ASE 2020-21 Advanced Notes

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Contents

§ 1 Algebraic Manipulations

Basically this topic is an assortment of problems with clever tricks. Some manipulation problems are about being fast, and some are about making harder ideas easier.

*The ASE playlist can be found here.

§ 1.1 Reading

An important part of manipulations is substitutions. You've seen substitutions before. For example, we can solve linear equations using substitution. Please take a look at this case if you are not familiar. Knowing how variables work is also mandatory.

§ 1.2 Warmup

A few problems using exponents and radicals:

Problem 1 (MathLeague Target 11322/2). Find the units digit of 147^{148}.

Problem 2. Find the last two digits of 6^{2020}.

Problem 3 (MathLeague Team 11321/7). What is the units digit of 2013^{2012}

Problem 4. Compute

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots+\frac{1}{\sqrt{99}+\sqrt{100}}.$$

There was one MathLeague Number Sense question on egyptian fractions. They aren't very important, but just for fun let's learn them:

Theorem 1 (Egyptian Fractions). For all a, b where $ab \neq 1$,

$$\frac{a}{ab-1}=\frac{1}{b(ab-1)}+\frac{1}{b}.$$

§ 1.3 Adding & Multiplying Equations

Example 1. Compute $a+b+c$, if $a+b=1, b+c=2, c+a=3$.

Solution. If we add the three equations, we get $2(a+b+c)=6 \implies a+b+c=\boxed{3}$. □

Factoring is also a great idea. Knowing how to prime factorize is mandatory.

Problem 5. If $a-b=1$, and $a+b=2$, find a^2-b^2.

Problem 6. If $a+b=20$ and $ab=64$, find $a-b$ if $a>b$.

§ 1.4 Multiplying Equations

Theorem 2 (Casework with 0). If $ab = 0$, then $a = 0$ or $b = 0$.

Problem 7. If $(x^2 - 2 - 3)(x^2 - 5x - 6) = 0$, find the possible values of x.

§ 1.5 Reciprocals

If something is in the form $x + \frac{1}{x}$, consider taking it to some power.

Problem 8 (MathLeague Sprint 11122/5). What is $\frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} \times \frac{6}{7}$?

Problem 9. If $x + \frac{1}{x} = 5$, find $x^4 + \frac{1}{x^4}$.

Problem 10. If $\frac{x^4+1}{x^2} = 47$, find all possible values of $\frac{x^2-1}{x}$.

§ 1.6 Substitutions

§ 1.6.1 Nested Radicals

Example 2. Find $\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{\dots}}}}$.

Solution. Let $S = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{\dots}}}}$. Then if we square both sides, we get

$$S^2 = 2 + \sqrt{2 + \sqrt{2 + \sqrt{\dots}}},$$

$$S^2 - 2 = \sqrt{2 + \sqrt{2 + \sqrt{\dots}}} = S.$$

Thus, $S^2 - S - 2 = 0 \implies (S-2)(S+1) = 0 \implies S = -1, 2$. Obviously something square rooted like $\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{\dots}}}}$ is positive, so $S = \boxed{2}$. □

Theorem 3 (Nested Radicals Formula). If $S = \sqrt[n]{x + \sqrt[n]{x + \sqrt[n]{x + \sqrt[n]{\dots}}}}$, then $S^n - S = x$.

Note that if $n = 1$, we can solve this with the quadratic formula.

Corollary 4 (Nested Square Roots Formula). If a is a positive integer, $\sqrt{a + \sqrt{a + \sqrt{\dots}}} = \frac{1+\sqrt{1+4a}}{2}$.

§ 1.6.2 Continued Fractions

Example 3. Let $\frac{55}{24} = a + \frac{1}{b+\frac{1}{c+\frac{1}{d}}}$, where a, b, c, d are positive integers. What is $a+b+c+d$.

Solution. Let's rewrite $\frac{55}{24}$ as $2\frac{7}{24}$. Then if we let $a = 2$, we get

$$\frac{7}{24} = \frac{1}{b+\frac{1}{c+\frac{1}{d}}} \implies \frac{24}{7} = 3\frac{3}{7} = b + \frac{1}{c+\frac{1}{d}}.$$

So if $b = 3$, then

$$\frac{3}{7} = \frac{1}{c+\frac{1}{d}} \implies \frac{7}{3} = 2\frac{1}{3} = c + \frac{1}{d}.$$

If $c = 2$, then

$$\frac{1}{3} = \frac{1}{d},$$

so $d = 3$. Thus, $a + b + c + d = 2 + 3 + 2 + 3 = \boxed{10}$. □

Theorem 5 (Continued Fractions Method). If we have some fraction $k = a_1 + \frac{1}{a_2+\frac{1}{\ddots}}$, then we pick the largest positive integer a_i such that when we subtract it we'll get a positive integer.

For example, when we had

$$3\frac{3}{7} = b + \frac{1}{c+\frac{1}{d}},$$

we picked $b = 3$ because $b = 4$ would have made the LHS a negative number ($3\frac{3}{7} - 4$ is negative).

Example 4. Find $\frac{1}{2+\frac{1}{2+\frac{1}{\ddots}}}$.

Solution. Let $S = \frac{1}{2+\frac{1}{2+\frac{1}{\ddots}}}$. Then

$$\frac{1}{S} = 2 + \frac{1}{2+\frac{1}{\ddots}} \implies \frac{1}{S} - 2 = \frac{1}{2+\frac{1}{\ddots}} = S,$$

so

$$1 - 2S = S^2 \implies S^2 + 2S - 1 = 0 \implies S = -1 \pm \sqrt{2},$$

and since $\frac{1}{2+\frac{1}{2+\frac{1}{\ddots}}}$ is obviously positive, $S = \boxed{\sqrt{2}-1}$. □

Theorem 6 (Continued Fraction Formula). If c, x are positive,

$$\frac{c}{x+\frac{c}{x+\frac{c}{\ddots}}} = \frac{-x+\sqrt{x^2+4c}}{2}.$$

§ 1.6.3 Methods of Substitution

You should substitute when:

1. There are large numbers
2. There are very tiny numbers (in general, bad numbers means substitution)
3. The problem is too complicated, and we need to simplify it

§ 1.7 Vieta's Formulas

I won't go into detail here. This is the basic version:

Theorem 7 (Simplified Vieta's Formulas). Let's say we have $ax^n + bx^{n-1} + \ldots + c$. In other words, the coefficient of the term with the largest exponent is a, the term with the second largest exponent is b, and the constant term is c. Then the sum of the roots is $\frac{-b}{a}$, and the product of the roots is $\frac{-(-1)^n c}{a}$.

§ 1.8 Arithmetic Sequences

If we subtract two arithmetic sequences, we will get a bunch of constant terms. This is why **matching them up** is important.

Example 5 (MathLeague Team 11021/3). What is the value of the sum of the first 100 natural numbers $(1+2+3+4+\ldots+100)$ minus the sum of the next 100 natural numbers $(101+102+103+104+\ldots+200)$?

Solution. $101-1=100, 102-2=100, \ldots, 200-100=100$, so if we subtract the two equations, we get 100 terms, all of which are 100. Thus, $100 \cdot 100 = \boxed{10000}$. □

§ 1.9 Patterns

There is something called **Engineer's Induction**, which basically means **guess the pattern**. Below, we will try some of these problems.

§ 1.10 Problems

These questions are not in order of difficulty.

Problem 11. What is the value of $(45 + 37 + 24 - 30 - 20 - 40)$?

Problem 12 (MathLeague Countdown 11022/30). What is the value of $(400 - 78 - 22 - 84 - 16 - 81 - 19 - 27)$?

Problem 13 (MathLeague Sprint 11324/6). Simplify: $2013 \times 2013 - 2012 \times 2012$.

Problem 14 (MathLeague Sprint 11121/6). If 3 BaBas equals 4 LaLas, 5 LaLas equals 8 GaGas, and 1 GaGa equals 9 YaYas, how many YaYas equals 5 BaBas?

Problem 15 (MathLeague Sprint 11021/9). In the land of Ah-oooga, 2 Oogies is worth 3 Boogies, and 5 Boogies is worth 7 Noogies. How many Oogies are worth 21 Noogies?

Problem 16 (MathLeague Sprint 11325/15). If $2A + 7 = 25$, find $4A + 14$.

Problem 17 (MathLeague Sprint 11325/16). Simplify: $2013 \times 2013 - 2012 \times 2013$.

Problem 18 (MathLeague Sprint 11122/15). What is the value of $46 \times 47 \times 48$?

Problem 19 (MathLeague Sprint 11121/22). What is the value of $(105 \times 107 \times 109)$?

Problem 20 (MathLeague Sprint 11121/30). What is the value of $(27 \times 27 \times 27) \div (3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3)$?

Problem 21 (MathLeague Team 11324/8). Four consecutive even integers have a sum of 68. What is the smallest integer?

Problem 22 (MathLeague Sprint 11324/30). Find the value of $\frac{1234 \times 1234 - 1}{1235}$.

Problem 23 (MathLeague Target 11022/5). If the sum of the first 111 natural numbers $(1 + 2 + 3 + 4 + \ldots + 111)$ is 6216, what is the sum of the natural numbers from 9 to 119 $(9 + 10 + 11 + 12 + \ldots + 119)$?

Problem 24 (MathLeague Team 11022/2). Miles adds up all the numbers from 1 through 25 $(1 + 2 + 3 + \ldots + 24 + 25)$, then multiplies his sum by 26. Sawyer adds up all the numbers from 1 through 26 $(1 + 2 + 3 + \ldots + 25 + 26)$, then multiplies his sum by 25. What is the positive difference between their two totals?

Problem 25. Solve for real x: $-7x^3 + 3x^2 + 3x + 1$.

Problem 26. Find x: $x^4 + 2x^2 + 1 = 0$.

Problem 27. Find x: $4^x - 3 \cdot 2^x + 2 = 0$.

Problem 28. Compute $\sqrt[4]{14 + \sqrt[4]{14 + \ldots}}$.

Problem 29. I chop a string into thirds and discard one of the thirds. Then, I chop the remaining string also into thirds, and I discard one of the thirds. If I repeat this process infinitely many times, how much string did I discard?

Problem 30. Find $1^2 + 3^2 + \ldots + 101^2$.

Problem 31 (MathLeague Sprint 11325/30). Simplify: $100^2 - 99^2 + 98^2 - 97^2 + \ldots + 2^2 - 1^1$.

Problem 32 (MathLeague Team 11326/6). Find a if $\sqrt{a} = \sqrt{2} + \sqrt{8} + \sqrt{18} + \sqrt{32}$.

Problem 33 (MathLeague Sprint 11326/30). The roots of the polynomial $x^3 + 7x + 8$ are r, s, and t. Find the value of $(r+s)^2(s+t)^2(t+r)^2$.

Problem 34 (MathLeague Sprint 11122/30). The number $111\ldots1$, where there are 2010 1's, is divided by 3. What is the number of zeroes in the quotient?