Math Level 2 Handouts Week 02

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Contents

§ 1 Sequences & Series

Sequences are pretty easy to understand: a group of numbers. **Series** are also pretty easy: the sum of a group of numbers. In all sequence problems, if you can find the pattern in the group of numbers, you have won. In all series problems, if you can find a nice property when you add some together, you have also won. My hope is to teach you the formulas you need to win, then it is your job to do more and more problems so that you can win fast and easily.

§ 1.1 Warm-Up

Problem 1. The sum of the odd positive integers from 1 to n is 9801. What is n?

§ 1.2 Sequences and Series Basics

§ 1.2.1 Arithmetic Sequences and Series

Definition 1 (Arithmetic Sequence). An **arithmetic sequence** is a sequence of numbers in which each term is given by adding a fixed value to the previous term.

For example, $-2, 1, 4, 7, 10, \ldots$ is an arithmetic sequence because each term is three more than the previous term. In this case, 3 is called the **common difference** of the sequence. More formally, an arithmetic sequence a_n is defined recursively by a first term a_0 and $a_n = a_{n-1} + d$ for $n \geq 1$, where d is the common difference. Explicitly, it can be defined as $a_n = a_0 + dn$.

Theorem 1 (Terms of an Arithmetic Sequence). The nth term in an arithmetic sequence is

$$a_n = a_1 + d(n-1),$$

where a_n is the nth term, a_1 is the first term, and d is the difference between consecutive terms.

Theorem 2 (Sum of an Arithmetic Sequence). The sum of the first n terms of an arithmetic sequences is

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2a_1 + (n-1)d).$$

Now for a few applications:

Theorem 3 (Sum of First n Positive Integers). For all positive integers n,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Corollary 4 (Sum of First n Odd Integers). For all positive integers n,

$$1 + 3 + 5 + \ldots + 2n - 1 = n^2.$$

Corollary 5 (Sum of First n Even Integers). For all positive integers n,

$$2 + 4 + 6 + \ldots + 2n = n(n+1).$$

Fact 1 (Cancellation Method). When you see an arithmetic sequence starting with $-a$, and somewhere later you see a, you can cancel all the terms in between, including $-a$ and a.

§ 1.2.2 Geometric Sequences and Series

Definition 2 (Geometric Sequence). A **geometric sequence** is a sequence of numbers in which each term is a fixed multiple of the previous term.

For example: $1, 2, 4, 8, 16, 32, \ldots$ is a geometric sequence because each term is twice the previous term. In this case, 2 is called the **common ratio** of the sequence. More formally, a geometric sequence may be defined recursively by:

$$a_n = r \cdot a_{n-1}, n > 1,$$

with a fixed first term a_1 and common ratio r. Using this definition, the nth term has the closed-form:

$$a_n = a_1 \cdot r^{n-1}.$$

Theorem 6 (Terms of a Geometric Sequence). The nth term in an geometric sequence is

$$a_n = a_1 r^{n-1},$$

where a_n is the nth term, a_1 is the first term, and r is the common ratio.

Theorem 7 (Sum of a Finite Geometric Sequence). The sum of the first n terms of a geometric sequence is given by

$$S_n = a_1 + a_2 + \cdots + a_n = a_1 \cdot \frac{r^n - 1}{r - 1},$$

where a_1 is the first term in the sequence, and r is the common ratio.

Definition 3 (Infinite Geometric Sequence). An **infinite geometric sequence** is a geometric sequence with an infinite number of terms.

Theorem 8 (Sum of an Infinite Geometric Sequence). Let $S = a + ar + ar^2 + \ldots$, where $|r| < 1$. Then

$$S = \frac{a}{1 - r}.$$

Example 1. Compute $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$.

Solution. Using the Sum of an Infinite Geometric Sequence formula, we get

$$\frac{1}{1 - \frac{1}{2}} = \boxed{2}.$$

□

Summing arithmetic sequences are sometimes known as **Gauss sums**.

Exercise 1. Compute $2 + 8 + 14 + \ldots + 152$. **Solution**: 1

Exercise 2. Determine $(-91) + (-84) + \ldots + 105$. **Solution**: 3

Exercise 3. An auditorium has weird rows. The first row has 1 seat, the second has 3, the third has 5, the fourth has 7, and so on. If there are n rows in the auditorium, how many seats are there in total? **Solution**: 2

§ 1.3 Telescoping Series

Definition 4 (Telescoping Series). A **telescoping** series is a series whose partial sums eventually only have a fixed number of terms after cancellation.

The cancellation technique, with part of each term cancelling with part of the next term, is known as the method of differences.

Example 2. Find the value of

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots.$$

Solution. If we split $\frac{1}{n(n+1)}$, we get

$$\frac{1}{n} - \frac{1}{n+1}.$$

Thus,

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots,$$

which results in 1. □

§ 1.4 Further Reading

1. Numerical Series Handout, Aritra12 (I recommend reading this during your free time, since it isn't something you **must** read, but it is something that can help with pattern recognition)

2. Partial Fraction Decomposition, Khan Academy (This is useful for telescoping series)

3. More Sequences and Series, Euclid eWorkshop

4. Arithmetic Sequences, Khan Academy

5. Geometric Sequences, Khan Academy

§ 1.5 Problems

There are some problems in here that are not part of the material above. This is because the topic of sequences is **very broad**.

Problem 2. Evaluate $1 + 2 + \ldots + 200$.

Problem 3 (AMC 10A 2015). How many terms are in the arithmetic sequence $13, 16, 19, \ldots, 70, 73$?

Problem 4 (AMC 10B 2015). What are the sign and units digit of the product of all the odd negative integers strictly greater than -2015?

Problem 5 (AMC 8 2013). What is the value of $4 \times (-1 + 2 - 3 + 4 - 5 + 6 - 7 + \ldots + 1000)$?

Problem 6 (AHSME 1997). Consider the sequence

$$1, -2, 3, -4, 5, -6, \ldots$$

whose nth term is $(-1)^{n+1} \cdot n$. What is the average of the first 200 terms of the sequence?

Problem 7 (AMC 10B 2013/19). The real numbers c, b, a form an arithmetic sequence with $a \geq b \geq c \geq 0$. The quadratic $ax^2 + bx + c$ has exactly one root. What is this root?

Problem 8. In a 10-team baseball league, each team plays each of the other teams 18 times. No game ends in a tie, and, at the end of the season, each team is the same positive number of games ahead of the next best team. What is the greatest number of games that the last place team could have won?

Problem 9. For how many positive integers n do there exist n consecutive integers that sum to -1?

Problem 10 (Mathcounts School 2015). The sum of the first $n + 2$ counting numbers is 43 more than the sum of the first n counting numbers. What is the value of n?

Problem 11. Alice and Bob decided to study math everyday. On the first day, Alice solved 3 problems. Starting from the second day, she solved two more problems than the day before. Bob, on the other hand, started by solving 7 problems on the first day, and starting from the second day, solved one more problem than the day before. After studying for a week, who solved more problems, and by how many?

Problem 12 (Mathcounts Chapter 2015). For some integer k, the first three terms of an arithmetic sequence are $k, 2k + 3$, and $4k + 1$, in that order. What is the integer value of the fourth term of the sequence?

Problem 13 (AMC 12B 2002). If a, b, c, d are positive real numbers such that a, b, c, d form an increasing arithmetic sequence, and a, b, d form a geometric sequence, then $\frac{a}{d}$ is __.

Problem 14 (FURMAN 2000). How many terms of the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots$$

must be added for the sum to first exceed 0.999?

Problem 15 (FURMAN 2004). Evaluate

$$\sum_{k=1}^{200}\frac{1}{k^2+k}.$$

Problem 16 (NC-SMC 2016/1). Compute $\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\dots\left(1-\frac{1}{n-1}\right)\left(1+\frac{1}{n}\right)$.

Problem 17 (AMC 12A 2009/17). Let $a+ar_1+ar_1^2+ar_1^3+\cdots$ and $a+ar_2+ar_2^2+ar_2^3+\cdots$ be two different infinite geometric series of positive numbers with the same first term. The sum of the first series is r_1, and the sum of the second series is r_2. What is r_1+r_2?

Problem 18 (AMC 10A 2000/6). The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \dots$ starts with two 1s, and each term afterwards is the sum of its two predecessors. Which one of the ten digits is the last to appear in the units position of a number in the Fibonacci sequence?

(A) 0 (B) 4 (C) 6 (D) 7 (E) 9

§ A Solutions

1. Let the first term be $a_1 = 2$ and the last term be $a_n = 152$. Then the common difference is $8-2=6$, so $a_n = a_1+(n-1)d \implies 152 = 2+(n-1)6 \implies n = 26$. Thus, $S_{26} = \frac{26}{2}(2+152) = \boxed{2002}$.

2. In the first row, there is 1 seat, in the second, there are 3, and so on, so it seems like in the kth row, there are $2k-1$ (you can check by writing out the pattern). Thus, in the nth row, there are $2n-1$ seats. Thus, $1+3+\ldots+2n-1 = \boxed{n^2}$.

3. Notice how we have $(-91)+(-84)+\ldots+(84)+(91)+(98)+(105)$. It is pretty obvious that -91 cancels with 91, -84 cancels with 84, and so on, so $(-91)+(-84)+\ldots+(84)+(91)$ will become 0. Thus, the answer is $98+105 = \boxed{203}$.