Master of Science in Aerospace engineering - First year

Worksheet 3 – Unconstrained Optimization : Part 1

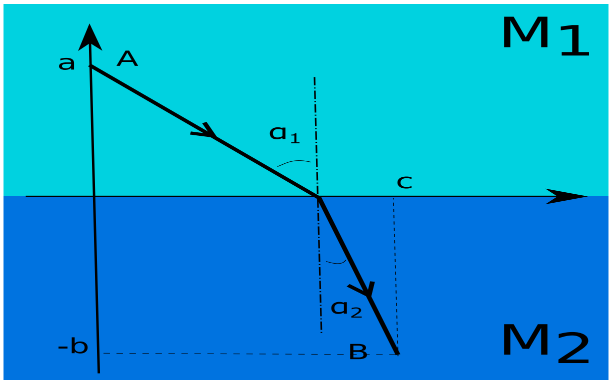
1 On the optimality conditions

For the following scenarios, can we declare the point $\boldsymbol{x}^* \in \mathbb{R}^n$ as a minimizer of $f : \mathbb{R}^n \to \mathbb{R}$? justify your answer.

1. The point $\boldsymbol{x}^*$ satisfies $\nabla f(\boldsymbol{x}) = 0$ and the condition $\nabla^2 f(\boldsymbol{x}) \succ 0$.
2. The point $\boldsymbol{x}^*$ satisfies $\nabla f(\boldsymbol{x}) = 0$ and the condition $\nabla^2 f(\boldsymbol{x}) \succeq 0$.
3. The point $\boldsymbol{x}^*$ satisfies $\nabla f(\boldsymbol{x}) = 0$ but not the condition $\nabla^2 f(\boldsymbol{x}) \succeq 0$.
4. The point $\boldsymbol{x}^*$ does not satisfy $\nabla f(\boldsymbol{x}) = 0$ nor $\nabla^2 f(\boldsymbol{x}) \succeq 0$.

2 Fermat's principle

Let a, b, and c be three positive scalars. Assume that we dispose of two environments $M_1 = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_2 > 0\}$ and $M_2 = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_2 < 0\}$ and the propagation speed of a ray is c_1 and c_2 on the environments M_1 and M_2, respectively. Fermat's principle states that a ray propagates in a straight line and that the path taken by a ray between two given points is the path that can be traversed in the least time.



Consider the ray trajectory between two point $A(0, a)$ and $B(c, -b)$

1. Rewrite Fermat's principale on the form of an optimization problem.
2. Using the optimality condition, show that Fermat's principale can be characterized by Snell's law

$$n_1 \sin(\alpha_1) = n_2 \sin(\alpha_2)$$

where $n_i = \frac{v}{c_i}$ is the refraction index on the environment M_i and v is the speed of light in vacuum.

3 Classification of stationary points

For each of the following functions, find all the stationary points and classify them according to whether they are saddle points, local minimum, or local maximum points :

1. $f(\boldsymbol{x}) = 2x_1^3 + 3x_2^2 + 3x_1^2 x_2 - 24x_2$.
2. $f(\boldsymbol{x}) = (x_1^2 + x_2^2 - 1)^2 + (x_2^2 - 1)^2$.

4 Minimizing a quadratic function

1. Find the solution of the minimization problem

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x}) = \frac{1}{2} \boldsymbol{x}^\top Q \boldsymbol{x} - \boldsymbol{a}^\top \boldsymbol{x}$$

   where the matrix $Q \in \mathbb{R}^n$ is symmetric positive definite and $\boldsymbol{a} \in \mathbb{R}^n$.

2. Let m, $n \in \mathbb{N}$ such that $m \geq n$, $A \in \mathbb{R}^{m \times n}$ be a full rank matrix (i.e., $\boldsymbol{rank}(A) = n$) and $\boldsymbol{b} \in \mathbb{R}^m$. Verify that the problem (known as linear least-squares problem)

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \frac{1}{2} \| A\boldsymbol{x} - \boldsymbol{b} \|_2^2$$

   is quadratic. Solve the problem.