Bases in Matroid Theory

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What is a Matroid?

Matroids are abstract structures in mathematics which generalize certain properties of structures like graphs, a set of vectors, bipartite graphs etc. For example, if the property being considered is independence, it would have a different notion for every different kind of structure. For a set of n vectors in $\mathbb{R}^n$, [linear] independence would be mean that no linear combination of any vectors would produce another vector in that set (or there is no nontrivial linear combination of the vectors that equals 0). For graphs, independence would mean a subset of edges that are acyclic (starting and ending vertices in a path are not the same). For a bipartite graph, independence would mean a subset of the edges in some matching (no two edges in that subset share any vertex).

Formally, in general, if E is a finite set and $\mathcal{I}$ is a family of subsets of E, then the family $\mathcal{I}$ forms the independent sets of a matroid M if:

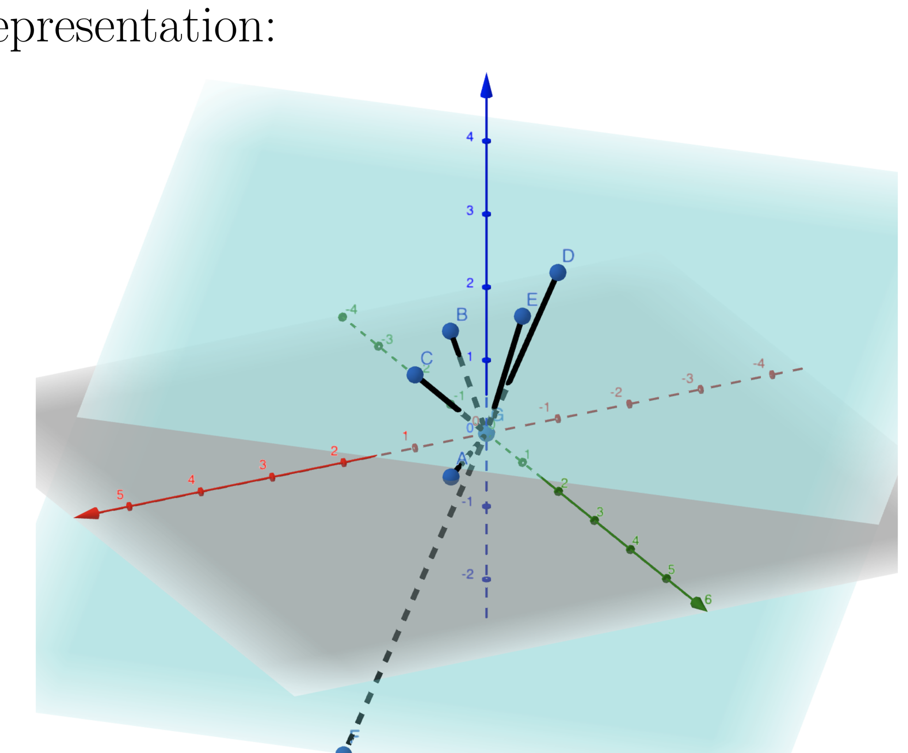
1. $\mathcal{I} \neq \emptyset$
2. if $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
3. if $I, J \in \mathcal{I}$ with $|I| < |J|$ there is some element $x \in J - I$ with $I \cup \{x\} \in \mathcal{I}$

Visualizing a Matroid

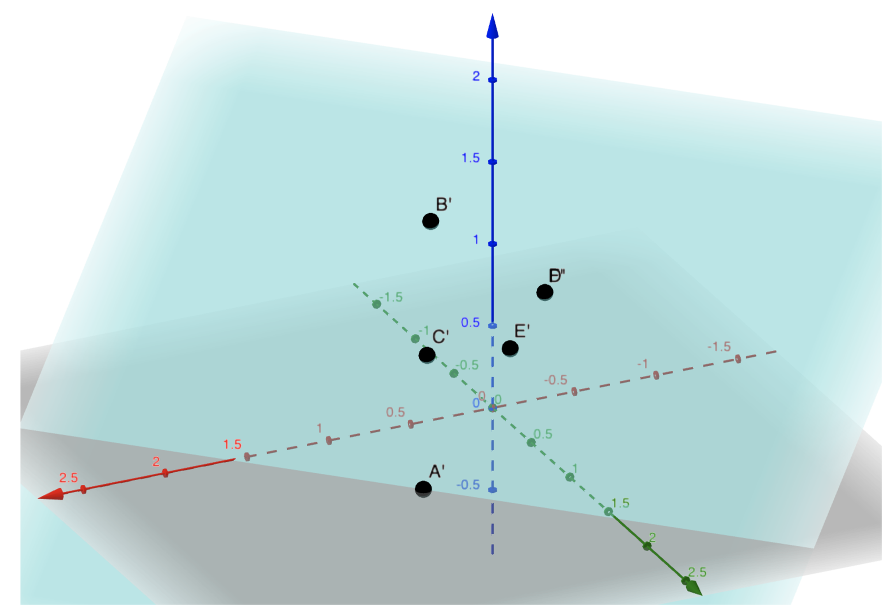
In terms of independence, for a set of vectors, a matroid should clearly represent the independence (and dependence) relations of the set of vectors. Let us take the following set of vectors:

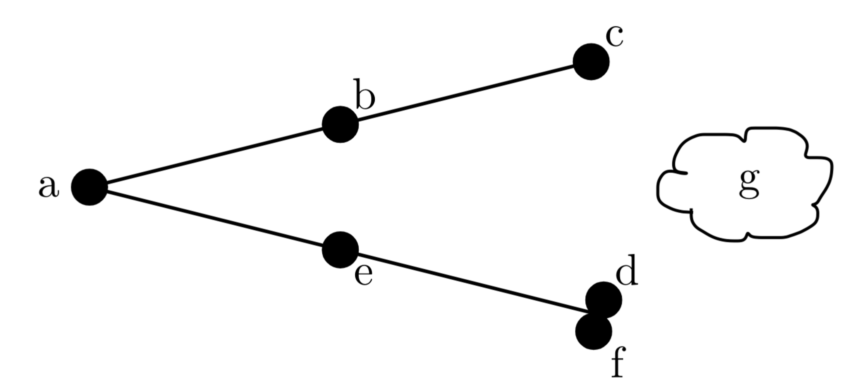
$$\begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 2 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & -4 & 0 \end{bmatrix} \end{array}$$

$\mathbb{R}^3$ representation:



If a plane that is not parallel to any of the vectors is added, all the vectors intersect with the plane at some point. If we project the vectors (via scaling) on to the plane (where they individually intersect the plane), we will get the points for our matroid.





Since $\vec{g}$ is the $\vec{0}$ vector, it will always be dependent (to all other vectors); hence we don't include it in the matroid. If three points are collinear, then they are dependent. As we can see from the matrix, vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly dependent, hence they are collinear in the matroid. Vectors $\vec{d}$ and $\vec{f}$ are scalar multiples of each other, hence they overlap at the same point on the matroid; to avoid clutter, we place them side by side on the matroid.

Bases Definition

As previously mentioned, matroids can be defined based on other abstract properties too. The one we focused on was bases (vectors: basis, graphs: spanning trees). If M is a matroid with independent sets $\mathcal{I}$, then B is a basis of the matroid M if B is a maximal independent set.

If E is a finite set and $\mathcal{B}$ is a family of subsets of E, then the family $\mathcal{B}$ forms the basis sets of a matroid M if:

1. $\mathcal{B} \neq \emptyset$
2. if $B_1,\ B_2 \in \mathcal{B}$ and $B_1 \subseteq B_2$, then $B_1 = B_2$
3. if $B_1,\ B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then there is an element $y \in B_2 - B_1$ so that $B_1 - x \cup \{y\} \in \mathcal{B}$

An *isthmus* is an element $x \in E$ that is in every basis. A *loop* is an element $l \in E$ that is in no basis.

Rank and Flats

- **Rank:** Let $M = (E, \mathcal{I})$ be a matroid and let $A \subseteq E$. The rank of A, written $r(A)$, is the size of the largest independent subset of A:
$$r(A) := \max_{I \subseteq A}\{|I| : I \in \mathcal{I}\}$$
The rank of the matroid $r(M)$ is just $r(E)$. Technically, the rank function r is a map from the set of all subsets of the ground set E of the matroid to the non-negative integers:
$$r : 2^E \to \mathbb{N} \cup \{0\}$$
- **Flats:** Let E be the ground set of the matroid M. A subset $F \subseteq E$ is a flat if $r(F \cup \{x\}) > r(F)$ for any $x \notin F$.
  - If a loop, l, is in the ground set E, then l is in every flat.
  - If x is an isthmus and F is a flat, then
  $x \notin F \implies F \cup \{x\}$ is a flat, and
  $x \in F \implies F - \{x\}$ is a flat.

Future Work

Let $q = p^k$ for some prime p and positive integer k. The points of the affine plane $AG(2, q)$ are the q^2 ordered pairs (x, y), where $x, y \in \mathbb{F}_q$ (field formed by $mod(q)$). The lines are collections of points (x, y) satisfying equations of the form $ax + by = c$, where $a, b, c \in \mathbb{F}_q$.
One can think of the points of $AG(2, q)$ as a collection of q^2 points in the Euclidean plane (we picked the dimension as 2 for ease of understanding), where certain subsets of points are lines. The total number of lines is $q^2 + q$ and every line has exactly q points.
The reason q is a power of a prime is because we want to deal with a unique finite field with $|\mathbb{F}_q| = q = p^k$.

Our next goal is to generalize the construction of the affine *plane* $AG(2, q)$ to n-dimensional affine space $AG(n, q)$. Further into the study of affine geometry, we would understand the role flats and rank plays in connecting affine geometry and matroids.
After affine geometry, we would like to study projective spaces, $PG(n, q)$.

The study of matroids, affine and projective geometry would finally result into working on the following project:

1. Suppose you are given an $n \times n$ matrix, with entries in the finite field $\mathbb{F}_q$. What is the probability your matrix is invertible?
2. Suppose you randomly choose a set B of $n + 1$ points from the rank $n + 1$ affine geometry $AG(n, q)$. What is the probability B is a basis?
3. Same question as (2), but for the rank $n + 1$ projective space $PG(n, q)$.

The goal of this project is to find formulas for each of these three probabilities, and then to establish a connection between them.

References

[1] Gary Gordon, Jennifer McNulty. *Matroids: A Geometric Introduction*. Cambridge University Press, 2012.

Acknowledgements

I would like to thank Dr.Morics for guiding me through out this research. Matroid Theory was a topic new to us both and I appreciate his willingness to venture on this new topic and learn together. I would also like to thank Zach Garcia, who's talk at the IIME event gave me motivation to explore Matroid Theory.
Being an international student, research opportunities are limited for me in the United States. Unlike other programs, the University of Redlands Summer Science Research Program doesn't have the citizenship eligibility criteria. Hence I was able to apply to program and got accepted. Thanks to such programs, we are able to get necessary and valuable research experience. Along with the program, I would like to thank the generous donors to this program without whom this research wouldn't have been possible.

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