3. a) $f(n) \notin o(g(n))$ & $f(n) \notin \omega(g(n)) \Rightarrow f(n) \in \Theta(g(n))$

False. Consider $f(n) = \begin{cases} e^n, & n \text{ is even} \\ 1 & n \text{ is odd} \end{cases}$

$g(n) = n$

~~f(g) doesn~~ It is evident that $f(n) \notin o(g(n))$ & $f(n) \notin \omega(g(n))$ as $\forall c > 0$, there exists no $n_0 > 0$ st $0 < f(n) < cg(n)$ or $0 \le cg(n) < f(n)$ & $\forall n \ge n_0$.

Also, $\exists$ no $c_1, c_2 > 0$ & $n_0 > 0$ st $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ $\forall n \ge n_0$ as $f(n)$ toggles between 1 & e^n based on parity (even/odd).
$\therefore f(n) \notin \Theta(g(n))$.

b) $f(n) \in \Theta(g(n))$ & $h(n) \in \Theta(g(n)) \Rightarrow \frac{f(n)}{g(n)} \in \Theta(1)$

$\exists c_1, n_0 > 0 \cdot 0 \le f(n) \le c_1 g(n)$ $\forall n \ge n_0$ (premise)
$\exists c_2, n_1 > 0 \cdot 0 \le h(n) \le c_2 g(n)$ $\forall n \ge n_1$

$$\frac{f(n)}{h(n)} \le \frac{c_1 \cancel{g(n)}}{c_2 \cancel{g(n)}} \le c_3 (1) \qquad \left[c_3 \ge \frac{c_1}{c_2}\right]$$

$$\Rightarrow \frac{f(n)}{h(n)} \in O(1) \qquad (1)$$

Similarly, $\exists c_4, n_2 > 0 \cdot$
$0 \le c_4 g(n) \le f(n)$ $\forall n \ge n_2$ (premise)
$\exists c_5, n_3 > 0 \cdot 0 \le c_5 g(n) \le h(n)$ $\forall n \ge n_3$

$$\frac{f(n)}{h(n)} \ge \frac{c_4 \cancel{g(n)}}{c_5 \cancel{g(n)}} \ge c_6 (1) \qquad \left[c_6 \le \frac{c_4}{c_5}\right]$$

$$\Rightarrow \frac{f(n)}{h(n)} \in \Omega(1) \qquad (2)$$

From (1) & (2), $\frac{f(n)}{h(n)} \in \Theta(1)$. □

c) $f(n) \in \Theta(g(n)) \Rightarrow 2^{f(n)} \in \Theta(2^{g(n)})$

<u>False</u>.

<u>Let</u> $f(n) = \log n^2$ & $g(n) = \log n$.

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\log n^2}{\log n} = \lim_{n\to\infty} \frac{2 \cancel{\log n}}{\cancel{\log n}} = 2$$

$\Rightarrow f(n) \in \Theta(g(n))$

Now, $2^{f(n)} = 2^{\log n^2} = n^2$
$2^{g(n)} = 2^{\log n} = n$

~~Let $p(n) = n^2$ $q(n) = n$~~ Let $p(n) = 2^{f(n)}$
$q(n) = 2^{g(n)}$

$$\lim_{n\to\infty} \frac{p(n)}{q(n)} = \lim_{n\to\infty} \frac{n^{\cancel{2}}}{\cancel{n}} = \infty$$

$\Rightarrow p(n) \in \omega(q(n))$
$\Rightarrow p(n) \notin O(q(n))$
$\Rightarrow p(n) \notin \Theta(q(n))$
$\Rightarrow 2^{f(n)} \notin \Theta(2^{g(n)})$

[Relationships b/w order notations from class]

d) $\min(f(n), g(n)) \in \Theta\left(\frac{f(n)\,g(n)}{f(n)+g(n)}\right)$

Assume $\min(f(n), g(n)) = f(n)$ w/o loss of generality.
& let $h(n) = \frac{f(n)\,g(n)}{f(n)+g(n)}$

$\exists\, c_1, c_2, n_0 > 0 \cdot 0 \le c_1 h(n) \le f(n) \le c_2 h(n) \;\forall n \ge n_0$.
[premise]

$$\frac{1}{2} f(n) = \frac{f(n)}{2} \cdot \frac{g(n)}{g(n)} \le \frac{f(n)\,g(n)}{f(n)+g(n)} \qquad \forall n > 0 \quad [f(n) \le g(n)]$$

$\Rightarrow f(n) \le 2h(n)$

$\therefore$ for $c_1 = 2$ & $n_0 = 1$, since $\exists\, c_1 > 0,\ n_0 > 0 \cdot 0 \le f(n) \le c_1 g(n)$
$\forall n \ge n_0$, $f(n) \in O(h(n))$.
(1)

Also, $\frac{f(n)g(n)}{f(n)+g(n)} \leq \frac{f(n)\cancel{g(n)}}{\cancel{g(n)}} = f(n)$ $\quad [f(n) \geq 0]$

$\therefore$ for $c_2 = 1$ & $n_1 = 1$, since $\exists\, c_2, n_1 > 0 \cdot 0 \leq c\,g(n) \leq f(n)$ $\forall\, n \geq n_1$, $f(n) \in \Omega(h(n))$. $\quad (2)$

From (1) & (2),
$f(n) \in \Theta(h(n))$

Similarly for $\min(f(n), g(n)) = g(n)$, $g(n) \in \Theta(h(n))$.

$\therefore \min(f(n), g(n)) = \Theta\left(\frac{f(n)\,g(n)}{f(n)+g(n)}\right)$. $\square$