MAT-MEK Mandatory Assignment 1

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Equations from the assignment

Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \tag{1.2}$$

Discretized version of the wave equation:

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = c^2 \left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2} \right) \tag{1.3}$$

Exact solution for the Dirichlet problem:

$$u(t, x, y) = \sin(k_x x) \sin(k_y y) \cos(\omega t) \tag{1.4}$$

1.2.1: Finding the dispersion coefficient

We find ω by solving Eq.1.2, inserted for u its value from Eq.1.4.

This gives us:

$$\begin{aligned} \frac{\partial u^2}{\partial t^2} &= \sin(k_x x) \sin(k_y y)(-\omega^2) \cos(\omega t) \\ &= -\omega^2 u \end{aligned}$$

From the opposite side of Eq.1.2, we have:

$$\begin{aligned} c^2 \nabla^2 u &= c^2(-k_x^2) \sin(k_x x) \sin(k_y y) \cos(\omega t) + c^2 \sin(k_x x)(-k_y^2) \sin(k_y y) \cos(\omega t) \\ &= c^2(-k_x^2 - k_y^2) u \end{aligned}$$

Hence, in order for the equation to hold, we must have:

$$\begin{aligned} -\omega^2 u &= c^2(-k_x^2 - k_y^2) u \\ \omega^2 &= c^2(k_x^2 + k_y^2) \\ \omega &= c\sqrt{k_x^2 + k_y^2} \end{aligned} \tag{2.1}$$

Here, $k_x = \pi m_x$ and $k_y = \pi m_y$.

1.2.3: Exact solution

The two stationary solutions to the wave equations are real and imaginary components of the waves

$$u(t, x, y) = e^{\imath(k_x x + k_y y - \omega t)} \tag{1.6}$$

where the imaginary unit $\imath = \sqrt{-1}$. Show that Eq.1.6 satisfies the wave equation.

We show that the wave equation holds by inserting the above expression of u. Note that we on line 3 use Eq.2.1 to replace ω:

$$\begin{aligned}
\frac{\partial u^2}{\partial t^2} &= e^{\imath k_x x} e^{\imath k_y y} (-\imath\omega)^2 e^{-\imath\omega t} \\
&= -\omega^2 e^{\imath k_x x} e^{\imath k_y y} e^{-\imath\omega t} \\
&= -c^2 (k_x^2 + k_y^2) e^{\imath k_x x} e^{\imath k_y y} e^{-\imath\omega t} \\
&= c^2 ((-k_x^2) e^{\imath k_x x} e^{\imath k_y y} e^{-\imath\omega t} + e^{\imath k_x x} (-k_y^2) e^{\imath k_y y} e^{-\imath\omega t}) \\
&= c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\
&= c^2 \nabla^2 u
\end{aligned}$$

1.2.4: Dispersion coefficient

Assume that $m_x = m_y$ such that $k_x = k_y = k$. A discrete version of Eq.1.6 will then read

$$u_{ij}^n = e^{\imath(kh(i+j) - \tilde{\omega}\Delta t)} \tag{1.7}$$

where $\tilde{\omega}$ is a numerical dispersion coefficient, i.e., the numerical approximation of the exact ω. Insert Eq.1.7 into the discretized Eq.1.3 and show that for CFL number $C = 1/\sqrt{(2)}$ we get $\tilde{\omega} = \omega$.

Inserting Eq.1.7 into Eq.1.3, we get:

$$\begin{aligned}
&e^{\imath(kh(i+j))} \left(e^{-\imath\tilde{\omega}(n+1)\Delta t} - 2e^{-\imath\tilde{\omega}n\Delta t} + e^{-\imath\tilde{\omega}(n-1)\Delta t} \right) \\
&\quad = C^2 e^{-\imath\tilde{\omega}n\Delta t} \left(2e^{\imath kh(i+j+1)} + 2e^{\imath kh(i+j-1)} - 4e^{\imath kh(i+j)} \right)
\end{aligned}$$

Dividing by $e^{\imath(kh(i+j))} e^{-\imath\tilde{\omega}n\Delta t}$ on both sides, we get:

$$e^{-\imath\tilde{\omega}\Delta t} - 2 + e^{\imath\tilde{\omega}\Delta t} = C^2 \left(2e^{\imath kh} + 2e^{-\imath kh} - 4 \right)$$

Using the trigonometric identity $2\cos(x) = e^{ix} + e^{-ix}$ for $x = \tilde{\omega}\Delta t$ and $x = kh$ on the left and right hand side of the equation respectively, we get:

$$\begin{aligned}
2\cos(\tilde{\omega}\Delta t) - 2 &= C^2 \left(4\cos kh - 4 \right) \\
\cos(\tilde{\omega}\Delta t) &= C^2 \left(2\cos kh - 2 \right) + 1
\end{aligned}$$

Inserting $C = 1/\sqrt{2}$, we get:

$$\begin{aligned}
\cos(\tilde{\omega}\Delta t) &= \cos kh - 1 + 1 \\
\tilde{\omega}\Delta t &= \cos^{-1}(\cos kh) \\
\tilde{\omega} &= \frac{kh}{\Delta t}
\end{aligned}$$

Now, $C = \frac{c\delta t}{h} = 1/\sqrt{2}$ implies that $\frac{h}{\Delta t} = c\sqrt{2}$. Furthermore, $k = \sqrt{k^2} = \sqrt{1/2(k_x^2 + k_y^2)} = 1/\sqrt{2}\sqrt{k_x^2 + k_y^2}$. Inserting this results in:

$$
\begin{aligned}
\tilde{\omega} &= \frac{kh}{\Delta t} \\
&= \frac{1}{\sqrt{2}}\sqrt{k_x^2 + k_y^2}c\sqrt{2} \\
&= c\sqrt{k_x^2 + k_y^2} \\
&= \omega
\end{aligned}
$$

where the last line is the formula for ω derived in 1.2.1. This concludes the proof.