Math Level 2.5 Handouts

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§ 1 Geometric Sequences

§ 1.1 Definitions

Definition 1 (Geometric Sequence). A sequence is **geometric** if the ratios of consecutive terms are the same.

A geometric sequence is sometimes also called a **geometric progression**.

Definition 2 (Common Ratio). The ratio between consecutive terms is known as the **common ratio**.

A geometric sequence is usually denoted $g_1, g_2, \ldots, g_n$.

Theorem 1 (nth Term of Geometric Sequence). If $g_1 = g$ and r is the common ratio, then the nth term is

$$g_n = gr^{n-1}.$$

Theorem 2 (Sum of Finite Geometric Sequence). The sum of a finite geometric sequence with n terms is

$$S_n = \frac{g(1-r^n)}{1-r}.$$

Theorem 3 (Sum of Infinite Geometric Sequence). The sum of an infinite geometric sequence if $|r| < 1$ is

$$S_\infty = \frac{a}{1-r}.$$

Example 1. The first term of a geometric sequence $\{a_n\}$ is 2 and $a_{n+1} = 2a_n$. What is the value of a_{10}?

Solution. The common ratio is 2 and the first term is 2, so using the formula we get $a_{10} = 2 \cdot 2^{10-1} = \boxed{1024}$. □

Example 2. In a geometric sequence, the second term is 3 and fifth term is 24. What is the 8th term?

Solution. Since $a_2 = ar = 3$ and $a_5 = ar^4 = 24$, so $\frac{a_5}{a_2} = r^3 = 8 \implies r = 2$. Thus, the 8th term is $a_8 = ar^7 = ar \cdot r^6 = 3 \cdot 2^6 = \boxed{192}$. □

Example 3. What is the sum of the infinite series $15 - 3 + \frac{3}{5} - \frac{3}{25} + \ldots$?

Solution. The common ratio is $\frac{-3}{15} = -\frac{1}{5}$ (and $0 < |-\frac{1}{5}| < 1$), and the first term is 15, so $S_\infty = \frac{15}{1-(-\frac{1}{5})} = \boxed{\dfrac{25}{2}}$. □

§ 1.2 Geometric Sequence Problems

Problem 1. Write down the first five terms of the geometric progression which has first term 1 and common ratio $\frac{1}{2}$.

Problem 2. Find the 10th and 20th terms of the geometric progression with first term 3 and common ratio 2.

Problem 3. Find $2 + 6 + 18 + \ldots + 486$.

Problem 4. Find $8 - 4 + 2 - 1 + \ldots + \frac{1}{32} - \frac{1}{64}$.

Problem 5. Compute

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots.$$

Problem 6 (MathFan). In a geometric sequence that begins with $7, 11, \frac{121}{7}, \ldots$, what is the geometric mean of the first and fifth numbers?

Now let's try some more general sequences and series problems:

Problem 7 (Mathcounts Chapter 2014). Each term in the sequence that begins $13, 9, 18, \ldots$, is the sum of three times the tens digit and two times the units digit of the previous number. What is the greatest value of any term in this sequence?

Problem 8 (Mathcounts State 2014). Consider the sequence $1, 3, 4, 7, 11, 18, 29, \ldots$ where each term is the sum of the two previous terms. How many of the first 100 terms are multiples of 5?

Problem 9 (AMC 10B 2003). The second and fourth terms of a geometric sequence are 2 and 6. Which of the following is a possible first term?

Problem 10 (Mathcounts State 2014). The nonnegative integers a, b, c, d, e form an arithmetic sequence. If their sum is 440, what is the largest possible value of e?

Problem 11 (Hard). Find

$$\frac{1}{2^0} + \frac{2}{2^1} + \frac{3}{2^2} + \frac{4}{2^3} + \dots.$$

(Hint: Let this sum be S. Then what is $\frac{S}{2}$, and when we pair up the terms correctly, what is $S - \frac{S}{2}$?)