Worksheet 1 – Introduction to Linear Algebra

1 Properties of $A^\top A$

Let $A \in \mathbb{R}^{m\times n}$ be a rectangular matrix, show the following properties of $A^\top A$:

1. $A^\top A$ is symmetric.
2. $A^\top A$ is positive semi-definite.
3. $\boldsymbol{ker}(A^\top A) = \boldsymbol{ker}(A)$.
4. $\boldsymbol{Im}(A^\top A) = \boldsymbol{Im}(A^\top)$.

2 On the SVD of a matrix of size 2×3

Consider the following rectangular matrix $A \in \mathbb{R}^{2\times 3}$ such that

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}.$$

Our goal is to write A of the form $U\Sigma V^\top$ where $U \in \mathbb{R}^{2\times 2}$ and $V \in \mathbb{R}^{3\times 3}$ are two orthogonal matrices and $\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 3}$ is a bloc diagonal matrix ($\sigma_1 > 0$ and $\sigma_2 > 0$ are the singular values).

Consider the matrix $B = A^\top A \in \mathbb{R}^{3\times 3}$.

1. Verify that $B = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$.
2. Compute the eigenvalues of the matrix B.
3. Let $\boldsymbol{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$, and $\boldsymbol{v}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$.
   Show that the vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ are the eigenvectors of the matrix B.
4. Verify that the vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ are orthonormal. Deduce the matrices V and Σ.
5. Let $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1$ and $\boldsymbol{u}_2 = \frac{1}{\sigma_2} A\boldsymbol{v}_2$. Compute $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and verify that are orthonormal. Deduce the matrix U.
6. Let $C \in \mathbb{R}^{3\times 2}$ such that
   $$C = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}.$$
   Give an SVD of the matrix C.

3 On the SVD of a matrix of size 3×2

Let $A \in \mathbb{R}^{3\times 2}$ such that

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

1. Show that the eigenvalues of $A^\top A$ are 18 and 0, with the corresponding unit vectors $\boldsymbol{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.
2. Find an SVD of the matrix A.

4 Pseudo inverse and linear systems

Consider the following linear system of 3 unknowns $\boldsymbol{x}$ and 2 linear equations given by

$$\begin{cases} 4x_1 & + & 11x_2 & + & 14x_3 & = & 1 \\ 8x_1 & + & 7x_2 & - & 2x_3 & = & 2 \end{cases}$$

1. Write the linear system on the form $A\boldsymbol{x} = \boldsymbol{b}$ where $A \in \mathbb{R}^{2\times 3}$ and $\boldsymbol{b} \in \mathbb{R}^2$.
2. Is there any $\boldsymbol{x} \in \mathbb{R}^3$ such that $A\boldsymbol{x} = \boldsymbol{b}$?
3. Using the SVD of A, compute A^+ the pseudo-inverse of A.
4. Find the solution with minimal norm of the problem $\min_{\boldsymbol{x}\in\mathbb{R}^2} \|A\boldsymbol{x} - \boldsymbol{b}\|$.

5 Matrix norm and singular values*

Let $A \in \mathbb{R}^{n\times m}$ be a rectangular matrix, the induced matrix ℓ_2-norm of A is given by

$$\|A\|_2 = \max_{\boldsymbol{x}\neq 0} \frac{\|A\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2},$$

where $\|\boldsymbol{x}\|_2 = \sqrt{\boldsymbol{x}^\top \boldsymbol{x}}$ and $\|A\boldsymbol{x}\|_2$ designate the Euclidian norm of the vectors $\boldsymbol{x}$ and $A\boldsymbol{x}$.

The Frobenius norm of $A = [A_{ij}]$ is given by

$$\|A\|_F = \sqrt{\boldsymbol{tr}(A^\top A)} = \sqrt{\sum_{i,j=1}^{n} A_{ij}^2}.$$

Let $U\Sigma V^\top$ be an SVD of the matrix $A \in \mathbb{R}^{n\times m}$ with $U = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m]$, $V = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]$, and $\sigma_i = [\Sigma]_{ii}$ for all $i \leq \min\{m, n\}$. Using the SVD of the matrix A, show that

$$\|A\|_F = \sqrt{\sum_{i=1}^{\min\{m,n\}} \sigma_i^2} \quad \text{and} \quad \|A\|_2 = \sigma_1,$$

where $\sigma_1 \geq \ldots \geq \sigma_{\min\{m,n\}} \geq 0$ are the singular values of A.