ASE 2020-21 Notes

Lecture Notes by Dylan Yu

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§0 Contents

§1 Sunday, 07/05/20

§1.1 Number Sense: Multiplying by 11 Trick

The simplest multiplication trick is the 11's trick. It is a mundane version of "moving down the line," where you add consecutive digits and record the answer. Here is an example:

$$523 \times 11 = \begin{array}{ll} \text{Ones:} & 1 \times 3 = \mathbf{3} \\ \text{Tens:} & 1 \times 2 + 1 \times 3 = \mathbf{5} \\ \text{Hundreds:} & 1 \times 5 + 1 \times 2 = \mathbf{7} \\ \text{Thousands:} & 1 \times 5 = \mathbf{5} \\ \text{Answer:} & \mathbf{5753} \end{array}$$

As one can see, the result can be obtained by subsequently adding the digits along the number you're multiplying. Be sure to keep track of the carries as well:

$$6798 \times 11 = \begin{array}{ll} \text{Ones:} & \mathbf{8} \\ \text{Tens:} & 9+8 = \mathit{1}\mathbf{7} \\ \text{Hundreds:} & 7+9+\mathit{1} = \mathit{1}\mathbf{7} \\ \text{Thousands:} & 6+7+\mathit{1} = \mathit{1}\mathbf{4} \\ \text{Ten Thousands:} & 6+\mathit{1} = \mathbf{7} \\ \text{Answer:} & \mathbf{74778} \end{array}$$

The trick can also be extended to 111 or 1111 (and so on). Where as in the 11's trick you are adding pairs of digits "down the line," for 111 you will be adding triples:

$$6543 \times 111 = \begin{array}{ll} \text{Ones:} & \mathbf{3} \\ \text{Tens:} & 4+3 = \mathbf{7} \\ \text{Hundreds:} & 5+4+3 = \mathit{1}\mathbf{2} \\ \text{Thousands:} & 6+5+4+\mathit{1} = \mathit{1}\mathbf{6} \\ \text{Ten Thousands:} & 6+5+\mathit{1} = \mathit{1}\mathbf{2} \\ \text{Hun. Thousands:} & 6+\mathit{1} = \mathbf{7} \\ \text{Answer:} & \mathbf{726273} \end{array}$$

Another common form of the 11's trick is used in reverse. For example:

$$1353 \div 11 =$$
$$\text{or}$$
$$11 \times x = 1353$$

Ones Digit of x is equal to the Ones Digit of 1353:		**3**
Tens Digit of x is equal to:	$5 = 3 + x_{tens}$	**2**
Hundreds Digit of x is equal to:	$3 = 2 + x_{hund}$	**1**
Answer:		**123**

Similarly you can perform the same procedure with 111, and so on. Let's look at an example:

$$46731 \div 111 =$$
$$\text{or}$$
$$111 \times x = 46731$$

Ones Digit of x is equal to the Ones Digit of 46731:		**1**
Tens Digit of x is equal to:	$3 = 1 + x_{tens}$	**2**
Hundreds Digit of x is equal to:	$7 = 2 + 1 + x_{hund}$	**4**
Answer:		**421**

The hardest part of the procedure is knowing when to stop. The easiest way I've found is to think about how many digits the answer *should* have. For example, with the above expression, we are dividing a 5-digit number by a roughly 100, leaving an answer which should be 3-digits, so after the third-digit you know you are done.

§1.1.1 Multiplying by 11 Problems

1. $11 \times 54 =$ ________________
2. $11 \times 72 =$ ________________
3. $11 \times 38 =$ ________________
4. $462 \times 11 =$ ________________
5. $11 \times 74 =$ ________________
6. $66 \times 11 =$ ________________
7. $2003 \times 111 =$ ________________
8. $3 \times 5 \times 7 \times 11 =$ ________________
9. $121 \times 121 =$ ________________
10. $\mathbf{33} \times \mathbf{1111} =$ ________________
11. $22 \times 32 =$ ________________
12. $44 \times 25 \times 11 =$ ________________
13. $55 \times 33 =$ ________________
14. (*) $32 \times 64 \times 16 \div 48 =$ ________________
15. $2002 \div 11 =$ ________________
16. $77 \times 88 =$ ________________

§1.2 Calculator: Percent Error

Let us start with the equation for percent error:

Theorem 1 (Percent Error). For all percent error questions,

$$\text{Percent Error} = 100 \times \left(\frac{\text{approximation}}{\text{exact value}} - 1\right)\%.$$

Let's try an example:

Example 1. What is the percent error in using $\frac{22}{7}$ for π?

Solution. The approximation is $\frac{22}{7}$ and the exact value is π, so

$$\text{Percent Error} = 100 \times \left(\frac{\frac{22}{7}}{\pi} - 1\right)\% = \boxed{.402}\%.$$

□

Remark 1. *Note that we did* **not** *include the % sign in the answer, and that we did not write 0.402, but instead .402. Make sure you are following the rules of the calculator test!*

Example 2. What is the percent error in using 365 days/year given that a year is defined as 365.256 days?

Solution. Using the formula, we get

$$\text{Percent Error} = 100 \times \left(\frac{365}{365.256} - 1\right)\% = \boxed{-.0701}\%.$$

□

Now let us consider **percent difference/change**:

Theorem 2 (Percent Difference). For all percent difference questions,

$$\text{Percent Difference} = 100 \times \left(\frac{\text{2nd Number}}{\text{1st Number}} - 1\right)\%.$$

Example 3. What is the percent difference in 28 students and 32 students?

Solution. Using the formula, we get

$$\text{Percent Difference} = 100 \times \left(\frac{32}{28} - 1\right) = \boxed{14.3}\%.$$

□

Example 4. What is the percent difference in 32 students and 28 students?

Solution. Using the formula, we get

$$\text{Percent Difference} = 100 \times \left(\frac{28}{32} - 1\right) = \boxed{-12.5}\%.$$

□

What about **percent increase/decrease**?

Theorem 3 (Percent Increase). For all percent increase questions,

$$\text{Percent Increase} = 100 \times \left(\frac{\text{Larger Number}}{\text{Smaller Number}} - 1\right).$$

Theorem 4 (Percent Decrease). For all percent decrease questions,

$$\text{Percent Decrease} = 100 \times \left(1 - \frac{\text{Smaller Number}}{\text{Larger Number}}\right).$$

Example 5. What is the percent increase for \$125 and \$100?

Solution. Using the formula, we get

$$\text{Percent Increase} = 100 \times \left(\frac{125}{100} - 1\right) = \boxed{25.0}\%.$$

□

Example 6. What is the percent decrease for \$125 and \$100?

Solution. Using the formula, we get

$$\text{Percent Decrease} = 100 \times \left(1 - \frac{100}{125}\right) = \boxed{20.0}\%.$$

□

As you have may have noticed the percent increase and percent decrease equations are set up so that the answers are **always positive**. This is a case where one of those rules in the calculator contest may not be quite the same as those taught at school.

§1.3 General Math: Quadratics

§1.3.1 Definitions

Definition 1 (Polynomial). A **polynomial** is an equation of the following form:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where

$$a_0, a_1, \ldots, a_n$$

are constants.

Definition 2 (Quadratic Equation). A **quadratic equation** is a polynomial with $n = 2$:

$$ax^2 + bx + c.$$

§1.3.2 Solving Quadratics

A common way to solve the equation $ax^2 + bx + c = 0$ is using the **Quadratic Formula**:

Theorem 5 (Quadratic Formula). For the equation $ax^2 + bx + c = 0$, the roots x_1, x_2 must be equal to

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Some people may ask, what if $b^2 - 4ac$ is less than 0? What do we do then? This leads us to the **Discriminant Test**:

Theorem 6 (Discriminant Test). For the equation $ax^2 + bx + c = 0$, we have the following cases:

- If $b^2 - 4ac > 0$, we have **two real solutions.**
- If $b^2 - 4ac = 0$, we have **one real solutions.**
- If $b^2 - 4ac < 0$, we have **zero real solutions.**

Using the Quadratic Formula, we can calculate the sum of roots and product of roots:

Theorem 7 (Sum and Product of Roots). For the equation $ax^2 + bx + c = 0$, the sum of roots s is

$$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a},$$

and the product of roots p is

$$x_1 x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}$$

Some problems will ask you to find the sum or product of the roots even for multi-degree polynomials. **Vieta's Formulas** state that the sum of the roots is still $-\frac{b}{a}$, and the product of the roots is still $\frac{c}{a}$:

Theorem 8 (Vieta's Formulas Simplified). For any polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

the **sum of roots** is $-\frac{\text{second coefficient}}{\text{first coefficient}} = -\frac{a_{n-1}}{a_n}$, and the **product of roots** is $\frac{\text{last coefficient}}{\text{first coefficient}} = \frac{a_0}{a_n}$.

For those who are looking for a more **advanced** and more **powerful** theorem, we can generalize this formula:

Theorem 9 (Vieta's Formulas). For any polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

let $r_1, r_2, \ldots, r_n$ (an n-degree equation has n different roots). Vieta's formulas state that

$$a_n = a_n$$

$$a_{n-1} = -a_n(r_1 + r_2 + \ldots + r_n)$$

$$a_{n-2} = a_n(r_1 r_2 + r_1 r_3 + \ldots + r_{n-1} r_n)$$

$$\vdots$$

$$a_0 = (-1)^n a_n \, (r_1 r_2 \ldots r_n).$$

Note that the parts in parenthesis are known as **symmetric sums**. The kth symmetric sum σ_k is equal to $(-1)^k \frac{a_{n-k}}{a_n}$.

Remark 2. *The theorem above is* **very** *hard to understand. We will spend not only this class but future classes understanding it, so don't worry if you don't get it immediately!*

§1.3.3 Quadratics Problems

Problem 1. Find the sum of roots to the equation $x^{2020} = 2020x^{2019} + 1$.

Problem 2. Find the product of the roots of the equation $x^3 = 9\pi x + x^2 + 1$.

Problem 3 (Mathcounts). What is the average value of the three roots of the equation $x^3 - 12x^2 - 4x + 48 = 0$?

Problem 4 (Mathcounts State Sprint 2014/27). The fourth degree polynomial equation $x^4 - 7x^3 + 4x^2 + 7x - 4 = 0$ has four real roots, a, b, c and d. What is the value of the sum $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$? Express your answer as a common fraction.

Remark 3. *The problem above is supposed to be a* **hard** *problem. But it has a very simple idea:* $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{abc+abd+acd+bcd}{abcd}$*! This shows that you should* **never** *back away form a problem.*

Problem 5. Solve the following system of equations for x, y:

$$2x - 3y = 11,$$

$$xy = -5$$

Problem 6. Let p, q, r be roots of the polynomial $x^3 - 2x^2 + 3x - 4$. Find $(p+1)(q+1)(r+1)$.

Problem 7 (AMC 10A 2003/18). What is the sum of the reciprocals of the roots of the equation $\frac{2003}{2004}x + 1 + \frac{1}{x} = 0$?

(A) $-\frac{2004}{2003}$ (B) -1 (C) $\frac{2003}{2004}$ (D) 1 (E) $\frac{2004}{2003}$

Problem 8. The polynomial $x^3 - ax + 15$ has three real roots. Two of these roots sum to 3. What is $|a|$?

Problem 9. What is the only real number that could be a multiple root of $x^3 + ax + 1 = 0$, if a is real?