# Assignment 6

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Download all latex-tikz codes from

https://github.com/GauthamBellamkonda/AI1103/tree/main/Assignment6

## 1 Problem

Let $X_1, X_2, \dots, X_n$ be a random sample of size $n$ ($\geq 2$) from a distribution having the probability density function

$$f(x;\theta) = \begin{cases} \frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right) & x > 0, \\ 0, & \text{otherwise,} \end{cases} \tag{1.0.1}$$

where $\theta \in (0, \infty)$. Let $X_{(1)} = \min\{X_1, X_2, \dots, X_n\}$ and $T = \sum_{i=1}^{n} X_i$. Then $E(X_{(1)}|T)$ equals .........

(A) $\frac{T}{n^2}$

(B) $\frac{T}{n}$

(C) $\frac{(n+1)T}{2n}$

(D) $\frac{(n+1)^2 T}{4n^2}$

## 2 Prerequisites

**Lemma 2.1** (Sum of Gamma random variables). *Suppose that $X_1, X_2, X_3, \dots X_n$ are $n$ gamma variables with parameters $k$ and $\theta$, $X_i \sim \Gamma(k, \theta)$. Then the sum $T = \sum_{i=1}^{n} X_i$ follows a gamma distribution with parameters $nk$ and $\theta$, $T \sim \Gamma(nk, \theta)$*

**Lemma 2.2** (Expectation of $X_{(1)}$). *Let $X_1, X_2, \dots, X_n$ be a random sample of size $n$ ($\geq 2$) from an exponential distribution with scale parameter $\theta$. Let $X_{(1)}$ be the min$\{X_1, X_2, \dots, X_n\}$. Then, expectation of $X_{(1)}$,*

$$E(X_{(1)}) = \frac{\theta}{n} \tag{2.0.1}$$

*Proof.* Let's find the probability distribution function and the expectation value of $X_{(1)}$ :

$$\Pr(X_{(1)} > x) = \Pr(X_1 > x) \dots \Pr(X_n > x) \tag{2.0.2}$$
$$= (1 - F_{X_1}(x)) \dots (1 - F_{X_n}(x)) \tag{2.0.3}$$
$$= (1 - F_{X_1}(x))^n \tag{2.0.4}$$
$$= \exp\left(-\frac{nx}{\theta}\right) \tag{2.0.5}$$
$$F_{X_{(1)}}(x) = 1 - \exp\left(-\frac{nx}{\theta}\right) \tag{2.0.6}$$
$$f_{X_{(1)}}(x) = \frac{n}{\theta}\exp\left(-\frac{nx}{\theta}\right) \tag{2.0.7}$$

Therefore, $X_{(1)}$ follows an exponential distribution with mean $\frac{\theta}{n}$.

$$E(X_{(1)}) = \frac{\theta}{n} \tag{2.0.8}$$

$\square$

**Definition 1** (Laplace transform). *Laplace transform is an integral transform that converts a real function of a real variable $t$ to a function of a complex variable $s$. The laplace transform of a function $f(t)$ evaluated at $s$ is defined by*

$$F(s) = \int_0^\infty f(t)e^{-st}dt \tag{2.0.9}$$

**Lemma 2.3.** *If the laplace transform of a function $f(t)$ at $s$ is 0 for all $s > 0$, then the function $f(t) = 0$ almost everywhere in $(0, \infty)$.*

**Definition 2** (Complete Statistic). *The statistic $T$ is said to be complete for the distribution of $X$ if, for every measurable function $g$, if*

$$E(g(T)) = 0 \ \forall\ \theta \Rightarrow \Pr(g(T) = 0) = 1 \tag{2.0.10}$$

**Definition 3** (Sufficient Statistic). *A statistic $t = T(X)$ is sufficient for underlying parameter $\theta$ precisely if the conditional probability distribution of the data $X$, given the statistic $t = T(X)$, does not depend on the parameter $\theta$.*

**Theorem 2.1** (Fischer-Neymann Factorisation theorem). *If the probability density function is $f_\theta(x)$, then $T$ is sufficient for $\theta$ if and only if nonnegative functions $g$ and $h$ can be found such that*

$$f_\theta(x) = h(x)g_\theta(T(x)) \tag{2.0.11}$$

**Lemma 2.4.**

$$T = \sum_{i=1}^{n} X_i \tag{2.0.12}$$

*is a complete and sufficient statistic.*

*Proof.* 1) *(Sufficiency)*

$$f_X(x_1, x_2, \ldots x_n) = f_{X_1}(x_1)f_{X_2}(x_2)\ldots f_{X_n}(x_n) \tag{2.0.13}$$

$$= \frac{1}{\theta}\exp\left(-\frac{x_1}{\theta}\right)\cdots\frac{1}{\theta}\exp\left(-\frac{x_n}{\theta}\right) \tag{2.0.14}$$

$$= 1 \cdot \frac{1}{\theta^n}\exp\left(-\frac{T}{\theta}\right) \tag{2.0.15}$$

$$= h(X) \cdot g(T, \theta) \tag{2.0.16}$$

with

$$g(T, \theta) = \frac{1}{\theta^n}\exp\left(-\frac{T}{\theta}\right) \tag{2.0.17}$$

$$h(X) = 1 \tag{2.0.18}$$

Therefore, $T$ is a sufficient statistic.

2) *(Completeness)*
$X_i$ follow a gamma distribution, $X_i \sim \Gamma\left(1, \frac{1}{\theta}\right)$. Hence, $T = \sum_{i=1}^{n} X_i$ follows a gamma distribution, $T \sim \Gamma\left(n, \frac{1}{\theta}\right)$. Therefore, the expected value of a function $g$ is

$$E(g(T)) = \int_0^\infty g(t)\frac{t^{n-1}e^{-\frac{t}{\theta}}}{\theta^n(n-1)!}dt \tag{2.0.19}$$

$$= \frac{1}{\theta^n(n-1)!}\int_0^\infty g(t)t^{n-1}e^{-\frac{t}{\theta}}dt \tag{2.0.20}$$

$$= \frac{1}{\theta^n(n-1)!}F\left(\frac{1}{\theta}\right) \tag{2.0.21}$$

$$= 0 \text{ for all } \theta > 0. \tag{2.0.22}$$

where,

$$f(t) = t^{n-1}g(t) \tag{2.0.23}$$

$$F\left(\frac{1}{\theta}\right) = \text{Laplace Transform of} f(t) \text{ at } \frac{1}{\theta} \tag{2.0.24}$$

Since, $E\left(g(T)\right) = 0$ for all $\theta > 0$,

$$F\left(\frac{1}{\theta}\right) = 0 \text{ for all } \theta > 0. \tag{2.0.25}$$

By Lemma 2.3 it follows from (2.0.25) that

$$f(t) = 0 \text{ almost everywhere in } (0, \infty) \tag{2.0.26}$$

$$g(t) = 0 \text{ almost everywhere in } (0, \infty) \tag{2.0.27}$$

$$\Pr\left(g(t) = 0\right) = 1 \tag{2.0.28}$$

Therefore, $T$ is a complete statistic.

□

**Theorem 2.2** (Lehmann–Scheffé theorem). *If $T$ is a complete sufficient statistic for $\theta$ and*

$$E(g(T)) = \tau(\theta) \tag{2.0.29}$$

*then $g(T)$ is the uniformly minimum-variance unbiased estimator (UMVUE) of $\tau(\theta)$.*

## 3 Solution

**Proposition 3.1.** *$E(X_{(1)}|T)$ is the uniformly minimum-variance unbiased estimator (UMVUE) of $E(X_{(1)})$*

*Proof.* By the law of total expectation,

$$E\left(E(X_{(1)}|T)\right) = E(X_{(1)}) \tag{3.0.1}$$

We know that $T = \sum_{i=1}^{n} X_i$ is a complete and sufficient statistic by 2.4. By Lehmann–Scheffé theorem, with

$$\theta = X_{(1)}, \tag{3.0.2}$$

$$\tau(x) = E(x), \tag{3.0.3}$$

$$g(T) = E(X_{(1)}|T). \tag{3.0.4}$$

it follows from (3.0.1) that $E(X_{(1)}|T)$ is the UMVUE of $E(X_{(1)})$. □

**Proposition 3.2.** *$\frac{T}{n^2}$ is uniformly minimum-variance unbiased estimator (UMVUE) of $E(X_{(1)})$*

*Proof.* $\frac{T}{n^2}$ is uniformly minimum-variance unbiased estimator (UMVUE) of $E(X_{(1)})$, since $E\left(\frac{T}{n^2}\right) = E(X_{(1)})$ :

Note that,

$$E\left(\frac{T}{n^2}\right) = E\left(\frac{\sum_{i=1}^{n} X_i}{n^2}\right) \tag{3.0.5}$$
$$= \frac{E(\sum_{i=1}^{n} X_i)}{n^2} \tag{3.0.6}$$
$$= \sum_{i=1}^{n} \frac{E(X_i)}{n^2} \tag{3.0.7}$$
$$= \sum_{i=1}^{n} \frac{\theta}{n^2} \tag{3.0.8}$$
$$= \frac{\theta}{n} \tag{3.0.9}$$
$$= E(X_{(1)}) \text{ (by Lemma 2.2)} \tag{3.0.10}$$

Therefore, by Lehmann–Scheffé theorem, with

$$\theta = X_{(1)}, \tag{3.0.11}$$
$$\tau(x) = E(x), \tag{3.0.12}$$
$$g(T) = \frac{T}{n^2}, \tag{3.0.13}$$

it follows that $\frac{T}{n^2}$ is UMVUE of $E(X_{(1)})$.

□

Since there exists a unique UMVUE for $E(X_{(1)})$, it follows from Proposition 3.1 and Proposition 3.2 that

$$E(X_{(1)}|T) = \frac{T}{n^2} \tag{3.0.14}$$

Hence, option A is correct.

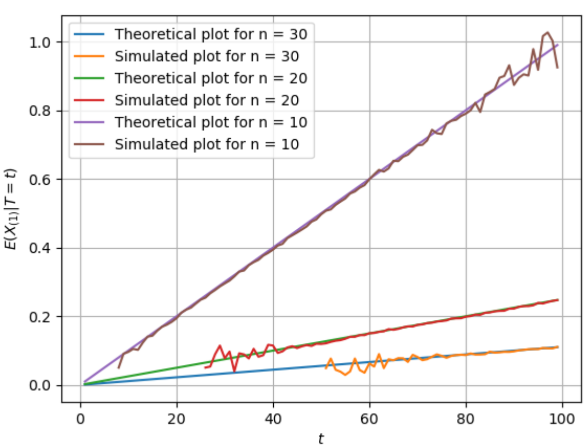


Fig. 2: Theory vs Simulated plot of $E(X_{(1)}|T)$