Learning Stochastic Models of Turbulence

Challenging the Log-Normal Hypothesis with Data

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Outline

The Challenge of Turbulence and K41 Theory

The Fundamental Goal

Understanding the statistical structure of fully developed turbulence remains a major challenge in classical physics.

- **K41 Theory (Kolmogorov, 1941):**
  - Hypothesized a universal local structure based on a constant mean energy dissipation rate $\langle \epsilon \rangle$.
  - Focuses on the longitudinal velocity increment at scale l:

$$\delta_l u(x) = u(x + l) - u(x)$$

  - **Prediction:** Variance scales as a power law $\langle (\delta_l u)^2 \rangle \sim l^{2/3}$ based on dimensional analysis.
- **The Limitation:** Experiments show that small-scale velocity fluctuations are strongly **non-Gaussian** (Intermittency), contradicting the uniform dissipation hypothesis.

K62 Refinement and the Log-Normal Hypothesis

Intermittency Phenomenon

Energy dissipation is not uniform but fluctuates significantly in space and time.

- **K62 Theory (Kolmogorov-Oboukhov):** The logarithm of the local energy dissipation rate, $\log \epsilon_r$, is assumed to follow a **Normal (Gaussian)** distribution.
- **Modern Formalism:** This hypothesis underpins the multifractal formalism used to capture intermittency.
- **Stochastic Model (Chevillard et al., 2012):** Velocity increments are modeled as a stochastic product:

$$\delta_l u \approx x_1 e^{x_2}$$

Where x_1 is a Gaussian noise and e^{x_2} is a log-normal multiplier.

Project Aim

Research Question

Are the components x_1 and x_2 in the relation $\delta_l u = x_1 e^{x_2}$ necessarily Gaussian?

Our Objective:

- Characterize the laws governing velocity increments by **challenging the classical log-normal hypothesis**.
- Adopt a **data-driven approach** to learn the probability distributions directly from experimental data.
- Investigate if a more complex, **non-Gaussian structure** is required to accurately describe turbulence intermittency.

Framework: Multifractal Random Walk (MRW)

We utilize the MRW framework where $\delta_l u \approx x_1 e^{x_2}$.

- x_1 **(Fractional Gaussian Noise):** Represents linear correlations.
- x_2 **(Log-dissipation):** Derived from the convolution of an uncorrelated source noise $n(x)$ over the integral scale L.

The Innovation

Standard theory assumes the source noise $n(x)$ is Gaussian. **We aim to learn its true probability density function (PDF) from data.**

Dataset:

- Modane wind tunnel ($R_\lambda = 2500$, $\langle v \rangle = 20.5$ m/s).
- 256 samples of velocity increment moments across 100 scales (4×10^{-5} s to 0.24 s).

Optimization Strategy

We minimize the Mean Squared Error (MSE) between computed and experimental moments in three stages:

1. **Baseline Log-Normal Model:** Assume $x_1, x_2 \sim \mathcal{N}$. Optimize physical parameters (c_1, c_2) to fit only the variance (S_2).
2. **Higher-Order Constraints:** Incorporate **flatness** into the loss function to test the Gaussian assumption against intermittency metrics.
3. **Learned Distribution (Non-Parametric):**
   - Relax the Gaussian assumption for x_2.
   - Use the **Gumbel-Softmax relaxation** technique to differentiably learn the optimal shape of the $n(x)$ distribution (discretized into 300 bins).

Optimization Phase 1: Variance Fitting

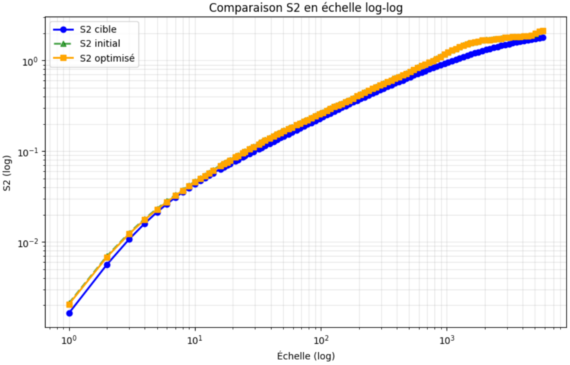


Figure: Variance spectrum S_2 (Log-Log).

Procedure:

- We assume standard Gaussian models for both x_1 and x_2.
- We optimize physical parameters c_1 and c_2 to minimize error on Variance (S_2).

Results:

- **Excellent Fit:** The optimized model (orange) perfectly overlaps the target (blue).
- **Parameters:** Converged to $c_1 \approx 0.33$ and $c_2 \approx 0.03$.
- **Conclusion:** The Gaussian model captures the energy scaling (2nd order) correctly.

Optimization Phase 2: Flatness Mismatch

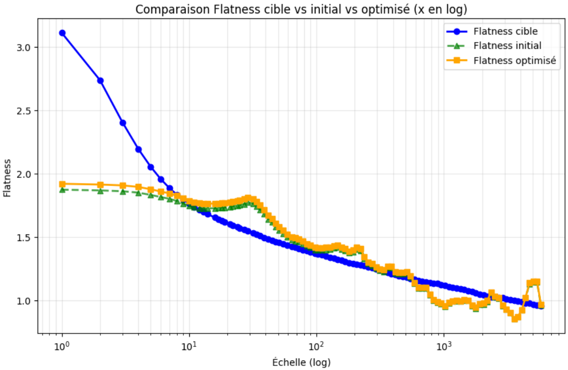


Figure: Flatness factor (Log-Linear).

The Problem:

- Even after optimizing c_1, c_2 on both variance and flatness:
- **Small Scales ($< 10^1$):** Significant deviation. The Gaussian model (orange/green) saturates around $\mathcal{F} \approx 1.9$, while experimental data (blue) reaches > 3.0.
- **Implication:** A purely Gaussian source noise $n(x)$ cannot generate strong enough intermittency.

Need for a learned, non-Gaussian distribution.

Major Result: The Learned Noise Distribution

When the gaussian assumption was relaxed for x_2, the optimization converged to a non-trivial distribution, deviating significantly from the Gaussian prior.

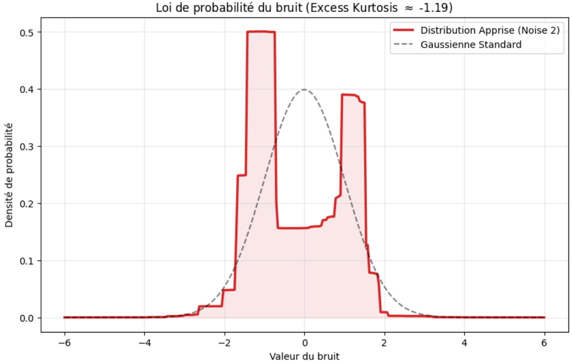


Figure 1: Learned PDF (red) vs Standard Gaussian (dashed).

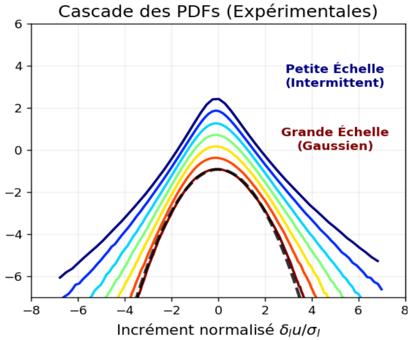
Key Observations:

- **Shape:** Trimodal uniform shape.
- **Statistics:** Negative Excess Kurtosis of -1.19.
- **Implication:** This structural deviation is a key feature required to capture intermittency at small scales.

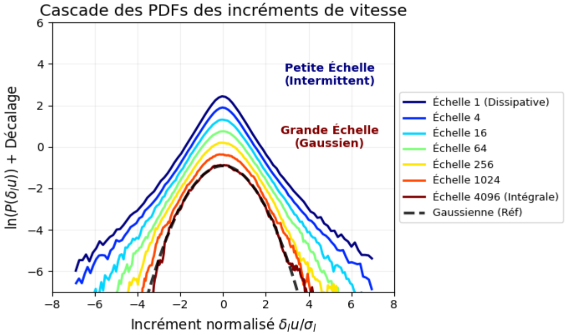
Visualizing Intermittency: The PDF Cascade

Comparison of velocity increment PDFs across scales.

Experimental Data (Target)



Learned MRW Model

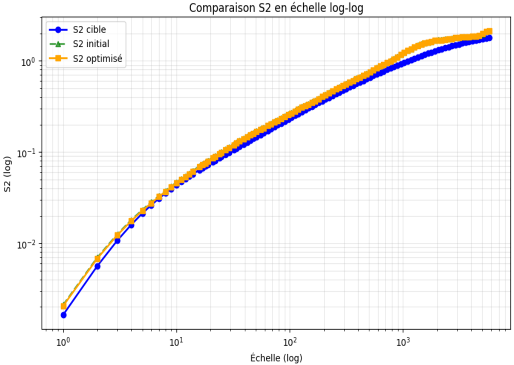


Transition from Gaussian to heavy-tailed PDFs.

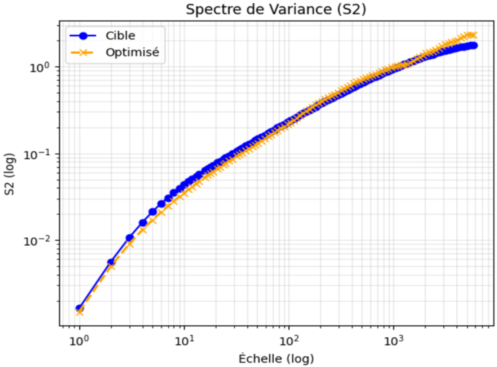
Model Performance: Variance and Flatness

The learned non-Gaussian model achieves remarkable agreement with experimental data.

Variance (Gaussian)



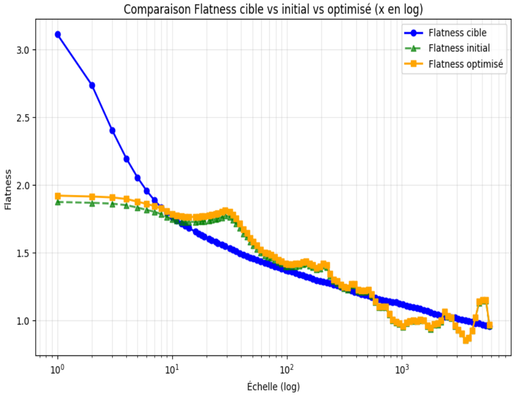
Variance (Learned)



Intermittency coefficient converged to $c_2 \approx 0.02$, consistent with turbulence phenomenology.

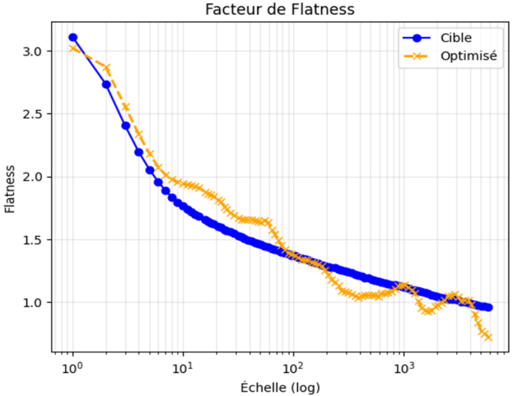
The learned non-Gaussian model achieves remarkable agreement with experimental data.

Flatness (Gaussian)



Significant deviation in low scales

Flatness (Learned)



Accurately captures scale-dependence

Conclusion

- **Challenging the Standard:** We questioned the classical log-normal hypothesis for the intermittency noise $n(x)$.
- **Data-Driven Insight:** By directly learning the PDF from Modane wind tunnel data, we identified a specific **trimodal distribution**.
- **Validation:** This non-Gaussian model successfully reproduces both the variance scaling and the complex behavior of flatness across scales.
- **Takeaway:** Accurate modeling of fully developed turbulence requires relaxing the Gaussian assumption in favor of learned, data-informed distributions.

Thank you for your attention.

Appendix: Learned Trimodal Distribution

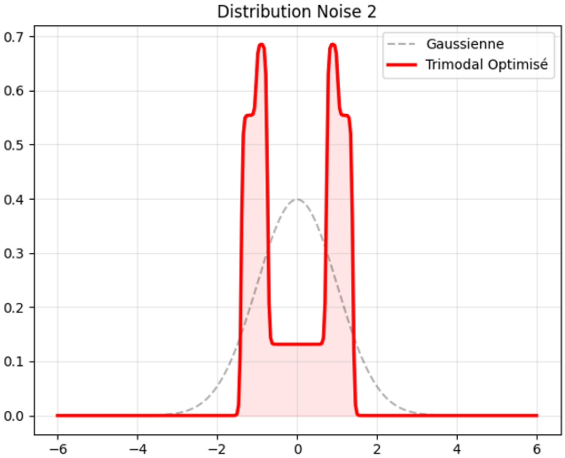


Figure: The optimized probability density function with a uniform-trimodal structure.

Appendix: Moments Fit (Trimodal Model)

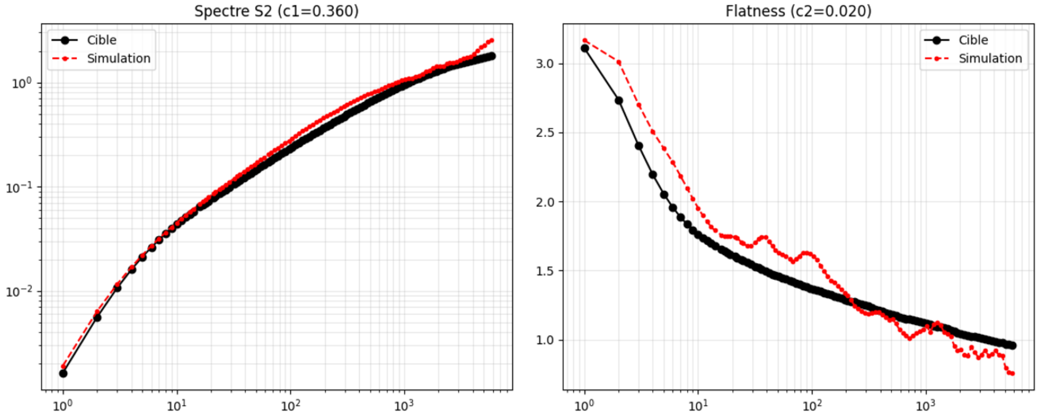


Figure: Comparison of Variance (S_2) and Flatness statistics between the target (black) and the trimodal simulation (red).