RPC Tyche

Gaussian Processes for Insurer Solvency

Presentation to the Gaussian Processes Cambridge Group

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Agenda

- Introduction
- Introduction to Life Insurance
- Capital Requirements and Proxy Modelling
- Neural Network Integrated Gaussian Covariance Function
- Summary

RPC Tyche – An Introduction

Formed in 2015 to embrace and transfer technological innovation to the insurance markets

Dedicated to the insurance industry (non-life and life) covering pricing, capital, risk, regulatory, asset, economic scenario, reserving and pension liability.

Focused on Tyche software and Tyche support services

Bespoke “Off the Shelf” solutions

RPCC has 97 staff now

29 Developers | 51 Consultants | 17 Business Support

Offices in London, Paris and Chicago

InsuranceERM Awards 2020 Winner
Actuarial modelling solution of the year

Who Am I

James Norman

Associate Partner, Head of Research

James Norman has over 15 years' experience in financial modelling of insurance companies and is Head of Research at RPC Tyche.

James develops cutting-edge solutions to quantitative problems for the insurance industry by leveraging the sophistication and computing power of the Tyche platform. He has extensive experience in non-life capital modelling and has led projects for some of the largest global insurance groups. Career highlights include:

- leading the technical design of a market-wide risk aggregation model for a multinational insurance market;
- validation of a number of firms' insurance and catastrophe models for PRA approval; and
- designing and delivering a full group internal model for regulatory and rating agency capital for a FTSE 100 company.

As Head of Research at RPCT, James coordinates R&D activity across the Life and Non-Life divisions and has a record of thought leadership and delivering innovative technical solutions to client problems. He has published a number of papers and presented at international conferences – including plenary sessions – on the subjects of actuarial modelling, statistics and capital management.

Actuary
Former Theoretical Physicist

Life Insurance 101*

- **Key products**
  - Protection (e.g. Term assurance, Critical illness)
  - Annuities
  - Unit-Linked savings
  - … products can get complicated
    - Embedded options & investment guarantees
- **Balance sheet**
  - Investment assets (A)
  - Insurance liabilities (L)
- **Risk factors (X)**
  - Interest rates, Mortality, Equity Markets, Credit, etc
- **Solvency**
  - EU Solvency II directive requires $P(A(X)-L(X)>0)>.995$

* This is a highly simplified description

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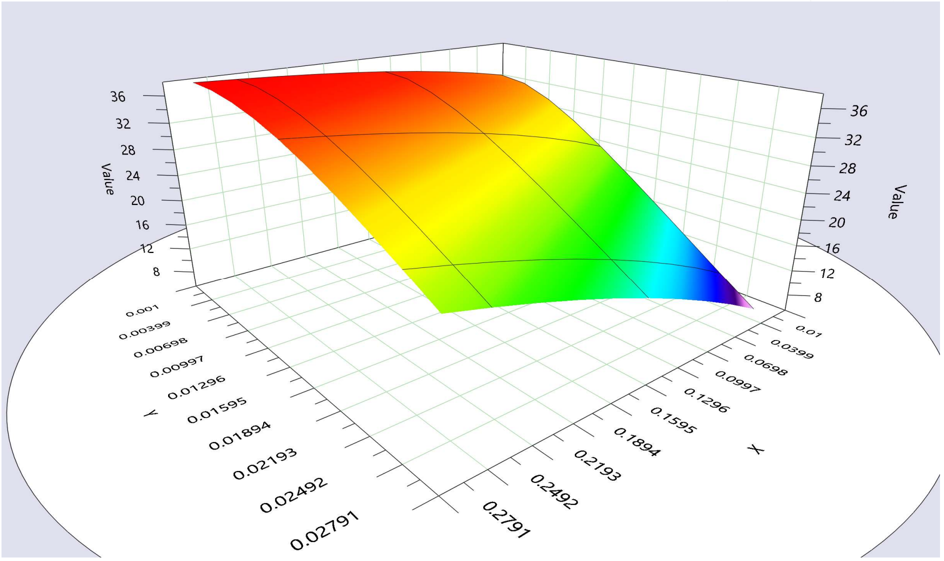
Typical Balance Sheet Function

- Smooth
- Non-linear
- Typically up to around 15 dimensional
- Interactions



Example A-L as function of two risk factors

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Proxy Modelling

- **Generate N simulations of risk factors X**
- **Evaluate A(X)-L(X) for each simulation**
- **Estimate P(A(X)-L(X)) from sample**
- **However calculating L(X) is computationally expensive**
  - each valuation can be several hours
  - need N>100,000 for Monte Carlo convergence
- **Therefore create proxy model for L(X) from a smaller number of training points**
  - Typically few hundred or thousand

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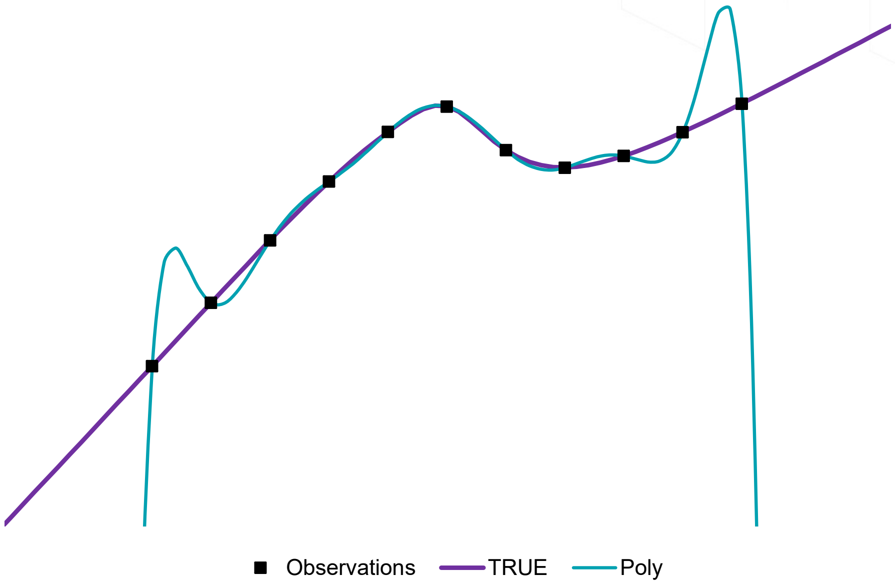
Polynomial approximations

- **Typically polynomials used to approximate balance sheet**
- **Well known issues with polynomials**
  - Oscillatory behaviour
  - Explosions when extrapolating
- **Multi-dimensions, cross terms**
  - 15 variables, 5^{th} order polynomial has >15,000 terms
  - Requires variable selection methods with p>>n



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GPs as alternative to Polynomials

- Relatively small number of observations
- Moderate dimensionality
- Well behaved, smooth but non-linear functions

Almost a perfect problem for Gaussian Processes!

Neural Network Gaussian Processes

- Single hidden layer neural network with activation function $h(x)$

$$f(\boldsymbol{x}) = \sum_i w_i h(a_i + \boldsymbol{b_i} \cdot \boldsymbol{x})$$

- With iid Gaussian prior on parameters $(a_i, \boldsymbol{b}_i) \sim N(0, \boldsymbol{\Sigma})$, $w_i \sim N(0, \sigma^2)$, in limit of infinite width $f(\boldsymbol{x})$ is Gaussian Process, with covariance

$$K(\boldsymbol{x}, \boldsymbol{x}') = \sigma^2 E[h(a + \boldsymbol{b} \cdot \boldsymbol{x}) h(a + \boldsymbol{b} \cdot \boldsymbol{x}')]$$

- Examples
  - Erf activation

$$K(\boldsymbol{x}, \boldsymbol{x}') = \frac{2\sigma^2}{\pi} \operatorname{asin}\left(\frac{2\overline{\boldsymbol{x}}^T \boldsymbol{\Sigma} \overline{\boldsymbol{x}}'}{\sqrt{(1+2\overline{\boldsymbol{x}}^T \boldsymbol{\Sigma} \overline{\boldsymbol{x}})(1+2\overline{\boldsymbol{x}}'^T \boldsymbol{\Sigma} \overline{\boldsymbol{x}}')}}\right) \qquad \overline{\boldsymbol{x}} = (1, \boldsymbol{x}^T)^T$$

  - ReLU activation

$$h(x) = \max(x, 0), \qquad K(\boldsymbol{x}, \boldsymbol{x}') = \frac{\sigma^2}{2\pi}\left\{\sqrt{(\overline{\boldsymbol{x}}^{\mathrm{T}} \boldsymbol{\Sigma}\, \overline{\boldsymbol{x}})(\overline{\boldsymbol{x}}'^{\mathrm{T}} \boldsymbol{\Sigma} \overline{\boldsymbol{x}}')} \sin(\theta) + \overline{\boldsymbol{x}}^T \boldsymbol{\Sigma} \overline{\boldsymbol{x}}'(\pi - \theta)\right\}$$

$$\theta = \operatorname{acos}\left(\frac{\overline{\boldsymbol{x}}^T \boldsymbol{\Sigma} \overline{\boldsymbol{x}}'}{\sqrt{(\overline{\boldsymbol{x}}^{\mathrm{T}} \boldsymbol{\Sigma}\, \overline{\boldsymbol{x}})(\overline{\boldsymbol{x}}'^{\mathrm{T}} \boldsymbol{\Sigma} \overline{\boldsymbol{x}}')}}\right)$$

Integrated Cumulative Gaussian Activation Function

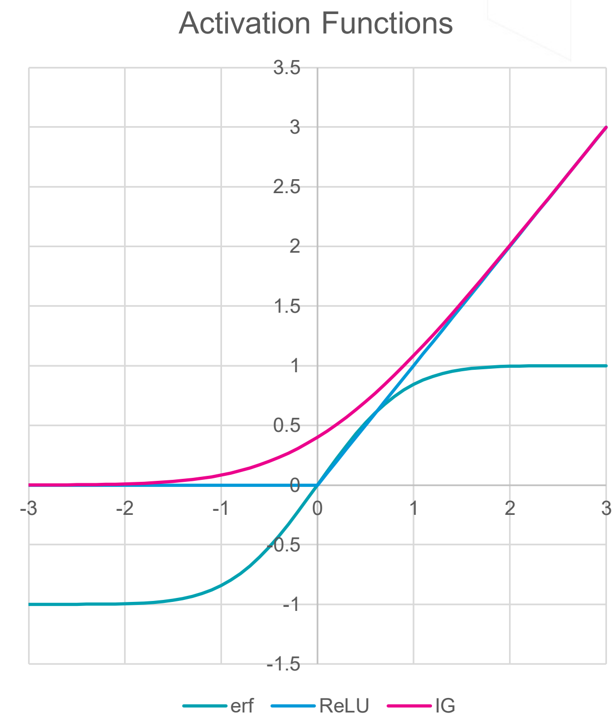
- Consider Integrated Cumulative Gaussian (ICG) function

$$h_{ICG}(x) = \int_{-\infty}^{x} \Phi(s)ds = x\Phi(x) + \phi(x)$$

$\Phi(x) = \int_{-\infty}^{x} \phi(s)ds$ is Gaussian CDF

$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ is Gaussian PDF

- ICG is a smoothed version of ReLU activation function

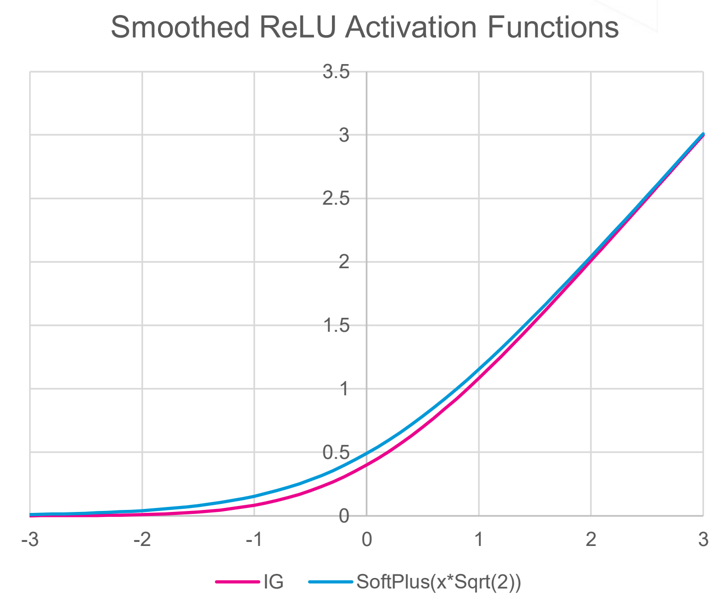


Smoothed ReLU Activation Functions

- ICG qualitatively similar shape to a scaled “SoftPlus”

$$h_{SoftPlus}(x) = \ln(1 + \exp(x))$$

- SoftPlus quicker to evaluate than ICG in Neural Network context
- However, ICG has analytically tractable GP limit



Integrated Cumulative Gaussian Covariance Function

For $h_{ICG}(x) = x\Phi(x) + \phi(x),\ (a, \boldsymbol{b}) \sim N(0, \boldsymbol{\Sigma})$

$$K(\boldsymbol{x}, \boldsymbol{x}') = \frac{\sigma^2}{2\pi}\left\{\sqrt{(1 + \overline{\boldsymbol{x}}^T\boldsymbol{\Sigma}\overline{\boldsymbol{x}})(1 + \overline{\boldsymbol{x}}'^{\mathrm{T}}\boldsymbol{\Sigma}\overline{\boldsymbol{x}}')}\,\sin\theta + \overline{\boldsymbol{x}}^T\boldsymbol{\Sigma}\overline{\boldsymbol{x}}'(\pi - \theta)\right\}$$

where $\overline{\boldsymbol{x}} = \left(1, \boldsymbol{x}^{\boldsymbol{T}}\right)^T$ and $\theta = \operatorname{acos}\left(\frac{\overline{\boldsymbol{x}}^T\boldsymbol{\Sigma}\overline{\boldsymbol{x}}\prime}{\sqrt{(1+\overline{\boldsymbol{x}}^T\boldsymbol{\Sigma}\overline{\boldsymbol{x}})(1+\overline{\boldsymbol{x}}\prime^T\boldsymbol{\Sigma}\overline{\boldsymbol{x}}\prime)}}\right)$

(hint, introduce parameters under the integral and differentiate)

Properties of the NN ICG Covariance Function

- Non-stationary
- Anisotropic
- Non-degenerate
- Infinitely differentiable
- Smooth samples
- For “large” x, x’, samples become approximately linear
- Derivatives of samples are samples from NN Erf GP

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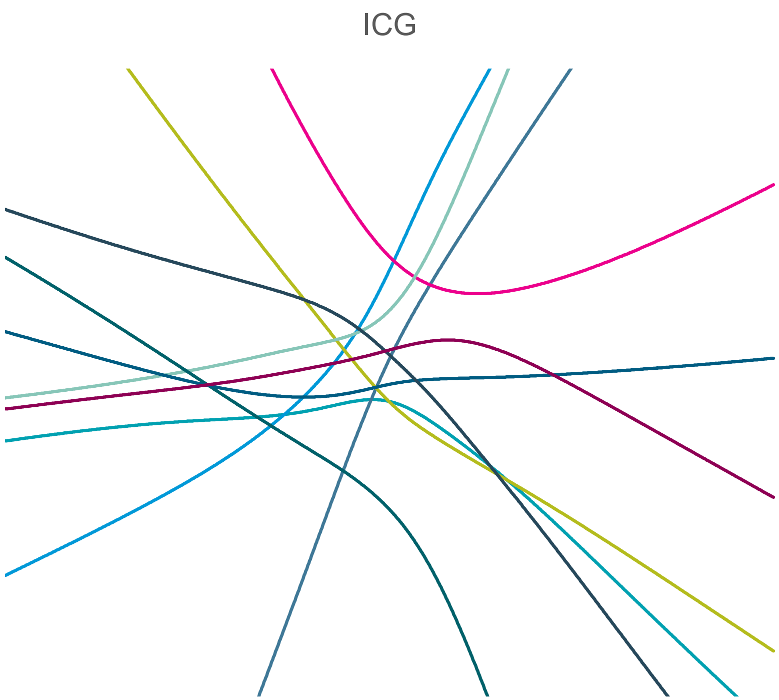
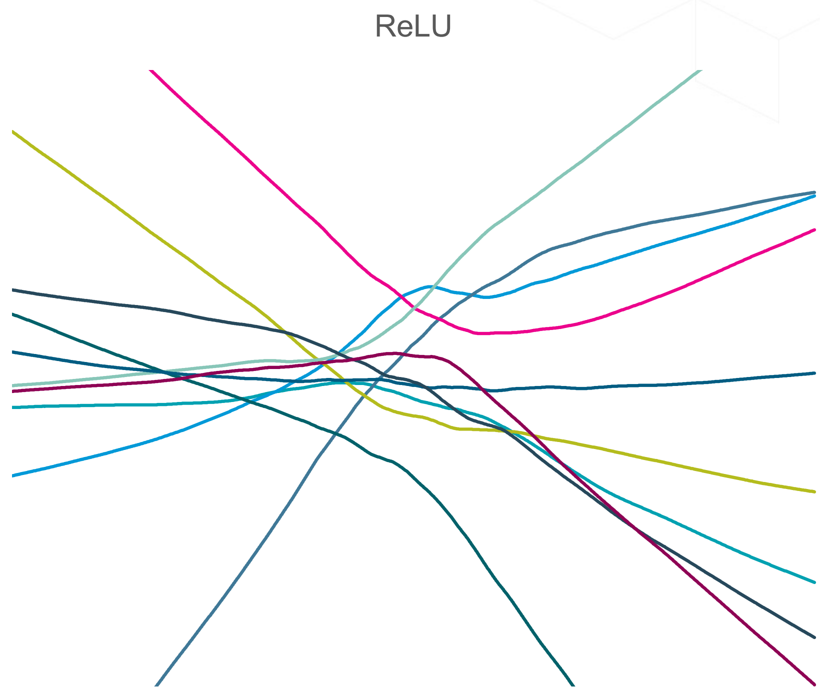
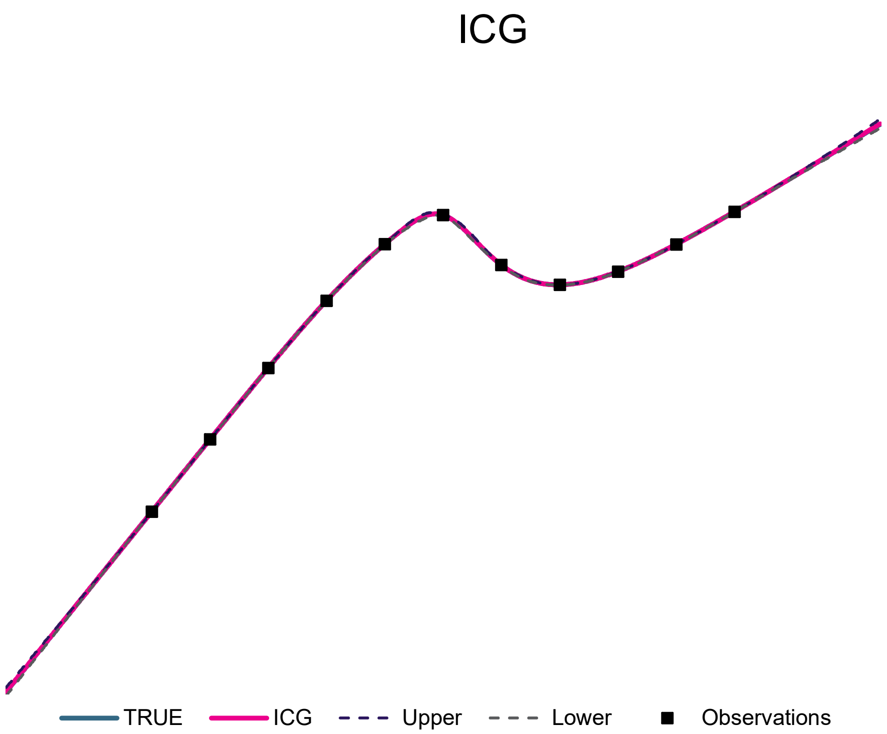
Prior Samples

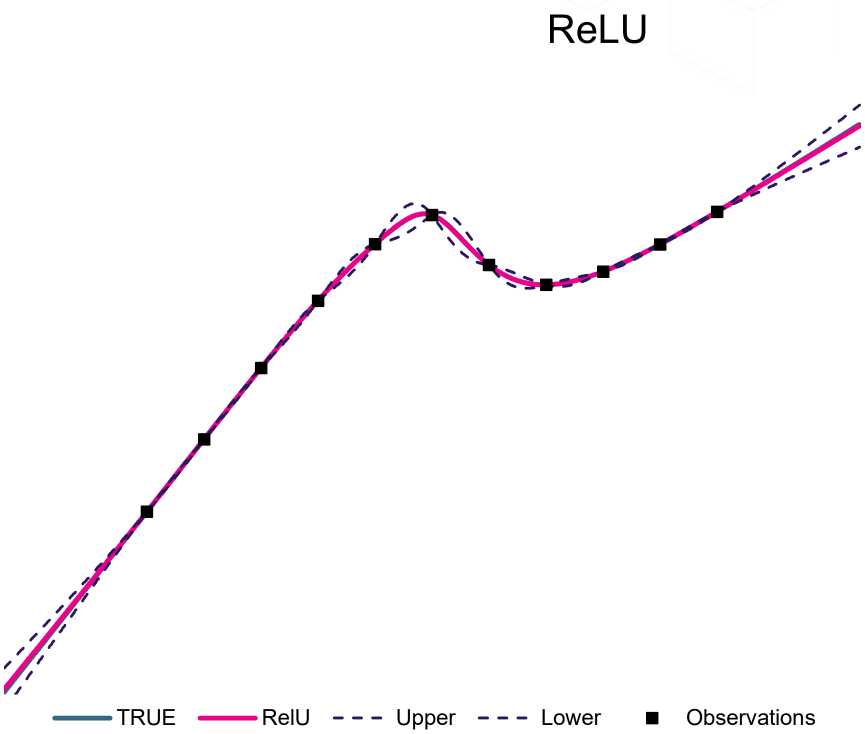




ICG has smoother samples than ReLU

Posterior Fit





Life insurance example

Synthetic example Life portfolio with 6 risk factors

- Two equity indices, mortality, short and long Term interest Rates, equity volatility
- Policies contain embedded investment options
- 256 Training Points

5th Order Polynomial, Stepwise Selection

Gaussian Process with Integrated Cumulative Gaussian Covariance

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Connection with Option Pricing

ICG activation function closely related to (Normal) Bachelier option pricing formula

$$V(S_0) = (S_0 - K)\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right)$$

$V(S_0)$ is price of European Call option

S_0 is current price

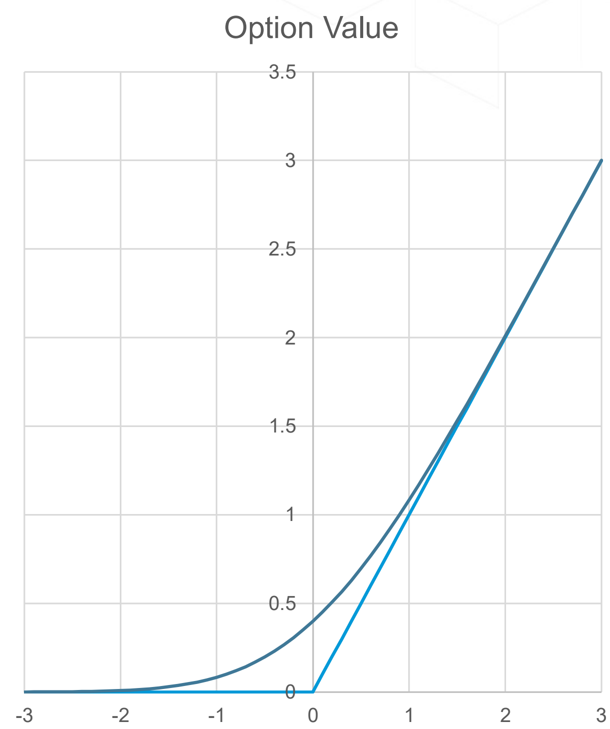
σ is volatility

T is time to maturity

K is strike price

Assuming risk free rate r=0 (similar formula, but more complicated, for $r \neq 0$)

Effectively using option pricing functions as basis functions!



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Summary

- Gaussian Processes are a powerful alternative to polynomials for creating proxy models of Life Insurance company balance sheets
- GPs achieve higher accuracy than polynomials
- Put another way, GPs can achieve same accuracy with fewer training points
  - Fewer training points = lower cost
  - Same number of training points = better model
- The Neural Network Integrated Gaussian Covariance function was derived for this use case, may be useful in other cases

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