Assignment 3

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Download all python codes from

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https://github.com/GauthamBellamkonda/AI1103/tree/main/
    Assignment3/Codes
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and latex-tikz codes from

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https://github.com/GauthamBellamkonda/AI1103/tree/main/
    Assignment3
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1 Problem

(GATE 45) Consider a discrete-time channel $Y = X+Z$, where the additive noise Z is signal dependent. In particular, given the transmitted symbol $X \in \{-a, a\}$ at any instant, the noise sample Z is chosen indepedently from a Gaussian distribution with mean βX and unit variance. Assume a threshold detector with zero threshold at the receiver. When $\beta = 0$, the BER was found to be $Q(a) = 1 \times 10^{-8}$.

$$\left(Q(v) = \frac{1}{\sqrt{2\pi}} \int_v^\infty e^{-\frac{u^2}{2}} du, \text{ and for } v > 1, \text{ use } Q(v) = e^{\frac{-v^2}{2}}\right) \tag{1.0.1}$$

When $\beta = -0.3$, BER is closest to

(A) 10^{-7}
(B) 10^{-6}
(C) 10^{-4}
(D) 10^{-2}

2 Solution

Given that $X \in \{-a, +a\}$ is a random variable.

$$\Pr(X = a) = \frac{n(X = a)}{2} = \frac{1}{2} \tag{2.0.1}$$
$$\Pr(X = -a) = \frac{n(X = -a)}{2} = \frac{1}{2} \tag{2.0.2}$$

Also, Z is chosen from Gaussian Distribution with mean βX and unit variance.

$$\therefore \Pr(Z = p) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(p - \beta X)^2}{2}\right) \tag{2.0.3}$$
$$\Leftrightarrow \Pr(Y = p + X) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(p - \beta X)^2}{2}\right) \tag{2.0.4}$$
$$\Leftrightarrow \Pr(Y = u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(u - X - \beta X)^2}{2}\right) \tag{2.0.5}$$
$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(u - X(1 + \beta))^2}{2}\right) \tag{2.0.6}$$
$$\therefore \Pr(Y = u|X = a) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(u - a(1 + \beta))^2}{2}\right) \tag{2.0.7}$$
$$\& \Pr(Y = u|X = -a) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(u + a(1 + \beta))^2}{2}\right) \tag{2.0.8}$$

Since $X \in \{-a, +a\}$ is also a random variable,

$$\Pr(Y = u) = \Pr(Y = u|X = a)\Pr(X = a) + \Pr(Y = u|X = -a)\Pr(X = -a) \tag{2.0.9}$$
$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(u - a(1 + \beta))^2}{2}\right) + \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(u + a(1 + \beta))^2}{2}\right) \tag{2.0.10}$$

Therefore, the resultant signal $Y = X + Z$ is comprised of X which can take either positive or negative value, and some noise Z. The detector (which has zero threshold) can give us incorrect bits when $X = +a$ and $Y < 0$ (BER_{+a}) or $X = -a$ and $Y < 0$ (BER_{-a}), as shown in the graph below.

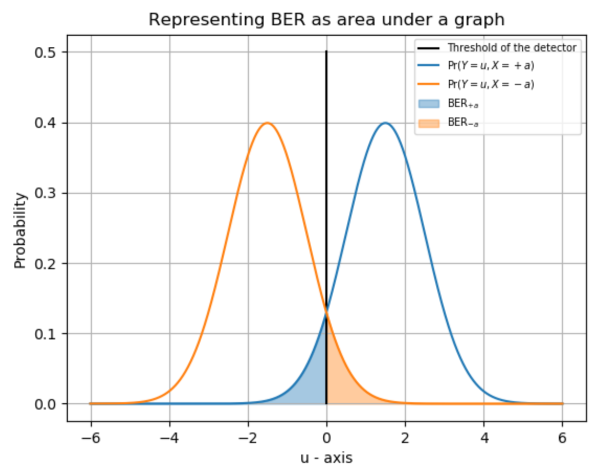

Fig. 1: The PDF of Y

$$\therefore \text{BER} = \text{BER}_{+a} + \text{BER}_{-a} \tag{2.0.11}$$
$$= \Pr(Y < 0, X = a) + \Pr(Y > 0, X = -a) \tag{2.0.12}$$
$$= \Pr(Y < 0|X = a)\Pr(X = a) + \Pr(Y > 0|X = -a)\Pr(X = -a) \tag{2.0.13}$$
$$= \int_{-\infty}^{0} \frac{1}{2} \cdot \Pr(Y = u|X = a)du + \int_{0}^{\infty} \frac{1}{2} \cdot \Pr(Y = u|X = -a)du \tag{2.0.14}$$

On substituting the values of $\Pr(Y = u|X = a)$ and $\Pr(Y = u|X = -a)$ from 2.0.7 and 2.0.8,

$$\text{BER} = \int_{-\infty}^{0} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(u - a(1+\beta))^2}{2}\right) du + \int_{0}^{\infty} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(u + a(1+\beta))^2}{2}\right) du \tag{2.0.15}$$

$$= \int_{0}^{\infty} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(u + a(1+\beta))^2}{2}\right) du + \int_{0}^{\infty} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(u + a(1+\beta))^2}{2}\right) du \tag{2.0.16}$$

$$= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(u + a(1+\beta))^2}{2}\right) du \tag{2.0.17}$$

From the definition of $Q(v)$, it is easy to see that the expression in 2.0.17 is equal to $Q(a(1+\beta))$

$$\therefore \text{BER} = Q(a(1+\beta)) \tag{2.0.18}$$

When $\beta = 0$, it is given that

$$\text{BER} = Q(a) = 10^{-8} \tag{2.0.19}$$

On computing, $Q(1) \approx 0.16$. Since $Q(a) < Q(1)$, it is easy to see that $a > 1$ (as $Q(x)$ is a decreasing function)

$$\therefore e^{-a^2/2} = 10^{-8} \tag{2.0.20}$$

$$\Leftrightarrow a \approx 6.069 \tag{2.0.21}$$

When $\beta = -0.3$,

$$\text{BER} = Q(a(1+\beta)) = Q(6.069 \times (1 - 0.3)) \tag{2.0.22}$$

$$= Q(6.069 \times 0.7) \tag{2.0.23}$$

$$= Q(4.249) \tag{2.0.24}$$

$$\approx \exp(-\frac{4.249^2}{2}) \tag{2.0.25}$$

$$\approx 1.2 \times 10^{-4} \tag{2.0.26}$$

Therefore, when $\beta = -0.3$, BER is closest to 10^{-4} and option (C) is correct.