π, again

If I ever get a tatoo, it would probably be π

Gavi Dhariwal — December 5, 2020

Before I get into why, if I ever get a tatoo, it would be a tatoo of π, let me talk about how my search began. I began my Google search with Ramanujan (an obvious choice). As I searched upon his prominent works, like partitions, Ramanujan Summation and the Ramanujan τ function, I was excited to learn what they were. I spent hours on YouTube, university websites and publication websites to understand his work but it was too advanced for me. I switched to Manjul Bhargava's work, but that too was complicated for me (his work on Gauss composition, and his ability to connect tabla notes and Sanskrit Sutras is really interesting). I guess what I realised was that I am still a novice in the world of Mathematics; there's SO much more for me to learn and so many more things that are too advanced for me to understand.

The topic in Number Theory I am about embark upon is: *Average of the number of representations of n as a sum of two squares.*

Let $r(n)$ be a function that gives all the number of ways to write n as the sum of squares. For example, for $n = 13$,

$$3^2 + 2^2 = 13 \quad , \quad -3^2 + 2^2 = 13$$

$$3^2 + (-2^2) = 13 \quad , \quad -3^2 + (-2^2) = 13$$

$$2^2 + 3^2 = 13 \quad , \quad -2^2 + 3^2 = 13$$

$$2^2 + (-3^2) = 13 \quad , \quad -2^2 + (-3^2) = 13$$

$$r(13) = 8$$

This can also be represented graphically,

Since $x^2 + y^2 = r^2$ is the equation of a circle, we can represent the topic at hand in terms of a circles

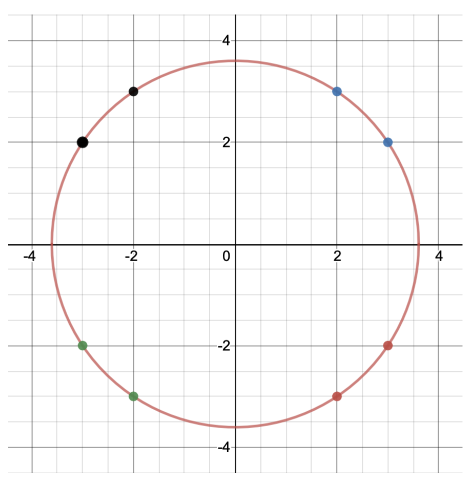

Figure 1: $x^2 + y^2 = 13 \mid r(13) = 8$

In the figure shown above, the equation of the circle is $x^2 + y^2 = 13$; circle of radius $\sqrt{13} = 3.6055$. If we go around each point on the circumference of the circle, those points will have an x and y coordinate who's sum of squares add up to 13 (since they lie on the curve of the function $x^2 + y^2 = 13$). We are only interested in integer values who's sum of squares add up to 13; those points have been marked on the graph which do match our previous count of $r(13) = 8$.

But we do have to be careful with the value of $r(n)$; not every value of n can be represented in terms of $x^2 + y^2$. For example, $r(3) = 0$. Graphically,

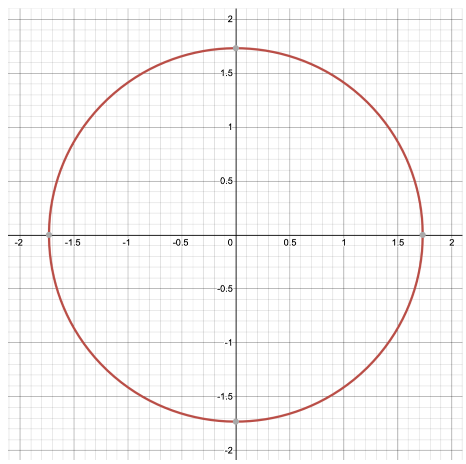

Figure 2: $x^2 + y^2 = 3 \;\mid\; r(3) = 0$

As you can see, there are no points on the circumference where the x and y coordinate is an integer.
If we make the circle into a disk and see all the points (where $x, y \in \mathbb{Z}$) inside the disk who's sum of squares add up to $\leq \sqrt{n}$, algebraically, that would be the sum of all the number of points who's sum of squares add up to $\leq \sqrt{n}$. For example, for $n = 5$, here are the number of possible combinations of integer values who's sum of squares would equal $0, 1, 2, 3, 4$ or 5:

$$0^2 + 0^2 = 0 \quad | \quad r(0) = 1$$

$$1^2 + 0^2 = 1 \quad , \quad 0^2 + 1^2 = 1 \quad , \quad -1^2 + 0 = 1 \quad , \quad 0^2 + (-1^2) = 1 \quad | \quad r(1) = 4$$

$$1^2 + 1^2 = 2 \quad , \quad 1^2 + (-1^2) = 2 \quad , \quad -1^2 + 1^2 = 2 \quad , \quad -1^2 + (-1^2) = 2 \quad | \quad r(2) = 4$$

$$2^2 + 0^2 = 4 \quad , \quad 0^2 + 2^2 = 4 \quad , \quad -2^2 + 0^2 = 4 \quad , \quad 0^2 + (-2^2) = 4 \quad | \quad r(4) = 4$$

$$2^2 + 1^2 = 5 \ , \ -2^2 + 1^2 = 5 \ , \ 2^2 + (-1^2) = 5 \ , \ -2^2 + (-1^2) = 5$$
$$1^2 + 2^2 = 5 \ , \ -1^2 + 2^2 = 5 \ , \ 1^2 + (-2^2) = 5 \ , \ -1^2 + (-2^2) = 5 \quad | \quad r(5) = 8$$

$$\therefore \sum_{k=0}^{5} r(k) = r(0) + r(1) + r(2) + r(3) + r(4) + r(5) = 21$$

This can be represented graphically,

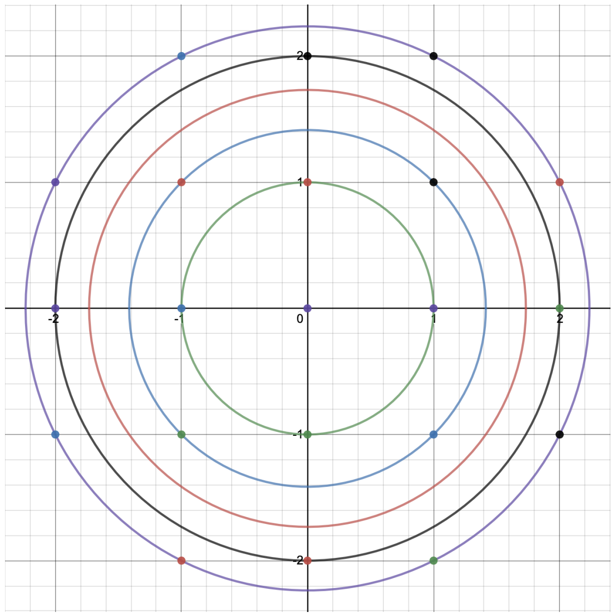

Figure 3: $x^2 + y^2 = 5 \;|\; \sum_{k=0}^{5} r(k) = 21$

Now if we were to draw boxes of length 1 unit around each of the marked points, we would get something like this:

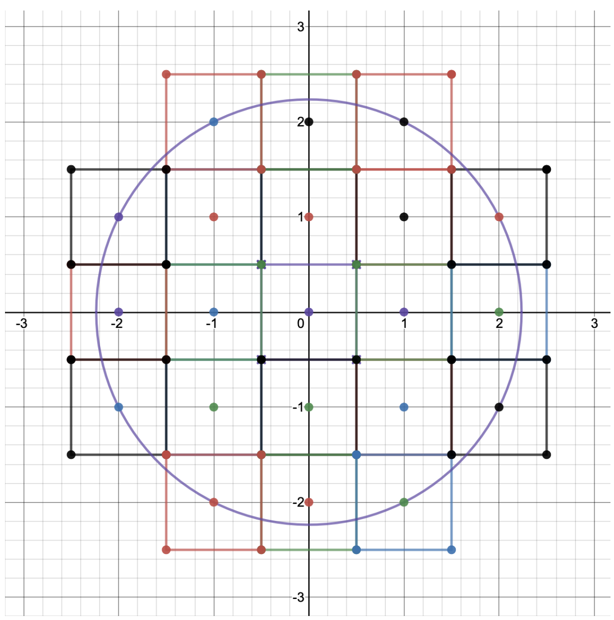

Figure 4: Box of area 1 $unit^2$ around each marked point

Important Note: The area of all those squares is 1. If we add up all those squares, we are basically adding up all the number of points ($x, y \in \mathbb{Z}$) who's sum of squares is $\leq \sqrt{n}$ (here, $n = 5$). *Reminder: $\sqrt{n}$ is being considered here because the radius of the circle is $\sqrt{n}$ $(x^2 + y^2 = n^2)$.*

Hence,

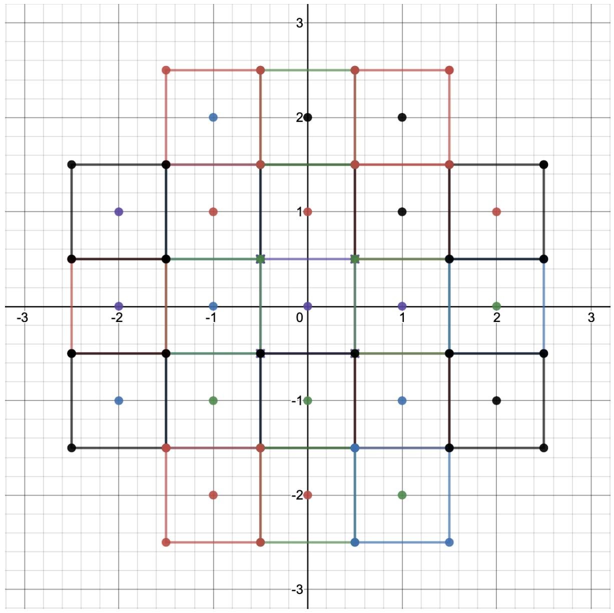

Figure 5: Area of all the squares $= \sum_{k=0}^{5} r(k)$

Let's try to give this area an upper and lower bound.

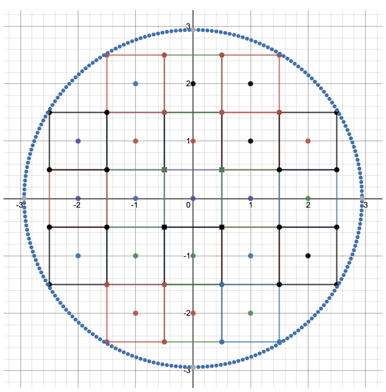

Figure 6: Upper Limit

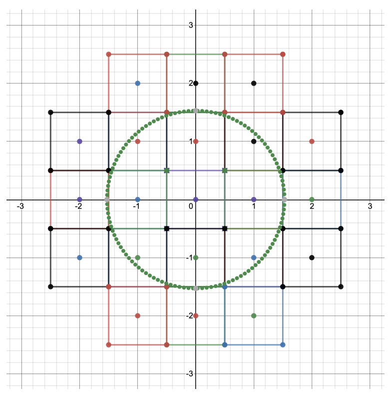

Figure 7: Lower Limit

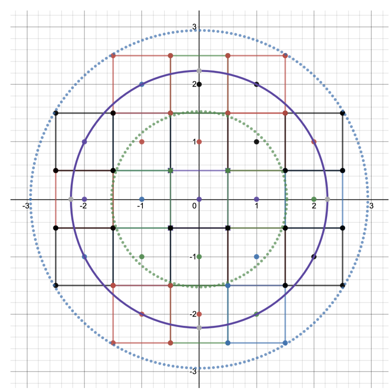

Figure 8: Upper and Lower Limit

How would we know the radii of these two circles? Pythagorean Theorem to the rescue!

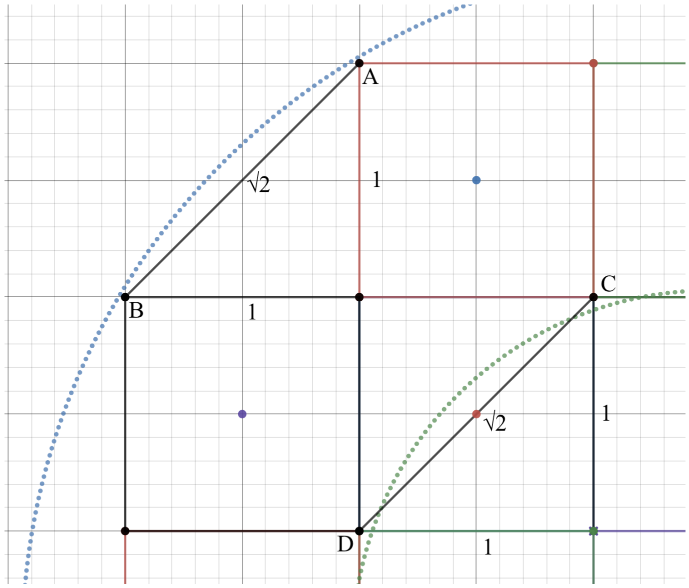

Figure 9: Hypotenuses $AB = CD = \sqrt{2}$

If we zoom in to the second quadrant of the graphs, we can use Pythagorean theorem to find the lengths of the hypotenuses AB and CD. Since the length of the sides of the triangles is 1, the hypotenuses turn out to be $\sqrt{2}$. Why do we need to find the lengths of AB and CD? Here's why,

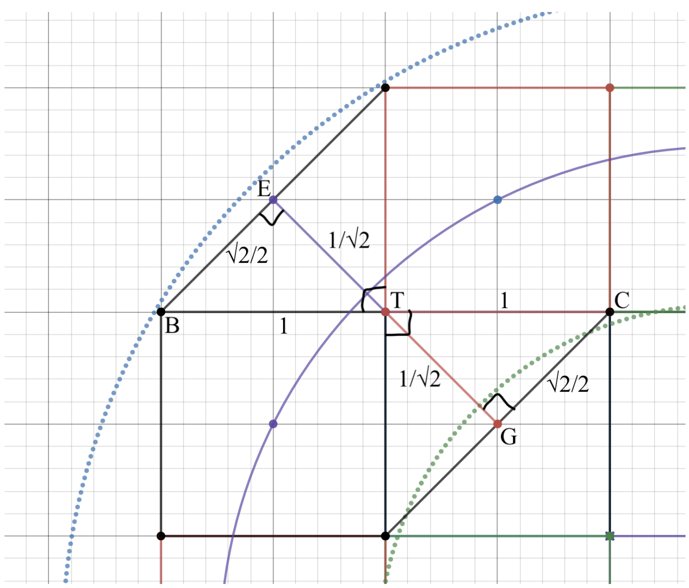

Figure 10: Approximate increment/reduction lengths, $TE = TG = \dfrac{1}{\sqrt{2}}$

If we draw lines from the point T, the lines cut the hypotenuses AB and CD in half at E and G respectively at $90°$ creating two small new right angle triangles ΔTEB and ΔTGC. Using the Pythagorean theorem, lines TE and TG turn out to be $\dfrac{1}{\sqrt{2}}$.

Reminder: The purple circle is the main circle with radius $\sqrt{5}$, the blue circle is the upper limit and the green circle is the lower limit.

TE and TG can be thought of as approximate increment and reduction in the radius of the main circle to reach the upper and lower limit. Hence, the radius of the upper limit is $\sqrt{5} + \dfrac{1}{\sqrt{2}}$ and the radius of the lower limit is $\sqrt{5} - \dfrac{1}{\sqrt{2}}$.

<u>Recall:</u>

$$\text{Area of the all the squares} = \sum_{k=0}^{5} r(k)$$

.

After finding the radii of the upper and lower limit, we can write this inequality:

$$\text{Area of the lower limit circle} \ \leq \ \text{Area of all the squares} \ \leq \ \text{Area of the upper limit circle}$$

$$\pi\left(\sqrt{5} - \frac{1}{\sqrt{2}}\right)^2 \ \leq \ \sum_{k=0}^{5} r(k) \leq \ \pi\left(\sqrt{5} + \frac{1}{\sqrt{2}}\right)^2$$

We can write this in general terms,

$$\pi\left(\sqrt{n} - \frac{1}{\sqrt{2}}\right)^2 \ \leq \ \sum_{k=0}^{n} r(k) \leq \ \pi\left(\sqrt{n} + \frac{1}{\sqrt{2}}\right)^2$$

Solving,

$$\pi\left(n + \frac{1}{2} - \frac{2\sqrt{n}}{\sqrt{2}}\right) \ \leq \ \sum_{k=0}^{n} r(k) \leq \ \pi\left(n + \frac{1}{2} + \frac{2\sqrt{n}}{\sqrt{2}}\right)$$

Dividing by n,

$$\pi\frac{\left(n + \frac{1}{2} - \frac{2\sqrt{n}}{\sqrt{2}}\right)}{n} \ \leq \ \frac{1}{n}\sum_{k=0}^{n} r(k) \leq \ \pi\frac{\left(n + \frac{1}{2} + \frac{2\sqrt{n}}{\sqrt{2}}\right)}{n}$$

$$\pi\left(1 + \frac{1}{2n} - \frac{2}{\sqrt{2}\sqrt{n}}\right) \ \leq \ \frac{1}{n}\sum_{k=0}^{n} r(k) \leq \ \pi\left(1 + \frac{1}{2n} + \frac{2}{\sqrt{2}\sqrt{n}}\right)$$

So what happens when n gets bigger and bigger? Let's take the limit! The approximation we made regarding the radius of the upper limit and lower limit also gets better.

$$\lim_{n\to\infty} \pi\left(1 + \frac{1}{2n} - \frac{2}{\sqrt{2}\sqrt{n}}\right) \ \leq \ \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} r(k) \leq \ \lim_{n\to\infty} \pi\left(1 + \frac{1}{2n} + \frac{2}{\sqrt{2}\sqrt{n}}\right)$$

Do you see what I see?

$$\pi \ \leq \ \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} r(k) \leq \ \pi$$

And (thank god I was awake and paying attention during this Real Analysis class), according to our beloved Squeeze Theorem,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} r(k) = \pi$$

You can email me at gavi_dhariwal@redlands.edu if that doesn't BLOW YOUR MIND!!!
To put this into words: *The average number of representations of a positive integer as a sum of squares is* π.

...and that's why if I ever get a tatoo, it would probably be π

References

[1] Video by Michael Penn explaining the concept

[2] Chapter 18 from "Number Theory in Context" by Karl-Dieter Crisman