ASE Handouts

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Home of the Dalmations - Anne Sullivan Elementary

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Part I

Level 1 Handouts

1 Casework

There are two main ways (that I know of) to keep your casework organized:

1. Lists
2. Tree Diagrams

[1]**Lists** are simply used to list out the possibilities one by one. **Tree diagrams** are shown to the side. They are used when some cases have multiple smaller cases.

1.1 Examples

In the example below, you can see the power of casework[2]:

Example. How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

(A) 41 (B) 42 (C) 43 (D) 44 (E) 45

Solution. If the middle digit is the average of the first and last digits, twice the middle digit must be equal to the sum of the first and last digits.
Let's do some **casework**:

- If the middle digit is 1, possible numbers range from 111 to 210. So there are 2 numbers in this case.
- If the middle digit is 2, possible numbers range from 123 to 420. So there are 4 numbers in this case.
- If the middle digit is 3, possible numbers range from 135 to 630. So there are 6 numbers in this case.
- If the middle digit is 4, possible numbers range from 147 to 840. So there are 8 numbers in this case.
- If the middle digit is 5, possible numbers range from 159 to 951. So there are 9 numbers in this case.
- If the middle digit is 6, possible numbers range from 369 to 963. So there are 7 numbers in this case.
- If the middle digit is 7, possible numbers range from 579 to 975. So there are 5 numbers in this case.
- If the middle digit is 8, possible numbers range from 789 to 987. So there are 3 numbers in this case.
- If the middle digit is 9, the only possible number is 999. So there is 1 number in this case.

So the total number of three-digit numbers that satisfy the property is $2 + 4 + 6 + 8 + 9 + 7 + 5 + 3 + 1 = 45 \Rightarrow E$

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[1] Either way is fine, but I usually prefer lists. However, I will teach you both ways.

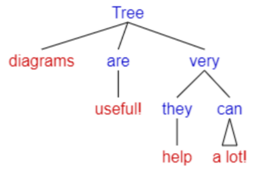
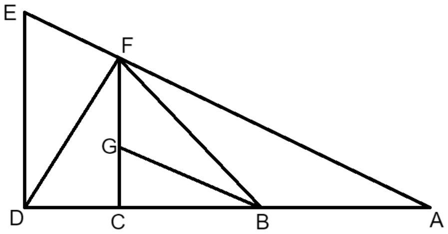


Figure 1. Tree Diagram

[2] However, you can also see how tedious case work is. You must be careful while using it, since missing one case could cost you the whole question. It is a lot better to use constructive counting (which is basically just counting directly), since it saves time. Only use casework when necessary.

Example. How many triangles are in the diagram below?



Solution. There are no fancy techniques for this problem but to tough it out and count the possibilities. However, this doesn't mean we have to just look for triangles, make tally marks for each one we find, and hope that we don't over count or miss anything. Instead, in this problem and every casework problem, the most important thing you can do is to have an organized system through which you can count.

For this problem, there are many ways to organize your counting, none more correct than another. However, in this solution, we will detail 1 specific method of organization.

First of all, there are 5 triangles that don't have any triangles inside of them. These triangles are ABF, BGF, BCG, DCF, and EFD. We will proceed by casework on which triangles contain each of these triangles. First of all, we can easily see that ABF is contained in 3 triangles (excluding itself). Next, in order to not over count, we can easily see that there are 2 triangles that contain BGF but not ABF (excluding BGF). There are 0 triangles that contain BCG but not ABF or BGF (excluding BCG). There are 0 triangles that contain DCF but not ABF, BGF, or BCG (excluding DCF). There are 0 triangles that contain FED but not $ABF, BGF, BCG, or DCF$ (excluding FED). Thus, there are a total of $5 + 3 + 2 = 10$ total triangles.

[3] Note that you can also do casework with probabilities. Simply count the probability of each one occurring, and add them up!

[3] Notice how we made it easier to count the triangles, avoided over counting easily, and we are certain that we counted each triangle, because we were organized, neat, and we had a method. Also, note that other concepts can be used in combination with casework, such as complementary counting and the Principle of Inclusion and Exclusion, in order to reduce the number of cases or to make the casework easier.

1.2 Problems

There won't be too many problems here, because the idea is simple - list the cases, and compute each one.

Problem 1.1. How many ways are there to seat three people in three different chairs?

Problem 1.2. How many ways are there to seat three people in four different chairs around a table, given rotations are the same?

Problem 1.3. How many ways are there to pick a three digit number that does **not** start with 3?

Problem 1.4. How many numbers less than or equal to 100 are divisible by 2 and 3 but not 4?

Problem 1.5. There are n people at a party. Each person shakes hands with every person (besides themselves) exactly once. How many handshakes occur?

Problem 1.6. How many different squares of any size are there on a 3×3 board?

Problem 1.7. Jamie has 2 dimes, 4 nickels and 8 pennies. In how many different ways can she make 26 cents?

Problem 1.8. A dresser has five drawers stacked vertically. To be able to reach the contents in an open drawer, the drawer that is directly above the open drawer may not be open at the same time. In how many ways can one or more drawers be open so that the contents in each of the open drawers can be reached?

Problem 1.9. How many non-congruent triangles are there with sides of integer length having at least one side of length five units and having no side longer than five units?

Problem 1.10. How many collections of six positive, odd integers have a sum of 18? Note that $1+1+1+3+3+9$ and $9+1+3+1+3+1$ are considered to be the same collection.