Math Level 2 Handout

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0 Introduction

0.1 Competitions

Throughout your years of math contests, the following are likely to be contests you all will attend at some point or another:

1. Mathcounts: Similar to Mathleague, in that it has
   - Sprint (30 questions in 40 minutes),
   - Target (8 questions in 24 minutes, divided into sets of 2 questions),
   - Team (10 questions [as a team] in 20 minutes), and
   - Countdown (two people go head to head for a couple of 45 second questions).

   The questions are quite hard at the higher divisions, and speed plays a big role into how well you do.
2. AMC8: 25 questions in 40 minutes
3. TMSCA: There are 3 math contests (and 1 science contest, discussed in Level 1 of the science class) in TMSCA:
   - Number Sense: 80 questions in 10 minutes - these are done by rapid mental math, and so you must be extremely fluent in your Number Sense Rules.
   - Calculator: 80 questions in 30 minutes done with a calculator - this competition is (in my opinion) not as important as the others, but to do well here you must be able to use a calculator extremely fast to solve problems.
   - General Math: 50 questions in 40 minutes - this is lower difficulty than Mathcounts, but definitely still a challenge. In order to do well, you must learn to the earlier problems faster as well as understand the theory behind the harder ones.

For now, when I refer to Mathcounts, I am directing it towards the Chapter competition, which is around the level of this class.

0.2 Fundamentals and Strategies of Each Competition

1. Mathcounts: Focus on training your speed and accuracy - these problems need to be done once and done correctly. I would recommend doing Chapter tests to test our speed, and reviewing each question to make sure you know how to do all of them.

2. AMC8: The questions are harder than in Mathcounts, but the time per problem ratio is larger. Simply doing past problems will increase your speed a lot, and even allow you to have time to check your work.

3. TMSCA: I will consider each contest individually, due to the diversity between them:

   - Number Sense: Bryant Heath has a great manual on almost every trick you need to know for this competiton (`http://bryantheath.com/files/2018/04/Heath_NSTricks_revA.pdf`), and simply by reviewing that you can score a near perfect score. This requires a certain amount of training to achieve, but by doing questions over and over again you can train yourself to attempt these problems extremely fast.
   - Calculator: Like Number Sense, you must do questions many, many times to develop the muscle memory to use your calculator fast. In terms of calculator choices: when I competed, I used a HP35, because it allowed me to do calculations extremely fast. These calculators funciton **differently** than a regular calculator, so it is necessary that you practice with them beforehand.
   - General Math: We will be focusing on this section of the TMSCA contest, because while I cannot teach you *speed*, I can teach you the *theory* to give you the knowledge necessary to do well in competitions.

1 Basic Algebra

1.1 Ratios and Proportions

A ratio shows the relative sizes of two or more values. For example, if there are 2 apples, 4 oranges, and 6 bananas, the *ratio* of apples to oranges to bananas is $2 : 4 : 6 = 1 : 2 : 3$. The : symbol represents the word "to".

A proportion is a statement that two *ratios* are equal. For example, the following is a proportion:

$$\frac{12}{15} = \frac{4}{5}.$$

Two numbers x, y are said to be directly proportional if

$$y = kx$$

for some constant k. For example, if $k = 5$, then if x is 3, y must be 15, and if x is 4, y must be 20. Two numbers a, b are said to be indirectly proportional if

$$ab = k$$

for some constant k. For example, if $k = 5$, then if x is 1, then y is 5, and if x is 2, then y is $\frac{5}{2}$.

1.2 d = rt

By now, most of you should be familiar with the formula

$$\text{distance} = \text{rate} \times \text{time}.$$

This implies that distance and rate are **directly proportional** when time is constant, distance and time are **directly proportional** when rate is constant, and rate and time are **indirectly proportional** when distance is constant.

1.3 Simon's Favorite Factoring Trick

Simon's Favorite Factoring Trick (SFFT) is best explained with an example:

Example 1. Find all positive integers x, y that satisfy

$$xy - 2x - 4y = 0.$$

Solution. Let us factor the first two terms:

$$x(y - 2) - 4y = 0.$$

We want to find *some way* we can turn the y into a $y - 2$. Let's see what happens if we do that:

$$x(y - 2) - 4(y - 2).$$

Now that we've done it, we can factor! However, because we added a -2 we also have to $+2$:

$$x(y-2)-4(y-2+2).$$

Remember, this is still equal to 0. Now, let's take out the $+2$ and distribute it to the other side:

$$x(y-2)-4(y-2)-8=0,$$

$$x(y-2)-4(y-2)=8.$$

Now, we can factor:

$$(x-4)(y-2)=8.$$

Because x, y are positive integers, we know that $x-4$ and $y-2$ are simply the positive factors of 8:

$$x-4=1, y-2=8,$$

$$x-4=2, y-2=4,$$

$$x-4=4, y-2=2,$$

$$x-4=8, y-2=1.$$

Solving, we get $\boxed{(x,y)=(5,10),(6,6),(8,4),(12,3)}$.

Now for the formal statement:

Theorem 1. (SFFT) For all real numbers (although commonly used only for integers) x, y, a, b,

$$xy+xa+yb+ab=(x+a)(y+bk).$$

Two special common cases are: $xy+x+y+1=(x+1)(y+1)$ and $xy-x-y+1=(x-1)(y-1)$.

1.4 Word Problems

These are the problems that use **words** instead of equations. For these types of problems - just convert back to equations! Examples will be given in the **Problems Section**.

1.5 Problems

The following problems are listed in (roughly) increasing order of difficulty / topic. Any problems we do not finish in class should be attempted at home.

Problem 1. (Mathcounts Ratio Warmup P1) What is $\frac{9}{30}$ as a common fraction?

Problem 2. (Mathcounts Ratio Warmup P2) In a pasture there are 12 white horses and the rest are black. If there are 52 horses in the pasture, what fraction are black? Express your answer as a common fraction.

Problem 3. (Mathcounts Ratio Warmup P5) There are three times the number of orange fish as blue fish in a tank at the pet store, and there are no other fish. What percentage of the fish are orange?

Problem 4. (Mathcounts Ratio Stretch P4-6) Fairy Godmother has granted wishes to Aurora, Belle and Cindi in the ratio $6:8:11$. What fraction of the ratios were granted to Belle? What percent of the wishes granted by Fairy Godmother were not granted to Aurora? What is the absolute difference between the percents of wishes Fairy Godmother has granted to Aurora and to Cindi?

Problem 5. (Mathcounts 2019 Chapter P8) After a brisk workout, Felicia counts 32 heartbeats in 15 seconds. Based on this count, what is Felicia's expected number of heartbeats in one minute?

Problem 6. (2018 AMC 8 P1) An amusement park has a collection of scale models, with a ratio $1:20$, of buildings and other sights from around the country. The height of the United States Capitol is 289 feet. What is the height in feet of its duplicate to the nearest whole number?

(A) 14 **(B)** 15 **(C)** 16 **(D)** 18 **(E)** 20

Problem 7. (2010 AMC8 P2) If $a@b = \frac{a \times b}{a+b}$ for a, b positive integers, then what is $5@10$?

(A) $\frac{3}{10}$ **(B)** 1 **(C)** 2 **(D)** $\frac{10}{3}$ **(E)** 50

Problem 8. (2004 AMC8 P3) Twelve friends met for dinner at Oscar's Overstuffed Oyster House, and each ordered one meal. The portions were so large, there was enough food for 18 people. If they shared, how many meals should they have ordered to have just enough food for the 12 of them?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 15 **(E)** 18

Problem 9. (2013-2014 TMSCA GM #5 P10)

10. Mike and Molly start a movie at 9 pm. The movie is 2.4 hours long. While they are watching their movie, the take a 15 minute break to make some popcorn and then another 15 minute break to go to the restroom. At what time will they have completed watching the movie?

A. 11:54 pm B. 10:59 pm C. 11:24 pm D. 11:39 pm E. 12:09 am

Problem 10. (2005-2006 TMSCA GM #6 P2)

2. How many complete miles can Daisy run in one hour if she can run one mile in 5 minutes and 20 seconds?

A. 11.25 B. 11 C. 12 D. 10 E. NOT

Problem 11. (2005-2006 TMSCA GM #6 P25)

25. An airplane flies 3000 miles in 4 hours with the wind, but it takes 5 hours to make the return trip against the wind. What is the speed of the wind?
A. 600 mph B. 675 mph C. 75 mph D. 750 mph E. NOT

Problem 12. (2007 AMC8 P6) The average cost of a long-distance call in the USA in 1985 was 41 cents per minute, and the average cost of a long-distance call in the USA in 2005 was 7 cents per minute. Find the approximate percent decrease in the cost per minute of a long- distance call.

(A) 7 (B) 17 (C) 34 (D) 41 (E) 80

Problem 13. (2010 AMC8 P8) As Emily is riding her bicycle on a long straight road, she spots Emerson skating in the same direction $1/2$ mile in front of her. After she passes him, she can see him in her rear mirror until he is $1/2$ mile behind her. Emily rides at a constant rate of 12 miles per hour, and Emerson skates at a constant rate of 8 miles per hour. For how many minutes can Emily see Emerson?

(A) 6 **(B)** 8 **(C)** 12 **(D)** 15 **(E)** 16

Problem 14. (2017 AMC8 P23) Each day for four days, Linda traveled for one hour at a speed that resulted in her traveling one mile in an integer number of minutes. Each day after the first, her speed decreased so that the number of minutes to travel one mile increased by 5 minutes over the preceding day. Each of the four days, her distance traveled was also an integer number of miles. What was the total number of miles for the four trips?

(A) 10 **(B)** 15 **(C)** 25 **(D)** 50 **(E)** 82

Problem 15. (Mathcounts 2019 Chapter P20) Jones is chasing a car 800 meters ahead of him. He is on a horse moving at 50 km/h. If Jones catches up to the car in 4 minutes, how fast was the car moving?

Problem 16. (2013 AMC8 P10) What is the ratio of the least common multiple of 180 and 594 to the greatest common factor of 180 and 594?

(A) 110 **(B)** 165 **(C)** 330 **(D)** 625 **(E)** 660

Problem 17. (2000 AMC12 P6) Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers could be obtained?

(A) 22 (B) 60 (C) 119 (D) 180 (E) 231

Problem 18. (2019 AMC10B P11) Two jars each contain the same number of marbles, and every marble is either blue or green. In Jar 1 the ratio of blue to green marbles is $9:1$, and the ratio of blue to green marbles in Jar 2 is $8:1$. There are 95 green marbles in all. How many more blue marbles are in Jar 1 than in Jar 2?

(A) 5 **(B)** 10 **(C)** 25 **(D)** 45 **(E)** 50