Assignment 3

Gautham Bellamkonda - CS20BTECH11017

Download all python codes from

https://github.com/GauthamBellamkonda/AI1103/tree/main/ Assignment3/Codes

and latex-tikz codes from

https://github.com/GauthamBellamkonda/AI1103/tree/main/Assignment3

1 Problem

(GATE 45) Consider a discrete-time channel Y = X + Z, where the additive noise Z is signal dependent. In particular, given the transmitted symbol $X \in \{-a, a\}$ at any instant, the noise sample Z is chosen indepedently from a Gaussian distribution with mean βX and unit variance. Assume a threshold detector with zero threshold at the receiver. When $\beta = 0$, the BER was found to be $Q(a) = 1 \times 10^{-8}$.

$$\left(Q(v) = \frac{1}{\sqrt{2\pi}} \int_{v}^{\infty} e^{-\frac{u^2}{2}} du, \text{ and for } v > 1, \text{ use } Q(v) = e^{\frac{-v^2}{2}}\right)$$
(1.0.1)

When $\beta = -0.3$, BER is closest to

- (A) 10^{-7}
- (B) 10^{-6}
- (C) 10^{-4}
- (D) 10^{-2}

2 Solution

Given that $X \in \{-a, +a\}$ is a random variable.

$$\Pr(X = a) = \frac{n(X = a)}{2} = \frac{1}{2}$$
 (2.0.1)

$$Pr(X = -a) = \frac{n(X = -a)}{2} = \frac{1}{2}$$
 (2.0.2)

Also, Z is chosen from Gaussian Distribution with mean βX and unit variance.

$$\therefore F_Z(z) = G\left(\frac{z - \beta X}{1}\right) \tag{2.0.3}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(\frac{-(z - \beta X)^2}{2}\right) dz \qquad (2.0.4)$$

On differentiating both the sides with respect to z, we get

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \frac{d}{dz} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp\left(\frac{-(z - \beta X)^2}{2}\right) dz \right]$$
(2.0.6)

$$=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(z-\beta X)^2}{2}\right) \tag{2.0.7}$$

$$\Leftrightarrow f_Y(z+X) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z-\beta X)^2}{2}\right)$$
 (2.0.8)

$$\Leftrightarrow f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y - X - \beta X)^2}{2}\right)$$
 (2.0.9)

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y - X(1+\beta))^2}{2}\right) \quad (2.0.10)$$

$$\therefore f_Y(y \mid X = a) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y - a(1 + \beta))^2}{2}\right) \quad (2.0.11)$$

&
$$f_Y(y \mid X = -a) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y + a(1+\beta))^2}{2}\right)$$
 (2.0.12)

Since $X \in \{-a, +a\}$ is also a random variable,

$$f_Y(y) = f_Y(y \mid X = a) \Pr(X = a) + f_Y(y \mid X = -a) \Pr(X = -a) = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y - a(1 + \beta))^2}{2}\right) + \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y + a(1 + \beta))^2}{2}\right)$$
(2.0.14)

Therefore, the resultant signal Y = X + Z is comprised of X which can take either positive or negative value, and some noise Z. The detector (which has zero threshold) can give us incorrect bits when X = +a and Y < 0 (BER_{+a}) or X = -a and Y < 0 (BER_{-a}), as shown in the graph below.

$$\therefore BER = BER_{+a} + BER_{-a}$$

$$= f_Y(y < 0, X = a) + f_Y(y > 0, X = -a)$$
(2.0.15)

 $= f_Y(y < 0 \mid X = a) \Pr(X = a)$

+
$$f_Y(y > 0 \mid X = -a) \Pr(X = -a)$$
 (2.0.17)

$$= \int_{-\infty}^{0} \frac{1}{2} \cdot f_Y(y \mid X = a) dy + \int_{0}^{\infty} \frac{1}{2} \cdot f_Y(y \mid X = -a) dy$$
 (2.0.18)

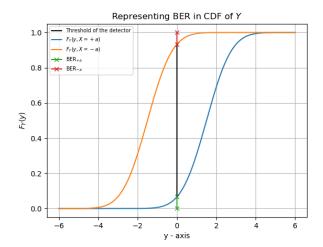


Fig. 1: The CDF of Y

On substituting the values of $f_Y(y \mid X = a)$ and $f_Y(y \mid X = -a)$ from 2.0.11 and 2.0.12,

BER =
$$\int_{-\infty}^{0} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(y - a(1+\beta))^{2}}{2}\right) dy$$
+
$$\int_{0}^{\infty} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(y + a(1+\beta))^{2}}{2}\right) dy \qquad (2.0.19)$$
=
$$\int_{0}^{\infty} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(y + a(1+\beta))^{2}}{2}\right) dy$$
+
$$\int_{0}^{\infty} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(y + a(1+\beta))^{2}}{2}\right) dy \qquad (2.0.20)$$
=
$$\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y + a(1+\beta))^{2}}{2}\right) dy \qquad (2.0.21)$$

From the definition of Q(v), it is easy to see that the expression in 2.0.21 is equal to $Q(a(1 + \beta))$

: BER =
$$Q(a(1 + \beta))$$
 (2.0.22)

When $\beta = 0$, it is given that

BER =
$$Q(a) = 10^{-8}$$
 (2.0.23)

On computing, $Q(1) \approx 0.16$. Since Q(a) < Q(1), it is easy to see that a > 1 (as Q(x) is a decreasing function)

$$\therefore e^{-a^2/2} = 10^{-8} \tag{2.0.24}$$

$$\Leftrightarrow a \approx 6.069 \tag{2.0.25}$$

When $\beta = -0.3$,

BER =
$$Q(a(1 + \beta)) = Q(6.069 \times (1 - 0.3))$$
 (2.0.26)

$$= Q(6.069 \times 0.7) \tag{2.0.27}$$

$$= Q(4.249) \tag{2.0.28}$$

$$\approx \exp(-\frac{4.249^2}{2})\tag{2.0.29}$$

$$\approx 1.2 \times 10^{-4} \tag{2.0.30}$$

Therefore, when $\beta = -0.3$, BER is closest to 10^{-4} and option (C) is correct.