Exponents

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February 12, 2021

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1 Introduction

These lecture notes will be short. In general, there are two (main) types of problems that appear on ML:

- **converting to the same base**, like changing 9 and 27 to 3^2 and 3^3, respectively, and
- **number-theoretic problems**, which mostly involve finding the last few digits of some number in the form a^b.

The NT problems require ***modular arithmetic***.

1.1 Definition

Exponent
An ***exponent*** is a symbol written above and to the right of a mathematical expression to indicate the operation of raising to a power.

In other words, in a^b, b is the exponent. An example of raising a number to an exponent or power is 3^2. All this means is that you are multiplying 3 by itself 2 times, so it's the same thing as $3 \cdot 3$. You can also have 3^{100} which is the same thing as $3 \cdot 3 \cdot 3 \cdot 3 \ldots$, but multiplying 3 by itself a total of 100 times. We can also do the same thing with variables. For example, x^3 is the same thing as $x \cdot x \cdot x$.

1.2 An (easy) example

Example 1.2
Compute 2^3.

Solution. This means 2 multiplied with itself 3 times, so $2 \times 2 \times 2 = \boxed{8}$. □

2 Exponent Rules

1. (Multiplication) $x^m \cdot x^n = x^{m+n}$
2. (Division) $\frac{x^m}{x^n} = x^{m-n}$
3. (Power Rule) $(x^m)^n = x^{m \cdot n}$
4. $\dfrac{1}{x^a} = x^{-a}$
5. $x^0 = 1$ when $x \neq 0$
6. $1^a = 1$
7. $x^{\frac{a}{b}} = x^{a \cdot \frac{1}{b}} = \sqrt[b]{x^a}$

Example 2.1
Compute $(2^3)^2$.

Solution. Using the power rule, we get $2^6 = \boxed{64}$. □

Example 2.2
Compute $\frac{(4x^2y^4)^3}{(8xy^3)^2}$.

Solution. The top is $64x^6y^{12}$ and the bottom is $64x^2y^6$, so the answer is $\boxed{x^4y^6}$. □

3 Exponents in Number Theory

Usually, you will see huge exponents and you'll be terrified of that. Here is how to deal with them:

Fact 3.1 (Large Exponents Method). If there is something in the form a^b, where b is large, and we are looking for the units digit, for example, try to find a pattern.

Example 3.2
Find the last two digits of $5^{20202020}$.

Solution. Through testing, we see that $5^2, 5^3, \ldots$ all end in 25, so the answer is $\boxed{25}$ (probably). □

We'll try some harder examples in the problem set.

3.1 Heuristics

A few strategies to apply:

- **follow a pattern**: usually, the problem asks for the last few digits, and these digits will eventually repeat. If you follow the pattern long enough, it should work out.
- **apply modular arithmetic**: in particular, try reducing the base to a number smaller than the modulo. Manipulation of the exponent requires heavier machinery, like Fermat's Little Theorem or Euler's Totient Theorem.