§2.8 The Existence and Uniqueness Theorem

Goal To prove that the IVP

$$\begin{cases} y' = F(t,y) \\ y(t_0) = y_0 \end{cases}$$

has a unique short-term solution when F and $\frac{\partial F}{\partial y}$ are continuous near (t_0, y_0).

Note It suffices by a linear change of variables to solve

$$\begin{cases} y' = F(t,y) \\ y(0) = 0 \end{cases} \quad (*)$$

By the FTC, ϕ solves $(*)$ if and only if $\phi(0) = 0$ and

$$\phi(t) = \phi(t) - \underbrace{\phi(0)}_{0} = \int_0^t F(s, \phi(s))\,ds \qquad (**)$$

We use the method of successive approximations to construct a function ϕ satisfying $(**)$:

Step I Let $\phi_0(t) = 0$.

Step II Define $\phi_{n+1}(t)$ recursively by

$$\phi_{n+1}(t) = \int_0^t F(s, \phi_n(s))\,ds.$$

Step III Let $\phi_n(t) = \lim_{n\to\infty} \phi_n(t)$.

[redacted] **Thm** On some [redacted] interval around t_0, $\phi_n(t) \to \phi(t)$, [redacted] where ϕ solves $(**)$ uniquely.

Proof Hard! Take a course in mathematical analysis. The essential computation:

Hard to justify!

$$\phi(t) = \lim_{n\to\infty} \phi_{n+1}(t) = \lim_{n\to\infty} \int_0^t F(s, \phi_n(s))\,ds = \int_0^t \lim_{n\to\infty} F(s, \phi_n(s))\,ds = \int_0^t F(s, \lim_{n\to\infty}\phi_n(s))\,ds = \int_0^t F(s, \phi(s))\,ds.$$

Ex 1 Solve the IVP

$$\begin{cases} y' = 2t(1+y) \\ y(0) = 0 \end{cases} \longleftarrow \text{Important!}$$

by the method of successive approximation.

Step I $\quad \phi_0(t) = 0 \qquad f(s,y) = 2t(1-y)$

Step II $\quad \phi_1(t) = \int_0^t f(s, \phi_0(s))\,ds$

$$= \int_0^t f(s,0)\,ds$$

$$= \int_0^t 2s(1+0)\,ds$$

$$= \int_0^t 2s\,ds$$

$$= [s^2]_0^t$$

$$= t^2$$

$$\phi_2(t) = \int_0^t f(s, \phi_1(s))\,ds$$

$$= \int_0^t f(s, s^2)\,ds$$

$$= \int_0^t 2s(1+s^2)\,ds$$

$$= \int_0^t (2s + 2s^3)\,ds$$

$$= [s^2 + \tfrac{1}{2}s^4]_0^t$$

$$= t^2 + \tfrac{1}{2}t^4$$

$$\phi_3(t) = \int_0^t f(s, \phi_3(s))\,ds$$

$$= \int_0^t 2s(1 + s^2 + \tfrac{1}{2}s^4)\,ds$$

$$= \int_0^t (2s + 2s^3 + s^5)\,ds$$

$$= [s^2 + \tfrac{1}{2}s^4 + \tfrac{1}{6}s^6]_0^t$$

$$= t^2 + \tfrac{1}{2}t^4 + \tfrac{1}{6}t^6$$

Etc. $\quad = t^2 + \frac{1}{2}t^4 + \frac{1}{2\cdot 3}t^6$

In general,

$$\phi_n(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots + \frac{t^{2n}}{n!} = \sum_{k=1}^{n} \frac{t^{2k}}{k!}$$

Step III $\quad \phi(t) = \lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{t^{2k}}{k!} = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$

This is a series solution, so we need to Find the interval of convergence:

Ratio test: $a_k = \frac{t^{2k}}{k!}$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{t^{2(k+1)}/(k+1)!}{t^{2k}/k!} \right| = \lim_{k\to\infty} \left| \frac{t^{2k+2}}{t^{2k}} \cdot \frac{k!}{(k+1)!} \right|$$

$$= \lim_{k\to\infty} \left| \frac{t^2}{k+1} \right| = 0 < 1 \quad \text{For all } t$$

$$\Longrightarrow \boxed{\phi(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!}} \text{ converges For all } t, \text{ i.e., on } (-\infty, \infty)$$

Note: One may check that $\phi' = 2t(1+\phi)$ by differentiating term-by-term:

check if this is my solution ?

$$\phi' = \sum_{k=1}^{\infty} \frac{2k\,t^{2k-1}}{k!} = 2 \sum_{k=1}^{\infty} \frac{t^{\overbrace{2k-1}^{2(k-1)+1}}}{(k-1)!} = 2 \sum_{k=0}^{\infty} \frac{t^{2k+1}}{k!}$$

$$2t(1+\phi) = 2t\left(1 + \sum_{k=1}^{\infty} \frac{t^{2k}}{k!}\right) = 2t\left(\sum_{k=0}^{\infty} \frac{t^{2k}}{k!}\right) = 2\sum_{k=0}^{\infty} \frac{t^{2k+1}}{k!}$$

So $\phi' = 2t(1+\phi)$!