

# Assignment 3

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Download all python codes from

<https://github.com/GauthamBellamkonda/AI1103/tree/main/Assignment3/Codes>

and latex-tikz codes from

<https://github.com/GauthamBellamkonda/AI1103/tree/main/Assignment3>

## 1 PROBLEM

(GATE 45) Consider a discrete-time channel  $Y = X + Z$ , where the additive noise  $Z$  is signal dependent. In particular, given the transmitted symbol  $X \in \{-a, a\}$  at any instant, the noise sample  $Z$  is chosen independently from a Gaussian distribution with mean  $\beta X$  and unit variance. Assume a threshold detector with zero threshold at the receiver. When  $\beta = 0$ , the BER was found to be  $Q(a) = 1 \times 10^{-8}$ .

$$\left( Q(v) = \frac{1}{\sqrt{2\pi}} \int_v^{\infty} e^{-\frac{u^2}{2}} du, \text{ and for } v > 1, \text{ use } Q(v) = e^{-\frac{v^2}{2}} \right) \quad (1.0.1)$$

When  $\beta = -0.3$ , BER is closest to

- (A)  $10^{-7}$
- (B)  $10^{-6}$
- (C)  $10^{-4}$
- (D)  $10^{-2}$

## 2 SOLUTION

Given that the threshold of the detector is zero. Define a detector function  $g$  such that

$$g(Y) = \begin{cases} a & Y > 0 \\ -a & Y < 0 \end{cases} \quad (2.0.1)$$

It is given that  $X \in \{-a, a\}$  is a random variable.

$$\therefore \Pr(X = a) = \frac{1}{2} \quad (2.0.2)$$

$$\& \Pr(X = -a) = \frac{1}{2} \quad (2.0.3)$$

Since the noise in the signal,  $Z$  is chosen independently from a Gaussian distribution with mean  $\mu = \beta X$  and unit variance, it follows that

$$F_Z(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z - \beta X)^2}{2}\right) dz \quad (2.0.4)$$

$$= \int_{-\infty}^{z - \beta X} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z'^2}{2}\right) dz' \quad (2.0.5)$$

$$= \int_{\beta X - z}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z'^2}{2}\right) dz' \quad (2.0.6)$$

$$= Q(\beta X - z) \quad (2.0.7)$$

Also, it is easy to see that

$$Q(-v) = 1 - Q(v) \quad \forall v \in \mathbb{R} \quad (2.0.8)$$

The detector can record erroneous bits in the signal iff

$$X > 0, g(Y) = -a \text{ (Call this BER}_{+a} \text{) or} \quad (2.0.9)$$

$$X < 0, g(Y) = a \text{ (Call this BER}_{-a} \text{)} \quad (2.0.10)$$

$$\therefore \text{BER}_{+a} = \Pr(g(Y) = -a, X = a) \quad (2.0.11)$$

$$= \Pr(g(Y) = -a | X = a) \Pr(X = a) \quad (2.0.12)$$

$$= \Pr(Y < 0 | X = a) \Pr(X = a) \quad (2.0.13)$$

$$= \frac{1}{2} \times \Pr(X + Z < 0 | X = a) \quad (2.0.14)$$

$$= \frac{1}{2} \times \Pr(a + Z < 0 | X = a) \quad (2.0.15)$$

$$= \frac{1}{2} \times \Pr(Z < -a | X = a) \quad (2.0.16)$$

$$= \frac{1}{2} \times F_Z(-a) \quad (2.0.17)$$

$$= \frac{1}{2} \times Q(\beta X + a) \text{ (From (2.0.7))} \quad (2.0.18)$$

$$= \frac{1}{2} \times Q(a\beta + a) \quad (2.0.19)$$

$$= \frac{1}{2} \times Q(a(1 + \beta)) \quad (2.0.20)$$

$$\& \text{BER}_{-a} = \Pr(g(Y) = a, X = -a) \quad (2.0.21)$$

$$= \Pr(g(Y) = a | X = -a) \Pr(X = -a) \quad (2.0.22)$$

$$= \Pr(Y > 0 | X = -a) \Pr(X = -a) \quad (2.0.23)$$

$$= \frac{1}{2} \times \Pr(X + Z > 0 | X = -a) \quad (2.0.24)$$

$$= \frac{1}{2} \times \Pr(Z - a > 0 | X = -a) \quad (2.0.25)$$

$$= \frac{1}{2} \times \Pr(Z > a | X = -a) \quad (2.0.26)$$

$$= \frac{1}{2} \times (1 - F_Z(a)) \quad (2.0.27)$$

$$= \frac{1}{2} \times (1 - Q(\beta X - a)) \text{ (From (2.0.7))} \quad (2.0.28)$$

$$= \frac{1}{2} \times (1 - Q(-a\beta - a)) \quad (2.0.29)$$

$$= \frac{1}{2} \times Q(a(1 + \beta)) \text{ (From (2.0.8))} \quad (2.0.30)$$

$$\therefore \text{BER} = \text{BER}_{+a} + \text{BER}_{-a} \quad (2.0.31)$$

$$= Q(a(1 + \beta)) \quad (2.0.32)$$

When  $\beta = 0$ , it is given that

$$\text{BER} = Q(a) = 10^{-8} \quad (2.0.33)$$

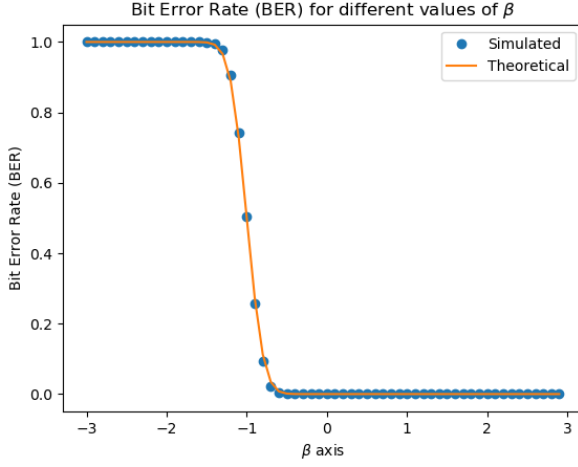


Fig. 1: Theory vs Simulated plot of BER

On computing,  $Q(1) \approx 0.16$ . Since  $Q(a) < Q(1)$ , it is easy to see that  $a > 1$  (as  $Q(x)$  is a decreasing function)

$$\therefore e^{-a^2/2} = 10^{-8} \quad (2.0.34)$$

$$\Leftrightarrow a \approx 6.069 \quad (2.0.35)$$

When  $\beta = -0.3$ ,

$$\text{BER} = Q(a(1 + \beta)) = Q(6.069 \times (1 - 0.3)) \quad (2.0.36)$$

$$= Q(6.069 \times 0.7) \quad (2.0.37)$$

$$= Q(4.249) \quad (2.0.38)$$

$$\approx \exp\left(-\frac{4.249^2}{2}\right) \quad (2.0.39)$$

$$\approx 1.2 \times 10^{-4} \quad (2.0.40)$$

Therefore, when  $\beta = -0.3$ , BER is closest to  $10^{-4}$  and option (C) is correct.