Math Level 2 Handouts Week 01

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§ 1 Polynomials

§ 1.1 Warm-Up

Problem 1 (Mathcounts). Find y: $\sqrt{19+3y}=7$.

Problem 2 (Alcumus). How many terms are in the expansion of $(a+b+c)(d+e+f+g)$?

§ 1.2 Linear Functions

Recall that a **polynomial function** of x with degree n is defined as follows:

$$P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x^1+a_0x^0.$$

Note that $a_1x^1=a_1x$ and $a_0x^0=a_0$.

Definition 1 (Constant Function). If $f(x)=a_0$, then $f(x)$ is a **constant function**.

Definition 2 (Linear Function). If $f(x) = a_1x + a_0$, then $f(x)$ is a **linear function**.

Definition 3 (Slope). The **slope** is the rate of change line describing the steepness and direction of a function at that point.

Definition 4 (Intercept). The **x-intercept** is where a function intersects the x-axis, and the **y-intercept** is where a function intersects the y-axis.

Theorem 1 (Slope of a Line). The slope of a line going through (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2-y_1}{x_2-x_1}$.

Example 1. Find the slope of a line going through $(1, 2)$ and $(2, 3)$.

Solution. Using our formula, we get $\frac{3-2}{2-1} = \boxed{1}$. □

Problem 3. Find the slope of the line through points $(3, -2)$ and $(-2, -3)$.

§ 1.3 Quadratic Function

Definition 5 (Quadratic Function). If $f(x) = ax^2 + bx + c$, then $f(x)$ is a **quadratic function**.

Definition 6 (Axis of Symmetry). The **axis of symmetry** of a parabola is the line such that reflecting one side of the parabola across the line will yield the other side.

Theorem 2 (Quadratic Minima/Maxima). If $a > 0$, f has a minimum at $x = -\frac{b}{2a}$. If $a < 0$, f has a maximum at $x = -\frac{b}{2a}$.

The **standard form** of a quadratic is $f(x) = ax^2 + bx + c$. The **factored form** is $f(x) = a(x - x_1)(x - x_2)$, where x_1 and x_2 are the roots of $f(x)$ (i.e. $f(x_1) = f(x_2) = 0$). The **vertex form is** $f(x) = a(x - h)^2 + k$, implying the **axis of symmetry** is $x = h$ and the **vertex** is (h, k). Note that the minimum/maximum is located at (h, k).

Theorem 3 (Vertex of a Quadratic). The vertex of $ax^2 + bx + c$ is $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$.

Now let's discuss **completing the square**:

Example 2. Find the roots of $x^2 + 8x + 12 = 0$.

Solution. Let's try to make the left hand side a **square**. The idea to do this is to take $x^2 + 8x$, which looks like $x^2 + 8x + 16 = (x + 4)^2$, but it is missing the 16, so $x^2 + 8x = (x + 4)^2 - 16$. Thus, $x^2 + 8x + 12 = (x + 4)^2 - 16 + 12 = 0$, so $(x + 4)^2 - 4 = 0$, which we can rearrange to get $(x + 4)^2 = 4, x + 4 = \pm 2$. Thus, the roots are $x = \boxed{-2, -6}$. □

This is the motivation behind completing the square.

Theorem 4 (Completing the Square). If we can write $f(x) = ax^2 + bx + c$ as $a(x - h)^2 + k$, then the roots of $f(x)$ are

$$x_1, x_2 = h \pm \sqrt{-k}.$$

Note that if $k > 0$, then the roots are not real.

What if we don't have a nice expression? Then let's find a general way to solve a quadratic:

Theorem 5 (Quadratic Formula). Let $ax^2 + bx + c = 0$. Then

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 3. Find the roots of $-x^2 + 4x + 5$.

Solution. Using the Quadratic Formula, we get

$$x_1, x_2 = \frac{-4 \pm \sqrt{4^2 - 4(-1)(5)}}{2(-1)} = \boxed{-1, 5}.$$

□

Po Shen Loh recently talked about a method for solving quadratics, which can be found here.

Problem 4. Solve $2x^2 - 8x + 6 = 0$ by completing the square.

Problem 5. Use the Quadratic Formula to solve $3x^2 - 4x + 1 = 0$.

Definition 7 (Discriminant). The **discriminant** Δ of the quadratic $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac.$$

Theorem 6 (Discriminant Test). Let Δ be the discriminant of a quadratic. Then:

1. If $\Delta > 0$, then the roots are real and unequal.
2. If $\Delta = 0$, then the roots are real and equal.
3. If $\Delta < 0$, then the roots are complex and unequal.

Exercise 1. If a quadratic equation $2x^2 - kx + 3 = 0$ have imaginary roots, what is the value of k? **Solution**: 1

Exercise 2. If $y = 3x^2 - 2x + k$ is positive for all x, then what is the smallest integral value of k? **Solution**: 3

§ 1.4 Factor and Remainder Theorem

Theorem 7 (Factor Theorem). If $p(a) = 0$, then $p(x)$ has a factor of $x - a$. Furthermore, $p(x) = (x - a)Q(x)$, where $Q(x)$ is the quotient (the remainder is 0).

Definition 8 (Degree). The **degree** of a polynomial is its largest exponent.

Theorem 8 (Remainder Theorem). When a polynomial $P(x)$ is divided by $x - a$, the remainder R is equal to $P(a)$. Furthermore, $P(x)$ can be expressed as follows:

$$P(x) = (x - a)Q(x) + R.$$

The identical equation is true for any value of x, especially $x = a$. Therefore,

$$P(a) = R.$$

Exercise 3. If a polynomial $f(x) = 2x^2 - 3x + 5$ is divided by $x - 1$, what is the remainder? **Solution**: 2

Example 4. Let $P(x)$ be a polynomial in terms of x. When $P(x)$ is divided by $x - 9$, the remainder is 5, and when $P(x)$ is divided by $x - 5$, the remainder is 9. What is the remainder when $P(x)$ is divided by $(x - 5)(x - 9)$?

Solution. Here, we are dividing by a quadratic rather than a linear term, so we cannot directly use the remainder theorem. The remainder polynomial will be a linear term, not a constant. Instead, let's see what happens when we divide by a quadratic:

$$P(x) = q(x) \cdot (x - 5)(x - 9) + r(x)$$

where $r(x)$ is the remainder polynomial. We know that $r(x)$ is a linear polynomial, so we have

$$P(x) = q(x) \cdot (x-5)(x-9) + ax + b.$$

Here, we see that just like the previous examples, we can plug in values for x. Plugging in $x = 5$, we have $P(5) = 5a + b$ and plugging in $x = 9$ gives $P(9) = 9a + b$. Also, we are given that the remainder, $ax + b$ is equal to 9 and 5 when dividing by $x-5$ and $x-9$, respectively. Therefore, we have $5a+b = 9$ and $9a+b = 5$. Solving the system, we get $(a, b) = (-1, 14)$, so the remainder is $ax + b = \boxed{-x+14}$. □

Problem 6 (AoPS). Let $f(x) = x^4 - 3x^3 + 7x^2 - x + 5$. What is the remainder when $f(x)$ is divided by $x-3$? What is $f(3)$?

Problem 7. Let $f(x) = x^9 + x^3 - 5x^2$. Find the remainder when $f(x)$ is divided by $3x - 6$.

§ 1.5 Long Division of Polynomials

Example 5. Find the quotient and remainder when $2x^4 + 4x^2 - 1$ is divided by $x + 1$.

Solution. Let's use long division:

$$\begin{array}{r l}
 & 2x^3 - 2x^2 + 6x - 6. \\
\cline{2-2}
x+1\big) & 2x^4 + 4x^2 - 1 \\
 & -2x^4 - 2x^3 \\
\cline{2-2}
 & -2x^3 + 4x^2 \\
 & 2x^3 + 2x^2 \\
\cline{2-2}
 & 6x^2 \\
 & -6x^2 - 6x \\
\cline{2-2}
 & -6x - 1 \\
 & 6x + 6 \\
\cline{2-2}
 & 5
\end{array}$$

Thus, the quotient is $\boxed{2x^3 - 2x^2 + 6x - 6}$ with remainder $\boxed{5}$. □

Problem 8. If $p(x) = 2x^2 - 3$ and $q(x) = 4x^3 + x^2 + 1$. Find $p(x) + q(x)$ and $p(x) \cdot q(x)$.

Problem 9 (Alcumus). Find the quotient and remainder when $x^6 - 3$ is divided by $x + 1$.

§ 1.6 Problems

Let's try some harder problems:

Problem 10 (Great Britain). Find the remainder when the polynomial $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$.

Problem 11. Find the remainder when $(x+3)^5 + (x+2)^8 + (5x+9)^{2020}$ is divided by $x + 2$.

Problem 12 (AHSME 1974/4). Find the remainder when $x^{51} + 51$ is divided by $x + 1$.

Problem 13 (AMC 12B 2003/9). Suppose that $P(x)$ is a linear polynomial with $P(6) - P(2) = 12$. What is $P(12) - P(2)$?

Problem 14 (MAΘ 1991). Find all values of m which make $x + 2$ a factor of $x^3 + 3m^2x^2 + mx + 4$.

Problem 15. Let m and n be the roots of the quadratic equation $4x^2 + 5x + 3 = 0$. Find $(m + 7)(n + 6)$?

§ A Solutions

1. The discriminant is $k^2 - 4 \cdot 2 \cdot 3 = k^2 - 24 < 0$, so $k^2 < 24 \implies \boxed{-2\sqrt{6} < k < 2\sqrt{6}}$.
2. The remainder is $f(1) = 2 \cdot 1^2 - 3 \cdot 1 + 5 = \boxed{4}$.
3. Since it is always possible, $3x^2 - 2x + k$ has no real solutions, implying all roots are complex. Thus, $4 - 12k < 0 \implies k > \frac{1}{3}$, so the smallest integral value of k is $k = \boxed{1}$.