Let's go step by step through the functions and provide reasoning to compare their growth rates. We will use standard asymptotic analysis techniques (Big-O and related concepts) to compare these functions. The goal is to show how each function grows as $n \to \infty$.

1 1. π (constant)

- **Explanation**: π is just a constant number (approximately 3.14159). It doesn't depend on n and thus remains fixed regardless of how large n becomes.
- **Growth**: Constant functions are considered $O(1)$. Therefore, for any n, a constant function grows slower than any non-constant function.

2 2. $\lg(\lg^* n)$ (iterated logarithm)

- **Explanation**: The iterated logarithm, $\lg^* n$, is the number of times the logarithm function needs to be applied to n before the result becomes less than or equal to 1. This function grows very slowly. For example:

$$\lg^*(10^{12}) = 5$$

$$\lg^*(10^{1000}) = 6$$

  Taking the logarithm of $\lg^* n$, which is already very slow-growing, makes $\lg(\lg^* n)$ grow extremely slowly.

- **Comparison**: Since $\lg^* n$ grows slower than any polynomial or exponential function, $\lg(\lg^* n)$ grows even slower. Therefore:

$$\lg(\lg^* n) = o(n^k) \quad \text{for any } k > 0$$

  Thus, it grows slower than any polynomial or exponential function.

3 3. e^{2n} (exponential function)

- **Explanation**: The exponential function grows rapidly. For large n, exponentials like e^{2n} grow much faster than polynomials or logarithms.
- **Comparison**: To compare exponential growth to other functions:

$$e^{2n} = \omega(n^k) \quad \text{for any } k > 0$$

  Exponentials grow faster than any polynomial but slower than factorial growth.

  Compared to $\lg(\lg^* n)$, the exponential e^{2n} grows dramatically faster:

$$\lg(\lg^* n) = o(e^{2n})$$

4 4. n^{2n} (super-exponential function)

- **Explanation**: The function n^{2n} is a super-exponential function. It grows much faster than a standard exponential function like e^{2n}. To see why:

  $$n^{2n} = (n^2)^n$$

  This means it is like raising n^2 to the power of n, which grows faster than e^{2n}, where the base is constant e.

- **Comparison**: n^{2n} grows faster than any exponential function because its base (which is n^2) increases with n:

  $$e^{2n} = o(n^{2n})$$

  Super-exponential growth dominates exponential growth.

5 5. $n!$ (factorial function)

- **Explanation**: The factorial function grows even faster than super-exponential functions. Stirling's approximation gives us an idea of how fast $n!$ grows:

  $$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

  As $n \to \infty$, the factorial grows faster than both n^{2n} and e^{2n}.

- **Comparison**:

  $$n^{2n} = o(n!)$$

  Factorial growth is the fastest of all the functions under consideration.

6 Final Sorted Order

Using the proofs above, the functions in increasing order of growth rate are:

1. π (constant)
2. $\lg(\lg^* n)$ (iterated logarithm)
3. e^{2n} (exponential)
4. n^{2n} (super-exponential)
5. $n!$ (factorial)

These comparisons prove that the given order is correct.