INTRODUCTION TO MACHINE LEARNING

CS-1390 / PHY-1390

HOMEWORK ASSIGNMENT - 1

TOTAL MARKS - 78 marks

BONUS MARKS - 28 marks

MARKS DISTRIBUTION

Q.1	-	10
Q.2	-	15
Q.3	-	5
Q.4	-	10
Q.5	-	10
Q.6	-	9
Q.7	-	8
Q.8	-	5
Q.9	-	6
Q.10	-	10

IML - A1

1. Consider the 5X5 matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Find an invertible matrix P such that PA is a row-reduced echelon matrix R.

(b) Find a basis for the row space W of R

(c) say which vectors $(b_1, b_2, b_3, b_4, b_5)$ are in W

(d) Find the coordinates matrix of each vector $(b_1, b_2, b_3, b_4, b_5) \in W$ in the ordered basis chosen in (b).

(e) Write each vector $(b_1, b_2, b_3, b_4, b_5) \in W$ as a linear combination of the rows of A.

(f) give an explicit description of the null space of A

(g) Find a basis for the null space

(h) For what column matrices Y does the equation $AX = Y$ have solutions X?

(i) Explicitly find the range space of A and find a basis.

(j) Verify the relations regarding the nullity, row rank and column rank of A.

2. Consider the space of square-integrable real functions on the interval $[-\pi, \pi]$, $L_2([-\pi, \pi])$, and the associated orthonormal basis given by

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}} \right\}, n = 1, 2, \ldots$$

Consider the following two subspaces: S - space of symmetric functions, that is, $f(x) = f(-x)$, on $[-\pi, \pi]$, and A - space of antisymmetric functions, $f(x) = -f(-x)$, on $[-\pi, \pi]$.

(a) Show how any function $f(x)$ from $L_2([-\pi, \pi])$ can be written as $f(x) = f_s(x) + f_a(x)$, where $f_s(x) \in S$ and $f_a(x) \in A$.

(b) Give orthonormal basis for S and A

(c) Verify that $L_2([-\pi, \pi]) = S \oplus A$

3. We have 2 opaque bags, each containing 2 balls. One bag has 2 black balls and the other has a black ball and a white ball. You pick a bag at random and then pick one of the balls in that bag at random. When you look at the ball, it is black. You now pick the second ball from that same bag. What is the probability that this ball is also black?

(a) 1/4
(b) 1/3
(c) 1/2
(d) 2/3
(e) 3/4

4. Consider the least squares problem:
Given $m \times n$ set of equations $Ax = b$, where $m > n$ and rank$(A) = n$. Show that the following (a-d) together constitute a least square solution.
(a) Find the SVD $A = U \Sigma V^T$
(b) Set $b' = U^T b$
(c) Find the vector y defined by $y_i = b'_i / \sigma_i$, where σ_i is the i^{th} diagonal entry of Σ
(d) The solution is $x = Vy$

5. Consider the least squares problem:
Given $m \times n$ set of equations $Ax = b$, where $m > n$ and rank$(A) = r < n$. Show that the following (a-e) together constitute a general least square solution.
(a) Find the SVD $A = U \Sigma V^T$
(b) Set $b' = U^T b$
(c) Find the vector y defined by $y_i = b'_i / \sigma_i$ for $i = 1, \ldots r$ and $y_i = 0$ otherwise
(d) The solution x of minimum norm $\|x\|$ is

Date
Page

Vy.

(e) The general solution is

~~$x = Vy + \lambda_r +$~~

$$x = Vy + \lambda_{r+1} V_{r+1} + \dots + \lambda_n V_n$$

where $V_{r+1}, \dots, V_n$ are the last $n-r$ columns of V.

6. Suppose you have a deck of 52 cards, and you draw cards from the deck with replacement uniformly at random independently. Let $X_1, X_2, \dots, X_{50}$ be the outcomes of the first 50 draws. Thus, each random variable X_i can take values $1, \dots, 52$ and the probability that it takes each of these values is $1/52$.

(a) What is $E[X_1]$?

(b) Let $Z = X_1 - 2X_2 + 3X_3$. What is $E[Z]$?

(c) Let $Y = X_1 - X_2 + X_3 - X_4 + \dots + X_{49} - X_{50}$.
What is $E[Y]$?

7. Let X and Y be random variables with the following joint distribution :

	X=1	X=2	X=3	X=4
Y=0	1/18	1/18	1/9	1/9
Y=1	1/12	1/12	1/6	0
Y=2	0	1/30	1/30	4/15

1. What are the marginal distributions of X and Y?
2. Are X and Y independent? Justify your answer.
3. What is the conditional distribution of X, given that Y=2? What is $E[X|Y=2]$?
4. Calculate $E[X]$, $E[Y]$ and $E[XY]$.

8. Let $A \in \mathbb{R}^{m \times n}$. Show that the set of all vectors X such that $AX=0$ is a subspace of $\mathbb{R}^n$. This subspace is called the nullspace of A and it's dimension is the nullity of A.

9. Let $v_1 = [1,-1,2,0]$, $v_2 = [1,0,1,1]$, $v_3 = [1,-2,3,-1]$ and $v_4 = [3,1,2,4]$
   1. Are v_1, v_2, v_3, v_4 linearly independent? Justify your answer.
   2. Let V be the 4×4 matrix whose rows are $v_1, \ldots v_4$. What is the rank of V? Justify your answer.
   3. Write down a basis of the null-space of V and a basis of the range of V.

10. Assume a finite dimensional space $\mathbb{R}^N$ and an orthonormal basis $\{x_1, x_2, \ldots, x_N\}$. Vector y can thus be written as $y = \sum_i a_i x_i$ where $a_i = \langle x_i, y \rangle$. Consider the best approximation to y in the least squares sense and living on the subspace spanned by the first K vectors $\{x_1, x_2, \ldots x_K\}$ ~~or~~ $\hat{y} = \sum_{i=1}^{K} \beta_i x_i$. Prove that $\beta_i = a_i$ for $i = 1, \ldots, K$

by showing that it minimizes $\|y - \hat{y}\|$
(Hint: Use Parseval's equality)