

Probability and Random Variables

G V V Sharma*

CONTENTS

1 AXIOMS OF PROBABILITY

1	Axioms of Probability	1
1.1	Boolean Logic	1
1.2	Independent Events	2
1.3	Conditional Probability . . .	2
2	Sum of Independent Random Variables	3
2.1	The Uniform Distribution . .	3
3	Cumulative Distribution Function	4
3.1	The Bernoulli Distribution .	4
3.2	The Binomial Distribution .	5
4	Central Limit Theorem: Gaussian Distribution	6
4.1	Bernoulli to Gaussian	6
4.2	Uniform to Gaussian	6
5	Stochastic Geometry	8
6	Transformation of Variables	8
6.1	Using Definition	8
6.2	Using Jacobian	9
7	Conditional Probability	10
8	Two Dimensions	10
9	Transform Domain: Moment Generating Function	12
10	Uniform to Other: Quantile function	12
11	Miscellaneous Distributions	13

Abstract—This book provides a simple introduction to probability and random variables. The contents are largely based on NCERT textbooks from Class 9-12.

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

1.1 Boolean Logic

If A and B are two events such that $\Pr(A) = \frac{1}{4}$, $\Pr(B) = \frac{1}{2}$ and $\Pr(AB) = \frac{1}{8}$. find $\Pr(\text{not } A \text{ and not } B)$.

1.1.1.

$$A'B' = (A + B)' \quad (1.1.1.1)$$

$$\Rightarrow \Pr(A'B') = \Pr((A + B)') \quad (1.1.1.2)$$

$$= 1 - \Pr(A + B) \quad (1.1.1.3)$$

1.1.2.

$$\because A + B = A(B + B') + B \quad (1.1.2.1)$$

$$= B(A + 1) + AB' \quad (1.1.2.2)$$

$$= B + AB' \quad (1.1.2.3)$$

$$\Rightarrow \Pr(A + B) = \Pr(B + AB') \quad (1.1.2.4)$$

$$= \Pr(B) + \Pr(AB') \quad (1.1.2.5)$$

$$\because B(AB') = 0 \quad (1.1.2.6)$$

1.1.3.

$$A = A(B + B') = AB + AB' \quad (1.1.3.1)$$

and

$$(AB)(AB') = 0, \because BB' = 0 \quad (1.1.3.2)$$

Hence, AB and AB' are mutually exclusive and

$$\Pr(A) = \Pr(AB) + \Pr(AB') \quad (1.1.3.3)$$

$$\Rightarrow \Pr(AB') = \Pr(A) - \Pr(AB) \quad (1.1.3.4)$$

1.1.4. Substituting (1.1.3.4) in (1.1.2.6),

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \quad (1.1.4.1)$$

1.1.5. Substituting (1.1.4.1) in (1.1.1.3)

$$\Pr(A'B') = 1 - \{\Pr(A) + \Pr(B) - \Pr(AB)\} \quad (1.1.5.1)$$

$$= 1 - \left(\frac{1}{4} + \frac{1}{2} - \frac{1}{8}\right) \quad (1.1.5.2)$$

$$= \frac{3}{8} \quad (1.1.5.3)$$

1.2 Independent Events

1.2.1. Prove that if E and F are independent events, then so are the events E and F' .

Solution: If E and F are independent,

$$\Pr(EF) = \Pr(E) \Pr(F) \quad (1.2.1.1)$$

From (1.1.3.2)

$$\Pr(EF') = \Pr(E) - \Pr(EF) \quad (1.2.1.2)$$

Substituting from (1.2.1.1) in (1.2.1.2),

$$\Pr(EF') = \Pr(E)(1 - \Pr(F)) = \Pr(E) \Pr(F') \quad (1.2.1.3)$$

$$\therefore FF' = 0, F + F' = 1 \quad (1.2.1.4)$$

$$\implies \Pr(F) + \Pr(F') = 1 \quad (1.2.1.5)$$

By definition, from (1.2.1.3), we conclude that E and F' are independent.

1.2.2. If A and B are two independent events, then the probability of occurrence of at least one of A and B is given by $1 - P(A')P(B')$

Solution:

$$\therefore (A + B)(A + B)' = 0 \quad (1.2.2.1)$$

$$\implies 1 = \Pr(A + B) + \Pr((A + B)') \quad (1.2.2.2)$$

$$\implies \Pr(A + B) = 1 - \Pr(A'B') \quad (1.2.2.3)$$

$$= 1 - \Pr(A') \Pr(B') \quad (1.2.2.4)$$

using the definition of independence.

1.3 Conditional Probability

1.3.1. Given that E and F are events such that $P(E) = 0.6$, $P(F) = 0.3$ and $P(EF) = 0.2$, find $\Pr(E|F)$ and $\Pr(F|E)$

Solution: By definition,

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{0.2}{0.3} = \frac{2}{3} \quad (1.3.1.1)$$

Similarly,

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{1}{3} \quad (1.3.1.2)$$

1.3.2. A fair die is rolled. Consider the events $E = (1, 3, 5)$, $F = (2, 3)$ and $G = (2, 3, 4, 5)$ Find

a) $\Pr(E|F)$ and $\Pr(F|E)$

b) $\Pr(E|G)$ and $\Pr(F|E)$

c) $\Pr((E + F)|G)$ and $\Pr(EF|G)$

Solution:

From the given information,

$$\Pr(E) = \frac{3}{6} = \frac{1}{2} \quad (1.3.2.1)$$

$$\Pr(F) = \frac{2}{6} = \frac{1}{3} \quad (1.3.2.2)$$

$$\Pr(G) = \frac{4}{6} = \frac{2}{3} \quad (1.3.2.3)$$

$$\Pr(EF) = \frac{1}{6} \quad (1.3.2.4)$$

$$\Pr(EG) = \frac{2}{6} = \frac{1}{3} \quad (1.3.2.5)$$

$$\Pr(FG) = \frac{2}{6} = \frac{1}{3} \quad (1.3.2.6)$$

$$\Pr(EFG) = \frac{1}{6} \quad (1.3.2.7)$$

a)

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} \quad (1.3.2.8)$$

$$\Pr(E|F) = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2} \quad (1.3.2.9)$$

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} \quad (1.3.2.10)$$

$$\Pr(F|E) = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3} \quad (1.3.2.11)$$

b)

$$\Pr(E|G) = \frac{\Pr(EG)}{\Pr(G)} \quad (1.3.2.12)$$

$$\Pr(E|G) = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2} \quad (1.3.2.13)$$

$$\Pr(G|E) = \frac{\Pr(GE)}{\Pr(G)} \quad (1.3.2.14)$$

$$\Pr(G|E) = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3} \quad (1.3.2.15)$$

$$(1.3.2.16)$$

c)

$$\begin{aligned} \Pr(E + F|G) &= \frac{\Pr(\{E + F\}G)}{\Pr(G)} \\ &= \frac{\Pr(EG + FG)}{\Pr(G)} \\ &= \frac{\Pr(EG) + \Pr(FG) - \Pr(EFG)}{\Pr(G)} \\ &= \frac{3}{4} \end{aligned} \quad (1.3.2.17)$$

and

$$\Pr(EF|G) = \frac{\Pr(EFG)}{\Pr(G)} = \frac{1}{4} \quad (1.3.2.18)$$

2 SUM OF INDEPENDENT RANDOM VARIABLES

2.1 The Uniform Distribution

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

2.1.1. *The Uniform Distribution:* Let $X_i \in \{1, 2, 3, 4, 5, 6\}, i = 1, 2$, be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \leq n \leq 6 \\ 0 & \text{otherwise} \end{cases} \quad (2.1.1.1)$$

The desired outcome is

$$X = X_1 + X_2, \quad (2.1.1.2)$$

$$\implies X \in \{1, 2, \dots, 12\} \quad (2.1.1.3)$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \quad (2.1.1.4)$$

2.1.2. *Convolution:* From (2.1.1.2),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \quad (2.1.2.1)$$

$$= \sum_k \Pr(X_1 = n - k | X_2 = k) p_{X_2}(k) \quad (2.1.2.2)$$

after unconditioning. $\because X_1$ and X_2 are independent,

$$\begin{aligned} \Pr(X_1 = n - k | X_2 = k) \\ = \Pr(X_1 = n - k) = p_{X_1}(n - k) \end{aligned} \quad (2.1.2.3)$$

From (2.1.2.2) and (2.1.2.3),

$$p_X(n) = \sum_k p_{X_1}(n - k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \quad (2.1.2.4)$$

where $*$ denotes the convolution operation. Substituting from (2.1.1.1) in (2.1.2.4),

$$p_X(n) = \frac{1}{6} \sum_{k=1}^6 p_{X_1}(n - k) = \frac{1}{6} \sum_{k=n-6}^{n-1} p_{X_1}(k) \quad (2.1.2.5)$$

$$\because p_{X_1}(k) = 0, \quad k \leq 1, k \geq 6. \quad (2.1.2.6)$$

From (2.1.2.5),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6} \sum_{k=1}^{n-1} p_{X_1}(k) & 1 \leq n-1 \leq 6 \\ \frac{1}{6} \sum_{k=n-6}^6 p_{X_1}(k) & 1 < n-6 \leq 6 \\ 0 & n > 12 \end{cases} \quad (2.1.2.7)$$

Substituting from (2.1.1.1) in (2.1.2.7),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \leq n \leq 7 \\ \frac{13-n}{36} & 7 < n \leq 12 \\ 0 & n > 12 \end{cases} \quad (2.1.2.8)$$

satisfying (2.1.1.4).

2.1.3. *The Z-transform:* The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n)z^{-n}, \quad z \in \mathbb{C} \quad (2.1.3.1)$$

From (2.1.1.1) and (2.1.3.1),

$$\begin{aligned} P_{X_1}(z) &= P_{X_2}(z) = \frac{1}{6} \sum_{n=1}^6 z^{-n} \\ &= \frac{z^{-1}(1 - z^{-6})}{6(1 - z^{-1})}, \quad |z| > 1 \end{aligned} \quad (2.1.3.2) \quad (2.1.3.3)$$

upon summing up the geometric progression.

$$\therefore p_X(n) = p_{X_1}(n) * p_{X_2}(n), \quad (2.1.3.4)$$

$$P_X(z) = P_{X_1}(z)P_{X_2}(z) \quad (2.1.3.5)$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (2.1.3.3) and (2.1.3.5),

$$P_X(z) = \left\{ \frac{z^{-1}(1 - z^{-6})}{6(1 - z^{-1})} \right\}^2 \quad (2.1.3.6)$$

$$= \frac{1}{36} \frac{z^{-2}(1 - 2z^{-6} + z^{-12})}{(1 - z^{-1})^2} \quad (2.1.3.7)$$

Using the fact that

$$p_X(n-k) \xleftrightarrow{\mathcal{H}} ZP_X(z)z^{-k}, \quad (2.1.3.8)$$

$$nu(n) \xleftrightarrow{\mathcal{H}} Z \frac{z^{-1}}{(1 - z^{-1})^2} \quad (2.1.3.9)$$

after some algebra, it can be shown that

$$\begin{aligned} & \frac{1}{36} [(n-1)u(n-1) - 2(n-7)u(n-7) \\ & \quad + (n-13)u(n-13)] \\ & \xleftrightarrow{\mathcal{H}} Z \frac{1}{36} \frac{z^{-2}(1 - 2z^{-6} + z^{-12})}{(1 - z^{-1})^2} \end{aligned} \quad (2.1.3.10)$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \quad (2.1.3.11)$$

From (2.1.3.1), (2.1.3.7) and (2.1.3.10)

$$\begin{aligned} p_X(n) &= \frac{1}{36} [(n-1)u(n-1) \\ & \quad - 2(n-7)u(n-7) + (n-13)u(n-13)] \end{aligned} \quad (2.1.3.12)$$

which is the same as (2.1.2.8). Note that (2.1.2.8) can be obtained from (2.1.3.10) using contour integration as well.

2.1.4. The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 2.1.4.1. The theoretical pmf obtained in (2.1.2.8) is plotted for comparison.



Fig. 2.1.4.1: Plot of $p_X(n)$. Simulations are close to the analysis.

2.1.5. The python code is available in

/codes/sum/dice.py

3 CUMULATIVE DISTRIBUTION FUNCTION

3.1 The Bernoulli Distribution

3.1.1. Find the probability of getting a head when a coin is tossed once. Also find the probability of getting a tail.

TABLE 3.1.3.1

Colour	X	Number
Blue	0	$n(X = 0)$
Green	1	$n(X = 1)$

codes/bernoulli/bernoulli.py

Solution: Let the random variable be $X \in \{0, 1\}$. Then

$$\Pr(X = 0) = \Pr(X = 1) = \frac{1}{2} \quad (3.1.1.1)$$

The following code simulates the event for 100 coin tosses

codes/bernoulli/coin.py

3.1.2. *Bernoulli Distribution:* In general the binomial distribution is defined using the PMF

$$p_X(n) = \begin{cases} p & n = 1 \\ 1 - p & n = 0 \\ \text{otherwise} \end{cases} \quad (3.1.2.1)$$

3.1.3. A jar contains 24 marbles, some are green and others are blue. If a marble is drawn at random from the jar, the probability that it is green is $\frac{2}{3}$. Find the number of blue balls (marbles) in the jar.

Solution: Let random variable $X \in \{0, 1\}$ denote the outcomes of the experiment of drawing a marble from a jar as shown in Table 3.1.3.1 From the given information,

$$p_X(1) = \frac{2}{3} \quad (3.1.3.1)$$

$$\Rightarrow p = 1 - p_X(1) = \frac{1}{3} \quad (3.1.3.2)$$

$$n(X = 0) + n(X = 1) = 24 \quad (3.1.3.3)$$

\therefore

$$p = \frac{n(X = 0)}{n(X = 0) + n(X = 1)}, \quad (3.1.3.4)$$

from (3.1.3.4) and (3.1.3.3),

$$n(X = 0) = p \{n(X = 0) + n(X = 1)\} \quad (3.1.3.5)$$

$$= \frac{1}{3} \times 24 = 8. \quad (3.1.3.6)$$

The following code generates the number of blue marbles

3.2 The Binomial Distribution

In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles?

3.2.1. Let $X_i \in \{0, 1\}$ represent the i th hurdle where 1 denotes a hurdle being knocked down. Then, X_i has a bernoulli distribution with parameter

$$p = 1 - \frac{5}{6} = \frac{1}{6} \quad (3.2.1.1)$$

3.2.2. *The Binomial Distribution:* Let

$$X = \sum_{i=1}^n X_i \quad (3.2.2.1)$$

where n is the total number of hurdles. Then X has a binomial distribution. Then, for

$$p_{X_i}(n) \stackrel{Z}{=} P_{X_i}(z), \quad (3.2.2.2)$$

yielding

$$P_{X_i}(z) = 1 - p + pz^{-1} \quad (3.2.2.3)$$

with Using the fact that X_i are i.i.d.,

$$P_X(z) = (1 - p + pz^{-1})^n \quad (3.2.2.4)$$

$$= \sum_{k=0}^n {}^nC_k p^k (1 - p)^{n-k} z^{-k} \quad (3.2.2.5)$$

$$\Rightarrow p_X(k) = \begin{cases} {}^nC_k p^{n-k} (1 - p)^k & 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases} \quad (3.2.2.6)$$

The cumulative distribution function of X is defined as

$$F_X(r) = \Pr(X \leq r) = \sum_{k=0}^r {}^nC_k p^k (1 - p)^{n-k} \quad (3.2.2.7)$$

upon substituting from (3.2.2.6).

3.2.3. *Evaluationg the Probability:* Substituting from (3.2.1.1) in (3.2.2.7),

$$\Pr(X < 2) = F_X(1) \quad (3.2.3.1)$$

$$= \sum_{k=0}^1 {}^nC_k \left(\frac{5}{6}\right)^{10-k} \left(\frac{1}{6}\right)^k \quad (3.2.3.2)$$

$$= 3 \left(\frac{5}{6}\right)^{10} = 0.4845167486695371 \quad (3.2.3.3)$$

which is the desired probability.

3.2.4. The following code verifies the above result.

```
codes/binomial/binomial.py
```

4 CENTRAL LIMIT THEOREM: GAUSSIAN DISTRIBUTION

4.1 Bernoulli to Gaussian

4.1.1 *Mean* : The mean of the bernoulli distribution is

$$\mu = E(X_i) = \sum_{k=0}^1 k p_{X_i}(k) = p = \frac{1}{6} \quad (4.1.1)$$

4.1.2 *Moment*: The moment of the distribution is defined as

$$E(X_i^r) = \sum_{k=0}^1 k^r p_{X_i}(k) = p = \frac{1}{6} \quad (4.2.1)$$

4.1.3 *Variance* : The variance of the bernoulli distribution is defined as

$$\sigma^2 = E(X - E(X))^2 = E(X^2) - E^2(X) \quad (4.3.1)$$

$$= p - p^2 = p(1 - p) = \frac{5}{36} \quad (4.3.2)$$

The standard deviation

$$\sigma = \sqrt{p(1 - p)} \quad (4.3.3)$$

4.1.4 *The Gaussian Distribution*: Define

$$G = \frac{1}{\sqrt{n}} \sum_{k=1}^n \frac{X_i - \mu}{\sigma} \quad (4.4.1)$$

4.1.5 *Approximating Binomial Using Gaussian*:

From (4.4.1) and (3.2.2.1),

$$X \approx \sigma \sqrt{n}G + n\mu \quad (4.5.1)$$

$$\Rightarrow F_X(k) = \Pr(\sigma \sqrt{n}G + n\mu \leq k) \quad (4.5.2)$$

$$= F_G\left(\frac{k - n\mu}{\sigma \sqrt{n}}\right) \approx \phi\left(\frac{k - n\mu}{\sigma \sqrt{n}}\right) \quad (4.5.3)$$

where

$$\phi_X(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < \infty \quad (4.5.4)$$

4.1.6 The probability density function (PDF) of G is

$$p_G(x) = \frac{d}{dx} F_X(x) \quad (4.6.1)$$

$$= \frac{1}{\sigma \sqrt{n}} \phi'\left(\frac{k - n\mu}{\sigma \sqrt{n}}\right) \quad (4.6.2)$$

For large n , G is a continuous distribution with probability density function (PDF)

$$p_G(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (4.6.3)$$

4.1.7 *Evaluationg the Probability*: From 4.5.3 and 4.6.2,

$$\Pr(X \leq 1) = F_G(1) = p_G(0) + p_G(1) \quad (4.7.1)$$

$$\approx 0.41299463887797094 \quad (4.7.2)$$

which is close to (3.2.3.3).

4.2 Uniform to Gaussian

4.2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (4.1.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

```
codes/cdf/exrand.c
codes/cdf/coeffs.h
```

4.2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?



Fig. 4.2: The CDF of X



Fig. 4.3: The PDF of X

Solution: The CDF of X is plotted in Fig. 4.2

4.2.3 Load `gau.dat` in python and plot the empirical PDF of X using the samples in `gau.dat`. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (4.3.1)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 4.3 using the code below

```
codes/clt/pdf_plot.py
```

4.2.4 Find the mean and variance of X by writing a C program.

Solution: Execute

```
codes/clt/gaussian_numbers.c
```

4.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (4.5.1)$$

repeat the above exercise theoretically.

Solution:

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx \quad (4.5.2)$$

$$= 0 \quad (\text{odd function}) \quad (4.5.3)$$

$$E(X^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx \quad (\text{even function}) \quad (4.5.4)$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} x^2 e^{-\frac{x^2}{2}} dx \quad (4.5.5)$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{2u} e^{-u} du \quad \left(\text{Let } \frac{x^2}{2} = u \right) \quad (4.5.6)$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u} u^{\frac{3}{2}-1} du \quad (4.5.7)$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \quad (4.5.8)$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) \quad (4.5.9)$$

$$= 1 \quad (4.5.10)$$

where we have used the fact that

$$\therefore \Gamma(n) = (n-1)\Gamma(n-1); \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad (4.5.11)$$

Thus, the variance is

$$\sigma^2 = E(X)^2 - E^2(X) = 1 \quad (4.5.12)$$

4.2.6 Let U be a uniform random variable between 0 and 1.

4.2.7 Load the `uni.dat` file into python and plot the empirical CDF of U using the samples in `uni.dat`. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (4.7.1)$$

Solution: The following code plots Fig. 7.1.1

```
codes/cdf/cdf_plot.py
```

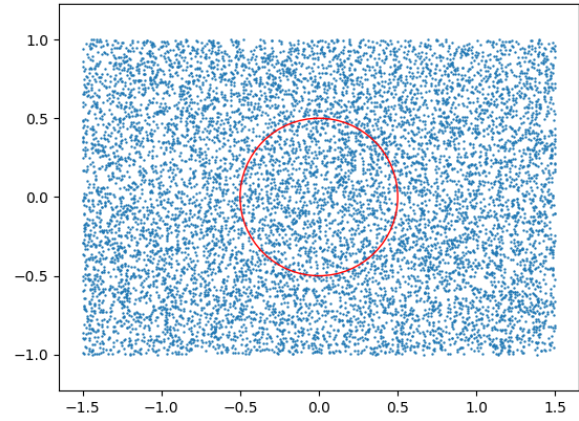
Fig. 4.7: The CDF of U 

Fig. 5.1.1

4.2.8 Find a theoretical expression for $F_U(x)$.

4.2.9 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (4.9.1)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (4.9.2)$$

Write a C program to find the mean and variance of U .

4.2.10 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (4.10.1)$$

5 STOCHASTIC GEOMETRY

Suppose you drop a die at random on the rectangular region shown in Fig. 5.1.1. What is the probability that it will land inside the circle with diameter 1m?

5.1. In Fig. 5.1.1, the sample size S is the area of the rectangle given by

$$S = 3 \times 2 = 6m^2 \quad (5.1.1)$$

The event size is the area of the circle given by

$$E = \pi \left(\frac{1}{2} \right)^2 = \frac{\pi}{4} m^2 \quad (5.1.2)$$

The probability of the dice landing in the circle is

$$\Pr(E) = \frac{E}{S} = \frac{\pi}{24} \quad (5.1.3)$$

5.2. The python code is available in

/codes/stochastic/rect.py

The python code generates 10,000 points uniformly within the rectangle of dimensions 3×2 and checks for the number of points within the circle of radius 0.5. The ratio of these is close to $\frac{\pi}{24}$. Note that each time the code is run, the ratio will change, but will still be close to $\frac{\pi}{24}$.

6 TRANSFORMATION OF VARIABLES

6.1 Using Definition

6.1.1. Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1.1.1)$$

6.1.2. If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.1.2.1)$$

find α .

6.1.3. Plot the CDF and PDF of

$$A = \sqrt{V} \quad (6.1.3.1)$$

6.1.4. Find an expression for $F_A(x)$ using the definition. Plot this expression and compare with the

result of problem 6.1.3.

Solution:

$$F_A(x) = \Pr(A \leq x) = \Pr(\sqrt{V} \leq x) \quad (6.1.4.1) \quad 6.2.3. \text{ Find}$$

$$= \Pr(V \leq x^2) = F_V(x^2) \quad (6.1.4.2)$$

From (6.1.2.1),

$$F_V(x^2) = \begin{cases} 1 - e^{-\alpha x^2} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.1.4.3)$$

Substituting

$$\alpha = \frac{1}{2} \quad (6.1.4.4)$$

$$F_V(x^2) = \begin{cases} 1 - e^{-\frac{x^2}{2}} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.1.4.5)$$

6.1.5. Find an expression for $p_A(x)$.

Solution: The PDF is obtained as

$$f_V(x^2) = \frac{d}{dx} F_V(x^2) \quad (6.1.5.1)$$

$$= \begin{cases} x e^{-\frac{x^2}{2}} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.1.5.2)$$

6.2 Using Jacobian

6.2.1. Evaluate the joint PDF of X_1, X_2 , given by

$$p_{X_1, X_2}(x_1, x_2) = p_{X_1}(x_1) p_{X_2}(x_2) \quad (6.2.1.1)$$

Solution: From (4.5.1)

$$p_{X_1}(x_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \quad (6.2.1.2) \quad 6.2.6. \text{ Are } V \text{ and } \Theta \text{ independent?}$$

$$p_{X_2}(x_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_2^2}{2}} \quad (6.2.1.3)$$

$$\Rightarrow p_{X_1, X_2}(x_1, x_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2 + x_2^2}{2}} \quad (6.2.1.4)$$

6.2.2. Let

$$X_1 = \sqrt{V} \cos \theta \quad (6.2.2.1) \quad 6.2.7. \text{ Find } p_A(x) \text{ using the Jacobian.}$$

$$X_2 = \sqrt{V} \sin \theta. \quad (6.2.2.2)$$

Evaluate the Jacobian

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} \\ \frac{\partial x_1}{\partial \theta} & \frac{\partial x_2}{\partial \theta} \end{vmatrix} \quad (6.2.2.3)$$

Solution:

$$J = \begin{vmatrix} \frac{1}{2\sqrt{v}} \cos \theta & \frac{1}{2\sqrt{v}} \sin \theta \\ -\sqrt{v} \sin \theta & \sqrt{v} \cos \theta \end{vmatrix} = \frac{1}{2} \quad (6.2.2.4)$$

$$p_{V, \Theta}(v, \theta) = |J| p_{X_1, X_2}(x_1, x_2) \quad (6.2.3.1)$$

Solution: From (6.2.2.4) and (6.2.1.4),

$$p_{V, \Theta}(v, \theta) = \frac{1}{4\pi} \exp -\frac{v}{2}, v \geq 0, 0 < \theta < 2\pi \quad (6.2.3.2)$$

6.2.4. Find $p_V(v)$.

Solution: For $v \geq 0$, from (6.2.3.2),

$$p_V(v) = \int_0^{2\pi} \frac{1}{4\pi} \exp -\frac{v}{2} d\theta \quad (6.2.4.1)$$

$$= (2\pi) \times \frac{1}{4\pi} \exp -\frac{v}{2} \quad (6.2.4.2)$$

$$= \frac{1}{2} \exp -\frac{v}{2} \quad (6.2.4.3)$$

$$\therefore p_V(v) = \begin{cases} \frac{1}{2} \exp -\frac{v}{2} & v \geq 0 \\ 0 & v < 0 \end{cases} \quad (6.2.4.4)$$

6.2.5. Find $p_\Theta(\theta)$.

Solution: For $0 \leq \theta \leq 2\pi$, from (6.2.3.2),

$$p_\Theta(\theta) = \int_0^\infty \frac{1}{4\pi} \exp -\frac{v}{2} dv \quad (6.2.5.1)$$

$$= \frac{1}{2\pi} \left[1 - e^{-\frac{v}{2}} \right]_0^\infty \quad (6.2.5.2)$$

$$= \frac{1}{2\pi} \quad (6.2.5.3)$$

$$\therefore p_V(v) = \begin{cases} \frac{1}{2\pi} & 0 \leq \theta \leq 2\pi \\ 0 & \text{otherwise} \end{cases} \quad (6.2.5.4)$$

Solution: Yes,

$$\therefore p_V(v) p_\Theta(\theta) = \frac{1}{2} \exp -\frac{v}{2} \times \frac{1}{2\pi} \quad (6.2.6.1)$$

$$= \frac{1}{4\pi} \exp -\frac{v}{2} \quad (6.2.6.2)$$

$$= p_{V, \Theta}(v, \theta) \quad (6.2.6.3)$$

Solution:

$$p_A(x) = \Pr(A = x) = \Pr(\sqrt{V} = x) \quad (6.2.7.1)$$

$$= \Pr(V = x^2) = p_V(x^2) \quad (6.2.7.2)$$

From (6.2.4.4), as $x^2 \geq 0$,

$$p_V(x^2) = \frac{1}{2} \exp -\frac{x^2}{2} \quad (6.2.7.3)$$

7 CONDITIONAL PROBABILITY

7.1. Plot

$$P_e = \Pr(\hat{X} = -1 | X = 1) \quad (7.1.1)$$

for

$$Y = AX + N, \quad (7.1.2)$$

where A is Raleigh with $E[A^2] = \gamma$, $N \sim \mathcal{N}(0, 1)$, $X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

Solution: The estimated value \hat{X} is given by

$$\hat{X} = \begin{cases} +1 & Y > 0 \\ -1 & Y < 0 \end{cases} \quad (7.1.3)$$

For $X = 1$,

$$Y = A + N \quad (7.1.4)$$

$$P_e = \Pr(\hat{X} = -1 | X = 1) \quad (7.1.5)$$

$$= \Pr(Y < 0 | X = 1) \quad (7.1.6)$$

$$= \Pr(A < -N) \quad (7.1.7)$$

$$= F_A(-N) \quad (7.1.8)$$

$$= \int_{-\infty}^{\infty} F_A(-n) \times f_N(n) dn \quad (7.1.9)$$

If $N > 0$, $F_A(-N) = 0$. Therefore,

$$P_e = \int_{-\infty}^0 F_A(-n) \times f_N(n) dn \quad (7.1.10)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \left(1 - \exp\left(-\frac{n^2}{2\sigma^2}\right)\right) \exp\left(-\frac{n^2}{2}\right) dn \quad (7.1.11)$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{n^2}{2}\right) dn - \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(1+\sigma^2)n^2}{2\sigma^2}\right) dn \quad (7.1.12)$$

$$= \frac{\sqrt{2\pi} - \sqrt{\frac{\pi(2\sigma^2)}{1+\sigma^2}}}{2\sqrt{2\pi}} \quad (7.1.13)$$

$$= \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\sigma^2}{1+\sigma^2}} \quad (7.1.14)$$

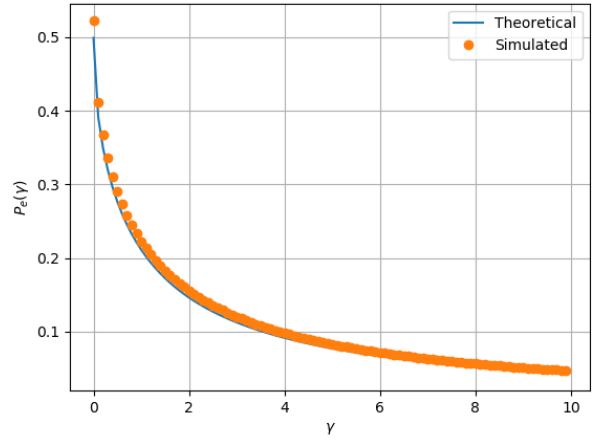


Fig. 7.1.1: The value of P_e for $0 \leq \gamma \leq 10$

For a Rayleigh Distribution with scale $= \sigma$,

$$E[A^2] = 2\sigma^2 \quad (7.1.15)$$

$$\gamma = 2\sigma^2 \quad (7.1.16)$$

$$\therefore P_e = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\gamma}{2+\gamma}} \quad (7.1.17)$$

7.2. Assuming that N is a constant, find an expression for P_e . Call this $P_e(N)$

7.3. For a function g ,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x) dx \quad (7.3.1)$$

Find $P_e = E[P_e(N)]$.

7.4. Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ . Comment.

8 TWO DIMENSIONS

8.1. Let

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}, \quad (8.1.1)$$

where

$$\mathbf{x} \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.1.2)$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \quad (8.1.3)$$

8.2. Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \quad (8.2.1)$$

on the same graph using a scatter plot.

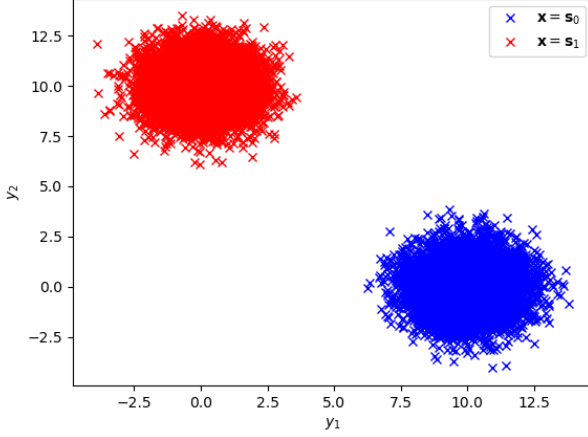


Fig. 8.2.1: Scatter plot of $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ for $A = 10$

Solution: The following python code plots the scatter plot when $\mathbf{x} = \mathbf{s}_0$ and $\mathbf{x} = \mathbf{s}_1$ in Fig. 8.2.1

```
codes/twoD/scatter_plot.py
```

8.3. For the above problem, find a decision rule for detecting the symbols \mathbf{s}_0 and \mathbf{s}_1 .

Solution: The multivariate Gaussian distribution is defined as

$$p_{\mathbf{x}}(x_1, \dots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp \left\{ -\frac{1}{2} (\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \right\} \quad (8.3.1)$$

where μ is the mean vector, $\Sigma = E[(\mathbf{x} - \mu)(\mathbf{x} - \mu)^T]$ is the covariance matrix and $|\Sigma|$ is the determinant of Σ . For a bivariate gaussian distribution,

$$p(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp \left[-\frac{1}{2(1-\rho^2)} \times \left\{ \frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} \right\} \right] \quad (8.3.2)$$

where

$$\mu = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}, \quad (8.3.3)$$

$$\rho = \frac{E[(x-\mu_x)(y-\mu_y)]}{\sigma_x\sigma_y}. \quad (8.3.4)$$

$$\mathbf{y}|s_0 = \begin{pmatrix} A + n_1 \\ n_2 \end{pmatrix} \quad (8.3.5)$$

$$\mathbf{y}|s_1 = \begin{pmatrix} n_1 \\ A + n_2 \end{pmatrix} \quad (8.3.6)$$

Substituting these values in (8.3.2),

$$p(\mathbf{y}|s_0) = \frac{1}{2\pi\sigma_{y_1}\sigma_{y_2}\sqrt{1-\rho_1^2}} \exp \left[-\frac{1}{2(1-\rho_1^2)} \times \left\{ \frac{(y_1-A)^2}{\sigma_{y_1}^2} + \frac{(y_2)^2}{\sigma_{y_2}^2} - \frac{2\rho_1(y_1-A)(y_2)}{\sigma_{y_1}\sigma_{y_2}} \right\} \right] \quad (8.3.7)$$

$$p(\mathbf{y}|s_1) = \frac{1}{2\pi\sigma_{y_1}\sigma_{y_2}\sqrt{1-\rho_2^2}} \exp \left[-\frac{1}{2(1-\rho_2^2)} \times \left\{ \frac{(y_1)^2}{\sigma_{y_1}^2} + \frac{(y_2-A)^2}{\sigma_{y_2}^2} - \frac{2\rho_2(y_1)(y_2-A)}{\sigma_{y_1}\sigma_{y_2}} \right\} \right] \quad (8.3.8)$$

where,

$$\begin{aligned} \rho_1 &= E[(y_1-A)(y_2)] = E[n_1n_2] = 0, \\ \rho_2 &= E[(y_1)(y_2-A)] = E[n_1n_2] = 0, \\ \sigma_{y_1} &= \sigma_{y_2} = 1 \end{aligned} \quad (8.3.9)$$

For equiprobably symbols, the MAP criterion is defined as

$$p(\mathbf{y}|s_0) \stackrel{s_0}{\underset{s_1}{\gtrless}} p(\mathbf{y}|s_1) \quad (8.3.10)$$

Using (8.3.7) and (8.3.8) and substituting the values from (8.3.9), we get

$$(y_1-A)^2 + y_2^2 \stackrel{s_1}{\underset{s_0}{\gtrless}} y_1^2 + (y_2-A)^2 \quad (8.3.11)$$

On simplifying, we get the decision rule is

$$y_1 \stackrel{s_0}{\underset{s_1}{\gtrless}} y_2 \quad (8.3.12)$$

8.4. Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \quad (8.4.1)$$

with respect to the SNR from 0 to 10 dB.

8.5. Obtain an expression for P_e . Verify this by comparing the theory and simulation plots on the same graph.

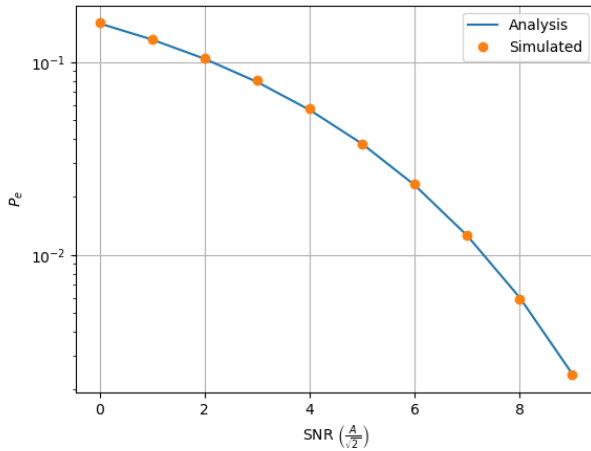


Fig. 8.4.1: P_e with respect to SNR from 0 to 10 dB

Solution:

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \quad (8.5.1)$$

Given that \mathbf{s}_0 was transmitted, the received signal is

$$\mathbf{y} | \mathbf{s}_0 = \begin{pmatrix} A \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \quad (8.5.2)$$

From (8.3.12), the probability of error is given by

$$P_e = \Pr(y_1 < y_2 | \mathbf{s}_0) = \Pr(A + n_1 < n_2) \quad (8.5.3)$$

$$= \Pr(n_2 - n_1 > A) \quad (8.5.4)$$

Note that $n_2 - n_1 \sim \mathcal{N}(0, 2)$. Thus,

$$P_e = \Pr(\sqrt{2}w > A) \quad (8.5.5)$$

$$\Pr\left(w > \frac{A}{\sqrt{2}}\right) \quad (8.5.6)$$

$$\Rightarrow P_e = Q\left(\frac{A}{\sqrt{2}}\right) \quad (8.5.7)$$

where $w \sim \mathcal{N}(0, 1)$. The following code plots the P_e curve in Fig. (8.4.1).

```
codes/twoD/ber_snr_plot.py
```

9 TRANSFORM DOMAIN: MOMENT GENERATING FUNCTION

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

9.1. Find $M_X(s) = E[e^{-sX}]$.

Solution: The MGF of X is

$$M_X(s) = \int_{-\infty}^{\infty} e^{-sX} p_X(x) dx \quad (9.1.1)$$

$$= e^{-s\mu} e^{-\frac{s^2\sigma^2}{2}} \quad (9.1.2)$$

9.2. Let

$$N = n_1 - n_2, \quad n_1, n_2 \sim \mathcal{N}(0, 1). \quad (9.2.1)$$

Find $M_N(s)$, assuming that n_1 and n_2 are independent.

Solution: Substituting from (9.1.2) and using independence,

$$M_N(s) = E[e^{-(n_1 - n_2)s}] = M_{n_1}(s)M_{n_2}(-s) \quad (9.2.2)$$

$$= e^{-s^2\sigma^2} = e^{-s^2} \quad (\because \sigma = 1) \quad (9.2.3)$$

9.3. Show that N is Gaussian. Find its mean and variance. Comment.

Solution: From (9.2.3) and (9.1.2), it is obvious that $X \sim \mathcal{N}(0, 2)$. Thus, the difference of two Gaussian random variables is also a Gaussian random variable.

10 UNIFORM TO OTHER: QUANTILE FUNCTION

10.1. Generate samples of

$$V = -2 \ln(1 - U) \quad (10.1.1)$$

and plot its CDF. Comment.

Solution: The CDF of V is defined as

$$F_V(v) = \Pr(V \leq v) \quad (10.1.2)$$

$$= \Pr(-2 \ln(1 - U) \leq v) \quad (10.1.3)$$

$$= \Pr\left(\ln(1 - U) \geq -\frac{v}{2}\right) \quad (10.1.4)$$

$$= \Pr\left(1 - U \geq \exp\left(-\frac{v}{2}\right)\right) \quad (10.1.5)$$

$$= \Pr\left(U \leq 1 - \exp\left(-\frac{v}{2}\right)\right) \quad (10.1.6)$$

$$= F_U\left(1 - \exp\left(-\frac{v}{2}\right)\right) \quad (10.1.7)$$

From (??),

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \quad (10.1.8)$$

Substituting the above in (10.1.7),

$$F_U\left(1 - \exp\left(-\frac{v}{2}\right)\right) = \begin{cases} 0 & 1 - \exp\left(-\frac{v}{2}\right) < 0 \\ 1 - \exp\left(-\frac{v}{2}\right) & 0 \leq 1 - \exp\left(-\frac{v}{2}\right) \leq 1 \\ 1 & 1 - \exp\left(-\frac{v}{2}\right) > 1 \end{cases} \quad (10.1.9)$$

After some algebra, the above conditions yield

$$F_V(v) = \begin{cases} 0 & v < 0 \\ 1 - \exp\left(-\frac{v}{2}\right) & v \geq 0 \end{cases} \quad (10.1.10)$$

which is the CDF of the exponential distribution with parameter $\frac{1}{2}$. The following code generates the CDF obtained using (10.1.1) and (10.1.10) in Fig. 10.1.1.

codes/quantile/ASSIGNMENT_1.py

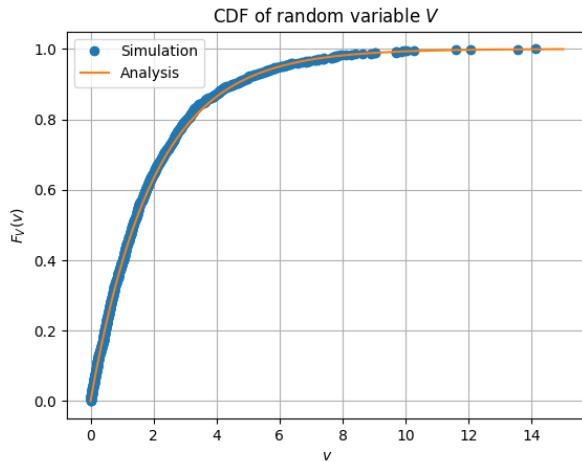


Fig. 10.1.1

Comment: For the uniform distribution, $0 < U < 1$. For any random variable V , $0 < F_V(x) < 1$. This similarity between U and $F_V(x)$ is used to generate the random variable V from U . Thus,

$$V = F_V^{-1}(U) \quad (10.1.11)$$

where F_V^{-1} is defined to be a *quantile* function.

10.2. Generate the Rayleigh distribution from Uniform. Verify your result through graphical plots.

11 MISCELLANEOUS DISTRIBUTIONS

- 11.1. A carton consists of 100 shirts of which 88 are good, 8 have minor defects and 4 have major defects. Jimmy, a trader, will only accept the shirts which are good, but Sujatha, another trader, will only reject the shirts which have major defects. One shirt is drawn at random from the carton. What is the probability that
- it is acceptable to Jimmy?
 - it is acceptable to Sujatha?

Solution: From the given information, Table 11.1 can be generated. Then

A Random variable which has 3 possible values	
$\Pr(X = 0) = \frac{4}{100} = 0.04$	out of 100, 4 have major defects shirts
$\Pr(X = 1) = \frac{8}{100} = 0.08$	out of 100, 8 are accepted minor defected shirts
$\Pr(X = 2) = \frac{88}{100} = 0.88$	out of 100, 88 are accepted good shirts

TABLE 11.1: Random variables

- a) The probability that the shirt is acceptable to Jimmy is

$$\Pr(X = 2) = \frac{88}{100} \quad (11.1.1)$$

- b) The probability that the shirt is acceptable to Sujatha is

$$1 - \Pr(X = 0) = 1 - \frac{4}{100} = \frac{96}{100} \quad (11.1.2)$$

The following code simulates the probability

codes/misc/discrete.ipynb