Assignment 3

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Download all python codes from

https://github.com/GauthamBellamkonda/AI1103/tree/main/Assignment3/Codes

and latex-tikz codes from

https://github.com/GauthamBellamkonda/AI1103/tree/main/Assignment3

1 Problem

(GATE 45) Consider a discrete-time channel $Y = X+Z$, where the additive noise Z is signal dependent. In particular, given the transmitted symbol $X \in \{-a, a\}$ at any instant, the noise sample Z is chosen indepedently from a Gaussian distribution with mean βX and unit variance. Assume a threshold detector with zero threshold at the receiver. When $\beta = 0$, the BER was found to be $Q(a) = 1 \times 10^{-8}$.

$$\left(Q(v) = \frac{1}{\sqrt{2\pi}}\int_v^\infty e^{-\frac{u^2}{2}}du, \text{ and for } v > 1, \text{ use } Q(v) = e^{\frac{-v^2}{2}}\right) \tag{1.0.1}$$

When $\beta = -0.3$, BER is closest to

(A) 10^{-7}
(B) 10^{-6}
(C) 10^{-4}
(D) 10^{-2}

2 Solution

Given that $X \in \{-a, +a\}$ is a random variable.

$$\Pr(X = a) = \frac{n(X = a)}{2} = \frac{1}{2} \tag{2.0.1}$$

$$\Pr(X = -a) = \frac{n(X = -a)}{2} = \frac{1}{2} \tag{2.0.2}$$

Also, Z is chosen from Gaussian Distribution with mean βX and unit variance.

$$\therefore F_Z(z) = \mathrm{G}\left(\frac{z - \beta X}{1}\right) \tag{2.0.3}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \exp\left(\frac{-(z - \beta X)^2}{2}\right)dz \tag{2.0.4}$$

On differentiating both the sides with respect to z, we get

$$f_Z(z) = \frac{d}{dz}F_Z(z) \tag{2.0.5}$$

$$= \frac{d}{dz}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \exp\left(\frac{-(z - \beta X)^2}{2}\right)dz\right] \tag{2.0.6}$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(z - \beta X)^2}{2}\right) \tag{2.0.7}$$

$$\Leftrightarrow f_Y(z + X) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(z - \beta X)^2}{2}\right) \tag{2.0.8}$$

$$\Leftrightarrow f_Y(y) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y - X - \beta X)^2}{2}\right) \tag{2.0.9}$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y - X(1 + \beta))^2}{2}\right) \tag{2.0.10}$$

$$\therefore f_Y(y \mid X = a) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y - a(1 + \beta))^2}{2}\right) \tag{2.0.11}$$

$$\& \ f_Y(y \mid X = -a) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y + a(1 + \beta))^2}{2}\right) \tag{2.0.12}$$

Since $X \in \{-a, +a\}$ is also a random variable,

$$f_Y(y) = f_Y(y \mid X = a)\Pr(X = a) + f_Y(y \mid X = -a)\Pr(X = -a) \tag{2.0.13}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y - a(1 + \beta))^2}{2}\right) + \frac{1}{2}\cdot\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y + a(1 + \beta))^2}{2}\right) \tag{2.0.14}$$

Therefore, the resultant signal $Y = X + Z$ is comprised of X which can take either positive or negative value, and some noise Z. The detector (which has zero threshold) can give us incorrect bits when $X = +a$ and $Y < 0$ (BER_{+a}) or $X = -a$ and $Y < 0$ (BER_{-a}), as shown in the graph below.

$$\therefore \mathrm{BER} = \mathrm{BER}_{+a} + \mathrm{BER}_{-a} \tag{2.0.15}$$

$$= f_Y(y < 0, X = a) + f_Y(y > 0, X = -a) \tag{2.0.16}$$

$$= f_Y(y < 0 \mid X = a)\Pr(X = a) + f_Y(y > 0 \mid X = -a)\Pr(X = -a) \tag{2.0.17}$$

$$= \int_{-\infty}^{0}\frac{1}{2}\cdot f_Y(y \mid X = a)dy + \int_{0}^{\infty}\frac{1}{2}\cdot f_Y(y \mid X = -a)dy \tag{2.0.18}$$

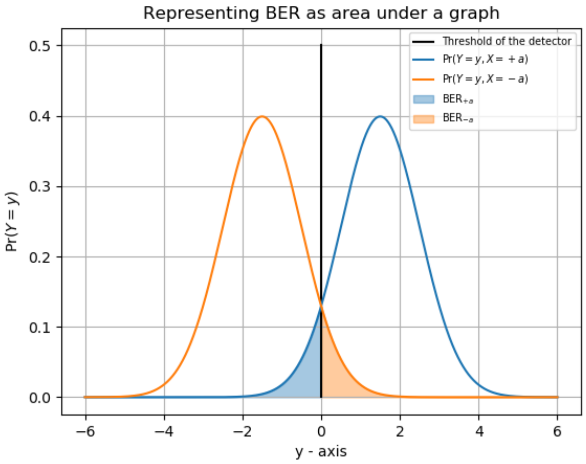

Fig. 1: The PDF of Y

On substituting the values of $f_Y(y \mid X = a)$ and $f_Y(y \mid X = -a)$ from 2.0.11 and 2.0.12,

$$\begin{aligned}
\mathrm{BER} &= \int_{-\infty}^{0} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(y - a(1+\beta))^2}{2}\right) dy \\
&\quad + \int_{0}^{\infty} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(y + a(1+\beta))^2}{2}\right) dy \qquad (2.0.19)\\
&= \int_{0}^{\infty} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(y + a(1+\beta))^2}{2}\right) dy \\
&\quad + \int_{0}^{\infty} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(y + a(1+\beta))^2}{2}\right) dy \qquad (2.0.20)\\
&= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y + a(1+\beta))^2}{2}\right) dy \qquad (2.0.21)
\end{aligned}$$

From the definition of $Q(v)$, it is easy to see that the expression in 2.0.21 is equal to $Q(a(1+\beta))$

$$\therefore \mathrm{BER} = Q(a(1+\beta)) \qquad (2.0.22)$$

When $\beta = 0$, it is given that

$$\mathrm{BER} = Q(a) = 10^{-8} \qquad (2.0.23)$$

On computing, $Q(1) \approx 0.16$. Since $Q(a) < Q(1)$, it is easy to see that $a > 1$ (as $Q(x)$ is a decreasing function)

$$\begin{aligned}
\therefore e^{-a^2/2} &= 10^{-8} \qquad (2.0.24)\\
\Leftrightarrow a &\approx 6.069 \qquad (2.0.25)
\end{aligned}$$

When $\beta = -0.3$,

$$\begin{aligned}
\mathrm{BER} = Q(a(1+\beta)) &= Q(6.069 \times (1 - 0.3)) \qquad (2.0.26)\\
&= Q(6.069 \times 0.7) \qquad (2.0.27)\\
&= Q(4.249) \qquad (2.0.28)\\
&\approx \exp(-\frac{4.249^2}{2}) \qquad (2.0.29)\\
&\approx 1.2 \times 10^{-4} \qquad (2.0.30)
\end{aligned}$$

Therefore, when $\beta = -0.3$, BER is closest to 10^{-4} and option (C) is correct.