Math Level 2 Week 5

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Contents

1 Basics

1.1 Definitions

Radius
The distance from the center to any point on a circle is the ***radius***.

This of course means this distance is constant.

Diameter
The largest distance between two points on a circle is the ***diameter***.

As it turns out, the largest distance d is $2r$.

Chord
A ***chord*** is a line segment with endpoints lying on a circle.

Tangent
A ***tangent*** is a line touching a circle at only one point.

Note that the line is perpendicular to the line that goes through the tangency point and the center of the circle.

1.2 Perimeter

Theorem 1.5 (Perimeter of a Circle)
The perimeter of a circle, also known as the ***circumference***, is given by $2\pi r$, where r is the radius.

1.3 Area

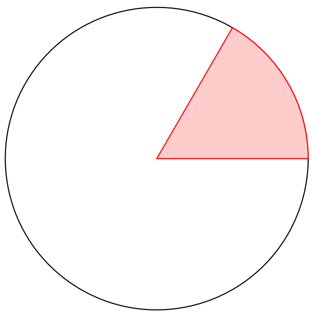
Theorem 1.6 (Area of a Circle)
The area of a circle is given by πr^2, where r is the radius.

Example 1.7
Find the area of a circle with diameter 12.

Solution. The radius is $\frac{12}{6}$. Therefore, the area is $(6^2)\pi = \boxed{36\pi}$. □

Sector
A ***sector*** is the area bounded between an ***arc*** and the radii that connect the ends of that arc.

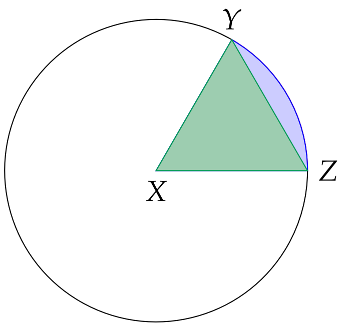
Below is an example of a sector.



Example 1.9
$\triangle XYZ$ is equilateral with side length 10. A circle is constructed with center X and radius 10, therefore passing through Y and Z. Find the area of sector YXZ of the circle.

Solution. To find the area of the sector, we need to first find what portion the sector is of the whole circle. Since $\triangle XYZ$ is equilateral, we must have $\angle X = 60^\circ$. This implies sector XYZ makes up $60^\circ/360^\circ = 1/6$ of the circle. Therefore, its area must also be $1/6$ the circle! Our answer is therefore

$$\frac{1}{6}(10^2\pi) = \boxed{\frac{50\pi}{3}}.$$

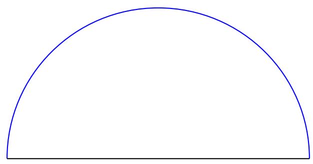


□

Theorem 1.10
We can apply the above method to find the area of any sector. For a given sector AOB, we have

$$\text{Area of sector } AOB = \frac{\angle AOB}{360^\circ} r^2 \pi.$$

Example 1.11
Farmer Tim has 50 feet of fence. He wants to enclose a semicircular area adjacent to his barn, using his barn as one side of the enclosure. What is the area of the space Farmer Tim can enclose?



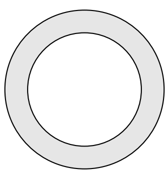
Solution. We know that if half the semicircle has perimeter 50 feet, then the whole circle must have a perimeter of 100 feet. We know that $2\pi r = C$, or $r = \frac{C}{2\pi} = \frac{100}{2\pi} = 50/\pi$. We also know the area of the semicircle is half the area of a whole circle. Therefore, our answer is

$$\frac{1}{2}\pi\left(\frac{50}{\pi}\right)^2 = \boxed{\frac{1250}{\pi}} \text{ square feet.}$$

□

Example 1.12
A man is standing on a lawn is wearing a circular sombrero of radius 3 feet. Unfortunately, the hat blocks the sunlight so effectively that the grass under it dies instantly. If the man walks in a circle of radius 5 feet, what is the area of the dead grass?

Solution. After the man walks in a circle, the resulting picture will look like this.

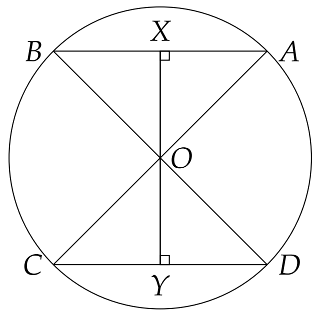


The outer ring is a total distance of $5+3=8$ feet from the center of the circle in which the man walks around. The small ring is a distance of $5-3=2$ feet from the center of the circle. Therefore, the area we want is the area of a circle with radius 8 minus the area of a circle with radius 2. This is just $8^2\pi - 2^2\pi = \boxed{60\pi}$. □

Example 1.13

A circle has two parallel chords of length x that are x units apart. If the part of the circle included between the chords has area $2+\pi$, find x.

Solution. We begin with a diagram.



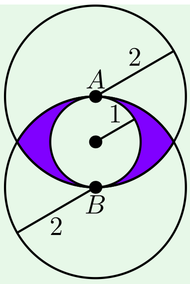
Let's show $\triangle BOA$ and $\triangle COA$ are right triangles. We note that $\triangle BOA$ and $\triangle COD$ are congruent because of SSS congruence. Thus, their corresponding altitudes are the same by CPCTC (congruent parts of corresponding triangles are congruent). Thus, $OY = OX = \frac{x}{2}$. Next, note that $\triangle BOX \cong \triangle AOX$ because of HL congruence (their hypotenuses are the radii and they share OX). Therefore, $\triangle BOX$ and $\triangle AOX$ are $45^\circ - 45^\circ - 90^\circ$ triangles. This means $\angle BOA$ must be a 90° angle. Notice that we can combine $\triangle BOA$ and $\triangle COD$ to create a square with side length r, where r is the radius of the circle. Thus, $\triangle BOA$ and $\triangle COD$ make up r^2. The other part region between the two segments BA and CD are sectors BOC and AOD, which comprise of $\frac{90+90}{360} = \frac{1}{2}$ the circle, or $\frac{1}{2}\pi r^2$. Thus, we have

$$\frac{\pi r^2}{2} + r^2 = 2 + \pi.$$

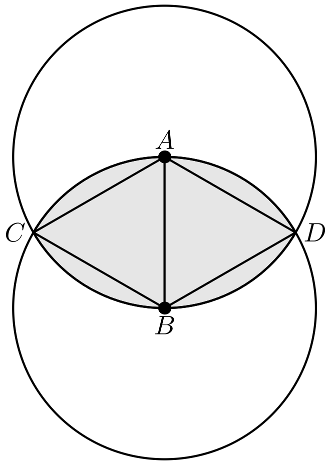
We notice that the $\frac{1}{2}\pi r^2$ on the left side must correspond to the π on the right side, so $r = \sqrt{2}$. Using properties of $45^\circ - 45^\circ - 90^\circ$ triangles, we see $2AX = 2 \times \frac{\sqrt{2}}{\sqrt{2}}) = \boxed{2}$. □

Example 1.14

A circle of radius 1 is internally tangent to two circles of radius 2 at points A and B, where AB is a diameter of the smaller circle. What is the area of the region, shaded in the picture, that is outside the smaller circle and inside each of the two larger circles?



Solution. This seems like a tough problem, mostly because it seems difficult to find the area of the region inside of both big circles. However, if we can just find this region, we can simply subtract the area of the smaller circle to find the area of the shaded region. Thus, we shall focus on finding the overlapping region between the two big circles.
In geometry problems, it is often helpful to connect points of intersection. Thus, let C and D be the intersections of the two large circles. Connect them to A and B to get the picture below:



We now see that $AC = AB = BC = 2$ (they are all radii), so $\triangle ABC$ is equilateral! The same logic applies to $\triangle ABD$ as well. Therefore, $\angle CBA = 60^\circ$. We see that our shaded region is made up of two equilateral triangles and 4 regions that are 60° sectors minus an equilateral triangle. The area of the equilateral triangles are

$$2\left(\frac{2^2\sqrt{3}}{4}\right) = 2\sqrt{3}.$$

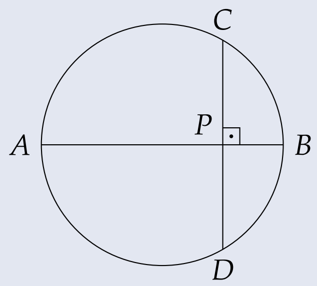
We can calculate the area of the 4 smaller regions by finding the area of 60° sector (which is just $1/6$ the area of a circle) and subtracting the area of the equilateral triangle within that sector and multiplying that area by 4.

$$4\left(\frac{2^2\pi}{6} - \frac{2^2\sqrt{3}}{4}\right) = \frac{8\pi}{3} - 4\sqrt{3}.$$

Therefore, the area of the overlapping regions between the two big circles is $8\pi/3 - 2\sqrt{3}$. However, we still need to subtract the area of the small circle of radius 1, which has area π. Therefore, our answer is $\boxed{5\pi/3 - 2\sqrt{3}}$. □

2 Length Properties

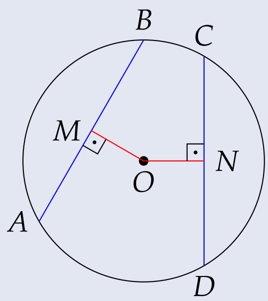
Theorem 2.1
A diameter perpendicular to a chord bisects the chord and its arc.



Note that:

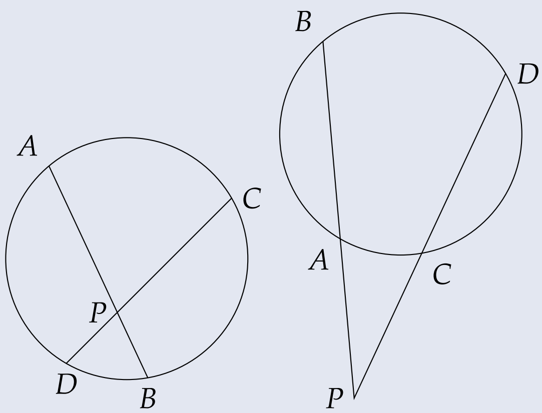
1. $\angle ACB = \angle ADB = 90°$
2. $CP^2 = AP \cdot PB$
3. $\triangle ACP \sim \triangle CBP \sim \triangle ABC$

Theorem 2.2
In a circle, congruent chords are equally distanced from the center.



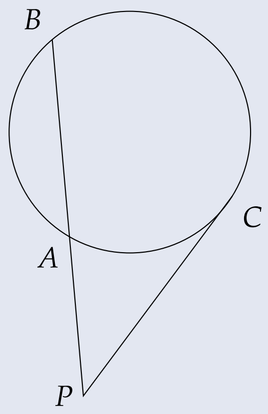
3 Power of a Point

Theorem 3.1 (Power of a Point)
Let A, B, C, D be points on a circle, and let AB and CD intersect at P. Then $PA \cdot PB = PC \cdot PD$.



There are two possibilities of this arrangement: the order of the points is A, B, C, D, or the order of the points is A, C, B, D. In either case, the theorem is the same.

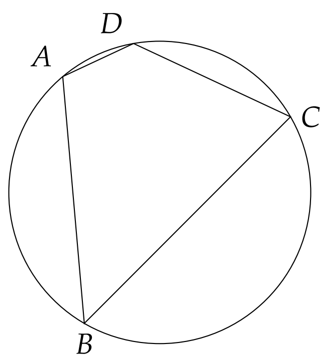
Corollary 3.2
Let P be a point outside the circle, and one line through P intersects the circle at A and B, whereas another is tangent to the circle at C. Then $PC^2 = PA \cdot PB$.



Note that because of this, tangents from a point outside the circle are the same length. Power of a Point is best seen through problems, and they will be introduced in the problems section.

4 Cyclic Quadrilaterals

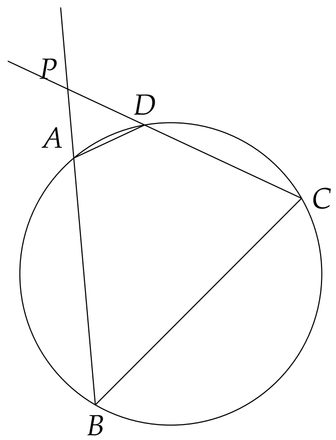
Consider the following quadrilateral:



Cyclic Quadrilateral
A ***cyclic quadrilateral*** is a quadrilateral inscribed in a circle.

Theorem 4.2 (Cyclic Quadrilateral Angle Condition)
In a cyclic quadrilateral $ABCD$, $\angle A + \angle C = \angle B + \angle D = 180°$.

Let's extend a few lengths:



Let the two lines meet at P. Then $\triangle PAD \sim \triangle PCB$. Furthermore, many angles in this configuration are equal. One pair is $\angle BAC = \angle BDC$, using our knowledge of ***inscribed angles***. This is sometimes useful in solving problems.

Theorem 4.3 (Ptolemy's Theorem)
Let $ABCD$ be a cyclic quadrilateral. Then

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

Theorem 4.4 (Brahmagupta's Formula)
The area of a cyclic quadrilateral with side lengths a, b, c, d is

$$\sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $s = \frac{a+b+c+d}{2}$.

Remark 4.5. This looks very similar to Heron's Formula, but one key difference is there is no s term inside the square root.

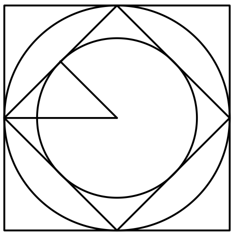
5 Problems

Problem 1 (AHSME 1950/33). The number of circular pipes with an inside diameter of 1 inch which will carry the same amount of water as a pipe with an inside diameter of 6 inches is n. Find n.

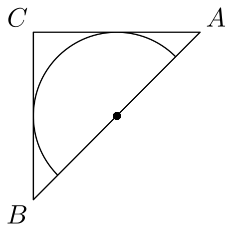
Problem 2 (AHSME 1973/25). A circular grass plot 12 feet in diameter is cut by a straight gravel path 3 feet wide, one edge of which passes through the center of the plot. What is the number of square feet in the remaining grass area?

Problem 3 (AMC 12A 2002/7). A 45° arc of circle A is equal in length to a 30° arc of circle B. What is the ratio of circle A's area and circle B's area?

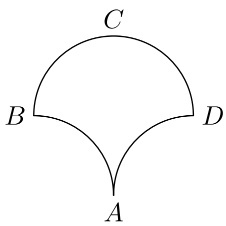
Problem 4 (AMC 10B 2005/7). A circle is inscribed in a square, then a square is inscribed in this circle, and finally, a circle is inscribed in this square. What is the ratio of the area of the smaller circle to the area of the larger square?



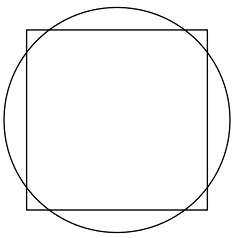
Problem 5 (AMC 8 2005/23). Isosceles right triangle ABC encloses a semicircle of area 2π. The circle has its center O on hypotenuse $\overline{AB}$ and is tangent to sides $\overline{AC}$ and $\overline{BC}$. What is the area of triangle ABC?



Problem 6. Three circular arcs of radius 5 units bound the region shown. Arcs AB and AD are quarter-circles, and arc BCD is a semicircle. What is the area, in square units, of the region?



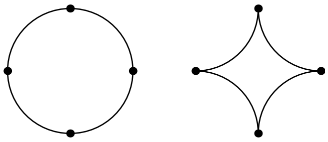
Problem 7. A square with side length 2 and a circle share the same center. The total area of the regions that are inside the circle and outside the square is equal to the total area of the regions that are outside the circle and inside the square. What is the radius of the circle?



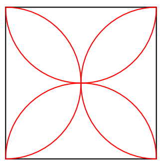
Problem 8 (AMC 8 2006/7). Circle X has a radius of π. Circle Y has a circumference of 8π. Circle Z has an area of 9π. List the circles in order from smallest to largest radius.

Problem 9 (AMC 8 2008/25). Margie's winning art design is shown. The smallest circle has radius 2 inches, with each successive circle's radius increasing by 2 inches. Which of the following is closest to the percent of the design that is black?

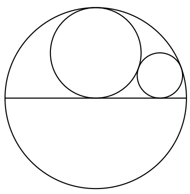
Problem 10 (AMC 8 2012/24). A circle of radius 2 is cut into four congruent arcs. The four arcs are joined to form the star figure shown. What is the ratio of the area of the star figure to the area of the original circle?



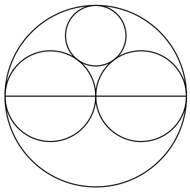
Problem 11. Find the area contained inside the red border below, if the square has a side length of 4:



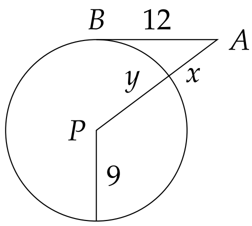
Problem 12. Find the radius of the smallest circle if the largest circle has a radius of 4.



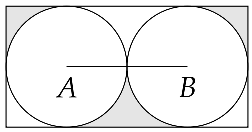
Problem 13. Find the radius of the smallest circle if the largest circle has a radius of 4.



Problem 14 (Mathcounts State Sprint 2000/22). Given that AB is tangent to circle P at B, what is the ratio $\frac{x}{x+y}$? Express your answer as a common fraction.



Problem 15 (Mathcounts State Sprint 1990). Find the area of the shaded portion of the rectangle, where $AB = 9$. Express the answer in terms of π.



Problem 16 (Mathcounts National Sprint 1996/25). In the diagram, circle Q is congruent to circle W, and both are tangent to circle O and to each other. Circle S and circle T are congruent and are tangent to circle O, to circle Q and to circle W. Find the ratio of the area of the smallest circle to the largest circle.