Circles

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Contents

1 Basics

1.1 Reading

1. Pages 185-192 of *Competitive Math for Middle School*, J. Batterson

1.2 Definitions

> **Radius**
> The distance from the center to any point on a circle is the ***radius***.

This of course means this distance is constant.

> **Diameter**
> The largest distance between two points on a circle is the ***diameter***

As it turns out, the largest distance d is $2r$.

> **Chord**
> A ***chord*** is a line segment with endpoints lying on a circle.

Tangent
A ***tangent*** is a line touching a circle at only one point.

Note that the line is perpendicular to the line that goes through the tangency point and the center of the circle.

1.3 Perimeter

Theorem 1.5 (Perimeter of a Circle)
The perimeter of a circle, also known as the ***circumference***, is given by $2\pi r$, where r is the radius.

1.4 Area

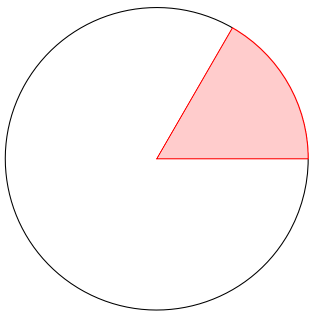
Theorem 1.6 (Area of a Circle)
The area of a circle is given by πr^2, where r is the radius.

Example 1.7
Find the area of a circle with diameter 12.

Solution. The radius is $\frac{12}{6}$. Therefore, the area is $(6^2)\pi = \boxed{36\pi}$. □

Sector
A ***sector*** is the area bounded between an ***arc*** and the radii that connect the ends of that arc.

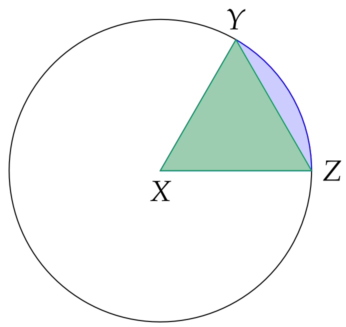
Below is an example of a sector.



Example 1.9
$\triangle XYZ$ is equilateral with side length 10. A circle is constructed with center X and radius 10, therefore passing through Y and Z. Find the area of sector YXZ of the circle.

Solution. To find the area of the sector, we need to first find what portion the sector is of the whole circle. Since $\triangle XYZ$ is equilateral, we must have $\angle X = 60^\circ$. This implies sector XYZ makes up $60^\circ/360^\circ = 1/6$ of the circle. Therefore, its area must also be $1/6$ the circle! Our answer is therefore

$$\frac{1}{6}(10^2\pi) = \boxed{\frac{50\pi}{3}}.$$

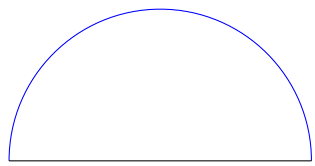


□

Theorem 1.10
We can apply the above method to find the area of any sector. For a given sector AOB, we have

$$\text{Area of sector } AOB = \frac{\angle AOB}{360^\circ} r^2 \pi.$$

Example 1.11
Farmer Tim has 50 feet of fence. He wants to enclose a semicircular area adjacent to his barn, using his barn as one side of the enclosure. What is the area of the space Farmer Tim can enclose?



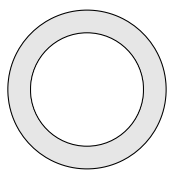
Solution. We know that if half the semicircle has perimeter 50 feet, then the whole circle must have a perimeter of 100 feet. We know that $2\pi r = C$, or $r = \frac{C}{2\pi} = \frac{100}{2\pi} = 50/\pi$. We also know the area of the semicircle is half the area of a whole circle. Therefore, our answer is

$$\frac{1}{2}\pi\left(\frac{50}{\pi}\right)^2 = \boxed{\frac{1250}{\pi}} \text{ square feet.}$$

□

Example 1.12
A man is standing on a lawn is wearing a circular sombrero of radius 3 feet. Unfortunately, the hat blocks the sunlight so effectively that the grass under it dies instantly. If the man walks in a circle of radius 5 feet, what is the area of the dead grass?

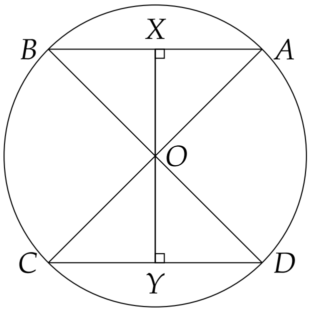
Solution. After the man walks in a circle, the resulting picture will look like this.



The outer ring is a total distance of $5+3=8$ feet from the center of the circle in which the man walks around. The small ring is a distance of $5-3=2$ feet from the center of the circle. Therefore, the area we want is the area of a circle with radius 8 minus the area of a circle with radius 2. This is just $8^2\pi - 2^2\pi = \boxed{60\pi}$. $\square$

Example 1.13
A circle has two parallel chords of length x that are x units apart. If the part of the circle included between the chords has area $2+\pi$, find x.

Solution. We begin with a diagram.

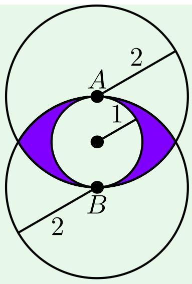


Let's show $\triangle BOA$ and $\triangle COA$ are right triangles. We note that $\triangle BOA$ and $\triangle COD$ are congruent because of SSS congruence. Thus, their corresponding altitudes are the same by CPCTC (congruent parts of corresponding triangles are congruent). Thus, $OY = OX = \frac{x}{2}$. Next, note that $\triangle BOX \cong \triangle AOX$ because of HL congruence (their hypotenuses are the radii and they share OX). Therefore, $\triangle BOX$ and $\triangle AOX$ are $45^\circ - 45^\circ - 90^\circ$ triangles. This means $\angle BOA$ must be a 90° angle. Notice that we can combine $\triangle BOA$ and $\triangle COD$ to create a square with side length r, where r is the radius of the circle. Thus, $\triangle BOA$ and $\triangle COD$ make up r^2. The other part region between the two segments BA and CD are sectors BOC and AOD, which comprise of $\frac{90+90}{360} = \frac{1}{2}$ the circle, or $\frac{1}{2}\pi r^2$. Thus, we have

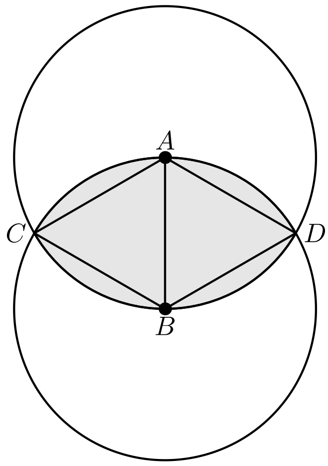
$$\frac{\pi r^2}{2} + r^2 = 2 + \pi.$$

We notice that the $\frac{1}{2}\pi r^2$ on the left side must correspond to the π on the right side, so $r = \sqrt{2}$. Using properties of $45^\circ - 45^\circ - 90^\circ$ triangles, we see $2AX = 2 \times \frac{\sqrt{2}}{\sqrt{2}}) = \boxed{2}$. $\square$

Example 1.14
A circle of radius 1 is internally tangent to two circles of radius 2 at points A and B, where AB is a diameter of the smaller circle. What is the area of the region, shaded in the picture, that is outside the smaller circle and inside each of the two larger circles?



Solution. This seems like a tough problem, mostly because it seems difficult to find the area of the region inside of both big circles. However, if we can just find this region, we can simply subtract the area of the smaller circle to find the area of the shaded region. Thus, we shall focus on finding the overlapping region between the two big circles.
In geometry problems, it is often helpful to connect points of intersection. Thus, let C and D be the intersections of the two large circles. Connect them to A and B to get the picture below:



We now see that $AC = AB = BC = 2$ (they are all radii), so $\triangle ABC$ is equilateral! The same logic applies to $\triangle ABD$ as well. Therefore, $\angle CBA = 60^\circ$. We see that our shaded region is made up of two equilateral triangles and 4 regions that are 60° sectors minus an equilateral triangle. The area of the equilateral triangles are

$$2\left(\frac{2^2\sqrt{3}}{4}\right) = 2\sqrt{3}.$$

We can calculate the area of the 4 smaller regions by finding the area of 60° sector (which is just $1/6$ the area of a circle) and subtracting the area of the equilateral triangle within that sector and multiplying that area by 4.

$$4\left(\frac{2^2\pi}{6} - \frac{2^2\sqrt{3}}{4}\right) = \frac{8\pi}{3} - 4\sqrt{3}.$$

Therefore, the area of the overlapping regions between the two big circles is $8\pi/3 - 2\sqrt{3}$. However, we still need to subtract the area of the small circle of radius 1, which has area π. Therefore, our answer is $\boxed{5\pi/3 - 2\sqrt{3}}$. $\square$

2 Angles

We say that an angle is ***inscribed*** in an arc if its vertex is on the circumference of the circle and its sides hit the circle at the ends of the arc.

Theorem 2.1 (Thale's Theorem)
Any angle inscribed in a semicircle is a right angle.

Example 2.2
Points A, B, and C, are on circle O such that $\overset{\frown}{AC} = 80^\circ$ and $\overset{\frown}{ACB} = 130^\circ$. Find $\angle ABC$.

Solution. We know that the measure of an arc equals the angle formed by the radii that cut off the arc (we call such an angle a **central angle**). Therefore, we draw radii to A, B, and C, thus forming some isosceles triangles. Since $\angle BOC = \overset{\frown}{BC} = \overset{\frown}{AB} - \overset{\frown}{AC} = 50^\circ$, we have

$$\angle OBC = \angle OCB = \frac{180^\circ - 50^\circ}{2} = 65^\circ$$

Similarly, $\angle AOB = \overset{\frown}{AB} = 130^\circ$, so

$$\angle OAB = \angle OBA = \frac{180^\circ - 130^\circ}{2} = 25^\circ$$

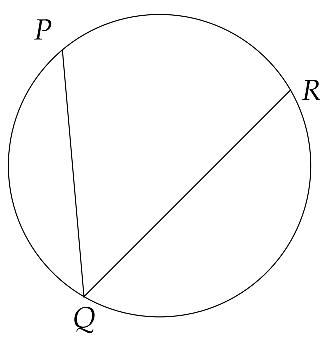
Therefore, $\angle ABC = \frac{\overset{\frown}{AC}}{2}$ and we wonder if this is always the case. We can try changing BC to see if that matters. If we let $\overset{\frown}{BC} = 64^\circ$, we can go through the same series of calculations as above to find that, indeed, $\angle ABC$ is still 40°. □

Now that we have a specific case as a guideline, we'll try to prove that an inscribed angle is always half the arc it intercepts. Unfortunately, to completely prove this, we'll need a number of cases. We'll try one of these cases here.

Theorem 2.3 (Inscribed Angle Theorem)
The measure of an inscribed angle is one half the measure of the arc it intercepts.

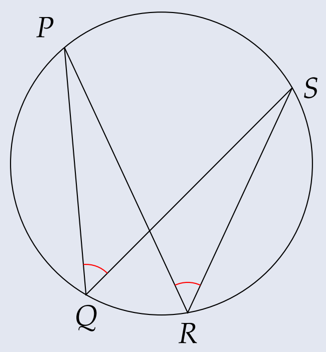
For example,

$$\angle PQR = \frac{\overset{\frown}{PR}}{2}.$$



Corollary 2.4

Any two angles that are inscribed in the same arc are equal.

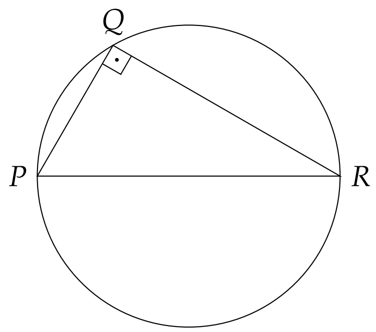


Corollary 2.5 (Thales' Theorem)

The measure of an inscribed angle is 90° if and only if the arc subtends the diameter.

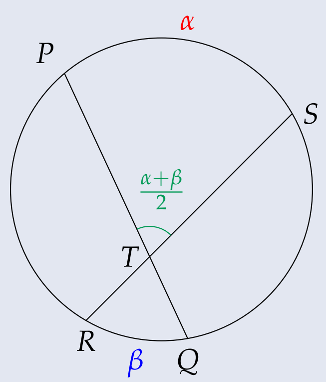
For example,

$$\angle PQR = 90^\circ.$$

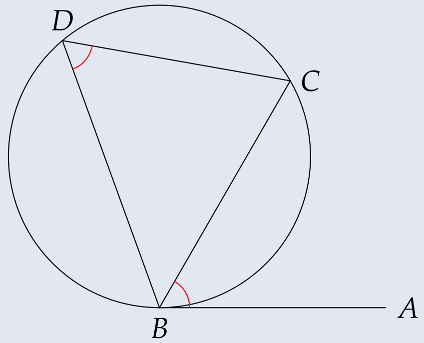


Theorem 2.6

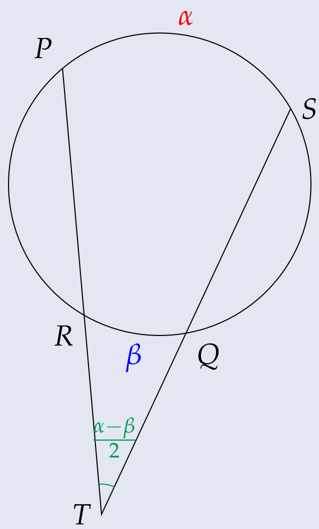
The measure of an angle formed by two secants which intersect outside the circle is equal to one-half the difference of the arcs intercepted by the secants.



Theorem 2.7
Let BC be a chord of a circle and A be a point outside the circle such that AB is tangent to the circle. If D is a point on the opposite side of BC to A, then $\angle ABC = \angle BDC$.



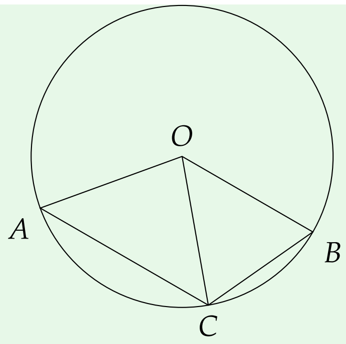
Theorem 2.8
The measure of the angle formed by two chords is one-half the sum of the intercepted arcs.



Example 2.9
Given that $\triangle ABC$ is inscribed in a circle, $\angle A = 70^\circ$, and $\overset{\frown}{AC} = 130^\circ$, find $\angle C$.

Solution. Since $\angle B$ is inscribed in $\overset{\frown}{AC}$, we have $\angle B = \frac{AC}{2} = 65^\circ$. Therefore, $\angle C = 180^\circ - \angle A - \angle B = 45^\circ$. □

Example 2.10
Points A, B, and C, are on circle O such that $\overset{\frown}{AC} = 80^\circ$ and $\overset{\frown}{ACB} = 130^\circ$. Find $\angle ABC$.



Solution. We know that the measure of an arc equals the angle formed by the radii that cut off the arc (we call such an angle a ***central angle***). Therefore, we draw radii to A, B, and C, thus forming some isosceles triangles. Since $\angle BOC = \widehat{BC} = \widehat{AB} - \widehat{AC} = 50^\circ$, we have

$$\angle OBC = \angle OCB = \frac{180^\circ - 50^\circ}{2} = 65^\circ$$

Similarly, $\angle AOB = \widehat{AB} = 130^\circ$, so

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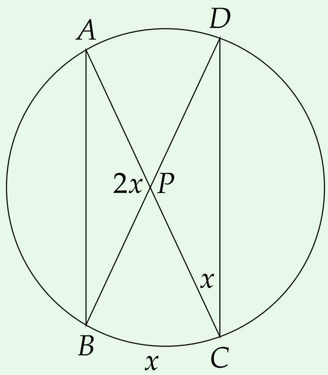
Therefore, $\angle ABC = \frac{\widehat{AC}}{2}$ and we wonder if this is always the case. We can try changing BC to see if that matters. If we let $\widehat{BC} = 64^\circ$, we can go through the same series of calculations as above to find that, indeed, $\angle ABC$ is still 40°. □

Let's use this in some problems.

Example 2.11
Find x given that $\angle APB = 2x$, $\angle ACD = x$, and $\widehat{BC} = x$.



Solution. Since $\angle B$ and $\angle C$ are inscribed in the same arc, they must be equal (since each equals half of the arc). Therefore, $\angle B = \angle C = x$. Since $\angle A$ is inscribed in $\widehat{BC}$, we have $\angle A = \frac{\widehat{BC}}{2} = \frac{x}{2}$. Now we can use $\triangle APB$ to write an equation for x.

$$\angle A + \angle APB + \angle B = 180^\circ$$

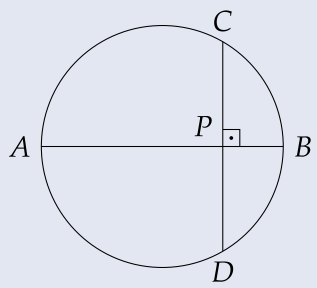
We have,

$$\frac{x}{2} + 2x + x = 180^\circ$$

Solving this equation gives $x = 51\frac{3}{7}^\circ$. □

3 Length Properties

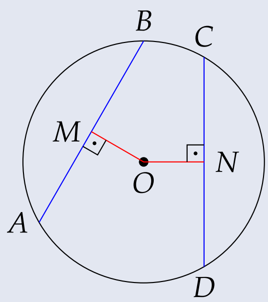
Theorem 3.1
A diameter perpendicular to a chord bisects the chord and its arc.



Note that:

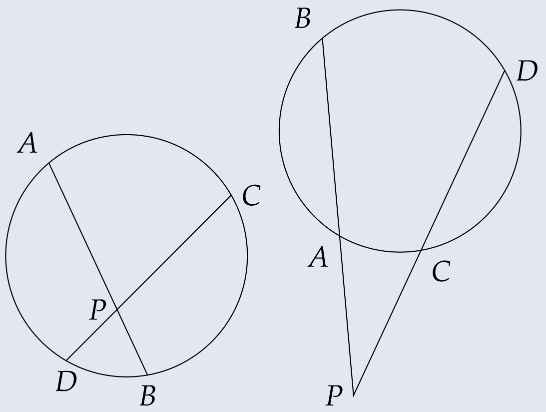
1. $\angle ACB = \angle ADB = 90^\circ$
2. $CP^2 = AP \cdot PB$
3. $\triangle ACP \sim \triangle CBP \sim \triangle ABC$

Theorem 3.2
In a circle, congruent chords are equally distanced from the center.



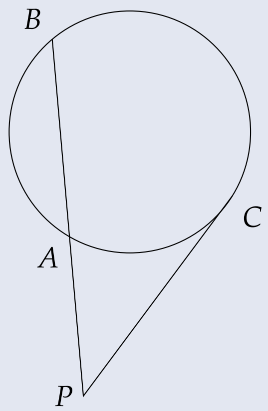
4 Power of a Point

Theorem 4.1 (Power of a Point)
Let A, B, C, D be points on a circle, and let AB and CD intersect at P. Then $PA \cdot PB = PC \cdot PD$.



There are two possibilities of this arrangement: the order of the points is A, B, C, D, or the order of the points is A, C, B, D. In either case, the theorem is the same.

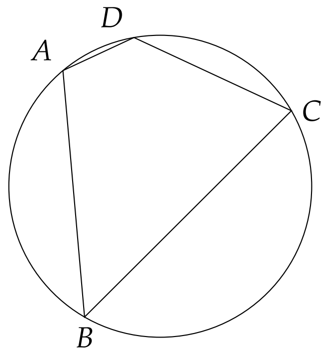
Corollary 4.2
Let P be a point outside the circle, and one line through P intersects the circle at A and B, whereas another is tangent to the circle at C. Then $PC^2 = PA \cdot PB$.



Note that because of this, tangents from a point outside the circle are the same length. Power of a Point is best seen through problems, and they will be introduced in the problems section.

5 Cyclic Quadrilaterals

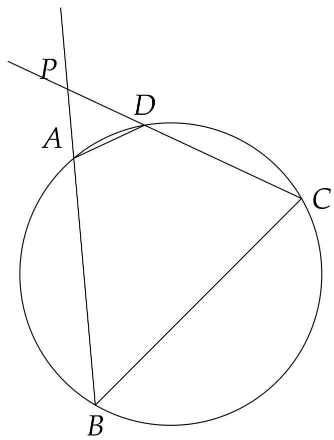
Consider the following quadrilateral:



Cyclic Quadrilateral
A ***cyclic quadrilateral*** is a quadrilateral inscribed in a circle.

Theorem 5.2 (Cyclic Quadrilateral Angle Condition)
In a cyclic quadrilateral, $\angle A + \angle C = \angle B + \angle D = 180^\circ$.

Let's extend a few lengths:



Let the two lines meet at P. Then $\triangle PAD \sim \triangle PCB$. Furthermore, many angles in this configuration are equal. One pair is $\angle BAC = \angle BDC$, using our knowledge of ***inscribed angles***. This is sometimes useful in solving problems.

Theorem 5.3 (Ptolemy's Theorem)
Let $ABCD$ be a cyclic quadrilateral. Then

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

Theorem 5.4 (Brahmagupta's Formula)
The area of a cyclic quadrilateral with side lengths a, b, c, d is

$$\sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $s = \frac{a+b+c+d}{2}$.

Remark 5.5. This looks very similar to Heron's Formula, but one key difference is there is no s term inside the square root.