Math Level 2.5 Handouts

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Contents

1 Units Digits

1.1 Introduction and Examples

Let's start with a few examples:

Example 1.1. Find the units digit of:

(a) $8 + 9$

(b) $8 + 19$

(c) $8 + 29$

(d) $18 + 9$

(e) $18 + 19$

(f) $438 + 1029$

What's the pattern?

Solution. Notice that all of these end in $\boxed{7}$. The reason is that the only thing that can affect the units digit is **other units digits**. $\square$

Example 1.2. Find the units digit of:

(a) 8×9

(b) 18×19

(c) 118×119

(d) 438×1029

What's the pattern?

Solution. Similarly, the units digit is always 2, because we only look at 8 and 9, which multiplied have a units digit of $\boxed{2}$. $\square$

1.2 Exponents of Units Digits

Example 1.3. Find the units digit of:

(a) 1^1

(b) 1^2

(c) 1^3

(d) 1^4

Solution. It is obvious always $\boxed{1}$. □

Example 1.4. Find the units digit of:

(a) 2^1

(b) 2^2

(c) 2^3

(d) 2^4

(e) 2^5

(f) 2^6

Notice a pattern?

Solution. It seems like the pattern is $\boxed{2, 4, 8, 6, 2, 4, 8, 6, \ldots}$. □

This gives us an important property: if we can find the pattern, we can find the units digit of large numbers! For example:

Example 1.5. Find the units digit of 2^{2020}.

Solution. The pattern is $2, 4, 8, 6, 2, 4, 8, 6, \ldots$. This means that it repeats every 4. Notice that $4, 8, 12, 16, \ldots$ all end in 6, and 2020 is also inside this sequence, so it must also end in $\boxed{6}$. □

Notice that **mod 10** gives us the **units digit**. This will be very important in the next problem!

Example 1.6. Karen is a teenager and the square of her age is equal to the number on her street address. If her age and the number on her street address have the same units digit, but do not add up to a multiple of 10, how old is Karen?

Solution. Let the age of Karen be k. This means that k^2 is her street address. This has the same units digit as k, so in mod 10,

$$k^2 \equiv k \pmod{10},$$

so if we test the possibilities, we get that $k \equiv 5, 6 \pmod{10}$ both work. However, if k ends in 5, so does k^2, so when we add them together, we will get a multiple of 10, which is not what we want! This means Karen's age ends in 6, and since she is a teenager, she has to be $\boxed{16}$. □

1.3 Units Digits in Different Bases

Example 1.7. Find the units digit of:

(a) $34_8 + 71_8$

(b) $131_4 + 2323_4$

(c) $121_3 \cdot 202_3$

Notice a pattern?

Solution. Just like in base 10, only the last digit matters! $\square$

1.4 Problems

Problem 1.1 (Mathcounts)**.** What is the digit in the units place of $(3^3)^5$?

Problem 1.2. Find the units digit of the following:

$$(972-269)(973-267)(974-214)(999-222)^3(42-43).$$

Problem 1.3. What are the only digits a perfect square can end with?

Problem 1.4. What are the only digits a perfect cube can end with?

Problem 1.5. If m is a whole number, what are the possible units digits of $2 \cdot 3^m$? What about $6 \cdot 3^m$?

Problem 1.6. How many positive divisors of 6^{2006} have a units digit of 6?

Problem 1.7. How many of the following have a units digit of 6?

$$2^1, 2^2, 2^3, \ldots, 2^{99}, 2^{100}$$

Problem 1.8. Find the units digit of n given that $mn = 21^6$ and m has a units digit of 7.

Problem 1.9. Alice and her younger brother Bob are both between 10 and 20 years old. The sum of their ages has a units digit of 6 and the difference between their ages is 2. If Bob's age is an even number, how old is Alice?

Problem 1.10. In how many bases b does $32_b + 32_b$ end in 4?

Problem 1.11 (Mathcounts)**.** The cube of the three-digit natural number $A7B$ is 10853133. What is $A + B$?

Problem 1.12. Find the units digit of $1! + 2! + 3! + \ldots + 1000!$.

Problem 1.13. Find the units digit of 3^{2006}.

Problem 1.14 (Mathcounts)**.** What is the units digit of $(133^{13})^3$?

Problem 1.15. Bob is reading a book and notices that the product of the numbers of the two pages his book is open to has a units digit of 6. What is the units digit of the sum of the two page numbers?