Math Level 2 Handouts Week 03

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Online Classes Season 3

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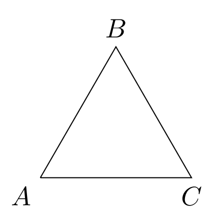
Contents

§ 1 Triangles

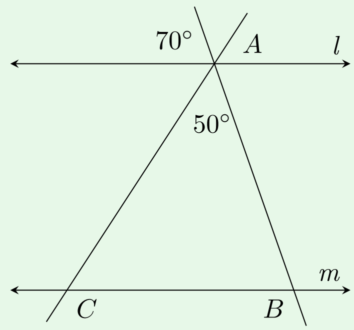
Definition 1 (Triangle). A **triangle** is a polygon with 3 sides and 3 angles.

There is no polygon that has less sides.

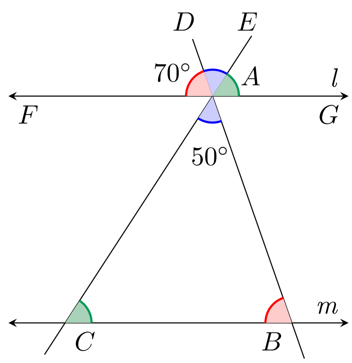


We call the points where the sides of the triangle meet **vertices**, so A, B and C are vertices (this term is not exclusive to triangles; every shape has vertices). There are many interesting properties to triangles, and we'll find one of the most interesting properties below.

Example 1. Find $\angle ABC$ and $\angle ACB$ below given line l and line m are parallel.



Solution. Let's mark equal angles in our diagram. Using the fact that corresponding angles and vertical angles are equal, we find



We notice that $\angle ABC = \angle DAF = 70^\circ$ and $\angle BAC = \angle DAE$. What about $\angle ACB$? Well, we can easily see that $\angle ACB = \angle EAG$. If we can just find $\angle EAG$, we are done. Looking around $\angle EAG$, we quickly see

$$\angle DAF + \angle DAE + \angle EAG = 180^\circ$$

$$70^\circ + 50^\circ + \angle EAG = 180^\circ \implies \angle EAG = 60^\circ.$$

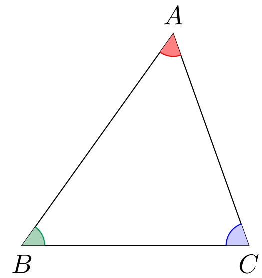
Since $\angle EAG = \angle ACB$, $\angle ACB = 60^\circ$. □

§ 1.1 Angles in a Triangle

From the above example, you might notice that the sum of the angles in $\triangle ABC$ add up to 180°. Turns out, this isn't a coincidence.

Theorem 1 (Sum of Angles of a Triangle). For a generic triangle, $\triangle ABC$

$$\angle A + \angle B + \angle C = 180^\circ.$$



Let's do a quick exercise to practice what you just learned.

Example 2. One angle in a triangle is twice another angle, and the third angle is $54°$. What is the measure of the smallest angle in the triangle?

Solution. We know that one angle is $54°$, and for the other two angles, one is twice the other. We can call the smaller angle x and the larger angle $2x$. Now that we know the angles in a triangle add up to $180°$, we have

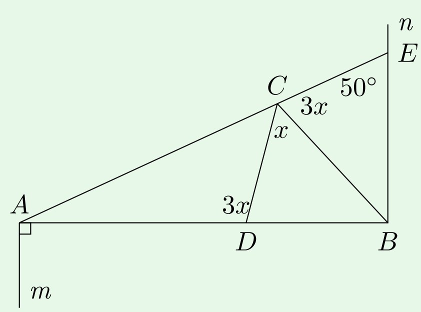
$$180° = 54° + x + 2x$$

Solving this, we find that $x = \boxed{42°}$. Therefore, the angles in our triangle are $42°$, $54°$, and $84°$. □

§ 1.2 Angle Chasing

Angle chasing is very common in math problems. Knowing how to do it is very important.

Example 3. In the diagram, $m \parallel n$, $\overline{AB} \perp m$, $\angle ADC = \angle BCE = 3x$, $\angle CEB = 50°$, and $\angle BCD = x$. Find x.



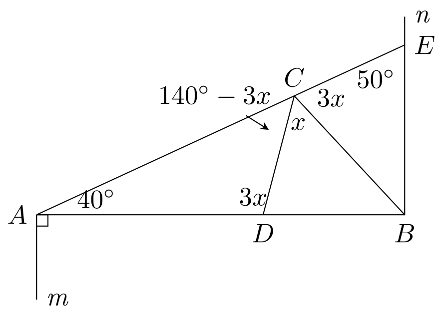
Solution. We can start with our parallel lines, and see that $\angle ABE = 90°$ because $m \parallel n$ and $m \perp \overline{AB}$. From $\triangle ABE$ we have $\angle EAB + \angle ABE + \angle AEB = 180°$.
Therefore,

$$\angle EAB = 180° - \angle ABE - \angle AEB = 40°$$

We can then use $\triangle ACD$ to find

$$\angle ACD = 180° - \angle EAB - \angle ADC = 180° - 40° - 3x = 140° - 3x$$

Now, our diagram looks like this:



We can see that there are now three angles at C which form a straight angle. Therefore, we have:

$$\angle ACD + \angle DCB + \angle BCE = 180^\circ$$

Substituting in the values for the angles gives us:

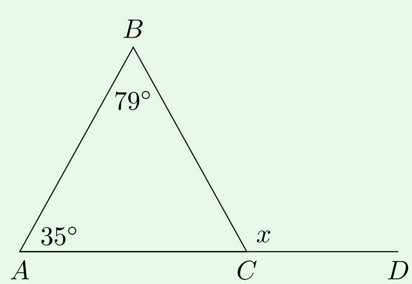
$$140^\circ - 3x + x + 3x = 180^\circ \implies x = 40^\circ.$$

□

§ 1.3 Exterior Angles

So far, we've focused on **interior angles**, angles inside a triangle. **Exterior angles** are angles that are formed by extending a side of the triangle. In the example below, x is an exterior angle. We can learn some useful things by studying exterior angles.

Example 4. Find x.



Solution. If we can find $\angle BCA$, we can use the fact that $\angle ACD$ equals 180° and do some simple algebra to find $\angle BCD$. We know that since the figure is a triangle,

$$\angle ABC + \angle BCA + \angle CAB = 180^\circ \implies \angle BCA = 180^\circ - \angle ABC - \angle BAC$$

Substituting values inside the equation yields

$$\angle BCA = 180^\circ - 79^\circ - 35^\circ \implies \angle BCA = 66^\circ.$$

We are almost done. Since $\angle ACD$ is a straight angle, $\angle BCA$ and x must add to 180°. Therefore,

$$x = 180^\circ - \angle BCA = 180^\circ - 66^\circ = 114^\circ.$$

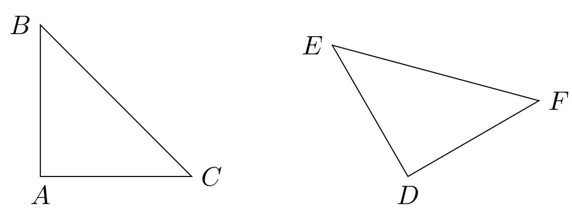
□

You might notice that x is the sum of the two interior angles furthest from it $(79^\circ + 35^\circ = 114^\circ)$.

Theorem 2 (Exterior Angle Theorem). An exterior angle of a triangle is equal to the sum of the two interior angles "furthest" from the exterior angle (these angles are called **remote interior angles**).

§ 1.4 Congruent Triangles and Angles

We say two angles are **congruent** if they are equal. Two figures are **congruent** if they are exactly the same. This means all corresponding sides and angles are congruent. Basically, if figure A and figure B were congruent, you would be able to lay B on top of A by any combination of sliding, spinning, rotating, or flipping. A easier way to understand this would be that congruent polygons are two identical copies, in different orientations. We use the symbol $\cong$ to denote congruence between figures. For instance, to describe the fact that $\triangle ABC$ and $\triangle DEF$ are congruent, we could say $\triangle ABC \cong \triangle DEF$.



It is important to note that **the order of the vertices matters**. Each vertex in $\triangle ABC$ matches onto each vertex in $\triangle DEF$ in that order. Therefore, $\triangle ABC \cong \triangle EFD$ would not always be a true statement. If two triangles are congruent, they must be identical to each other. Therefore, all of their corresponding side lengths and angle measures must be the same as well. Conversely, if all the sides and all the angles of two triangles are the same, then the two triangles must be congruent.

Theorem 3 (SSS Congruence). If two triangles have three congruent (equal) sides, they are congruent.

Another way to show congruence between two triangles is to prove that two corresponding sides are congruent and that the angle **between** them is congruent. We will show you why it is crucial that you realize that the angle you are comparing is between the triangles later on.

Theorem 4 (SAS Congruence). If two sides of one triangle and the angle between them are congruent to the corresponding sides and angle of another triangle, then the two triangles are congruent.

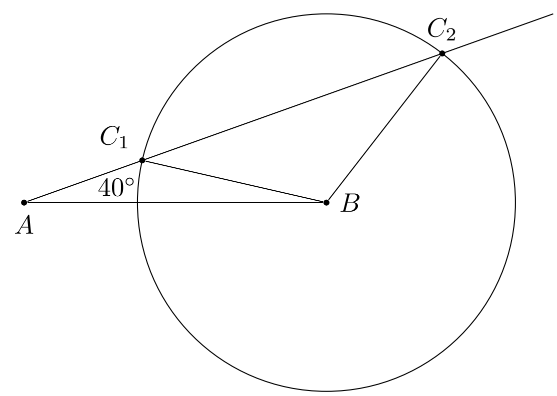
ASA and AAS are basically the same thing - two angles and a side are given. The order that these are given in don't really matter.

Theorem 5 (AAS Congruence). If two angles and a side of one triangle equal the corresponding angles and side in another triangle, then the two triangles are congruent.

One might think that SSA might work too – after all, its using the same components. However, SSA is most definitely **not** a valid congruence theorem.

Example 5. Construct $\triangle ABC$, where $AB = 1.8$ cm, $BC = 1.5$ cm, and $\angle BAC = 40^\circ$.

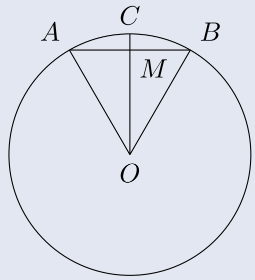
Solution. Let's assume $\overline{AB}$ has a length of 1.8 cm. Since $BC = 1.5$ cm, the last point C must be on a circle of radius 1.5 cm with center B. We also know that $\angle BAC = 40^\circ$, so C must be on a ray from A at a 40° angle. We quickly see the problem – the ray intersects the circle at two points. This means that there are two possible locations for C to be, and obviously $\triangle ABC_1$ is not congruent to $\triangle ABC_2$.



□

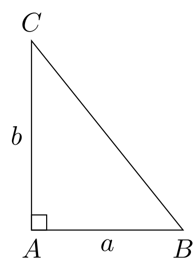
Fact 1. SSA Congruence does **not** exist.

Theorem 6. Let A, B, C lie on a circle with center O. Then if $AM = BM$, $\overline{OC} \perp \overline{AB}$.



§ 1.5 Right Triangles

A right triangle is simply a triangle with a 90° angle. In the picture below, $\overline{AC}$ and $\overline{AB}$ are the **legs** of the right triangle. The side opposite the right angle is called the **hypotenuse**. The height of the triangle is known as the **altitude** (this applies to all triangles, not just right triangles). We can find the area of a right triangle quite easily.



Theorem 7 (Area of Right Triangle). A right triangle has legs of length a and b. Then the area of the right triangle is

$$\frac{1}{2}ab.$$

Right triangles are probably a mathematician's favorite triangle, and it's for a good reason. Many interesting properties of them exist, and we'll prove the most important one of them below.

Theorem 8 (Pythagorean Theorem). If the hypotenuse of a right triangle is c, and the legs are a and b,

$$a^2 + b^2 = c^2.$$

There also exists a unique congruence theorem for only right triangles.

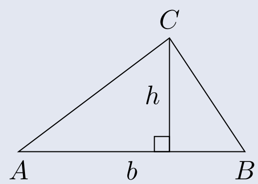
Theorem 9 (HL Congruence). If two right triangles have two congruent corresponding sides, then the two triangles are congruent.

This congruence theorem is essentially a special case of SSS congruence.

§ 1.6 Area

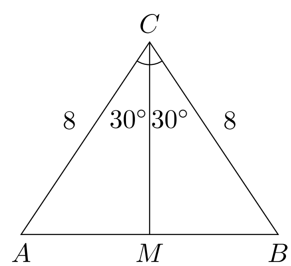
Most of you probably already know the area of a triangle. But, where does it come from? Maybe we can use things we already learned to our advantage here.

Theorem 10. The area of a triangle is given by $\frac{1}{2}bh$, where b is the base and h is the height.



Example 6. Find the area of $\triangle ABC$ below if $\angle C = 60^\circ$ and $AC = BC = 8$.

Solution. First, let's draw the triangle.



We see that since the triangle is isosceles, $\angle A = \angle B = 60^\circ$. Therefore, the triangle is equilateral and $\overline{AB} = 8$. Drawing an angle bisector (divides an angle into two equal angles) from C to $\overline{AB}$, we see $\triangle AMC \cong \triangle BMC$ by SAS. Therefore, $\angle CMB = \angle CMA = 90^\circ$ and $AM = MB = 4$. Now we can use the Pythagorean Theorem to find the height.

$$CM^2 + 16 = 64 \implies CM = 4\sqrt{3}.$$

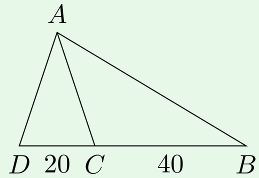
Therefore, the area of the triangle is

$$\frac{1}{2} \cdot 8 \cdot 4\sqrt{3} = 16\sqrt{3}.$$

□

Sometimes, it becomes a hassle to keep saying "find the area of $\triangle ABC$", so we denote the area of the triangle by putting the vertices in square brackets ($[ABC]$). Also note this convention applies to all shapes, not just triangles.

Example 7. Given that $BC = 40$ and $CD = 20$ in the figure, what are $\frac{[ABC]}{[ACD]}$ and $\frac{[ABC]}{[ABD]}$?



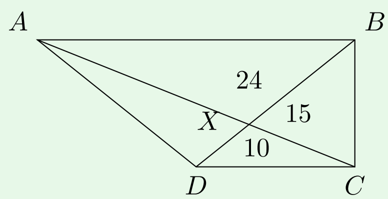
Solution. You might not think this is possible because we don't have the heights, but it turns out we don't need to. The key insight is to realize that the height of all triangles is the same (drop an altitude from A and see for yourself). Let this h be this height. We see that

$$\frac{[ABC]}{[ABD]} = \frac{\frac{1}{2} \cdot 40 \cdot h}{\frac{1}{2} \cdot 20 \cdot h} = 2.$$

□

Notice how the $\frac{1}{2}$ and h cancels out in the ratio of area, so we are only left with ratio of the bases. Therefore, the **ratio of the areas of triangles with equal heights is simply the ratio of their bases** (this same concept applies to triangles with same bases but different altitudes).

Example 8. $\overline{AC}$ and $\overline{BD}$ meet at X as shown. Given $[ABX] = 24$, $[BCX] = 15$, and $[CDX] = 10$, find $[ADX]$.



Solution. We notice that $\triangle ABX$ and $\triangle BXC$ have the same altitude. Therefore

$$\frac{[ABX]}{[BXC]} = \frac{24}{15} = \frac{8}{5} = \frac{AX}{CX}.$$

Now we apply the same concept to the other triangle!

$$\frac{[ADX]}{[CXD]} = \frac{AX}{CX} = \frac{8}{5} \implies [ADX] = \frac{8}{5}[CXD] = \frac{8 \cdot 10}{5} = 16.$$

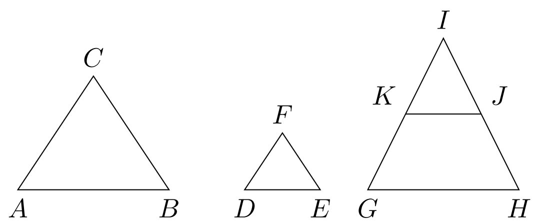
□

Theorem 11 (Heron's Formula). If the sides of a triangle are a, b, and c, and $s = (a+b+c)/2$, then the area of the triangle is $\sqrt{s(s-a)(s-b)(s-c)}$.

This is not always the fastest way to find the area of a triangle, but if we end up having all three side lengths of a triangle, this is a sure way to get it. This formula also shows the power of the Pythagorean theorem, as we were able to derive it only using algebra and the Pythagorean theorem. When in doubt, always try to construct right triangles in your problems. They can usually help lead you to the solution.

§ 1.7 Similarity

We call two figures **similar** if one is simply a blown up, and possible rotated and/or flipped version of the original figure. In the figure below, $\triangle ABC \sim \triangle DEF$ and $\triangle IJK \sim \triangle IHG$.



Similarity between figures is shown by the symbol $\sim$, so we could say something like $\triangle ABC \sim \triangle DEF$. While congruency means the both the size and shape of the figures have to be identical, similarity only deals with the shape of the figure. This means that similar triangles will have the same angles, and all their side lengths will be in the same ratio (though not necessarily the same size). The converse holds true, so if both triangles have the same side ratio and angle measures, they must be similar.

Theorem 12 (SAS Similarity). If two sides in one triangle are in the same ratio as two sides in another triangle, and the angle between those sides are equal, then the triangles are similar.

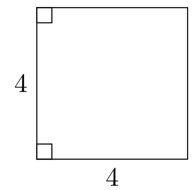
Theorem 13 (SSS Similarity). If each side of one triangle is the same constant multiple of the corresponding side of another triangle, then the triangles are similar (meaning their corresponding angles are equal as well).

This type of similarity is used less often than AA and SAS Similarity, as it is generally rare to have all the sides of both triangles in a similarity problem. However, just like SSS congruence, SSS similarity is quite obvious. If the ratio of the sides of both triangles are the same, one is just a blown up version of the other. They are then going to have to have the same angle measures, thus satisfying the definition of similarity.

Theorem 14 (Side-Area Ratio Relation). The ratio of the areas of two similar triangles is equal to the square of the ratio of their sides.

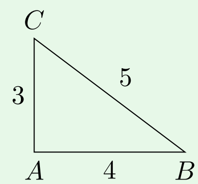
§ 1.8 Perimeter

The **perimeter** of an object is simply the sum of all the lengths of a shape's sides. For example, the rectangle below has a perimeter of 16.



Although most of might know what perimeter is, you might be surprised of how often a simple concept appears in geometry problems.

Example 9 (AMC 8 2017/16). In the figure below, choose point D on $\overline{BC}$ so that $\triangle ACD$ and $\triangle ABD$ have equal perimeters. What is the area of $\triangle ABD$?



Solution. Say the length of AD is l and the length of $DB = x$. Then $DC = 5 - x$. Setting the perimeters equal, we see

$$l + 4 + x = l + 3 + (5 - x) \implies x = 2.$$

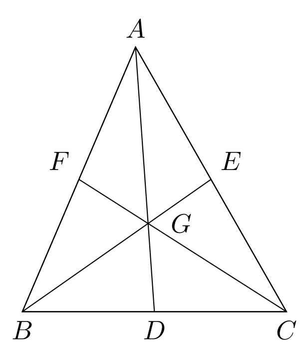
Now imagine drawing a line from D perpendicular to AB such that it meets AB at point E. By AA similarity (they share a right angle and vertex angle), $\triangle BDE$ is similar to $\triangle BCA$. Similarity tells us corresponding sides must be proportional, so

$$\frac{BD}{BC} = \frac{2}{5} = \frac{DE}{CA} = \frac{DE}{3} \implies DE = \frac{6}{5}.$$

We now have the height $\triangle BDA$. We can quickly see that the area of the $\triangle BDA$ $\frac{1}{2}(4)(\frac{6}{5}) = \frac{12}{5}$. $\square$

§ 1.9 Medians

A **median** of a triangle is a segment from a vertex to the midpoint of the opposite side. In the diagram, $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are all medians.



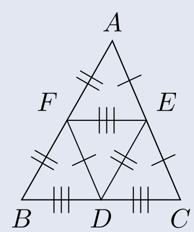
The medians in a triangle are always **concurrent**, meaning they all meet at a single point. The spot where the three medians of a triangle meet is called the **centroid**.

Theorem 15. The medians of a triangle divide the triangle into six smaller triangles of equal area.

Theorem 16. The centroid of a triangle cuts its medians into a 2:1 ratio.

Now, let's say we connect the midpoints to each other instead of to the opposite vertex. The triangle formed is called the **medial triangle**, as shown in the below example.

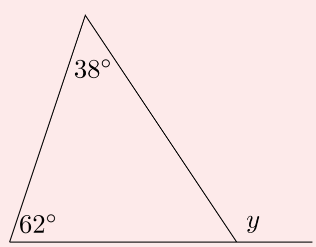
Theorem 17. Given that $\triangle DEF$ is the medial triangle of $\triangle ABC$, we have:

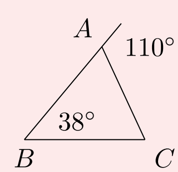


1. $\triangle DEF \sim \triangle ABC, \triangle DEF \cong \triangle FBD \cong \triangle AFE \cong \triangle EDC$.
2. $\frac{EF}{BC} = \frac{DE}{AB} = \frac{DF}{AC} = \frac{1}{2}$
3. $\overline{DF} \parallel \overline{AC}, \overline{EF} \parallel \overline{BC}, \overline{DE} \parallel \overline{AB}$.

§ 1.10 Problems

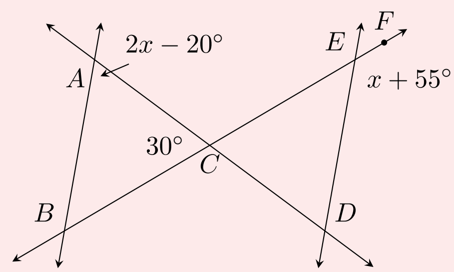
Problem 1. Find y in the left figure below. Find $\angle C$ in the right figure below.



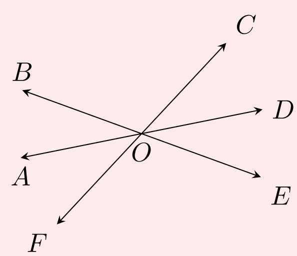


Problem 2. The exterior angles of a triangle are in the ratio $2 : 3 : 4$. What are the (interior) angles of the triangle?

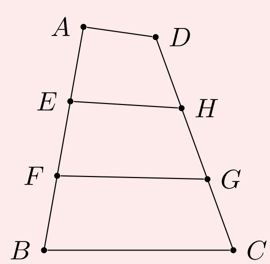
Problem 3. In the diagram below, $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$, $\angle BAC = 2x - 20^\circ$, $\angle ACB = 30^\circ$, and $\angle DEF = x + 55^\circ$. Find $\angle CED$.



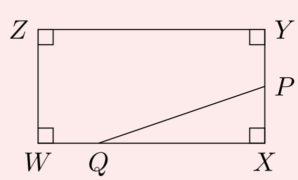
Problem 4. Three straight lines intersect at O and $\angle COD = \angle DOE$ in the diagram below. The ratio of $\angle COB$ to $\angle BOF$ is $7 : 2$. What is the number of degrees in $\angle COD$?



Problem 5. Let $ABCD$ be a quadrilateral (a shape with four sides). The points E and F are on AB such that $AE = EF = BF = \frac{1}{3}AB$ and G and H are on CD such that $CG = GH = HG = \frac{1}{3}CD$. Show that $[EFGH] = \frac{1}{3}[ABCD]$.



Problem 6. Find QX in the diagram below given that $PY = PX$, $WX = 8$, and $[PQX] = [WXYZ]/6$.



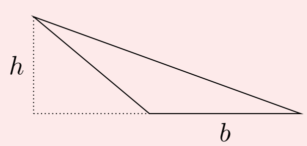
Problem 7 (AHSME 1951/3). If the diagonal (the line segment connecting opposite vertices) of a square is of length is $a + b$, what is the area of the square in terms of a and b?

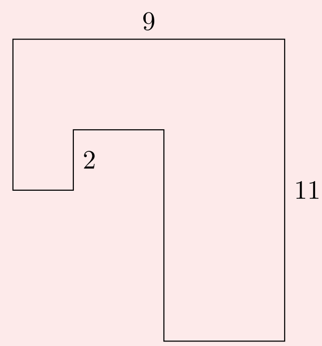
Problem 8 (AHSME Problem 1951/4). A barn with a roof is rectangular in shape, 10 yd. wide, 13 yd. long and 5 yd. high. It is to be painted outside and inside but not on the ceiling or floor. How many square yds. of paint are needed?

Problem 9 (AMC 10B 2006/5). A 2×3 rectangle and a 3×4 rectangle are contained within a square without overlapping at any point, and the sides of the square are parallel to the sides of the two given rectangles. What is the smallest possible area of the square?

Problem 10. We have $\triangle ABC$, with side lengths 13, 14, and 15. Find $[ABC]$.

Problem 11. Prove that the area of an obtuse triangle is still $\frac{1}{2}bh$, using the left figure below.





Problem 12. Find the perimeter of the figure above right.

Problem 13. Prove that the area of an equilateral triangle with side length s is given by $\dfrac{s^2\sqrt{3}}{4}$.

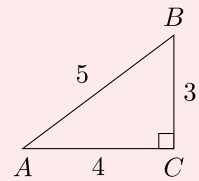
Problem 14. Find the area of a $30° - 60° - 90°$ triangle given s is the length of the shortest side in the triangle.

Problem 15. In triangle ABC, $AB = AC$, $BC = 1$, and $\angle BAC = 36°$. Let D be the point on side $\overline{AC}$ such that $\angle ABD = \angle CBD$.

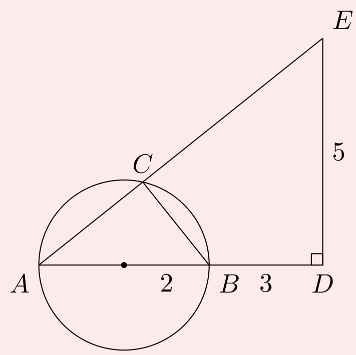
1. Prove that triangles ABC and BCD are similar.
2. Find $\overline{AB}$.

Problem 16. The length of one side of a rectangle is 4 less than 3 times an adjacent side. The perimeter of the rectangle is 64. Find the area of the rectangle.

Problem 17 (AMC 10A 2018/13). A paper triangle with sides of lengths 3, 4, and 5 inches, as shown, is folded so that point A falls on point B. What is the length in inches of the crease?



Problem 18 (AMC 10B 2017/22). The diameter $\overline{AB}$ of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and line ED is perpendicular to line AD. Segment $\overline{AE}$ intersects the circle at a point C between A and E. $\overline{BC}$ is perpendicular to $\overline{AE}$. What is the area of $\triangle ABC$?



Problem 19 (AMC 12A 2009/8). Four congruent rectangles are placed as shown. The area of the outer square is 4 times that of the inner square. What is the ratio of the length of the longer side of each rectangle to the length of its shorter side?