7.4 Theory of System of First Order Linear Equations

The general form of a system of n first order linear equations is

$$\begin{aligned} x_1' &= p_{11}(t)\, x_1 + \cdots + p_{1n}(t) x_n + g_1(t) \\ \vdots &= \vdots \\ x_n' &= p_{n1}(t)\, x_1 + \cdots + p_{nn}(t) x_n + g_n(t) \end{aligned}$$

We can write it in matrix form

$$\vec{x}' = P(t)\, \vec{x} + \vec{g}(t)$$

The corresponding homogeneous system is

$$\vec{x}' = P(t)\, \vec{x}.$$

Principle of Superposition If the vector functions $\mathbf{x}^{(1)}(t), \cdots, \mathbf{x}^{(n)}(t)$ are solutions of the homogeneous system, then

~~the~~ $c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \cdots + c_n \vec{x}_n(t)$ is also a solution for any constant c_i

The Wronskian of these n functions are

$$W(\vec{x}_1, \vec{x}_2, \cdots \vec{x}_n) = \det \left[\vec{x}^{(1)}(t) \;\; \vec{x}_2^{(2)}(t) \; \cdots \; \vec{x}^{(n)}(t) \right]$$

We say the vector functions $\mathbf{x}^{(1)}(t), \cdots, \mathbf{x}^{(n)}(t)$ are solutions form **a fundamental set of solutions** if they are linearly independent. ~~at each point~~

or the Wronskian is not zero.

In this case, each solution $\mathbf{x}(t)$ of the homogeneous system can be express as

$$\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + \cdots + c_n \vec{x}^{(n)}(t)$$

If $\mathbf{x}_p(t)$ is a particular solution of the nonhomogeneous system, the general solution is

$$\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + \cdots + c_n \vec{x}^{(n)}(t) + \vec{x}_p(t)$$