Math Level 2.5 Handouts

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§1   Rate Problems

§1.1 $d = rt$

There is only one theorem you need to know:

Theorem 1 ($d = rt$). The distance travelled equals the rate multiplied by the time.

Another form of this formula involves finding out how long it takes multiple people working together to complete a task.

Corollary 2 (Completing a Task). If it takes person 1 t_1 time to finish a task, person 2 t_2 time to finish a task, and so on, and there are n people (meaning person n takes t_n time), then if they all work together it will take

$$\frac{1}{\frac{1}{t_1} + \frac{1}{t_2} + \ldots + \frac{1}{t_2}}$$

time to finish the task.

Example 1 (AHSME 1998/21). In an h-meter race, Sunny is exactly d meters ahead of Windy when Sunny finishes the race. The next time they race, Sunny sportingly starts d meters behind Windy, who is at the starting line. Both runners run at the same constant speed as they did in the first race. How many meters ahead is Sunny when Sunny finishes the second race?

Solution. Let s and w be the speeds of Sunny and Windy. From the first race we know that $\frac{s}{w} = \frac{h}{h-d}$. In the second race, Sunny's track length is $h + d$. She will finish this track in $\frac{h+d}{s}$. In this time, Windy will run the distance $w \cdot \frac{h+d}{s} = \frac{(h+d)(h-d)}{h}$. This is less than h, therefore Sunny is ahead. The exact distance between Windy and the finish is $h - \frac{(h+d)(h-d)}{h} = \frac{h^2 - (h^2 - d^2)}{h} = \boxed{\frac{d^2}{h}}$. $\square$

§1.2   Problems

Note that rate problems do not only refer to problems about going certain distances. It is anything that involves a rate and a time.

Exercise 1 (Mathcounts Minis). Armond the Ant crawls one foot every 10 seconds when not carrying anything, and he crawls one foot every 15 seconds when he's carrying a crumb. He carries nothing as he walks 10 feet from a wall to the place on the floor where you dropped a cookie. He picks up a crumb and then walks back to the wall along the same path. How many minutes does this entire trip take? Express your answer as a mixed number.

Exercise 2 (Mathcounts Minis). If I drive 30 miles an hour from work to home, then my trip will take 24 minutes. How fast do I have to drive to make the trip in 12 minutes?

Exercise 3 (Mathcounts Minis). Wally the wandering walrus swims 6 miles north, then 3 miles east, then 3 miles north, then 15 miles west. How many miles is he from where he started?

Exercise 4 (Mathcounts Minis). A frog is at the bottom of a 12-meter well. Each morning he climbs up 5 meters. Each night he slides down 3 meters. If he starts climbing on a Sunday, on which day will he reach the top of the well and escape?

More problems from Mathcounts Minis can be found here.

Exercise 5 (Mathcounts Handbook). A car averages 20 miles per gallon of gas in city driving and 30 miles per gallon in highway driving. At these rates, how many gallons of gas will the car use on a 300-mile trip if $\frac{4}{5}$ of the trip distance is highway driving and the rest is city driving?

Exercise 6 (Mathcounts Handbook). Chris and Sandy ran a half-mile race. Sandy ran the race at a steady pace of $\frac{1}{6}$ mile per minute, and Chris ran at a steady pace of $\frac{1}{5}$ mile per minute. How many seconds after Chris finished the race did Sandy cross the finish line?

Exercise 7 (Mathcounts Handbook). Selina can run around the track 3 times in 8 minutes. Marta can run around the same track 2 times in 5 minutes. If Selina and Marta begin at the same time and the same place, what is the combined number of laps the girls both will have run when they next reach the starting point at the same time?

Exercise 8 (AMC 8 2009). Austin and Temple are 50 miles apart along Interstate 35. Bonnie drove from Austin to her daughter's house in Temple, averaging 60 miles per hour. Leaving the car with her daughter, Bonnie rod a bus back to Austin along the same route and averaged 40 miles per hour on the return trip. What was the average speed for the round trip, in miles per hour?

Exercise 9 (AMC 8 2005). Joe had walked half way from home to school when he realized he was late. He ran the rest of the way to school. He ran 3 times as fast as he walked. Joe took 6 minutes to walk half way to school. How many minutes did it take Joe to get from home to school?

Exercise 10 (Mathcounts Handbook). A hose could fill a small pool in 50 minutes if the pool did not leak. Alas, the pool leaks at a steady rate that can drain it completely in 300 minutes. How many minutes will it take the hose to fill the leaky pool?

Exercise 11 (Mathcounts Handbook). Working together, 2 groomers can brush 8 horses in 3 hours. How many hours would it take 3 groomers to brush 12 horses at this rate?

Exercise 12 (Mathcounts Handbook). Larry and Curly are trying to fill a sandbox with sand. Working alone, Larry can fill an empty sandbox in 4 hours, and Curly can do the same job in 5 hours. Moe is trying to empty the sandbox. Working alone, Moe can empty a full sandbox in 6 hours. If the sandbox is half full at the time Larry and Curly begin filling the sandbox and Moe begins emptying it, how many minutes will it take for the sandbox to be filled? Express your answer to the nearest whole number.

Exercise 13 (Mathcounts Handbook). Danielle and Jennifer can do a job in 2 hours working together. Danielle could do it in 3 hours alone. How many hours would it take Jennifer to do the job alone?

Exercise 14. Three runners Dirk, Edith, and Foley all start at the same time for a 24 km race, and each of them runs at a constant speed. When Dirk finishes the race, Edith is 8 km behind, and Foley is 12 km behind. When Edith finishes the race, how far behind is Foley, in km?

Exercise 15. A cyclist travels at 20 kilometers per hour when cycling uphill, 24 kilometers per hour when cycling on flat ground, and 30 kilometers per hour when cycling downhill. On a sunny day, they cycle the hilly road from Aopslandia to Beast Island before turning around and cycling back to Aopslandia. What was their average speed during the entire round trip?

Exercise 16 (Alcumus). Jennifer rode her bike at 12 miles per hour for 4 hours, then put her bike down and began walking at 3 miles per hour. Thomas started walking at 4 miles per hour along the same path at the same time that Jennifer started biking. When he reached the bike, he began riding it at 10 miles per hour. How many hours will it have been since they started when Thomas catches up to Jennifer?

Exercise 17 (AHSME 1996/13). Sunny runs at a steady rate, and Moonbeam runs m times as fast, where m is a number greater than 1. If Moonbeam gives Sunny a head start of h meters, how many meters must Moonbeam run to overtake Sunny?

Exercise 18 (AHSME 1967/27). Two candles of the same length are made of different materials so that one burns out completely at a uniform rate in 3 hours and the other in 4 hours. At what time P.M. should the candles be lighted so that, at 4 P.M., one stub is twice the length of the other?

Exercise 19 (Mathcounts Minis). Every day, Zuleica's mother Wilma drives from home to the train station, arriving right when Zuleica's train from school gets to the station. Then Wilma drives Zuleica home. They always return home at 5:00 p.m. One day Zuleica left school early and got to the train station an hour early. She then started walking home. Wilma left home at her usual time to pick Zuleica up, and they met along the route between the train station and their house. Wilma picked Zuleica up and then drove home, arriving at 4:46 p.m. For how many minutes had Zuleica been walking before Wilma picked her up?

Exercise 20 (AMC 12B 2008). Bricklayer Brenda would take 9 hours to build a chimney alone, and bricklayer Brandon would take 10 hours to build it alone. When they work together they talk a lot, and their combined output is decreased by 10 bricks per hour. Working together, they build the chimney in 5 hours. How many bricks are in the chimney?