Challenge Problem

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1 Problem

GATE 2019 (ST) , Q.49 (Statistics section)
Let X be a random variable with characteristic function $\phi_X(\cdot)$ such that $\phi_X(2\pi) = 1$.Let $\mathbb{Z}$ denote the set of integers.Then $P(X \in \mathbb{Z})$ is equal to ...

2 Solution

We know that,

$$\phi_X(t) = \int_{\mathbb{R}} e^{itx} f_X(x)\, dx \tag{2.0.1}$$

$$\phi_X(2\pi) = \int_{\mathbb{R}} e^{2\pi ix} f_X(x)\, dx \tag{2.0.2}$$

$$= \int_{\mathbb{R}} \cos(2\pi x) f_X(x)\, dx + i \int_{\mathbb{R}} \sin(2\pi x) f_X(x)\, dx \tag{2.0.3}$$

$$\because \phi_X(2\pi) = 1, \int_{\mathbb{R}} \sin(2\pi x) f_X(x)\, dx = 0 \tag{2.0.4}$$

$$1 = \phi_X(2\pi) \tag{2.0.5}$$

$$= \int_{\mathbb{R}} \cos(2\pi x) f_X(x)\, dx \tag{2.0.6}$$

Assume that $\cos(2\pi x) \neq 1$.

$$\therefore 1 = \int_{\mathbb{R}} \cos(2\pi x) f_X(x)\, dx \tag{2.0.7}$$

$$< \int_{\mathbb{R}} 1 \cdot f_X(x)\, dx \tag{2.0.8}$$

$$< \int_{\mathbb{R}} f_X(x)\, dx \tag{2.0.9}$$

$$< 1. \tag{Contradiction}$$

Hence, our assumption that $\cos(2\pi x) \neq 1$ is incorrect.

$$\therefore \cos(2\pi x) = 1, \text{ for all } X = x \tag{2.0.10}$$

$$\Rightarrow X \in \mathbb{Z} \tag{2.0.11}$$

$$\Rightarrow \Pr(X \in \mathbb{Z}) = 1 \tag{2.0.12}$$