Math Level 2 Handout

Dylan Yu

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1 Advanced Analytic Geometry

1.1 When to Use Coordinates

Use coordinate geometry when you have weird intersections of figures. For example, if a circle intersects with 3 other circles at various points and you are given the position of their centers, use coordinate geometry!

1.2 Importance of Drawing a Good Figure

Sometimes it is *very* important that your figure be good. There are a few reasons why:

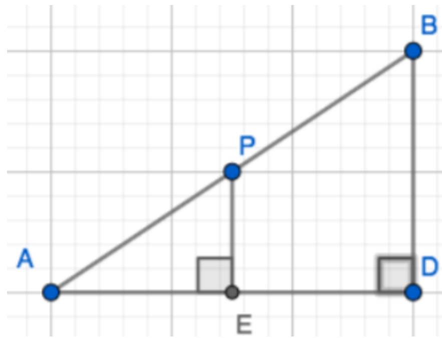
1. You can read your work better, and therefore won't accidentally think one thing is something else. This also means drawing a big diagram when you're stuck.
2. You can check your work better, instead of having to redo the problem all over again because you can't read your own solution.
3. You can find things out about the problem. When you draw a random triangle, and you realize that after you draw a few lines something is a right angle, it is sometimes not a coincidence! Having 2-3 figures will help you determine if this is true.

1.3 Advanced Coordinates

Theorem 1.1. (Ratio Point Theorem) Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$. The point

$$P = (rx_1 + (1-r)x_2, ry_1 + (1-r)y_2)$$

lies on the line between A and B, and splits the segment AB into $1-r : r$ ratio, given $0 < r < 1$. If $r < 1-r$, P is closer to B than A.



Theorem 1.2. (Distance from a Point to a Line) The distance between point $A(x_1, y_1)$ and the line $ax_2 + by_2 + c = 0$ is given by the expression

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

Moreover, the point on the line that is closest to A (called the **projection** of A onto the line) has coordinates

$$(x, y) = \left(\frac{b(bx_1 - ay_1) - ac}{a^2 + b^2}, \frac{a(-bx_1 + ay_1) - bc}{a^2 + b^2} \right).$$

I do not recommend remembering the formula above, but the first formula given is good to memorize.

Theorem 1.3. (Shoelace Theorem) The area of a n-sided polygon with vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ listed in clockwise or counterclockwise order is given by

$$\frac{1}{2}|(x_1y_2 + x_2y_3 + \ldots + x_ny_1) - (y_1x_2 + y_2x_3 + \ldots + y_nx_1)|.$$

This theorem is called **Shoelace Theorem** because you list the vertices vertically:

$$\begin{gathered}(x_1, y_1)\\(x_2, y_2)\\\vdots\\(x_n, y_n)\\(x_1, y_1)\end{gathered}$$

and drawing ”shoelaces” between the coordinates, as I will demonstrate in class.

Theorem 1.4. (Pick's Theorem) The area of a figure in the coordinate plane is given by $\frac{b}{2} + i - 1$, where b is the number of lattice points on the border of the figure and i is the number of lattice points in the interior (excluding the border) of the figure.

1.4 The Strategy for Coordinate Bashing

To coordinate bash is to turn a normal geometry problem into a problem in the xy-plane. A few things you must do to use coordinate bash:

1. Choose a reference point. You can put any point anywhere you want, but where you put it could simplify the problem a lot. This means choosing a point that is a nice integer, and possibly at the origin.
2. Make sure the lines and circles are nice. You don't want a circle with center $(\sqrt{22}, \sqrt{35})$.
3. Make sure you can convert back to the original problem.

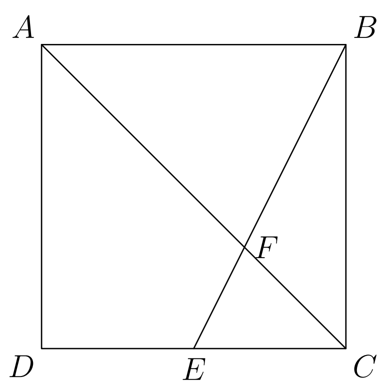
Of course, if the problem is already in coordinates, use coordinate geometry!

1.5 Problems

Problem 1.1. Let $ABCD$ be a rectangle with $AB = 1$ and $BC = 2$. In addition, let E be a point on diagonal BD such that it trisects the diagonal and is closer to B than D. Find the length of segment AE.

Problem 1.2. Find the area of the triangle with vertices at $(2, 2), (5, 4)$ and $(4, 1)$.

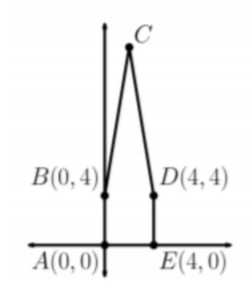
Problem 1.3. (2018 AMC8 P22) Point E is the midpoint of side $\overline{CD}$ in square $ABCD$, and $\overline{BE}$ meets diagonal $\overline{AC}$ at F. The area of quadrilateral $AFED$ is 45. What is the area of $ABCD$?



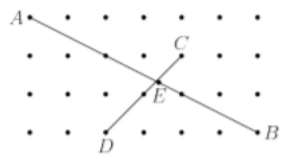
(A) 100 **(B)** 108 **(C)** 120 **(D)** 135 **(E)** 144

Problem 1.4. (Mathcounts) When a triangle with vertices at $(2, 0), (10, 2)$ and $(6, 6)$ is rotated 360 degrees about the point $(0, -3)$, the sides and vertices of the triangle sweep out a region in the shape of an annulus (ring). What is the area of the annulus? Note: this problem is extremely hard, so don't be afraid to not know how to do this one!

Problem 1.5. Pentagon $ABCDE$ has a vertical line of symmetry and has an area of 40 square units. How many lattice points are in the interior of the pentagon?



Problem 1.6. The diagram shows 28 lattice points, each one unit from its nearest neighbors. Segment AB meets segment CD at E. Find the length of segment AE.



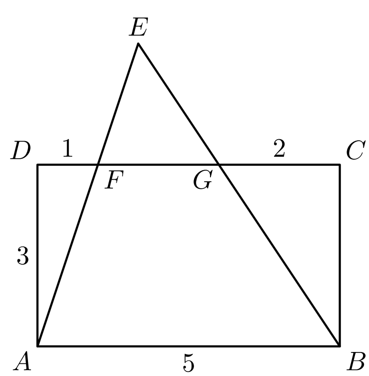
Problem 1.7. (2018-19 School Handbook P27) A segment with endpoints $G(-2,3)$ and $H(4,7)$ is dilated by a scale factor of $\frac{2}{3}$ with center of dilation $(0,0)$. What is the sum of all the coordinates of G' and H'?

Problem 1.8. The point $(-2,10)$ lies on the circle $(x-3)^2+(y+2)^2=169$. What is the equation of the line tangent to that circle at that point?

Problem 1.9. The two circles, $x^2+y^2=25$ and $(x-7)^2+(y-7)^2=25$, have a common chord. What is the equation of the line containing that chord? What is the length of this chord?

Problem 1.10. Triangle ABC has vertices $A(3,2), B(-2,1)$, and $C(6,-5)$. What is the equation of the line containing the altitude from vertex A to side BC?

Problem 1.11. (2003 AMC10B P20) In rectangle $ABCD, AB=5$ and $BC=3$. Points F and G are on $\overline{CD}$ so that $DF=1$ and $GC=2$. Lines AF and BG intersect at E. Find the area of $\triangle AEB$.



(A) 10 **(B)** $\frac{21}{2}$ **(C)** 12 **(D)** $\frac{25}{2}$ **(E)** 15

Problem 1.12. (2018 National Countdown P21) In square units, what is the area of the triangle with vertices $P(-2,1), Q(3,8)$ and $R(9,3)$? Express your answer as a decimal to the nearest tenth.

Problem 1.13. (Mathcounts Problem of the Week 4/15/19) Molly has sketched a design for her new kite on a coordinate grid. The sides of her quadrilateral kite design are given by the equations $y = \frac{2}{3}x + 30$, $y = -\frac{2}{3}x + 50$, $y = -2x + 30$ and $y = 2x - 30$. How many square units are in the area of Molly's design?