# ASE Handouts

ASE Math Club

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Anne Sullivan Elementary

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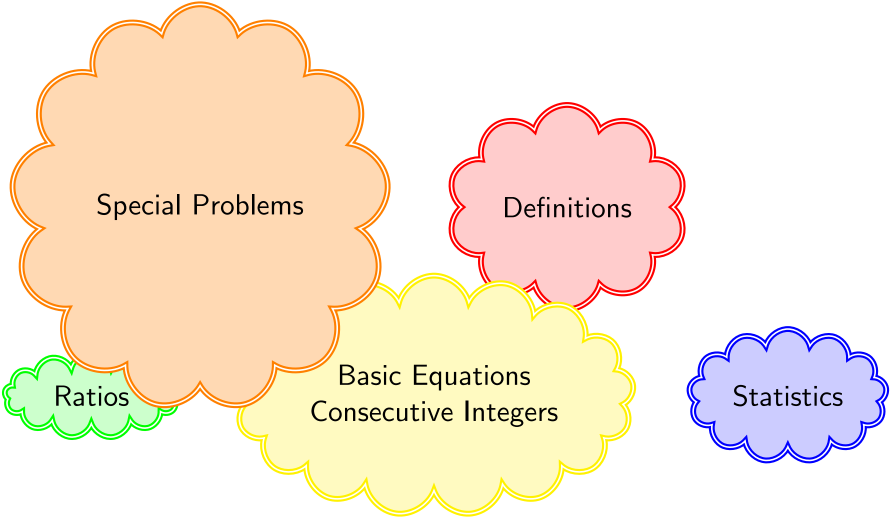

# Definitions

## Definition (Variable)

*A* **variable** *is something that can change. It can also be a symbol for a number we don't know yet.*

There are **independent** and **dependent** variables. *Independent* variables do not affect one another (for example, $x = 5, y = 6$ do not affect each other, but *dependent* variables do affect each other (for example, $x = 5y$).

## Definition (Consecutive)

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# Statistics

**Definition (Mean)**

*The **mean** (or average) of a set of numbers is the sum of the numbers divided by the number of numbers.*

Note that the mean refers to the **arithmetic mean**, not the **geometric mean**.

**Theorem (Arithmetic Mean)**

*Let $a_1, a_2, a_3, \ldots, a_n$ be a set of $n$ numbers. Then the arithmetic mean of these numbers is*

$$\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}.$$

The **median** is the number in the middle and the **mode** is the number that appears the most. Note that there could be more than one mode. For example, in the set $1, 2, 2, 2, 3, 3, 3$, there are two modes of (size) 3, and the most popular values were 2 and 3.

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# Consecutive Integer

Let's take a look at an example.

**Example**

Let's say we have 7 consecutive integers that sum to 28. What is the smallest integer of these 7?

*Solution.* By trial and error we can find the sequence is 1 to 7, so the answer is 1. However, is there a better way? Answer: there is!

**Theorem**

*Given $n$ consecutive integers that sum to $k$, the smallest number is $\frac{k}{n} - \frac{n-1}{2}$, and the largest number is $\frac{k}{n} + \frac{n-1}{2}$.*

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# Ratios and Proportions

A **ratio** shows the relative sizes of two or more values. For example, if there are 2 apples, 4 oranges, and 6 bananas, the *ratio* of apples to oranges to bananas is $2:4:6=1:2:3$. The : symbol represents the word "to".
A **proportion** is a statement that two *ratios* are equal. For example, the following is a proportion:

$$\frac{12}{15}=\frac{4}{5}.$$

Two numbers $x, y$ are said to be *directly proportional* if

$$y=kx$$

for some constant $k$. For example, if $k=5$, then if $x$ is 3, $y$ must be 15, and if $x$ is 4, $y$ must be 20. Two numbers $a, b$ are said to be *inversely proportional* if

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# Moles Digging Holes

**Example**

10 moles dig 10 holes in 10 hours, how many holes do 20 moles dig in 20 hours?

*Solution.* There are twice as many moles, so that doubles the number of holes, and twice as many hours also doubles the number of holes. Thus, the answer is $10 \times 2 \times 2 = 40$.

**Theorem**

*If we have $a$ moles, $b$ holes, and $c$ hours, then*

- *$a$ varies directly with $b$*
- *$b$ varies directly with $c$*
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# $d = rt$

There is only one theorem you need to know:

**Theorem ($d = rt$)**

*Like the title of this theorem says,*

$$\textit{distance} = \textit{rate} \times \textit{time}.$$

Even though this is the only necessary formula, let us list one more to speed up computation:

**Theorem**

*Let person 1 work at a speed of one object per $t_1$ time, person 2 works at a speed of one object per $t_2$ time, and so on, all the way to person $n$ who works at a speed of one object per $t_n$ time. Then if they work together, they will finish in $\frac{1}{\frac{1}{t_1}+\frac{1}{t_2}+\ldots+\frac{1}{t_n}}$ time.*

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Let's try an example:

**Example**

Dylan can paint a house in 3 hours. Cody can paint a house in $x$ hours. If they work together, it takes then 2 hours to paint the house. What is $x$?

*Solution.* Using the theorem, we get

$$\frac{1}{\frac{1}{3}+\frac{1}{x}} = 2,$$

$$\frac{1}{x} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$

$$x = 6.$$

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$$x = 6.$$

# Simon's Favorite Factoring Trick I

**Simon's Favorite Factoring Trick** (SFFT) is best explained with an example:

## Example

Find all positive integers $x, y$ that satisfy

$$xy - 2x - 4y = 0.$$

*Solution.* Let us factor the first two terms:

$$x(y-2) - 4y = 0.$$

We want to find *some way* we can turn the $y$ into a $y - 2$. Let's see what happens if we do that:

$$x(y-2) - 4(y-2).$$

# Simon's Favorite Factoring Trick II

## Example

Now that we've done it, we can factor! However, because we added a $-2$ we also have to $+2$:

$$x(y-2)-4(y-2+2).$$

Remember, this is still equal to 0. Now, let's take out the $+2$ and distribute it to the other side:

$$x(y-2)-4(y-2)-8=0,$$

$$x(y-2)-4(y-2)=8.$$

Now, we can factor:

$$(x-4)(y-2)=8.$$

Because $x, y$ are positive integers, we know that $x-4$ and $y-2$ are simply the positive factors of 8:

# Simon's Favorite Factoring Trick III

## Example

$$x - 4 = 1, y - 2 = 8,$$

$$x - 4 = 2, y - 2 = 4,$$

$$x - 4 = 4, y - 2 = 2,$$

$$x - 4 = 8, y - 2 = 1.$$

Solving, we get $\boxed{(x, y) = (5, 10), (6, 6), (8, 4), (12, 3)}$.

# Simon's Favorite Factoring Trick IV

Now for the formal statement:

**Theorem (SFFT)**

*For all real numbers (although commonly used only for integers) $x, y, a, b$,*

$$xy + xa + yb + ab = (x + b)(y + a).$$

Two special common cases are: $xy + x + y + 1 = (x + 1)(y + 1)$ and $xy - x - y + 1 = (x - 1)(y - 1)$.

# Word Problems

These are the problems that use **words** instead of equations. For these types of problems - just convert back to equations! Examples will be given in the **Problems Section**.

## Example

Sandwiches at Joe's Fast Food cost $3$ each and sodas cost $2$ each. How many dollars will it cost to purchase $5$ sandwiches and $8$ sodas?

$$\text{(A) } 31 \qquad \text{(B) } 32 \qquad \text{(C) } 33 \qquad \text{(D) } 34 \qquad \text{(E) } 35$$

*Solution.* The $5$ sandwiches cost $5 \cdot 3 = 15$ dollars. The $8$ sodas cost $8 \cdot 2 = 16$ dollars. In total, the purchase costs $15 + 16 = 31$ dollars. The answer is $\text{(A)}$.

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# Problems I

## Problem

*Find the smallest and largest number of a sequence of 7 consecutive integers that sum to 49.*

# Problems II

**Problem (Mathcounts Ratio Warmup)**

*What is $\frac{9}{30}$ as a common fraction?*

# Problems III

## Problem (Mathcounts Ratio Warmup)

*In a pasture there are 12 white horses and the rest are black. If there are 52 horses in the pasture, what fraction are black? Express your answer as a common fraction.*

# Problems IV

**Problem (Mathcounts Ratio Warmup)**

*There are three times the number of orange fish as blue fish in a tank at the pet store, and there are no other fish. What percentage of the fish are orange?*

# Problems V

## Problem (Mathcounts Ratio Stretch)

*Fairy Godmother has granted wishes to Aurora, Belle and Cindi in the ratio* $6 : 8 : 11$*. What fraction of the ratios were granted to Belle? What percent of the wishes granted by Fairy Godmother were not granted to Aurora? What is the absolute difference between the percents of wishes Fairy Godmother has granted to Aurora and to Cindi?*

# Problems VI

**Problem (Mathcounts Chapter 2019/8)**

*After a brisk workout, Felicia counts 32 heartbeats in 15 seconds. Based on this count, what is Felicia's expected number of heartbeats in one minute?*

# Problems VII

## Problem

*Dylan and Cody are working together again. Dylan can eat one apple in 1 minute. Cody can eat one apple in 2 minutes. How long will it take them working together to eat 15 apples?*

# Problems VIII

## Problem

*Alice can mow the lawn in 3 hours, Bob can mow the lawn in 4 hours, and Chris can mow the lawn in 5 hours. Working together, how long will it take them to mow one lawn?*

# Problems IX

## Problem

*Alice mows half the lawn (at the same rate as before), and then Bob joins in to mow a fourth of the lawn, and then Chris joins in to finish the job. How long does it take them in total?*

# Problems X

## Problem (AMC 8 2018/1)

*An amusement park has a collection of scale models, with a ratio* $1:20$*, of buildings and other sights from around the country. The height of the United States Capitol is 289 feet. What is the height in feet of its duplicate to the nearest whole number?*

**(A)** $14$ **(B)** $15$ **(C)** $16$ **(D)** $18$ **(E)** $20$

# Problems XI

## Problem (AMC 8 2010/2)

*If* $a@b = \frac{a \times b}{a+b}$ *for* $a, b$ *positive integers, then what is* $5@10$*?*

**(A)** $\frac{3}{10}$ **(B)** $1$ **(C)** $2$ **(D)** $\frac{10}{3}$ **(E)** $50$

# Problems XII

## Problem (AMC 8 2004/3)

*Twelve friends met for dinner at Oscar's Overstuffed Oyster House, and each ordered one meal. The portions were so large, there was enough food for* $18$ *people. If they shared, how many meals should they have ordered to have just enough food for the* $12$ *of them?*

**(A)** $8$ **(B)** $9$ **(C)** $10$ **(D)** $15$ **(E)** $18$

# Problems XIII

## Problem (AMC 8 2007/6)

*The average cost of a long-distance call in the USA in* $1985$ *was* $41$ *cents per minute, and the average cost of a long-distance call in the USA in* $2005$ *was* $7$ *cents per minute. Find the approximate percent decrease in the cost per minute of a long- distance call.*

$$\text{(A) } 7 \qquad \text{(B) } 17 \qquad \text{(C) } 34 \qquad \text{(D) } 41 \qquad \text{(E) } 80$$

# Problems XIV

## Problem (AMC 8 2010/6)

*As Emily is riding her bicycle on a long straight road, she spots Emerson skating in the same direction* $1/2$ *mile in front of her. After she passes him, she can see him in her rear mirror until he is* $1/2$ *mile behind her. Emily rides at a constant rate of* $12$ *miles per hour, and Emerson skates at a constant rate of* $8$ *miles per hour. For how many minutes can Emily see Emerson?*

**(A)** $6$ **(B)** $8$ **(C)** $12$ **(D)** $15$ **(E)** $16$

# Problems XV

## Problem (AMC 8 2017/23)

*Each day for four days, Linda traveled for one hour at a speed that resulted in her traveling one mile in an integer number of minutes. Each day after the first, her speed decreased so that the number of minutes to travel one mile increased by* $5$ *minutes over the preceding day. Each of the four days, her distance traveled was also an integer number of miles. What was the total number of miles for the four trips?*

**(A)** $10$ **(B)** $15$ **(C)** $25$ **(D)** $50$ **(E)** $82$

# Problems XVI

## Problem (Mathcounts Chapter 2019/20)

*Jones is chasing a car 800 meters ahead of him. He is on a horse moving at 50 km/h. If Jones catches up to the car in 4 minutes, how fast was the car moving?*

# Problems XVII

## Problem (AMC 8 2013/10)

*What is the ratio of the least common multiple of 180 and 594 to the greatest common factor of 180 and 594?*

**(A)** $110$ **(B)** $165$ **(C)** $330$ **(D)** $625$ **(E)** $660$

# Problems XVIII

## Problem (AMC 12 2000/6)

*Two different prime numbers between* $4$ *and* $18$ *are chosen. When their sum is subtracted from their product, which of the following numbers could be obtained?*

$$\text{(A) } 22 \qquad \text{(B) } 60 \qquad \text{(C) } 119 \qquad \text{(D) } 180 \qquad \text{(E) } 231$$

# Problems XIX

## Problem (AMC 10 B 2019/11)

*Two jars each contain the same number of marbles, and every marble is either blue or green. In Jar* $1$ *the ratio of blue to green marbles is* $9:1$, *and the ratio of blue to green marbles in Jar* $2$ *is* $8:1$. *There are* $95$ *green marbles in all. How many more blue marbles are in Jar* $1$ *than in Jar* $2$*?*

**(A)** $5$ **(B)** $10$ **(C)** $25$ **(D)** $45$ **(E)** $50$