Math Level 2 Handout

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March-April 2020

1 Basic Counting Techniques

1.1 Introduction and Definitions

Definition 1. (Factorial) The factorial of n is denoted as $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$.

Definition 2. (Order) When **order matters**, that means ABC is different from CBA. When **order doesn't matter**, that means ABC is the same as CBA. For example, if you have an ice cream cone with a scoop of strawberry, vanilla, and chocolate, it probably doesn't matter what order it's in. However, what does matter is that the cherry is on top and not inside an ice cream scoop.

Definition 3. (Combination) The number of ways to choose k elements from n total elements when **order doesn't matter** is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Definition 4. (Permutation) The number of ways to choose k elements from n total elements when **order matters** is $\frac{n!}{(n-k)!}$.

1.2 Permutations

There are two types of permutations:

1. Distinguishable Permutations
2. Indistinguishable Permutations

Distinguishable permutations occur when you are looking for the number of permutations of something with size n taken k at a time (in other words, $P(n, k)$).

Indistinguishable permutations can be shown in the following theorem:

Theorem 1.1. A word with N letters and the 1st distinct letter has frequency n_1, 2nd distinct letter has frequency n_2, and so on until the last (let's call it kth) (distinct) letter has frequency n_k. The number of ways to permute it is

$$\frac{N!}{(n_1)!(n_2)!\dots(n_k)!}.$$

1.3 Combinations

Combinations are very useful, because they not only help reduce notation they also help simplify other quantities.

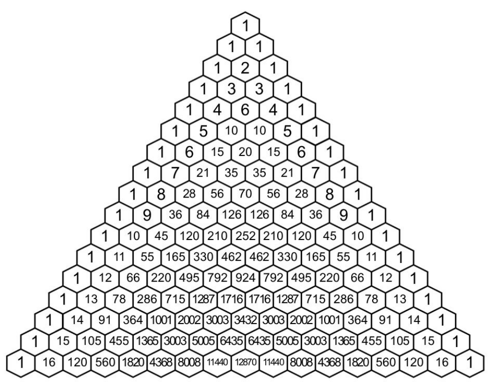
Theorem 1.2. (Binomial Theorem) For real or complex a, b, and non-negative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-1} a b^{n-1} + b^n,$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is a binomial coefficient. In other words, the coefficients when $(a+b)^n$ is expanded and like terms are collected are the same as the entries in the nth row of Pascal's Triangle.

For example, $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, with coefficients $1 = \binom{5}{0}$, $5 = \binom{5}{1}$, $10 = \binom{5}{2}$, etc.

So what is Pascal's Triangle? Well, it looks like this:



A few properties of Pascal's Triangle:

1. The sum of the two terms above a term equal that term. This means $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.
2. The number in the nth row (starting from row 0) and k to the right is $\binom{n}{k}$.
3. The sum of the values in the nth row is 2^n.

4. The 1st downward diagonal is a row of 1's, the 2nd downward diagonal on each side consists of the natural numbers, the 3rd diagonal the triangular numbers, and the 4th the pyramidal numbers.

Theorem 1.3. **(Hockey Stick Identity)** For nonnegative integers n, k,

$$\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}.$$

1.4 Constructive Counting

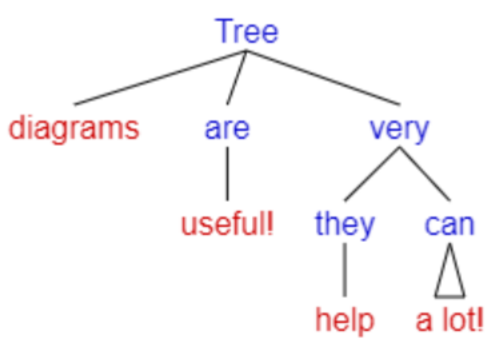
Constructive counting is basically just calculating several pieces of the answer and combining them through addition and multiplication (and occasionally subtraction).

1.5 Casework

There are two main ways (that I know of) to keep your casework organized:

1. Lists
2. Tree Diagrams

Either way is fine, but I usually prefer lists. However, I will teach you both ways.
Lists are simply used to list out the possibilities one by one. **Tree diagrams** look like the following:



They are used when some cases have multiple smaller cases.

Example 1.1. How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

(A) 41 (B) 42 (C) 43 (D) 44 (E) 45

Solution. If the middle digit is the average of the first and last digits, twice the middle digit must be equal to the sum of the first and last digits.

Let's do some **casework**:

- If the middle digit is 1, possible numbers range from 111 to 210. So there are 2 numbers in this case.
- If the middle digit is 2, possible numbers range from 123 to 420. So there are 4 numbers in this case.
- If the middle digit is 3, possible numbers range from 135 to 630. So there are 6 numbers in this case.
- If the middle digit is 4, possible numbers range from 147 to 840. So there are 8 numbers in this case.
- If the middle digit is 5, possible numbers range from 159 to 951. So there are 9 numbers in this case.
- If the middle digit is 6, possible numbers range from 369 to 963. So there are 7 numbers in this case.
- If the middle digit is 7, possible numbers range from 579 to 975. So there are 5 numbers in this case.
- If the middle digit is 8, possible numbers range from 789 to 987. So there are 3 numbers in this case.
- If the middle digit is 9, the only possible number is 999. So there is 1 number in this case.

So the total number of three-digit numbers that satisfy the property is $2+4+6+8+9+7+5+3+1 = 45 \Rightarrow E$

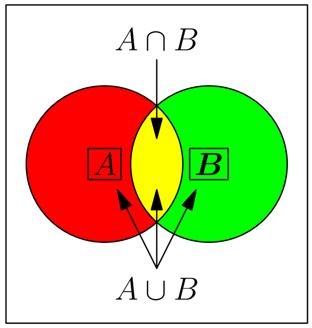
1.6 Complementary Counting

Instead of counting what we **want**, we count what we **don't want** and subtract from the **total number of possibilities**.

1.7 Venn Diagrams and PIE

A set is a group of elements, which are simply whatever you want them to be. You could have the elements "Harry Potter", "Batman Comics", and "Math Textbook", and if you put them in one set it would be a set of books. Usually, when we say "set" we mean "set of numbers", where the elements are numbers. We use curly brackets to show the elements of a set. If we have a set S, then we could have $S = \{1, 2, 3, 4, 5, 6, 7\}$, so the elements are $1, 2, 3, 4, 5, 6, 7$.

A Venn diagram is a diagram that shows the intersection and union of two sets. The **intersection** of two sets is the part they have in common, and the **union** of two sets is the part they have in total. Below is a representation of two sets A and B:



Now for PIE, which stands for Principle of Inclusion-Exclusion.

Theorem 1.4. (PIE Simplified) Let A and B be two sets, and let $A \cap B$ be the intersection of the two sets, and let $A \cup B$ be the union of the two sets. Then,

$$\#(A \cup B) = \#A + \#B - \#(A \cap B),$$

where $\#$ means the number of elements in a set.

Theorem 1.5. (PIE Generalized) Let $A_1, A_2, \ldots, A_n$ be n sets. Then,

$$\begin{aligned}\#(A_1 \cup A_2 \cup \ldots \cup A_n) = \#A_1 + \#A_2 + \ldots + \#A_n - (\#(A_1 \cap A_2) + \#(A_1 \cap A_3) \\ + \ldots + (A_{n-1} \cap A_n)) + \ldots - (-1)^n \#(A_1 \cap A_2 \cap \ldots \cap A_n).\end{aligned}$$

The super hard way to state this is:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots \ + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

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PIE is extremely useful for overcounting, or in other words when you first get rid of a constraint and count over the amount you need, then subtract to get under the amount you need, and so on until you have considered all the cases.

Example 1.2. (2002 AIME I P1) Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution 1. Using complementary counting, we count all of the license plates that do not have the desired property. In order to not be a palindrome, the first and third characters of each string must be different. Therefore, there are $10 \cdot 10 \cdot 9$ three digit non-palindromes, and there are $26 \cdot 26 \cdot 25$ three letter non palindromes. As there are $10^3 \cdot 26^3$ total three-letter three-digit arrangements, the probability that a license plate does not have the desired property is $\frac{10 \cdot 10 \cdot 9 \cdot 26 \cdot 26 \cdot 25}{10^3 \cdot 26^3} = \frac{45}{52}$. We subtract this from 1 to get $1 - \frac{45}{52} = \frac{7}{52}$ as our probability. Therefore, our answer is $7 + 52 = \boxed{059}$.

Solution 2. Consider the three-digit arrangement, $\overline{aba}$. There are 10 choices for a and 10 choices for b (since it is possible for $a = b$), and so the probability of picking the palindrome is $\frac{10 \times 10}{10^3} = \frac{1}{10}$. Similarly, there is a $\frac{1}{26}$ probability of picking the three-letter palindrome.
By the Principle of Inclusion-Exclusion, the total probability is

$$\frac{1}{26} + \frac{1}{10} - \frac{1}{260} = \frac{35}{260} = \frac{7}{52} \quad \Longrightarrow \quad 7 + 52 = \boxed{059}$$

Example 1.3. (2017 AMC10B P13) There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

$$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 5$$

Solution. By PIE (Property of Inclusion/Exclusion), we have

$$|A_1 \cup A_2 \cup A_3| = \sum |A_i| - \sum |A_i \cap A_j| + |A_1 \cap A_2 \cap A_3|.$$

Number of people in at least two sets is $\sum |A_i \cap A_j| - 2|A_1 \cap A_2 \cap A_3| = 9$. So, $20 = (10 + 13 + 9) - (9 + 2x) + x$, which gives $x = \boxed{\textbf{(C)}\ 3}$.

1.8 Common Types of Problems

The following are problems that appear a lot in basic counting:

1. People in a Club (PIE)
2. Handshakes
3. Standing in a Line
4. Sitting in a Circle
   - Beads in a Necklace
   - Keychains
5. Digit / Password / Telephone Permutation Counting
6. Card Counting

1.9 Problems

Problem 1.1. Calculate $\frac{100!}{98!}$.

Problem 1.2. Calculate $\binom{100}{2}$.

Problem 1.3. How many ways are there to seat three people in three different chairs?

Problem 1.4. How many ways are there to seat three people in three different chairs around a table, given rotations are the same?

Problem 1.5. How many numbers less than or equal to 100 are divisible by either 2 or 3?

Problem 1.6. **(2011 AMC8 P6)** In a town of 351 adults, every adult owns a car, motorcycle, or both. If 331 adults own cars and 45 adults own motorcycles, how many of the car owners do not own a motorcycle?

(A) 20 **(B)** 25 **(C)** 45 **(D)** 306 **(E)** 351

Problem 1.7. **(Mathcounts 2008 National Target P1)** How many numbers less than or equal to 100 are divisible by 2 or 3 but not 4?

Problem 1.8. How many ways are there to pick a three digit number that does **not** start with 3?

Problem 1.9. There are n people at a party. Each person shakes hands with every person (besides themselves) exactly once. How many handshakes occur?

Problem 1.10. How many diagonals does an n-sided polygon have?

Problem 1.11. How many different words can be formed with the letters $ABBCD$? (Not necessarily meaningful words)

Problem 1.12. What is $(x+2)^3$?