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ORIGINAL PAPER

On Half-Way AZ-Style Identities

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Abstract The Ahlswede–Zhang identity is an elegant sharpening of the famous LYM-inequality. Recently, we have found a parametrised identity which implies the AZ identity and characterizes deficiencies of other inequalities in combinatorics. In this paper, we show identities of half-way extraction from AZ-style identities. These identities aim to characterize more clearly terms participating in AZ identities or LYM-style inequalities.

Keywords Poset · LYM inequality · AZ identity

Mathematics Subject Classification (2000) 05D05

1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$, $2^{[n]}$ denote the family of all subsets of $[n]$, $\mathcal{P}_k$ be the family of all subsets of $[n]$ of size k, and $\emptyset$ be the empty set. If $\emptyset \neq \mathcal{F} \subseteq 2^{[n]}$ and $A \not\subset B$ for all $A, B \in \mathcal{F}$, $A \neq B$, then $\mathcal{F}$ is called a *Sperner family* or an *antichain*. The well-known LYM inequality (Lubell, Yamamoto, Meshalkin) is

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1,$$

where $\mathcal{F}$ is an arbitrary antichain.

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The LYM inequality can be sharpened or generalized in several ways. For example, by adding to the left hand side of the LYM inequality suitable terms [5,6,10] or by proposing a stronger inequality implying the LYM inequality [8], by unifying LYM-style inequalities with a common generalization [4], by studying LYM-style inequalities for other types of posets [4,8,11]. Recently, Katona has given a revised presentation of the LYM inequality with linear or non-linear profile-vectors [13].

Especially, Ahlswede and Zhang [1] found a powerful identity (called *AZ identity*) which is a sharpening of the original LYM inequality. Moreover, the generalisation of the AZ identity [2] implies Bollobás inequality for two set systems [7]. Several results on AZ-style identities are also proposed; for example, AZ-style identities for other posets [3], pseudo-LYM inequalities and AZ identities [11], duals of AZ-identities [9,12], AZ type identity for k-chains in k-Sperner families [14], extremal cases of the Ahlswede–Cai identity [15], parametrised AZ identity [17].

In this paper, we present identities of *"half-way extraction"* from the recent generalization of AZ identities [17]. It means that summation in these identities is over subsets whose cardinalities are not exceedings $\frac{n}{2}$. The *half-way identities* characterize more clearly terms in LYM-style inequalities or AZ-style identities.

Let $\mathcal{G}$ be the family of all $\mathcal{F}$ such that $\emptyset \neq \mathcal{F} \subseteq 2^{[n]}$. For every $\mathcal{F} \in \mathcal{G}$ the downset is defined

$$\mathcal{D}(\mathcal{F}) = \{D \subset [n] : D \subset F \text{ } for \text{ } some \text{ } F \in \mathcal{F}\},$$

and the upset is

$$\mathcal{U}(\mathcal{F}) = \{U \subset [n] : U \supset F \text{ } for \text{ } some \text{ } F \in \mathcal{F}\}.$$

Then for every $X \subseteq [n]$, we define

$$Z_{\mathcal{F}}(X) = \begin{cases} \emptyset & if \ X \notin \mathcal{U}(\mathcal{F}), \\ \bigcap\limits_{X \supseteq F \in \mathcal{F}} F & otherwise. \end{cases}$$

In fact, the case $X \notin \mathcal{U}(\mathcal{F})$ is trivial; we are concerned only with the value of $Z_{\mathcal{F}}(X)$ whenever $X \in \mathcal{U}(\mathcal{F})$.

Theorem 1 (Thu [17]) *Let m be an integer, $\emptyset \notin \mathcal{A} \in \mathcal{G}$. If $|A| + m > 0$ for each $A \in \mathcal{A}$, then*

$$\sum_{X \in \mathcal{U}(\mathcal{A})} \frac{m + |Z_{\mathcal{A}}(X)|}{(m + |X|) \binom{m+n}{m+|X|}} = 1. \tag{1}$$

The following inequality of Bollobás relates to intersecting Sperner families.

Theorem 2 (Bollobás [7]) *Let $\mathcal{A}$ be an intersecting antichain of subsets of $[n]$ such that $|A| \leq \frac{n}{2}$ for each $A \in \mathcal{A}$. Then*

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n-1}{|A|-1}} \leq 1. \tag{2}$$

We get the original AZ identity [1] by setting $m = 0$ in (1). On the other hand, for every antichain $\mathcal{A}$ with $|A| > 1$ for each $A \in \mathcal{A}$, the case $m = -1$ gives the following identity

$$\sum_{A\in\mathcal{A}} \frac{1}{\binom{n-1}{|A|-1}} + \sum_{X\in\mathcal{U}(\mathcal{A})\setminus\mathcal{A}} \frac{|Z_{\mathcal{A}}(X)|-1}{(|X|-1)\binom{n-1}{|X|-1}} = 1,$$

which is a generalization of inequality (2) for intersecting families, as well as the Tuza's inequality for Helly families [18].

In Sect. 2 of the paper, we introduce *half-way identities* for a set system. Section 3 presents some applications and the relationship between these identities and LYM-style inequalities. Finally, the case of two set systems is considered in Sect. 4.

2 Main Theorem

Theorem 3 *Let m be an integer, $\emptyset \notin \mathcal{A} \in \mathcal{G}$. If $|A| + m > 0$ for each $A \in \mathcal{A}$, then*

$$\sum_{X\in\mathcal{U}(\mathcal{A}),|X|\le\lfloor\frac{n}{2}\rfloor} \frac{m+|Z_{\mathcal{A}}(X)|}{(m+|X|)\binom{m+n}{m+|X|}} = \frac{\left|\mathcal{U}(\mathcal{A})\cap\mathcal{P}_{\lfloor\frac{n}{2}\rfloor}\right|}{\binom{m+n}{\lceil\frac{n}{2}\rceil}}. \tag{3}$$

The identity (3) is found when we consider thoroughly terms in the identity (1). Now the same technique as in [16,17] is used to give an induction proof of (3).

Lemma 1 *Let r, s, n be positive integers such that $\lceil\frac{n}{2}\rceil \le r < n$. Then*

$$\sum_{k=0}^{r-\lceil\frac{n}{2}\rceil} \binom{r}{k} \frac{s}{(s+k)\binom{s+r}{s+k}} = \frac{\binom{r}{\lceil\frac{n}{2}\rceil}}{\binom{s+r}{\lceil\frac{n}{2}\rceil}}. \tag{4}$$

Proof Let $L(n)$ and $R(n)$ be the left and the right hand side of (4) respectively. We prove (4) by induction on n. The condition on r, n implies $n \ge 2$. When $n = 2$, r must be 1 and $L(2) = R(2) = \frac{1}{s+1}$. We assume (4) holds for $n \ge 2$ and consider the case of $n+1$. Observe that for n odd we have $\lceil\frac{n+1}{2}\rceil = \lceil\frac{n}{2}\rceil$, so $L(n+1) = L(n)$, $R(n+1) = R(n)$ and hence, by the induction hypothesis, $L(n+1) = R(n+1)$. Suppose now n is even and put $n = 2m$. Then we have

$$L(n+1) = L(n) - \binom{r}{r-m} \frac{s}{(s+r-m)\binom{s+r}{s+r-m}}.$$

Replacing $L(n)$ by $R(n)$ and then evaluating the latter expression we get $L(n+1) = R(n+1)$. □

Lemma 2 [9,16] *Let* $\emptyset \notin \mathcal{A} \in \mathcal{G}, \emptyset \notin \mathcal{B} \in \mathcal{G}$ *and put* $\mathcal{A} \vee \mathcal{B} = \{A \cup B : A \in \mathcal{A}$ *and* $B \in \mathcal{B}\}$.

Then for each $\emptyset \neq X \subset [n]$ *we have*

$$|Z_{\mathcal{A}\cup\mathcal{B}}(X)| = |Z_{\mathcal{A}}(X)| + |Z_{\mathcal{B}}(X)| - |Z_{\mathcal{A}\vee\mathcal{B}}(X)|.$$

Induction proof of Theorem 3. We follow a similar process of the proof of Theorem 3 in [17]. Put $W(X) = (m+|X|)\binom{m+n}{m+|X|} > 0$ for all $X \in \mathcal{U}(\mathcal{A})$.

Case 1 $|\mathcal{A}| = 1$ so $\mathcal{A} = \{A\},\ A \neq \emptyset,\ |A| + m > 0$. For each $[n] \supseteq X \supseteq A$ we have $Z_{\mathcal{A}}(X) = A$. Put $a = |A| > 0$. If $a > \lfloor \frac{n}{2} \rfloor$ then both sides of (3) are zero. So we can suppose $a \leq \lfloor \frac{n}{2} \rfloor$, then the left hand side of (3) is

$$\text{LHS (3)} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - a} \binom{n-a}{k} \frac{a+m}{(a+k+m)\binom{n+m}{a+k+m}}$$

$$= \sum_{k=0}^{r-\lceil \frac{n}{2} \rceil} \binom{r}{k} \frac{s}{(s+k)\binom{s+r}{s+k}},$$

where $r = n - a$ and $s = a + m$ which imply $m + n = s + r$ and $\lceil \frac{n}{2} \rceil \leq r < n,\ s > 0$. By Lemma 1, we have

$$\text{LHS (3)} = \frac{\binom{r}{\lceil \frac{n}{2} \rceil}}{\binom{s+r}{\lceil \frac{n}{2} \rceil}} = \frac{\binom{n-a}{\lceil \frac{n}{2} \rceil}}{\binom{m+n}{\lceil \frac{n}{2} \rceil}} = \frac{\binom{n-a}{\lfloor \frac{n}{2} \rfloor - a}}{\binom{m+n}{\lceil \frac{n}{2} \rceil}}.$$

Since the numerator of the last term is $\left|\mathcal{U}(\mathcal{A}) \cap \mathcal{P}_{\lfloor \frac{n}{2} \rfloor}\right|$ in this case, we deduce immediately LHS(3) $= RHS(3)$.

Case 2 We assume (3) holds for $1 \le |\mathcal{A}| < h$ and consider the case $\mathcal{A} = \{A_1, A_2, \ldots, A_h\}$ with $h > 1$ and $|A_i| + m > 0$ for each $1 \le i \le h$. Let $\mathcal{B} = \{A_1, \ldots, A_{h-1}\}$ and $\mathcal{D} = \{A_h\}$. Then we write LHS(3) as (by using Lemma 2 in the same way as in [17]):

$$\text{LHS}(3) = \sum_{\substack{X \in \mathcal{U}(\mathcal{B}) \\ |X| \le \lfloor \frac{n}{2} \rfloor}} \frac{|Z_{\mathcal{B}}(X)| + m}{W(X)} + \sum_{\substack{X \in \mathcal{U}(\mathcal{D}) \\ |X| \le \lfloor \frac{n}{2} \rfloor}} \frac{|Z_{\mathcal{D}}(X)| + m}{W(X)}$$

$$- \sum_{\substack{X \in \mathcal{U}(\mathcal{B} \vee \mathcal{D}) \\ |X| \le \lfloor \frac{n}{2} \rfloor}} \frac{|Z_{\mathcal{B} \vee \mathcal{D}}(X)| + m}{W(X)}.$$

Now $\mathcal{B}$, $\mathcal{D}$, $\mathcal{B} \vee \mathcal{D}$ satisfy the condition $|A| + m > 0$ for each set A belonging to them, and have cardinalities less than h. By the induction hypothesis, LHS(3) now is

$$\text{LHS}(3) = \frac{\left|\mathcal{U}(\mathcal{B}) \cap \mathcal{P}_{\lfloor \frac{n}{2} \rfloor}\right|}{\binom{m+n}{\lceil \frac{n}{2} \rceil}} + \frac{\left|\mathcal{U}(\mathcal{D}) \cap \mathcal{P}_{\lfloor \frac{n}{2} \rfloor}\right|}{\binom{m+n}{\lceil \frac{n}{2} \rceil}} - \frac{\left|\mathcal{U}(\mathcal{B} \vee \mathcal{D}) \cap \mathcal{P}_{\lfloor \frac{n}{2} \rfloor}\right|}{\binom{m+n}{\lceil \frac{n}{2} \rceil}}.$$

Then using $\mathcal{U}(\mathcal{B} \vee \mathcal{D}) = \mathcal{U}(\mathcal{B}) \cap \mathcal{U}(\mathcal{D})$ to write $\mathcal{U}(\mathcal{B} \vee \mathcal{D}) \cap \mathcal{P}_{\lfloor \frac{n}{2} \rfloor}$ as

$$\left[\mathcal{U}(\mathcal{B}) \cap \mathcal{P}_{\lfloor \frac{n}{2} \rfloor}\right] \cap \left[\mathcal{U}(\mathcal{D}) \cap \mathcal{P}_{\lfloor \frac{n}{2} \rfloor}\right]$$

and rewrite LHS(3) as

$$\frac{\left|\left[\mathcal{U}(\mathcal{B}) \cap \mathcal{P}_{\lfloor \frac{n}{2} \rfloor}\right] \cup \left[\mathcal{U}(\mathcal{D}) \cap \mathcal{P}_{\lfloor \frac{n}{2} \rfloor}\right]\right|}{\binom{m+n}{\lceil \frac{n}{2} \rceil}} = \frac{\left|[\mathcal{U}(\mathcal{B}) \cup \mathcal{U}(\mathcal{D})] \cap \mathcal{P}_{\lfloor \frac{n}{2} \rfloor}\right|}{\binom{m+n}{\lceil \frac{n}{2} \rceil}}.$$

Since $\mathcal{U}(\mathcal{B}) \cup \mathcal{U}(\mathcal{D}) = \mathcal{U}(\mathcal{B} \cup \mathcal{D}) = \mathcal{U}(\mathcal{A})$, the last term is $RHS(3)$. □

We deduce from (1) and (3) the following consequence.

Corollary 1 *Let m be an integer, $\emptyset \notin \mathcal{A} \in \mathcal{G}$. If $|A| + m > 0$ for each $A \in \mathcal{A}$, then*

$$\sum_{X \in \mathcal{U}(\mathcal{A}), |X| > \lfloor \frac{n}{2} \rfloor} \frac{m + |Z_{\mathcal{A}}(X)|}{(m + |X|)\binom{m+n}{m+|X|}} = 1 - \frac{\left|\mathcal{U}(\mathcal{A}) \cap \mathcal{P}_{\lfloor \frac{n}{2} \rfloor}\right|}{\binom{m+n}{\lceil \frac{n}{2} \rceil}}. \tag{5}$$

In the same way as in [9], we get the dual of the above identity as follows.

Corollary 2 *Let $\mathcal{A} \in \mathcal{G}$ and q be a positive integer. If $|A| < q$ for each $A \in \mathcal{A}$, then*

$$\sum_{X \in \mathcal{D}(\mathcal{A}), |X| > \lceil \frac{n}{2} \rceil} \frac{q - \left|Z_{\mathcal{A}}^{*}(X)\right|}{(q - |X|)\binom{q}{|X|}} = \frac{\left|\mathcal{D}(\mathcal{A}) \cap \mathcal{P}_{\lceil \frac{n}{2} \rceil}\right|}{\binom{q}{\lceil \frac{n}{2} \rceil}},$$

and

$$\sum_{X \in \mathcal{D}(\mathcal{A}), |X| \le \lceil \frac{n}{2} \rceil} \frac{q - \left|Z_{\mathcal{A}}^{*}(X)\right|}{(q - |X|)\binom{q}{|X|}} = 1 - \frac{\left|\mathcal{D}(\mathcal{A}) \cap \mathcal{P}_{\lceil \frac{n}{2} \rceil}\right|}{\binom{q}{\lceil \frac{n}{2} \rceil}},$$

where

$$Z_{\mathcal{A}}^{*}(X) = \begin{cases} [n] & if\ X \notin \mathcal{D}(\mathcal{A}), \\ \bigcup_{X \subseteq A \in \mathcal{A}} A & otherwise. \end{cases}$$

3 Some Consequences

By setting $m = 0$ in (3) and (5), we obtain

$$\sum_{\substack{X \in \mathcal{U}(\mathcal{A}) \\ |X| \le \lfloor \frac{n}{2} \rfloor}} \frac{|Z_{\mathcal{A}}(X)|}{|X|\binom{n}{|X|}} = \frac{\left|\mathcal{U}(\mathcal{A}) \cap \mathcal{P}_{\lfloor \frac{n}{2} \rfloor}\right|}{\binom{n}{\lceil \frac{n}{2} \rceil}}$$

and

$$\sum_{\substack{X \in \mathcal{U}(\mathcal{A}) \\ |X| > \lfloor \frac{n}{2} \rfloor}} \frac{|Z_{\mathcal{A}}(X)|}{|X|\binom{n}{|X|}} = 1 - \frac{\left|\mathcal{U}(\mathcal{A}) \cap \mathcal{P}_{\lfloor \frac{n}{2} \rfloor}\right|}{\binom{n}{\lceil \frac{n}{2} \rceil}},$$

which are half-way types of the original AZ identity. If $\mathcal{A}$ is an antichain then $Z_{\mathcal{A}}(A) = A$ for all $A \in \mathcal{A}$. So we get

$$\sum_{A \in \mathcal{A}, |A| \le \lfloor \frac{n}{2} \rfloor} \frac{1}{\binom{n}{|A|}} \le \frac{\left|\mathcal{U}(\mathcal{A}) \cap \mathcal{P}_{\lfloor \frac{n}{2} \rfloor}\right|}{\binom{n}{\lceil \frac{n}{2} \rceil}}$$

and

$$\sum_{A\in\mathcal{A},|A|>\lfloor\frac{n}{2}\rfloor}\frac{1}{\binom{n}{|A|}}\le 1-\frac{\left|\mathcal{U}(\mathcal{A})\cap\mathcal{P}_{\lfloor\frac{n}{2}\rfloor}\right|}{\binom{n}{\lceil\frac{n}{2}\rceil}},$$

which are the half-way type of the LYM-inequality. If $|A|\le\lfloor\frac{n}{2}\rfloor$ for all $A\in\mathcal{A}$ then $\sum_{A\in\mathcal{A}}\frac{1}{\binom{n}{|A|}}\le\frac{\left|\mathcal{U}(\mathcal{A})\cap\mathcal{P}_{\lfloor\frac{n}{2}\rfloor}\right|}{\binom{n}{\lceil\frac{n}{2}\rceil}}\le 1.$

On the other hand, we put $m=-1$ to get the following identity when $\mathcal{A}$ is an antichain of sets with $1<|A|\le\lfloor\frac{n}{2}\rfloor$ for every $A\in\mathcal{A}$:

$$\sum_{A\in\mathcal{A}}\frac{1}{\binom{n-1}{|A|-1}}+\sum_{\substack{X\in\mathcal{U}(\mathcal{A})\setminus\mathcal{A}\\|X|\le\lfloor\frac{n}{2}\rfloor}}\frac{|Z_{\mathcal{A}}(X)|-1}{(|X|-1)\binom{n-1}{|X|-1}}=\frac{\left|\mathcal{U}(\mathcal{A})\cap\mathcal{P}_{\lfloor\frac{n}{2}\rfloor}\right|}{\binom{n-1}{\lceil\frac{n}{2}\rceil}}.$$

So the deficiency of the Bollobás inequality (see Theorem 2) is

$$\Delta(\mathcal{A})=1-\frac{\left|\mathcal{U}(\mathcal{A})\cap\mathcal{P}_{\lfloor\frac{n}{2}\rfloor}\right|}{\binom{n-1}{\lceil\frac{n}{2}\rceil}}+\sum_{\substack{X\in\mathcal{U}(\mathcal{A})\setminus\mathcal{A}\\|X|\le\lfloor\frac{n}{2}\rfloor}}\frac{|Z_{\mathcal{A}}(X)|-1}{(|X|-1)\binom{n-1}{|X|-1}},$$

where $\mathcal{A}$ is an intersecting antichain of sets whose cardinalities are not exceeding $\lfloor\frac{n}{2}\rfloor$. The Bollobás inequality (2) is equivalent with $\Delta(\mathcal{A})\ge 0$.

Since $\binom{n-1}{|A|-1}\le\binom{n-1}{k-1}\le\binom{n-1}{\lfloor\frac{n}{2}\rfloor-1}=\binom{n-1}{\lceil\frac{n}{2}\rceil}$ for $1\le|A|\le k\le\frac{n}{2}$ and $Z_{\mathcal{A}}(X)\supset\bigcap_{A\in\mathcal{A}}A$ for all $X\in\mathcal{U}(\mathcal{A})$, the following corollary can be deduced immediately from the above identity.

Corollary 3 *Let $\mathcal{A}$ be an antichain and $\bigcap_{A\in\mathcal{A}}A\neq\emptyset$, k be an integer such that $|A|\le k\le\frac{n}{2}$ for every $A\in\mathcal{A}$. Put $a=|\bigcap_{A\in\mathcal{A}}A|$. Then*

(a) $\sum_{A\in\mathcal{A}}\frac{1}{\binom{n-1}{|A|-1}}\le\frac{|\mathcal{U}(\mathcal{A})\cap\mathcal{P}_{\lfloor\frac{n}{2}\rfloor}|}{\binom{n-1}{\lceil\frac{n}{2}\rceil}}\le\frac{\binom{n-a}{\lceil\frac{n}{2}\rceil}}{\binom{n-1}{\lceil\frac{n}{2}\rceil}}$, *and*

(b) $|\mathcal{A}|\le\frac{|\mathcal{U}(\mathcal{A})\cap\mathcal{P}_{\lfloor\frac{n}{2}\rfloor}|}{\binom{n-1}{\lceil\frac{n}{2}\rceil}}\binom{n-1}{k-1}\le\frac{\binom{n-a}{\lceil\frac{n}{2}\rceil}}{\binom{n-1}{\lceil\frac{n}{2}\rceil}}\binom{n-1}{k-1}\le\binom{n-1}{k-a}.$

4 Half-Way Identities for Two Set Families

Theorem 4 *Let $A_1, A_2, \ldots, A_q, B_1, B_2, \ldots, B_q$ be subsets of $[n]$ such that $A_i \subset B_j$ if and only if $i = j$, and let m be an integer such that $m + |A_i| > 0$ for $1 \le i \le q$. Put $a_i = |A_i|, b_i = |B_i|, \mathcal{A} = \{A_1, A_2, \ldots, A_q\}$, and $\mathcal{B} = \{B_1, B_2, \ldots, B_q\}$. If $|B_i| \le \lfloor \frac{n}{2} \rfloor$ for all $1 \le i \le q$ then*

$$\sum_{\substack{X \in \mathcal{U}(\mathcal{A}) \setminus \mathcal{D}(\mathcal{B}) \\ |X| \le \lfloor \frac{n}{2} \rfloor}} \frac{m + |Z_{\mathcal{A}}(X)|}{(m + |X|)\binom{m+n}{m+|X|}} = \frac{\left|\mathcal{U}(\mathcal{A}) \cap \mathcal{P}_{\lfloor \frac{n}{2} \rfloor}\right|}{\binom{m+n}{\lceil \frac{n}{2} \rceil}} - \sum_{i=1}^{q} \frac{1}{\binom{m+n-b_i+a_i}{m+a_i}}, \qquad (6)$$

and

$$\sum_{\substack{X \in \mathcal{U}(\mathcal{A}) \setminus \mathcal{D}(\mathcal{B}) \\ |X| > \lfloor \frac{n}{2} \rfloor}} \frac{m + |Z_{\mathcal{A}}(X)|}{(m + |X|)\binom{m+n}{m+|X|}} = 1 - \frac{\left|\mathcal{U}(\mathcal{A}) \cap \mathcal{P}_{\lfloor \frac{n}{2} \rfloor}\right|}{\binom{m+n}{\lceil \frac{n}{2} \rceil}}. \qquad (7)$$

To prove Theorem 4, we reuse a lemma in [17].

Lemma 3 [17] *Let a, b, c be integers such that $a \ge 0$, $b > 0$, and $c \ge a + b$. Then*

$$\sum_{k=0}^{a} \binom{a}{k} \frac{1}{(b+k)\binom{c}{b+k}} = \frac{1}{b\binom{c-a}{b}}.$$

Proof of Theorem 4 The identity (7) is deduced immediately from (6) and Theorem 5 in [17]. Now we present a proof of (6). By using Theorem 3 we can rewrite the identity (6) as

$$\sum_{\substack{X \in \mathcal{U}(\mathcal{A}) \setminus \mathcal{D}(\mathcal{B}) \\ |X| \le \lfloor \frac{n}{2} \rfloor}} \frac{m + |Z_{\mathcal{A}}(X)|}{(m + |X|)\binom{m+n}{m+|X|}} = \sum_{\substack{X \in \mathcal{U}(\mathcal{A}) \\ |X| \le \lfloor \frac{n}{2} \rfloor}} \frac{m + |Z_{\mathcal{A}}(X)|}{(m + |X|)\binom{m+n}{m+|X|}} - \sum_{i=1}^{q} \frac{1}{\binom{m+n-b_i+a_i}{m+a_i}}$$

which is equivalent to the following identity

$$\sum_{\substack{X \in \mathcal{U}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B}) \\ |X| \leq \lfloor \frac{n}{2} \rfloor}} \frac{m + |Z_{\mathcal{A}}(X)|}{\left(m + |X|\right) \binom{m+n}{m+|X|}} = \sum_{i=1}^{q} \frac{1}{\binom{m+n-b_i+a_i}{m+a_i}}. \tag{8}$$

For each $X \in \mathcal{U}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$, there exist $i, j \in \{1, 2, \ldots, q\}$ such that $A_i \subset X \subset B_j$ which implies $A_i \subset B_j$: by the hypothesis of Theorem 4 we must have i=j, i.e $A_i \subset X \subset B_i$. If $A_k \subset X \subset B_k$ for some k, then $A_i \subset X \subset B_k$ which implies $A_i \subset B_k$, so i=k. Therefore, there exists a unique $i \in \{1, 2, \ldots, q\}$ such that $A_i \subset X \subset B_i$ for each $X \in \mathcal{U}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$. Besides, if $A_k \subset X$, then $A_k \subset X \subset B_i$ and so i=k and finally $Z_{\mathcal{A}}(X) = A_i$. Hence, the left hand side of (8) is

$$\mathrm{LHS}(8) = \sum_{i=1}^{q} \sum_{\substack{A_i \subset X \subset B_i \\ |X| \leq \lfloor \frac{n}{2} \rfloor}} \frac{m + a_i}{(m + |X|) \binom{m+n}{m+|X|}}.$$

Since $|B_i| \leq \lfloor \frac{n}{2} \rfloor$, the condition $A \subset X \subset B$ also implies $|X| \leq \lfloor \frac{n}{2} \rfloor$. Finally, we use Lemma 3 to obtain

$$\begin{aligned}
\mathrm{LHS}(8) &= \sum_{i=1}^{q} \sum_{A_i \subset X \subset B_i} \frac{m + a_i}{(m + |X|) \binom{m+n}{m+|X|}} \\
&= \sum_{i=1}^{q} \sum_{k=0}^{b_i - a_i} \binom{b_i - a_i}{k} \frac{m + a_i}{(m + a_i + k) \binom{m+n}{m+a_i+k}} \\
&= \sum_{i=1}^{q} (m + a_i) \frac{1}{(m + a_i) \binom{m+n-(b_i-a_i)}{m+a_i}} = RHS(8). \qquad \square
\end{aligned}$$

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References

1. Ahlswede, R., Zhang, Z.: An identity in combinatorial extremal theory. Adv. Math **80**, 137–151 (1990)
2. Ahlswede, R., Cai, N.: A generalisation of the AZ identity. Combinatorica **13**, 241–247 (1993)
3. Ahlswede, R., Cai, N.: Incomparability and intersection properties of Boolean interval lattices and chain posets. Europ. J. Comb. **17**, 677–687 (1996)
4. Beck, M., Wang, X., Zaslavsky, T.: A unifying generalization of Sperner's theorem, More Sets, Graphs and Numbers: a Salute to Vera Sos and Andras Hajnal. In: Gyori, E., Katona, G.O.H., Lovasz, L. (eds.) Bolyai Society Mathematical Studies 15, pp. 9–24. Springer, Berlin, and Janos Bolyai Mathematical Society, Budapest (2006)

5. Bey, C.: Polynomial LYM inequalities. Combinatorica **25**, 19–38 (2005)
6. Bey, C.: Quadratic LYM-type inequalities for intersecting Sperner families, Discrete Mathematics and Theoretical Computer Science (DMTCS), Nancy, France (2005)
7. Bollobás, B.: Combinatorics: Set systems, Hypergraphs, Families of vectors, and Combinatorial probability. pp. 95–97. Cambridge University Press, Cambridge (1986)
8. Chudak, F., Griggs, J.R.: A new extension of Lubell's inequality to the lattice of divisors. Stud. Sci. Math. Hung. **35**, 347–351 (1999)
9. Daykin, D.E., Thu, T.D.: The dual of Ahlswede–Zhang identity. J. Combin. Theory Ser. A **68**, 246–249 (1994)
10. Erdös, P.L., Frankl, P., Kleitman, D.J., Saks, M.E., Székely, L.A.: Sharpening the LYM inequality. Combinatorica **12**, 287–293 (1992)
11. Erdös, P.L., Székely, L.A.: Pseudo-LYM inequalities and AZ identities. Adv. Appl. Math **19**, 431–443 (1997)
12. Hilton, A.J.W., Stirling, D.S.G.: The dual of an identity of Ahlswede and Zhang. Congr. Numer. **130**, 113–126 (1998)
13. Katona, G.O.H.: Back to the old LYM (YBLM) inequality, Fifth Cracow Conference on Graph Theory, Ustron, Poland, September 11–15 (2006)
14. Patkós, B.: AZ-type identities, www published Nov. 2005. http://web.ceu.hu/math/Research/Preprints_Archives/Patkos3.pdf Accessed 17 Mar 2011
15. Radcliffe, A.J., Szaniszló, Z.: Extremal Cases of the Ahlswede–Cai Inequality. J. Combin. Theory Ser. A **76**, 108–120 (1996)
16. Thu, T.D.: An Induction proof of the Ahlswede–Zhang identity. J. Combin. Theory Ser. A **62**, 168–169 (1993)
17. Thu, T.D.: An AZ-style identity and Bollobas deficiency. J. Combin. Theory Ser. A **114**(8), 1504–1514 (2007)
18. Tuza, Z.: Helly-type hypergraphs and Sperner families. Eur. J. Combin. **5**, 185–187 (1984)