Math Level 2 Week 1

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Contents

1 Warm-Up

Problem 1 (Mathcounts). Find y: $\sqrt{19+3y} = 7$.

Problem 2 (Alcumus). How many terms are in the expansion of $(a+b+c)(d+e+f+g)$?

2 Linear Functions

Recall that a ***polynomial function*** of x with degree n is defined as follows:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x^1 + a_0 x^0.$$

Note that $a_1 x^1 = a_1 x$ and $a_0 x^0 = a_0$.

Constant Function
If $f(x) = a_0$, then $f(x)$ is a ***constant function***.

Linear Function
If $f(x) = a_1 x + a_0$, then $f(x)$ is a ***linear function***.

Slope
The ***slope*** is the rate of change line describing the steepness and direction of a function at that point.

Intercept
The ***x-intercept*** is where a function intersects the x-axis, and the ***y-intercept*** is where a function intersects the y-axis.

Theorem 2.5 (Slope of a Line)
The slope of a line going through (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Example 2.6
Find the slope of a line going through $(1, 2)$ and $(2, 3)$.

Solution. Using our formula, we get $\frac{3-2}{2-1} = \boxed{1}$. □

3 Quadratics

Degree
The ***degree*** of a polynomial is its largest exponent.

Quadratic Function
If $f(x) = ax^2 + bx + c$, then $f(x)$ is a ***quadratic function***.

Axis of Symmetry
The ***axis of symmetry*** of a parabola is the line such that reflecting one side of the parabola across the line will yield the other side.

Theorem 3.4 (Quadratic Minima/Maxima)
If $a > 0$, f has a minimum at $x = -\frac{b}{2a}$. If $a < 0$, f has a maximum at $x = -\frac{b}{2a}$.

The **standard form** of a quadratic is $f(x) = ax^2 + bx + c$. The **factored form** is $f(x) = a(x - x_1)(x - x_2)$, where x_1 and x_2 are the roots of $f(x)$ (i.e. $f(x_1) = f(x_2) = 0$). The **vertex form is** $f(x) = a(x - h)^2 + k$, implying the **axis of symmetry** is $x = h$ and the ***vertex*** is (h, k). Note that the minimum/maximum is located at (h, k).

Theorem 3.5 (Vertex of a Quadratic)
The vertex of $ax^2 + bx + c$ is $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$.

Now let's discuss ***completing the square***:

Example 3.6
Find the roots of $x^2 + 8x + 12 = 0$.

Solution. Let's try to make the left hand side a **square**. The idea to do this is to take $x^2 + 8x$, which looks like $x^2 + 8x + 16 = (x + 4)^2$, but it is missing the 16, so $x^2 + 8x = (x + 4)^2 - 16$. Thus, $x^2 + 8x + 12 = (x + 4)^2 - 16 + 12 = 0$, so $(x + 4)^2 - 4 = 0$, which we can rearrange to get $(x + 4)^2 = 4, x + 4 = \pm 2$. Thus, the roots are $x = \boxed{-2, -6}$. □

This is the motivation behind completing the square.

Theorem 3.7 (Completing the Square)
If we can write $f(x) = ax^2 + bx + c$ as $a(x - h)^2 + k$, then the roots of $f(x)$ are

$$x_1, x_2 = h \pm \sqrt{-k}.$$

Note that if $k > 0$, then the roots are not real.

What if we don't have a nice expression? Then let's find a general way to solve a quadratic:

Theorem 3.8 (Quadratic Formula)
Let $ax^2 + bx + c = 0$. Then

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 3.9
Find the roots of $-x^2 + 4x + 5$.

Solution. Using the Quadratic Formula, we get

$$x_1, x_2 = \frac{-4 \pm \sqrt{4^2 - 4(-1)(5)}}{2(-1)} = \boxed{-1, 5}.$$

□

Po Shen Loh recently talked about a method for solving quadratics, which can be found here.

Discriminant
The ***discriminant*** Δ of the quadratic $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac.$$

Theorem 3.11 (Discriminant Test)
Let Δ be the discriminant of a quadratic. Then:

1. If $\Delta > 0$, then the roots are real and unequal.
2. If $\Delta = 0$, then the roots are real and equal.
3. If $\Delta < 0$, then the roots are complex and unequal.

Example 3.12
If a quadratic equation $2x^2 - kx + 3 = 0$ have imaginary roots, what is the value of k?

Solution. The discriminant is $k^2 - 4 \cdot 2 \cdot 3 = k^2 - 24 < 0$, so $k^2 < 24$, implying

$$\boxed{-2\sqrt{6} < k < 2\sqrt{6}}.$$

□

Example 3.13
If $y = 3x^2 - 2x + k$ is positive for all x, then what is the smallest integral value of k?

Solution. Since it is always possible, $3x^2 - 2x + k$ has no real solutions, implying all roots are complex. Thus, $4 - 12k < 0 \implies k > \frac{1}{3}$, so the smallest integral value of k is $k = \boxed{1}$. □

4 General Polynomials

4.1 Factor and Remainder Theorem

Theorem 4.1 (Factor Theorem)
If $p(a) = 0$, then $p(x)$ has a factor of $x - a$. Furthermore, $p(x) = (x - a)Q(x)$, where $Q(x)$ is the quotient (the remainder is 0).

Theorem 4.2 (Remainder Theorem)
When a polynomial $P(x)$ is divided by $x - a$, the remainder R is equal to $P(a)$. Furthermore, $P(x)$ can be expressed as follows:

$$P(x) = (x - a)Q(x) + R.$$

The identical equation is true for any value of x, especially $x = a$. Therefore,

$$P(a) = R.$$

Example 4.3
If a polynomial $f(x) = 2x^2 - 3x + 5$ is divided by $x - 1$, what is the remainder?

Solution. The remainder is $f(1) = 2 \cdot 1^2 - 3 \cdot 1 + 5 = \boxed{4}$. □

Example 4.4
Let $P(x)$ be a polynomial in terms of x. When $P(x)$ is divided by $x - 9$, the remainder is 5, and when $P(x)$ is divided by $x - 5$, the remainder is 9. What is the remainder when $P(x)$ is divided by $(x - 5)(x - 9)$?

Solution. Here, we are dividing by a quadratic rather than a linear term, so we cannot directly use the remainder theorem. The remainder polynomial will be a linear term, not a constant. Instead, let's see what happens when we divide by a quadratic:

$$P(x) = q(x) \cdot (x - 5)(x - 9) + r(x)$$

where $r(x)$ is the remainder polynomial. We know that $r(x)$ is a linear polynomial, so we have

$$P(x) = q(x) \cdot (x-5)(x-9) + ax + b.$$

Here, we see that just like the previous examples, we can plug in values for x. Plugging in $x = 5$, we have $P(5) = 5a + b$ and plugging in $x = 9$ gives $P(9) = 9a + b$. Also, we are given that the remainder, $ax + b$ is equal to 9 and 5 when dividing by $x - 5$ and $x - 9$, respectively. Therefore, we have $5a + b = 9$ and $9a + b = 5$. Solving the system, we get $(a, b) = (-1, 14)$, so the remainder is $ax + b = \boxed{-x + 14}$. □

4.2 Long Division of Polynomials

Example 4.5
Find the quotient and remainder when $2x^4 + 4x^2 - 1$ is divided by $x + 1$.

Solution. Let's use long division:

$$\begin{array}{r|l}
 & \quad 2x^3 - 2x^2 + 6x - 6. \\ \hline
x+1 & \quad 2x^4 \qquad\quad + 4x^2 \qquad - 1 \\
 & -2x^4 - 2x^3 \\ \cline{2-2}
 & \qquad -2x^3 + 4x^2 \\
 & \qquad\; 2x^3 + 2x^2 \\ \cline{2-2}
 & \qquad\qquad\quad 6x^2 \\
 & \qquad\qquad -6x^2 - 6x \\ \cline{2-2}
 & \qquad\qquad\qquad -6x - 1 \\
 & \qquad\qquad\qquad\;\; 6x + 6 \\ \cline{2-2}
 & \qquad\qquad\qquad\qquad\quad 5
\end{array}$$

Thus, the quotient is $\boxed{2x^3 - 2x^2 + 6x - 6}$ with remainder $\boxed{5}$. □

5 Problems

Problem 3. Find the slope of the line through points $(3, -2)$ and $(-2, -3)$.

Problem 4. Solve $2x^2 - 8x + 6 = 0$ by completing the square.

Problem 5. Use the Quadratic Formula to solve $3x^2 - 4x + 1 = 0$.

Problem 6 (AoPS). Let $f(x) = x^4 - 3x^3 + 7x^2 - x + 5$. What is the remainder when $f(x)$ is divided by $x - 3$? What is $f(3)$?

Problem 7. Let $f(x) = x^9 + x^3 - 5x^2$. Find the remainder when $f(x)$ is divided by $3x - 6$.

Problem 8. If $p(x) = 2x^2 - 3$ and $q(x) = 4x^3 + x^2 + 1$. Find $p(x) + q(x)$ and $p(x) \cdot q(x)$.

Problem 9 (Alcumus). Find the quotient and remainder when $x^6 - 3$ is divided by $x + 1$.

Let's try some harder problems:

Problem 10 (Great Britain). Find the remainder when the polynomial $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$.

Problem 11. Find the remainder when $(x+3)^5 + (x+2)^8 + (5x+9)^{2020}$ is divided by $x + 2$.

Problem 12 (AHSME 1974/4). Find the remainder when $x^{51} + 51$ is divided by $x + 1$.

Problem 13 (AMC 12B 2003/9). Suppose that $P(x)$ is a linear polynomial with $P(6) - P(2) = 12$. What is $P(12) - P(2)$?

Problem 14 (MAΘ 1991). Find all values of m which make $x + 2$ a factor of $x^3 + 3m^2x^2 + mx + 4$.

Problem 15. Let m and n be the roots of the quadratic equation $4x^2 + 5x + 3 = 0$. Find $(m+7)(n+6)$?