Master of Science in Aerospace engineering - First year

Worksheet 2 – Basics of Optimization

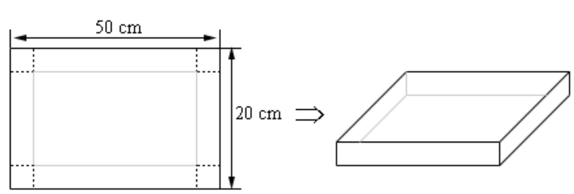
1 Optimization problems in practice

Rewrite the following practical optimization problems using the following canonical form

$$\min_{\boldsymbol{x}\in\Omega} f(\boldsymbol{x}), \quad \Omega = \{\boldsymbol{x} \in \mathbb{R}^n |\ \boldsymbol{h}(\boldsymbol{x}) = 0,\ \boldsymbol{g}(\boldsymbol{x}) \leq 0\}$$

where $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m$ represents m equality constraints ($\boldsymbol{h}(\boldsymbol{x}) = [h_i(\boldsymbol{x})]_{i=1}^m$) and $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^p$ represents p inequality constraints ($\boldsymbol{g}(\boldsymbol{x}) = [g_j(\boldsymbol{x})]_{j=1}^p$).

1. Find two positive numbers whose sum is 300 and whose product is a maximum.
2. Find two positive numbers whose product is 750 and for which the sum of one and 10 times the other is a minimum.
3. We are going to fence in a rectangular field. If we look at the field from above the cost of the vertical sides are \$10/ft, the cost of the bottom is \$2/ft and the cost of the top is \$7/ft. If we have \$700 determine the dimensions of the field that will maximize the enclosed area.
4. We have $45m^2$ of material to build a box with a square base and no top. Determine the dimensions of the box that will maximize the enclosed volume.
5. We want to build a box whose base length is 6 times the base width and the box will enclose 20 m^3. The cost of the material of the sides is \$$3/m^2$ and the cost of the top and bottom is \$$15/m^2$. Determine the dimensions of the box that will minimize the cost.
6. We want to construct a cylindrical can with a bottom but no top that will have a volume of $30cm^3$. Determine the dimensions of the can that will minimize the amount of material needed to construct the can.
7. We have a piece of cardboard that is 50 cm by 20 cm and we are going to cut out the corners and fold up the sides to form a box. Determine the height of the box that will give a maximum volume.



2 Gradient and Hessian

For each of the following functions, compute the gradient and the Hessian of f.

1. $f(\boldsymbol{x}) = 2x_1^3 + 3x_2^2 + 3x_1^2 x_2 - 24x_2$.
2. $f(\boldsymbol{x}) = (x_1^2 + x_2^2 - 1)^2 + (x_2^2 - 1)^2$.
3. $f(\boldsymbol{x}) = -2x_1^2 + x_1 x_2^2 + 4x_1^4$.
4. $f(\boldsymbol{x}) = \frac{x_1 + x_2}{x_1^2 + x_2^2 + 1}$.

3 On the use of the chain rule

1. Let $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^2$ be a differentiable function over $\mathbb{R}^3$ and $\boldsymbol{g} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{f}(\cos(x_1) + \sin(x_2), \sin(x_1) + \cos(x_2), x_1 - x_2)$. Assuming that the value of the Jacobian of $\boldsymbol{f}$ at the point $[1, 1, 0]^\top$ is $\begin{bmatrix} 1 & -2 & 2 \\ 3 & 0 & 1 \end{bmatrix}$, compute the Jacobian of $\boldsymbol{g}$ at the point $[0, 0]^\top$.
2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function over $\mathbb{R}^n$. Let $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{d} \in \mathbb{R}^n$, consider the function $\phi(t) = f(\boldsymbol{x} + t\boldsymbol{d})$ where $t \in \mathbb{R}$. Show that

$$\phi'(t) = \nabla f(\boldsymbol{x} + t\boldsymbol{d})^\top \boldsymbol{d}.$$