ASE 2020-21 Notes

Lecture Notes by Dylan Yu

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§0 Contents

§1 Sunday, 07/26/20

§1.1 Number Sense: Double and Half Trick

This trick involves multiplying by a clever version of 1. Let's look at an example:

$$\begin{aligned}15 \times 78 &= \frac{2}{2} \times 15 \times 78 \\ &= (15 \times 2) \times \frac{78}{2} \\ &= 30 \times 39 = \mathbf{1170}\end{aligned}$$

So the procedure is you double one of the numbers and half the other one, then multiply. This trick is exceptionally helpful when multiplying by 15 or any two-digit number ending in 5. Another example is:

$$35 \times 42 = 70 \times 21 = \mathbf{1470}$$

It is also good whenever you are multiplying an even number in the teens by another number:

$$18 \times 52 = 9 \times 104 = \mathbf{936}$$

or

$$14 \times 37 = 7 \times 74 = \mathbf{518}$$

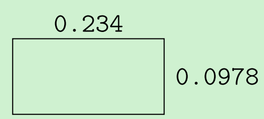
The purpose of this trick is to save time on calculations. It is a lot easier to multiply a single-digit number than a two-digit number.

§1.1.1 Double and Half Trick Problems

1. $1.5 \times 5.2 =$ ____________
2. $4.8 \times 15 =$ ____________
3. $64 \times 1.5 =$ ____________
4. $15 \times 48 =$ ____________
5. $14 \times 203 =$ ____________
6. $14 \times 312 =$ ____________
7. $\mathbf{24} \times \mathbf{35} =$ ____________
8. $312 \times 14 =$ ____________
9. A rectangle has a length of 2.4 and a width of 1.5. Its area is ____________
10. $18 \times 112 =$ ____________
11. $27 \times 14 =$ ____________
12. $\mathbf{21} \times \mathbf{15} \times \mathbf{14} =$ ____________
13. $33.75 = 1.5\times$ ____________
14. $345 \times 12 =$ ____________
15. $\mathbf{1.2} \times \mathbf{1.25} =$ ____________
16. 24% of 44 = ____________
17. $14 \times 25 + 12.5 \times 28 =$ ____________

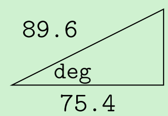
§1.2 Calculator: Geometry Problems

Example 1. `Perimeter = ?`



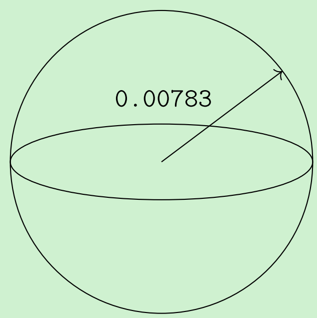
Solution. Using the perimeter formula, we get $2(0.234 + 0.0978) = \boxed{6.64 \times 10^{-1}}$. ☐

Example 2. `Angle = ?`



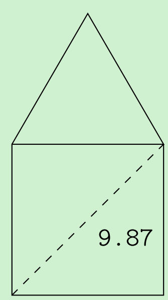
Solution. Let the angle be θ. Then we know that $\cos\theta = \frac{75.4}{89.6} \implies \theta = \boxed{32.7^\circ}$. ☐

Example 3. `Surface Area = ?`



Solution. Using the formula, we get $4\pi \times 0.00783^2 = \boxed{7.70 \times 10^{-4}}$. ☐

Example 4. `Area = ?`



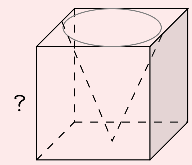
Solution. The side length of the square and the equilateral triangle are both $s = \frac{9.87}{\sqrt{2}}$. Thus, the area is

$$s^2 + \frac{s^2\sqrt{3}}{4} = \boxed{69.8}.$$

☐

§1.2.1 Calculator Geometry Problems

Problem 1. A cone is taken out of a cube. The remaining volume is 100. What is the side length of the cube?



Problem 2. Find the area of a regular pentagon with side length 3.

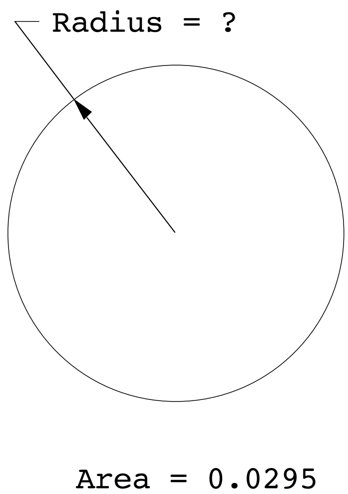
Problem 3. Find the area of a regular hexagon with side length 3.

Problem 4. Find the area of a octagon with side length 3.

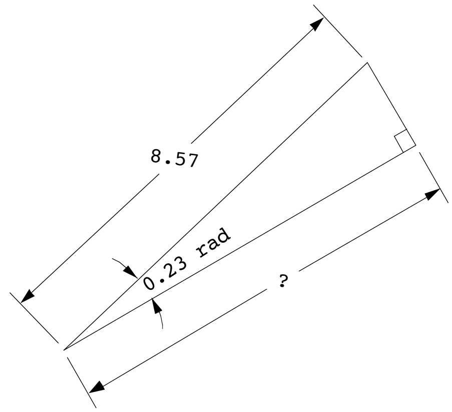
I have included some extra problems on the next page. **Please skip the problem in the bottom row and middle column and the problem in the bottom row and right column. They require hard methods, such as calculus.**

"Complementary" angles are two angles whose sum equals 90°. "Supplementary" angles are two angles whose

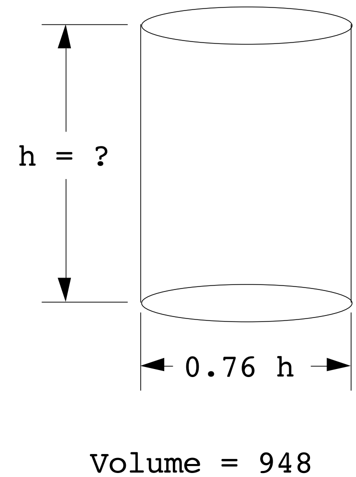
Page 1, Problems 9, 10
CIRCLE



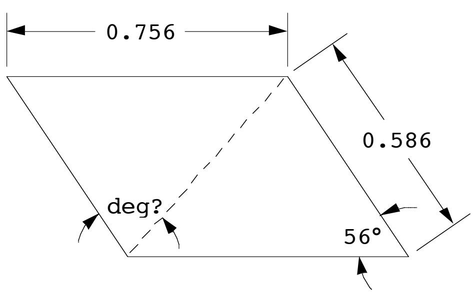
Page 2, Problems 19, 20
RIGHT TRIANGLE



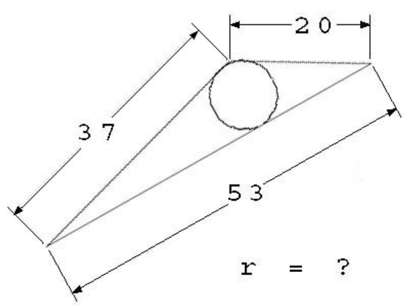
Page 3, Problems 29, 30
CYLINDER



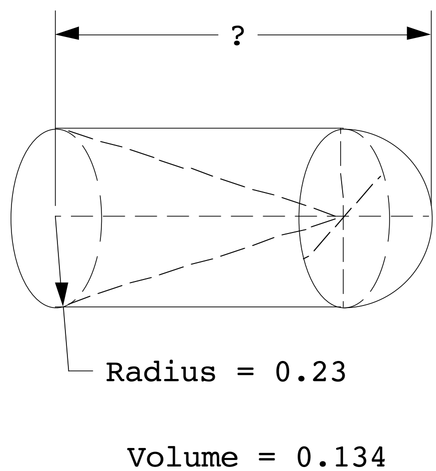
Page 4, Problem 39
PARALLELOGRAM



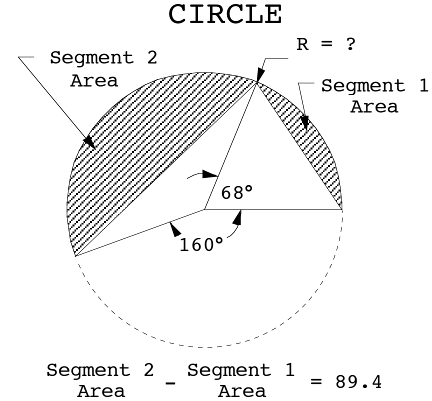
Page 4, Problem 40
SCALENE TRIANGLE



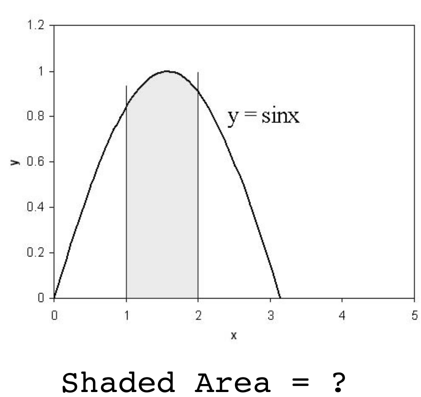
Page 5, Problems 49, 50
CYLINDER WITH CONICAL
CAVITY AND HEMISPHERE



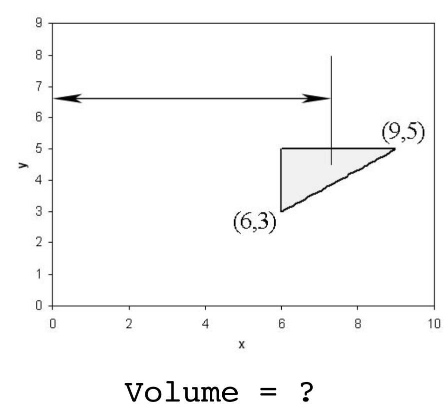
Page 6, Problem 59
Page 7, Problems 69, 70
CIRCLE



Page 6, Problem 60



Page 6, Problem 60
Solid of Revolution



§1.3 General Math: Sequences and Series

§1.3.1 Arithmetic Sequences and Series

Definition 1 (Arithmetic Sequence). An **arithmetic sequence** is a sequence of numbers in which each term is given by adding a fixed value to the previous term.

For example, $-2, 1, 4, 7, 10, \ldots$ is an arithmetic sequence because each term is three more than the previous term. In this case, 3 is called the **common difference** of the sequence. More formally, an arithmetic sequence a_n is defined recursively by a first term a_0 and $a_n = a_{n-1} + d$ for $n \geq 1$, where d is the common difference. Explicitly, it can be defined as $a_n = a_0 + dn$.

Theorem 1 (Terms of an Arithmetic Sequence). The nth term in an arithmetic sequence is described

$$a_n = a_1 + d(n-1),$$

where a_n is the nth term, a_1 is the first term, and d is the difference between consecutive terms.

Theorem 2 (Sum of an Arithmetic Sequence). The sum of the first n terms of an arithmetic sequences is

$$s_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2a_1 + (n-1)d).$$

Now for a few applications:

Theorem 3 (Sum of First n Positive Integers). For all positive integers n,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Theorem 4 (Sum of First n Even Integers). For all positive integers n,

$$2 + 4 + \ldots + 2n = n(n+1).$$

Theorem 5 (Sum of First n Odd Integers). For all positive integers n,

$$1 + 3 + 5 + \ldots + 2n - 1 = n^2.$$

§1.3.2 Geometric Sequences and Series

Definition 2 (Geometric Sequence). A **geometric sequence** is a sequence of numbers in which each term is a fixed multiple of the previous term.

For example: $1, 2, 4, 8, 16, 32, \ldots$ is a geometric sequence because each term is twice the previous term. In this case, 2 is called the common ratio of the sequence. More formally, a geometric sequence may be defined recursively by:

$$a_n = r \cdot a_{n-1}, n > 1,$$

with a fixed first term a_1 and common ratio r. Using this definition, the nth term has the closed-form:

$$a_n = a_1 \cdot r^{n-1}.$$

Theorem 6 (Sum of a Finite Geometric Sequence). The sum of the first n terms of a geometric sequence is given by

$$S_n = a_1 + a_2 + \cdots + a_n = a_1 \cdot \frac{r^n - 1}{r - 1},$$

where a_1 is the first term in the sequence, and r is the common ratio.

Definition 3 (Infinite Geometric Sequence). An **infinite geometric sequence** is a geometric sequence with an infinite number of terms.

If the common ratio is small, the terms will approach 0 and the sum of the terms will approach a fixed limit. In this case, "small" means $|r| < 1$. We say that the sum of the terms of this sequence is a convergent sum.

Theorem 7 (Sum of an Infinite Geometric Sequence). The general formula for the sum of such a sequence is

$$S = \frac{a_1}{1 - r}.$$

Example 5. Compute $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$.

Solution. Using the Sum of an Infinite Geometric Sequence formula, we get

$$\frac{1}{1 - \frac{1}{2}} = \boxed{2}.$$

□

One common instance of summing infinite geometric sequences is the decimal expansion of most rational numbers. For instance, $0.33333\ldots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \ldots$ has first term $a_0 = \frac{3}{10}$ and common ratio $\frac{1}{10}$, so the infinite sum has value $S = \frac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{1}{3}$, just as we would have expected.

§1.3.3 Sequences and Series Problems

Problem 1. Compute $1 + 2 + \ldots + 1000$.

Problem 2. Find the 16th term in the arithmetic sequence $1, 4, 7, 10, \ldots$.

Problem 3. Compute $2 + 4 + 6 + \ldots + 200$.

Problem 4. Compute $1 + 5 + 9 + 13 + \ldots + 101$.

Problem 5. Compute $(1+9+17+25+\ldots+97)-(2+3+4+\ldots+77)$.

Problem 6. Find the value of
$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots.$$

Problem 7. Verify that
$$0.11111\ldots=\frac{1}{9},$$
using $0.11111\ldots=\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\ldots$.

Problem 8. Find the 20th term in the geometric sequence $1024, 512, 256, \ldots$.