Math Level 2 Handouts Week 06

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§ 1 Conditional Probability

§ 1.1 Basics

Theorem 1 (Law of Probability). The sum of all the probabilities is 1.

Definition 1 (Sample Set). A **sample set** is the set of all possible outcomes.

Definition 2 (Events). An **event** is a set of outcomes out of the total sample set.

Definition 3 (Conditional Probability). The **conditional probability** of an event A occurring given event B has occurred can be written as $P(A \mid B)$. Then

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Fact 1. The probabilities can be rewritten as the number of possibilities, since the total is the same:

$$P(A \mid B) = \frac{|A \cap B|}{|B|}.$$

Example 1. Suppose you flip a fair coin 7 times. Given that the first 3 flips are heads, what is the probability of getting exactly 4 heads total?

Solution. Using our definition, if A is the event we get 4 heads, and B is the event the first 3 flips are heads, then

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{4}{2^7}}{\frac{1}{2^3}} = \boxed{\frac{1}{4}}.$$

□

§ 1.2 Law of Total Probability

Theorem 2 (Law of Total Probability). The probability an event A occurs is the sum of, for all events B, the probability A occurs given B multiplied by the probability B occurs. In other words,

$$P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \ldots + P(A \mid B_n)P(B_n).$$

Example 2. A test for a certain rare disease is assumed to be correct 95% of the time: if a person has the disease, the test results are positive with probability 0.95, and if the person does not have the disease, the test results are negative with probability 0.95. The disease affects 1 out of every 1000 people in the population randomly. Given that the person just tested positive, what is the probability of actually having the disease?

Solution. Let A be the probability the person has the disease, and B be the probability he was test positive. Then

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

The probability that the person has the disease and he tests positive is $\frac{1}{1000} \cdot 0.95$. The probability he tests positive can be split into two parts: he has the disease, and he tests positive, or he doesn't have the disease, but still tests positive. If it is the first case, the probability is

$$\frac{1}{1000} \cdot 0.95,$$

and in the second case, the probability is

$$\frac{999}{1000} \cdot 0.05.$$

Thus,

$$P(B) = \frac{50.9}{1000}.$$

Thus,

$$P(A \mid B) = \frac{\frac{0.95}{1000}}{\frac{50.9}{1000}} \approx \boxed{1.87\%}.$$

Note how we used the Law of Total Probability to find $P(B)$. □

§ 1.3 Bayes' Theorem

Theorem 3 (Bayes' Theorem). Let A, B be events such that $P(B) > 0$. Then

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}.$$

Example 3. I have two boxes of apples and oranges. In box 1, there are 5 oranges and 6 apples, in box 2 there are 6 oranges and 5 apples. I randomly pick a box and then in this box randomly pick a fruit. What is the probability that I picked box 1 given that I picked an orange?

Solution. Let A be the event that box 1 was picked, and B be the event that an orange was picked. Then using Bayes' Theorem, we get

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}.$$

The probability $P(B \mid A)$ represents the probability that an orange is picked given we picked box 1, which equals $\frac{5}{5+6} = \frac{5}{11}$. The probability box 1 is picked is $P(A) = \frac{1}{2}$. The probability that an orange was picked can be split into two cases using the Law of Total Probability: box 1 was picked, then an orange was picked, or box 2 was picked, then an orange was picked. Thus,

$$P(B) = \frac{1}{2} \cdot \frac{5}{11} + \frac{1}{2} \cdot \frac{6}{11} = \frac{1}{2}.$$

Therefore,

$$P(A \mid B) = \frac{\frac{5}{11} \cdot \frac{1}{2}}{\frac{1}{2}} = \boxed{\frac{5}{11}}.$$

□

Sometimes it is a good idea to write out what is $P(A \mid B), P(B \mid A), P(A), P(B)$, so that when you apply the formulas, plugging in will be easier.

§ 1.4 Further Reading

1. Conditional Probability, Berkeley Math Circle
2. Mathcounts Minis, AoPS

§ 1.5 Problems

Problem 1 (Folklore). A family has two children.

1. Given that at least one of the children is a boy, what is the probability that both children are boys?
2. Given that at least one of the children is a boy born on a Tuesday, what is the probability that both children are boys?

You may assume throughout this problem that children are male and female with equal probability, that children are born on each day of the week with equal probability, and that gender and birth day are independent.

Problem 2. Two dice are rolled and the sum of the numbers showing up taken. We know that the sum is even. What is the probability that the sum is also prime?

Problem 3. Brice has five mice. He randomly distributes a total of seven grains of rice to them. Given that each mouse has at least one grain of rice, what is the probability that no mouse has more than three grains of rice?

Problem 4 (AMC 10A 2011/21). Two counterfeit coins of equal weight are mixed with 8 identical genuine coins. The weight of each of the counterfeit coins is different from the weight of each of the genuine coins. A pair of coins is selected at random without replacement from the 10 coins. A second pair is selected at random without replacement from the re- maining 8 coins. The combined weight of the first pair is equal to the combined weight of the second pair. What is the probability that all 4 selected coins are genuine?

Problem 5. Three standard six-sided dice are rolled. Given that the product of the three numbers rolled is a multiple of 3, determine the probability that the sum of these three numbers is a multiple of 3.

Problem 6. We flip 10 fair coins, all lined up in a row. You cannot see the outcome yet, but I inform you that the number of heads is even. What is the probability that we get heads in at least 6 of the 10 flips?

Problem 7. A box contains six cards. Three of the cards are black on both sides, one card is black on one side and red on the other, and two of the cards are red on both sides. You pick a card uniformly at random from the box and look at a random side. Given that the side you see is red, what is the probability that the other side is red?

Problem 8. About $\frac{2}{3}$ of drivers use their cell phone while driving. Suppose that you are 5 times more likely to get into an accident if you text and drive, and if you don't use your cell phone, you have a 1% chance of getting into an accident. What is the probability that someone was texting given that they got into an accident?

Problem 9 (AoPS). Suppose the following facts to be true:

- The probability of a random kindergartener having chicken pox at any given time is 2%.
- Among kindergarteners who have chicken pox, 75% have red spots.
- Among kindergarteners who do not have chicken pox, 1% have red spots.

Given that Sanjay, a kindergartener, has red spots, what is the probability that Sanjay has chicken pox?

Problem 10. Bag A has white and black marbles. Bag B has white and black marbles. Suppose we draw a marble at random from Bag A and put it in Bag B. After doing this, we draw a marble at random from Bag B, which turns out to be white. Given this information, what is the probability that the marble we moved from Bag A to Bag B is white?

Problem 11 (AoPS). The inhabitants of Liarland tell the truth 40% of the time. One-third of the inhabitants have six toes on their right foot. You ask four inhabitants of Liarland whether the King of Liarland has six toes on his right foot. They've all seen the King's feet, since the King always goes barefoot, and they all say yes. What is the probability that the King's right foot has six toes?

Problem 12 (AoPS). In Freedonia, every day is either cloudy or sunny (not both). If it's sunny on any given day, then the probability that the next day will be sunny is $\frac{3}{4}$. If it's cloudy on any given day, then the probability that the next day will be cloudy is $\frac{2}{3}$.

1. In the long run, what fraction of days are sunny?
2. Given that a consecutive Saturday and Sunday had the same weather in Freedonia, what is the probability that that weather was sunny?

Problem 13 (Mandelbrot). On average, one in five Martians is a compulsive liar, and the rest always tell the truth. It rains 30% of the time on Mars. If three randomly-chosen Martians tell Astronaut Sue that it is raining, then what is the probability that it is actually raining?

Problem 14 (Monty Hall Problem). Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

Problem 15. A jar has 499 fair pennies and one penny with heads on both sides. A penny is chosen from the jar at random and flipped 9 times. It comes up heads every time. What is the probability that the coin is the two-headed coin?

Problem 16. A coin is flipped 20 times in a row. Given that exactly 14 heads appeared, find the probability that no two consecutive coin flips were both tails.