# Modelling Joint Distributions

TEAM RENEGADES

## 1 Digital Options

The price of entering a call option contract can be given as:

$$\begin{aligned} C(X,K,T) &= \int_0^{\infty} max(X_T - K, 0) f_x(X_T) dX_T \\ &= \int_K^{\infty} (X_T - K) f_x(X_T) dX_T \end{aligned} \tag{1.1}$$

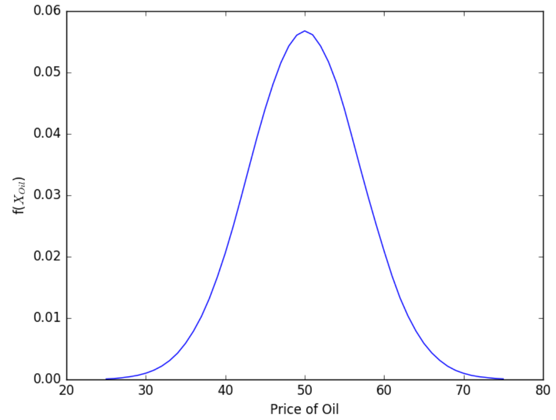
Differentiating both sides w.r.t K and using Leibniz Rule [1]:

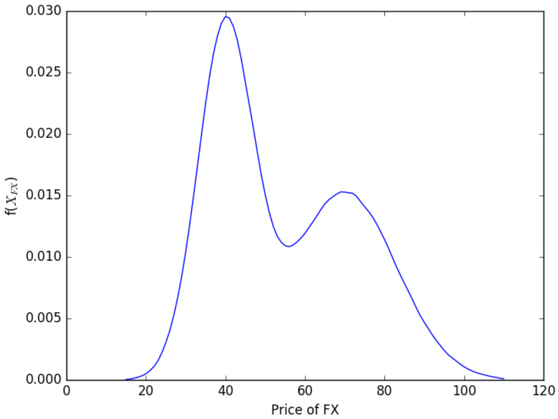
$$\frac{\partial C}{\partial K} = \int_K^{\infty} (-1) f_x(X_T) dX_T \tag{1.2}$$

Differentiating both sides w.r.t K and using Leibniz Rule again:

$$\frac{\partial^2 C}{\partial K^2} = f_x(K) \tag{1.3}$$

Thus, probability density function (p.d.f.) of the underlying asset can be obtained for any $X_T$, simply by taking the second derivative, given the strike (K) and the call prices (C).
So, using the data given in "Oil Call Option Prices" we can get the p.d.f. of the underlying Oil asset. Similarly, using the "FX Call Option Prices" we can get the p.d.f. of the underlying Exchange Rate asset.
The probability density function of Oil and FX assets are plotted below:




Now we need to price the digital options. The price of a digital Oil option is:

$$\begin{aligned} D(K,T;X) &= E[I(X_T > K)] \\ &= \int_K^{\infty} f_x(X_T)dX_T \end{aligned} \tag{1.4}$$

Thus knowing the p.d.f of the underlying Oil asset is enough to price the digital option. The same is done for pricing the digital FX option. We know that the price of a digital option can never be greater than 1, which is the case with our result.

## 2 Exotic Options

In case of exotic options we have the price of the contract as:

$$\begin{aligned} P(K,T;X) &= E[(max(X_T - K, 0))^2] \\ &= \int_K^{\infty} (X_T - K)^2 f_x(X_T)dX_T \end{aligned} \tag{2.1}$$

Using the probability density functions obtained in question 1 we can price the both oil exotic and FX exotic options.

## 3 Joint Options

In these type of options, the payoff is dependent on the prices of two underlying assets. Thus to accurately predict the payoff we need the joint probability density function. We have the marginal density function of Oil asset and FX asset from question 1. We now make an attempt to estimate the joint density using these marginals.

We have used the concept of Copulas to estimate the joint density. Two random variables $T_1$ and $T_2$ are joined by a copula function, C, if their joint cdf can be written as:

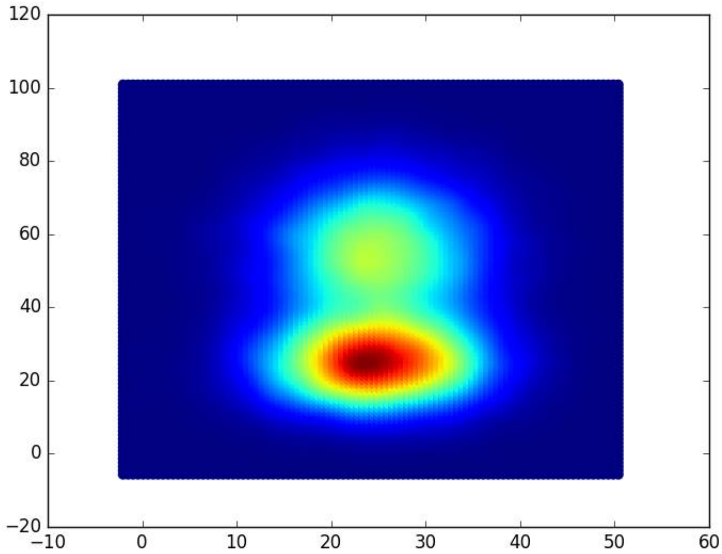
$$F(T_1, T_2) = C(F_1(t_1), F_2(t_2)) \tag{3.1}$$

This result is known as Sklar's Theorem.[2]
We now need to find the copula for the marginal distribution of Oil prices and FX prices. We use the Frank's copula [2] for this purpose.We have used the 'ambhas'[3] package in python for finding the copulas.

STEPS:

1. Obtain the marginal distribution for Oil ($X_1$) and FX ($X_2$) prices from question 1.
2. Generate discrete random variables (R1, R2) that follow this distribution.
3. Use these generated random variables the get the Frank's copula.
4. This copula function captures the joint characteristics of the two variables.
5. Now generate random pairs using this copula function.
6. These random pairs follow the joint distribution we are interested in.
7. Use a Gaussian Kernel Density Estimator to get the required joint distribution.

We now have the joint distribution of oil and FX prices.The joint distribution is plotted below:



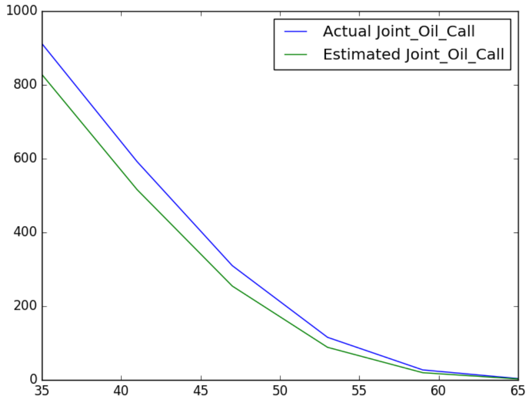

We can verify if this is a good estimate by evaluating the Joint FX Put and Joint Oil Call using the obtained joint distribution and comparing it with the given data in the input file.

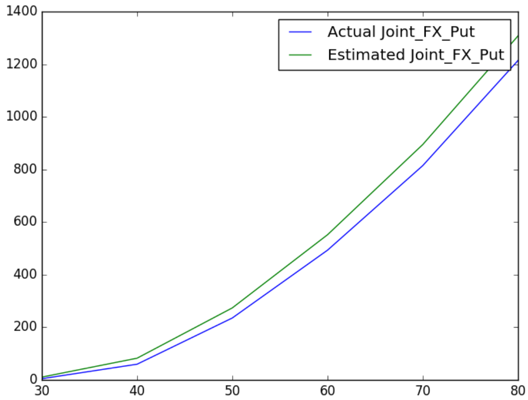
$$\begin{aligned} Q_1(X_1, X_2; B_2) &= E[X_1 \times max((B_2 - X_2), 0)] \\ &= \int_0^\infty \int_0^{B_2} X_1(B_2 - X_2) f(X_1, X_2)\, dX_2\, dX_1 \end{aligned} \tag{3.2}$$

Similarly:

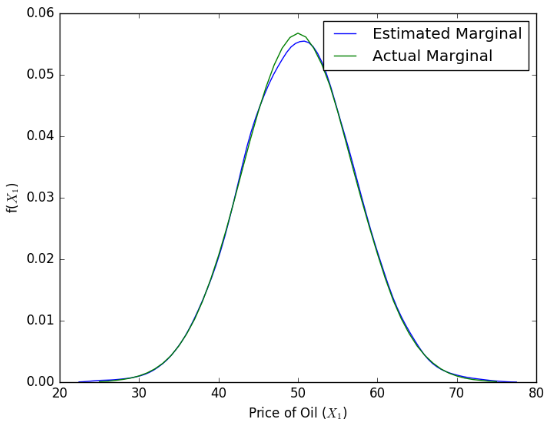
$$\begin{aligned} Q_2(X_1, X_2; B_1) &= E[X_2 \times max((X_1 - B_1), 0)] \\ &= \int_0^\infty \int_{B_1}^\infty X_1(B_2 - X_2) f(X_1, X_2)\, dX_1\, dX_2 \end{aligned} \tag{3.3}$$

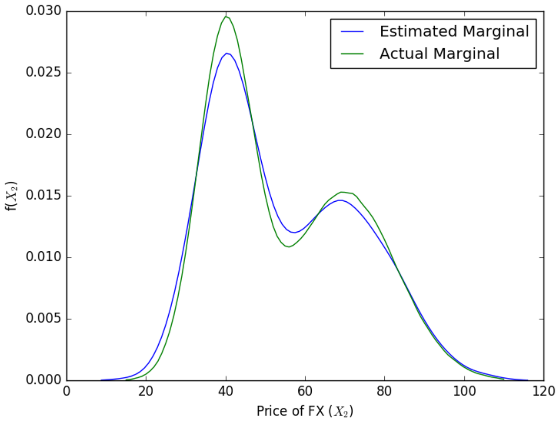
Here f($X_1$,$X_2$) is the estimated joint probability density function obtained using the copula method. The results are plotted below:




We can also verify the obtained joint distribution by back calculating the marginals.The results are plotted below:






Thus,the joint distribution obtained in this question would also be able to recover the input data:'Oil Call Option Prices' and 'FX Call Option Prices'. Finally the required OilCall_FXPut can be calculated using:

$$\begin{aligned} F(X_1, X_2; B_1, B_2) &= E[max((X_1 - B_1), 0) \times max((B_2 - X_2), 0)] \\ &= \int_{B_1}^{\infty} \int_{0}^{B_2} (X_1 - B_1)(B_2 - X_2) f(X_1, X_2)\, dX_1\, dX_2 \end{aligned} \tag{3.4}$$

By using this copula method the estimate of joint distribution function is better than what we would have obtained if we had assumed that the Oil and FX prices were independent. We have used the Frank's copula for all results here, but we can also use the Clayton's copula.

# 4 References

[1] https://en.wikipedia.org/wiki/Leibniz_integral_rule
[2] Bouye, Eric, et al. "Copulas for finance- A reading guide and some applications." Available at SSRN 1032533 (2000).
[3] https://pypi.python.org/pypi/ambhas/0.4.0