

Assignment 6

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Download all latex-tikz codes from

<https://github.com/GauthamBellamkonda/AI1103/tree/main/Assignment6>

1 PROBLEM

Let X_1, X_2, \dots, X_n be a random sample of size n (≥ 2) from a distribution having the probability density function

$$f(x; \theta) = \begin{cases} \frac{1}{\theta} \exp\left(-\frac{x}{\theta}\right) & x > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (1.0.1)$$

where $\theta \in (0, \infty)$. Let $X_{(1)} = \min \{X_1, X_2, \dots, X_n\}$ and $T = \sum_{i=1}^n X_i$. Then $E(X_{(1)}|T)$ equals

- (A) $\frac{T}{n^2}$
- (B) $\frac{T}{n}$
- (C) $\frac{(n+1)T}{2n}$
- (D) $\frac{(n+1)^2 T}{4n^2}$

2 SOLUTION

Lehmann–Scheffé theorem :

If T is a complete sufficient statistic for θ and

$$E(g(T)) = \tau(\theta) \quad (2.0.1)$$

then $g(T)$ is the uniformly minimum-variance unbiased estimator (UMVUE) of $\tau(\theta)$.

We know that

$$T = \sum_{i=1}^n X_i \quad (2.0.2)$$

is a complete and sufficient statistic. By the law of total expectation,

$$E(E(X_{(1)}|T)) = E(X_{(1)}) \quad (2.0.3)$$

By Lehmann–Scheffé theorem, with

$$\theta = X_{(1)}, \quad (2.0.4)$$

$$\tau(x) = E(x), \quad (2.0.5)$$

$$g(T) = E(X_{(1)}|T). \quad (2.0.6)$$

it follows from (2.0.3) that $E(X_{(1)}|T)$ is the UMVUE of $E(X_{(1)})$.

$$\Pr(X_{(1)} > x) = \Pr(X_1 > x) \dots \Pr(X_n > x) \quad (2.0.7)$$

$$= (1 - F_{X_1}(x)) \dots (1 - F_{X_n}(x)) \quad (2.0.8)$$

$$= (1 - F_{X_1}(x))^n \quad (2.0.9)$$

$$= \exp\left(-\frac{nx}{\theta}\right) \quad (2.0.10)$$

$$F_{X_{(1)}}(x) = 1 - \exp\left(-\frac{nx}{\theta}\right) \quad (2.0.11)$$

$$f_{X_{(1)}}(x) = \frac{n}{\theta} \exp\left(-\frac{nx}{\theta}\right) \quad (2.0.12)$$

Therefore, $X_{(1)}$ follows an exponential distribution with mean $\frac{\theta}{n}$.

$$E(X_{(1)}) = \frac{\theta}{n} \quad (2.0.13)$$

Note that,

$$E\left(\frac{T}{n^2}\right) = E\left(\frac{\sum_{i=1}^n X_i}{n^2}\right) \quad (2.0.14)$$

$$= \frac{E(\sum_{i=1}^n X_i)}{n^2} \quad (2.0.15)$$

$$= \sum_{i=1}^n \frac{E(X_i)}{n^2} \quad (2.0.16)$$

$$= \sum_{i=1}^n \frac{\theta}{n^2} \quad (2.0.17)$$

$$= \frac{\theta}{n} \quad (2.0.18)$$

$$= E(X_{(1)}) \quad (2.0.19)$$

Therefore, by Lehmann–Scheffé theorem, with

$$\theta = X_{(1)}, \quad (2.0.20)$$

$$\tau(x) = E(x), \quad (2.0.21)$$

$$g(T) = \frac{T}{n^2}, \quad (2.0.22)$$

it follows that $\frac{T}{n^2}$ is UMVUE of $E(X_{(1)})$.

Since there exists a unique UMVUE for $E(X_{(1)})$, it follows that

$$E(X_{(1)}|T) = \frac{T}{n^2} \quad (2.0.23)$$

Hence, option A is correct.