Assignment 6

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Download all latex-tikz codes from

https://github.com/GauthamBellamkonda/AI1103/tree/main/Assignment6

1 Problem

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n (≥ 2) from a distribution having the probability density function

$$f(x;\theta) = \begin{cases} \frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right) & x > 0, \\ 0, & \text{otherwise,} \end{cases} \tag{1.0.1}$$

where $\theta \in (0, \infty)$. Let $X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$ and $T = \sum_{i=1}^{n} X_i$. Then $E(X_{(1)}|T)$ equals ....

(A) $\dfrac{T}{n^2}$

(B) $\dfrac{T}{n}$

(C) $\dfrac{(n+1)T}{2n}$

(D) $\dfrac{(n+1)^2 T}{4n^2}$

2 Solution

Lehmann–Scheffé theorem :
If T is a complete sufficient statistic for θ and

$$E(g(T)) = \tau(\theta) \tag{2.0.1}$$

then $g(T)$ is the uniformly minimum-variance unbiased estimator (UMVUE) of $\tau(\theta)$.

We know that

$$T = \sum_{i=1}^{n} X_i \tag{2.0.2}$$

is a complete and sufficient statistic. By the law of total expectation,

$$E\left(E(X_{(1)}|T)\right) = E(X_{(1)}) \tag{2.0.3}$$

By Lehmann–Scheffé theorem, with

$$\theta = X_{(1)}, \tag{2.0.4}$$
$$\tau(x) = E(x), \tag{2.0.5}$$
$$g(T) = E(X_{(1)}|T). \tag{2.0.6}$$

it follows from (2.0.3) that $E(X_{(1)}|T)$ is the UMVUE of $E(X_{(1)})$.

$$\Pr(X_{(1)} > x) = \Pr(X_1 > x)\ldots\Pr(X_n > x) \tag{2.0.7}$$
$$= (1 - F_{X_1}(x))\ldots(1 - F_{X_n}(x)) \tag{2.0.8}$$
$$= (1 - F_{X_1}(x))^n \tag{2.0.9}$$
$$= \exp\left(-\frac{nx}{\theta}\right) \tag{2.0.10}$$
$$F_{X_{(1)}}(x) = 1 - \exp\left(-\frac{nx}{\theta}\right) \tag{2.0.11}$$
$$f_{X_{(1)}}(x) = \frac{n}{\theta}\exp\left(-\frac{nx}{\theta}\right) \tag{2.0.12}$$

Therefore, $X_{(1)}$ follows an exponential distribution with mean $\frac{\theta}{n}$.

$$E(X_{(1)}) = \frac{\theta}{n} \tag{2.0.13}$$

Note that,

$$E\left(\frac{T}{n^2}\right) = E\left(\frac{\sum_{i=1}^{n} X_i}{n^2}\right) \tag{2.0.14}$$
$$= \frac{E(\sum_{i=1}^{n} X_i)}{n^2} \tag{2.0.15}$$
$$= \sum_{i=1}^{n}\frac{E(X_i)}{n^2} \tag{2.0.16}$$
$$= \sum_{i=1}^{n}\frac{\theta}{n^2} \tag{2.0.17}$$
$$= \frac{\theta}{n} \tag{2.0.18}$$
$$= E(X_{(1)}) \tag{2.0.19}$$

Therefore, by Lehmann–Scheffé theorem, with

$$\theta = X_{(1)}, \tag{2.0.20}$$

$$\tau(x) = E(x), \tag{2.0.21}$$

$$g(T) = \frac{T}{n^2}, \tag{2.0.22}$$

it follows that $\frac{T}{n^2}$ is UMVUE of $E(X_{(1)})$.

Since there exists a unique UMVUE for $E(X_{(1)})$, it follows that

$$E(X_{(1)}|T) = \frac{T}{n^2} \tag{2.0.23}$$

Hence, option A is correct.

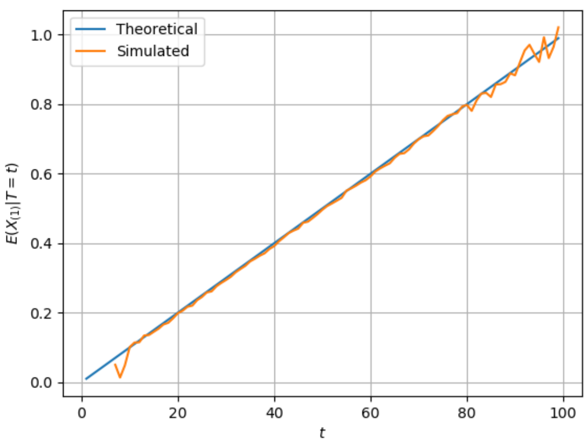

Fig. 4: Theory vs Simulated plot of $E(X_{(1)}|T)$