ASE 2020-21 Special Notes

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§ 1 Quadratics

§ 1.1 Reading

1. Simple linear equations, Math Centre
2. Solving linear equations, Math Centre
3. Simultaneous equations, Math Centre
4. Pages 35-43 of *Competitive Math for Middle School*, J. Batterson

§ 1.2 Definitions

*The ASE playlist can be found here.

Definition 1 (Trinomial). A **trinomial** in the form $ax^2 + bx + c$ is easiest to factor into a product of binomials (if it is factorable) when the value of a is 1.

For example $x^2 + 2x - 15$ is a trinomial that can be factored as $(x-3)(x+5)$.

Definition 2 (Quadratic). A trinomial in the form $ax^2 + bx + c$ where $a \neq 0$ is called a **quadratic**.

The ax^2 term is called the **quadratic term**, the bx term is called the **linear term**, and the c term is called the **constant term**. a, b, c are **coefficients** of the quadratic.

§ 1.3 Warm-Up

Problem 1. Find the values of x that satisfy: $x^2 = 16$ and $2x^2 - 576 = 0$.

Problem 2. Expand $x(x-6)$. What makes $x(x-6)$ equal to 0? Find the solutions to $x^2 - 6x = 0$.

Problem 3. Expand each of the following:

1. $(x-3)(x+9)$
2. $(y+4)(y+2)$
3. $\left(x - \frac{1}{3}\right)\left(x + \frac{5}{3}\right)$

§ 1.4 Factoring Strategies

One strategy is to look for the greatest common factor of each term.

Example 1. Factor $4x^2 - 16x$.

Solution. Note that $4x$ divides both $4x^2$ and $-16x$. Thus, $4x^2 - 16x = \boxed{4x(x-4)}$. □

If we have $x^2 + ax + b$, we can try finding s and t so that $s + t = a$ and $st = b$. If we can find these two, then $x^2 + ax + b = (x+s)(x+t)$.

Example 2. Factor $x^2 + 7x - 18$.

Solution. We are trying to find s and t, so that $s + t = 7$ and $st = -18$. We see that $s = -2$ and $t = 9$ works, so $x^2 + 7x - 18 = \boxed{(x-2)(x+9)}$. □

Sometimes the quadratic is a perfect square.

Example 3. Factor $4x^2 + 12x + 9$.

Solution. Notice how the coefficient of the x^2 is 4 and the constant term is 9, both of which are perfect squares. This means that there is a good chance that this quadratic equals $\sqrt{4}x + \sqrt{9}$ squared. As it turns out, $(2x + 3)^2 = 4x^2 + 12x + 9$, so we can indeed factor it as $\boxed{(2x + 3)^2}$. □

We can also factor difference of squares, which usually are in the form $a^2x^2 - b^2$.

Example 4. Factor $4x^2 - 81$.

Solution. Note that if $a = 2x$ and $b = 9$, we have that $4x^2 - 81 = a^2 - b^2 = (a - b)(a + b)$. Thus, $4x^2 - 81 = \boxed{(2x - 9)(2x + 9)}$. □

You can also factor by grouping.

Example 5. Factor $6x^2 - 3x - 8x + 4$.

Solution. $6x^2 - 3x$ can be factored as $3x(2x - 1)$. Furthermore, $-8x + 4$ can be factored as $-4(2x - 1)$. Thus, $3x(2x - 1) - 4(2x - 1) = \boxed{(3x - 4)(2x - 1)}$. □

Example 6. Factor $6x^2 - 23x - 18$.

Solution. We can split up $-23x$ into $-27x$ and $4x$. Thus, $6x^2 - 27x + 4x - 18 = 3x(2x - 9) + 2(2x - 9) = \boxed{(3x + 2)(2x - 9)}$. □

§ 1.5 Quadratic Formula

Theorem 1 (Quadratic Formula). If $ax^2 + bx + c = 0$, where a, b, c are real, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 7. Find x: $2x^2 - 4x - 7 = 0$.

Solution. Using the quadratic formula, we get

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-7)}}{2(2)} = \boxed{\frac{2 \pm 3\sqrt{2}}{2}}.$$

□

§ 1.6 Word Problems

Some problems may appear as word problems. Let's try one here:

Example 8. Ryan has forgotten the day and month of his father's birthday. His father gives him the following hints for the date: the day is 10 more than twice the month. The month times the day equals 72. On what date was Ryan's dad born?

Solution. Let d be the day and m be the month. Then

$$d = 10 + 2m,$$

$$dm = 72.$$

Substituting, we get

$$(10 + 2m)m = 72 \implies m^2 + 5m - 36 = 0 \implies (m - 4)(m + 9) = 0,$$

so $m = 4, -9$. Of course, the month cannot be negative, so $m = 4, d = 18$. Thus, his dad was born on $\boxed{\text{April 18th}}$. □

More will be included in the problem set.

§ 1.7 Sum and Product of the Roots

Theorem 2 (Sum and Product of the Roots). Let x^2+ax+b. Then the sum of the roots of this quadratic is $-a$, and the product of the roots is b.

Example 9. The roots of the quadratic $3x^2 - 7x - 18$ are m and n. Find $\frac{1}{m} + \frac{1}{n}$.

Solution. Note that $\frac{1}{m} + \frac{1}{n} = \frac{m+n}{mn}$. The sum of the roots is $m + n = \frac{7}{3}$, and the product of the roots is $mn = \frac{-18}{3} = -6$, so

$$\frac{1}{m} + \frac{1}{n} = \frac{\frac{7}{3}}{-6} = \boxed{-\frac{7}{18}}.$$

□

§ 1.8 Problem Solving Strategies

1. When faced with an equation you can't solve, think about how expressions in the equation can be created from simpler expressions
2. When you solve an equation, plug your solutions back in to check that they work (this is also true for substitutions)

3. Organize expressions into familiar forms
4. Factoring quadratics is useful for more than just finding the roots – always try to factor
5. Look for common factors
6. Sometimes you should write a quadratic as x^2+bx+c and sometimes you should write it as $(x-r)(x-s)$
7. Sometimes you don't need to find each root – for example, if you know b in x^2+bx+c, you don't need to find the specific roots to find the sum of roots
8. Substitution is a good idea to turn large polynomials into quadratics
9. Try plugging in values if you don't understand what to do

§ 1.9 Problems

More on factorization can be found here.

Problem 4. Solve $2x^2 - 8x + 6 = 0$ by completing the square.

Problem 5. Use the Quadratic Formula to solve $3x^2 - 4x + 1 = 0$.

Problem 6. Solve $x^2 + 2x + 1$.

Problem 7. Solve for x: $(x + 3)^2 = 4x - 3$.

Problem 8. Factor $16x^2 - 38x - 5$.

Problem 9. The perimeter of a rectangle is 16, and the area of the same rectangle is 8. What is the diagonal length of the rectangle?

Problem 10. The difference between the squares of two numbers is 80. If the sum of the two numbers is 16, what is their positive difference?

Problem 11. Factor completely: $256x^8 - 1$.

Problem 12. Solve for x: $\frac{2x^2-5x-12}{2x+3} = 9$.

Problem 13. Two circles are internally tangent at A, with diameter AB intersecting the smaller circle at C. The shaded region has an area of 9π, and $BC = 4$. What is the sum of the radii of the two circles?

Problem 14. The diagonal length of a rectangle is $\sqrt{46}$, and its area is 9. Find its perimeter.

Problem 15. The sum of two positive numbers is 6, and the sum of their squares is 22. What is the sum of their reciprocals? Express your answer as a common fraction.

Problem 16. Completely factorize $2^{20} - 1$.

Problem 17. For positive integers a and b, $(a + \sqrt{b})^2 = 9 + 4\sqrt{5}$, what is $(a + \sqrt{b})^3$?

Problem 18. The product of two numbers is 9 and their sum is 12. Express their positive difference in simplest radical form.

Problem 19. 48 feet of fencing is used to create a pen enclosure which consists of three congruent rectangular pens as shown, with a combined area of 72 ft^2. What is the perimeter of the enclosure?

Problem 20. A photograph is 4.25 inches wide and 7.75 inches tall. It is mounted in a frame with a border x inches wide on all slides. If the area of the shaded border is 45 in^2, what is its width x?

Problem 21. The solutions of $2x^2 - 5x + 1 = 0$ are j and k. What is the value of $(j - 1)(1 - k)$?

Problem 22. Distribute these polynomials:

1. $(x - 7)(x + 9) =$ ________________
2. $(x - 5)(x + 8) =$ ________________
3. $(x - 5)(x + 8) =$ ________________
4. $(x - 8)(x + 7) =$ ________________
5. $(x - 10)(x + 1) =$ ________________
6. $(x - 1)(x + 5) =$ ________________

Problem 23. Factor these polynomials:

1. $x^2 - 6x - 16 =$ ________________
2. $x^2 - x - 72 =$ ________________
3. $x^2 + x - 27 =$ ________________
4. $x^2 - 2x - 80 =$ ________________
5. $x^2 - x - 56 =$ ________________
6. $x^2 - 8x - 9 =$ ________________
7. $x^2 - x - 56 =$ ________________
8. $x^2 + x - 70 =$ ________________
9. $x^2 - 3x - 70 =$ ________________
10. $x^2 + x - 18 =$ ________________

Problem 24 (ML Sprint 11323/27). How many ordered pairs of real numbers (a, b) satisfy the equations $|ab| = \sqrt{3}$ and $\frac{a}{b} = \frac{b}{a}$?

Problem 25 (ML Sprint 12027/29). What is the sum of all the absolute values of all solutions to $x^4 - 25x^2 + 144 = 0$?

Problem 26 (ML Sprint 12028/28). A right triangle has side lengths that are whole numbers. The perimeter of the triangle is 154, and the hypotenuse has length 65. What is the area of the triangle?

Problem 27 (ML Sprint 12028/29). There is one whole number x for which $\sqrt{x+1010} - \sqrt{x-1010} = 10$. What is the sum of the digits of x?

Problem 28 (ML Sprint 12029/29). What is the sum of the digits of $\sqrt{1 + 2019 \cdot 2020 \cdot 2021 \cdot 2022}$?

Problem 29 (ML Sprint 12020.2/9). What is the value of $99 \times 100 \times 101$? **Generalization**: What is the value of $(x-1)x(x+1)$?

Problem 30 (ML Sprint 12020.2/22). If $a \heartsuit b = a^2 - 2ab + b^2$, what is the value of $(8 \heartsuit 6) \heartsuit (7 \heartsuit 6)$?

Problem 31 (AMC 10 2001/7). When the decimal point of a certain positive decimal number is moved four places to the right, the new number is four times the reciprocal of the original number. What is the original number?

Problem 32 (Mandelbrot). The mathematical constant e is approximately equal to 2.71828. Is $e^2 - 5e + 6$ positive, zero, or negative?

Problem 33 (AMC 10B 2006/14). Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of the equation $x^2 - px + q = 0$. What is q?

Problem 34. Find the sum and product of the solutions of the equation $-32x^2 + 84x + 135 = 0$.

Problem 35 (AHSME 1986/13). If m and n are the roots of $x^2 + mx + n = 0, m \neq 0, n \neq 0$, then what is the sum of the roots?

Problem 36 (AMC 10B 2019/6). There is a positive integer n such that $(n+1)! + (n+2)! = n! \cdot 440$. What is the sum of the digits of n?

Problem 37 (AMC 10B 2020/9). How many ordered pairs of integers (x, y) satisfy the equation

$$x^{2020} + y^2 = 2y?$$

Problem 38. Find all x such that $\frac{x^2-2x-3}{x+1}+\frac{x^2+5x-24}{x-3}=13$.

Problem 39 (AMC 10B 2002/12). For which of the following values of k does the equation $\frac{x-1}{x-2}=\frac{x-k}{x-6}$ have no solution for x?

Problem 40 (UNCC). Find all solutions to the equation $\sqrt{x+1}=1-x$.

Problem 41 (UNCC). If the sides of a square are each increased by 12 inches, the area is increased by 200 square inches. What is the lenght of a side of the original square?

Problem 42 (AHSME 1997/14). The number of geese in a flock increases so that the difference between the populations in year $n+2$ and year n is directly proportional to the population in year $n+1$. If the populations in the years 1994, 1995, and 1997 were 39, 60, and 123, respectively, then what was the population in 1996?

Problem 43 (UNCC). Compute the sum of all the roots of $(2x+3)(x-4)+(2x+3)(x-6)=0$.

Problem 44. Find all x such that $\frac{10}{x^2}+\frac{22}{x}+4=0$. Try to find a solution using substitution and another by multiplying by x^2.

Problem 45 (AHSME 1966/27). At his usual rate a man rows 15 miles downstream in five hours less time than it takes him to return. If he doubles his usual rate, the time downstream is only one hour less than the time upstream. In miles per hour, what is the rate of the stream's current?

Problem 46. Find all triples (a,b,c) such that all three of the following equations are satisfied:

$$a(b+c-5)=7,$$

$$b(a+c-5)=7,$$

$$a^2+b^2=50.$$