Math Level 2 Week 2

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January 9, 2021

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1 Difference Of Squares

Theorem 1.1 (Difference Of Squares)
For any x, y, we have $x^2 - y^2 = (x - y)(x + y)$.

A few examples when this is useful:

- factoring polynomials and other groups of variables, and
- prime factorizations of various numbers.

> *Remark 1.2.* Difference of squares is extremely helpful for computations.

Corollary 1.3
For any a, b, we have $a \cdot b = (\frac{a+b}{2})^2 - (\frac{a-b}{2})^2$.

Here, instead of going from differences to products, we go from products to differences. This is very useful in number theory – you can use this to express the product of two factors as a difference.

1.1 Examples

Example 1.4 (Mathcounts Chapter Sprint 2017/26)
Find the value of x that makes

$$55 \times 59 - 53 \times 57 = x^2 - 1$$

true.

Walkthrough. A few hints:

1. How can you rewrite 55×59 and 53×57 using what we've learned?
2. Note that the 2^2 cancels. What can you do to $57^2 - 55^2$ to easily compute it without multiplying out 57^2 and 55^2?
3. Find $x^2 - 1$.

Example 1.5 (ARML Local Individual 2020/1)
There are four distinct prime factors of 999039. Compute the sum of these four prime factors.

Some motivation: the first step here is to notice that 999039 is a very large number. If a contest problem asks you to explicitly factorize some large number, then it likely has a simple trick. Note that 999039 is very close to $1000000 = 1000^2$. Because of this, we might think of difference of squares. Checking, we see that the difference between 1000^2 and 999039 is $961 = 31^2$.

Solution. Applying difference of squares, we get

$$999039 = 1000^2 - 31^2 = 1031 \cdot 969.$$

Now, 969 is obviously a multiple of 3 because its digits sum to 24. Then, $969 = 323 \cdot 3$. If you memorized your squares, note that 323 is very close to $324 = 18^2$. Thus, $323 = 18^2 - 1 = 17 \cdot 19$. We're given that there are 4 distinct prime factors and we already found 3: $3, 17, 19$. This implies 1031 is the last prime factor.

Summing these up, we get $1031 + 3 + 17 + 19 = 1060$. □

2 Quadratics

We're going to prove the quadratic formula by ***completing the square***.

Theorem 2.1 (Quadratic Formula)
The solutions to $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Proof. We do a series of manipulations

$$ax^2 + bx + c = 0,$$

$$a^2x^2 + abx + ac = 0,$$

$$(ax + \frac{b}{2})^2 + ac - \frac{b^2}{4} = 0,$$

$$(ax + \frac{b}{2})^2 = \frac{b^2 - 4ac}{4},$$

$$\begin{aligned} ax + \frac{b}{2} &= \pm\sqrt{\frac{b^2-4ac}{4}}, \\ ax &= \frac{-b}{2} \pm \frac{\sqrt{b^2-4ac}}{2}, \\ x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}, \end{aligned}$$

giving us the formula. □

You will want to use the formula if the quadratic is tricky to factor or just doesn't have integer factors. The quadratic formula is usually an intermediate step of a problem.

2.1 Vieta's Formulas

Theorem 2.2 (Sum and Product of Quadratic Roots)
The sum of the roots of $ax^2 + bx + c = 0$ is $-\frac{b}{a}$. The product of the roots of $ax^2 + bx + c = 0$ is $\frac{c}{a}$.

Proof. Let the roots of $ax^2 + bx + c = 0$ be r_1, r_2. Then, we have

$$ax^2 + bx + c = a(x - r_1)(x - r_2) = ax^2 - a(r_1 + r_2)x + ar_1r_2.$$

Two polynomials are identical if and only if the coefficients of each monomial x^k are also identical. Thus we have $-a(r_1 + r_2)x = b$, which implies $r_1 + r_2 = -\frac{b}{a}$, and $ar_1r_2 = c$, which implies $r_1r_2 = \frac{c}{a}$. □

These are useful when you get information about roots from the coefficients or vice versa. This is the **heart of algebraic manipulation**: find the value of some combination of variables without knowing the values of the variables directly.

> *Remark 2.3.* Note that the sum and products count multiplicities. For example, the sum of the roots of $(x-3)^2 = x^2 - 6x + 9 = 0$ is considered 6 even though there's only one distinct root of 3. The 3 is counted twice because $x - 3$ divides $x^2 - 6x + 9$ twice. Furthermore, it counts complex roots, not just real roots.

2.2 Examples

Example 2.4 (AHSME 1984/24)
If a and b are positive real numbers and each of the equations $x^2 + ax + 2b = 0$ and $x^2 + 2bx + a = 0$ has real roots, then find the smallest possible value of $a + b$.

Solution. By the Quadratic Formula Corollary that we looked at before, we have that $a^2 - 8b \geq 0$ and $4b^2 - 4a \geq 0$. Now, note that $4b^2 - 4a \geq 0$ implies $b^2 \geq a$, or $b^4 \geq a^2$.[1] Also note that $a^2 - 8b \geq 0$ implies $a^2 \geq 8b$. Thus $b^4 \geq a^2 \geq 8b$, which implies that $b^3 \geq 8$ or $b \geq 2$.[2]

So, we now have $a^2 \geq 8b \geq 16 \implies a \geq 4$ (again, because a is positive). So, the minimum a and b are apparently $(2, 4)$. You should double-check that these work by plugging them into the equations. So, our final answer is $2 + 4 = 6$. □

Many of our algebraic manipulations are only true because a, b were given as positive numbers. Check to make sure what you're doing is valid!

[1] This is because a, b are positive. This does not necessarily hold if a, b are negative.
[2] Again, note that we can only go from $b^4 \geq 8b$ to $b^3 \geq 8$ because b is positive.

Example 2.5 (AMC 10A 2020/5)
What is the sum of all real numbers x for which $|x^2 - 12x + 34| = 2$?

Walkthrough. The absolute value is annoying. Let's start by getting rid of it:

1. This gives you two quadratics. What are those two?
2. Before diving into the Quadratic Formula, think about factoring.
3. After finding the factors, add them up. We're done!
4. Think about why you can't just apply Vieta's. (Hint: Multiplicity!) This is a very important distinction that could cost you on contests.

3 Simon's Favorite Factoring Trick

Simon's Favorite Factoring Trick, a.k.a. **completing the rectangle**, is often abbreviated as ***SFFT***.

Theorem 3.1 (SFFT)
For any x, y, a, b, we have $xy + ax + by = (x+b)(y+a) - ab$.

While this identity is true for all x, y, a, b, it's mainly useful in number theory problems where you want to count the number of integer (x, y) satisfy some equation or find what kind of (x, y) that satisfy some equation.

Example 3.2 (AMC 10A 2015/20)
A rectangle with positive integer side lengths in cm has area A cm^2 and perimeter P cm. What is the least integer greater than or equal to 100 that can't be written as $A + P$?

Walkthrough. Note this is a NT problem disguised as geometry. It is easy to tell this by the key words "integer side lengths".

1. If you let l, h be the length and height of the rectangle, what are A and P?
2. How can you rewrite $A + P$ using SFFT?
3. If $A + P = k$ for some positive integer k, how would you find l, h?
4. Then, for what k is it impossible for $A + P = k$ for any l or h? That is, what k will make the process for finding l, h you discovered before fail?
5. Now, we just find the smallest k larger than or equal to 100 with that property to get the answer. We're done!

4 Problems

Problem 1 (AHMSE 1962/11). Find the difference between the larger root and the smaller root of $x^2 - px + \frac{p^2-1}{4} = 0$.

Problem 2. Let x, y be nonnegative real numbers such that $x + y = 5$ and $xy = 7$. Find $\frac{x}{y-1} + \frac{y}{x-1}$.

Problem 3 (Mathcounts State Sprint 2013/7). What is the value of $\frac{444^2-111^2}{444-111}$?

Problem 4. Find the value of $1000 \cdot 20 + 492 \cdot 488$.

Problem 5 (AMC 10A 2008/7). The fraction

$$\frac{\left(3^{2008}\right)^2 - \left(3^{2006}\right)^2}{\left(3^{2007}\right)^2 - \left(3^{2005}\right)^2}$$

simplifies to what value?

Problem 6 (AMC 10B 2017/2). Sofia ran 5 laps around the 400-meter track at her school. For each lap, she ran the first 100 meters at an average speed of 4 meters per second and the remaining 300 meters at an average speed of 5 meters per second. How many seconds did it take Sofia to run the 5 laps?

Problem 7. How many polynomials of the form $x^2 + mx + 28$ are there with positive integer roots?

Problem 8. If $x + \frac{1}{x} = 3$, find $x^2 + \frac{1}{x^2}$.

Problem 9 (AHMSE 1993/19). How many ordered pairs (m, n) of positive integers are solutions to

$$\frac{4}{m} + \frac{2}{n} = 1?$$

Problem 10 (BmMT 2018). Let x be a positive real number so that $x - \frac{1}{x} = 1$. Compute $x^8 - \frac{1}{x^8}$.

Problem 11 (djmathman's Mock AMC 2013/9). Let p and q be numbers with $|p| < 1$ and $|q| < 1$ such that

$$p + pq + pq^2 + pq^3 + \cdots = 2 \text{ and } q + qp + qp^2 + \cdots = 3.$$

What is $100pq$?

Problem 12 (HMMT November 2012/1). What is the sum of all of the distinct prime factors of $25^3 - 27^2$?

Problem 13 (AMC 10A 2018/10). Suppose that real number x satisfies

$$\sqrt{49 - x^2} - \sqrt{25 - x^2} = 3.$$

What is the value of $\sqrt{49 - x^2} + \sqrt{25 - x^2}$?