Assignment 3

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Download all python codes from

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https://github.com/GauthamBellamkonda/AI1103/tree/main/
    Assignment3/Codes
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and latex-tikz codes from

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https://github.com/GauthamBellamkonda/AI1103/tree/main/
    Assignment3
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1 Problem

(GATE 45) Consider a discrete-time channel $Y = X+Z$, where the additive noise Z is signal dependent. In particular, given the transmitted symbol $X \in \{-a, a\}$ at any instant, the noise sample Z is chosen indepedently from a Gaussian distribution with mean βX and unit variance. Assume a threshold detector with zero threshold at the receiver. When $\beta = 0$, the BER was found to be $Q(a) = 1 \times 10^{-8}$.

$$\left(Q(v) = \frac{1}{\sqrt{2\pi}} \int_v^\infty e^{-\frac{u^2}{2}} du, \text{ and for } v > 1, \text{ use } Q(v) = e^{\frac{-v^2}{2}}\right) \tag{1.0.1}$$

When $\beta = -0.3$, BER is closest to

(A) 10^{-7}
(B) 10^{-6}
(C) 10^{-4}
(D) 10^{-2}

2 Solution

Given that the threshold of the detector is zero. Define a detector function g such that

$$g(Y) = \begin{cases} a & Y > 0 \\ -a & Y < 0 \end{cases} \tag{2.0.1}$$

It is given that $X \in \{-a, a\}$ is a random variable.

$$\therefore \Pr(X = a) = \frac{1}{2} \tag{2.0.2}$$

$$\& \ \Pr(X = -a) = \frac{1}{2} \tag{2.0.3}$$

Since the noise in the signal, Z is chosen independently from a Gaussian distribution with mean $\mu = \beta X$ and unit variance, it follows that

$$F_Z(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(z-\beta X)^2}{2}\right) dz \tag{2.0.4}$$

$$= \int_{-\infty}^{z-\beta X} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz \tag{2.0.5}$$

$$= \int_{\beta X - z}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz \tag{2.0.6}$$

$$= Q(\beta X - z) \tag{2.0.7}$$

Also, it is easy to see that

$$Q(-v) = 1 - Q(v) \ \forall \ v \in \mathbb{R} \tag{2.0.8}$$

The detector can record erroneous bits in the signal iff

$$X > 0 \ , \ g(Y) = -a \text{ (Call this BER}_{+a}\text{) or} \tag{2.0.9}$$

$$X < 0 \ , \ g(Y) = a \text{ (Call this BER}_{-a}\text{)} \tag{2.0.10}$$

$$\therefore \text{BER}_{+a} = \Pr(g(Y) = -a \ , \ X = a) \tag{2.0.11}$$

$$= \Pr(g(Y) = -a \mid X = a) \Pr(X = a) \tag{2.0.12}$$

$$= \Pr(Y < 0 \mid X = a) \Pr(X = a) \tag{2.0.13}$$

$$= \frac{1}{2} \times \Pr(X + Z < 0 \mid X = a) \tag{2.0.14}$$

$$= \frac{1}{2} \times \Pr(a + Z < 0 \mid X = a) \tag{2.0.15}$$

$$= \frac{1}{2} \times \Pr(Z < -a \mid X = a) \tag{2.0.16}$$

$$= \frac{1}{2} \times F_Z(-a) \tag{2.0.17}$$

$$= \frac{1}{2} \times Q(\beta X + a) \ \ \text{(From (2.0.7))} \tag{2.0.18}$$

$$= \frac{1}{2} \times Q(a\beta + a) \tag{2.0.19}$$

$$= \frac{1}{2} \times Q(a(1+\beta)) \tag{2.0.20}$$

$$\& \ \text{BER}_{-a} = \Pr(g(Y) = a \ , \ X = -a) \tag{2.0.21}$$

$$= \Pr(g(Y) = a \mid X = -a) \Pr(X = -a) \tag{2.0.22}$$

$$= \Pr(Y > 0 \mid X = -a) \Pr(X = -a) \tag{2.0.23}$$

$$= \frac{1}{2} \times \Pr(X + Z > 0 \mid X = -a) \tag{2.0.24}$$

$$= \frac{1}{2} \times \Pr(Z - a > 0 \mid X = -a) \tag{2.0.25}$$

$$= \frac{1}{2} \times \Pr(Z > a \mid X = -a) \tag{2.0.26}$$

$$= \frac{1}{2} \times (1 - F_Z(a)) \tag{2.0.27}$$

$$= \frac{1}{2} \times (1 - Q(\beta X - a)) \ \ \text{(From (2.0.7))} \tag{2.0.28}$$

$$= \frac{1}{2} \times (1 - Q(-a\beta - a)) \tag{2.0.29}$$

$$= \frac{1}{2} \times Q(a(1+\beta)) \ \ \text{(From (2.0.8))} \tag{2.0.30}$$

$$\therefore \text{BER} = \text{BER}_{+a} + \text{BER}_{-a} \tag{2.0.31}$$

$$= Q(a(1+\beta)) \tag{2.0.32}$$

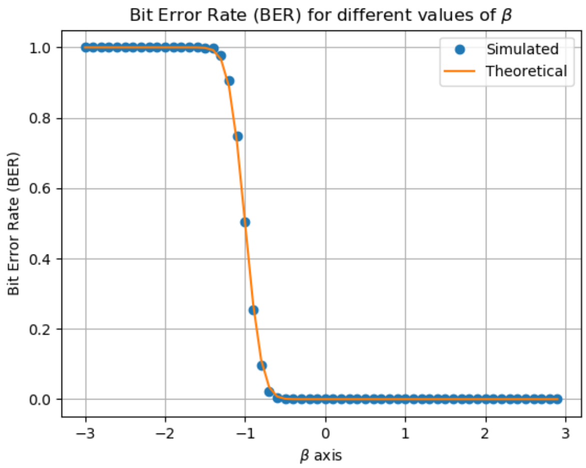

Fig. 1: Theory vs Simulated plot of BER

When $\beta = 0$, it is given that

$$\text{BER} = Q(a) = 10^{-8} \tag{2.0.33}$$

On computing, $Q(1) \approx 0.16$. Since $Q(a) < Q(1)$, it is easy to see that $a > 1$ (as $Q(x)$ is a decreasing function)

$$\therefore e^{-a^2/2} = 10^{-8} \tag{2.0.34}$$

$$\Leftrightarrow a \approx 6.069 \tag{2.0.35}$$

When $\beta = -0.3$,

$$\text{BER} = Q(a(1+\beta)) = Q(6.069 \times (1 - 0.3)) \tag{2.0.36}$$

$$= Q(6.069 \times 0.7) \tag{2.0.37}$$

$$= Q(4.249) \tag{2.0.38}$$

$$\approx \exp(-\frac{4.249^2}{2}) \tag{2.0.39}$$

$$\approx 1.2 \times 10^{-4} \tag{2.0.40}$$

Therefore, when $\beta = -0.3$, BER is closest to 10^{-4} and option (C) is correct.