分析力学的哈密顿体系

本节导读

- 勒让德变换与哈密顿正则方程
- 泊松括号
- 正则变换

哈密顿正则方程

拉格朗日方程:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_{\alpha}}-\frac{\partial L}{\partial q_{\alpha}}=0 \quad (\alpha=1,2,\cdots,s)$$

S个二阶微分方程，S等于体系的自由度

可否化为一阶微分方程？

$$\begin{cases} \dfrac{d}{dt}p_{\alpha}-\dfrac{\partial L}{\partial q_{\alpha}}=0 \\ p_{\alpha}=\dfrac{\partial L}{\partial \dot{q}_{\alpha}} \end{cases}$$

2S个一阶微分方程

可否写为关于q和p的2s个微分方程组？

哈密顿正则方程

当认为*L*是广义坐标、 广义速度和时间的函数时

$$\mathrm{d}L=\sum_{\alpha=1}^{s}\left(\frac{\partial L}{\partial q_\alpha}\mathrm{d}q_\alpha+\frac{\partial L}{\partial \dot{q}_\alpha}\mathrm{d}\dot{q}_\alpha\right)+\frac{\partial L}{\partial t}\mathrm{d}t$$

考虑广义动量的定义, 得

$$\mathrm{d}L=\sum_{\alpha=1}^{s}\left(p_\alpha \mathrm{d}q_\alpha+p_\alpha \mathrm{d}\dot{q}_\alpha\right)+\frac{\partial L}{\partial t}\mathrm{d}t$$

对于哈密顿量 $H(p,q,t)=-L+\sum_{\alpha=1}^{s}p_\alpha \dot{q}_\alpha$

可得

$$\mathrm{d}H=-\mathrm{d}L+\sum_{\alpha=1}^{s}\left(p_\alpha \mathrm{d}\dot{q}_\alpha+\dot{q}_\alpha \mathrm{d}p_\alpha\right)=\sum_{\alpha=1}^{s}\left(-\dot{p}_\alpha \mathrm{d}q_\alpha+\dot{q}_\alpha \mathrm{d}p_\alpha\right)-\frac{\partial L}{\partial t}\mathrm{d}t$$

H作为广义动量, 广义坐标和时间的函数, 又有

$$\mathrm{d}H=\sum_{\alpha=1}^{s}\left(\frac{\partial H}{\partial q_\alpha}\mathrm{d}q_\alpha+\frac{\partial H}{\partial p_\alpha}\mathrm{d}p_\alpha\right)+\frac{\partial H}{\partial t}\mathrm{d}t$$

由于动量, 坐标和时间都是独立的, 所以

$$\begin{cases}\dot{q}_\alpha=\dfrac{\partial H}{\partial p_\alpha}\\ \dot{p}_\alpha=-\dfrac{\partial H}{\partial q_\alpha}\end{cases}\qquad(\alpha=1,2,\cdots,s)$$

——哈密顿正则方程

相应的广义动量, 坐标叫做正则变量, 它们组成的2s维空间叫相空间, 一组数值对应相空间中一点, 叫相点.

勒让德变换

$$f = f(x,y)$$
$$df = udx + vdy$$
$$u = \frac{\partial f}{\partial x}, v = \frac{\partial f}{\partial y}$$

$\xrightarrow{x,y,u,v}$

$$x = x(u,y), v = v(u,y)$$
$$\bar{f}(u,y) = f(x(u,y),y)$$

$$\frac{\partial \bar{f}}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} = v + u\frac{\partial x}{\partial y}$$
$$\frac{\partial \bar{f}}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} = u\frac{\partial x}{\partial u} = \frac{\partial ux}{\partial u} - x$$

$$v = -\frac{\partial}{\partial y}(-\bar{f} + ux) = -\frac{\partial g}{\partial y}$$
$$x = \frac{\partial}{\partial u}(-\bar{f} + ux) = \frac{\partial g}{\partial u}$$

勒让德变换:

新函数

=不要的变量×新变量−原来函数

哈密顿量即是拉格朗日量的勒让德变换

例1 一维弹簧振子的运动

$$L = T - V$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

- 哈密顿量 $H=T+V$

$$H = T + V = m\dot{x}^2/2 + kx^2/2$$

$$= \frac{p^2}{2m} + \frac{1}{2}kx^2$$

p 为广义动量

x为广义位移

哈密顿正则方程:

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} = \dfrac{p}{m} & \text{————} \quad \text{动量定义} \\ \dot{p} = -\dfrac{\partial H}{\partial x} = -kx & \text{————} \quad \text{牛顿第二定律} \end{cases}$$

$\longrightarrow$ $m\ddot{x} = -kx$ 即: $m\ddot{x} + kx = 0$

在拉格朗日动力学中，从拉格朗日函数可以直接写出动力学方程即拉格朗日方程. 在哈密顿动力学中，必须从拉格朗日函数转到哈密顿函数，才可写出动力学方程即哈密顿正则方程,从哈密顿正则方程消去广义动量的结果其实不过是从另一条路径达到拉格朗日方程，所以哈密顿动力学不如拉格朗日动力学简便.

哈密顿动力学的优点之一是便于量子化. 另一个优点在变量的变换中比较自由：拉格朗日动力学采用的变量广义坐标和广义动量并不对等，只能对广义坐标进行变换 而广义速度也随之而变 哈密顿动力学采用的变量坐标和动量是完全对等的，不仅可以对广义坐标进行变换，而且可以坐标和动量一起变换 这个问题在后面正则变换中将进一步分析.

例2 分别用笛卡儿坐标、柱面坐标和球面坐标写出一个自由质点在势场$V(r)$中的哈密顿函数H.

解：体系为质点，自由度数$s=3$

(1) 在笛卡儿坐标系中, 取x, y, z为广义坐标, 则拉格朗日函数L为

$$L=T-V=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)-V(x,y,z)$$

$$\left.\begin{aligned} p_x&=\frac{\partial L}{\partial \dot{x}}=m\dot{x}\\ p_y&=\frac{\partial L}{\partial \dot{y}}=m\dot{y}\\ p_z&=\frac{\partial L}{\partial \dot{z}}=m\dot{z}\end{aligned}\right\} \Rightarrow \left.\begin{aligned} \dot{x}&=\frac{p_x}{m}\\ \dot{y}&=\frac{p_y}{m}\\ \dot{z}&=\frac{p_z}{m}\end{aligned}\right\}$$

$$H=-L+\sum_{\alpha}p_{\alpha}\dot{q}_{\alpha}$$

$$=-\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)+V(x,y,z)+p_x\dot{x}+p_y\dot{y}+p_z\dot{z}$$

$$H=\frac{1}{2m}(p_x^2+p_y^2+p_z^2)+V(x,y,z)$$

(2)　在柱面坐标系中

$$T=\frac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\varphi}^2+\dot{z}^2)$$

$$L=T-V\qquad=\frac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\varphi}^2+\dot{z}^2)-V(\rho,\varphi,z)$$

$$\left.\begin{aligned} p_\rho &= \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho} \\ p_\varphi &= \frac{\partial L}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi} \\ p_z &= \frac{\partial L}{\partial \dot{z}} = m\dot{z} \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} \dot{\rho} &= \frac{p_\rho}{m} \\ \dot{\varphi} &= \frac{p_\varphi}{m\rho^2} \\ \dot{z} &= \frac{p_z}{m} \end{aligned}\right\}$$

$$H = -L + p_\rho\dot{\rho} + p_\varphi\dot{\varphi} + p_z\dot{z}$$

$$H = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} + p_z^2\right) + V(\rho, \varphi, z)$$

(3) 在球面坐标系中

$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2 \sin^2\theta) \quad ,V=V(r, \theta, \varphi)$$

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2 \sin^2\theta) - V(r, \theta, \varphi)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \; p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi} \sin^2\theta$$

$$\dot{r} = \frac{p_r}{m}, \quad \dot{\theta} = \frac{p_\theta}{mr^2}, \quad \dot{\varphi} = \frac{p_\varphi}{mr^2 \sin^2\theta}$$

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta} \right) + V(r, \theta, \varphi)$$