Let's go step by step through the functions and provide reasoning to compare their growth rates. We will use standard asymptotic analysis techniques (Big-O and related concepts) to compare these functions. The goal is to show how each function grows as $n \to \infty$.

1 1. π (constant)

- Explanation: π is just a constant number (approximately 3.14159). It doesn't depend on n and thus remains fixed regardless of how large n becomes.
- **Growth**: Constant functions are considered O(1). Therefore, for any n, a constant function grows slower than any non-constant function.

2 2. $\lg(\lg^* n)$ (iterated logarithm)

• **Explanation**: The iterated logarithm, $\lg^* n$, is the number of times the logarithm function needs to be applied to n before the result becomes less than or equal to 1. This function grows very slowly. For example:

$$\lg^*(10^{12}) = 5$$

$$\lg^*(10^{1000}) = 6$$

Taking the logarithm of $\lg^* n$, which is already very slow-growing, makes $\lg(\lg^* n)$ grow extremely slowly.

• Comparison: Since $\lg^* n$ grows slower than any polynomial or exponential function, $\lg(\lg^* n)$ grows even slower. Therefore:

$$\lg(\lg^* n) = o(n^k)$$
 for any $k > 0$

Thus, it grows slower than any polynomial or exponential function.

3 3. e^{2n} (exponential function)

- Explanation: The exponential function grows rapidly. For large n, exponentials like e^{2n} grow much faster than polynomials or logarithms.
- Comparison: To compare exponential growth to other functions:

$$e^{2n} = \omega(n^k)$$
 for any $k > 0$

Exponentials grow faster than any polynomial but slower than factorial growth.

Compared to $\lg(\lg^* n)$, the exponential e^{2n} grows dramatically faster:

$$\lg(\lg^* n) = o(e^{2n})$$

4 4. n^{2n} (super-exponential function)

• Explanation: The function n^{2n} is a super-exponential function. It grows much faster than a standard exponential function like e^{2n} . To see why:

$$n^{2n} = (n^2)^n$$

This means it is like raising n^2 to the power of n, which grows faster than e^{2n} , where the base is constant e.

• Comparison: n^{2n} grows faster than any exponential function because its base (which is n^2) increases with n:

$$e^{2n} = o(n^{2n})$$

Super-exponential growth dominates exponential growth.

5 5. n! (factorial function)

• Explanation: The factorial function grows even faster than super-exponential functions. Stirling's approximation gives us an idea of how fast n! grows:

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

As $n \to \infty$, the factorial grows faster than both n^{2n} and e^{2n} .

• Comparison:

$$n^{2n} = o(n!)$$

Factorial growth is the fastest of all the functions under consideration.

6 Final Sorted Order

Using the proofs above, the functions in increasing order of growth rate are:

- 1. π (constant)
- 2. $\lg(\lg^* n)$ (iterated logarithm)
- 3. e^{2n} (exponential)
- 4. n^{2n} (super-exponential)
- 5. n! (factorial)

These comparisons prove that the given order is correct.