AMC8

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Contents

1 AMC 8 Information

There are 25 questions and 40 minutes. It is multiple choice, with no penalty for incorrect answers.

1.1 Test Taking Strategies

There are a few strategies for guessing:

- Substitute in answer choices
- Process of elimination
- Engineer's Induction (find a pattern)
- For NT problems, try eliminating odd answer choices, or even answer choices, etc. (this involves taking a mod)
- Draw a good figure (use graph paper, rulers, protractors, etc.)
- **Guess every problem**

To avoid silly mistakes:

- Reread the question
- Check your work when your answer doesn't match a choice
- Plug your answer back into the equation
- Sanity check (for example, cars don't travel at 1000 mph)
- Keep your scratch work neat (it's OK to use a lot of paper)
- Try doing the problem in different ways when checking

1.2 Useful Properties of 2020

1. Prime factorization of 2020: $2^2 \times 5 \times 101$
2. Prime factorization of 2019: 3×673
3. Prime factorization of 2021: 43×47
4. Number of factors of 2020: 12
5. Sum of factors of 2020: 4284
6. Product of factors of 2020: 2020^6

1.3 General Manipulations

The following are formulas you should definitely memorize:

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a-b)^2 = a^2 - 2ab + b^2$
- $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
- $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ca)$
- $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$
- $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab+bc+ca}{abc}$

> *Remark 1.1 (Warning!).* This review is **not** comprehensive. These are just outlines of some topics that might show up on the AMC 8.

2 Algebra

2.1 Arithmetic

1. PEMDAS
2. Introduction to variables

Everyone knows PEMDAS: we apply operations in the order of:

1. **P**arenthesis
2. **E**xponents
3. **M**ultiplication/**D**ivision
4. **A**ddition/**S**ubtraction

Variable
A ***variable*** is a placeholder for a value we don't know.

In other words, it *varies*. There are quite a few things we can do with variables, and these things make up the subject of Algebra.

Problem 1 (AMC 10 2002/6). Cindy was asked by her teacher to subtract 3 from a certain number and then divide the result by 9. Instead, she subtracted 9 and then divided the result by 3, giving an answer of 43. What would her answer have been had she worked the problem correctly?

2.2 Fractions, Decimals, & Percentages

1. Types of fractions (proper/improper, mixed, simplest form)
2. Repeating/terminating digits
3. Conversions
4. Operations with fractions/decimals/percentages
5. Simple/compound interest
6. Word problems (with fractions/decimals/percentages)

For anyone who has done Number Sense, these ideas should be common. I will assume everyone understands these.

Problem 2 (UNB Grade 9 2016/12). A collection of coins was shared. Mary received $\frac{1}{3}$ of the coins, Amir received $\frac{1}{5}$ of the coins, and Samita received $\frac{1}{6}$ of the coins. The remaining 36 coins were given to Troy. How many coins were in the entire collection?

Problem 3 (CEMC Gauss 2000/13). Karl had his salary reduced by 10%. He was later promoted and his salary increased by 10%. If his original salary was \$20,000, what is his present salary?

2.3 Exponents, Logarithms, & Radicals

1. Basic properties (multiplication, division, raising an exponent)
2. Negative/fractional exponents
3. Rationalizing and simplifying

Exponent
An ***exponent*** is a symbol written above and to the right of a mathematical expression to indicate the operation of raising to a power.

Logarithm
A ***logarithm*** indicates how many of a number we need to multiply to get that number.

Radical
If $c = a^b$, then $\sqrt[b]{c} = a$ is the ***radical***.

Some exponent rules:

1. (Multiplication) $x^m \cdot x^n = x^{m+n}$
2. (Division) $\frac{x^m}{x^n} = x^{m-n}$
3. (Power Rule) $(x^m)^n = x^{m \cdot n}$
4. $\dfrac{1}{x^a} = x^{-a}$
5. $x^0 = 1$ when $x \neq 0$
6. $1^a = 1$
7. $x^{\frac{a}{b}} = x^{a \cdot \frac{1}{b}} = \sqrt[b]{x^a}$

Some logarithm rules:

1. (Identity) $\log_a a = 1$
2. $\log_a 1 = 0$
3. (Addition) $\log_a b + \log_a c = \log_a (bc)$
4. (Subtraction) $\log_a b - \log_a c = \log_a \left(\frac{b}{c}\right)$
5. (Power Rule) $\log_a (b^c) = c \log_a b$
6. (Change of Base) $\log_a b = \frac{\log_c b}{\log_c a}$

Some radical rules:

1. (Radical to Exponent) If $\sqrt[n]{x} = r$, then $x = r^n$
2. $a\sqrt{b} + c\sqrt{b} = (a+c)\sqrt{b}$
3. $a\sqrt{b} - c\sqrt{b} = (a-c)\sqrt{b}$
4. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$
5. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$
6. (Simplifying Radicals) $\frac{1}{\sqrt{a}} = \frac{\sqrt{a}}{a}$ and $\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$

Problem 4 (AMC 10A 2002/3). According to the standard convention for exponentiation,

$$2^{2^{2^2}} = 2^{(2^{(2^2)})} = 2^{16} = 65536.$$

If the order in which the exponentiations are performed is changed, how many other values are possible?

Problem 5. Compute the value of $(\sqrt{12}+\sqrt{75}+\sqrt{147})^2$.

Problem 6 (ML ES Target 11322/2). Find the units digit of 147^{148}.

2.4 Word Problems

1. Converting to math equations

Word problems are common, and the idea is always to **turn the words into equations**. We'll see this in all types of problems, not just Algebra ones.

Problem 7 (UNB Grade 9 2010/21). Farmer Fred said to Farmer John: "If you sell me 45 hectares of land, I will have twice as much land as you." Then Farmer John said to Farmer Fred: "If you sell me 45 hectares of land, I will have just as much land as you." How many hectares of land does farmer Fred have?

2.5 Rate Problems

1. Unit conversions
2. $d = rt$
3. Average speed
4. Amount of work/output done

For unit conversions, just know your units! A common one is 1 mile is 5280 feet.

Theorem 2.5 (Rate Formula)
The distance travelled equals the rate of travel times the time travelled. In other words,

$$d = rt.$$

Harmonic Mean
The ***harmonic mean*** of $a_1, a_2, \ldots, a_n$ is

$$\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}.$$

Average Speed
Average speed is the total distance divided by the total time.

Remark 2.8. It is very common for there to be a ***system of equations*** when solving harder rate problems.

Problem 8 (AMC 8 2017/23). Each day for four days, Linda traveled for one hour at a speed that resulted in her traveling one mile in an integer number of minutes. Each day after the first, her speed decreased so that the number of minutes to travel one mile increased by 5 minutes over the preceding day. Each of the four days, her distance traveled was also an integer number of miles. What was the total number of miles for the four trips?

2.6 Statistics

1. Mean, median, mode, range
2. Weighted average

Mean
Mean is the average of all numbers.

The mean is also sometimes known as the ***arithmetic mean***.

Median
Median is the "middle number" in a sorted list.

For example, 3 is the median in $3, 1, 2, 4, 5$, and 3.5 is the median in $3, 1, 2, 4, 5, 6$ (you sort the list, take the middle two numbers, and average them).

Mode
Mode is the most common number.

Range
Range is the largest number minus the smallest number.

Weighted Average
Weighted average is the average of numbers that are not worth the same.

Weighted average rarely appears, but it should be noted. Basically, if there is a $\frac{2}{3}$ chance we get a 1 and a $\frac{1}{3}$ chance we get a 2, then the weighted average is

$$\frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 2 = \frac{4}{3}.$$

This is actually the underlying idea behind expected value.

Problem 9 (CEMC Gauss 2016/16). The mean (average) of a set of six numbers is 10. If the number 25 is removed from the set, what is the mean of the remaining numbers?

2.7 Sequences

1. Arithmetic/geometric sequences (finite and infinite)
2. Recursive sequences
3. Telescoping series

Arithmetic Sequence
An ***arithmetic sequence*** is a sequence of numbers in which each term is given by adding a fixed value to the previous term.

Theorem 2.15 (Terms of an Arithmetic Sequence)
The nth term in an arithmetic sequence is

$$a_n = a_1 + d(n-1),$$

where a_n is the nth term, a_1 is the first term, and d is the difference between consecutive terms.

Theorem 2.16 (Sum of an Arithmetic Sequence)
The sum of the first n terms of an arithmetic sequences is

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2a_1 + (n-1)d).$$

Theorem 2.17 (Sum of First n Positive Integers)
For all positive integers n,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Corollary 2.18 (Sum of First n Odd Integers)
For all positive integers n,

$$1 + 3 + 5 + \ldots + 2n - 1 = n^2.$$

Corollary 2.19 (Sum of First n Even Integers)
For all positive integers n,

$$2 + 4 + 6 + \ldots + 2n = n(n+1).$$

Fact 2.20 (Cancellation Method). When you see an arithmetic sequence starting with $-a$, and somewhere later you see a, you can cancel all the terms in between, including $-a$ and a.

Geometric Sequence
A ***geometric sequence*** is a sequence of numbers in which each term is a fixed multiple of the previous term.

Theorem 2.22 (Terms of a Geometric Sequence)
The nth term in an geometric sequence is

$$a_n = a_1 r^{n-1},$$

where a_n is the nth term, a_1 is the first term, and r is the common ratio.

Theorem 2.23 (Sum of a Finite Geometric Sequence)
The sum of the first n terms of a geometric sequence is given by

$$S_n = a_1 + a_2 + \cdots + a_n = a_1 \cdot \frac{r^n - 1}{r - 1},$$

where a_1 is the first term in the sequence, and r is the common ratio.

Infinite Geometric Sequence
An ***infinite geometric sequence*** is a geometric sequence with an infinite number of terms.

Theorem 2.25 (Sum of an Infinite Geometric Sequence)
Let $S = a + ar + ar^2 + \ldots$, where $|r| < 1$. Then

$$S = \frac{a}{1 - r}.$$

Recursive Sequence
A ***recursive sequence*** is a sequence defined upon previous terms.

The most famous example is the ***Fibonacci Sequence***:

$$F_n = F_{n-1} + F_{n-2},$$

where $F_0 = F_1 = 1$.

Telescoping Sequence
A ***telescoping sequence*** is a sequence where some terms cancel as a result of manipulating the sequence in some way.

Problem 10 (Folklore). Compute

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{2019 \cdot 2020}.$$

Problem 11 (AMC 8 2008/12). A ball is dropped from a height of 3 meters. On its first bounce it rises to a height of 2 meters. It keeps falling and bouncing to $\frac{2}{3}$ of the height it reached in the previous bounce. On which bounce will it not rise to a height of 0.5 meters?

2.8 Functions & Operations

1. Definitions of function, domain, range
2. Linear functions
3. Piecewise-defined functions
4. Absolute value, floor/ceiling functions
5. Operators

Function
A ***function*** is a relation that gives an output for every input.

Domain
The ***domain*** of a function is all the values that go into it.

Range
The ***range*** of a function is all the values that come out of it.

Linear Function
A ***linear function*** is a function of the form $f(x) = mx + b$, where m, b are constants.

In other words, a linear equation.

Piecewise-Defined Function
A ***piecewise-defined function*** is a group of functions combined together.

For example,

$$f(x) = \begin{cases} 3 & x < 0 \\ x & 0 < x < 5 \\ x^2 & x > 5 \end{cases}$$

is a piecewise function.

Absolute value
The ***absolute value*** of a number is the distance of the number from 0.

In other words, positive numbers remain positive and negative numbers become positive. It is written as $|x|$.

Floor
The ***floor*** of a number is the largest integer below the number.

This is denoted as $\lfloor x \rfloor$.

Ceiling
The ***ceiling*** of a number is the smallest integer above the number.

This is denoted as $\lceil x \rceil$.

Problem 12. Let $a\#b = a^2 - 2ab + b^2$. Find $(2\#4)\#3$.

2.9 Polynomials

1. Polynomials of a single variable
2. Definitions of degree, coefficient, root, term
3. Quadratics
4. Finding roots, sum and product of roots (Vieta's formulas)

Polynomial
A ***polynomial*** of x with degree n is defined as follows:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x^1 + a_0 x^0.$$

It is basically a bunch of variables of various powers summed together.

Degree
The ***degree*** of a polynomial is its largest exponent.

Coefficient
The ***coefficient*** of a term is the constant multiplied to a variable.

Root
A ***root*** of a function is a value that, when plugged in, makes it 0.

There are many tricks for finding roots.

Term
A ***term*** in a polynomial is a part consisting of a constant and a variable to some power.

Quadratic
A ***quadratic*** is a polynomial with degree 2.

We usually write these in the form $ax^2 + bx + c$. We will use this notation for quadratics unless otherwise specified.

Theorem 2.42 (Quadratic Minima/Maxima)
If $a > 0$, f has a minimum at $x = -\frac{b}{2a}$. If $a < 0$, f has a maximum at $x = -\frac{b}{2a}$.

Theorem 2.43 (Completing the Square)
If we can write $f(x) = x^2 + bx + c$ as $(x-h)^2 + k$, then the roots of $f(x)$ are

$$x_1, x_2 = h \pm \sqrt{-k}.$$

Note that if $k > 0$, then the roots are not real.

Theorem 2.44 (Quadratic Formula)
Let $ax^2 + bx + c = 0$. Then

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Discriminant
The ***discriminant*** Δ of the quadratic $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac.$$

Theorem 2.46 (Discriminant Test)
Let Δ be the discriminant of a quadratic. Then:

1. If $\Delta > 0$, then the roots are real and unequal.
2. If $\Delta = 0$, then the roots are real and equal.
3. If $\Delta < 0$, then the roots are complex and unequal.

Let's get into polynomials now.

Theorem 2.47 (Factor Theorem)
If $p(a) = 0$, then $p(x)$ has a factor of $x - a$. Furthermore, $p(x) = (x - a)Q(x)$, where $Q(x)$ is the quotient (the remainder is 0).

Theorem 2.48 (Remainder Theorem)
When a polynomial $P(x)$ is divided by $x - a$, the remainder R is equal to $P(a)$. Furthermore, $P(x)$ can be expressed as follows:

$$P(x) = (x - a)Q(x) + R.$$

The identical equation is true for any value of x, especially $x = a$. Therefore,

$$P(a) = R.$$

Theorem 2.49 (Fundamental Theorem of Algebra)
Every non-constant single-variable polynomial with complex coefficients has at least one complex root.

Theorem 2.50 (Rational Root Theorem)
Given a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ with integral coefficients, $a_n \neq 0$. The Rational Root Theorem states that if $P(x)$ has a rational root $r = \pm\frac{p}{q}$ with p, q relatively prime positive integers, p is a divisor of a_0 and q is a divisor of a_n.

Theorem 2.51 (Sum and Product of Roots Formula)
For a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, then the sum of the roots is $-\frac{a_{n-1}}{a_n}$, and the product of the roots is $(-1)^n \frac{a_0}{a_n}$, where n is the degree of the polynomial.

Problem 13. Let m and n be the roots of the polynomial $x^2 - 2020x + 202$. Find the polynomial with roots $-m$ and $-n$ and leading coefficient 2.

Problem 14. Let s and t be the roots of $x^2 - 20x + 202 = 0$. What is $s^2 + t^2$?

3 Combinatorics

3.1 Counting

1. Addition/multiplication
2. Rule of sums/products

Addition and multiplication are your best friends in combinatorics.

Theorem 3.1 (Rule of Sum)
If there are a_1 ways to do a first thing, a_2 ways to do a second thing, and so on, until a_n ways to do the nth thing, and none of them can be done at the same time, then there are

$$a_1 + a_2 + \ldots + a_n$$

actions possible.

Theorem 3.2 (Rule of Product)
If there are a_1 ways to do a first thing, a_2 ways to do a second thing, and so on, until a_n ways to do the nth thing, then there are

$$a_1 \cdot a_2 \cdot \ldots \cdot a_n$$

actions possible to perform all n things.

The rule of product is sometimes called the ***Fundamental Principle of Counting***. There aren't many theorems in combinatorics, which means more logic is required.

Problem 15 (AMC 8 2002/2). How many different combinations of \$5 bills and \$2 bills can be used to make a total of \$17? Order does not matter in this problem.

Problem 16 (UNB Grade 9 2008/16). How many ways can the number $1, 2, 3, 4$ and 5 be placed in a line so that neither 1 nor 5 occupy either the first or the last place in the sequence?

3.2 Permutations & Combinations

1. Factorials
2. Permutations
3. Combinations
4. Combinatorial formulas

Factorial
The ***factorial*** $n! = n \times (n-1) \times (n-2) \ldots \times 2 \times 1$.

Theorem 3.4 (Arranging Objects on a Line/Circle)
The number of ways to arrange n objects on a line is $n!$, and on a circle is $(n-1)!$ (where rotations are considered the same).

Theorem 3.5 (Word Rearrangements)
Let the number of letters in a word be n. If there are k distinct letters, with the 1st letter appearing d_1 times, the 2nd appearing d_2 times, and so on, then the number of ways to arrange the letters is

$$\frac{n!}{d_1!d_2!\dots d_k!}.$$

Permutation
The number of ways to assign k distinct positions to n things is

$$P(n,k) = \frac{n!}{(n-k)!}.$$

Words that imply permutation: **permute, order does matter**. Words that imply combination: **choose, select, order doesn't matter**.

Combination
The number of ways to choose k objects from n objects is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Problem 17 (CEMC Gauss 2006/19). Bethany, Chun, Dominic, and Emily go to the movies. They choose a row with four consecutive empty seats. If Dominic and Emily must sit beside each other, in how many different ways can the four friends sit?

Problem 18. 10 books are labeled a number from 1-10. How many ways can Bob choose 3 books to bring to school?

3.3 Casework

1. Splitting problems into cases
2. When to use casework

Casework problems are solved, of course, by **splitting up into cases**.

Problem 19 (AMC 10A 2005/14). How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

3.4 Complementary Counting

1. Techniques of complementary counting and when to use it

Complementary Counting

Complementary counting is the problem solving technique of counting the opposite of what we want and subtracting that from the total number of cases.

The keyword to look for is **at least**.

Problem 20 (AMC 8 2016/17). An ATM password at Fred's Bank is composed of four digits from 0 to 9, with repeated digits allowable. If no password may begin with the sequence $9, 1, 1$, then how many passwords are possible?

3.5 Overcounting and Sets

1. Overcounting with subtraction
2. Overcounting with division
3. Overcounting with pairs/symmetry
4. Definitions of set, subset, size, union, and intersection
5. PIE

Overcounting

Overcounting is the process of counting more than what you need and then systematically subtracting the parts which do not belong.

Problem 21. Mrs. Sanders has three grandchildren, who call her regularly. One calls her every three days, one calls her every four days, and one calls her every five days. All three called her on December 31, 2016. On how many days during the next year did she not receive a phone call from any of her grandchildren?

3.6 Path Walking

1. Counting the number of paths in a lattice grid
2. Counting shapes

Theorem 3.10 (Grid Walking Formula)
The number of ways to get from $(0, 0)$ to (x, y) in a grid is

$$\binom{x+y}{x}.$$

Remark 3.11. This is equivalent to the number of ways to go from the top left corner to the bottom right corner of a $x \times y$ grid.

Counting shape usually relies on **casework**. In general, I recommend starting with the smallest shapes and working your way up (for example, if you are asked how many triangles are in a figure, try counting the smallest triangles then move to larger ones). However, there are a few formulas:

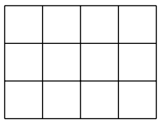
Theorem 3.12 (Number of Rectangles in a Grid Formula)
The number of rectangles in a $m \times n$ grid is

$$\binom{m+1}{2} \cdot \binom{n+1}{2}.$$

Corollary 3.13 (Number of Squares in a Square Grid Formula)
The number of squares in a $n \times n$ grid is

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Problem 22. How many rectangles are in the figure below?



3.7 Digit Counting

1. Palindromic numbers
2. Problems involving digits

Theorem 3.14 (Number of n-Digit Palindromes)
If n is odd, there are $9 \times 10^{\frac{n}{2}-1}$ n-digit palindromes. If n is even, there are $9 \times 10^{\frac{n-1}{2}}$ n-digit palindromes.

Problem 23 (BmMT Team 2016/8). A seven digit number is called “bad" if exactly four of its digits are 0 and the rest are odd. How many seven digit numbers are bad?

3.8 Stars and Bars

1. Applying stars and bars (or balls and boxes or sticks and stones) to counting problems
2. Distinguishability

Theorem 3.15 (Stars and Bars For Nonnegative # of Items)
If we need to split n identical items to k distinguishable groups, where each group can have 0 or more items, the total number of ways to distribute it is $\binom{n+k-1}{k-1}$.

Theorem 3.16 (Stars and Bars For Positive # of Items)
If we have to split n identical items into k distinguishable groups, where every group has at least one item, the number of ways to do so is $\binom{n-1}{k-1}$.

Problem 24. How many positive integer solutions are there to $a+b+c+d=5$?

Problem 25. How many solutions are there to $a+b+c+d=100$, if a, b, c, d are all positive even integers? What if they are all odd?

Problem 26. How many solutions are there to $a+b+c+d \leq 10$, if a, b, c, d are all positive integers?

3.9 Pascal's Triangle

1. Binomial Theorem
2. Pascal's triangle and properties
3. Relation to combinations

Pascal's Triangle
Pascal's Triangle is a triangle of numbers where each number is the two numbers directly above it added together, except for the edges, which are all 1.

Here is row 0 to row 4 (the row with one 1 is known as *row 0*):

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Let's overlay this with **binomial coefficients**:

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Interestingly, Pascal's Triangle is made up of binomial coefficients!

Theorem 3.18 (Pascal's Identity)
For integers $0 \le k \le n$, we have

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Theorem 3.19 (Binomial Theorem)
For a nonnegative integer n, the expression $(x+y)^n$ can be written in the form

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Theorem 3.20 (Sum of Binomial Coefficients)
Let n be a nonnegative integer. Then

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

Problem 27. Simplify

$$\binom{100}{0} + 2\binom{100}{1} + 4\binom{100}{2} + \ldots + 2^{100}\binom{100}{100}.$$

Problem 28. What is the 3rd element to the right in the 8th row of Pascal's triangle?

3.10 Probability

1. Definition of probability, sample space, independent/dependent events, disjoint events
2. Probability with cards, coins, etc.

3. Geometric probability
4. Conditional probability

Probability
Probability is the likelihood of something happening.

To calculate probability, you need to know how many possible options or outcomes there are and how many right combinations there are.

Theorem 3.22 (Probability Formula)
The probability of an event occurring is equal to the number of outcomes in which the event occurs divided by the total number of outcomes.

Theorem 3.23 (Geometric Probability Formula)
The probability of landing on a certain region is equal to the area of the desired region divided by the area of the total region.

Problem 29. Four fair six-sided dice are rolled. What is the probability that the largest number rolled is at least 4? Express your answer as a common fraction in reduced form.

Problem 30 (AHSME 1973/23). There are two cards; one is red on both sides and the other is red on one side and blue on the other. The cards have the same probability $\left(\frac{1}{2}\right)$ of being chosen, and one is chosen and placed on the table. If the upper side of the card on the table is red, then what is the probability that the under-side is also red?

Problem 31. x and y are two positive numbers chosen randomly and uniformly in the interval $[0, 1]$. What is the probability that $x^2 + y^2 \leq 1$ and $x \leq y$?

4 Geometry

4.1 Angles

1. Definitions of acute, right, obtuse, complementary, supplementary, parallel and perpendicular
2. Sum of the degrees in a triangle
3. Inscribed angles and arcs in circles
4. Angle chasing

Acute
An angle is ***acute*** if it is less than 90°.

Right
An angle is ***right*** if it is equal to 90°.

Obtuse
An angle is ***obtuse*** if it is greater than 90°.

Complementary
Two angles are ***complementary*** if they add up to 90°.

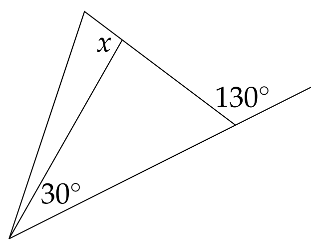
Supplementary
Two angles are ***supplementary*** if they add up to 180°.

Parallel
Two lines are ***parallel*** if they never intersect.

In other words, you can shift one up vertically to get the other.

Perpendicular
Two lines are ***perpendicular*** if they intersect at a right angle.

Problem 32 (UNB Grade 9 2018/3). Find the measure of the angle labeled x in the diagram.



4.2 Triangles

1. Equilateral, isosceles, and scalene triangles
2. Acute, obtuse, and right triangles
3. Pythagorean Theorem and Pythagorean triples
4. $30^\circ - 60^\circ - 90^\circ$ and $45^\circ - 45^\circ - 90^\circ$ triangles

Theorem 4.8 (Area of a Triangle)
A triangle with base b and height h has an area of

$$\frac{bh}{2}.$$

Theorem 4.9 (Area of an Equilateral Triangle)
An equilateral triangle with side length s has an area of

$$\frac{s^2\sqrt{3}}{4}.$$

The height of an equilateral triangle with side length s is $\frac{s\sqrt{3}}{2}$.

Theorem 4.10 (Pythagorean Theorem)
A right triangle with legs a, b and a hypotenuse c satisfies the following condition:

$$a^2 + b^2 = c^2.$$

Common Pythagorean triples:

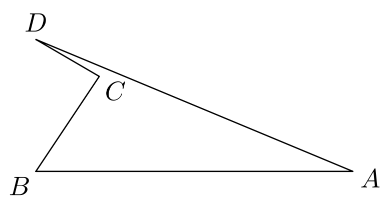
1. $3, 4, 5$
2. $5, 12, 13$
3. $7, 24, 25$
4. $8, 15, 17$

Note that these are **primitive** Pythagorean triples. We could easily scale the sides up (for example, $6, 8, 10$ and $16, 30, 34$ are also Pythagorean triples).
There are quite a few special right triangles:

1. $\mathbf{45^\circ - 45^\circ - 90^\circ}$: if the two legs have side lengths s, then the hypotenuse has length $s\sqrt{2}$.
2. $\mathbf{30^\circ - 60^\circ - 90^\circ}$: if the smaller leg has length s, then the other leg has length $s\sqrt{3}$ and the hypotenuse has length $2s$.

There are also special non-right triangles. For example, $\mathbf{13 - 14 - 15}$ is a special triangle because the area is an integer (84). Furthermore, the height from the side of length 14 has length 12.

Problem 33 (AMC 8 2017/18). In the non-convex quadrilateral $ABCD$ shown below, $\angle BCD$ is a right angle, $AB = 12$, $BC = 4$, $CD = 3$, and $AD = 13$.



What is the area of quadrilateral $ABCD$?

4.3 Quadrilaterals

1. Types of quadrilaterals
2. Cyclic quadrilaterals

3. Area formulas

There are many types of quadrilaterals: squares, rectangles, rhombi, parallelograms, trapezoids, kites, etc.

Theorem 4.11 (Area of a Square)
The area of a square of side length s is s^2.

The two diagonals each have length $s\sqrt{2}$.

Theorem 4.12 (Area of a Rectangle)
The area of a rectangle with length l and width w is lw.

The two diagonals each have length $\sqrt{l^2+w^2}$.

Theorem 4.13 (Area of a Rhombus)
The area of a rhombus with diagonals d_1 and d_2 is $\frac{d_1 d_2}{2}$.

The two diagonals are perpendicular and bisect each other.

Theorem 4.14 (Area of a Parallelogram)
The area of a parallelogram with base b and height h is bh.

The two diagonals bisect each other.

Theorem 4.15 (Area of a Trapezoid)
The area of a trapezoid with bases b_1 and b_2 and height h is $\frac{(b_1+b_2)h}{2}$.

The two bases are parallel to each other.

Theorem 4.16 (Area of a Kite)
The area of a kite with diagonals d_1 and d_2 is $\frac{d_1 d_2}{2}$.

The two diagonals are perpendicular to each other, and one diagonal bisects the other.

Cyclic Quadrilateral
A ***cyclic quadrilateral*** is a quadrilateral inscribed in a circle.

Theorem 4.18 (Cyclic Quadrilateral Angle Condition)
In a cyclic quadrilateral $ABCD$, $\angle A + \angle C = \angle B + \angle D = 180^\circ$.

4.4 Polygons

1. Definition of polygon and regular polygon
2. Angles of polygons
3. Regular polygon

Polygon
A ***polygon*** is a plane figure with at least three straight sides and angles.

Regular Polygon
A ***regular polygon*** is a polygon with equal sides and angles.

Theorem 4.21 (Sum of Interior Angles)
The sum of the interior angles in a polygon is $180°(n-2)$.

This means an interior angle in a regular polygon is $\frac{180°(n-2)}{n}$.

Theorem 4.22 (Sum of Exterior Angles)
The sum of the exterior angles in a polygon is $360°$.

This means an exterior angle in a polygon is $\frac{360°}{n}$.

4.5 Circles

1. Definitions of radius, diameter, chord, tangent
2. Perimeter and area of a circle
3. Inscribed angles
4. Power of a point

Radius
The distance from the center to any point on a circle is the ***radius***.

This of course means this distance is constant.

Diameter
The largest distance between two points on a circle is the ***diameter***.

As it turns out, the largest distance d is $2r$.

Chord
A ***chord*** is a line segment with endpoints lying on a circle.

Tangent
A ***tangent*** is a line touching a circle at only one point.

Note that the line is perpendicular to the line that goes through the tangency point and the center of the circle.

Sector
A ***sector*** is the area bounded between an ***arc*** and the radii that connect the ends of that arc.

Theorem 4.28 (Perimeter of a Circle)
The perimeter of a circle, also known as the ***circumference***, is given by $2\pi r$, where r is the radius.

Theorem 4.29 (Area of a Circle)
The area of a circle is given by πr^2, where r is the radius.

Theorem 4.30 (Area of a Sector)
For a given sector AOB, we have

$$\text{Area of sector } AOB = \frac{\angle AOB}{360^\circ} r^2 \pi.$$

Theorem 4.31 (Inscribed Angle Theorem)
The measure of an inscribed angle is one half the measure of the arc it intercepts.

Corollary 4.32
Any two angles that are inscribed in the same arc are equal.

Corollary 4.33 (Thales' Theorem)
The measure of an inscribed angle is 90° if and only if the arc subtends the diameter.

Theorem 4.34
The measure of an angle formed by two secants which intersect outside the circle is equal to one-half the difference of the arcs intercepted by the secants.

Theorem 4.35
Let BC be a chord of a circle and A be a point outside the circle such that AB is tangent to the circle. If D is a point on the opposite side of BC to A, then $\angle ABC = \angle BDC$.

Theorem 4.36
The measure of the angle formed by two chords is one-half the sum of the intercepted arcs.

Theorem 4.37 (Power of a Point)
Let A, B, C, D be points on a circle, and let AB and CD intersect at P. Then $PA \cdot PB = PC \cdot PD$.

There are two possibilities of this arrangement: the order of the points is A, B, C, D, or the order of the points is A, C, B, D. In either case, the theorem is the same.

Corollary 4.38
Let P be a point outside the circle, and one line through P intersects the circle at A and B, whereas another is tangent to the circle at C. Then $PC^2 = PA \cdot PB$.

4.6 Similarity & Congruence

1. Congruence and similarity theorems (SSS, SAS, ASA, AA)

2. SSA is not a theorem

3. Angle Bisector Theorem

Similarity
Two objects are ***similar*** if they are similar in every aspect except possibly size or orientation.

Congruency
Two geometric figures are ***congruent*** if one of them can be turned and/or flipped and placed exactly on top of the other, with all parts lining up perfectly with no parts on either figure left over.

In plain language, two objects are congruent if they have the same size and shape. We use $\sim$ to denote similarity.

Theorem 4.41 (Similarity Criterion)
There are 3 types of similarity:

1. **AA**: if we know that 2 angles of the triangle are the same, then we know all the angles of the triangle are the same, making them similar. This is the most common way to identify similar triangles.

2. **SSS**: if all sides of the triangle are proportional, then they are similar.

3. **SAS**: if 2 of the triangles' sides have the same ratio and the angle between the sides are equal, the triangles are similar.

Two polygons are similar if their corresponding angles are equal and corresponding sides are in a fixed ratio. Note that for polygons with 4 or more sides, both of these conditions are necessary. For instance, all rectangles have the same angles, but not all rectangles are similar.

Theorem 4.42 (Ratio of Similarity)
If the ratio between corresponding sides of similar objects is r, then if the objects are 2D, the ratio of their *areas* is

$$r^2.$$

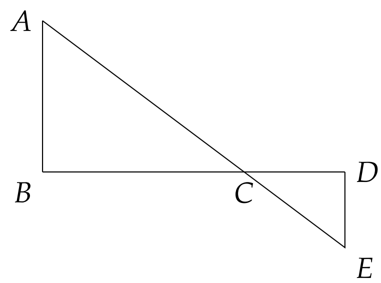
If the objects are 3D, the ratio of their *volumes* is

$$r^3.$$

Theorem 4.43 (Congruency Criterion)
There are 4 types of congruency:

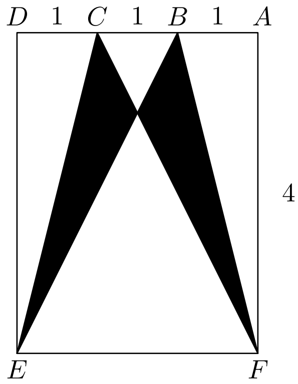
1. **SSS Congruence:** If the three sides of one triangle are congruent to the corresponding sides of of another triangle, then congruence between the two triangles is established.
2. **SAS Congruence**: If two corresponding side congruences hold and the angle between the two sides is equal on both triangles, then the other two angles of the triangle are equal.
3. **ASA Congruence**: If two corresponding angles are equal and the side between the two angles is equal on both triangles, then the other two sides of the triangle are equal.
4. **HL Congruence**: If the both the hypotenuse and leg of one right triangle are congruent to that of another, the two triangles are congruent. Similarly, **LL Congruence** also holds for right triangles (if two legs are congruent, all angles and sides are congruent).

Problem 34 (CEMC Gauss 2011/15). In the diagram, AE and BD are straight lines that intersect at C. If $BD = 16$, $AB = 9$, $CE = 5$, and $DE = 3$, then what is the length of AC?



Problem 35 (AMC 8 2018/22). Point E is the midpoint of side $\overline{CD}$ in square $ABCD$, and $\overline{BE}$ meets diagonal $\overline{AC}$ at F. The area of quadrilateral $AFED$ is 45. What is the area of $ABCD$?

Problem 36. Rectangle $DEFA$ below is a 3×4 rectangle with $DC = CB = BA = 1$. The area of the "bat wings" (shaded area) is



Problem 37 (AMC 10B 2007/21). Right $\triangle ABC$ has $AB = 3, BC = 4$, and $AC = 5$. Square $XYZW$ is inscribed in $\triangle ABC$ with X and Y on $\overline{AC}$, W on $\overline{AB}$, and Z on $\overline{BC}$. What is the side length of the square?

4.7 Length Bashing

1. Perimeter/area formulas
2. Triangle inequality
3. Funky areas

Funky Area
A ***funky area*** is the area of an irregular shape.

There are many of these problems, but the main idea is to **reduce the figure to simpler shapes**.

Theorem 4.45 (Triangle Inequality)
Let the side lengths of a triangle be a, b, c. Then

$$a + b > c \qquad b + c > a \qquad c + a > b.$$

Problem 38. A triangle has perimeter 14. What is the largest possible integer side length of the triangle?

Problem 39. What is the perimeter of a right triangle with legs 15 and 20?

Problem 40. There are two chords BC and DE in a circle that intersect at point A. If $AB = 1, AC = 8$, and $AE = 2$, what is the length of chord DE?

Problem 41. The area of a rectangle is 27. If the length is three times the length of the width, What is the length of the rectangle?

Problem 42 (CEMC Gauss 2002/24). $PQRS$ is a square with side length 8. X is the midpoint of side PQ, and Y and Z are the midpoints of XS and XR, respectively, as shown. What is the area of trapezoid $YZRS$?

Problem 43. A circle is inscribed inside a square of side length 8. What is the total area of the region inside the square but outside the circle?

4.8 Analytic Geometry

1. Cartesian coordinate system
2. Forms of lines
3. Midpoint and distance formulas
4. Solving problems using coordinates (coordinate bashing)
5. Equation of a circle
6. Area of polygons on the coordinate plane

The ***Cartesian coordinate system*** is a coordinate system that specifies each point uniquely in a plane by a set of numerical coordinates, which are the signed distances to the point from two fixed perpendicular oriented lines, measured in the same unit of length.

Theorem 4.47 (Midpoint Formula)
The midpoint of (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Theorem 4.48 (Relation of Two Lines)
For two linear functions $y = m_1x + b_1$ and $y = m_2x + b_2$:

1. (Inconsistent) If $m_1 = m_2$ and $b_1 \neq b_2$, then these two lines are **parallel**.
2. (Dependent) If $m_1 = m_2$ and $b_1 = b_2$, then these two lines **coincide**.
3. If $m_1 \cdot m_2 = -1$, then these two lines are **perpendicular**.
4. (Consistent) If $m_1 \neq m_2$, these two lines are **intersecting**.

Theorem 4.49 (Equation of Circle)
The standard form of the equation of a circle whose center is at (h, k) and whose radius is r is

$$(x-h)^2 + (y-k)^2 - r^2.$$

Problem 44 (BmMT Ciphering 2012/26). The lines $y = 3x$ and $x = 4$ form a right triangle with the x-axis. Find the slope of a line through the origin that bisects the triangle into two portions of equal area.

Problem 45 (PPP Volume 5/11). Find the coordinates of all points in the Cartesian plane that are equidistant from the x-axis, y-axis, and the point $(2, 1)$.

4.9 3D Geometry

1. Volume and surface area formulas
2. Cross sections

Theorem 4.50 (Volume/Surface Area of a Cube)
A cube with side length s has volume s^3 and surface area $6s^2$.

Note that a ***face diagonal*** (a diagonal of a square on one of the faces) has length $s\sqrt{2}$ and a ***space diagonal*** (a diagonal from one vertex to the opposite vertex) has length $s\sqrt{3}$.

Theorem 4.51 (Volume/Surface Area of a Rectangular Prism)
A rectangular prism with side lengths l, w, h has volume lwh and surface area $2(lw + wh + hl)$.

Theorem 4.52 (Volume/Surface Area of a Cylinder)
A cylinder with radius r and height h has volume $\pi r^2 h$ and surface area $2\pi r(r + h)$.

Theorem 4.53 (Volume/Surface Area of a Cone)
A cone with radius r and height h has volume $\frac{1}{3}\pi r^2 h$ and surface area $\pi r(r + \sqrt{r^2 + h^2})$.

Theorem 4.54 (Volume/Surface Area of a Sphere)
A sphere with radius r has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$.

Theorem 4.55 (Volume of a Pyramid)
A pyramid with a base of area A and height h has volume $\frac{1}{3}Ah$.

For hard 3D Geometry problems, take a *cross section*. Basically, we take a plane and examine the intersection of the plane and the 3D shape.

Problem 46 (BmMT Ciphering 2012/20). What is the surface area of a cube inscribed in a sphere with surface area 8π?

Problem 47 (BmMT Ciphering 2012/24). An 8×11 sheet of paper is rolled up so that the 11-inches align. Find the volume of the resulting cylinder.

5 Number Theory

5.1 Primes

1. Definition of prime and composite numbers
2. Primality testing

Prime
A ***prime*** is a positive integer with exactly two factors – one and itself.

Composite
A ***composite*** is a positive integer with more than two factors.

Fact 5.3. 1 is the only positive integer that is **not** composite nor prime.

Primality testing just refers to checking if a number is prime.

Theorem 5.4 (Simple Primality Testing)
A positive integer n is prime if there are no factors from 2 to $\sqrt{n}$.

5.2 Prime Factorization

1. Factor trees

Problem 48 (CEMC Gauss 2008/7). The product of three *different* positive integers is 72. What is the smallest possible sum of these integers?

5.3 Divisibility

1. Divisibility rules
2. Other divisibility tricks
3. Divisors
4. Number/sum/product of divisors

There are a few rules to keep in mind:

1. **2**: check if the last digit is even
2. **3**: check if the sum of the digits is divisible by 3 (repeat if necessary)
3. **4**: check if the last 2 digits is a 2-digit multiple of 4
4. **5**: check if the last digit is 0 or 5

5. **6**: use the rule for 2 and 3
6. **7**: take the last digit, multiply it by 2, and subtract it from the rest of the digits (repeat if necessary)
7. **8**: check if the last 3 digits is a 3-digit multiple of 8
8. **9**: check if the sum of the digits is divisible by 9 (repeat if necessary)
9. **10**: check if the last digit is 0
10. **11**: start from the last digit, then skip every other digit and up the ones you didn't skip; next, take all the digits you skipped and add those together; check if the difference between the first sum and the second sum is a multiple of 11

You can expand upon this list by using a quick product trick: if you are checking if a number is divisible by 15, try using the rule for 3 and 5 (because $15 = 3 \times 5$).

Divisor
A ***divisor*** of n is a number that divides n.

Theorem 5.6 (Number of Factors Formula)
The number of factors of a number $n = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}$ is

$$(e_1 + 1)(e_2 + 1) \ldots (e_k + 1).$$

Theorem 5.7 (Sum of Factors Formula)
The sum of factors of a number $n = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}$ is

$$\frac{p_1^{e_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{e_2+1} - 1}{p_2 - 1} \cdot \ldots \cdot \frac{p_k^{e_k+1} - 1}{p_k - 1}.$$

Theorem 5.8 (Product of Factors Formula)
The product of factors of a numb,er $n = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}$ is

$$n^{\frac{d(n)}{2}},$$

where $d(n)$ is the number of divisors of n.

Problem 49. Does 3 divide 11111199?

Problem 50. Find the number of divisors of 210.

Problem 51. What is the sum of all odd factors of 162?

Problem 52. The four-digit number $AB37$ is divisible by 99. What is the product AB?

5.4 Factoring

1. Differences of squares
2. SFFT

There are quite a few ways to factor. We could use $a^2 - b^2 = (a-b)(a+b)$ (along with the manipulations listed earlier), or:

Theorem 5.9 (SFFT)
For real x, y, a, b,

$$xy + xb + ya + ab = (x+a)(y+b).$$

Usually, we use this when we have something of the form $xy + xb + ya = c$, then add ab to both sides to factor.

Remark 5.10. This formula is usually only useful for **integer** x, y, a, b.

Problem 53. What is 993×1007?

5.5 Bases

1. Representing integers in other bases (conversions)
2. Operations in other bases

Here is a tutorial for converting between bases.

Problem 54. Compute $726_8 + 145_8$. Express your answer in base 8.

5.6 GCD & LCM

1. Definitions of GCD and LCM
2. Computing the GCD and LCM

GCD
The ***greatest common divisor***, or ***GCD***, is the largest divisor of two integers.

LCM
The ***least common multiple***, or ***LCM***, is the smallest multiple of two integers.

Theorem 5.13 (GCD & LCM Formula)
For two numbers

$$m = 2^{m_1} \cdot 3^{m_2} \cdot 5^{m_3} \cdot \ldots,$$
$$n = 2^{n_1} \cdot 3^{n_2} \cdot 5^{n_3} \cdot \ldots,$$

we have

$$\gcd(m,n) = 2^{\min(m_1,n_1)} \cdot 3^{\min(m_2,n_2)} \cdot 5^{\min(m_3,n_3)} \cdot \ldots,$$
$$\operatorname{lcm}(m,n) = 2^{\max(m_1,n_1)} \cdot 3^{\max(m_2,n_2)} \cdot 5^{\max(m_3,n_3)} \cdot \ldots.$$

There are a few methods to find the GCD and LCM, namely the **Cake Method**. You can also use the ***Euclidean Algorithm***:

Theorem 5.14 (Euclidean Algorithm)
Let a, b be integers, with $b \neq 0$, and let q, r be the unique integers such that $a = qb + r$. Then
$$\gcd(a, b) = \gcd(b, r).$$

Theorem 5.15 (Product of GCM & LCM)
For any two positive integers m, n,
$$\gcd(m, n) \cdot \text{lcm}(m, n) = mn.$$

Problem 55. Compute the greatest common divisor of 2021 and 2881.

Problem 56 (PiMC Final Round Team 2018/8). What is the least common multiple of $24, 28, 32$, and 36?

5.7 Modular Arithmetic

1. Definition of modulo and congruence
2. Basic properties of modular arithmetic
3. Finding patterns with remainders

The idea behind modular arithmetic is a cyclic pattern of integers.

Modulo
The ***modulo*** is the number that divides other numbers to get the remainder.

Congruent
An integer a is said to be ***congruent*** to an integer b in mod m if $m \mid a - b$.

Problem 57. Find the last digit of $1^{2020} + 2^{2020} + 3^{2020} + 4^{2020} + 5^{2020} + 6^{2020} + 7^{2020} + 8^{2020} + 9^{2020}$.