

Challenge Problem

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1 PROBLEM

GATE 2019 (ST) , Q.49 (Statistics section)

Let X be a random variable with characteristic function $\phi_X(\cdot)$ such that $\phi_X(2\pi) = 1$. Let \mathbb{Z} denote the set of integers. Then $P(X \in \mathbb{Z})$ is equal to ...

2 SOLUTION

We know that,

$$\phi_X(t) = \int_{\mathbb{R}} e^{itx} f_X(x) dx \quad (2.0.1)$$

$$\phi_X(2\pi) = \int_{\mathbb{R}} e^{2\pi ix} f_X(x) dx \quad (2.0.2)$$

$$= \int_{\mathbb{R}} \cos(2\pi x) f_X(x) dx + i \int_{\mathbb{R}} \sin(2\pi x) f_X(x) dx \quad (2.0.3)$$

$$\because \phi_X(2\pi) = 1, \int_{\mathbb{R}} \sin(2\pi x) f_X(x) dx = 0 \quad (2.0.4)$$

$$1 = \phi_X(2\pi) \quad (2.0.5)$$

$$= \int_{\mathbb{R}} \cos(2\pi x) f_X(x) dx \quad (2.0.6)$$

$$\leq \int_{\mathbb{R}} 1 \cdot f_X(x) dx \quad (2.0.7)$$

$$\leq \int_{\mathbb{R}} f_X(x) dx \quad (2.0.8)$$

$$\leq 1. \quad (2.0.9)$$

Therefore equality must occur throughout.

$$\therefore \cos(2\pi x) = 1, \text{ for all } X = x \quad (2.0.10)$$

$$\Rightarrow X \in \mathbb{Z} \quad (2.0.11)$$

$$\Rightarrow \Pr(X \in \mathbb{Z}) = 1 \quad (2.0.12)$$