# Math Level 2.5 Handouts

Dylan Yu

July 1, 2020

## §0  Contents

# §1 Stars and Bars

The ball-and-urn technique, also known as stars-and-bars, is a commonly used technique in combinatorics.

It is used to solve problems of the form: how many ways can one distribute $k$ indistinguishable objects into $n$ distinguishable bins? We can imagine this as finding the number of ways to drop $k$ balls into $n$ urns, or equivalently to arrange $k$ balls and $n-1$ dividers. For example,

$$* * * * \,||$$

$$* * *|\,*\,|$$

$$*|\,* *|*$$

represent the ways to put $k = 4$ objects in $n = 3$ bins.

> **Theorem 1.1 (Stars and Bars with Nonnegative Integers).** The number of ways to complete the above task is $\binom{n+k-1}{k}$, or $\binom{n+k-1}{n-1}$.

When something is **indistinguishable** from another object, that means it looks identical to that object. If something is **distinguishable**, that means you can tell which one is which.

> **Theorem 1.2 (Stars and Bars with Positive Integers).** The number of ways to put $n$ indistinguishable balls into $k$ distinguishable urns is
> $$\binom{n-1}{k-1}.$$

## §1.1 Stars and Bars Strategies

The trick for Stars and Bars is to **reduce to a simpler problem**. For example, if we have $a+b+c+d = 10$ for non-negative integers $a, b, c, d$, we can simply apply the formula. If $a, b, c, d$ are positive integers, we can simply replace $a$ with $a'+1$, $b$ with $b'+1$, and so on and resolve. If $a, b, c, d$ are all even, we can simply replace $a$ with $2a'$ and so on. As long as we can keep **reducing**, the problem becomes easy.

## §1.2 Problems

> **Problem 1.1.** If $a, b, c, d$ are nonnegative integers, how many ways can $a + b + c + d = 7$?

> **Problem 1.2.** If $a, b, c, d$ are positive integers, how many ways can $a + b + c + d = 7$?

> **Problem 1.3.** If $a, b, c, d$ are positive odd integers, how many ways can $a + b + c + d = 8$?

**Problem 1.4.** If $a, b, c, d$ are nonnegative even integers, how many ways can $a + b + c + d = 8$?

**Problem 1.5.** How many ways can we place 2 bars between 9 stars in a row such that the stars are partitioned into three groups of at least one star each?

**Problem 1.6.** How many ways can we place 9 indistinguishable balls into 3 distinguishable urns such that every urn has at least one ball in it? (Also, what is the difference between this problem and the last one?)

**Problem 1.7.** Amber especially loves this time of year, when the leaves on the trees in her yard begin changing color to display an array of vibrant colors. There are yellow hickory tree leaves, orange maple tree leaves, purple cherry tree leaves, yellow ash tree leaves and scarlet dogwood tree leaves. Before raking, Amber collected some of the fallen leaves for a craft project. If the leaves Amber collected include at least one leaf from each type of tree, at least two purple leaves and at least three yellow leaves, how many such collections of 10 leaves are possible?

**Problem 1.8.** How many ways can you buy 8 fruit if your options are apples, bananas, pears, and oranges?

**Problem 1.9.** How many ways can you give 10 cookies to 4 friends if each friend gets at least 1 cookie?

**Problem 1.10.** Find the number of nonnegative integers $a, b, c, d$ such that

$$a + b + c + d < 5.$$