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Section 2.8 and 3.1
Existance theorems:

Consider $y' = f(t,y)$, $y(0) = 0$

Thm: If $f$ and $\frac{\partial f}{\partial y}$ are continuous

on $|t| < a$ and $|y| < b$

then there is an interval

$|t| \leq h \leq a$ on why $y = \phi(t)$ has

a unique solution.

to find solutions compute a

sequence of functions

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\, dt$$

Recall if

$$F(t) = \int_{t_0}^{t} g(s)\, ds$$

then $F'(t) = g(t)$.

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solve $\frac{dy}{dt} = f(t, y)$

start $\phi_0(t) = 0$

$$\phi_1(t) = \int_0^t f(s, \phi_0(s))\, ds$$

$$\phi_2(t) = \int_0^t f(s, \phi_1(s))\, ds$$

$$\vdots$$

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\, ds$$

then $\lim_{n \to \infty} \phi_n(t) = \phi(t)$

If the seq. stops we are done:

$$\cancel{\phi_{n+1}(t) = \int_0^t f(s, \phi_{n+1}(s))}$$

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_{n+1}(s))\, ds$$

$$\frac{d}{dt} \phi_{n+1}(t) = f(t, \phi_{n+1}(t))$$

Let $y = \phi_{n+1}(t)$

then

$$\frac{dy}{dt} = f(t, y) \qquad \text{solution.}$$

Pb #4 approximate

2.8 p1

(3)

$y' = -\frac{1}{2}y + t \qquad y(0)=0,$

$\phi_0 = 0 = \varphi_0(t)$

$\phi_1 = \int_0^t (0+s)\,dt$

$= \frac{t^2}{2} = y$

$\phi_2 = \int_0^t \left(-\frac{1}{2}\left(\frac{s^2}{2}\right) + s\right) ds$

$= \frac{-s^3}{3\cdot 2^2} + \frac{s^2}{2}$

$\phi_3 = \int_0^t -\frac{1}{2}\left[\frac{-s^3}{3\cdot 2^2} + \frac{s^2}{2}\right] + s \,] ds$

$= \frac{t^4}{4\cdot 3\cdot 2^3} - \frac{1}{2}\frac{t^3}{3\cdot 2} + \frac{t^2}{2}$

$= \frac{t^4}{4!\ 2^2} - \frac{t^3}{3!\cdot 2} + \frac{t^2}{2!}$

#4

2.8 p2

$$\phi_4 = \int_0^t -\frac{1}{2}\left[\frac{s^4}{4!\,2^2} - \frac{s^3}{3!\cdot 2} + \frac{s^2}{2!}\right] + s \, ds$$

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$$= \frac{-s^5}{5\cdot 4!\,2^3} + \frac{s^4}{4!\,2^2} - \frac{s^3}{3!\cdot 2} + \frac{s^2}{2}\Big|_0^t$$

$$= \frac{-t^5}{5!\cdot 2^3} + \frac{t^4}{4!\,2^2} - \frac{t^3}{3!\cdot 2} + \frac{t^2}{2} = \phi_4$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, t^{k+1}}{(k+1)!\; 2^k} = \phi(t)$$

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④ a

3.1 homogeneous diff eq with constant co-efficients.

homogeneous means $=0$

Example: Solve $y''-y=0$

Solve $y'=y$, $y=e^t$, $''$?

try $e^{-t}$

$$(e^{-t})''-y=(-e^{-t})'-e^{-t}=e^{-t}-e^{-t}=0$$

so $y''-y$ has 2 solutions; $y=e^t$, $y=e^{-t}$

$5e^t$ is also a solution.

in fact if $c_1$, $c_2$ are constants

$$\boxed{y=c_1e^t+c_2e^{-t}}$$ is

a solution to $y''-y=0$

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Specify initial conditions:

$y(0) = 2$; 2 constants

$y'(0) = -1$

Start $y = c_1 e^{t} + c_2 e^{-t}$

then $c_1 e^0 + c_2 e^0 = 2$

$c_1 e^0 - c_2 e^0 = -1$ (derivative)

so $c_1 + c_2 = 2$

$c_1 - c_2 = -1$

add $2c_1 = 1$ ; $c_1 = 1/2$

$c_2 = 3/2$

the solution is

$$y = \frac{1}{2} e^{t} + \frac{3}{2} e^{-t}$$

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general problem:

solve $ay'' + by' + cy = 0$

$a, b, c$ are constants.

char. polynomial $ar^2 + br + c = 0$

guess $y = e^{r_1 t}$ ; $y = e^{r_2 t}$

$$a(e^{r_1 t})'' + b(e^{r_1 t})' + ce^{r_1 t} = 0$$

$$ar_1^2 e^{r_1 t} + br_1 e^{r_1 t} + ce^{r_1 t} = 0$$

factor

$$e^{r_1 t}(ar_1^2 + br_1 + c) = 0$$

if $e^{r_1 t}$ and $e^{r_2 t}$ are solutions $r_1$ and $r_2$ satisfy the char. poly.

For constant $c_1$ and $c_2$

$c_1 e^{r_1 t} + c_2 e^{r_2 t}$ are solutions

to $ay'' + by' + cy = 0 \leftarrow$ need "0"

Suppose $y(t_0) = y_0$ ; $y'(t_0) = y_0'$

are initial conditions:

to find the solution need

$c_1$ and $c_2$ with.

$$c_1 e^{r_1 t} + c_2 e^{r_2 t} = y_0$$

$$c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t} = y_0'$$

$$c_1 \cdot -r_2 e^{r_1 t_0} + \overset{-r_2}{c_2} e^{r_2 t_0} = -r_2 y_0$$

$$c_1 (r_1 - r_2) e^{r_1 t_0} = y_0' - r_1 y_0$$

finally

$$C_1 = \left(\frac{y_0' - r_2 y_0}{r_1 - r_2}\right) e^{-r_1 t_0}$$

$$C_2 = \left(\frac{y_0 r_1 - y_0'}{r_1 - r_2}\right) e^{-r_2 t_0}$$

So we can solve explicitly given initial conditions.

 

**#16** $y'' - y' - 2y = 0$

$y(0) = \alpha$; $y'(0) = 2$

Find $\alpha$ so that the solution goes to 0 as $t \to \infty$.

Compute $r^2 - r - 2 = 0$

$$(r-2)(r+1) = 0$$

$e^{2t}$ and $e^{-t}$ are solutions.

Find $c_1$ and $c_2$

$$y = c_1 e^{2t} + c_2 e^{-t} = y$$

$$y' = 2c_1 e^{2t} - c_2 e^{-t}$$

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$$2c_1 - c_2 = 2$$

$$c_1 + c_2 = \alpha$$

so $c_1 = \dfrac{\alpha + 2}{3}$

$$c_2 = \alpha - \left(\frac{\alpha+2}{3}\right) = \frac{2\alpha - 2}{3}$$

the solution is

$$y = \frac{\alpha+2}{3} e^{2t} + \frac{2\alpha-2}{3} e^{-t}$$

we want $y(t) \to 0$ as $t \to \infty$

the $e^{-t} \to 0$ as $t \to \infty$

get rid of $e^{2t}$

$\dfrac{\alpha+2}{3} = 0$ and $\alpha = -2$