

Assignment 3

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Download all python codes from

<https://github.com/GauthamBellamkonda/AI1103/tree/main/Assignment3/Codes>

and latex-tikz codes from

<https://github.com/GauthamBellamkonda/AI1103/tree/main/Assignment3>

1 PROBLEM

(GATE 45) Consider a discrete-time channel $Y = X + Z$, where the additive noise Z is signal dependent. In particular, given the transmitted symbol $X \in \{-a, a\}$ at any instant, the noise sample Z is chosen independently from a Gaussian distribution with mean βX and unit variance. Assume a threshold detector with zero threshold at the receiver. When $\beta = 0$, the BER was found to be $Q(a) = 1 \times 10^{-8}$.

$$\left(Q(v) = \frac{1}{\sqrt{2\pi}} \int_v^\infty e^{-\frac{u^2}{2}} du, \text{ and for } v > 1, \text{ use } Q(v) = e^{-\frac{v^2}{2}} \right) \quad (1.0.1)$$

When $\beta = -0.3$, BER is closest to

- (A) 10^{-7}
- (B) 10^{-6}
- (C) 10^{-4}
- (D) 10^{-2}

2 SOLUTION

Given that $X \in \{-a, +a\}$ is a random variable.

$$\Pr(X = a) = \frac{n(X = a)}{2} = \frac{1}{2} \quad (2.0.1)$$

$$\Pr(X = -a) = \frac{n(X = -a)}{2} = \frac{1}{2} \quad (2.0.2)$$

Also, Z is chosen from Gaussian Distribution with mean βX and unit variance.

$$\therefore F_Z(z) = G\left(\frac{z - \beta X}{1}\right) \quad (2.0.3)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp\left(-\frac{(z - \beta X)^2}{2}\right) dz \quad (2.0.4)$$

On differentiating both the sides with respect to z , we get

$$f_Z(z) = \frac{d}{dz} F_Z(z) \quad (2.0.5)$$

$$= \frac{d}{dz} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp\left(-\frac{(z - \beta X)^2}{2}\right) dz \right] \quad (2.0.6)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z - \beta X)^2}{2}\right) \quad (2.0.7)$$

$$\Leftrightarrow f_Y(z + X) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z - \beta X)^2}{2}\right) \quad (2.0.8)$$

$$\Leftrightarrow f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y - X - \beta X)^2}{2}\right) \quad (2.0.9)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y - X(1 + \beta))^2}{2}\right) \quad (2.0.10)$$

$$\therefore f_Y(y | X = a) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y - a(1 + \beta))^2}{2}\right) \quad (2.0.11)$$

$$\& f_Y(y | X = -a) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y + a(1 + \beta))^2}{2}\right) \quad (2.0.12)$$

Since $X \in \{-a, +a\}$ is also a random variable,

$$f_Y(y) = f_Y(y | X = a) \Pr(X = a) + f_Y(y | X = -a) \Pr(X = -a) \quad (2.0.13)$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y - a(1 + \beta))^2}{2}\right) + \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y + a(1 + \beta))^2}{2}\right) \quad (2.0.14)$$

Therefore, the resultant signal $Y = X + Z$ is comprised of X which can take either positive or negative value, and some noise Z . The detector (which has zero threshold) can give us incorrect bits when $X = +a$ and $Y < 0$ (BER_{+a}) or $X = -a$ and $Y > 0$ (BER_{-a}), as shown in the graph below.

$$\therefore \text{BER} = \text{BER}_{+a} + \text{BER}_{-a} \quad (2.0.15)$$

$$= f_Y(y < 0, X = a) + f_Y(y > 0, X = -a) \quad (2.0.16)$$

$$= f_Y(y < 0 | X = a) \Pr(X = a) + f_Y(y > 0 | X = -a) \Pr(X = -a) \quad (2.0.17)$$

$$= \int_{-\infty}^0 \frac{1}{2} \cdot f_Y(y | X = a) dy + \int_0^{\infty} \frac{1}{2} \cdot f_Y(y | X = -a) dy \quad (2.0.18)$$

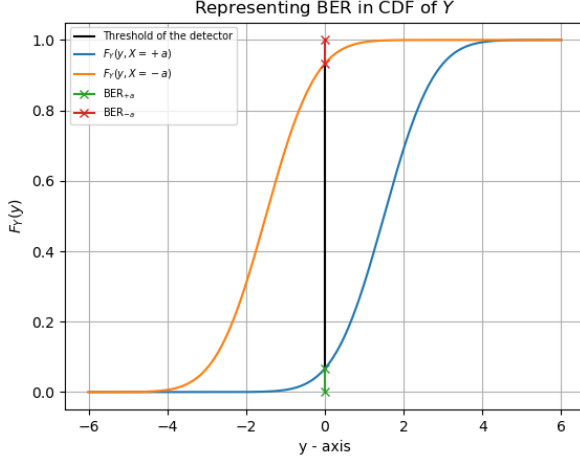


Fig. 1: The PDF of Y

On substituting the values of $f_Y(y | X = a)$ and $f_Y(y | X = -a)$ from 2.0.11 and 2.0.12,

$$\begin{aligned} \text{BER} &= \int_{-\infty}^0 \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(y - a(1 + \beta))^2}{2}\right) dy \\ &+ \int_0^{\infty} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(y + a(1 + \beta))^2}{2}\right) dy \quad (2.0.19) \end{aligned}$$

$$\begin{aligned} &= \int_0^{\infty} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(y + a(1 + \beta))^2}{2}\right) dy \\ &+ \int_0^{\infty} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(y + a(1 + \beta))^2}{2}\right) dy \quad (2.0.20) \end{aligned}$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y + a(1 + \beta))^2}{2}\right) dy \quad (2.0.21)$$

From the definition of $Q(v)$, it is easy to see that the expression in 2.0.21 is equal to $Q(a(1 + \beta))$

$$\therefore \text{BER} = Q(a(1 + \beta)) \quad (2.0.22)$$

When $\beta = 0$, it is given that

$$\text{BER} = Q(a) = 10^{-8} \quad (2.0.23)$$

On computing, $Q(1) \approx 0.16$. Since $Q(a) < Q(1)$, it is easy to see that $a > 1$ (as $Q(x)$ is a decreasing function)

$$\therefore e^{-a^2/2} = 10^{-8} \quad (2.0.24)$$

$$\Leftrightarrow a \approx 6.069 \quad (2.0.25)$$

When $\beta = -0.3$,

$$\text{BER} = Q(a(1 + \beta)) = Q(6.069 \times (1 - 0.3)) \quad (2.0.26)$$

$$= Q(6.069 \times 0.7) \quad (2.0.27)$$

$$= Q(4.249) \quad (2.0.28)$$

$$\approx \exp\left(-\frac{4.249^2}{2}\right) \quad (2.0.29)$$

$$\approx 1.2 \times 10^{-4} \quad (2.0.30)$$

Therefore, when $\beta = -0.3$, BER is closest to 10^{-4} and option (C) is correct.