Master of Science in Aerospace engineering - First year

Worksheet 4 – Unconstrained Optimization : Part 2

1 On the descent direction

1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(\boldsymbol{x}) = x_1^3 + x_2^2 + 2x_2$. Give some descent directions for f at the point $[1, 1]^\top$. Is there any descent direction at the point $[0, -1]^\top$? Justify the answer.
2. Let $f : \mathbb{R}^n \to \mathbb{R}$. For the following cases specify if the direction $\boldsymbol{d}$ is a descent one for f at $\boldsymbol{x}$ or not (Justify the answer).
   (a) $\boldsymbol{d} = -\nabla f(\boldsymbol{x})$
   (b) $\boldsymbol{d} = -[1, \ldots, 1]^\top$
   (c) $\boldsymbol{d} = -A\nabla f(\boldsymbol{x})$ where $A \succ 0$.
   (d) $\boldsymbol{d} = -[\frac{\partial f}{\partial x_1}(\boldsymbol{x}), 2\frac{\partial f}{\partial x_2}(\boldsymbol{x}), 3\frac{\partial f}{\partial x_3}(\boldsymbol{x}), \ldots, n\frac{\partial f}{\partial x_n}(\boldsymbol{x})]^\top$.

2 On the optimal step-size

1. Consider the following optimization problem

$$\min_{\boldsymbol{x} \in \mathbb{R}^2} f(\boldsymbol{x}) = \frac{1}{2}x_1^2 + \frac{9}{2}x_2^2.$$

   Compute the optimal step-size at $[5, 1]^\top$ following the direction $\boldsymbol{d} = -\nabla f\left([5, 1]^\top\right)$.
2. Consider the quadratic minimization problem

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top Q\boldsymbol{x} - \boldsymbol{a}^\top \boldsymbol{x}$$

   where the matrix $Q \in \mathbb{R}^n$ is symmetric positive definite and $\boldsymbol{a} \in \mathbb{R}^n$.
   Given a point $\boldsymbol{x} \in \mathbb{R}^n$ and a direction $\boldsymbol{d} \in \mathbb{R}^n$ (such that $\boldsymbol{d} \neq 0_{\mathbb{R}^2}$), compute the optimal step-size α^* given by

$$\alpha^* = \arg\min_{\alpha \in \mathbb{R}} f(\boldsymbol{x} + \alpha\boldsymbol{d}).$$

3 On the Newton method

Consider the following optimization problem

$$\min_{\boldsymbol{x} \in \mathbb{R}^2} f(\boldsymbol{x})$$

1. First let $f(\boldsymbol{x}) = \frac{1}{2}x_1^2 + \frac{9}{2}x_2^2$
   (a) For a given starting point $\boldsymbol{x}^0 = [x_1^0, x_2^0]^\top$, compute the Newton direction at this point.
   (b) Show that such direction is a descent direction at the point $\boldsymbol{x}^0$.
   (c) Compute the first iterate $\boldsymbol{x}^1$ of the Newton method. Comment the obtained result.
2. Assume now that $f(\boldsymbol{x}) = x_1^4 + 2x_2^2$ and answer questions (a), (b) and (c).