ASE 2020-21 Notes

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§1 Sunday, 07/12/20

§1.1 Number Sense: Multiplying by 101 Trick

In the same spirit as the multiplying by 11's trick, multiplying by 101 involves adding gap connected digits. Let's look at an example:

$438 \times 101 =$	Ones:	1×8	**8**
	Tens:	1×3	**3**
	Hundreds:	$1 \times 4 + 1 \times 8$	*1***2**
	Thousands:	$1 \times 3 + \mathit{1}$	**4**
	Tens Thousands:	1×4	**4**
	Answer:	**44238**	

So you see, immediately you can write down the ones/tens digits (they are the same as what you are multiplying 101 with). Then you sum gap digits and move down the line. Let's look at another example:

$8234 \times 101 =$

Ones/Tens:	**34**	**34**
Hundreds:	$2+4$	**6**
Thousands:	$8+3$	*1***1**
Tens Thousands:	$2+\mathit{1}$	**3**
Hundred Thousands:	**8**	**8**
Answer:	**831634**	

§1.1.1 Multiplying by 101 Problems

1. $1234 \times 101 =$ ________________
2. $10.1 \times 234 =$ ________________
3. $369 \times 101 =$ ________________
4. $34845 \div 101 =$ ________________
5. $22422 \div 101 =$ ________________
6. $202 \times 123 =$ ________________
7. If 6 balls cost \$6.06, then 15 balls cost: \$ ________
8. $\mathbf{404} \times \mathbf{1111} =$ ________________
9. (*) $(48+53) \times 151 =$ ________________
10. (*) $8888 \times 62.5\% \times \frac{5}{11} =$ ________________

§1.2 Calculator: Stated Problems

There are usually around fourteen stated problems. There are many types of problems, and here are few:

1. Rate
2. Probability
3. Pythagorean Theorem
4. Simple Geometry Figures
5. Money

We have talked about rate problems before, so I will skip it. Probability questions will be discussed later throughout the year. There are problems where the formula for some process is described and the student is required to interpret the words to come up with the formula in order to solve the problem. For example:

Example 1. Bob has 13 lit candles. On each step, he blows out 3 and lights 2. After how many steps are all the lights first blown out?

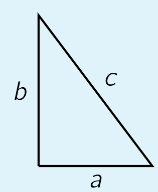
Remark 1. *This isn't a real calculator problem, but it is a question that involves a* **process**.

Solution. Each step, he blows out 3 and lights 2, which means at the end of each step 1 candle was blown out. This might seem to mean the answer is 13 (one candle at each step), but it is not. Let's say he makes 10 steps, and now has 3 candles left. After one more step, all the candles are out! This is the first time this happens, so the answer is $\boxed{11}$. □

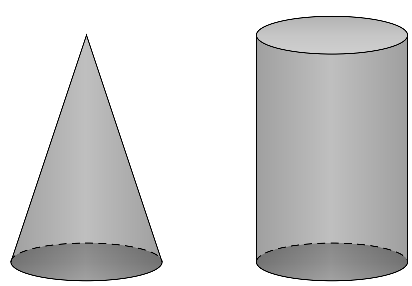
All of you should be familiar with the **Pythagorean Theorem**:

Theorem 1 (Pythagorean Theorem). If the legs of a right triangle are a and b and the hypotenuse is c, then

$$a^2 = b^2 = c^2.$$



For simple geometric figures, typical geometric figures would be circles, squares, triangles (isosceles, equilateral, right, scalene), rectangles, rhombus, parallelograms, trapezoids, cubes, rectangular boxes, spheres, right cylinders, right cones and square-base pyramids.



You can easily search up the surface areas and volumes online. For example, to find the volume of a cube, type `Volume of a Cube` into Google. Sometimes there are **funky areas** to find, such as hemispheres or even sliced up cubes. We can try these types of problems in the future.

Money problems are just that – problems about money. These just require reading the question carefully, and **knowing when to add the money and when to subtract it**.

§1.3 Basic Equation Solving

§1.3.1 Algebraic Manipulation

To introduce the topic of **algebraic manipulation**, let us start with a little known example: **egyptian fractions**.

Theorem 2 (Egyptian Fractions). For all a, b where $ab \neq 1$,

$$\frac{a}{ab-1} = \frac{1}{b(ab-1)} + \frac{1}{b}.$$

From here, we can see that putting things together (**factoring**) is just as important as taking them apart (**distributing**). Now, let us turn the power of **products**:

Example 2. For positive real numbers a, b,

$$a + \frac{1}{b} = 4,$$

$$b + \frac{1}{a} = 5.$$

Find $ab + \frac{1}{ab}$.

Solution. It is very easy to get lost in the problem if we directly try to solve for a and b. Instead, let us **multiply** the equations:

$$(a + \frac{1}{b})(b + \frac{1}{a}) = 4 \cdot 5 = 20,$$

$$ab + \frac{a}{a} + \frac{b}{b} + \frac{1}{ab} = ab + \frac{1}{ab} + 2 = 20,$$

$$ab + \frac{1}{ab} = \boxed{18}.$$

□

There are way too many types of manipulations to include in this one lecture. Instead, we will do some examples and hopefully the idea behind them will become clear.

Example 3. Let $x + y = \frac{3+\sqrt{5}}{2}$, and $x - y = \frac{3-\sqrt{5}}{2}$. Find $x^2 - y^2$.

Solution. If we multiply these equations like in this example, we get

$$x^2 - y^2 = \frac{(3^2) - (\sqrt{5})^2}{4} = \frac{4}{4} = \boxed{1}.$$

Can you imagine if we tried to solve for x and y? □

The following is a very common manipulation:

Theorem 3 (Common Manipulations). Let x, y be nonzero real numbers such that $x + y = a$ and $xy = b$. Then:

- $x^2 + y^2 = a^2 - 2b$
- $x^2y + xy^2 = ab$
- $x^3 + y^3 = a^3 - 3ab$
- $(x+1)(y+1) = a + b + 1$
- $|x - y| = \sqrt{a^2 - 4b}$
- $\frac{1}{x} + \frac{1}{y} = \frac{a}{b}$

This is very easily proven, but still useful nonetheless.

§1.3.2 Memorization

The following are formulas you should definitely memorize:

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a-b)^2 = a^2 - 2ab + b^2$
- $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
- $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
- $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$
- $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab+bc+ca}{abc}$

§1.3.3 Linear Equations

Definition 1 (Linear Equation). A **linear equation** is an equation with variables that have degree 1, meaning they are not squared, or cubed, square rooted, etc.

A **system of linear equations** is a set of two or more linear equations.

Definition 2 (Slope). The **slope** of a linear equation is the rate of change.

For example, if a point is moved two to the right and four up, we say the slope is $\frac{4}{2} = 2$. In general, for a point (x_1, y_1) and another point (x_2, y_2), the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope usually uses the variable m. There are a couple of ways you will see two-variable linear equations written:

1. Standard Form: $ax + by + c = 0$, or sometimes $ax + by = c$, where a, b, c are constants, and $-\frac{a}{b}$ is the slope.
2. Slope-Intercept Form: $y = mx + b$, where m is the slope and b is the y-intercept, which is where the y-axis intersects the line. Similarly the x-intercept is the point where the x-axis intersects the line.
3. Point-Slope Form: $y - y_0 = m(x - x_0)$, where (x_0, y_0) is a point on the line, and m is the slope of the line.

The following is a list of ways to solve these linear equations.

Theorem 4 (Substitution Method). Given two equations, solve for one variable in one equation, and replace that variable with the other variable in all the other equations.

For example, if you have the equations

$$4x + 2y = 6,$$

$$5x + y = 6,$$

you can solve for y in the first one:

$$y = 3 - 2x,$$

then plug it into the second equation:

$$5x + 3 - 2x = 6,$$

$$3x + 3 = 6,$$

$$x = 1,$$

$$y = 3 - 2 \cdot 1 = 1.$$

Theorem 5 (Elimination Method). If we subtract two equations, and one of the variables has the same **coefficient** (which means the number in front of the variable); for example, $5x$ has a coefficient of 5) in both equations, then it will cancel out and we will be left with one less variable.

For example, if you have the equations

$$4x + 2y = 6,$$

$$5x + y = 6,$$

then if you multiply equation 2 by 2, you get

$$10x + 2y = 12,$$

and if we subtract equation 1 from equation 2, you get

$$10x + 2y - (4x + 2y) = 12 - 6 = 6,$$

$$10x - 4x + 2y - 2y = 6,$$

$$6x + 0 = 6,$$

$$x = 1.$$

Plugging this back into the second equation, we get

$$5 \cdot 1 + y = 6,$$

$$y = 1.$$

Theorem 6 (Graphing Method). This method is less common, because it requires you to be extremely **precise**. The method is just what it sounds like – you graph the equations, and see where they intersect.

A method we did not talk about was **the matrix method**. This will likely not appear in Mathcounts, AMC8, or TMSCA; it certainly hasn't appeared for me. The only time a *matrix* will appear is at the very end of a General Math test, and even then there is only ever one question.
There a few possibilities when solving a pair of linear equations. You can either get **0 solutions**, **1 solution**, or **infinitely many solutions**. We will consider each case below:

- <u>0 Solutions:</u> This means the lines are **parallel**.
- <u>1 Solution:</u> This is usually the type of problems you will see on competitions. This just means the lines **intersect**.
- <u>Infinitely Many Solutions:</u> This can only occur for linear equations if they are the **same line**.

§1.3.4 Equation Problems

Problem 1. Find $32^2 + 2 \cdot 32 \cdot 68 + 68^2$.

Problem 2. Find $2^3 - 3 \cdot 2^2 \cdot 8 + 3 \cdot 2 \cdot 8^2 - 8^3$.

Problem 3. Find the slope of the line that goes through $(5, 7)$ and $(6, 8)$.

Problem 4. Find the line with a slope of 3 and y-intercept of -2.

Problem 5. Find the equation of the line going through $(2, 3)$ and $(7, 13)$.

Problem 6. Find the slope of the line with a y-intercept of 3 and a x-intercept of 4.

Problem 7. At what point do the lines $2x + 9y = 7$ and $x = 32 - 4.5y$ intersect?

Problem 8. Find the intersection of the lines $y = ax + b$ and $y = cx + d$ in terms of a, b, c, d, given that they are not parallel.

Problem 9 (Mathcounts). Chris graphs the line $y = 3x + 7$ in the coordinate plane, while Sebastian graphs the line $y = ax + b$, for some numbers a and b. The x-intercept and y-intercept of Sebastian's line are double the x-intercept and y-intercept of Chris's line, respectively. What is the value of the sum $a + b$?