# Calculating the value of $\pi$ via series

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## 1 Taylor Series of $\tan^{-1} x$

Taylor Series for $\tan^{-1} x$ expanded around $x = 0$ :

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

General Sum:

$$\sum_{n=0}^{\infty} \frac{-1^{n+1}\ 1^{2n+1}}{2n+1}$$

Using the Alternating Series Test when $x = 1$:

$$c_n = \frac{1^{2n+1}}{2n+1}$$
$$c_{n+1} = \frac{1^{2n+3}}{2n+3}$$

i. $c_{n+1} \leq c_n$

$$\boxed{\frac{1^{2n+3}}{2n+3} \leq \frac{1^{2n+1}}{2n+1}}$$

ii. $\lim_{n\to\infty} c_n = 0$

$$\boxed{\lim_{n\to\infty} \frac{1^{2n+1}}{2n+1} = \lim_{n\to\infty} \frac{1}{2n+1} = 0}$$

$\boxed{\therefore \text{The series converges by the Alternating Series test where } x = 1}$

**b.** First ten terms in the Taylor Series of $\tan^{-1}(x)$ :

$$x - \frac{x}{3} + \frac{x}{5} - \frac{x}{7} + \frac{x}{9} - \frac{x}{11} + \frac{x}{13} - \frac{x}{15} + \frac{x}{17} - \frac{x}{19}$$

First ten terms in the Taylor Series of $4\tan^{-1}(x)$ :

$$4 - \frac{x}{3} + \frac{4x}{5} - \frac{4x}{7} + \frac{4x}{9} - \frac{4x}{11} + \frac{4x}{13} - \frac{4x}{15} + \frac{4x}{17} - \frac{4x}{19}$$

$$\therefore 4\tan^{-1}(1) = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} - \frac{4}{15} + \frac{4}{17} - \frac{4}{19}$$

$4\tan^{-1}(1) \approx 3.041839619$

$$\boxed{\therefore \pi \approx 3.041839619}$$

Based on what we know the value of $\pi$ to be (3.14159), only <u>one</u> digit is correct in this approximation

**c.** Taylor Series being considered: $4\tan^{-1}(x)$

General Sum:

$$\sum_{n=0}^{\infty} \frac{4\left(-1^{n+1}\ 1^{2n+1}\right)}{2n+1}$$

Error for $4\tan^{-1}(x)$, $E_n$:

$$4\tan^{-1}(x) - P_n(x)$$

where $P_n(x)$ is the Taylor series for $4\tan^{-1}(x)$ to the $n^{th}$ term

Given: In an alternating series, the error in using the first n terms is always less than the absolute value of the $(n+1)^{st}$ term

$$\therefore 4\tan^{-1}(x) - P_n(x) < \left| \frac{4\left(-1^{n+1+1}\ 1^{2n+1+1}\right)}{2n+1+1} \right|$$

$$E_n < \left| \frac{4}{2n+3} \right|$$

For the error to be less than $1 \times 10^{-8}$:

$$\frac{4}{2n+3} < 1 \times 10^{-8}$$

$$\frac{4}{1 \times 10^{-8}} < 2n+3$$

$$4 \times 10^{8} < 2n+3 \quad \rightarrow \quad 400000000 < 2n+3$$

$$400000000 - 3 < 2n \quad \rightarrow \quad \frac{399999997}{2} < n$$

$$199999998.5 < n$$

$$\boxed{\therefore \text{ the smallest } n \text{ possible for the error to be less than } 1 \times 10^{-8} \text{ is } 199999999}$$

## 2 Machin's Formula

$$\frac{\pi}{4} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}$$

Given formula:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

**a.** Let $A = \tan^{-1}\left(\frac{120}{119}\right)$ and $B = -\tan^{-1}\left(\frac{1}{239}\right)$

$$\tan(A+B) = \tan\left(\tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right)\right)$$

Using the given formula:

$$\tan(A+B) = \frac{\tan\left\{\tan^{-1}\left(\frac{120}{119}\right)\right\} + \tan\left\{-\tan^{-1}\left(\frac{1}{239}\right)\right\}}{1 - \left[\tan\left\{\tan^{-1}\left(\frac{120}{119}\right)\right\} \cdot \tan\left\{-\tan^{-1}\left(\frac{1}{239}\right)\right\}\right]}$$

Now since tan is an odd function, $\tan(-x) = -\tan(x)$

$$\therefore \tan\left\{-\tan^{-1}\left(\frac{1}{239}\right)\right\} = -\tan\left\{\tan^{-1}\left(\frac{1}{239}\right)\right\} = -\frac{1}{239}$$

$$\tan(A+B) = \frac{\left(\dfrac{120}{119} - \dfrac{1}{239}\right)}{1 + \left(\dfrac{120}{119} \times \dfrac{1}{239}\right)}$$

$$= \frac{\left(\dfrac{28680 - 119}{28441}\right)}{\left(1 + \dfrac{120}{28441}\right)} = \frac{\left(\dfrac{28561}{28441}\right)}{\left(\dfrac{28441 + 120}{28441}\right)}$$

$$= \frac{28561}{28441} \times \frac{28441}{28561} = 1$$

$$\underline{\therefore \tan(A+B) = 1}$$

$$\tan(A+B) = \tan\left(\tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right)\right) = 1$$

Taking $\tan^{-1}$ on both sides

$$\tan^{-1}\left[\tan\left(\tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right)\right)\right] = \tan^{-1} 1$$

$$\boxed{\therefore \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right) = \tan^{-1} 1}$$

**b. i.** Given, $A = B = \tan^{-1}\left(\frac{1}{5}\right)$

Using the formula for $\tan(A+B)$:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A+B) = \frac{\tan\left\{\tan^{-1}\left(\frac{1}{5}\right)\right\} + \tan\left\{\tan^{-1}\left(\frac{1}{5}\right)\right\}}{1 - \left[\tan\left\{\tan^{-1}\left(\frac{1}{5}\right)\right\} \cdot \tan\left\{\tan^{-1}\left(\frac{1}{5}\right)\right\}\right]}$$

$$= \frac{\left(\frac{1}{5} + \frac{1}{5}\right)}{1 - \left(\frac{1}{5} \times \frac{1}{5}\right)} = \frac{\left(\frac{2}{5}\right)}{\left(1 - \frac{1}{25}\right)} = \frac{2}{5} \times \frac{25}{24} = \frac{5}{12}$$

$$\therefore \tan(A+B) = \frac{5}{12}$$

$$A + B = \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{5}\right) = 2\tan^{-1}\left(\frac{1}{5}\right)$$

$$\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right)\right\} = \frac{5}{12}$$

Taking $\tan^{-1}$ on both sides

$$\tan^{-1}\left[\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right)\right\}\right] = \tan^{-1}\left(\frac{5}{12}\right)$$

$$\boxed{\therefore 2\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{5}{12}\right)}$$

**ii.** $4\tan^{-1}\left(\frac{1}{5}\right) = 2\tan^{-1}\left(\frac{1}{5}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = (A+B)+(A+B) = 2A+2B$

Using the formula for $\tan(A+B)$,

$$\tan(2A+2B) = \frac{\tan(2A)+\tan(2B)}{1-\tan(2A)\tan(2B)}$$

$$\tan(A+B) = \frac{\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right)\right\} + \tan\left\{2\tan^{-1}\left(\frac{1}{5}\right)\right\}}{1-\left[\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right)\right\}\cdot\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right)\right\}\right]}$$

$$= \frac{\tan\left\{\tan^{-1}\left(\frac{5}{12}\right)\right\} + \tan\left\{\tan^{-1}\left(\frac{5}{12}\right)\right\}}{1-\tan\left\{\tan^{-1}\left(\frac{5}{12}\right)\right\}\cdot\tan\left\{\tan^{-1}\left(\frac{5}{12}\right)\right\}} \quad \text{.... } \mathit{from}\ \mathbf{b.\ i.}$$

$$= \frac{\left(\frac{5}{12}+\frac{5}{12}\right)}{1-\left(\frac{5}{12}\times\frac{5}{12}\right)} = \frac{\left(\frac{10}{12}\right)}{\left(1-\frac{25}{144}\right)} = \frac{10}{12}\times\frac{144}{119} = \frac{120}{119}$$

$$\underline{\therefore \tan(2A+2B) = \frac{120}{119}}$$

$$\tan\left\{\tan^{-1}\left(\frac{1}{5}\right)\right\} = \frac{120}{119}$$

Taking $\tan^{-1}$ on both sides

$$\tan^{-1}\left[\tan\left\{\tan^{-1}\left(\frac{1}{5}\right)\right\}\right] = \tan^{-1}\left(\frac{120}{119}\right)$$

$$\boxed{\therefore 4\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{120}{119}\right)}$$

**c.** From a.

$$\tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right) \; = \; \tan^{-1} 1$$

$$\therefore \tan^{-1}\left(\frac{1}{239}\right) \; = \; \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}(1)$$

$$\therefore 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$= \; \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{120}{119}\right) + \tan^{-1}(1) \;\; .... \; from \; \textbf{b. ii.}$$

$$= \; \tan^{-1}(1) \; = \; \frac{\pi}{4}$$

$$\boxed{\therefore \frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)}$$

Machin's Formula Derived!

**d.**

$$\pi = 16\tan^{-1}\left(\frac{1}{5}\right) - 4\tan^{-1}\left(\frac{1}{239}\right)$$

**i.** Using the first 4 terms:

$$= 16\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\right) - 4\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\right)$$

$$= 16\left((1/5) - \frac{(1/5)^3}{3} + \frac{(1/5)^5}{5} - \frac{(1/5)^7}{7}\right)$$
$$- 4\left((1/239) - \frac{(1/239)^3}{3} + \frac{(1/239)^5}{5} - \frac{(1/239)^7}{7}\right)$$

$$= 3.15832807619 - 0.0167363040083$$
$$= \boxed{3.14159177218}$$

**ii.** Using the first 5 terms:

$$= 16\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9}\right) - 4\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9}\right)$$

$$= 16\left((1/5) - \frac{(1/5)^3}{3} + \frac{(1/5)^5}{5} - \frac{(1/5)^7}{7} + \frac{(1/5)^9}{9}\right)$$
$$-4\left((1/239) - \frac{(1/239)^3}{3} + \frac{(1/239)^5}{5} - \frac{(1/239)^7}{7} + \frac{(1/239)^9}{9}\right)$$

$$= 3.15832898641 - 0.0167363040083$$
$$= \boxed{3.1415926824}$$

**iii.** Using the first 6 terms:

$$= 16\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11}\right) - 4\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11}\right)$$

$$= 16\left((1/5) - \frac{(1/5)^3}{3} + \frac{(1/5)^5}{5} - \frac{(1/5)^7}{7} + \frac{(1/5)^9}{9} - \frac{(1/5)^{11}}{11}\right)$$
$$-4\left((1/239) - \frac{(1/239)^3}{3} + \frac{(1/239)^5}{5} - \frac{(1/239)^7}{7} + \frac{(1/239)^9}{9} - \frac{(1/239)^{11}}{11}\right)$$

$$= 3.15832895662 - 0.0167363040083$$
$$= \boxed{3.14159265262}$$

**iv.** Using the first 7 terms:

$$= 16\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \frac{x^{13}}{13}\right) - 4\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \frac{x^{13}}{13}\right)$$

$$= 16\left((1/5) - \frac{(1/5)^3}{3} + \frac{(1/5)^5}{5} - \frac{(1/5)^7}{7} + \frac{(1/5)^9}{9} - \frac{(1/5)^{11}}{11} + \frac{(1/5)^{13}}{13}\right)$$
$$-4\left((1/239) - \frac{(1/239)^3}{3} + \frac{(1/239)^5}{5} - \frac{(1/239)^7}{7} + \frac{(1/239)^9}{9} - \frac{(1/239)^{11}}{11} + \frac{(1/239)^{13}}{13}\right)$$

$$= 3.15832895763 - 0.0167363040083$$
$$= \boxed{3.14159265362}$$

**e.** Approximation for $\pi$ based on the calculations made in **d**, $\boxed{\pi = 3.14159}$
The reason behind including only those digits is because those are the only digits that are common with all four approximations made for $\pi$ using the first 4, 5, 6 and 7 terms of the $\tan^{-1}(x)$ series.

## 3 Srinivasa Ramanujan's Formula

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!\,(1103 + 26,390n)}{(n!)\,396^{4n}}$$

**a.** Using the Ratio Test:

$$a_n = \frac{(4n)!\,(1103 + 26,390n)}{(n!)^4\,396^{4n}} \qquad a_{n+1} = \frac{(4n+4)!\,(1103 + 26,390(n+1))}{((n+1)!^4)\,396^{4n+4}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(4n+4)!\,(1103 + 26,390n + 26,390)}{((n+1)!^4)\,396^{4n+4}}}{\dfrac{(4n)!\,(1103 + 26,390n)}{(n!)^4\,396^{4n}}}$$

$$= \lim_{n\to\infty} \frac{(4n+4)(4n+3)(4n+2)(4n+1)\,(4n)!\,(1103 + 26390n + 26390)}{(n+1)^4 \cdot (n!)^4 \cdot 396^{4n} \cdot 396^4} \times \frac{(n!)^4 \cdot 396^{4n}}{(4n)!\,(1103 + 26390n)}$$

$$= \lim_{n\to\infty} \frac{(256n^4 + 640n^3 + 560n^2 + 200n + 24) \cdot n\,(1103 + 26390n + 26390)}{(n^4 + 4n^3 + 6n^2 + 4n + 1)\,(1103 + 26390n) \cdot 396^4}$$

$$= \lim_{n\to\infty} \frac{n^4 \left(256 + \dfrac{640}{n} + \dfrac{560}{n^2} + \dfrac{200}{n^3} + \dfrac{24}{n^4}\right) \cdot n \left(\dfrac{1103}{n} + 26390 + \dfrac{26390}{n}\right)}{n^4 \left(1 + \dfrac{4}{n} + \dfrac{6}{n^2} + \dfrac{4}{n^3} + \dfrac{1}{n^4}\right) \cdot n \left(\dfrac{1103}{n} + 26390\right) \cdot 396^4}$$

all fractions with $n$ in the denominator go to *zero* because, as the limit of n approaches $\infty$, all those fractions get very small and tend to go towards *zero*

$$= \lim_{n\to\infty} \frac{256 \cdot (26390)}{(26390) \cdot 396^4} = \frac{256}{396^4} = \frac{1}{96059601}$$

$$\boxed{\because \frac{1}{96059601} < 1 \quad (L < 1), \ \text{ the series converges by the Ratio Test}}$$

**b.** First term of the series, n = 0:

$$\frac{0! \cdot (1103 + 0)}{(0!)^4 \, 396^0} = 1103$$

Plugging this into the given Ramanujan's Formula:

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \cdot 1103$$

$$\frac{1}{\pi} = \frac{2206\sqrt{2}}{9801}$$

$$\boxed{\therefore \pi = 3.141592730013305660, \text{ when using the first term}}$$

First and second term of the series:

$$n = 0 \quad \to \quad \frac{0! \cdot (1103 + 0)}{(0!)^4 \, 396^0} = 1103$$

$$n = 1 \quad \to \quad \frac{(4!)\,(1103 + 26390)}{(1!)^4 \cdot 396^4} = \frac{27493}{1024635744}$$

Plugging these first two terms into the given Ramanujan's Formula:

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \cdot \left(1103 + \frac{27493}{1024635744}\right)$$

$$\frac{1}{\pi} = \frac{1130173253125}{2510613731736\sqrt{2}}$$

$$\boxed{\therefore \pi = 3.14159265358979387, \text{ when using the first two terms}}$$

The accuracy of the approximation <u>does</u> increase by 8 digits as the number of terms being used are increased by one!

This is absolutely breathtaking and humbling.