1\cdot3)x^2 + (2\cdot4)x^3 + (3\cdot5)x^4 + (4\cdot6)x^5 + \cdots$$

$$\Rightarrow \frac{x(3-x)}{(1-x)^3} = (1\cdot3)x + (2\cdot4)x^2 + (3\cdot5)x^3 + (4\cdot6)x^4 + \cdots$$

$$\therefore \boxed{h(x) = \frac{x(3-x)}{(1-x)^3}}$$

23) $h(x) = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$

$$\Rightarrow h(x)(1-x) = (a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)(1-x)$$

$$= (a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) - (a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \cdots)$$

$$= a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 + (a_3 - a_2)x^3 + \cdots$$

coefficient

$x^r \longrightarrow a_r - a_{r-1}$; $r \geq 1$

$x^0 \longrightarrow a_0$

24) $h(x) = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$S_r = \sum_{k=r+1}^{\infty} a_k = (a_{r+1} + a_{r+2} + \cdots)$

Let $g(x)$ be the generating f^n for S_r

i.e, $S_0 = a_1 + a_2 + a_3 + \cdots$

$S_1 = a_2 + a_3 + a_4 + \cdots$

$S_2 = a_3 + a_4 + a_5 + \cdots$

and so on ...

Now, $h(x) = a_0 + (S_0 - S_1)x + (S_1 - S_2)x^2 + (S_2 - S_3)x^3 + \cdots$

$\Rightarrow -[h(x) - a_0] + S_0 = S_0 + (S_1 - S_0)x + (S_2 - S_1)x^2 + (S_3 - S_2)x^3 + \cdots$

$= g(x)(1-x) \longrightarrow$ using 23)

$\Rightarrow g(x) = \dfrac{a_0 + S_0 - h(x)}{1-x}$

$= \dfrac{(a_0 + a_1 + a_2 + a_3 + \cdots) - h(x)}{1-x}$

$= \dfrac{h(1) - h(x)}{1-x}$

$\therefore \boxed{g(x) = \dfrac{h(1) - h(x)}{1-x}}$

29) Example 7 (pg. 287)

34) Example 10 (pg. 289)

26) $S_{n-1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \cdots\cdots$

$S_{n-2} = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots$

$\Rightarrow S_{n-1} + S_{n-2} = \binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \cdots = S_n$ ✓

$\because$

Recall: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

30) A → chicken salad
B → tuna sandwich
C → turkey wrap

~~a_n = no. of ways of arranging n~~

a_n = no. of strings of length n with elements $\in \{A, B, C\}$
s.t AAA doesn't appear

i) $\boxed{B/C}\ \underbrace{\square \cdots \square}_{n-1}$: $2a_{n-1}$

ii) ① $\boxed{A}\ \boxed{B/C}\ \underbrace{\square \cdots \square}_{n-2}$: $2a_{n-2}$

② $\boxed{A}\ \boxed{A}\ \boxed{B/C}\ \underbrace{\square \cdots \square}_{n-3}$: $2a_{n-3}$

$a_0 = 0,\ a_1 = 3,\ a_2 = 9$

$a_n = 2a_{n-1} + 2a_{n-2} + 2a_{n-3}, \quad n \geq 3$

32) a_n = no. of n-digit $\{0, 1\}$ sequences with even 0's, even 1's

b_n = even 0's, odd 1's

c_n = odd 0's, even 1's

$2^n - a_n - b_n - c_n$ = odd 0's, odd 1's

a_n:

$\boxed{0}\,\underbrace{\square \cdots \square}_{n-1}$: c_{n-1}

$\boxed{1}\,\underbrace{\square \cdots \square}_{n-1}$: b_{n-1}

$\Rightarrow a_n = b_{n-1} + c_{n-1}$

b_n:

$\boxed{0}\,\underbrace{\square \cdots \square}_{n-1}$: $2^{n-1} - a_{n-1} - b_{n-1} - c_{n-1}$

$\boxed{1}\,\underbrace{\square \cdots \square}_{n-1}$: a_{n-1}

$\Rightarrow b_n = 2^{n-1} - b_{n-1} - c_{n-1}$

c_n:

$\boxed{0}\,\underbrace{\square \cdots \square}_{n-1}$: a_{n-1}

$\boxed{1}\,\underbrace{\square \cdots \square}_{n-1}$: $2^{n-1} - a_{n-1} - b_{n-1} - c_{n-1}$

$\Rightarrow c_n = 2^{n-1} - b_{n-1} - c_{n-1}$

$$a_n = b_{n-1} + c_{n-1}$$
$$b_n = c_n = 2^{n-1} - b_{n-1} - c_{n-1}$$

$\Rightarrow$

$$\boxed{\begin{aligned} a_n &= 2b_{n-1} \\ b_n &= 2^{n-1} - 2b_{n-1} \\ a_1 &= 0,\ b_1 = 1 \end{aligned}}$$

33) a_n = no. of n-digit $\{0,1\}$ sequences with exactly one pair of consecutive 0's

b_n = no. of n-digit $\{0,1\}$ sequences with no consecutive 0's

$$a_n \to \begin{cases} \boxed{1}\,\underbrace{\square \cdots \square}_{n-1} : a_{n-1} \\ \boxed{0}\,\boxed{1}\,\underbrace{\square \cdots \square}_{n-2} : a_{n-2} \\ \boxed{0}\,\boxed{0}\,\boxed{1}\,\underbrace{\square \cdots \square}_{n-3} : b_{n-3} \end{cases} \Rightarrow a_n = a_{n-1} + a_{n-2} + b_{n-3}$$

$$b_n \to \begin{cases} \boxed{1}\,\underbrace{\square \cdots \square}_{n-1} : b_{n-1} \\ \boxed{0}\,\boxed{1}\,\underbrace{\square \cdots \square}_{n-2} : b_{n-2} \end{cases} \Rightarrow b_n = b_{n-1} + b_{n-2}$$

$$a_n = a_{n-1} + a_{n-2} + b_{n-3}$$
$$b_n = b_{n-1} + b_{n-2}$$

$a_1 = 0, \quad b_1 = 2$

$a_k = 0, \quad \forall k \le 0$

$b_k = 1, \quad \forall k \le 0$

27) (sketch) $F_n = \begin{cases} 1, & \text{if } n \in \{0,1\} \\ F_{n-1}+F_{n-2}, & \text{if } n \geq 2 \end{cases}$

We'll prove all the results by induction for $n = n+1$:

a) $\underbrace{F_0+F_1+\cdots+F_n}_{=F_{n+2}-1} + F_{n+1} \overset{\text{to}}{\underset{\text{show}}{=}} F_{n+3}-1$

$$LHS = (F_{n+1}+F_{n+2})-1 = F_{n+3}-1 \quad ✓$$

b) To show: $\underbrace{F_0^2+F_1^2+\cdots+F_n^2}_{F_nF_{n+1}} + F_{n+1}^2 = F_{n+1}F_{n+2}$

$$LHS = F_{n+1}(F_n+F_{n+1}) = F_{n+1}F_{n+2} \quad ✓$$

c) To show: $\underbrace{F_0+F_2+\cdots+F_{2n}}_{F_{2n+1}} + F_{2n+2} = F_{2n+3}$

$$LHS = F_{2n+1}+F_{2n+2} = F_{2n+3} \quad ✓$$

d) To show: $F_{n+1}F_{n+3} = F_{n+2}^2 + (-1)^{n+1}$

$$\begin{aligned} LHS &= (F_{n+2}-F_n)(F_{n+2}+F_{n+1}) \\ &= F_{n+2}^2 + F_{n+2}F_{n+1} - F_nF_{n+2} - F_nF_{n+1} \\ &= F_{n+2}^2 + (F_{n+1}+F_n)F_{n+1} - F_{n+1}^2 - (-1)^n - F_nF_{n+1} \\ &= F_{n+2}^2 + (-1)^{n+1} \quad ✓ \end{aligned}$$

e) To show: $\underbrace{F_1-F_2+F_3-F_4+\cdots-F_{2n}}_{-F_{2n-1}} + F_{2n+1} - F_{2n+2} = -F_{2n+1}$

$$LHS = -F_{2n-1} + \cancel{F_{2n+1}} - \cancel{F_{2n+1}} - F_{2n} = -F_{2n+1} \quad ✓$$

35) a) $a_n = 3a_{n-1} + 4a_{n-2}$, $n \geq 2$ $a_0 = a_1 = 1$

$a_n = \alpha^n \Rightarrow \alpha^n = 3\alpha^{n-1} + 4\alpha^{n-2}$

$\Rightarrow \alpha^2 = 3\alpha + 4$

$\Rightarrow (\alpha-4)(\alpha+1) = 0 \Rightarrow \alpha = 4, -1$

$a_n = A_1 \cdot 4^n + A_2 \cdot (-1)^n$

$a_0 = 1 \Rightarrow A_1 + A_2 = 1$
$a_1 = 1 \Rightarrow 4A_1 - A_2 = 1$
$\Rightarrow A_1 = \frac{2}{5}, \ A_2 = \frac{3}{5}$

$\therefore a_n = \frac{2}{5} \cdot 4^n + \frac{3}{5} \cdot (-1)^n$

i.e, $\boxed{a_n = \frac{2 \cdot 4^n + 3 \cdot (-1)^n}{5}, \quad n \geq 0}$

b) $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$, $n \geq 3$ $a_0 = a_1 = 1$, $a_2 = 2$

$a_n = \alpha^n \Rightarrow \alpha^n = 3\alpha^{n-1} - 3\alpha^{n-2} + \alpha^{n-3}$

$\Rightarrow \alpha^3 = 3\alpha^2 - 3\alpha + 1$

$\Rightarrow (\alpha-1)^3 = 0 \Rightarrow \alpha = 1, 1, 1$

$a_n = A_1 \cdot 1^n + A_2 \cdot n \cdot 1^n + A_3 \cdot n^2 \cdot 1^n$

i.e, $a_n = A_1 + nA_2 + n^2 A_3$

$a_0 = 1 \Rightarrow A_1 = 1$
$a_1 = 1 \Rightarrow A_1 + A_2 + A_3 = 1$
$a_2 = 2 \Rightarrow A_1 + 2A_2 + 4A_3 = 2$
$\Rightarrow A_2 = -\frac{1}{2}, \ A_3 = \frac{1}{2}$

$\therefore \boxed{a_n = 1 - \frac{n}{2} + \frac{n^2}{2}, \quad n \geq 0}$

36) $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

$a_n = \alpha^n \Rightarrow \alpha^n = c_1 \alpha^{n-1} + c_2 \alpha^{n-2}$

$\Rightarrow \alpha^2 - c_1 \alpha + c_2 = 0$

$\Rightarrow c_1 = 3+6 = 9, \quad c_2 = 3 \cdot 6 = 18$

$\therefore \boxed{c_1 = 9, \quad c_2 = 18}$

28) $F_0 = F_1 = 1; \quad F_n = F_n + F_{n-1}, \quad n \geq 2$

$F_{n+1} = F_n + F_{n-1}$

$\Rightarrow \frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}$

$\Rightarrow \underbrace{\lim_{n\to\infty} \frac{F_{n+1}}{F_n}}_{\substack{\parallel \\ K \text{ (say)}}} = 1 + \lim_{n\to\infty} \frac{1}{F_n/F_{n-1}}$

$\Rightarrow K = 1 + \frac{1}{\lim_{n\to\infty} F_n/F_{n-1}} = 1 + \frac{1}{K}$

$\Rightarrow K^2 = K+1 \Rightarrow K^2 - K + 1 = 0 \Rightarrow K = \frac{\sqrt{5}+1}{2}$ ✓

37) a) $a_n = a_{n-1} + \underbrace{n(n-1)}_{\substack{\parallel \\ f(n)}}, \quad a_0 = 3$

$$\begin{aligned} a_n &= a_{n-1} + f(n) \\ &= [a_{n-2} + f(n-1)] + f(n) \\ &= [a_{n-3} + f(n-2)] + f(n-1) + f(n) \\ &\vdots \\ &= a_0 + f(1) + f(2) + \cdots + f(n) \end{aligned}$$

$$\Rightarrow a_n = 3 + \sum_{k=1}^{n} k(k-1)$$

$$= 3 + \left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right]$$

$$= 3 + \left[n(n+1)\left\{\frac{2n+1-3}{6}\right\}\right]$$

$$= 3 + \frac{n(n+1)(n-1)}{3}$$

$$\therefore \boxed{a_n = 3 + \frac{(n-1)n(n+1)}{3}, \quad n \geq 1}$$ ✓

b) $a_n = 2a_{n-1} + (-1)^n$, $a_0 = 2$

put $a_n = \alpha^n$ to $a_n = 2a_{n-1}$ $\Rightarrow \alpha^n = 2\alpha^{n-1}$

$\Rightarrow \alpha = 2$

general sol^n to homogeneous $\rightarrow a_n = A \cdot 2^n$

Putting $a_n^* = B(-1)^n$,

$B(-1)^n = 2B(-1)^{n-1} + (-1)^n$

$\Rightarrow -B(-1)^{n-1} = 2B(-1)^{n-1} - (-1)^{n-1}$

$\Rightarrow (-1)^{n-1} = 3B(-1)^{n-1} \Rightarrow B = 1/3$

particular sol^n to inhomogeneous $\rightarrow a_n^* = \frac{(-1)^n}{3}$

Thus, general sol^n to inhomogeneous : $a_n = A \cdot 2^n + \frac{(-1)^n}{3}$

$a_0 = 2 \Rightarrow A + \frac{1}{3} = 2 \Rightarrow A = 5/3$

$$\therefore \boxed{a_n = \frac{5 \cdot 2^n + (-1)^n}{3}, \quad n \geq 1}$$ ✓

39) $a_n = -n\, a_{n-1} + n!$, $a_0 = 1$

$\Rightarrow \dfrac{a_n}{n!} = \dfrac{-a_{n-1}}{(n-1)!} + 1$

Let $b_n = \dfrac{a_n}{n!}$. Then,

$b_n = -b_{n-1} + 1$, $b_0 = 1$

Putting $b_n = \alpha^n$, we've : $\alpha^n = -\alpha^{n-1}$
to $b_n = -b_{n-1}$

$\Rightarrow \alpha = -1$

So, general solⁿ to homogeneous : $b_n = A \cdot (-1)^n$

Putting $b_n^* = B$, we've :

$B = -B + 1 \Rightarrow B = 1/2$

So, particular solⁿ to inhomogeneous : $b_n^* = 1/2$

Thus, general solⁿ to inhomogeneous : $b_n = A \cdot (-1)^n + \frac{1}{2}$

$b_0 = 1 \Rightarrow A + 1/2 = 1 \Rightarrow A = 1/2$

Thus, $b_n = \dfrac{(-1)^n + 1}{2}$, $n \geq 1$

$$\boxed{\begin{array}{ll} a_n = \dfrac{n!\,[(-1)^n + 1]}{2} & n \geq 1 \\ a_0 = 1 & \end{array}}$$

40> a> $a_n = a_{n-1} + 2 \;,\; n \geq 1 \;,\; a_0 = 1$

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$\Rightarrow g(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n$$

$$\Rightarrow g(x) - 1 = \sum_{n=1}^{\infty} (a_{n-1} + 2)\, x^n$$

$$= x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + 2 \sum_{n=1}^{\infty} x^n$$

$$= x\, g(x) + \frac{2x}{1-x}$$

$$\Rightarrow g(x)(1-x) = 1 + \frac{2x}{1-x} = \frac{1+x}{1-x}$$

$$\Rightarrow g(x) = \frac{1+x}{(1-x)^2} = \frac{1}{(1-x)^2} + \frac{x}{(1-x)^2}$$

NOTE : Recall $\rightarrow$ $(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{r+n-1}{r} x^r$

$$\Rightarrow (1-x)^{-2} = \sum_{r=0}^{\infty} \binom{r+1}{r} x^r = \sum_{r=0}^{\infty} (r+1)\, x^r$$

coefficient of x^r in $(1-x)^{-2}$ is $(r+1)$

" x^{r-1} " " is r

$\Rightarrow$ coefficient of x^r in $g(x) = r+1+r = 2r+1$

$$\Rightarrow g(x) = \sum_{r=0}^{\infty} (2r+1)\, x^r$$

$\therefore$ $\boxed{a_n = 2n+1}$ ✓

b) $a_n = 3a_{n-1} - 2a_{n-2} + 2$, $a_0 = a_1 = 1$

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$\Rightarrow g(x) - a_0 - a_1 x = \sum_{n=2}^{\infty} a_n x^n$$

$$\Rightarrow g(x) - 1 - x = \sum_{n=2}^{\infty} \left[3a_{n-1} - 2a_{n-2} + 2\right] x^n$$

$$= 3x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} - 2x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} + 2\sum_{n=2}^{\infty} x^n$$

$$= 3x\left[g(x) - 1\right] - 2x^2 g(x) + \frac{2x^2}{1-x}$$

$$= (3x - 2x^2)\, g(x) + \frac{2x^2}{1-x} - 3x$$

$$\Rightarrow (2x^2 - 3x + 1)\, g(x) = \frac{2x^2}{1-x} - 3x + 1 + x$$

$$\Rightarrow (2x-1)(x-1)\, g(x) = \frac{2x^2}{1-x} + 1 - 2x = \frac{4x^2 - 3x + 1}{1-x}$$

$$\Rightarrow -g(x) = \frac{4x^2 - 3x + 1}{(x-1)^2 (2x-1)}$$

$$= \frac{(2x-1)^2 + (x-1) + 1}{(x-1)^2 (2x-1)}$$

$$= \frac{2(x-1)+1}{(x-1)^2} + \frac{(2x-1) - 2(x-1)}{(x-1)(2x-1)} + \frac{(2x-1) - 2(x-1)}{(x-1)^2 (2x-1)}$$

$$= \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x-1} - \frac{2}{2x-1} + \frac{1}{(x-1)^2} - \frac{2\left[(2x-1) - 2(x-1)\right]}{(x-1)(2x-1)}$$

$$= \frac{3}{x-1} + \frac{2}{(x-1)^2} - \frac{2}{2x-1} - \frac{2}{x-1} + \frac{4}{2x-1}$$

$$= \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{2}{(2x-1)}$$

$\Rightarrow \quad g(x) = \frac{1}{1-x} - \frac{2}{(1-x)^2} + \frac{2}{1-2x}$

coeff of x^r in $(1-x)^{-1} = 1$

coeff of x^r in $(1-x)^{-2} = \binom{r+1}{r} = r+1$

coeff of x^r in $(1-2x)^{-1} = 2^r$

$\therefore$ **Recall** : $\boxed{(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{r+n-1}{r} x^r}$

$$\Rightarrow g(x) = \sum_{r=0}^{\infty} \left[1 - 2(r+1) + 2 \times 2^r\right] x^r$$

$$= \sum_{r=0}^{\infty} \left[2^{r+1} - 2r - 1\right] x^r$$

$$\therefore \boxed{a_n = 2^{n+1} - 2n - 1} \quad ✓$$

25) $a_n = a_{n-1} + a_{n-2}$; $a_1 = 1, a_2 = 3$

$a_n = \alpha^n \Rightarrow \alpha^n = \alpha^{n-1} + \alpha^{n-2}$

$\Rightarrow \alpha^2 - \alpha - 1 = 0 \Rightarrow \alpha = \frac{1 \pm \sqrt{5}}{2}$

$$a_n = A_1 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n + A_2 \left(\frac{1+\sqrt{5}}{2}\right)^n$$

$$\left.\begin{aligned} a_1 = 1 &\Rightarrow A_1\left(\frac{1-\sqrt{5}}{2}\right) + A_2\left(\frac{1+\sqrt{5}}{2}\right) = 1 \\ a_2 = 3 &\Rightarrow A_1\left(\frac{6-2\sqrt{5}}{4}\right) + A_2\left(\frac{6+2\sqrt{5}}{4}\right) = 3 \end{aligned}\right\} \Rightarrow A_1 = A_2 = 1$$

$$\therefore \boxed{a_n = \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^n, \quad n \geq 3} \quad ✓$$

Claim: $a_n < (1.75)^n \quad \forall n \geq 1$

pf (strong induction on n):

Base Case ($n=1,2$): $a_1 = 1 < 1.75$ ✓

$a_2 = 3 < 1.75^2 = 3.0625$ ✓

Suppose true for $1, 2, \ldots, n-1$

Now, $a_n = a_{n-1} + a_{n-2}$

$< (1.75)^{n-1} + (1.75)^{n-2}$ → using induction hypothesis

$= 2.75 \times (1.75)^{n-2}$

$< \underbrace{1.75^2}_{= 3.0625} \times (1.75)^{n-2} = (1.75)^n$ ✓