Math Level 2 Handouts Week 05

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§ 1 Overcounting

§ 1.1 Introduction

Sometimes, it is much easier to hone in on the answer (i.e. adding and subtracting) than to count the answer directly. This is when we use **overcounting**. Let us try a motivating example:

> **Example 1.** How many numbers less than or equal to 100 are divisible by either 2 or 3?

Solution. Clearly, there are 50 numbers less than 100 that are divisible by 2, and 33 that are divisible by 3. However, we note that we overcount several numbers, such as 12, which is divisible by both 2 and 3. To correct for this overcounting, we must subtract out the numbers that are divisible by both 2 and 3, as we have counted them twice. A number that is divisible by both 2 and 3 must be divisible by 6, and there are 16 such numbers. Thus, there are $50 + 33 - 16 = \boxed{67}$ numbers that are divisible by either 2 or 3. □

Note that it is not a coincidence that 67 is close to two thirds of 100! We can approach this problem in a constructive way, building the set based on the remainders when divided by 3, but that is a different subject.

§ 1.2 Venn Diagrams

Definition 1 (Set). A **set** is a well-defined collection of distinct objects, considered as an object in its own right.

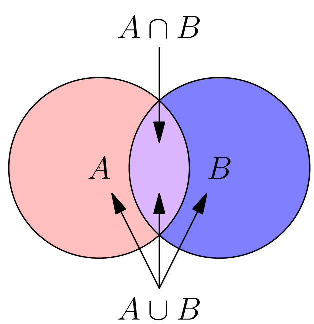
Definition 2 (Element). An **element**, or member, of a set is any one of the distinct objects that make up that set.

For example, if we have a set S, then we could have $S = \{1, 2, 3, 4, 5, 6, 7\}$, so the elements are $1, 2, 3, 4, 5, 6, 7$.

Definition 3 (Venn Diagram). A **Venn diagram** is a diagram that shows the *intersection* (represented with $\cap$) and *union* (represented with $\cup$) of two sets.

Definition 4 (Intersection & Union). The **intersection** of two sets is the part they have in common, and the **union** of two sets is the part they have in total.

Below is a representation of two sets A and B:



Example 2 (AoPS). How many possible distinct arrangements are there of the letters in the word BALL?

Solution. Let's pretend the two L's are different. In other words, in the word BAL_1L_2, there are $4! = 24$ ways to arrange the letters. However, notice that in reality the two L's **are** the same. This means that every time we do BAL_1L_2, we get the same word as BAL_2L_1. This means that we are always counting **twice** the number of arrangements, so we must divide by 2 to get $\frac{24}{2} = \boxed{12}$. □

§ 1.3 Counting in Pairs

Sometimes, if we count the numbers in pairs, we can solve the problem easier. Here are a few examples:

Example 3. Find a formula for the sum of the first n positive integers.

Solution. Let's write

$$S = 1 + 2 + \ldots + n.$$

If we reverse the order, we get

$$S = n + (n-1) + \ldots + 1.$$

If we add these two up, and match the numbers, we get

$$2S = (n+1) + (n+1) + \ldots + (n+1).$$

Now we have to count how many $n+1$'s there are. This is easy, since there were n terms, so

$$2S = n(n+1),$$

so

$$S = \boxed{\frac{n(n+1)}{2}}.$$

□

This is actually how Gauss solved $1 + 2 + \ldots + 100 = 5050$ so fast when he was 10!

§ 1.4 Symmetric Counting

This is actually a generalization of Counting in Pairs. A lot of the examples here are regarding **circles**.

Example 4 (AoPS). In how many distinct ways can 4 keys be placed on a keychain? Two arrangements are not considered different if the keys are in the same order. Note that the keychain can be flipped, but this isn't considered different from the original arrangement.

Solution. Due to *rotational symmetry*, we must first fix one of the keys, then arrange the rest. Thus, we get $3! = 6$. However, for every arrangement, there is also a reflection. Thus, the true answer is $\frac{6}{2} = \boxed{3}$. □

§ 1.5 Principle of Inclusion-Exclusion

Principle of Inclusion-Exclusion, abbreviated as PIE, is used to systematically overcount and undercount.

Theorem 1 (PIE for Two Variables). Let A and B be two sets, and let $A \cap B$ be the intersection of the two sets, and let $A \cup B$ be the union of the two sets. Then,

$$\#(A \cup B) = \#A + \#B - \#(A \cap B),$$

where # means the number of elements in a set.

Theorem 2 (PIE Generalized). Let $A_1, A_2, \ldots, A_n$ be n sets. Then,

$$\#(A_1 \cup A_2 \cup \ldots \cup A_n) = \#A_1 + \#A_2 + \ldots + \#A_n - (\#(A_1 \cap A_2) + \#(A_1 \cap A_3) + \ldots + (A_{n-1} \cap A_n)) + \ldots - (-1)^n \#(A_1 \cap A_2 \cap \ldots \cap A_n).$$

Example 5 (AIME I 2002/1). Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Consider the three-digit arrangement, $\overline{aba}$. There are 10 choices for a and 10 choices for b (since it is possible for $a = b$), and so the probability of picking the palindrome is $\frac{10\times 10}{10^3} = \frac{1}{10}$. Similarly, there is a $\frac{1}{26}$ probability of picking the three-letter palindrome. By the Principle of Inclusion-Exclusion, the total probability is

$$\frac{1}{26} + \frac{1}{10} - \frac{1}{260} = \frac{35}{260} = \frac{7}{52} \implies 7 + 52 = \boxed{059}.$$

□

Example 6 (AMC 10B 2017/13). There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

Solution. By PIE, we have

$$|A_1 \cup A_2 \cup A_3| = \sum |A_i| - \sum |A_i \cap A_j| + |A_1 \cap A_2 \cap A_3|.$$

Number of people in at least two sets is $\sum |A_i \cap A_j| - 2|A_1 \cap A_2 \cap A_3| = 9$. So, $20 = (10+13+9) - (9+2x) + x$, which gives $x = \boxed{3}$. □

Let's try a super hard example:

Example 7 (AMC 12 2001/16). A spider has one sock and one shoe for each of its eight legs. In how many different orders can the spider put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe?

Solution. Let the spider try to put on all 16 things in a random order. Each of the 16! permutations is equally probable. For any fixed leg, the probability that he will first put on the sock and only then the shoe is clearly $\frac{1}{2}$. Then the probability that he will correctly put things on all legs is $\frac{1}{2^8}$. Therefore the number of correct permutations must be $\boxed{\frac{16!}{2^8}}$. □

§ 1.6 Problems

Problem 1 (AoPS). How many possible distinct arrangements are there of the letters in the word PAPA?

Problem 2 (AoPS). How many possible distinct arrangements are there of the letters in the word TATTER?

Problem 3. How many possible distinct arrangements are there of the letters in the word BANANA?

Problem 4 (AoPS). A round-robin tennis tournament consists of each player playing ever other player exactly once. How many matches will be held during an 8-person round-robin tennis tournament? What is wrong with the following idea?:

Each of the eight players plays 7 games, so there are $8 \times 7 = 56$ total games played.

Problem 5 (AoPS). A club has 15 members and needs to choose 2 members to be co-presidents. In how many ways can the club choose its co-presidents?

Problem 6 (AoPS). Find a formula for the sum of the first:

(a) n even integers, and

(b) n odd integers.

Problem 7 (AoPS). A sports conference has 14 teams in two divisions of 7. How many games are in a complete season for the conference if each team must play every other team in its own division twice and every team in the other division once?

Problem 8 (Mathcounts National Target 2008/1). How many numbers less than or equal to 100 are divisible by 2 or 3 but not 4?

Problem 9 (AoPS). Find the number of diagonals in a n-sided convex polygon.

Problem 10 (AoPS). In how many different ways can 6 people be seated at a round table? Two seating arrangements are considered the same if, for each person, the person to his or her left is the same in both arrangements. Can we develop a formula?

Problem 11 (AoPS). In how many distinct ways can 5 keys be placed on a keychain? Two arrangements are not considered different if the keys are in the same order. Note that the keychain can be flipped, but this isn't considered different from the original arrangement. From here, can we develop a formula?

Problem 12 (AoPS). A Senate committee has 5 Democrats and 5 Republicans. In how many ways can they sit around a circular table:

(a) without restrictions?

(b) if all the members of each party all sit next to each other?

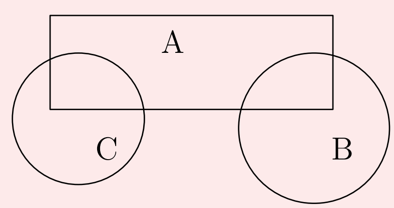
(c) if each members sits next to two members of the other party?

Problem 13 (AoPS). In how many ways can we seat 8 people around a table if Alice and Bob:

(a) want to sit next to each other?

(b) refuse to sit next to each other?

Problem 14 (AMC 8 2009/16). Three flower beds overlap as shown. Bed A has 500 plants, bed B has 450 plants, and bed C has 350 plants. Beds A and B share 50 plants, while beds A and C share 100. What is the total number of plants?



Problem 15 (AMC 8 2011/6). In a town of 351 adults, every adult owns a car, motorcycle or both. If 331 adults own cars 45 adults own motorcycles, how many of the car owners do not own a motorcycle?