Counting Basics

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Contents

1 Introduction

1.1 Arithmetic

Let's do some counting.

Theorem 1.1 (Number of Consecutive Integers)
In the list $a, a+1, a+2, \ldots, b-1, b$, there are $b-a+1$ numbers.

Theorem 1.2 (Number of Integers in an Arithmetic Sequence)
In the list $a, a+d, a+2d, \ldots, b$, there are $\frac{b-a}{d}+1$ numbers.

Example 1.3
How many multiples of 3 are between 62 and 215?

Solution. The list is $63, 66, \ldots, 210, 213$, so there are $\frac{213-63}{3}+1=\boxed{51}$ numbers. □

1.2 Basic Principles

Theorem 1.4 (Addition Principle)
If event 1 can happen in a_1 ways, event 2 can happen in a_2 ways, etc., and event n can happen in a_n ways, and these n events **cannot** occur at the same time, then the number of ways events 1 through n can occur is $a_1 + a_2 + \ldots + a_n$ ways.

Example 1.5
Dylan will either choose one appetizer or one dessert, but not both appetizer and dessert. The restaurant offers four choices for appetizers and three choices for desserts. How many different choices does Dylan have for appetizer or dessert?

Solution. By the addition principle, we have $4 + 3 = \boxed{7}$. □

Theorem 1.6 (Multiplication Principle)
If event 1 can happen in a_1 ways, event 2 can happen in a_2 ways, etc., and event n can happen in a_n ways, and these n events **can** occur at the same time, then the number of ways events 1 through n can occur is $a_1 a_2 \ldots a_n$ ways.

Example 1.7
If there are 3 options for breakfast, 4 options for lunch, and 5 options for dinner, how many breakfast-lunch-dinner combos are there?

Solution. Using our multiplication principle, we get $3 \cdot 4 \cdot 5 = \boxed{60}$. □

1.3 What is constructive counting?

Constructive counting is when: instead of counting every single possible value, we try to consider individual parts and in the end combine them in some clever way.

For example, in the example about breakfast, lunch, and dinner above, it is not smart to try every single possibility. It is smarter to consider the options of breakfast individually, then lunch, then dinner.

1.4 What is casework?

Casework is the counting technique of splitting up a problem into a number of cases and determining the results of each case. Casework is used even outside of combinatorics.

1.5 What is complementary counting?

Complementary counting refers to the technique where we count the number of outcomes we don't need to count.

For example, say I have 12 apples and 5 are green and the rest are red. We would use complementary counting to say that there are 7 red apples.

2 Indications

The following is a list of when to use these techniques.

> *Remark 2.1.* Often, you will find that you need to combine these strategies. For example, if there are a lot of cases, you might want to count the cases that don't work and do casework on that instead.

2.1 Constructive Counting

Use constructive counting when:

- the problem can be split into parts,
- there is a general pattern, or
- each part (e.g. digit, letter, etc.) does not depend on each other.

2.2 Casework

Use casework when:

- there are many possibilities that fit some category, or
- you need to be careful in your counting.

2.3 Complementary Counting

Use complementary counting when:

- there are too many cases to check,
- the word "not" appears, or
- you want to check your work.

3 Walkthroughs

Example 3.1
How many 3-letter words can be formed from the standard 26-letter alphabet, if the first letter must be a vowel?

Walkthrough.

1. Do the letters depend on each other? In other words, does what I pick for the first letter change the second letter?
2. It does not! So count each part individually.
3. Multiply your parts.

Example 3.2 (Mathcounts)
How many different three-digit security codes are using the digits 1-5, if the second digit cannot be the same as the first, and third digit cannot be the same as the second?

Walkthrough.

1. How many ways are there to pick the first digit?
2. After we've picked the first digit, how many ways are there to pick the second?
3. What about third? Remember: the first and third digit can be the same.
4. Multiply these possibilities for the digits.

Example 3.3
How many sequences

$$x_1, x_2, x_3, \ldots, x_7$$

can be formed in which all the x_i are integers greater than 0 and less than 6, and no two adjacent x_i are equal?

Walkthrough. Let's consider each x_i individually.

1. Let's start by making a sequence from left to right. This means every time we find the number of possibilities for one of the x_i, we only have to consider the term before it (x_{i-1}).
2. How many possible values work for x_1?
3. What about x_2?
4. What about x_3?
5. The number of possibilities are the same for $x_2, x_3, \ldots, x_7$. Why?
6. Multiply the possibilities to get the final answer.

Example 3.4
In how many ways can you spell the word NOON in the grid below? You can start on any letter, then on each step you can move one letter in any direction (up, down, left, right, or diagonally). You cannot visit the same letter twice.

NNNN
NOON
NOON
NNNN

Walkthrough.

1. There are 12 possible N's to start at. Do you see any symmetry?
2. There are 2 cases: vertex N's and edge N's. Count each of these cases.
3. The rest of this problem isn't too hard. Be careful about counting though!

Example 3.5
How many of the natural numbers from 1 to 600, inclusive, contain the digit 5 at least once? (The numbers 152 and 553 are two natural numbers that contain the digit 5 at least once, but 430 is not.)

Walkthrough.

1. The “at least" is a clue to try complementary counting.
2. In other words, we count number of numbers with no 5’s at all, and subtract this from 600, since there are 600 numbers from 1 to 600.