Math Level 2 Week 6

Dylan Yu

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1 Introduction

1.1 Common Notation

In a triangle ABC:

- the side lengths are $a = BC, b = CA, c = AB$,
- the circumradius is R,
- the inradius is r, and
- the semiperimeter is $s = \frac{a+b+c}{2}$.

The area of an object X is $[X]$. For example, the area of $\triangle ABC$ is $[ABC]$.

> *Remark 1.1.* All of the above notation will be assumed for the tools below unless otherwise stated.

1.2 Importance of a good figure

Sometimes it is important that your figure be good. There are a few reasons why:

- You can read your work better, and therefore won't accidentally think one thing is something else. This also means drawing a big diagram when you're stuck.
- You can check your work better, instead of having to redo the problem all over again because you can't read your own solution.
- You can find things out about the problem. When you draw a random triangle, and you realize that after you draw a few lines something is a right angle, it is sometimes not a coincidence! Having 2-3 figures will help you determine if this is true.

2 Tools

2.1 Length Tools

Theorem 2.1 (Pythagorean Theorem)
Let ABC be a triangle with $\angle C = 90^\circ$. Then

$$a^2 + b^2 = c^2.$$

The Pythagorean Theorem is **by far** the most important. Don't forget it!

Theorem 2.2 (Extended Law of Sines)
Let ABC be a triangle. Then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

It is important to remember the following:

Fact 2.3. For all angles θ,

$$\sin\theta = \sin(180^\circ - \theta).$$

Theorem 2.4 (Law of Cosines)
Let ABC be a triangle. Then

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Note that if $C = 90^\circ$ we get the Pythagorean Theorem.

Theorem 2.5 (Stewart's Theorem)
Given a triangle ABC, if cevian AD is drawn so that $BD = m$, $DC = n$ and $AD = d$, we have that
$$b^2m + c^2n = amn + d^2a.$$

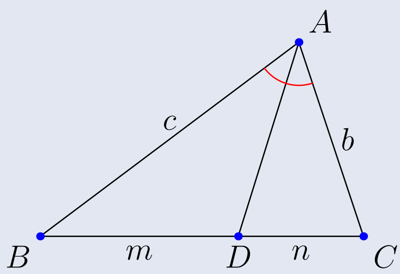
Alternatively, this can be written as

$$man + dad = bmb + cnc,$$

which can be memorized as "A man and his dad put a bomb in the sink."

Theorem 2.6 (Angle Bisector Theorem)
Given triangle ABC and angle bisector AD, where D is on side BC, then

$$\frac{c}{m} = \frac{b}{n}.$$



We can combine this with Stewart's to get

$$AD^2 = bc - mn.$$

Theorem 2.7 (Ptolemy's Theorem)
Given $ABCD$ is a cyclic quadrilateral, then

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

2.2 Area Tools

The area of a triangle can be written as:

- $\frac{1}{2}$ of base times height,
- $\sqrt{s(s-a)(s-b)(s-c)}$
- $\frac{1}{2}ab\sin C$,
- sr, and
- $\frac{abc}{4R}$.

Fact 2.8 (Similar Figures). If two similar figures have corresponding lengths in the ratio $a : b$, then their areas are in the ratio $a^2 : b^2$.

Fact 2.9 (Same Base-Same Height). Two triangles with the same base and the same height have equal area.

2.3 Coordinate Bashing

The idea of ***coordinate bashing*** is basically to plot a geometry figure into the coordinate plane. A few notes:

- Choose a good origin. Some good choices:
  - the center of a circle/regular polygon,
  - a vertex of a triangle, or
  - the intersection of two lines.
- Try to make as many ***lattice points*** (points with integer coordinates) as possible.
- Similarly, try to put as many points on the x- and y-axes as possible.

Theorem 2.10 (Ratio Point Theorem)
Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$. The point

$$P = (rx_1 + (1-r)x_2, ry_1 + (1-r)y_2)$$

lies on the line between A and B, and splits the segment AB into $1-r : r$ ratio, given $0 < r < 1$. If $r < 1-r$, P is closer to B than A.

Theorem 2.11 (Point to Line Distance)
The distance between point $A(x_1, y_1)$ and the line $ax_2 + by_2 + c = 0$ is given by the expression

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

Theorem 2.12 (Shoelace Formula)
Suppose the polygon P has vertices (a_1, b_1), (a_2, b_2), ... , (a_n, b_n), listed in clockwise order. Then

$$[P] = \frac{1}{2}\left|(a_1b_2 + a_2b_3 + \cdots + a_nb_1) - (b_1a_2 + b_2a_3 + \cdots + b_na_1)\right|$$

Shoelace gets its name because if one lists the coordinates in a column,

$$\begin{gathered}(a_1, b_1)\\(a_2, b_2)\\\vdots\\(a_n, b_n)\\(a_1, b_1)\end{gathered}$$

and marks the pairs of coordinates to be multiplied,

$$A = \frac{1}{2}\left| \begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \\ a_1 & b_1 \end{array} \qquad - \qquad \begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \\ a_1 & b_1 \end{array} \right|$$

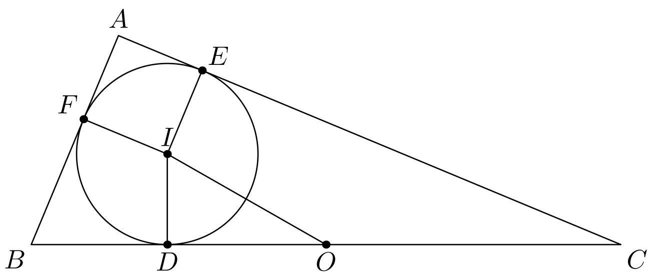
the resulting image looks like laced-up shoes.

3 Examples

3.1 Coordinate Bashing

Problem 1 (AMC 10B 2004/22). A triangle with sides of length $5, 12$, and 13 has both an inscribed and a circumscribed circle. What is the distance between the centers of those circles?

Solution. This is obviously a right triangle. Pick a coordinate system so that the right angle is at $(0, 0)$ and the other two vertices are at $(12, 0)$ and $(0, 5)$.



As this is a right triangle, the center of the circumcircle is in the middle of the hypotenuse, at $(6, 2.5)$.

The radius r of the inscribed circle can be computed using the well-known identity $\frac{rP}{2} = S$, where S is the area of the triangle and P its perimeter. In our case, $S = \frac{5\cdot 12}{2} = 30$ and $P = 5 + 12 + 13 = 30$. Thus, $r = 2$. As the inscribed circle touches both legs, its center must be at $(r, r) = (2, 2)$.

The distance of these two points is then $\sqrt{(6-2)^2 + (2.5-2)^2} = \sqrt{16.25} = \sqrt{\frac{65}{4}} = \boxed{\frac{\sqrt{65}}{2}}$. □

Example 3.1 (AMC 10B 2009/18)
Rectangle $ABCD$ has $AB = 8$ and $BC = 6$. Point M is the midpoint of diagonal $\overline{AC}$, and E is on AB with $\overline{ME} \perp \overline{AC}$. What is the area of $\triangle AME$?

Solution. Set A to $(0, 0)$. Since M is the midpoint of the diagonal, it would be $(4, -3)$. The diagonal AC would be the line $y = -\frac{3x}{4}$. Since ME is perpendicular to AC, its line would be in the form $y = \frac{4x}{3} + b$. Plugging in 4 and -3 for x and y would give $b = \frac{25}{3}$. To find the x-intercept of $y = \frac{4x}{3} + \frac{25}{3}$ we plug in 0 for y and get $x = \frac{25}{4}$. Then, using the Shoelace Formula for $(0,0)$, $(4,-3)$, and $(\frac{25}{4}, 0)$, we find the area is $\frac{75}{8}$. □

3.2 Trigonometry

Example 3.2 (AMC 10 A 2012/21)
Let points $A = (0,0,0)$, $B = (1,0,0)$, $C = (0,2,0)$, and $D = (0,0,3)$. Points E, F, G, and H are midpoints of line segments $\overline{BD}$, $\overline{AB}$, $\overline{AC}$, and $\overline{DC}$ respectively. What is the area of $EFGH$?

(A) $\sqrt{2}$ **(B)** $\frac{2\sqrt{5}}{3}$ **(C)** $\frac{3\sqrt{5}}{4}$ **(D)** $\sqrt{3}$ **(E)** $\frac{2\sqrt{7}}{3}$

Solution. Computing the points of $EFGH$ gives

$$E(0.5, 0, 1.5), F(0.5, 0, 0), G(0, 1, 0), H(0, 1, 1.5).$$

The vector EF is $(0, 0, -1.5)$, while the vector HG is also $(0, 0, -1.5)$, meaning the two sides EF and GH are parallel. Similarly, the vector FG is $(-0.5, 1, 0)$, while the vector EH is also $(-0.5, 1, 0)$. Again, these are equal in both magnitude and direction, so FG and EH are parallel. Thus, figure $EFGH$ is a parallelogram.

Computation of vectors EF and HG is sufficient evidence that the figure is a parallelogram, since the vectors are not only point in the same direction, but are of the same magnitude, but the other vector FG is needed to find the angle between the sides.

Taking the dot product of vector EF and vector FG gives $0 \cdot -0.5 + 0 \cdot 1 + -1.5 \cdot 0 = 0$, which means the two vectors are perpendicular. (Alternately, as above, note that vector EF goes directly down on the z-axis, while vector FG has no z-component and lie completely in the xy-plane.) Thus, the figure is a parallelogram with a right angle, which makes it a rectangle. With the distance formula in three dimensions, we find that $EF = \frac{3}{2}$ and $FG = \frac{\sqrt{5}}{2}$, giving an area of $\frac{3}{2} \cdot \frac{\sqrt{5}}{2} = \boxed{\textbf{(C) } \frac{3\sqrt{5}}{4}}$. □

Example 3.3 (AIME 1983/4)
A machine-shop cutting tool has the shape of a notched circle, as shown. The radius of the circle is $\sqrt{50}$ cm, the length of AB is 6 cm and that of BC is 2 cm. The angle ABC is a right angle. Find the square of the distance (in centimeters) from B to the center of the circle.

Solution. Draw segment OB with length x, and draw radius OQ such that OQ bisects chord AC at point M. This also means that OQ is perpendicular to AC. By the Pythagorean Theorem, we get that $AC = \sqrt{(BC)^2 + (AB)^2} = 2\sqrt{10}$, and therefore $AM = \sqrt{10}$. Also by the Pythagorean theorem, we can find that $OM = \sqrt{50 - 10} = 2\sqrt{10}$.

Next, find $\angle BAC = \arctan\left(\frac{2}{6}\right)$ and $\angle OAM = \arctan\left(\frac{2\sqrt{10}}{\sqrt{10}}\right)$. Since $\angle OAB = \angle OAM - \angle BAC$, we get

$$\angle OAB = \arctan 2 - \arctan\frac{1}{3}$$

$$\tan(\angle OAB) = \tan(\arctan 2 - \arctan\frac{1}{3})$$

By the subtraction formula for tan, we get

$$\tan(\angle OAB) = \frac{2 - \frac{1}{3}}{1 + 2 \cdot \frac{1}{3}}$$

$$\tan(\angle OAB) = 1$$

$$\cos(\angle OAB) = \frac{1}{\sqrt{2}}$$

Finally, by the Law of Cosines on $\triangle OAB$, we get

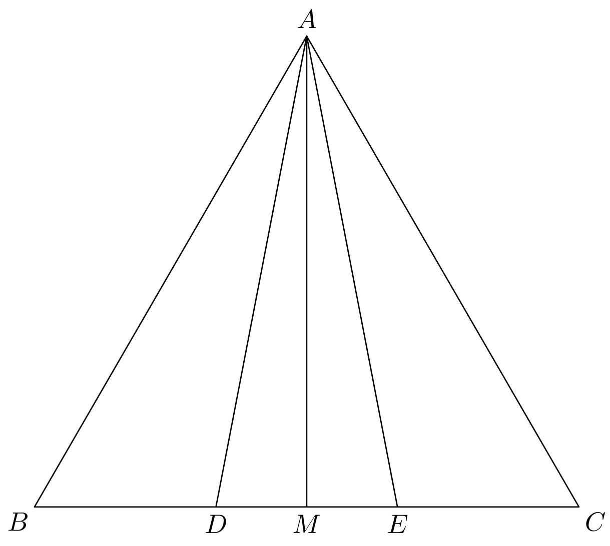
$$x^2 = 50 + 36 - 2(6)\sqrt{50}\frac{1}{\sqrt{2}}$$

$$x^2 = \boxed{026}.$$

□

Example 3.4 (AIME II 2013/5)
In equilateral $\triangle ABC$ let points D and E trisect $\overline{BC}$. Then $\sin(\angle DAE)$ can be expressed in the form $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is an integer that is not divisible by the square of any prime. Find $a + b + c$.

Solution. Without loss of generality, assume the triangle sides have length 3. Then the trisected side is partitioned into segments of length 1, making your computation easier.



Let M be the midpoint of $\overline{DE}$. Then $\triangle MCA$ is a 30-60-90 triangle with $MC = \frac{3}{2}$, $AC = 3$ and $AM = \frac{3\sqrt{3}}{2}$. Since the triangle $\triangle AME$ is right, then we can find the length of $\overline{AE}$ by Pythagorean theorem, $AE = \sqrt{7}$. Therefore, since $\triangle AME$ is a right triangle, we can easily find $\sin(\angle EAM) = \frac{1}{2\sqrt{7}}$ and $\cos(\angle EAM) = \sqrt{1 - \sin(\angle EAM)^2} = \frac{3\sqrt{3}}{2\sqrt{7}}$. So we can use the double angle formula for sine, $\sin(\angle EAD) = 2\sin(\angle EAM)\cos(\angle EAM) = \frac{3\sqrt{3}}{14}$. Therefore, $a + b + c = \boxed{020}$. □

4 Parting Notes

4.1 When and when NOT to use coordinates

- Use coordinate geometry when you have weird intersections of figures. For example, if a circle intersects with 3 other circles at various points and you are given the position of their centers, use coordinate bashing!
- Don't use coordinates when the problem has a weird regular polygon (like a pentagon) or heavily involves angles.

4.2 A few extra formulas

Some formulas to keep in mind, but aren't necessary.

Theorem 4.1 (Breitschneider's Formula)
The area of quadrilateral $ABCD$ is

$$[ABCD] = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\frac{\angle A + \angle C}{2}}.$$

If $\angle A + \angle C = 180^\circ$, then we have a ***cyclic quadrilateral***, implying the following:

Corollary 4.2 (Brahmagupta's Formula)
The area of cyclic quadrilateral $ABCD$ is

$$[ABCD] = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

Theorem 4.3 (British Flag Theorem)
Let P be a point inside rectangle $ABCD$. Then

$$AP^2 + CP^2 = BP^2 + DP^2.$$

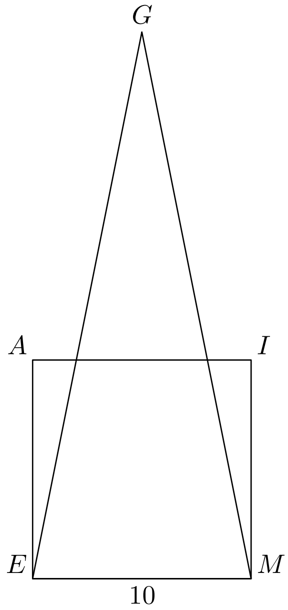
5 Problems

5.1 Coordinate Bashing

Problem 2. In right triangle ABC, we have $\angle ACB = 90^\circ$, $AC = 2$, and $BC = 3$. Medians AD and BE are drawn to sides BC and AC, respectively. AD and BE intersect at point F. Find the area of $\triangle ABF$.

Problem 4 (ARML Individual 1999). In circle O, $PO \perp OB$, and PO equals the length of the diameter of circle O. Compute $\frac{PA}{AB}$.

Problem 5 (AIME I 2008/2). Square $AIME$ has sides of length 10 units. Isosceles triangle GEM has base EM, and the area common to triangle GEM and square $AIME$ is 80 square units. Find the length of the altitude to EM in $\triangle GEM$.



Problem 6 (SMT 2012). ABC is an equilateral triangle with side length 1. Point D lies on AB, point E lies on AC, and points G and F lie on BC, such that $DEFG$ is a square. What is the area of $DEFG$?

Problem 7 (AMC 10B 2018/24). Let $ABCDEF$ be a regular hexagon with side length 1. Denote by X, Y, and Z the midpoints of sides $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$, respectively. What is the area of the convex hexagon whose interior is the intersection of the interiors of $\triangle ACE$ and $\triangle XYZ$?

5.2 Trigonometry

Problem 8. In $\triangle ABC$, $\angle B = 3\angle C$. If $AB = 10$ and $AC = 15$, compute the length of BC.

Problem 9 (AIME II 2004/1). A chord of a circle is perpendicular to a radius at the midpoint of the radius. The ratio of the area of the larger of the two regions into which the chord divides the circle to the smaller can be expressed in the form $\frac{a\pi + b\sqrt{c}}{d\pi - e\sqrt{f}}$, where a, b, c, d, e, and f are positive integers, a and e are relatively prime, and neither c nor f is divisible by the square of any prime. Find the remainder when the product $abcdef$ is divided by 1000.

Problem 10 (AIME 2001/4). In triangle ABC, angles A and B measure 60 degrees and 45 degrees, respectively. The bisector of angle A intersects $\overline{BC}$ at T, and $AT = 24$. The area of triangle ABC can be written in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

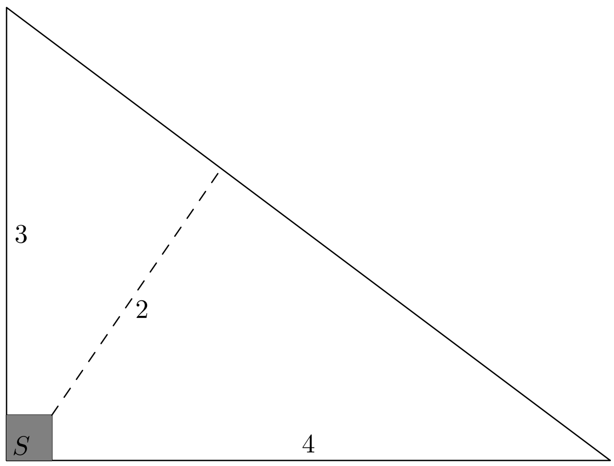
5.3 Miscellaneous

Problem 11 (AMC 10A 2008/20). Trapezoid $ABCD$ has bases $\overline{AB}$ and $\overline{CD}$ and diagonals intersecting at K. Suppose that $AB = 9$, $DC = 12$, and the area of $\triangle AKD$ is 24. What is the area of trapezoid $ABCD$?

Problem 12. Let $ABCD$ be a quadrilateral and let the midpoints of AB, BC, CD, DA be E, F, G, H, respectively. Compute $\frac{[EFGH]}{[ABCD]}$.

Problem 13 (AMC 10B 2017/19). Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

Problem 14 (AMC 10A 2018/23). Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?



Problem 15 (MATHCOUNTS School Handbook 2018/27). A segment with endpoints $G(-2, 3)$ and $H(4, 7)$ is dilated by a scale factor of $\frac{2}{3}$ with center of dilation $(0, 0)$. What is the sum of all the coordinates of G' and H'?

Problem 16 (MATHCOUNTS National Countdown 2018/21). In square units, what is the area of the triangle with vertices $P(-2, 1)$, $Q(3, 8)$ and $R(9, 3)$? Express your answer as a decimal to the nearest tenth.