Math Level 2 Handout

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1 Basic Equation Solving

1.1 Algebraic Manipulation

To introduce the topic of **algebraic manipulation**, let us start with a little known example: **egyptian fractions**.

Theorem 1. (Egyptian Fractions) For all a, b where $ab \neq 1$,

$$\frac{a}{ab-1} = \frac{1}{b(ab-1)} + \frac{1}{b}.$$

From here, we can see that putting things together (factoring) is just as important as taking them apart (distributing). Now, let us turn the power of **products**:

Example 1.1. For positive real numbers a, b,

$$a + \frac{1}{b} = 4,$$

$$b + \frac{1}{a} = 5.$$

Find $ab + \frac{1}{ab}$.

Solution. It is very easy to get lost in the problem if we directly try to solve for a and b. Instead, let us **multiply** the equations:

$$(a + \frac{1}{b})(b + \frac{1}{a}) = 4 \cdot 5 = 20,$$

$$ab + \frac{a}{a} + \frac{b}{b} + \frac{1}{ab} = ab + \frac{1}{ab} + 2 = 20,$$

$$ab + \frac{1}{ab} = \boxed{18}.$$

There are way too many types of manipulations to include in this one lecture. Instead, we will do some examples and hopefully the idea behind them will become clear.

Example 1.2. Let $x + y = \frac{3+\sqrt{5}}{2}$, and $x - y = \frac{3-\sqrt{5}}{2}$. Find $x^2 - y^2$.

Solution. If we multiply these equations like in **Example 1**, we get

$$x^2 - y^2 = \frac{(3^2) - (\sqrt{5})^2}{4} = \frac{4}{4} = \boxed{1}.$$

Can you imagine if we tried to solve for x and y?

The following is a very common manipulation:

Theorem 2. Let x, y be nonzero real numbers such that $x + y = a$ and $xy = b$. Then,

$$x^2 + y^2 = a^2 - 2b,$$

$$(x+1)(y+1) = a + b + 1,$$
$$x^2y + xy^2 = ab,$$
$$|x - y| = \sqrt{a^2 - 4b},$$
$$x^3 + y^3 = a^3 - 3ab,$$
$$\frac{1}{x} + \frac{1}{y} = \frac{a}{b}.$$

This is very easily proven, but still useful nonetheless. In fact, you will see it a lot when we discuss **Vieta's Formulas**.

1.2 Memorization

The following are formulas you should definitely memorize:

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a-b)^2 = a^2 - 2ab + b^2$
- $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
- $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
- $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$
- $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab+bc+ca}{abc}$

1.3 Quadratic Equations

A polynomial is an equation of the following form:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where

$$a_0, a_1, \ldots, a_n$$

are constants. A quadratic equation is a polynomial with $n = 2$:

$$ax^2 + bx + c.$$

A common way to solve the equation $ax^2 + bx + c = 0$ is using the **Quadratic Formula**:

Theorem 3. (Quadratic Formula) For the equation $ax^2 + bx + c = 0$, the roots x_1, x_2 must be equal to

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$
$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Some people may ask, what if $b^2 - 4ac$ is less than 0? What do we do then? This leads us to the **Discriminant Test**:

Theorem 4. For the equation $ax^2 + bx + c = 0$, we have the following cases:

- If $b^2 - 4ac > 0$, we have **two real solutions.**
- If $b^2 - 4ac = 0$, we have **one real solutions.**
- If $b^2 - 4ac < 0$, we have **zero real solutions.**

Using the Quadratic Formula, we can calculate the sum of roots and product of roots:

Theorem 5. (Sum and Product of Roots) For the equation $ax^2 + bx + c = 0$, the sum of roots s is

$$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a},$$

and the product of roots p is

$$x_1 x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}$$

Some problems will ask you to find the sum or product of the roots even for multi-degree polynomials. **Vieta's Formulas** state that the sum of the roots is still $-\frac{b}{a}$, and the product of the roots is still $\frac{c}{a}$:

Theorem 6. (Vieta's Formulas Simplified) For any polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

the **sum of roots** is $-\frac{\text{second coefficient}}{\text{first coefficient}} = -\frac{a_{n-1}}{a_n}$, and the **product of roots** is $\frac{\text{last coefficient}}{\text{first coefficient}} = \frac{a_0}{a_n}$.

For those who are looking for a more **advanced** and more **powerful** theorem, we can generalize this formula:

Theorem 7. (Vieta's Formulas) For any polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

let $r_1, r_2, \ldots, r_n$ (an n-degree equation has n different roots). Vieta's formulas state that

$$a_n = a_n$$

$$a_{n-1} = -a_n(r_1 + r_2 + \ldots + r_n)$$
$$a_{n-2} = a_n(r_1 r_2 + r_1 r_3 + \ldots + r_{n-1} r_n)$$
$$\vdots$$
$$a_0 = (-1)^n a_n \ (r_1 r_2 \ldots r_n).$$

Note that the parts in parenthesis are known as **symmetric sums**. The kth symmetric sum σ_k is equal to $(-1)^k \frac{a_{n-k}}{a_n}$.

1.4 Problems

Problem 1.1. Let x be a real number such that $x + \frac{1}{x} = \sqrt{2020}$. What is $x^2 + \frac{1}{x^2}$?

Problem 1.2. (2000 AMC12 P11) Two non-zero real numbers, a and b, satisfy $ab = a - b$. Which of the following is a possible value of $\frac{a}{b} + \frac{b}{a} - ab$?

$$\text{(A)}\ -2 \qquad \text{(B)}\ \frac{-1}{2} \qquad \text{(C)}\ \frac{1}{3} \qquad \text{(D)}\ \frac{1}{2} \qquad \text{(E)}\ 2$$

Problem 1.3. Let x, y be nonnegative real numbers such that $x + y = 5$ and $xy = 7$. Find $\frac{x}{y-1} + \frac{y}{p-1}$.

Problem 1.4. Let a, b be real numbers such that

$$\frac{1}{a(b+1)} + \frac{1}{b(a+1)} = \frac{1}{(a+1)(b+1)}.$$

Compute $\frac{1}{a} + \frac{1}{b}$.

Problem 1.5. Find the sum of roots to the equation $x^{2020} = 2020x^{2019} + 1$.

Problem 1.6. Find the product of the roots of the equation $x^3 = 9\pi x + x^2 + 1$.

Problem 1.7. (Mathcounts) What is the average value of the three roots of the equation $x^3 - 12x^2 - 4x + 48 = 0$?

Problem 1.8. (2014 Mathcounts State Sprint P27) The fourth degree polynomial equation $x^4 - 7x^3 + 4x^2 + 7x - 4 = 0$ has four real roots, a, b, c and d. What is the value of the sum $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$? Express your answer as a common fraction.

Problem 1.9. Solve the following system of equations for x, y:

$$2x - 3y = 11,$$

$$xy = -5$$

Problem 1.10. Let p, q, r be roots of the polynomial $x^3 - 2x^2 + 3x - 4$. Find $(p+1)(q+1)(r+1)$.

Problem 1.11. (2003 AMC10A P18) What is the sum of the reciprocals of the roots of the equation $\frac{2003}{2004}x + 1 + \frac{1}{x} = 0$?

(A) $-\frac{2004}{2003}$ (B) -1 (C) $\frac{2003}{2004}$ (D) 1 (E) $\frac{2004}{2003}$

Problem 1.12. The polynomial $x^3 - ax + 15$ has three real roots. Two of these roots sum to 3. What is $|a|$?

Problem 1.13. What is the only real number that could be a multiple root of $x^3 + ax + 1 = 0$, if a is real?