Assignment 3

Gautham Bellamkonda - CS20BTECH11017

Download all python codes from

https://github.com/GauthamBellamkonda/AI1103/tree/main/ Assignment3/Codes

and latex-tikz codes from

https://github.com/GauthamBellamkonda/AI1103/tree/main/ Assignment3

1 Problem

(GATE 45) Consider a discrete-time channel Y = X + Z, where the additive noise Z is signal dependent. In particular, given the transmitted symbol $X \in \{-a, a\}$ at any instant, the noise sample Z is chosen indepedently from a Gaussian distribution with mean βX and unit variance. Assume a threshold detector with zero threshold at the receiver. When $\beta = 0$, the BER was found to be $Q(a) = 1 \times 10^{-8}$.

$$\left(Q(v) = \frac{1}{\sqrt{2\pi}} \int_{v}^{\infty} e^{-\frac{u^2}{2}} du, \text{ and for } v > 1, \text{ use } Q(v) = e^{\frac{-v^2}{2}}\right)$$
(1.0.1)

When $\beta = -0.3$, BER is closest to

- (A) 10^{-7}
- (B) 10^{-6}
- (C) 10^{-4}
- (D) 10^{-2}

2 Solution

Given that the threshold of the detector is zero. Define a detector function g such that

$$g(Y) = \begin{cases} +a & Y > 0 \\ -a & Y < 0 \end{cases}$$
 (2.0.1)

It is given that $X \in \{-a, a\}$ is a random variable.

$$\therefore \Pr(X = a) = \Pr(X = -a) = \frac{1}{2}$$
 (2.0.2)

Since the noise in the signal, Z is chosen independently from a Gaussian distribution with mean $\mu = \beta X$ and unit variance, it follows that

$$F_Z(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(z - \beta X)^2}{2}\right) dz$$
 (2.0.3)

$$= \int_{-\infty}^{z-\beta X} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz \tag{2.0.4}$$

$$= \int_{\beta X-z}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz \tag{2.0.5}$$

$$=Q(\beta X-z) \tag{2.0.6}$$

Also, it is easy to see that

$$Q(-v) = 1 - Q(v) \ \forall \ v \in \mathbb{R}$$
 (2.0.7)

The detector can record erroneous bits in the signal iff

$$X > 0$$
, $g(Y) = -a$ (Call this BER_{+a}) or (2.0.8)

$$X < 0$$
, $g(Y) = a$ (Call this BER_{-a}) (2.0.9)

:. BER_{+a} =
$$Pr(g(Y) = -a \mid X = a) Pr(X = a)$$
 (2.0.10)

$$= \Pr(Y < 0 \mid X = a) \Pr(X = a)$$
 (2.0.11)

$$= \frac{1}{2} \times \Pr(X + Z < 0 \mid X = a)$$
 (2.0.12)

$$= \frac{1}{2} \times F_Z(-a) \tag{2.0.13}$$

$$= \frac{1}{2} \times Q(\beta X + a) \text{ (From (2.0.6))}$$
 (2.0.14)

$$= \frac{1}{2} \times Q(a(1+\beta))$$
 (2.0.15)

$$BER_{-a} = Pr(g(Y) = a \mid X = -a) Pr(X = -a)$$
 (2.0.16)

$$= \Pr(Y > 0 \mid X = -a) \Pr(X = -a)$$
 (2.0.17)

$$= \frac{1}{2} \times \Pr(X + Z > 0 \mid X = -a)$$
 (2.0.18)

$$= \frac{1}{2} \times (1 - F_Z(a)) \tag{2.0.19}$$

$$= \frac{1}{2} \times (1 - Q(\beta X - a)) \quad \text{(From (2.0.6))} \quad (2.0.20)$$

$$= \frac{1}{2} \times Q(a(1+\beta)) \text{ (From (2.0.7))}$$
 (2.0.21)

$$\therefore BER = BER_{+a} + BER_{-a} \qquad (2.0.22)$$

$$= Q(a(1+\beta)) \tag{2.0.23}$$

When $\beta = 0$, it is given that

BER =
$$Q(a) = 10^{-8}$$
 (2.0.24)

On computing, $Q(1) \approx 0.16$. Since Q(a) < Q(1), it is easy to see that a > 1 (as Q(x) is a decreasing function)

$$\therefore e^{-a^2/2} = 10^{-8} \tag{2.0.25}$$

$$\Leftrightarrow a \approx 6.069 \tag{2.0.26}$$

When $\beta = -0.3$,

BER =
$$Q(a(1 + \beta)) = Q(6.069 \times (1 - 0.3))$$
 (2.0.27)

$$= Q(6.069 \times 0.7) \tag{2.0.28}$$

$$= Q(4.249) \tag{2.0.29}$$

$$\approx \exp(-\frac{4.249^2}{2})\tag{2.0.30}$$

$$\approx 1.2 \times 10^{-4} \tag{2.0.31}$$

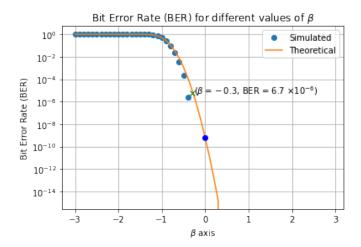


Fig. 1: Theory vs Simulated plot of BER

Therefore, when $\beta = -0.3$, BER is closest to 10^{-4} and option (C) is correct.