Math Level 2 Handout

Dylan Yu

March-April 2020

1 Basic Analytic Geometry

Don't worry - **analytic geometry** is just a fancy way of saying "geometry with coordinates". We will be looking at a few formulas, definitions, and theorems today, as well as attempting a few problems.

1.1 Linear Equations

A linear equation is an equation with variables that have degree 1, meaning they are not squared, or cubed, square rooted, etc. A system of linear equations is a set of two or more linear equations. The slope of a linear equation is the rate of change. For example, if a point is moved two to the right and four up, we say the slope is $\frac{4}{2} = 2$. In general, for a point (x_1, y_1) and another point (x_2, y_2), the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope usually uses the variable m. There are a couple of ways you will see two-variable linear equations written:

1. Standard Form: $ax + by + c = 0$, or sometimes $ax + by = c$, where a, b, c are constants, and $-\frac{a}{b}$ is the slope.
2. Slope-Intercept Form: $y = mx + b$, where m is the slope and b is the y-intercept, which is where the y-axis intersects the line. Similarly the x-intercept is the point where the x-axis intersects the line.
3. Point-Slope Form: $y - y_0 = m(x - x_0)$, where (x_0, y_0) is a point on the line, and m is the slope of the line.

The following is a list of ways to solve these linear equations.

Method 1. (Substitution Method) Given two equations, solve for one variable in one equation, and replace that variable with the other variable in all the other equations. For example, if you have the equations

$$4x + 2y = 6,$$

$$5x + y = 6,$$

you can solve for y in the first one:

$$y = 3 - 2x,$$

then plug it into the second equation:

$$5x + 3 - 2x = 6,$$

$$3x + 3 = 6,$$

$$x = 1,$$

$$y = 3 - 2 \cdot 1 = 1.$$

Method 2. (Elimination Method) If we subtract two equations, and one of the variables has the same coefficient (which means the number in front of the variable; for example, $5x$ has a coefficient of 5) in both equations, then it will cancel out and we will be left with one less variable. For example, if you have the equations

$$4x + 2y = 6,$$

$$5x + y = 6,$$

then if you multiply equation 2 by 2, you get

$$10x + 2y = 12,$$

and if we subtract equation 1 from equation 2, you get

$$10x + 2y - (4x + 2y) = 12 - 6 = 6,$$

$$10x - 4x + 2y - 2y = 6,$$

$$6x + 0 = 6,$$

$$x = 1.$$

Plugging this back into the second equation, we get

$$5 \cdot 1 + y = 6,$$

$$y = 1.$$

Method 3. (Graphing Method) This method is less common, because it requires you to be extremely **precise**. The method is just what it sounds like - you graph the equations, and see where they intersect.

A method we did not talk about was **the matrix method**. This will likely not appear in Mathcounts, AMC8, or TMSCA; it certainly hasn't appeared for me. The only time a *matrix* will appear is at the very end of a General Math test, and even then there is only ever one question.

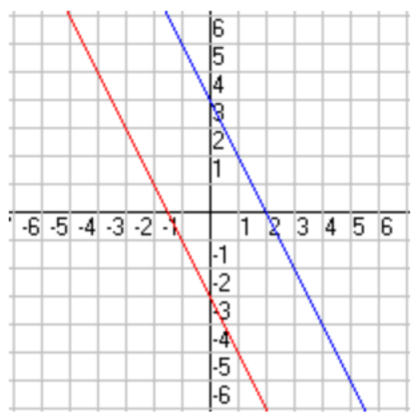
There a few possibilities when solving a pair of linear equations. You can either get **0 solutions**, **1 solution**, or **infinitely many solutions**. We will consider each case below:

- <u>0 Solutions:</u> This means the lines are **parallel**. For example, if we have the equations

$$y = -2x + 4,$$

$$y = -2x - 3,$$

because they have the same **slope**, the equations are parallel. If you think about this intuitively, if they increase at the same rate, of course they can never reach each other! For example, if in a race between two people, someone has a head start, then the person without the head start can never catch up. The blue line below is for $y = -2x + 4$, and the red line below is for $y = -2x - 3$.

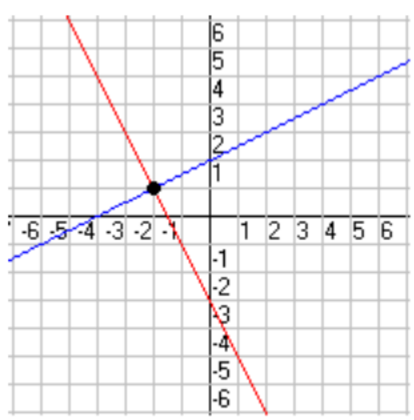


- <u>1 Solution:</u> This is usually the type of problems you will see on competitions. The following shows the graph for

$$y = -0.5x + 2,$$

and

$$y = -2x - 3.$$

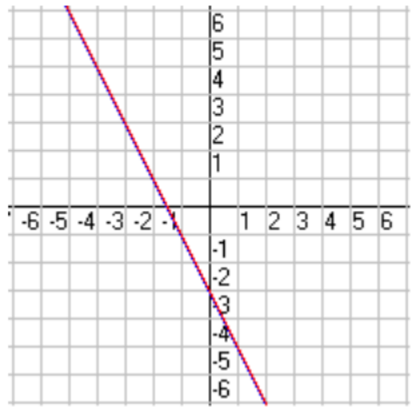


- <u>Infinitely Many Solutions:</u> This can only occur for linear equations if **they are the same line**. For example, in the lines

$$y = -2x - 4,$$

$$y + 4 = -2x,$$

they may seem different, but by subtracting 4 in the second equation they become the same equation. The following is the graph for this:



1.2 Graphing Common Objects

There are many functions and shapes you can graph on the **Cartesian Coordinate**, which is basically just the xy-plane. However, we will talk about the ones that most commonly show up on tests:

- Point
- Line (already covered!)
- Circle
- Parabola

Let us start with the **point**.

Theorem 1.1. (Distance between Two Points) The distance between two points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Theorem 1.2. (Midpoint of Two Points) The midpoint of two points (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Theorem 1.3. (Perpendicular and Parallel Lines) Two lines are **perpendicular** if and only if the product of the slopes is -1. Two lines are **parallel** if and only if their slopes are the same.

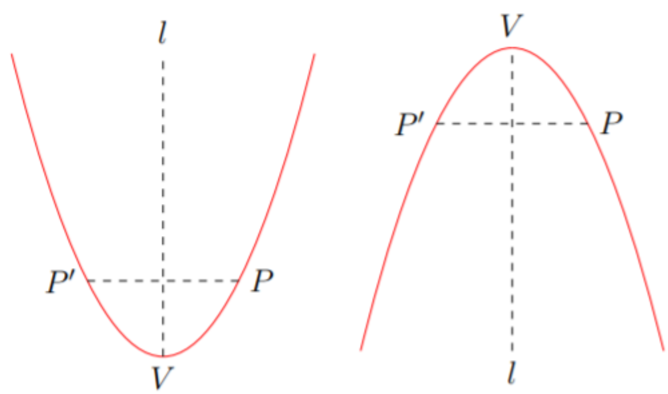
The circle moves us to squares.

Theorem 1.4. (Graph of a Circle) The graph of a circle is $(x-h)^2+(y-k)^2=r^2$, where (h,k) is the center of the circle, and r is the radius of the circle. **Tangent lines** are perpendicular to the radius, which means given a point on the circle and the center of the circle, you can find the tangent line and the equation of the circle.

In the last section, we talked about **quadratics**. Well, a parabola is the graph of a quadratic. To graph a quadratic, follow these steps:

1. Determine the **vertex** of the parabola.
2. Find a point on the parabola (for example, plug in $x=0$ and see if that yields a nice value). Note that a parabola is **symmetric**, which means reflecting the point on the parabola across the vertical line going through the vertex will give us another point.
3. Determine the **sign** of the parabola. If $a>0$, the sign is positive, and the parabola *opens up*. If $a<0$, the sign is negative, and the parabola *opens down*.
4. Draw a "U" through the vertex and the three points we just determined if $a>0$, and if $a<0$, draw an upside down "U" through these three points.

The steps above are illustrated in the next two diagrams:



In these two diagrams, V is the vertex, P is the point on the parabola, P' is its reflection, and l is the vertical line going through the vertex.

1.3 Problems

Problem 1.1. Find the slope of the line that goes through $(5,7)$ and $(6,8)$.

Problem 1.2. Find the slope of the line with a y-intercept of 3 and a x-intercept of 4.

Problem 1.3. At what point do the lines $2x+9y=7$ and $x=32-4.5y$ intersect?

Problem 1.4. Find the intersection of the lines $y = ax + b$ and $y = cx + d$ in terms of a, b, c, d, given that they are not parallel.

Problem 1.5. (Mathcounts) Chris graphs the line $y = 3x + 7$ in the coordinate plane, while Sebastian graphs the line $y = ax + b$, for some numbers a and b. The x-intercept and y-intercept of Sebastian's line are double the x-intercept and y-intercept of Chris's line, respectively. What is the value of the sum $a + b$?

Problem 1.6. Let r_1 and r_2 be the roots of the quadratic $x^2 - 5x + 6$. Let the parabola $x^2 - 5x + 6$ go through the x-axis at $(p_1, 0)$ and $(p_2, 0)$. What is $r_1 + r_2 - p_1 - p_2$?

Problem 1.7. Find the center and radius of the circle with equation

$$x^2 + y^2 = 6x + 8y.$$

At what points does it intersect the x-axis and y-axis?

Problem 1.8. A tangent line of a circle is a line that intersects a circle at exactly one point. What is the equation of a tangent line that is tangent to a circle with equation $x^2 + y^2 = 6x + 8y$ at $(0, 0)$? This requires you know that a tangent line is perpendicular to the line that goes through the tangent point and the center of the circle.

Problem 1.9. Find the equation of the line in which all the points on the line are equidistant (meaning same distance) from the points $(1, 2)$ and $(3, 4)$.

Problem 1.10. What is the distance between the points $(1, 2)$ and $(4, 6)$?

Problem 1.11. Let the vertices of a rectangle be at the points $(-1, -1), (-1, 1), (3, 1), (3, -1)$. Find the coordinates of the intersection of the diagonals. (This is a preview of tomorrow's lesson on **Advanced Coordinate Geometry**.)