Fourier Analysis (formulas and exercises)

enric.meinhardt@ens-paris-saclay.fr

0. Prerequisites

0.1. Differential and integral calculus of one and several variables

0.2. Complex numbers: $e^{i\theta} = \cos\theta + i\sin\theta$, etc

0.3. Taylor series: $f(x) = c_0 + c_1 x + c_2 + x^2 + \cdots \quad \Longrightarrow \quad c_n = \frac{f^{(n)}(0)}{n!}$

0.4. Linear algebra, orthonormal bases: $\mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n \quad \Longrightarrow \quad a_i = \langle \mathbf{v}, \mathbf{e}_i \rangle$

0.5. Given the graph of a function, draw the graphs of its derivative and integral.

1. Fourier transform of functions $f : \mathbf{R} \to \mathbf{C}$

1.1 Definitions. The Fourier transform of an integrable function $f : \mathbf{R} \to \mathbf{C}$ is the function $\hat{f} : \mathbf{R} \to \mathbf{C}$ defined by

$$\widehat{f}(y) := \int f(x) e^{-ixy} \mathrm{d}x$$

and its inverse Fourier transform is defined by

$$\check{f}(y) := \frac{1}{2\pi} \int f(x) e^{ixy} \mathrm{d}x$$

With some care, these definitions are extended to all square-integrable functions, not necessarily integrable. Then, the Fourier inversion theorem says that these two operations are inverses of each other. Thus, we have a representation of f as an (infinite) linear combination of sinusoidal functions, where the coefficients of the linear combination are given by $\widehat{f}(y)$:

$$f(x) = \frac{1}{2\pi} \int \widehat{f}(y) e^{ixy} \mathrm{d}y$$

1.2. Energy conservation (Plancherel's theorem)

$$\int \left|f(x)\right|^2 \mathrm{d}x = \frac{1}{2\pi} \int \left|\widehat{f}(y)\right|^2 \mathrm{d}y$$

1.3. Convolution theorems

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g} \qquad\qquad \widehat{fg} = \frac{1}{2\pi} \widehat{f} * \widehat{g}$$

Where the convolution of two functions is defined by

$$(f * g)(y) = \int f(x) g(y - x) \mathrm{d}x$$

not to be confused with the correlation

$$(f \star g)(y) = \int \overline{f(x)} g(y + x) \mathrm{d}x$$

1.4. General properties

f	$\widehat{f}$
real even	real even
real odd	imaginary odd
real	hermitian $\overline{\hat{f}(y)} = \hat{f}(-y)$
$\lambda f + \mu g$	$\lambda \widehat{f} + \mu \widehat{g}$
$f(x/a)$	$\lvert a\rvert \widehat{f}(ay)$
$f(x-a)$	$e^{-iay}\widehat{f}(y)$
$f'(x)$	$+iy\widehat{f}(y)$
$-ixf(x)$	$\widehat{f}\,'(y)$

1.5. Examples of transforms

$f(x)$	$\widehat{f}(y)$
$\mathbf{1}_{[-a,a]}(x)$	$2a \text{ sinc }(ay)$
$e^{-ax}\mathbf{1}_{[0,+\infty[}(x)$	$\dfrac{1}{a+iy}$
$e^{-a\lvert x\rvert}$	$\dfrac{2a}{a^2+y^2}$
e^{-ax^2}	$\sqrt{\dfrac{\pi}{a}}e^{-\frac{y^2}{4a}}$
$\text{sech }(ax)$	$\dfrac{\pi}{a}\text{sech}\left(\dfrac{\pi}{2a}y\right)$

$f(x)$	$\widehat{f}(y)$ (in the sense of distributions)
1	$2\pi\delta(y)$
$\delta(x)$	1
e^{iax}	$2\pi\delta(y-a)$
$\cos(ax)$	$\pi\delta(y-a)+\pi\delta(y+a)$
$\sin(ax)$	$-i\pi\delta(y-a)+i\pi\delta(y+a)$
x^n	$2\pi i^n\delta^{(n)}(y)$
$\sum_{n\in\mathbf{Z}}\delta(x-na)$	$\dfrac{2\pi}{a}\sum_{k\in\mathbf{Z}}\delta\left(y-\dfrac{2\pi k}{a}\right)$

2. Fourier series of functions $f : \mathbf{T} \to \mathbf{C}$

Let $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ be the periodization of the interval $[0,2\pi]$. The functions $f : \mathbf{T} \to \mathbf{C}$ are identified with the the 2π-periodic functions on $\mathbf{R}$. They can be expressed as Fourier series, which is a linear combination of sinusoidal functions of integer frequencies:

$$f(\theta) = \sum_{n\in\mathbf{Z}} \widehat{f}(n) \exp^{in\theta} \tag{1}$$

the coefficients $\widehat{f}(n)$ are

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}\mathrm{d}\theta \tag{2}$$

(this formula follows by multiplying (1) with the function $e^{-in\theta}$ and integrating with respect to θ.)

Fourier series have analogous properties to Fourier integrals (see 1.4. above). Only the property for $f(x/a)$ has no direct equivalent, because it does not preserve the periodicity of the function. The energy conservation property $\int_0^{2\pi} \lvert f\rvert^2 = 2\pi\sum_{n\in\mathbf{Z}}\left\lvert\widehat{f}(n)\right\rvert^2$ can be used to compute the sums of many series.

3. Discrete Fourier transform of functions $f : \mathbf{P}_\mathrm{N} \to \mathbf{C}$

Let $\mathbf{P}_\mathrm{N} = \mathbf{Z}/\mathrm{N}\mathbf{Z}$ be the integers modulo N. The functions $f : \mathbf{P}_\mathrm{N} \to \mathbf{C}$ are the vectors of $\mathbf{C}^\mathrm{N}$. The set of vectors

$$\mathbf{e}_n = \left(e^{\frac{2\pi i k n}{\mathrm{N}}} \right)_{k=0,\dots,\mathrm{N}-1}$$

for $n = 0, \dots, \mathrm{N}-1$, is an orthogonal basis of $\mathbf{C}^\mathrm{N}$, indeed $\mathbf{e}_p \cdot \mathbf{e}_q = \mathrm{N}\delta_{pq}$.

The discrete Fourier transform (DFT) is the expression of $\mathbf{C}^\mathrm{N}$ vectors in this basis. Thus, a vector $\mathbf{v} = (v_0, \dots, v_{\mathrm{N}-1})$, can be expressed as

$$v_k = \sum_{n=0}^{\mathrm{N}-1} \widehat{v}_n e^{\frac{2\pi i k n}{\mathrm{N}}}$$

and the coefficients $\widehat{v}_n$ are recovered by computing $\mathbf{v} \cdot \mathbf{e}_n$:

$$\widehat{v}_n = \frac{1}{\mathrm{N}} \sum_{k=0}^{\mathrm{N}-1} v_k e^{-\frac{2\pi i k n}{\mathrm{N}}}$$

The discrete Fourier transform has analogous properties to the other transforms. Here all the relationships are trivial to check using linear algebra (there are not convergence problems as in the previous cases).

4. Sampling

Consider a trigonometric polynomial of the form $f(\theta) = a_0 + a_1 e^{i\theta} + \dots + a_{\mathrm{N}-1} e^{i(\mathrm{N}-1)\theta}$. The coefficients a_k can be obtained from $f(\theta)$ by computing the Fourier series of the function $f : \mathbf{T} \to \mathbf{C}$, which is a finite sum.

Sampling theory provides another way to compute these coefficients. First, we evaluate the function f at N equally spaced points $\mathbf{v} := \big(f(2\pi k/\mathrm{N})\big)_{k=0,\dots,\mathrm{N}-1}$. Then, the vector of coefficients $\mathbf{a} = (a_0, \dots, a_{\mathrm{N}-1})$ is the DFT of the vector $\mathbf{v}$.

There are similar relationships between the other transforms described above, describing how each transform commutes with sampling and interpolation operators.

5. Aliasing

Continuing with the sampling example above, suppose that $\mathrm{N} = 2\mathrm{P}+1$ is odd, and notice that the two trigonometric polynomials

$$f(\theta) = a_0 + a_1 e^{i\theta} + \dots + a_{\mathrm{N}-1} e^{i(\mathrm{N}-1)\theta}$$

and

$$\tilde{f}(\theta) = a_0 + a_1 e^{i\theta} + \dots + a_\mathrm{P} e^{i\mathrm{P}\theta} + a_{\mathrm{P}+1} e^{-i\mathrm{P}\theta} + a_{\mathrm{P}+2} e^{-i(\mathrm{P}-1)\theta} \dots + a_{\mathrm{P}+\mathrm{P}} e^{-2i\theta} + a_{\mathrm{P}+\mathrm{P}+1} e^{-i\theta}$$

take exactly the same values at the points $\theta_k = \frac{2\pi k}{\mathrm{N}}$ for $k = 0, \dots, \mathrm{N}-1$. This happens because they have the same frequencies modulo N and the functions $e^{ik\theta}$ are 2π–periodic. The second choice, with frequencies symmetric around zero, is much more natural (since it is the only way to obtain real-valued signals!), and it is typically written as

$$f(\theta) = \sum_{k=-\lfloor \mathrm{N}/2 \rfloor}^{\lfloor \mathrm{N}/2 \rfloor - 1} a_k e^{ik\theta}$$

where the indices of the coefficients a_k are to be understood modulo N.

This is the simplest case of aliasing: different functions having the same samples. In the general case, we sample a function $f(\theta) = \sum_{n=0}^{\mathrm{N}-1} a_n e^{in\theta}$ at M points $\theta_k = \frac{2\pi k}{\mathrm{M}}$. The case $\mathrm{M} = \mathrm{N}$ is the perfect sampling rate (corresponding to the Shannon-Nyquist condition), and the coefficients a_n are the DFT of the samples.

The case $\mathrm{M} > \mathrm{N}$ is called *oversampling*, *zero-padding* or *zoom-in*, depending on the context. In that case we have

$$f(\theta_k) = \sum_{n=0}^{\mathrm{N}-1} a_n e^{\frac{2\pi i k n}{\mathrm{M}}} + \sum_{n=\mathrm{N}}^{\mathrm{M}-1} 0 \cdot e^{\frac{2\pi i k n}{\mathrm{M}}} = \sum_{n=0}^{\mathrm{M}-1} \mathrm{ZP}(a)_n e^{\frac{2\pi i k n}{\mathrm{M}}}$$

where $\mathrm{ZP}(a_0,a_1,\dots,a_{N-1}) = (a_0,\dots,a_{N-1},0,\dots,0)$ is the zero-padding of the vector a to length M.

The case $M < N$ is called *subsampling*, *aliasing*, *decimation* or *zoom-out*, depending on the context. In that case we have

$$f(\theta_k) = \sum_{n=0}^{N-1} a_n e^{\frac{2\pi i k(n\%M)}{M}} = \sum_{m=0}^{M-1}\left(\sum_{n\%M=m} a_n\right) e^{\frac{2\pi i k m}{M}} = \sum_{m=0}^{M-1} \mathrm{AL}(a)_m e^{\frac{2\pi i k m}{M}}$$

where $\mathrm{AL}(a)$ is the vector *a folded* to length M by summing the positions that are equal modulo M.

6. Fourier transform in several dimensions

If $f : \mathbf{R}^N \to \mathbf{C}$ is an integrable function, its Fourier transform is defined as:

$$\widehat{f}(\mathbf{y}) = \int_{\mathbf{R}^n} f(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{y}}\,\mathrm{d}\mathbf{x}$$

and the inverse transform is

$$\check{f}(\mathbf{y}) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \widehat{f}(\mathbf{x}) e^{i\mathbf{x}\cdot\mathbf{y}}\,\mathrm{d}\mathbf{x}$$

Its properties are directly derived from those of the 1-dimensional Fourier transform by separability.

7. (Optional) Generalization: Pontryagin duality

The transforms described above are particular cases of a general construction called Pontryagin duality, which works on locally compact abelian groups G. On such a group, there is a natural measure called the *Haar measure* that allows to compute integrals of functions $f : G \to \mathbf{C}$ in a translation-invariant way. The *dual* of a group G is the set of morphisms $\mu : G \to \mathbf{C}^*$ (here C^* is the complex unit circle). The dual G^* is itself a group under the point-wise product of functions. Moreover, it is locally compact and abelian, and it has its own Haar integral. Finally, the Fourier transform of an integrable function $f : G \to \mathbf{C}$ is a function $\hat{f} : G^* \to \mathbf{C}$ defined as

$$\hat{f}(y) = \int_G f(x)\overline{y(x)}\mathrm{d}x \tag{3}$$

And, likewise, given an integrable function $g : G^* \to \mathbf{C}$, its inverse transform is

$$\check{f}(x) = \int_{G^*} f(y)y(x)\mathrm{d}y \tag{4}$$

And the Pontryagin duality theorem states that these two operations are inverses of each other. Notice that since $y(x)$ is a unit complex number, it is typically written as e^{iyx}.

On the table below we find the typical transforms as particular cases

	spatial domain	frequency domain	analysis	synthesis
general case	G	G^*	$\hat{f}(y) = \int_G f(x)e^{-iyx}\mathrm{d}x$	$\check{f}(x) = \int_{G^*} f(y)e^{iyx}\mathrm{d}y$
Fourier series	$\mathbf{T}$	$\mathbf{Z}$	$\hat{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}\mathrm{d}\theta$	$\check{f}(\theta) = \sum_{n\in\mathbf{Z}} f_n e^{inx}$
Fourier transform	$\mathbf{R}$	$\mathbf{R}$	$\hat{f}(y) = \int_{\mathbf{R}} f(x)e^{-iyx}\mathrm{d}x$	$\check{f}(x) = \frac{1}{2\pi}\int_{\mathbf{R}} f(y)e^{iyx}\mathrm{d}y$
DFT	$\mathbf{Z}_N$	$\mathbf{Z}_N$	$\hat{f}_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{-ikn}$	$\check{f}_n = \sum_{n=0}^{N-1} f_k e^{ikn}$
DTFT	$\mathbf{Z}$	$\mathbf{T}$	$\hat{f}(\theta) = \sum_{n\in\mathbf{Z}} f_n e^{-inx}$	$\check{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{in\theta}\mathrm{d}\theta$

Notice that the placement of many multiplicative factors in this table seems arbitrary. Indeed the Haar measure is unique up to a multiplicative constant, and this leads to different conventions for the factors.

8. (Optional) Generalization: Laplace-Beltrami spectrum

Another generalization of Fourier series and integrals is found in differential geometry. Given a manifold Ω (for example, a subset of the plane, or an arbitrary curved surface), a standard geometric construction is the Laplace-Beltrami operator $\Delta_\Omega : C^\infty(\Omega) \to C^\infty(\Omega)$. This is a negative-definite linear operator, and when Ω is compact, it has a numerable set of eigenfunctions φ_k, satisfying $\Delta_\Omega\varphi = -\lambda_k\varphi_k$, with $0 \le \lambda_1 \le \lambda_2 \le \ldots$. Under mild regularity conditions, these functions form an orthogonal basis of $L^2(\Omega)$, thus providing a generalization of "Fourier series" for functions defined on the manifold: $f(x) = \sum_n f_n\varphi_n(x)$, with $f_n = \int_\Omega f\varphi_n$.

The eigenfunctions φ_k are called the *vibration modes* of Ω, and the numbers $\sqrt{\lambda_k}$ are called the *partials* or *overtones*, with $\sqrt{\lambda_1}$ being the *fundamental frequency*. In the case of a solid body Ω, they correspond to the modes of vibration of the object. For example, if Ω is the skin of a drum, the functions φ_k describe the shapes in which the skin can vibrate, and the numbers λ_k determine the frequency at which they vibrate. Any vibration pattern can be expressed as a linear combination of functions f_k. The zero-level sets $\varphi_n^{-1}(\{0\}) \subset \Omega$ are called the *nodal curves* or *Chladni figures* of each vibration pattern. They can be found experimentally by pouring sand on the drum and making it resonate to the corresponding frequency.

Notice that when Ω is one-dimensional, the Laplace-Beltrami operator is minus the second derivative, and the solutions of $-f'' = \lambda^2 f$ are the functions of the form $f(x) = c_1\sin(\lambda x) + c_2\cos(\lambda y)$. On the table below we find some particular cases

	Ω	φ_n	$-\lambda_n$
interval	$[0,2\pi]$	$\sin\left(\frac{nx}{2}\right)$	$n^2/4$
circle	S^1	$\sin(n\theta), \cos(n\theta)$	n^2
square	$[0,2\pi]^2$	$\sin\left(\frac{nx}{2}\right)\sin\left(\frac{m\theta}{2}\right)$	$\frac{n^2+m^2}{4}$
torus	$(S^1)^2$	$\sin(nx)\sin(my), \ldots$	n^2+m^2
disk	$\lvert r\rvert \le 1$	$\sin, \cos(n\theta) J_n(\rho_{m,n}r)$	$\rho_{m,n}$ roots of J_n
sphere	S^2	$Y_l^m(\theta,\varphi)$	l^2+l

Several geometric properties of Ω can be interpreted in terms of the Laplace-Beltrami spectrum. For example, if Ω has k connected components, the first k eigenfuntions will be supported successively on each connected component. On a connected manifold M, the first vibration mode can be taken to be positive $\varphi_1 \ge 0$, thus all the other modes have non-constant signs (because they are orthogonal to φ_1). In particular, the sign of φ_2 cuts Ω in two parts in an optimal way, it is the Cheeger cut of Ω, maximizing the perimeter/area ratio of the cut. Generally, symmetries of Ω arise as multiplicities of eigenvalues. The Laplace-Beltrami spectrum $\lambda_1, \lambda_2, \lambda_3, \ldots$ is closely related, but not identical, to the geodesic length spectrum, that measures the sequence of lengths of all closed geodesics of Ω. The grand old man of this theory is Yves Colin de Verdière, student of Marcel Berger.

Geometry is not in general a spectral invariant, but non-isometric manifolds with the same spectrum are difficult to come by. The first pair of distinct but isospectral manifolds was wound in 1964 by John Milnor, in dimension 16. The first example in dimension 2 was found in 1992 by Gordon, Webb and Wolperd, and it answered negatively the famous question of Marc Kac "Can you hear the shape of a drum?'. In 2018, we have many ways to construct discrete and continuous families of isospectral manifolds in dimensions two and above.

9. (Optional) Spectral geometry on graphs

The discrete version of a manifold is a finite graph $G = (V, E)$, of n vertices and m edges. There are three matrices naturally associated with a graph; they are typically sparse. The *adjacency* matrix A is of size $n \times n$ and has an entry of 1 at position a, b if the vertex $(a, b) \in E$. The signed *incidence* matrix B has m rows and n columns, with entries 1 and -1 on row e at positions a and b if the e-the edge joins vertices a and b. Finally, the *Laplacian* matrix $L = -B^T B$. The Laplacian matrix is the analogous of the Laplace-Beltrami operator on a manifold: it can be used to compute "second derivatives" of functions defined on the set of vertices V. Its eigenvalues are nonnegative, and the signs of its eigenvectors provide a sequence of binary segmentations of the graph called *spectral clustering*. The first partition (for a connected graph) is the *Cheeger cut* of the graph, the optimal partition of the graph into two parts.

10. Application : solving linear PDE

Given a function $f : \mathbf{R}^2 \to \mathbf{R}$, we want to find a function u satisfying the following PDE:

$$u - \alpha^2 \Delta u = f$$

This is a linear PDE with constant coefficients. We can solve it by applying the Fourier transform on each side of the equation to obtain an equivalent relation:

$$\hat{u} + \alpha^2(\xi^2 + \eta^2)\hat{u} = \hat{f}$$

And solving for $\hat{u}$

$$\hat{u} = \frac{1}{1 + \alpha^2(\xi^2 + \eta^2)} \cdot \hat{f}$$

we find the Fourier transform of the solution. Thus the solution is the convolution of the datum f with a positive kernel of type Laplace.

More generally, a linear PDE has the form

$$\mathrm{P}\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) u = f$$

where P is a polynomial in n variables. Applying the Fourier transform on both sides, we obtain

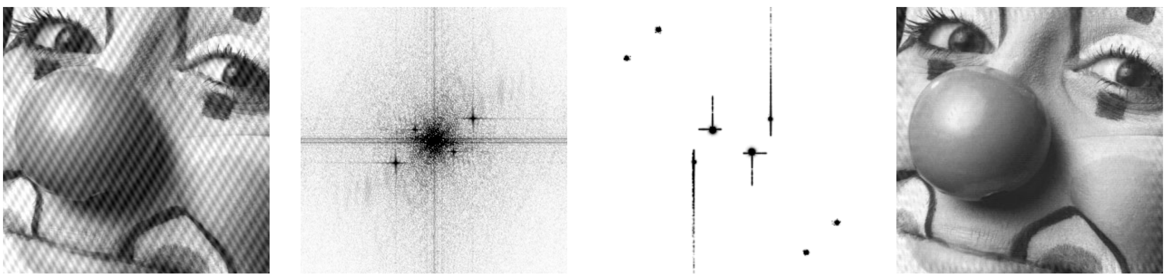
$$\mathrm{P}\left(i\xi_1, \ldots, i\xi_n\right) \hat{u} = \hat{f}$$

Which gives the solution immediately.

11. Application : image processing

The Fourier transform in dimension 2 is an important tool in image processing.

A simple application is the removal of periodic noise in images. The frequencies of periodic noise on an image I appear as local maxima in the image $|\widehat{\mathrm{I}}|$, and can be removed manually (setting them to zero using an image editor).



I | $\log\left(1 + |\widehat{\mathrm{I}}|\right)$ | mask for $\widehat{\mathrm{I}}$ | reconstruction

12. Bibliography

12.0. T. W. Körner : *Fourier analysis*
(learn all the history in well-written, very short lessons)

12.1. R. Bracewell : *The Fourier Transform and its Applications*
(lean to compute Fourier transforms visually and graphically)

12.2. C. Gasquet, P. Witomski : *Analyse de Fourier et applications*
(the complete mathematical theory in dimension 1, from the elementary to the fancier)

12.3. R. C. Gonzalez, R. E. Woods : *Digital Image Processing*
(applications to image processing, with practical emphasis on programming)

13. Exercices (stars denote relative difficulty) (red: indications for the solution)

E0. Is it possible to sample pure sinusoidal wave of frequency 10.000Hz (a very high pitched sound) so that you hear a pure sinusoidal wave of frequency 440Hz (the middle A). If that is the case, what is the necessary sampling rate?
Il faut voir s'il est possible d'échantillonner les deux fonctions

$$f(\theta) = e^{10.000i\theta} \qquad \text{et} \qquad g(\theta) = e^{440i\theta}$$

sur points de la forme $\theta_k = \frac{2\pi k}{N}$, $k = 0, \dots, N-1$ de façon que les valeurs de chaque fonction aux points d'échantillonnage coïncident (ce qui veut dire que f et g sont des polynômes intérpolateurs des mêmes valeurs), et que la fréquence d'échantillonnage N est plus grande que 880Hz (fréquence de Nyquist pour g). Ceci est bien possible, puisque la fonction exponentielle est $2\pi i$-périodique, donc les échantillons $f(\theta_k)$ ne changent pas si on rajoute un multiple entier de $2\pi i$ aux θ_k. Autrement, dit, il faut trouver une factorisation de $10.000 - 440 = \mathrm{NM}$ avec $\mathrm{N} > 880$, par exemple $9560 = 4780 \times 2 = 2390 \times 4 = 1195 \times 8 = 1912 \times 5$.

E1. Is the definition of the Fourier transform (section 1.1) correct? Do all integrable functions (i.e., those that $\int_{\mathbf{R}} |f| < \infty$) have a well-defined Fourier transform? Is $\hat{f}(\xi)$ a bounded function? Is it continuous?
If f is integrable this means that $\int |f| < +\infty$ thus $\widehat{f}(y)$ is a well-defined, finite number, by the triangle inequality:$|\widehat{f}(y)| \leq \int |f(x)e^{-ixy}|\mathrm{d}x = \int |f(x)|\mathrm{d}x < +\infty$. To see that $y \mapsto \widehat{f}(y)$ is continuous, apply the dominated convergence theorem to the sequence of functions $x \mapsto f(x)e^{-ixy_n}$ which is dominated by the integrable function $|f|$.

E2. Check the general properties stated on section 1.4.
The proofs are all straightforward verifications from the definition of $\widehat{f}$ and elementary properties of the integral. For example, if $g(x) = f(x-a)$ then by a linear change of variable we have $\widehat{g}(y) = \int f(x-a)e^{-ixy}\mathrm{d}x = \int g(t)e^{-i(a+t)y}\mathrm{d}t = e^{-iay}\widehat{f}(y)$.

E3. Check the validity of the convolution theorem (recall the definition of convolution $(f * g)(x) := \int_{\mathbf{R}} f(x-y)g(y)\mathrm{d}y$). What hypotheses are needed on f and g to assure that the statement makes sense?
If both f and g are integrable, then $f * g$ is well defined and integrable. This follows from the integrability of the function $(x,y) \mapsto f(x)g(y-x)$ by the triangle inequality and Fubini's theorem.

E4. Check the first three Fourier transforms on the table 1.5.
All these transforms can be computed by elementary means. Let $a > 0$ and $f(x) = \mathbf{1}_{[-a,a]}(x)$. The Fourier transform of f can be written as a suitably scaled *cardinal sine*, $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$:

$$\widehat{f}(y) = \int_{-a}^{a} e^{-ixy}\,\mathrm{d}x = \left[\frac{e^{-ixy}}{-iy}\right]_{-a}^{a} = 2\frac{\sin(ay)}{y} = 2a\mathrm{sinc}(ay)$$

For $g(x) = e^{-ax}\mathbf{1}_{[0,+\infty[}(x)$, a similar computation gives

$$\widehat{g}(y) = \int_{0}^{+\infty} e^{-ax-ixy}\mathrm{d}x = \left[\frac{e^{-ax-ixy}}{-a-iy}\right]_{0}^{+\infty} = \frac{1}{a+iy}$$

For $h(x) = e^{-a|x|}$ we have that $h(x) = g(x) + g(-x)$ and by linearity of the transform the result follows: $\widehat{h}(y) = \widehat{g}(y) + \widehat{g}(-y) = \frac{1}{a+iy} + \frac{1}{a-iy} = \frac{2a}{a^2+y^2}$.

E5. Discuss the following reasoning. Let f be an integrable function. By the inversion theorem, the function f it the Fourier transform of $\check{f}$. Thus, f is continuous (by exercise E1).
The argument in incorrect because $\widehat{f}$ need not be integrable. For example, the Fourier transform of the discontinuous function $f(x) = \mathbf{1}_{[-a,a]}(x)$ is the non-integrable function $\widehat{f}(y) = 2a\mathrm{sinc}(ay)$.

***E6.** The goal of this exercice is to compute that the Fourier transform of a gaussian function is another gaussian function. Let $a > 0$ and $\psi(x) = e^{-ax^2}$. (a) Prove that $\psi(x)$ and $\psi(x)'$ are integrable. (b) Compute $\widehat{\psi}(0)$ (This is a classical result that you *must* know). (c) Assume that $\widehat{\psi}$ is derivable. Compute $\widehat{\psi}\,'$ and $\widehat{\psi'}$ to obtain a differential equation for $\widehat{\psi}$. (d) Write an explicit formula for $\widehat{\psi}(\xi)$.

We start with a basic observation:
Lemma (Gauss ODE) The ordinary differential equation

$$\begin{cases} u'(x) = -\beta x u(x) \\ u(0) = \gamma \end{cases} \tag{5}$$

has a unique solution for $\beta, \gamma > 0$ given by $u(x) = \gamma e^{-\beta x^2/2}$. *Proof. The given function satisfies both conditions, and it is unique due to Cauchy-Lipschitz/Picard-Lindelöf theorem.*

Now let $g(x) = e^{-ax^2}$. We will compute $\widehat{g}$ as the solution of the equation 5.

$$\begin{aligned}
g(x) &= e^{-ax^2} && \text{by definition of } g \\
g'(x) &= -2axg(x) && \text{by taking derivatives} \\
\widehat{g'}(y) &= \widehat{x \mapsto -2axg(x)}(y) && \text{by taking the Fourier transform} \\
\widehat{g'}(y) &= \left(\tfrac{2a}{i}\right) \widehat{x \mapsto -ixg(x)}(y) && \text{by shuffling constants} \\
iy\widehat{g}(y) &= \left(\tfrac{2a}{i}\right) \widehat{g}'(y) && \text{table 1.4, last two lines} \\
\widehat{g}'(y) &= -\tfrac{1}{2a} y \widehat{g}(y) && \text{by rearranging}
\end{aligned}$$

Now, together with $\widehat{g}(0) = \sqrt{\frac{\pi}{a}}$ we see that $\widehat{g}$ satisfies Gauss' ODE with $\beta = \frac{1}{2a}$ and $\gamma = \sqrt{\frac{\pi}{a}}$, thus

$$\widehat{g}(y) = \sqrt{\frac{\pi}{a}} e^{-y^2/4a}$$

****E7.** Give a closed form expression for the sum $g(x) = \displaystyle\sum_{n\geq 1} \frac{1}{n^2 + x^2}$. Check that $\displaystyle\lim_{x\to 0} = \frac{\pi^2}{6}$.
Hint: compute the Parseval identity for the Fourier series of the function $f(\theta) = e^{\alpha\theta}$.
On trouve aisément les coefficients de Fourier de f:

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} e^{\alpha x} e^{-inx} \mathrm{d}x = \frac{1}{2\pi} \left[\frac{e^{(\alpha - in)x}}{\alpha - in} \right]_0^{2\pi} = \frac{1}{2\pi} \frac{e^{2\pi\alpha} - 1}{\alpha - in}$$

et l'identité de Parseval dit alors (après arrangement)

$$\sum_{n\in\mathbf{Z}} \frac{1}{\alpha^2 + n^2} = \frac{\pi}{\alpha} \coth(\pi\alpha)$$

et on retrouve dans cette expression la fonction $g(\alpha)$, puisque $\sum_{n\in\mathbf{Z}} \frac{1}{\alpha^2+n^2} = \frac{1}{\alpha^2} + 2g(\alpha)$.
On peut calculer directement la limite suivante

$$2g(0) = \lim_{\alpha\to 0} \left(\frac{\pi}{\alpha} \frac{e^{2\pi\alpha} + 1}{e^{2\pi\alpha} - 1} - \frac{1}{\alpha^2} \right) = \lim_{\alpha\to 0} \frac{\pi\alpha e^{2\pi\alpha} + \pi\alpha - \alpha e^{2\pi\alpha} + 1}{\alpha^2 e^{2\pi\alpha} - \alpha} = \frac{\pi^2}{3}$$

c'est une limite indeterminée que on calcule par la règle de l'Hôpital. Autrement, on peut se servir du développement asymptotique de la fonction coth:

$$\coth(x) = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \dots$$

pour obtenir plus rapidement le même résultat.

****E8.** Define the functions $H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$. Check that each H_k is a polynomial of degree k; they are called the Hermite polynomials, for example $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2 \dots$. Now define the (non-normalized) Hermite functions as $\varphi_n(x) = e^{-x^2/2} H_n(x)$. Prove that the Hermite functions are eigenfunctions of the Fourier transform: $\widehat{\varphi_n} = (-i)^n \varphi_n$.
This is a classic exercice. Notice that you have to use the unitary convention $\widehat{u}(y) = \frac{1}{\sqrt{2\pi}} \int u(x) e^{-ixy} \mathrm{d}x$ to obtain the correct scalar factors.

**E9. *[Principle of Radar Focusing].* A *chirp*, also called *glissando*, is a wave whose instantaneous frequency increases very fast. More concretely, let us call the function $g(x) = e^{i\pi x^2}$ a *linear chirp*. Linear chirps are commonly used in echolocation and in radar because they are a sort of "square root" of the impulse response $g \star g = \delta$, which eases the separation of a reflected signal.

(a) Explain why the function $g(x) = e^{i\pi x^2}$ is called a linear chirp and not otherwise (e.g., a quadratic chirp).

La fonction $t \mapsto e^{i\omega t}$ est une onde de fréquence constante ω. Si ω varie le long du temps selon $\omega(t)$, on dit que la fonction $t \mapsto e^{i\omega(t)t}$ est une onde de fréquence instantanée $\omega(t)$. Le chirp linéaire est un cas particulier où $\omega(t)$ varie de forme linéaire selon t. (b) Is the function $g \star g$ well-defined?

Let T, K be strictly positive real numbers. We define the linear chirp of support T and rate K as the function $h(x) = \mathbf{1}_{[-T,T]}(x) e^{iKx^2}$.

La fonction $g(x)$ n'étant pas intégrable sur $\mathbf{R}$, l'auto-corrélation $g \star g$ n'est pas définie.

Observation: on peut voir que, au sens des distributions, on aura $g \star g = \delta$. Ceci vient du fait que $\widehat{1} = \sqrt{2\pi}\,\delta$, et donc

$$g \star g(y) = \int e^{-i\pi x^2} e^{i\pi(x+y)^2} \mathrm{d}x = e^{i\pi y^2} \int 1 \cdot e^{2i\pi xy} \mathrm{d}x = e^{i\pi y^2} \sqrt{2\pi}\delta(-2\pi y) = e^{i\pi y^2}\delta(y) = \delta(y)$$

(c) Prove that the function $s(y) = (h \star h)(y)$ is well-defined.

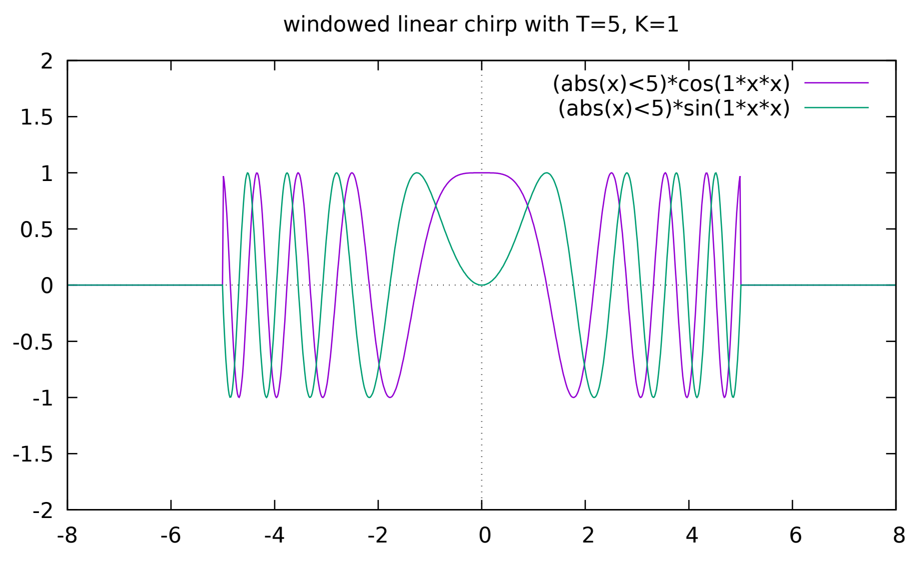
La fonction $h(x)$ étant bornée et à support compact, l'intégrale qui définit $h \star h(y)$ est bien définie pour tout valeur de $y \in \mathbf{R}$. (d) Prove that $s(y)$ is continuous, real-valued, even, and identify its support.

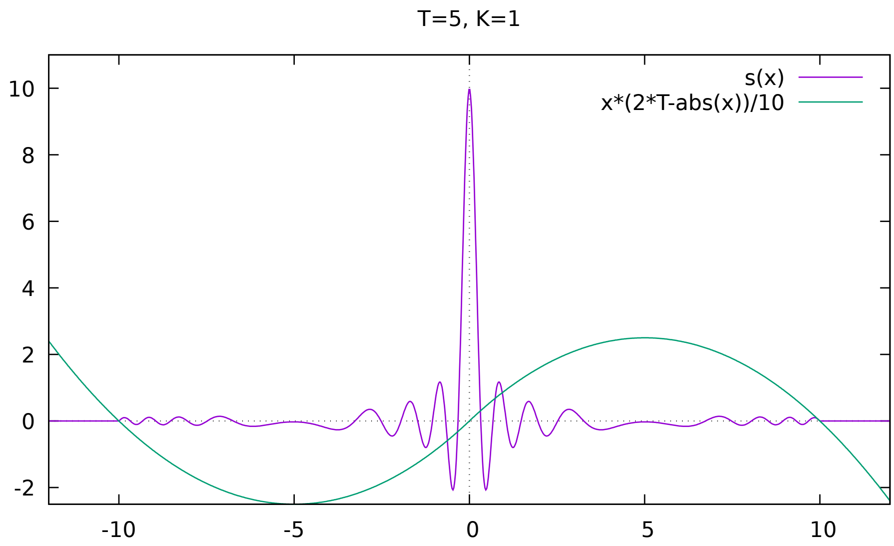
Le support est $[-2T, 2T]$. Les autres propriétés sont élémentaires (e.g., en regardant l'expression suivante). (e) Give an expression for $s(y)$ in terms of elementary functions applied to y, T, K.

On a $s(y) = \mathbf{1}_{[-2T,2T]}(y)\,\left(2T - |y|\right)\,\mathrm{sinc}\left(Ky\left(2T - y\right)\right)$

(calcul par des méthodes élémentaires). (f) Draw the graph of $s(y)$ for a reasonable choice of T, K. Do you think $s(y)$ will converge to $\delta(y)$ as some (which?) parameter goes to infinity?

La fonction $s(y)$ est le produit d'une fonction porte "triangulaire" $\mathbf{1}_{[-2T,2T]}(y)\,\left(2T - y\right)$ fois le sinus cardinal évalué sur $Ky(2T - y)$.





E10. Check that the vectors $\mathbf{e}_n$ of section 3 are orthogonal. *(Hint: there is a geometric interpretation.)*
The expression $\langle \mathbf{e}_n, \mathbf{e}_m \rangle = \sum_{k=0}^{N-1} (\mathbf{e}_n)_k \overline{(\mathbf{e}_m)_k} = \sum_{k=0}^{N-1} e^{\frac{2\pi i(n-m)k}{N}}$ is a sum of N roots of 1. When $n = m$ all the terms are equal to 1 and the sum is N. When $n \neq m$ the sum is zero since it is the barycenter of a regular polygon. (Note: this is also a geometric sum that can be evaluated explicitly.)

E11. Prove the sampling result stated on section 4 (relating the DFT to Fourier series via sampling).
If you take a trigonometric polynomial $f(\theta) = \sum_{n=0}^{N-1} F_n e^{in\theta}$ and evaluate it at N regularly spaced positions $\theta_k = \frac{2\pi k}{N}$, $k = 0, \ldots, N-1$ you obtain values $v_k = f(\theta_k) = \sum_{n=0}^{N-1} F_n e^{\frac{2\pi i n k}{N}}$. Notice that the vector of samples $\mathbf{v}$ is the IDFT of the vector of coefficients $\mathbf{F}$. Or, equivalently, the Fourier coefficients are the discrete Fourier transform of the samples.

E12. A *left coordinate shift* of a positions is the map $s_a : \mathbf{R}^N \to \mathbf{R}^N$ defined by

$$s_a(x_0, x_1, \ldots, x_{N-1}) := (x_a, x_{a+1}, \ldots, x_{a+N-1})$$

where the indices are to be interpreted modulo N. This definition makes sense only when $a \in \mathbf{Z}$. How would you define a coordinate shift for $a \in \mathbf{R}$? *(Hint: look at the various properties of the DFT).*

In the spectral domain, the effect of an a-shift is the multiplication by a phase of frequency a:

$$\widehat{s_a\mathbf{x}}_k = \frac{1}{N}\sum_{n=0}^{N-1} x_{n+a} e^{\frac{-2\pi i n k}{N}} = \frac{1}{N}\sum_{t=a}^{(a-1 \bmod N)} x_t e^{\frac{-2\pi i (t-a) k}{N}} = e^{\frac{2\pi i a k}{N}} \widehat{\mathbf{x}}_k$$

Notice that this expression makes sense for arbitrary $a \in \mathbf{R}$ (not only for $a \in \mathbf{Z}$ as in the definition of s_a). Thus, we can use this formula as the *definition* for a shift of arbitrary size a. Conceptually, it is the evaluation at a-shifted positions of a trigonometric polynomial that interpolates the original samples $\mathbf{x}$.

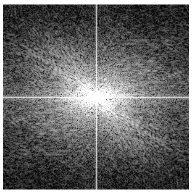
****E13.** The previous exercice gives a discrete implementation of a shift $f(x-a)$ for $a \in \mathbf{R}$. Given $a > 1$, how would you define a *zoom-in* $f(ax)$ and a *zoom-out* $f(x/a)$?
We retake the idea of evaluating the trigonometric polynomial that interpolates the discrete data at new positions. For $a > 1$, the zoom-in corresponds to an oversampling of a trigonometric polynomial, and in the frequency domain it is just adding new frequencies with zero amplitude. The zoom-out entails a loss of information or *aliasing*. To avoid aliasing artifacts the high frequencies must be filtered out before the re-interpolation.

E14. A *digital image* is an array of $W \times H$ real numbers. The indexes of the array are called *pixels* and the value of each pixel is called its *gray level*. How would you define the Fourier transform of an image? How would you display it?
The 2D discrete Fourier transform is defined by separability. If $f(x,y)$ is an array of $W \times H$ numbers, then we define the transformed array as

$$\widehat{f}(p,q) = \frac{1}{WH}\sum_{x=0}^{W-1}\sum_{y=0}^{H-1} f(x,y) e^{\frac{-2\pi i p x}{N}} e^{\frac{-2\pi i q y}{M}}$$

Notice that this expression is W-periodic on p and H-periodic on q. Thus it represents a tiling of the discrete plane $\mathbb{Z}^2$ by rectangles of size $W \times H$. Typically we would display a single tile centered on the origin, and show the values of $\left|\widehat{f}\right|$ using a gray-scale palette.

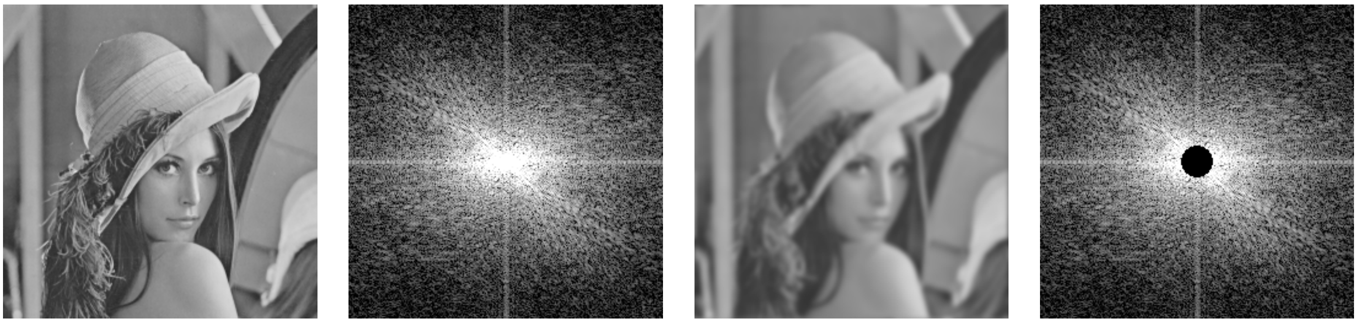
E15. The following figure shows an image and its Fourier transform. What is the value of the central pixel? Why do you see a cross around it? *(Hint: how would the original image look after a coordinate shift?)*



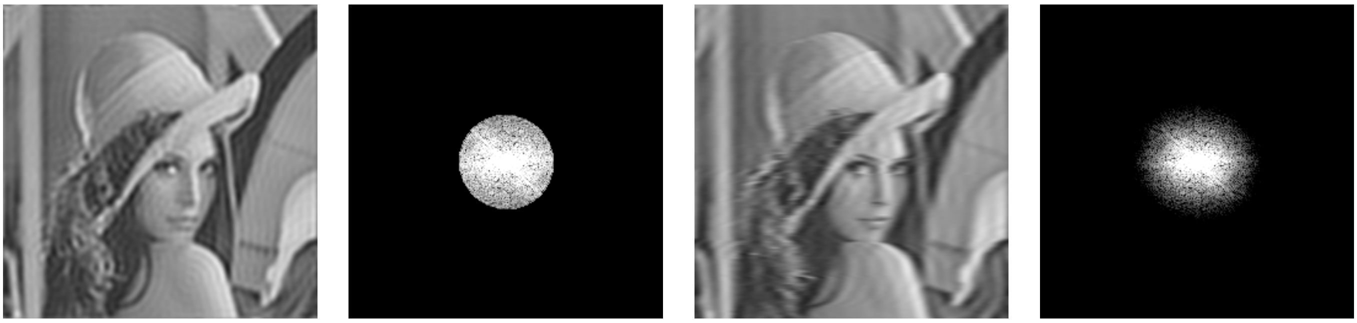
The value at the central pixel is $\widehat{f}(0,0) = \frac{1}{WH}\sum_{x=0}^{W-1}\sum_{y=0}^{H-1} f(x,y)$, that is, the average gray-level of the image f. The "cross" is due to the discontinuities at the boundary of the image, when intepreted as a periodic tiling on the plane. Since these discontinuities are in the vertical and horizontal directions, this imposes a slower decay for the amplitudes of the form $\widehat{f}(p,0)$ and $\widehat{f}(0,q)$.

Notice that the following "shifted" image has exactly the same amplitudes than the original one. Only the phases are different. But here the discontinuities are clearly visible:

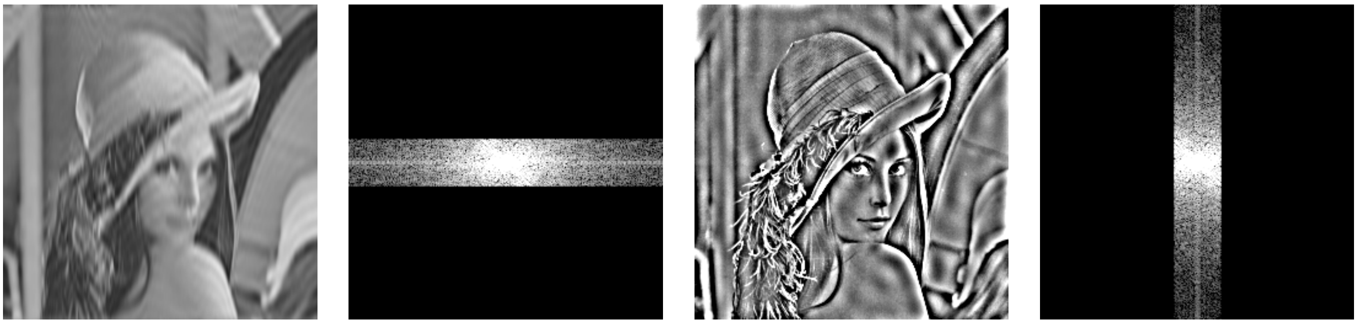
***E16.** Below you see six images and their six Fourier transforms (not necessarily in the same order). Look *very attentively* at the images. Which image corresponds to each spectrum?



(1) (a) (2) (b)



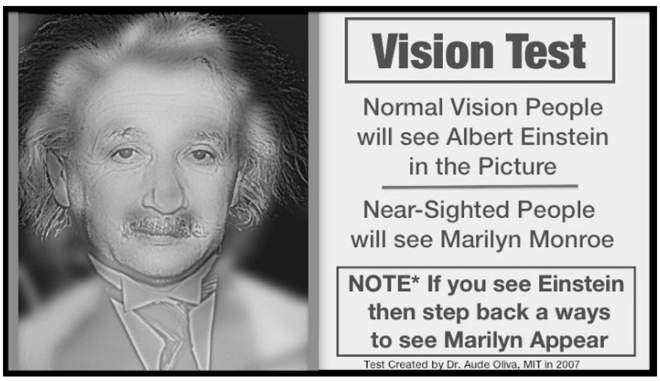
(3) (c) (4) (d)



(5) (e) (6) (f)

(1)–(a) : base case
(6)–(b) : high-pass filter, average values are lost, only fine texture remains
(3)–(c) : low-pass filter, discontinuous frequency cut results in visible ringing artifacts
(2)–(d) : smooth blur, smooth frequency cut
(5)–(e) : removal of high horizontal frequencies produces horizontal ringing, remaining high vertical frequencies can still represent vertical discontinuities (e.g., left boundary of the hat, vertical flowing hair)
(4)–(f) : removal of high vertical frequencies produces vertical ringing, remaining high horizontal frequencies can still represent horizontal discontinuities (e.g., lips, top boundary of the hat)

*****E17**. Explain the following optical illusion and formalize it in terms of Fourier analysis.



This is an hybrid image produced by adding the high frequencies of Albert to the low frequencies of Marilyn. When looked from far away (or by a short-sighted person), the high frequencies below the resolution of the eyes are lost, and only the data from Marilyn remains.