ASE Handouts

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Contents

Level 1 Handouts

1 Variables

1.1 Definitions

> **Definition 1.1** (Variable). A **variable** is something that can change. It can also be a symbol for a number we don't know yet.

There are **independent** and **dependent** variables.[1]

> **Definition 1.2** (Consecutive). When two numbers are **consecutive**, we mean they are right next to each other.

For example, 3 is consecutive to 4, 4 is consecutive to 5, 8 is consecutive to 9, 9 is consecutive to 8, etc.[2]

1.2 Statistics

> **Definition 1.3** (Mean). The **mean** (or average) of a set of numbers is the sum of the numbers divided by the number of numbers.

Note that the mean refers to the **arithmetic mean**, not the **geometric mean**.[3]

> **Theorem 1.1** (Arithmetic Mean). Let $a_1, a_2, a_3, \ldots, a_n$ be a set of n numberse. Then the *arithmetic mean* of these numbers is
> $$\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}.$$

> **Definition 1.5** (Median). The number in the middle.

> **Definition 1.6** (Mode). The most common number.

Note that there could be more than one mode. For example, in the set $1, 2, 2, 2, 3, 3, 3$, there are two modes of (size) 3, and the most popular values were 2 and 3.[4]

1.3 Basics

In this lesson, we will only be dealing with basic functions (like linear[5] functions).

> **Example.** If $a + b = 3$, and $a = 1$, what is b?
> *Solution.* Plugging in 1, we get $b = 2$.

1.4 Consecutive Integers

Let's take a look at an example.[6]

> **Example.** Let's say we have 7 consecutive integers that sum to 28. What is the smallest integer of these 7?
> *Solution.* By trial and error we can find the sequence is 1 to 7, so the answer is 1.

Contents

[1] *Independent* variables do not affect one another (for example, $x = 5, y = 6$ do not affect each other, but *dependent* variables do afftect each other (for example, $x = 5y$).

[2] We can also have consecutive **even** integers. For example, if we only consider even numbers, 2 is consecutive to 4. If the problem doesn't specify, then we just mean consecutive integers.

[3]

> **Definition 1.4** (Geometric Mean). The product of the numbers taken to the root of the number of numbers.

[4] When a problem says there is a **unique** mode, that means there is only one mode.

[5] *Linear* means a line. For example, $y = 3x + 4$.

[6] It is always important to look at examples to motivate a problem. This will help explain *why* you are learning this problem.

| However, is there a better way? Answer: there is!

Theorem 1.2. Given n consecutive integers that sum to k, the smallest number is

$$\frac{k}{n} - \frac{n-1}{2},$$

and the largest number is

$$\frac{k}{n} + \frac{n-1}{2}.$$

Note that if you ever get an answer that is not an integer, there are **no solutions**.

1.5 Ratios and Proportions

A **ratio** shows the relative sizes of two or more values. For example, if there are 2 apples, 4 oranges, and 6 bananas, the *ratio* of apples to oranges to bananas is $2 : 4 : 6 = 1 : 2 : 3$. The : symbol represents the word "to".

A **proportion** is a statement that two *ratios* are equal. For example, the following is a proportion:

$$\frac{12}{15} = \frac{4}{5}.$$

Two numbers x, y are said to be *directly proportional* if

$$y = kx$$

for some constant k. For example, if $k = 5$, then if x is 3, y must be 15, and if x is 4, y must be 20. Two numbers a, b are said to be *inversely proportional* if

$$ab = k$$

for some constant k. For example, if $k = 5$, then if x is 1, then y is 5, and if x is 2, then y is $\frac{5}{2}$. *Joint proportions* occur when $x = kyz$, for some constant k. In this case, x and y are joint.

1.6 Moles Digging Holes

Example. 10 moles dig 10 holes in 10 hours, how many holes do 20 moles dig in 20 hours?

Solution. There are twice as many moles, so that doubles the number of holes, and twice as many hours also doubles the number of holes. Thus, the answer is $10 \times 2 \times 2 = 40$.

Theorem 1.3. If we have a moles, b holes, and c hours, then

- a varies directly with b

- b varies directly with c

- c varies inversely with a

1.7 d = rt

There is only one theorem you need to know:

Theorem 1.4 ($d = rt$)**.** Like the title of this theorem says,

$$\text{distance} = \text{rate} \times \text{time}.$$

This implies that distance and rate are **directly proportional** when time is constant, distance and time are **directly proportional** when rate is constant, and rate and time are **inversely proportional** when distance is constant. Even though this is the only necessary formula, let us list one more to speed up computation:

Theorem 1.5. Let person 1 work at a speed of one object per t_1 time, person 2 works at a speed of one object per t_2 time, and so on, all the way to person n who works at a speed of one object per t_n time. Then if they work together, they will finish in

$$\frac{1}{\frac{1}{t_1} + \frac{1}{t_2} + \ldots + \frac{1}{t_n}}$$

time.

Example. Dylan can paint a house in 3 hours. Cody can paint a house in x hours. If they work together, it takes then 2 hours to paint the house. What is x?
Solution. Using the theorem, we get

$$\frac{1}{\frac{1}{3} + \frac{1}{x}} = 2,$$

$$\frac{1}{x} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$

$$x = 6.$$

1.8   Simon's Favorite Factoring Trick

Simon's Favorite Factoring Trick (SFFT) is best explained with an example:

Example. Find all positive integers x, y that satisfy

$$xy - 2x - 4y = 0.$$

Solution. Let us factor the first two terms:

$$x(y - 2) - 4y = 0.$$

We want to find *some way* we can turn the y into a $y - 2$. Let's see what happens if we do that:

$$x(y - 2) - 4(y - 2).$$

Now that we've done it, we can factor! However, because we added a -2 we also have to $+2$:

$$x(y - 2) - 4(y - 2 + 2).$$

Remember, this is still equal to 0. Now, let's take out the $+2$ and distribute it to the other side:

$$x(y - 2) - 4(y - 2) - 8 = 0,$$

$$x(y - 2) - 4(y - 2) = 8.$$

Now, we can factor:
$$(x-4)(y-2) = 8.$$

Because x, y are positive integers, we know that $x-4$ and $y-2$ are simply the positive factors of 8:
$$x-4 = 1, y-2 = 8,$$
$$x-4 = 2, y-2 = 4,$$
$$x-4 = 4, y-2 = 2,$$
$$x-4 = 8, y-2 = 1.$$

Solving, we get $\boxed{(x, y) = (5, 10), (6, 6), (8, 4), (12, 3)}$.

Now for the formal statement:

Theorem 1.6 (SFFT)**.** For all real numbers (although commonly used only for integers) x, y, a, b,
$$xy + xa + yb + ab = (x+b)(y+a).$$

Two special common cases are: $xy + x + y + 1 = (x+1)(y+1)$ and $xy - x - y + 1 = (x-1)(y-1)$.

1.9 Word Problems

These are the problems that use **words** instead of equations. For these types of problems - just convert back to equations! Examples will be given in the **Problems Section**.

Example. Sandwiches at Joe's Fast Food cost 3 each and sodas cost 2 each. How many dollars will it cost to purchase 5 sandwiches and 8 sodas?

(A) 31   (B) 32   (C) 33   (D) 34   (E) 35

Solution. The 5 sandwiches cost $5 \cdot 3 = 15$ dollars. The 8 sodas cost $8 \cdot 2 = 16$ dollars. In total, the purchase costs $15 + 16 = 31$ dollars. The answer is (A).

1.10 Problems

The following problems are listed in (roughly) increasing order of difficulty / topic. Any problems we do not finish in class should be attempted at home.

Problem 1.1. Find the smallest and largest number of a sequence of 7 consecutive integers that sum to 49.

Problem 1.2 (Mathcounts Ratio Warmup)**.** What is $\frac{9}{30}$ as a common fraction?

Problem 1.3 (Mathcounts Ratio Warmup)**.** In a pasture there are 12 white horses and the rest are black. If there are 52 horses in the pasture, what fraction are black? Express your answer as a common fraction.

Problem 1.4 (Mathcounts Ratio Warmup)**.** There are three times the number of orange fish as blue fish in a tank at the pet store, and there are no other fish. What percentage of the fish are orange?

Problem 1.5 (Mathcounts Ratio Stretch)**.** Fairy Godmother has granted wishes to Aurora, Belle and Cindi in the ratio 6 : 8 : 11. What fraction of the ratios were granted to Belle? What percent of the wishes granted by Fairy Godmother were not granted to Aurora? What is the absolute difference between the percents of wishes

Fairy Godmother has granted to Aurora and to Cindi?

Problem 1.6 (Mathcounts Chapter 2019/8). After a brisk workout, Felicia counts 32 heartbeats in 15 seconds. Based on this count, what is Felicia's expected number of heartbeats in one minute?

Problem 1.7. Dylan and Cody are working together again. Dylan can eat one apple in 1 minute. Cody can eat one apple in 2 minutes. How long will it take them working together to eat 15 apples?

Problem 1.8. Alice can mow the lawn in 3 hours, Bob can mow the lawn in 4 hours, and Chris can mow the lawn in 5 hours. Working together, how long will it take them to mow one lawn?

Problem 1.9. Alice mows half the lawn (at the same rate as before), and then Bob joins in to mow a fourth of the lawn, and then Chris joins in to finish the job. How long does it take them in total?

Problem 1.10 (AMC 8 2018/1). An amusement park has a collection of scale models, with a ratio 1 : 20, of buildings and other sights from around the country. The height of the United States Capitol is 289 feet. What is the height in feet of its duplicate to the nearest whole number?

$$\textbf{(A) } 14 \qquad \textbf{(B) } 15 \qquad \textbf{(C) } 16 \qquad \textbf{(D) } 18 \qquad \textbf{(E) } 20$$

Problem 1.11 (AMC 8 2010/2). If $a@b = \frac{a \times b}{a+b}$ for a, b positive integers, then what is 5@10?

$$\textbf{(A) } \frac{3}{10} \qquad \textbf{(B) } 1 \qquad \textbf{(C) } 2 \qquad \textbf{(D) } \frac{10}{3} \qquad \textbf{(E) } 50$$

Problem 1.12 (AMC 8 2004/3). Twelve friends met for dinner at Oscar's Overstuffed Oyster House, and each ordered one meal. The portions were so large, there was enough food for 18 people. If they shared, how many meals should they have ordered to have just enough food for the 12 of them?

$$\textbf{(A) } 8 \qquad \textbf{(B) } 9 \qquad \textbf{(C) } 10 \qquad \textbf{(D) } 15 \qquad \textbf{(E) } 18$$

Problem 1.13 (AMC 8 2007/6). The average cost of a long-distance call in the USA in 1985 was 41 cents per minute, and the average cost of a long-distance call in the USA in 2005 was 7 cents per minute. Find the approximate percent decrease in the cost per minute of a long- distance call.

$$\text{(A) } 7 \qquad \text{(B) } 17 \qquad \text{(C) } 34 \qquad \text{(D) } 41 \qquad \text{(E) } 80$$

Problem 1.14 (AMC 8 2010/6). As Emily is riding her bicycle on a long straight road, she spots Emerson skating in the same direction 1/2 mile in front of her. After she passes him, she can see him in her rear mirror until he is 1/2 mile behind her. Emily rides at a constant rate of 12 miles per hour, and Emerson skates at a constant rate of 8 miles per hour. For how many minutes can Emily see Emerson?

$$\textbf{(A) } 6 \qquad \textbf{(B) } 8 \qquad \textbf{(C) } 12 \qquad \textbf{(D) } 15 \qquad \textbf{(E) } 16$$

Problem 1.15 (AMC 8 2017/23). Each day for four days, Linda traveled for one hour at a speed that resulted in her traveling one mile in an integer number of minutes. Each day after the first, her speed decreased so that the number of minutes to travel one mile increased by 5 minutes over the preceding day. Each of the four days, her distance traveled was also an integer number of miles. What was the total number of

miles for the four trips?

<div align="center">

(A) 10   **(B)** 15   **(C)** 25   **(D)** 50   **(E)** 82

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Problem 1.16 (Mathcounts Chapter 2019/20)**.** Jones is chasing a car 800 meters ahead of him. He is on a horse moving at 50 km/h. If Jones catches up to the car in 4 minutes, how fast was the car moving?

Problem 1.17 (AMC 8 2013/10)**.** What is the ratio of the least common multiple of 180 and 594 to the greatest common factor of 180 and 594?

<div align="center">

(A) 110   **(B)** 165   **(C)** 330   **(D)** 625   **(E)** 660

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Problem 1.18 (AMC 12 2000/6)**.** Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers could be obtained?

<div align="center">

(A) 22   (B) 60   (C) 119   (D) 180   (E) 231

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Problem 1.19 (AMC 10 B 2019/11)**.** Two jars each contain the same number of marbles, and every marble is either blue or green. In Jar 1 the ratio of blue to green marbles is 9 : 1, and the ratio of blue to green marbles in Jar 2 is 8 : 1. There are 95 green marbles in all. How many more blue marbles are in Jar 1 than in Jar 2?

<div align="center">

(A) 5   **(B)** 10   **(C)** 25   **(D)** 45   **(E)** 50

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