# Math Level 2 Week 7

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## 1 Introduction

### 1.1 Definitions

When something is ***indistinguishable*** from another object, that means it looks identical to that object. If something is ***distinguishable***, that means you can tell which one is which.

> *Remark 1.1.* Some people call stars and bars "stones and sticks" as well as "balls and urns". Just note these are the same so that you don't get confused.

The ***ball and urn*** technique, also known as ***stars and bars***, is a commonly used technique in combinatorics. It is used to solve problems of the form: how many ways can one distribute $k$ indistinguishable objects into $n$ distinguishable bins? We can imagine this as finding the number of ways to drop $k$ balls into urns, or equivalently to arrange $k$ balls and $n-1$ dividers. For example,

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represent the ways to put $k = 4$ objects in $n = 3$ bins.

## 1.2 A Motivating Example

**Example 1.2**
Dr. Evil's 5th grade class has 6 students. Dr. Evil has 6 identical candies to give to the 6 students. However, as Dr. Evil is evil, he doesn't necessarily want to give 1 to each student. How many ways are there for him to give these candies to the students?

When reading this problem, it's easy to notice two things. First, this follows our guideline of identical balls and distinguishable boxes. In this case, the candies are identical, and evidently, the students are distinguishable. Secondly, if we were to physically count them by listing them all out, it would (a) be extremely time-consuming and (b) be extremely easy to miscount or make a stupid mistake. Therefore, we need a more efficient solution.

Let us imagine each of the identical candies to be circles/balls as shown below.

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Let us imagine putting lines or bars in between gaps of the circles to see how many candies every student gets. A small example is demonstrated below.

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In the configuration above, we have divided the candies into six partitions using five lines. Each partition has size one, meaning every student has one candy. Now, let us look at another configuration and base some conclusions on that.

○ | | ○ ○ | | ○ ○ | ○ ○

In the configuration shown above, we again have divided the candies into six partitions using five lines. Yet, the partition sizes are varying, meaning that each student gets a varied amount of candies. The number of candies are $1, 0, 2, 0, 2, 1$ per student from left to right. These two configurations inform us of several things. First of all, we can now be sure that the balls are identical, and that the boxes are distinguishable. The balls are identical because we are partitioning the **interchangeable** balls. In addition, with the sticks, we can repeatedly change the candy distribution (to account for all the various cases). For example, in the second configuration shown, the distribution of candies is 1-0-2-0-2-1, yet the distribution could also be 2-0-1-2-1-0. Here, we see that the individual numbers can be identical, yet the order is different. Therefore, the boxes have to be distinctive for us to use this technique.

Adding to above, putting the bars in different gaps between the balls will yield us every possible arrangement, thus ensuring we won't be over-counting or under-counting. Now let us attempt to solve this example.

*Solution.* Let us first draw out our filled circles and regular circles.

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As we saw above, we just need to find the number of ways we can arrange the filled circles. As there are 11 total positions and 5 filled circles, the total number of ways is $\binom{11}{5} = \boxed{462}$. □

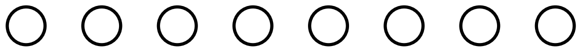
## 2 Variations of Stars and Bars

Let's try another example:

**Example 2.1**
How many ways are there for Jones to give eight coins to his four friends such that each of his friends gets at least one coin?

In this question, we will try to use our stars and bars strategy. However, we will need to be a little careful because everyone gets at least one coin.

*Solution.* Let us initially draw eight balls to represent the eight coins.



Let's mark all the gaps which we can use, yet we have to be careful not to mark extra gaps, as no person can get 0 coins.

Just like the previous problem, we know we have to use three lines to break the eight balls into four different pieces. As we have $8 - 1 = 7$ different gaps we can put them in, the answer is $\binom{7}{3} = \boxed{35}$. □

> *Remark 2.2.* Although this strategy is remarkably powerful, it isn't difficult to make stupid mistakes. You should understand exactly what the question is asking, as it ensures you will fulfill all the restrictions in the problem.

We have also derived the formula for the number of ways to split $n$ identical items into $k$ different groups, such that every group has at least 1 item.

**Theorem 2.3 (Stars and Bars For Positive # of Items)**
If we have to split $n$ identical items into $k$ distinguishable groups, where every group has at least one item, the number of ways to do so is $\binom{n-1}{k-1}$.

Now that we've derived this, we need to derive the formula if every group can have 0 or more items. We recall that we used the shaded circles to split up the candies in the first example. We had $k - 1$ shaded circles for splitting into $k$ groups. There were also $n$ unfilled circles representing the candies. Thus, the total number of gaps are $n + k - 1$. As there are $k - 1$ shaded balls, the total number of ways is $\binom{n+k-1}{k-1}$.

**Theorem 2.4 (Stars and Bars For Nonnegative # of Items)**
If we need to split $n$ identical items to $k$ distinguishable groups, where each group can have 0 or more items, the total number of ways to distribute it is $\binom{n+k-1}{k-1}$.

### 2.1 Stars and Bars Strategies

The trick for Stars and Bars is to **reduce to a simpler problem**. For example, if we have $a+b+c+d=10$ for non-negative integers $a, b, c, d$, we can simply apply the formula. If $a, b, c, d$ are positive integers, we can simply replace $a$ with $a'+1$, $b$ with $b'+1$, and so on and resolve. If $a, b, c, d$ are all even, we can simply replace $a$ with $2a'$ and so on. As long as we can keep **reducing**, the problem becomes easy.

## 3 Hard Examples

**Example 3.1 (AIME 1998/7)**
Let $n$ be the number of ordered quadruples $(x_1, x_2, x_3, x_4)$ of positive odd integers that satisfy $\sum_{i=1}^{4} x_i = 98$. Find $\frac{n}{100}$.

*Solution.* Imagine a line of 98 stones. We want to place the stones into 4 boxes so that each box has an odd number of stones. We then proceed by placing one stone in each box to begin with, ensuring that we have a positive number in every box. Now we have 94 stones left. Because we want an odd number in each box, we pair the stones, creating 47 sets of 2. Every time we add a pair to one of the boxes, the number of stones in the box remains odd, because (an odd number) + (an even number) = (an odd number).

Our problem can now be restated: how many ways are there to partition a line of 47 stones? We can easily solve this by using 3 sticks to separate the stones into 4 groups, and this is the same as arranging a line of 3 sticks and 47 stones.

$$\frac{50!}{47! \cdot 3!} = 19600$$

$$\frac{50 \cdot 49 \cdot 48}{3 \cdot 2} = 19600$$

Our answer is therefore $\frac{19600}{100} = \boxed{196}$ □

**Example 3.2 (AIME II 2000/5)**
Given eight distinguishable rings, let $n$ be the number of possible five-ring arrangements on the four fingers (not the thumb) of one hand. The order of rings on each finger is significant, but it is not required that each finger have a ring. Find the leftmost three nonzero digits of $n$.

*Solution.* There are $\binom{8}{5}$ ways to choose the rings, and there are $5!$ distinct arrangements to order the rings (we order them so that the first ring is the bottom-most on the first finger that actually has a ring, and so forth). The number of ways to distribute the rings among the fingers is equivalent the number of ways we can drop five balls into 4 urns, or similarly dropping five balls into four compartments split by three dividers. The number of ways to arrange those dividers and balls is just $\binom{8}{3}$.

Multiplying gives the answer: $\binom{8}{5}\binom{8}{3}5! = 376320$, and the three leftmost digits are $\boxed{376}$. □

**Example 3.3 (AIME I 2007/10)**
In a $6 \times 4$ grid (6 rows, 4 columns), 12 of the 24 squares are to be shaded so that there are two shaded squares in each row and three shaded squares in each column. Let $N$ be the number of shadings with this property. Find the remainder when $N$ is divided by 1000.

*Solution.* Consider the first column. There are $\binom{6}{3} = 20$ ways that the rows could be chosen, but without loss of generality let them be the first three rows (change the order of the rows to make this true). We will multiply whatever answer we get by 20 to get our final answer.

Now consider the $3 \times 3$ that is next to the 3 boxes we have filled in. We must put one ball in each row (since there must be 2 balls in each row and we've already put one in each). We split into three cases:

1. **All three balls are in the same column.** In this case, there are 3 choices for which column that is. From here, the bottom half of the board is fixed.

2. **Two balls are in one column, and one is in the other.** In this case, there are 3 ways to choose which column gets 2 balls and 2 ways to choose which one gets the other ball. Then, there are 3 ways to choose which row the lone ball is in. Now, what happens in the bottom half of the board? Well, the 3 boxes in the column with no balls in the top half must all be filled in, so there are no choices here. In the column with two balls already, we can choose any of the 3 boxes for the third ball. This forces the location for the last two balls. So we have $3 \cdot 2 \cdot 3 \cdot 3 = 54$.

3. **All three balls are in different columns.** Then there are 3 ways to choose which row the ball in column 2 goes and 2 ways to choose where the ball in column 3 goes. (The location of the ball in column 4 is forced.) Again, we think about what happens in the bottom half of the board. There are 2 balls in each row and column now, so in the $3 \times 3$ where we still have choices, each row and column has one square that is not filled in. But there are 6 ways to do this. So in all there are 36 ways.

So there are $20(3 + 54 + 36) = 1860$ different shadings, and so the answer is $\boxed{860}$. □

## 4 Problems

**Problem 1.** Suppose you have five friends and 10 indistinguishable candy bars. How many ways are there for you to give the 10 candy bars to your friends so that each friend gets at least one candy bar?

**Problem 2.** If $a, b, c, d$ are nonnegative integers, how many ways can $a + b + c + d = 7$?

**Problem 3.** If $a, b, c, d$ are positive integers, how many ways can $a + b + c + d = 7$?

**Problem 4.** If $a, b, c, d$ are positive odd integers, how many ways can $a + b + c + d = 8$?

**Problem 5.** If $a, b, c, d$ are nonnegative even integers, how many ways can $a + b + c + d = 8$?

**Problem 6.** How many ways can you buy 8 fruit if your options are apples, bananas, pears, and oranges?

**Problem 7 (AMC 8 2019/25).** Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the three people has at least two apples?

**Problem 8 (AMC 10A 2018/11).** When 7 standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as $\frac{n}{6^7}$, where $n$ is a positive integer. What is $n$?

**Problem 9 (Mathcounts Chapter 2008).** During football season, 25 teams are ranked by three reporters (Alice, Bob and Cecil). Each reporter assigned all 25 integers (1 through 25) when ranking the twenty-five teams. A team earns 25 points for each first-place ranking, 24 points for each second- place ranking, and so on, getting one point for a 25th place ranking. The Hedgehogs earned 27 total points from the three reporters. How many different ways could the three reporters have assigned their rank- ings for the Hedgehogs?

**Problem 10 (Mathcounts Chapter Team 2020/5).** An ordered triple $(a, b, c)$ is randomly chosen from the set of all ordered triples for which $a, b$ and $c$ are non negative integers that satisfy $a + b + c = 22$. What is the probability that $a < b < c$?