Math Level 2.5 Handouts

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July 29, 2020

§0 Contents

§1 Advanced Sequences and Series

§1.1 Sequences and Series Basics

§1.1.1 Arithmetic Sequences and Series

Definition 1 (Arithmetic Sequence). An **arithmetic sequence** is a sequence of numbers in which each term is given by adding a fixed value to the previous term.

For example, $-2, 1, 4, 7, 10, \ldots$ is an arithmetic sequence because each term is three more than the previous term. In this case, 3 is called the **common difference** of the sequence. More formally, an arithmetic sequence a_n is defined recursively by a first term a_0 and $a_n = a_{n-1} + d$ for $n \geq 1$, where d is the common difference. Explicitly, it can be defined as $a_n = a_0 + dn$.

Theorem 1 (Terms of an Arithmetic Sequence). The nth term in an arithmetic sequence is described

$$a_n = a_1 + d(n-1),$$

where a_n is the nth term, a_1 is the first term, and d is the difference between consecutive terms.

Theorem 2 (Sum of an Arithmetic Sequence). The sum of the first n terms of an arithmetic sequences is

$$s_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2a_1 + (n-1)d).$$

Now for a few applications:

Theorem 3 (Sum of First n Positive Integers). For all positive integers n,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

§1.1.2 Geometric Sequences and Series

Definition 2 (Geometric Sequence). A **geometric sequence** is a sequence of numbers in which each term is a fixed multiple of the previous term.

For example: $1, 2, 4, 8, 16, 32, \ldots$ is a geometric sequence because each term is twice the previous term. In this case, 2 is called the common ratio of the sequence. More formally, a geometric sequence may be defined recursively by:

$$a_n = r \cdot a_{n-1}, n > 1,$$

with a fixed first term a_1 and common ratio r. Using this definition, the nth term has the closed-form:

$$a_n = a_1 \cdot r^{n-1}.$$

Theorem 4 (Sum of a Finite Geometric Sequence). The sum of the first n terms of a geometric sequence is given by

$$S_n = a_1 + a_2 + \cdots + a_n = a_1 \cdot \frac{r^n - 1}{r - 1},$$

where a_1 is the first term in the sequence, and r is the common ratio.

Definition 3 (Infinite Geometric Sequence). An **infinite geometric sequence** is a geometric sequence with an infinite number of terms.

Theorem 5 (Sum of an Infinite Geometric Sequence). Let $S = a + ar + ar^2 + \ldots$, where $|r| < 1$. Then

$$S = \frac{a}{1 - r}.$$

Example 1. Compute $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$.

Solution. Using the Sum of an Infinite Geometric Sequence formula, we get

$$\frac{1}{1 - \frac{1}{2}} = \boxed{2}.$$

□

Summing arithmetic sequences are sometimes known as **Gauss sums**.

§1.2 Problems

Today, we will be doing more **arithmetic sequence** problems than **geometric sequence** problems. Later on, we can return to geometric sequence problems. Here is a list of exercises, **all of the same variety**, but each one is different from the others. Instead of giving the answers like I normally do, I won't provide any answers to avoid spoiling the fun.

Exercise 1. Evaluate $1 + 2 + \ldots + 200$.

Exercise 2. Compute $2 + 8 + 14 + \ldots + 152$.

Exercise 3. Determine $-91 + -84 + \ldots + 105$.

Exercise 4. An auditorium has weird rows. The first row has 1 seat, the second has 3, the third has 5, the fourth has 7, and so on. If there are n rows in the auditorium, how many seats are there in total?

Exercise 5 (AMC 10A 2015). How many terms are in the arithmetic sequence $13, 16, 19, \dots, 70, 73$?

Exercise 6 (AMC 10B 2015). What are the sign and units digit of the product of all the odd negative integers strictly greater than -2015?

Exercise 7 (AMC 8 2013). What is the value of $4 \times (-1 + 2 - 3 + 4 - 5 + 6 - 7 + \ldots + 1000)$?

Exercise 8 (AHSME 1997). Consider the sequence

$$1, -2, 3, -4, 5, -6, \ldots$$

whose nth term is $(-1)^{n+1} \cdot n$. What is the average of the first 200 terms of the sequence?

Exercise 9 (AMC 8 2013). When Clara totaled her scores, she inadvertently reversed the units digit and the tens digit of one score. By which of the following might her incorrect sum have differed from the correct one?

(A) 45 (B) 46 (C) 47 (D) 48 (E) 49

Exercise 10 (AMC 8 2013). The sum of six consecutive positive integers is 2013. What is the largest of these six integers?

Exercise 11 (AMC 10A 2011). Let X and Y be the following sums of arithmetic sequences:

$$X = 10 + 12 + 14 + \ldots + 100,$$

$$Y = 12 + 14 + 16 + \ldots + 102.$$

What is the value of $Y - X$?

Exercise 12 (Mathcounts School 2015). The sum of the first $n + 2$ counting numbers is 43 more than the sum of the first n counting numbers. What is the value of n?

Exercise 13 (Mathcounts School 2015). What is the sum of the first 200 terms of the arithmetic sequence $5, 12, 19, 26, \ldots$?

Exercise 14 (Mathcounts Chapter 2015). For some integer k, the first three terms of an arithmetic sequence are $k, 2k+3$, and $4k+1$, in that order. What is the integer value of the fourth term of the sequence?

Exercise 15 (AHSME 1997). How many two-digit positive integers N have the property that the sum of N and the number obtained by reversing the order of the digits of N is a perfect square?

Exercise 16 (Mathcounts State 2015). Consider an arithmetic sequence with $a_3 = 165$ and $a_{12} = 615$. For what value of n is $a_n = 2015$?

Exercise 17 (AHSME 1999). It took 600 digits to label the pages of a book starting with page one. How many pages does the book have?

Exercise 18. Find $\underbrace{333\ldots333}_{2020} \cdot \underbrace{666\ldots666}_{2020}$. (Hint: Try smaller cases!)

Exercise 19 (Mathcounts Chapter 2015). If the sum of an arithmetic progression of six positive integer terms is 78 what is the greatest possible difference between consecutive terms?

Exercise 20 (Mathcounts State 2015). An arithmetic sequence has first term a and common difference d. If the sum of the first ten terms is half the sum of the next ten terms, what is the ratio $\frac{a}{d}$? Express your answer as a common fraction.

Exercise 21. Determine

$$1+2-3+4+5-6+7+8-9+\ldots+97+98-99+100.$$

Exercise 22 (AMC 12B 2002). If a, b, c, d are positive real numbers such that a, b, c, d form an increasing arithmetic sequence, and a, b, d form a geometric sequence, then $\frac{a}{d}$ is __.

Exercise 23 (FURMAN 2000). How many terms of the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$$

must be added for the sum to first exceed 0.999?

This next problem is **hard**. Good luck!

Exercise 24 (FURMAN 2004). Evaluate

$$\sum_{k=1}^{200} \frac{1}{k^2 + k}.$$