ASE 2020-21 Notes

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§0 Contents

§1 Sunday, 06/21/20

§1.1 Number Sense: FOIL

Multiplication is at the heart of every Number Sense test. Slow multiplication hampers how far you are able to go on the test as well as making you prone to making more errors. To help beginners learn how to speed up multiplying, the concept of FOILing, learned in beginning algebra classes, is introduced as well as some exercises to help in speeding up multiplication. What is nice about the basic multiplication exercises is that *anyone* can make up problems, so practice is unbounded.

When multiplying two two-digit numbers ab and cd swiftly, a method of FOILing – or more accurately named LIOFing (Last-Inner+Outer-First) – is used. To understand this concept better, lets take a look at what we do when we multiply $ab \times cd$:

$$ab = 10a + b \text{ and } cd = 10c + d$$

$$(10a + b) \times (10c + d) = 100(ac) + 10(ad + bc) + bd$$

A couple of things can be seen by this:

1. The one's digit of the answer is simply bd or the *Last* digits (by *Last* I mean the least significant digit) of the two numbers multiplied.
2. The ten's digit of the answer is $(ad + bc)$ which is the sum of the *Inner* digits multiplied together plus the *Outer* digits multiplied.
3. The hundred's digit is ac which are the *First* digits (again, by *First* I mean the most significant digit) multiplied with each other.
4. If in each step you get more than a single digit, you carry the extra (most significant digit) to the next calculation. For example:

$$74 \times 23 = \begin{array}{ll} \text{Units:} & 3 \times 4 = \mathit{1}\mathbf{2} \\ \text{Tens:} & 3 \times 7 + 2 \times 4 + \mathit{1} = \mathit{3}\mathbf{0} \\ \text{Hundreds:} & 2 \times 7 + \mathit{3} = \mathbf{17} \\ \text{Answer:} & \mathbf{1702} \end{array}$$

   Where the bold represents the answer and the italicized represents the carry.

Similarly, you can extend this concept of LIOFing to multiply any n-digit number by m-digit number in a procedure I call "moving down the line." Let's look at an example of a 3-digit multiplied by a 2-digit:

$$493 \times 23 = \begin{array}{ll} \text{Ones:} & 3 \times 3 = \mathbf{9} \\ \text{Tens:} & 3 \times 9 + 2 \times 3 = \mathit{3}\mathbf{3} \\ \text{Hundreds:} & 3 \times 4 + 2 \times 9 + \mathit{3} = \mathit{3}\mathbf{3} \\ \text{Thousands:} & 2 \times 4 + \mathit{3} = \mathbf{11} \\ \text{Answers:} & \mathbf{11339} \end{array}$$

As one can see, you just continue multiplying the two-digit number "down the line" of the three-digit number, writing down what you get for each digit then moving on (always remembering to carry when necessary). The following are exercises to familiarize you with this process of multiplication:

§1.1.1 FOIL Problems

$95 \times 30 =$	$90 \times 78 =$	$51 \times 11 =$	$83 \times 51 =$
$64 \times 53 =$	$65 \times 81 =$	$92 \times 76 =$	$25 \times 46 =$
$94 \times 92 =$	$27 \times 64 =$	$34 \times 27 =$	$11 \times 77 =$
$44 \times 87 =$	$86 \times 63 =$	$54 \times 92 =$	$83 \times 68 =$
$72 \times 65 =$	$81 \times 96 =$	$57 \times 89 =$	$25 \times 98 =$
$34 \times 32 =$	$88 \times 76 =$	$22 \times 11 =$	$36 \times 69 =$
$35 \times 52 =$	$15 \times 88 =$	$62 \times 48 =$	$56 \times 40 =$
$62 \times 78 =$	$57 \times 67 =$	$28 \times 44 =$	$80 \times 71 =$
$51 \times 61 =$	$81 \times 15 =$	$64 \times 14 =$	$47 \times 37 =$
$79 \times 97 =$	$99 \times 87 =$	$49 \times 54 =$	$29 \times 67 =$

§1.2 Calculator: Scientific Notation

TMSCA Calculator tests use **three significant digits scientific notation**.

Definition 1 (Scientific Notation). **Scientific notation** is a way of expressing numbers that are too big or too small to be conveniently written in decimal form.

All digits are **significant figures** *except*:

- All leading zeroes. For example, in 013, 1 and 3 are significant, and 0 is not.
- **Trailing zeroes** when they are merely placeholders to indicate the scale of the number. For example, the 0's in 1.00 are **not** significant, but the 0's 1.002 are.
- **Spurious zeroes**, or in other words the zeroes in front of the first nonzero digit. For example, 000032 has 4 spurious zeroes in front of it, and 0.05 has 2 spurious zeroes in front of it (why?).

You may wonder why 0.05 has 2 spurious zeroes – without them, the number would be 5. The reason they are spurious is because in scientific notation, we would write this as 5×10^{-2}. If we had written $0.05 \times 100 = 005 \implies 0.05 = 005 \times 10^{-2}$, that wouldn't make much sense. Instead, **just because there is a decimal point doesn't mean it isn't spurious**. From here, let's see a few examples of scientific notation:

- $0.32 \implies 3.2 \times 10^{-1}$
- $500 \implies 5 \times 10^{2}$
- $123456 \implies 1.23456 \times 10^{5}$

Notice as the numbers get very large or very small, really only the first few digits matter. For example, although 32003 and 32000 are different numbers, they basically are the same – that small difference doesn't matter much. This is where three significant digits comes in handy – let's look at our examples again:

- $0.32 \implies 3.2 \times 10^{-1} = 3.20 \times 10^{-1}$
- $500 \implies 5 \times 10^{2} = 5.00 \times 10^{2}$
- $123456 \implies 1.23456 \times 10^{5} \implies 1.23 \times 10^{5}$

Let's try a few problems.

§1.2.1 Scientific Notation Problems

Convert the following to scientific notation with **two significant digits**:

$3.232 =$	$2.32 =$	$4.1 =$	$2.34 =$
$9.9 =$	$12345 =$	$123 =$	$456 =$

Convert the following to scientific notation with **three significant digits** (remember: this is used on **calculator**!):

$3.31 =$	$2.23 =$	$66.1 =$	$234.5 =$
$9.911 =$	$12414 =$	$434 =$	$321 =$

§1.3 General Math: Variables

§1.3.1 Definitions

Definition 2 (Variable). A **variable** is something that can change. It can also be a symbol for a number we don't know yet.

There are **independent** and **dependent** variables. *Independent* variables do not affect one another (for example, $x = 5, y = 6$ do not affect each other, but *dependent* variables do affect each other (for example, $x = 5y$).

Definition 3 (Consecutive). When two numbers are **consecutive**, we mean they are right next to each other.

For example, 3 is consecutive to 4, 4 is consecutive to 5, 8 is consecutive to 9, 9 is consecutive to 8, etc. We can also have consecutive **even** integers. For example, if we only consider even numbers, 2 is consecutive to 4. If the problem doesn't specify, then we just mean consecutive integers.

§1.3.2 Statistics

Definition 4 (Mean). The **mean** (or average) of a set of numbers is the sum of the numbers divided by the number of numbers.

Note that the mean refers to the **arithmetic mean**.

Theorem 1 (Arithmetic Mean). Let $a_1, a_2, a_3, \dots, a_n$ be a set of n numbers. Then the *arithmetic mean* of these numbers is

$$\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}.$$

Definition 5 (Median). The **median** is the number in the middle.

Definition 6 (Mode). The **mode** is the most common number.

Note that there could be more than one mode. For example, in the set $1, 2, 2, 2, 3, 3, 3$, there are two modes of (size) 3, and the most popular values were 2 and 3. When a problem says there is a **unique** mode, that means there is only one mode.

§1.3.3 Basics

In this lesson, we will only be dealing with basic functions (like linear functions).

Example 1. If $a + b = 3$, and $a = 1$, what is b?

Solution. Plugging in 1, we get $b = 2$. □

§1.3.4 Consecutive Integers

Let's take a look at an example.

Example 2. Let's say we have 7 consecutive integers that sum to 28. What is the smallest integer of these 7?

Solution. By trial and error we can find the sequence is 1 to 7, so the answer is 1. However, is there a better way? Answer: there is! □

Theorem 2. Given n consecutive integers that sum to k, the smallest number is

$$\frac{k}{n} - \frac{n-1}{2},$$

and the largest number is

$$\frac{k}{n} + \frac{n-1}{2}.$$

Note that if you ever get an answer that is not an integer, there are **no solutions**.

§1.3.5 Ratios and Proportions

Definition 7 (Ratio). A **ratio** shows the relative sizes of two or more values.

For example, if there are 2 apples, 4 oranges, and 6 bananas, the *ratio* of apples to oranges to bananas is $2:4:6 = 1:2:3$. The : symbol represents the word "to".

Definition 8 (Proportion). A **proportion** is a statement that two *ratios* are equal.

For example, the following is a proportion:

$$\frac{12}{15} = \frac{4}{5}.$$

Two numbers x, y are said to be **directly proportional** if

$$y = kx$$

for some constant k. For example, if $k = 5$, then if x is 3, y must be 15, and if x is 4, y must be 20. Two numbers a, b are said to be **inversely proportional** if

$$ab = k$$

for some constant k. For example, if $k = 5$, then if x is 1, then y is 5, and if x is 2, then y is $\frac{5}{2}$. **Joint proportions** occur when $x = kyz$, for some constant k. In this case, x and y are joint.

§1.3.6 Moles Digging Holes

Example 3. 10 moles dig 10 holes in 10 hours, how many holes do 20 moles dig in 20 hours?

Solution. There are twice as many moles, so that doubles the number of holes, and twice as many hours also doubles the number of holes. Thus, the answer is $10 \times 2 \times 2 = 40$. □

Theorem 3. If we have a moles, b holes, and c hours, then

- a varies directly with b
- b varies directly with c
- c varies inversely with a

§1.3.7 d = rt

There is only one theorem you need to know:

Theorem 4 ($d = rt$). Like the title of this theorem says,

$$\text{distance} = \text{rate} \times \text{time}.$$

This implies that distance and rate are **directly proportional** when time is constant, distance and time are **directly proportional** when rate is constant, and rate and time are **inversely proportional** when distance is constant. Even though this is the only necessary formula, let us list one more to speed up computation:

Theorem 5. Let person 1 work at a speed of one object per t_1 time, person 2 works at a speed of one object per t_2 time, and so on, all the way to person n who works at a speed of one object per t_n time. Then if they work together, they will finish in

$$\frac{1}{\frac{1}{t_1}+\frac{1}{t_2}+\ldots+\frac{1}{t_n}}$$

time.

Example 4. Dylan can paint a house in 3 hours. Cody can paint a house in x hours. If they work together, it takes then 2 hours to paint the house. What is x?

Solution. Using the theorem, we get

$$\frac{1}{\frac{1}{3}+\frac{1}{x}} = 2,$$
$$\frac{1}{x} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$
$$x = 6.$$

□

§1.3.8 Word Problems

These are the problems that use **words** instead of equations. For these types of problems - just convert back to equations! Examples will be given in the **Problems Section**.

Example 5. Sandwiches at Joe's Fast Food cost 3 each and sodas cost 2 each. How many dollars will it cost to purchase 5 sandwiches and 8 sodas?

(A) 31 (B) 32 (C) 33 (D) 34 (E) 35

Solution. The 5 sandwiches cost $5 \cdot 3 = 15$ dollars. The 8 sodas cost $8 \cdot 2 = 16$ dollars. In total, the purchase costs $15 + 16 = 31$ dollars. The answer is (A). □

§1.3.9 Variable Problems

Problem 1. Find the smallest and largest number of a sequence of 7 consecutive integers that sum to 49.

Problem 2 (Mathcounts Ratio Warmup). What is $\frac{9}{30}$ as a common fraction?

Problem 3 (Mathcounts Ratio Warmup). In a pasture there are 12 white horses and the rest are black. If there are 52 horses in the pasture, what fraction are black? Express your answer as a common fraction.

Problem 4 (Mathcounts Ratio Warmup). There are three times the number of orange fish as blue fish in a tank at the pet store, and there are no other fish. What percentage of the fish are orange?

Problem 5 (Mathcounts Ratio Stretch). Fairy Godmother has granted wishes to Aurora, Belle and Cindi in the ratio $6 : 8 : 11$. What fraction of the ratios were granted to Belle? What percent of the wishes granted by Fairy Godmother were not granted to Aurora? What is the absolute difference between the percents of wishes Fairy Godmother has granted to Aurora and to Cindi?

Problem 6 (Mathcounts Chapter 2019/8). After a brisk workout, Felicia counts 32 heartbeats in 15 seconds. Based on this count, what is Felicia's expected number of heartbeats in one minute?

Problem 7. Dylan and Cody are working together again. Dylan can eat one apple in 1 minute. Cody can eat one apple in 2 minutes. How long will it take them working together to eat 15 apples?

Problem 8. Alice can mow the lawn in 3 hours, Bob can mow the lawn in 4 hours, and Chris can mow the lawn in 5 hours. Working together, how long will it take them to mow one lawn?