高等数学积分表推导过程

(一)含有 ax+b 的积分

1. $\int \frac{dx}{\mathrm{ax+b}} = \frac{1}{a}\int \frac{1}{ax+b}d(ax+b) = \frac{1}{a}\ln|ax+b| + C$

2. $\int (\mathrm{ax+b})^u dx = \frac{1}{a}\int (\mathrm{ax+b})^u d(ax+b) = \frac{(ax+b)^u}{a(u+1)} + C$

3. $\int \frac{x}{ax+b}dx = \frac{1}{a}\int \frac{ax}{ax+b}dx = \frac{1}{a}\int \frac{ax+b-b}{ax+b}dx = \frac{1}{a^2}\int \frac{ax+b}{ax+b}dax - \frac{b}{a^2}\int \frac{d(ax+b)}{ax+b} = \frac{1}{a^2}(ax+b-b\ln|ax+b|) + C$

4. $\int \frac{x^2}{ax+b}dx = \frac{1}{a^2}\int \frac{a^2x^2}{ax+b}dx = \frac{1}{a^2}\int \frac{(ax+b)^2-2abx-b^2}{ax+b}dx = \frac{1}{a^3}\left[\int (ax+b)d(ax+b) - 2b\int d(ax+b) - b^2\int \frac{d(ax+b)}{ax+b}\right]$

$= \frac{1}{a^3}\left[\frac{1}{2}(ax+b)^2 - 2b(ax+b) - b^2\ln|ax+b|\right] + C$

5. $\int \frac{dx}{x(ax+b)} = \frac{-a}{b}\int \left(\frac{1}{ax+b} - \frac{1}{ax}\right)dx = -\frac{1}{b}\left(\ln|ax+b| - \ln|ax|\right) + C_1 = -\frac{1}{b}\ln|ax+b| + \frac{1}{b}\ln|a| + C_1 + \ln|x| = -\frac{1}{b}\ln\left|\frac{ax+b}{x}\right| + C$

6. $\int \frac{dx}{x^2(ax+b)} = \frac{1}{b}\int \left[\frac{1}{x^2} + \frac{a^2}{b(ax+b)} - \frac{a}{bx}\right]dx = \frac{1}{b}\int \frac{1}{x^2}dx + \frac{a^2}{b^2}\int \frac{dx}{ax+b} - \frac{a}{b^2}\int \frac{dx}{x} = -\frac{1}{bx} + \frac{a}{b^2}\ln|ax+b| - \frac{a}{b^2}\ln|x| + C$

$= \frac{1}{bx} + \frac{a}{b^2}\ln\left|\frac{ax+b}{x}\right| + C$

7. $\int \frac{xdx}{(ax+b)^2} = \int \left[\frac{1}{a(ax+b)} - \frac{b}{a(ax+b)^2}\right]dx = \frac{1}{a}\int \frac{dx}{ax+b} - \frac{b}{a}\int \frac{dx}{(ax+b)^2} = \frac{1}{a^2}\ln|ax+b| + \frac{b}{a^2}\cdot\frac{1}{ax+b} + C$

$= \frac{1}{a^2}\left(\ln|ax+b| + \frac{b}{ax+b}\right) + C$

8. $\int \frac{x^2}{(ax+b)^2}dx = \int \frac{\frac{1}{a}(ax+b)^2 - \frac{2bx}{a} - \frac{b^2}{a^2}}{(ax+b)^2}dx = \frac{x}{a^2} - \frac{2b}{a}\int \frac{xdx}{(ax+b)^2} - \frac{b^2}{a^2}\int \frac{dx}{(ax+b)^2} = \frac{1}{a^3}\left(ax - 2b\ln|ax+b| - \frac{b^2}{ax+b}\right) + C$

9. $\int \frac{dx}{x(ax+b)^2} = -\int \frac{adx}{b(ax+b)^2} - \frac{a}{b^2}\int \frac{dx}{ax+b} + \frac{1}{b^2}\int \frac{dx}{x} = \frac{1}{b(ax+b)} - \frac{1}{b^2}\ln|ax+b| + \frac{\ln|x|}{b^2} + C = \frac{1}{b(ax+b)} - \frac{1}{b^2}\ln\left|\frac{ax+b}{x}\right| + C$

（二）含有 $\sqrt{ax+b}$ 的积分

10. $\int \sqrt{ax+b}dx = \frac{1}{a}\int \sqrt{ax+b}d(ax+b) = \frac{2}{3a}\sqrt{(ax+b)^3} + C$

11. $\int x\sqrt{ax+b}dx = \frac{1}{a}\int (ax+b)\sqrt{ax+b}dx - \frac{b}{a}\int \sqrt{ax+b}dx = \frac{2}{5a^2}\sqrt{(ax+b)^5} - \frac{2b}{3a^2}\sqrt{(ax+b)^3} + C = \frac{2}{15a^2}(3ax-2b)\sqrt{(ax+b)^3} + C$

12. $\int x^2\sqrt{ax+b}dx = \frac{1}{a^2}\int (ax+b)^2\sqrt{ax+b}dx - \frac{2b}{a}\int x\sqrt{ax+b}dx - \frac{b^2}{a^2}\int \sqrt{ax+b}dx = \frac{2}{7a^3}\sqrt{(ax+b)^7} - \frac{2b}{a}[\frac{2}{15a^2}(3ax-2b)\cdot$

$\sqrt{(ax+b)^3}] - \frac{2b^2}{3a^3}\sqrt{(ax+b)^3} + C = \frac{2}{105a^3}(15a^2x^2 - 12abx + 8b^2)\sqrt{(ax+b)^3} + C$

13. $\int\frac{xdx}{\sqrt{ax+b}}=\frac{1}{a}\int\frac{ax+b}{\sqrt{ax+b}}dx-\frac{b}{a}\int\frac{dx}{\sqrt{ax+b}}=\frac{1}{a}\int\sqrt{ax+b}dx-\frac{b}{a^2}\int\frac{d(ax+b)}{\sqrt{ax+b}}=\frac{2}{3a^2}\sqrt{(ax+b)^3}-\frac{2b}{a^2}\sqrt{ax+b}+C=$

$=\frac{2}{3a^2}(ax-2b)\sqrt{ax+b}+C$

14. $\int\frac{x^2}{\sqrt{ax+b}}dx=\frac{1}{a^2}\int\frac{(ax+b)^2}{\sqrt{ax+b}}dx-\frac{2b}{a}\int\frac{xdx}{\sqrt{ax+b}}-\frac{b^2}{a^2}\int\frac{dx}{\sqrt{ax+b}}=\frac{1}{a^3}\int(ax+b)^{\frac{3}{2}}d(ax+b)-\frac{2b}{a^3}\int\sqrt{ax+b}d(ax+b)+$

$\frac{b^2}{a^2}\int\frac{dx}{\sqrt{ax+b}}=\frac{2}{15a^3}(3a^2x^2-4bx+8b^2)\sqrt{ax+b}+C$

15. $\int\frac{dx}{x\sqrt{ax+b}}$ 当 b>0 时，有

$$\int\frac{dx}{x\sqrt{ax+b}}=\int\frac{a}{2\sqrt{b}\sqrt{ax+b}}\left(\frac{1}{\sqrt{ax+b}-\sqrt{b}}-\frac{1}{\sqrt{ax+b}+\sqrt{b}}\right)d\sqrt{ax+b}=\frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right|+C$$

当 b<0 时，令 ax+b=t, 则 dx= $d\frac{t-b}{a}=\frac{1}{a}dt$

$$\int\frac{dx}{x\sqrt{ax+b}}=\int\frac{\frac{1}{a}dt}{\frac{t-b}{a}\sqrt{t}}=\int\frac{2}{t-b}d\sqrt{t}=-2\int b\frac{1}{1+\left(\sqrt{\frac{t}{b}}\right)^2}d\sqrt{t}=-\frac{2}{b}\int\sqrt{-b}\cdot\frac{1}{1+\left(\sqrt{\frac{t}{b}}\right)^2}d\sqrt{\frac{t}{-b}}=\frac{2}{\sqrt{-b}}\arctan\sqrt{\frac{t}{-b}}+C$$

$$=\frac{2}{\sqrt{-b}}\arctan\sqrt{\frac{ax+b}{-b}}+C$$

所以 $\int\frac{dx}{x\sqrt{ax+b}}=\begin{cases}\frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b-\sqrt{b}}}{\sqrt{ax+b+\sqrt{b}}}\right|+C(b>0)\\ \frac{1}{\sqrt{-b}}\arctan\sqrt{\frac{ax+b}{-b}}+C(b<0)\end{cases}$

16. $\int\frac{dx}{x^2\sqrt{ax+b}}=\int\frac{ax+2b}{2bx^2\sqrt{ax+b}}dx-\frac{a}{2b}\int\frac{dx}{x\sqrt{ax+b}}=\int\frac{\frac{ax+2b}{2\sqrt{ax+b}}}{bx^2}dx-\frac{a}{2b}\int\frac{dx}{x\sqrt{ax+b}}=-\int\frac{\frac{abx}{2\sqrt{ax+b}}-b\sqrt{ax+b}}{b^2x^2}dx-\frac{a}{2b}\int$

$\frac{dx}{x\sqrt{ax+b}}=-\int\frac{u'v-uv'}{v^2}dx-\frac{a}{2b}\int\frac{dx}{x\sqrt{ax+b}}=-\frac{\sqrt{ax+b}}{bx}-\frac{a}{2b}\int\frac{dx}{x\sqrt{ax+b}}$

17. $\int\frac{\sqrt{ax+b}}{x}dx=\int(\frac{a}{\sqrt{ax+b}}+\frac{b}{x\sqrt{ax+b}})dx=\int\frac{a}{\sqrt{ax+b}}dx+b\int\frac{dx}{x\sqrt{ax+b}}=2\sqrt{ax+b}+b\int\frac{dx}{x\sqrt{ax+b}}$

18. $\int\frac{\sqrt{ax+b}}{x^2}dx=\int(\frac{a}{x\sqrt{ax+b}}+\frac{b^2}{x\sqrt{ax+b}})dx=b\int\frac{dx}{x^2\sqrt{ax+b}}+a\int\frac{dx}{x\sqrt{ax+b}}=-b\frac{\sqrt{ax+b}}{bx}-b\cdot a\frac{1}{2b}\int\frac{dx}{x\sqrt{ax+b}}$

$$+a\int\frac{dx}{x\sqrt{ax+b}}=-\frac{\sqrt{ax+b}}{x}+\frac{a}{2}\int\frac{dx}{x\sqrt{ax+b}}$$

（三）含有 $x^2\pm a^2$ 的积分

19. $\int\frac{dx}{x^2+a^2}$ 设 x=atant（$-\frac{\pi}{2}<t<\frac{\pi}{2}$）那么 $x^2+a^2=a^2\sec^2 t$ $dx=a\sec^2 tdt$ 于是

$$\int\frac{dx}{x^2+a^2}=\int\frac{a\sec^2 t}{a^2\sec^2 t}dt=\frac{1}{a}\int dt+C=\frac{1}{a}\arctan\frac{x}{a}+C$$

20. $\int\frac{dx}{\left(x^2+a^2\right)^n}$ 用分部积分法，n>1 时有

$$\int\frac{dx}{\left(x^2+a^2\right)^n}=\frac{x}{\left(x^2+a^2\right)^{n-1}}+2(n-1)\int\frac{x^2}{\left(x^2+a^2\right)^n}dx=\frac{x}{\left(x^2+a^2\right)^{n-1}}+2(n-1)\int\left[\frac{1}{\left(x^2+a^2\right)^{n-1}}-\frac{a^2}{\left(x^2+a^2\right)^n}\right]dx$$

即 $I_{n-1}=\frac{x}{\left(x^2+a^2\right)^{n-1}}+2(n-1)\left(I_{n-1}-a^2I_n\right)$ 于是 $I_n=\frac{1}{2a^2(n-1)}\left[\frac{x}{\left(x^2+a^2\right)^{n-1}}+(2n-3)I_{n-1}\right]$

由此作递推公式并由 $I_1=\frac{1}{a}\arctan\frac{x}{a}+C$ 即可得 I_n $\therefore\int\frac{dx}{\left(x^2+a^2\right)^n}=\frac{1}{2(n-1)a^2\left(x^2+a^2\right)^{n-1}}+\frac{2n-3}{2(n-1)a^2}\int\frac{dx}{\left(x^2+a^2\right)^{n-1}}$

21. $\int\frac{dx}{x^2-a^2}=\int\frac{1}{x+a}\cdot\frac{1}{x-a}dx=\frac{1}{2a}\int\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|=C$

(四)含有 $ax^2+b(a>0)$ 的积分

22. $\int\frac{dx}{ax^2+b}=\frac{1}{b}\int\frac{dx}{1+\frac{a}{b}x^2}=\frac{1}{b}\sqrt{\frac{a}{b}}\int\frac{d\sqrt{\frac{a}{b}}x}{1+\left(\sqrt{\frac{a}{b}}x\right)^2}=\frac{1}{\sqrt{ab}}\arctan\sqrt{\frac{a}{b}}x+C(b>0)$

$$\int\frac{dx}{ax^2+b}=\int\frac{dx}{\left(\sqrt{a}x+\sqrt{-b}\right)\left(\sqrt{a}x-\sqrt{-b}\right)}=\frac{1}{2\sqrt{-ab}}\int\left(\frac{1}{\sqrt{a}x-\sqrt{-b}}-\frac{1}{\sqrt{a}x+\sqrt{-b}}\right)dx=\frac{1}{2\sqrt{-ab}}\ln\left|\frac{\sqrt{a}x-\sqrt{-b}}{\sqrt{a}x+\sqrt{-b}}\right|+C(b<0)$$

23. $\int\frac{xdx}{ax^2+b}=\frac{1}{2}\int\frac{dx^2}{ax^2+b}=\frac{1}{2}\ln\left|ax^2+b\right|+C$

24. $\int\frac{1}{\left(ax^2+b\right)x}dx=\int\left[\frac{1}{bx}-\frac{ax}{b\left(ax^2+b\right)}\right]dx=\frac{1}{b}\ln|x|-\frac{a}{b}\cdot\frac{1}{2a}\ln\left|ax^2+b\right|+C=\frac{1}{2b}\ln\frac{x^2}{\left|ax^2+b\right|}+C$

25. $\int\frac{x^2dx}{ax^2+b}=\frac{1}{a}\int\frac{ax^2+b}{ax^2+b}dx-\frac{b}{a}\int\frac{1}{ax^2+b}dx=\frac{x}{a}-\frac{b}{a}\int\frac{dx}{ax^2+b}$

26. $\int\frac{dx}{x^2\left(ax^2+b\right)}=\int\frac{dx}{bx^2}-\frac{a}{b}\int\frac{dx}{ax^2+b}=-\frac{1}{bx}-\frac{a}{b}\int\frac{dx}{ax^2+b}+C$

27. $\int\frac{dx}{x^3(ax^2+b)}=\int\left[\frac{1}{bx^3}+\frac{a^2x}{b^2(ax^2+b)}-\frac{a}{b^2x}\right]dx=-\frac{1}{2bx^2}-\frac{a}{b^2}\ln|x|+\frac{a^2}{b^2}\cdot\frac{1}{2a}\ln\left|ax^2+b\right|+C=-\frac{1}{2bx^2}-\frac{a}{2b^2}\ln x^2+\frac{a}{2b^2}\ln$

$\left|ax^2+b\right|+C=\frac{a}{2b^2}\ln\frac{\left|ax^2+b\right|}{x^2}-\frac{1}{2bx^2}+C$

28. $\int\frac{dx}{(ax^2+b)^2}=\int\left[\frac{b-ax^2}{2b(ax^2+b)^2}+\frac{1}{2b(ax^2+b)}\right]dx=\frac{1}{2b}\int\frac{b-ax^2}{(ax^2+b)^2}dx+\frac{1}{2b}\int\frac{dx}{ax^2+b}=\frac{1}{2b}\int\frac{u'(ax^2+b)-2axu}{(ax^2+b)^2}dx+\frac{1}{2b}\int\frac{dx}{ax^2+b}$

$au'x^2+bu'-2axu=b-ax^2\qquad u'b=b\qquad u'=1\qquad u=x$

$\int\frac{dx}{(ax^2+b)^2}=\int\frac{x'(ax^2+b)-2ax\cdot x}{(ax^2+b)^2}dx+\frac{1}{2b}\int\frac{dx}{ax^2+b}=\frac{1}{2b}\cdot\frac{x}{ax^2+b}+\frac{1}{2b}\int\frac{dx}{ax^2+b}$

(五)含有 $ax^2+bx+c(a>0)$ 的积分

29. $\int\frac{dx}{ax^2+bx+c}=\int\left[a\left(x+\frac{a}{2b}\right)^2-\frac{b^2-4ac}{4a}\right]^{-1}dx$ 　　当 $b^2<4ac$ 时，有

$\int\frac{dx}{ax^2+bx+c}=\int\left[a\left(x+\frac{a}{2b}\right)^2-\frac{b^2-4ac}{4a}\right]^{-1}dx=\int\frac{\frac{4a}{4ac-b^2}dx}{\left[\frac{2a\left(x+\frac{b}{2a}\right)}{\sqrt{4ac-b^2}}\right]^2+1}$ 　　令 $t=\frac{2a\left(x+\frac{b}{2a}\right)}{\sqrt{4ac-b^2}}$ 　则

$dt=\frac{2a}{\sqrt{4ac-b^2}}dx$ 则原式$=\frac{4a}{4ac-b^2}\cdot\frac{\sqrt{4ac-b^2}}{2a}\int\frac{dt}{1+t^2}=\frac{1}{\sqrt{4ac-b^2}}\arctan t+C=\frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2a\left(x+\frac{b}{2a}\right)}{\sqrt{4ac-b^2}}+C$

当 $b^2>4ac$ 时有 $\int\frac{dx}{ax^2+bx+c}=\int\frac{-\frac{4a}{4ac-b^2}}{1-\frac{4a^2\left(x+\frac{b}{2a}\right)^2}{b^2-4ac}}dx$ 　令 $t=\frac{2a\left(x+\frac{b}{2a}\right)}{\sqrt{b^2-4ac}}$ 　则 $dt=\frac{2a}{\sqrt{b^2-4ac}}dx$

则原式$=\frac{4a}{b^2-4ac}\frac{\sqrt{b^2-4ac}}{2a}\int\frac{dt}{1-t^2}=\frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|+C$

综上所述 $\int\frac{dx}{ax^2+bx+c}=\begin{cases}\frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}}+C\left(b^2<4ac\right)\\ \frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|+C(b^2>4ac)\end{cases}$

30. $\int \frac{xdx}{ax^2+bx+c} = \int \frac{2ax+b}{2a\left(ax^2+bx+c\right)}dx - \int \frac{bdx}{2a\left(ax^2+bx+c\right)} = \frac{1}{2a}\int \frac{2ax+b}{ax^2+bx+c}dx - \frac{b}{2a}\int \frac{dx}{ax^2+bx+c} = \frac{1}{2a}\ln\left|ax^2+bx+c\right|$

$-\frac{b}{2a}\int \frac{dx}{ax^2+bx+c}$　　其中 $\left\{\left[\ln\left(ax^2+bx+c\right)\right]' = \frac{\left(ax^2+bx+c\right)'}{\left(ax^2+bx+c\right)} = \frac{2ax+b}{ax^2+bx+c}\right\}$

（六）含有 $\sqrt{x^2+a^2}\left(a>0\right)$ 的积分

31. $\int \frac{dx}{\sqrt{x^2+a^2}}$ 由于 $1+\tan^2 t = \sec^2 t$，不妨设 $x = a\tan t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，那么 $\sqrt{x^2+a^2} = a\sec t$，$dx = a\sec^2 tdt$

于是 $\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2 t}{a\sec t}dt = \int \sec tdt$，利用例 17 的结果得 $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|\sec t+\tan t\right|+C$

作图可知 $\tan t = \frac{x}{a}$，$\sec t = \frac{\sqrt{x^2+a^2}}{a}$，且 $\sec t+\tan t>0$，因此 $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left[\frac{x}{a}+\frac{\sqrt{x^2+a^2}}{a}\right]+C_1 = \ln\left(x+\sqrt{x^2+a^2}\right)+C$

32. $\int \frac{dx}{\sqrt{\left(x^2+a^2\right)^3}}$　　设 $x = a\tan t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，那么 $\sqrt{x^2+a^2} = a\sec t$，$dx = a\sec^2 tdt$，于是

$\int \frac{dx}{\sqrt{\left(x^2+a^2\right)^3}} = \int \frac{a\sec^2 t}{a^3\sec^3 t}dt = \frac{1}{a^2}\int \frac{1}{\sec t}dt = -\frac{1}{a^2}\sin t+C$　　$\tan t = \frac{x}{a}$，$\frac{1}{\sec t} = \frac{a}{\sqrt{a^2+x^2}} = \cos t$，

$\sin t = \tan t\cdot\cos t = \frac{x}{\sqrt{x^2+a^2}}$，$\int \frac{dx}{\sqrt{\left(x^2+a^2\right)^3}} = -\frac{x}{a^2\sqrt{x^2+a^2}}+C$

33. $\int \frac{xdx}{\sqrt{x^2+a^2}}$，不妨设 $x = a\tan t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，那么 $\sqrt{x^2+a^2} = a\sec t$，$dx = a\sec^2 tdt$，于是

$\int \frac{xdx}{\sqrt{x^2+a^2}} = \int \frac{a\tan t}{a\sec t}a\sec^2 tdt = a\int \sec t\tan tdt = a\sec t+C = \sqrt{x^2+a^2}+C$

34. $\int \frac{xdx}{\sqrt{\left(x^2+a^2\right)^3}}$　设 $x = a\tan t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，那么 $\sqrt{\left(x^2+a^2\right)^3} = a^3\sec^3 t$，$dx = a\sec^2 tdt$，于是

$\int \frac{xdx}{\sqrt{\left(x^2+a^2\right)^3}} = \int \frac{a\tan t}{a^3\sec^3 t}a\sec^2 tdt = \frac{1}{a}\int \frac{\tan t}{\sec t}dt = -\frac{1}{a}\cos t+C = -\frac{1}{\sqrt{x^2+a^2}}+C$

35. $\int \frac{x^2dx}{\sqrt{x^2+a^2}} = \int \sqrt{x^2+a^2}dx - a^2\int \frac{dx}{\sqrt{x^2+a^2}} = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})-a^2\ln(x+\sqrt{x^2+a^2})+C = \frac{x}{2}\sqrt{x^2+a^2}-$

$\frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)+C$

36. $\int \frac{x^2dx}{\sqrt{(x^2+a^2)^3}} = \int \frac{x^2+a^2-a^2}{\sqrt{(x^2+a^2)^3}}dx = \int \frac{dx}{\sqrt{x^2+a^2}} - a^2\int \frac{dx}{\sqrt{(x^2+a^2)^3}} = \ln\left(x+\sqrt{x^2+a^2}\right) - \frac{x}{\sqrt{x^2+a^2}} + C$

37. $\int \frac{dx}{x\sqrt{x^2+a^2}}$　设 $x = a\tan t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，那么 $\sqrt{(x^2+a^2)^3} = a^3\sec^3 t$，$dx = a\sec^2 tdt$，于是

$$\int \frac{dx}{x\sqrt{x^2+a^2}} = \int \frac{a\sec^2 tdt}{a\sec t\cdot a\tan t} = \frac{1}{a}\int \frac{dt}{\sin t} = \frac{1}{a}\ln\left|\csc t-\cot t\right| + C = \frac{1}{a}\ln\frac{\sqrt{x^2+a^2}-a}{|x|} + C$$

38. $\int \frac{dx}{x^2\sqrt{x^2+a^2}}$　设 $x = a\tan t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，那么 $\sqrt{(x^2+a^2)^3} = a^3\sec^3 t$，$dx = a\sec^2 tdt$，于是

$$\int \frac{dx}{x^2\sqrt{x^2+a^2}} = \int \frac{a\sec^2 tdt}{a\tan^2 t\cdot a\sec t} = \frac{1}{a^2}\int \frac{\cos t}{\sin^2 t}dt = -\frac{1}{a^2\sin t} + C = -\frac{\sqrt{a^2+x^2}}{a^2x} + C$$

39. $\int \sqrt{x^2+a^2}dx$　设 $x = a\tan t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，那么 $\sqrt{(x^2+a^2)^3} = a^3\sec^3 t$，$dx = a\sec^2 tdt$，于是

$$\int \sqrt{x^2+a^2}dx = \int a\sec t\cdot a\sec^2 tdt = a^2\int \sec^3 tdt = a^2\int \sec td\tan t = a^2\sec t\cdot\tan t - a^2\int \sec t\cdot\tan^2 tdt = a^2\sec t\cdot\tan t -$$

$$a^2\int \sec t\left(\sec^2 t-1\right)dt = a^2\sec t\cdot\tan t - a^2\int \sec^3 tdt + a^2\ln\left|\sec t+\tan t\right| + C_1$$

$$\therefore \int \sqrt{x^2+a^2}dx = \frac{a^2}{2}\left(\sec t+\tan t+\ln\left|\sec t+\tan t\right|\right) + C = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + C$$

40. $\int \sqrt{(x^2+a^2)^3}dx$　设 $x = a\tan t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，那么 $\sqrt{(x^2+a^2)^3} = a^3\sec^3 t$，$dx = a\sec^2 tdt$，于是

$$\int \sqrt{(x^2+a^2)^3}dx = a^4\int \sec^5 tdt \qquad \int \sec^5 tdt = \int \sec^3 td\tan t = \sec^3 t\tan t - 3\int \tan t\sec^2 t\cdot\sec t\tan tdt$$

$$= \sec^3 t\tan t - 3\int \sec^3 t\tan^2 tdt = \sec^3 t\tan t - 3\int \sec^5 tdt + 3\int \sec^3 tdt$$

$$\because \int \sec^3 tdt = \frac{1}{2}\left(\sec t+\tan t+\ln\left|\sec t+\tan t\right|\right) + C_1$$

$$\therefore \int \sec^5 tdt = \sec^3 t\tan t - 3\int \sec^5 tdt + \frac{3}{2}\left(\sec t+\tan t+\ln\left|\sec t+\tan t\right|\right) + C_1$$

$$\therefore \int \sqrt{(x^2+a^2)}dx = a^4\int \sec^5 tdt = \frac{a^4}{4}\sec^3 t\tan t + \frac{3a^4}{8}\left(\sec t\tan t+\ln\left|\sec t+\tan t\right|\right) + C_1 = \frac{a^4}{4}\frac{\sqrt{(x^2+a^2)^3}}{a^3}\cdot\frac{x}{a}$$

$$+ \frac{3a^4}{8}\left(\frac{x\sqrt{a^2+x^2}}{a^2} + \ln\frac{x+\sqrt{x^2+a^2}}{a}\right) + C_1 = \frac{x}{8}\left(2x^2+5a^2\right)\sqrt{x^2+a^2} + \frac{3}{8}a^4\ln\left(x+\sqrt{x^2+a^2}\right) + C$$

41. $\int x\sqrt{x^2+a^2}dx$　设 $x = a\tan t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，那么 $\sqrt{(x^2+a^2)^3} = a^3\sec^3 t$，$dx = a\sec^2 tdt$，于是

$$\int x\sqrt{x^2+a^2}dx=\int a\tan t\cdot a\sec t\cdot a\sec^2 tdt=a^3\int \tan t\sec^3 tdt=-a^3\int\frac{d\cos t}{\cos^4 t}=\frac{a^3}{3\cos^3 t}+C=\frac{1}{3}\sqrt{\left(x^2+a^2\right)^3}+C$$

42. $\int x^2\sqrt{x^2+a^2}dx=\int\left[\left(x^2+a^2\right)\sqrt{x^2+a^2}-a^2\sqrt{x^2+a^2}\right]dx=\int\sqrt{\left(x^2+a^2\right)^3}dx-a^2\int\sqrt{x^2+a^2}dx=\frac{x}{8}\left(2x^2+5a^2\right)\sqrt{x^2+a^2}$

$$+\frac{3}{8}a^4\ln\left(x+\sqrt{x^2+a^2}\right)-a^2\left[\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)\right]+C=\frac{x}{8}\left(2x^2+a^2\right)\sqrt{x^2+a^2}-\frac{a^4}{8}\ln\left(x+\sqrt{x^2+a^2}\right)+C$$

43. $\int\frac{\sqrt{x^2+a^2}}{x}dx$　设 $x=a\tan t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，那么 $\sqrt{\left(x^2+a^2\right)^3}=a^3\sec^3 t$，$dx=a\sec^2 tdt$，于是

$$\int\frac{\sqrt{x^2+a^2}}{x}dx=\int\frac{a\sec t}{a\tan t}a\sec^2 tdt=a\int\frac{dt}{\sin t\cos^2 t}=a\int\frac{\sin t}{\cos^2 t}dt+a\int\frac{dt}{\sin t}=\frac{a}{\cos t}+a\ln\left|\csc t-\cot t\right|+C=\sqrt{a^2+x^2}+a\ln\frac{\sqrt{x^2+a^2}-a}{|x|}+C$$

44. $\int\frac{\sqrt{x^2+a^2}}{x^2}dx$　设 $x=a\tan t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，那么 $\sqrt{\left(x^2+a^2\right)^3}=a^3\sec^3 t$，$dx=a\sec^2 tdt$，于是

$$\int\frac{\sqrt{x^2+a^2}}{x^2}dx=\int\frac{a\sec t}{a^2\tan^2 t}a\sec^2 tdt=\int\sec^3 t\cot^2 tdt=\int\frac{dt}{\sin^2 t\cos t}=\int\sec tdt+\int\frac{d\sin t}{\sin^2 t}=\ln\left|\sec t+\tan t\right|-\frac{1}{\sin t}+C$$

$$=\ln\left(x+\sqrt{x^2+a^2}\right)-\frac{\sqrt{x^2+a^2}}{x}+C$$

（七）含有 $\sqrt{x^2-a^2}\left(a>0\right)$的积分

45. $\int\frac{dx}{\sqrt{x^2-a^2}}$　当 $x>a$ 时，设 $x=a\sec t\left(0<t<\frac{\pi}{2}\right)$，那么 $\sqrt{x^2-a^2}=a\sqrt{\sec^2 t-1}=a\tan t$，$dx=a\sec t\tan tdt$，于是

$$\int\frac{dx}{\sqrt{x^2-a^2}}=\int\frac{a\sec t\tan t}{a\tan t}dt=\int\sec tdt=\ln\left(\sec t+\tan t\right)+C=\ln\left(\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right)+C_1=\ln\left(x+\sqrt{x^2-a^2}\right)+C$$

当 $x<-a$ 时，令 $x=-u$，那么 $u>a$，由上段结果有 $\int\frac{dx}{\sqrt{x^2-a^2}}=-\int\frac{du}{\sqrt{u^2-a^2}}=-\ln\left(u+\sqrt{u^2-a^2}\right)+C_1==-\ln\left(-x+\sqrt{x^2-a^2}\right)$

$$+C_1=\ln\frac{1}{\sqrt{x^2-a^2}-x}+C_1=\ln\frac{-x-\sqrt{x^2-a^2}}{a^2}+C_1=\ln\left(-x-\sqrt{x^2-a^2}\right)+C$$

综上所述，$\int\frac{dx}{\sqrt{x^2-a^2}}=\ln\left|x+\sqrt{x^2-a^2}\right|+C$

46. $\int \frac{dx}{\sqrt{\left(x^2-a^2\right)^3}}$，设 $x=a\sec t\left(0<t<\frac{\pi}{2}\right)$，则 $\sqrt{\left(x^2-a^2\right)^3}=a^3\tan^3 t$，$dx=a\sec t\tan tdt$，于是

$$\int \frac{dx}{\sqrt{\left(x^2-a^2\right)^3}}=\int \frac{a\sec t\tan t}{a^3\tan^3 t}dt=\frac{1}{a^2}\int \frac{\sec t}{\tan^2 t}dt=\frac{1}{a^2}\int \frac{\cos t}{\sin^2 t}dt=-\frac{1}{a^2\sin t}+C=-\frac{x}{a^2\sqrt{x^2-a^2}}+C$$

47. $\int \frac{xdx}{\sqrt{x^2-a^2}}$，设 $x=a\sec t\left(0<t<\frac{\pi}{2}\right)$，则 $\sqrt{x^2-a^2}=a\tan t$，$dx=a\sec t\tan tdt$，于是

$$\int \frac{xdx}{\sqrt{x^2-a^2}}=\int \frac{a\sec t}{a\tan t}a\sec t\tan tdt=a\int \sec^2 tdt=a\tan t+C=\sqrt{x^2-a^2}+C$$

48. $\int \frac{xdx}{\sqrt{\left(x^2-a^2\right)^3}}$，设 $x=a\sec t\left(0<t<\frac{\pi}{2}\right)$，则 $\sqrt{x^2-a^2}=a\tan t$，$dx=a\sec t\tan tdt$，于是

$$\int \frac{xdx}{\sqrt{\left(x^2-a^2\right)^3}}=\int \frac{a\sec t}{a^3\tan^3 t}a\sec t\tan tdt=\frac{1}{a}\int \frac{1}{\sin^2 t}dt=-\frac{1}{a}\cot t+C=-\frac{1}{\sqrt{x^2-a^2}}+C$$

49. $\int \frac{x^2dx}{\sqrt{x^2-a^2}}=\int \frac{x^2-a^2}{\sqrt{x^2-a^2}}dx+\int \frac{a^2dx}{\sqrt{x^2-a^2}}=\frac{x^2}{2}\sqrt{x^2-a^2}-\frac{1}{2}a^2\ln\left|x+\sqrt{x^2-a^2}\right|+a^2\ln\left|x+\sqrt{x^2-a^2}\right|+C$

$$=\frac{x^2}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C$$

50. $\int \frac{x^2dx}{\sqrt{\left(x^2-a^2\right)^3}}=\int \frac{x^2-a^2}{\sqrt{\left(x^2-a^2\right)^3}}dx+a^2\int \frac{dx}{\sqrt{\left(x^2-a^2\right)^3}}=\int \frac{dx}{\sqrt{x^2-a^2}}+a^2\int \frac{dx}{\sqrt{\left(x^2-a^2\right)^3}}=\ln\left|x+\sqrt{x^2-a^2}\right|+a^2\left(\frac{-1}{a^2}\frac{x}{\sqrt{x^2-a^2}}\right)$

$$+C=\ln\left|x+\sqrt{x^2-a^2}\right|-\frac{x}{\sqrt{x^2-a^2}}+C$$

51. $\int \frac{dx}{x\sqrt{x^2-a^2}}$，设 $x=a\sec t\left(0<t<\frac{\pi}{2}\right)$，则 $\sqrt{x^2-a^2}=a\tan t$，$dx=a\sec t\tan tdt$，于是，当 $x>0$ 时有

$$\int \frac{dx}{x\sqrt{x^2-a^2}}=\int \frac{a\sec t\tan t}{a\sec t\tan t}dt=\frac{t}{a}+C=\arccos\frac{a}{x}+C$$

当 $x<0$ 时有，$\int \frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\arccos\frac{a}{-x}+C$，综上所述，有 $\int \frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\arccos\frac{a}{|x|}+C$

52. $\int \frac{dx}{x^2\sqrt{x^2-a^2}}$，设 $x=a\sec t\left(0<t<\frac{\pi}{2}\right)$，则 $\sqrt{x^2-a^2}=a\tan t$，$dx=a\sec t\tan tdt$，于是

$$\int \frac{dx}{x^2\sqrt{x^2-a^2}}=\int \frac{a\sec t\tan t}{a^2\sec^2 t\cdot a\tan t}dt=\frac{1}{a^2}\int \frac{dt}{\sec t}=\frac{1}{a^2}\sin t+C=\frac{\sqrt{x^2-a^2}}{a^2x}+C$$

53. $\int\sqrt{x^2-a^2}dx$，设 $x=a\sec t\left(0<t<\frac{\pi}{2}\right)$，则 $\sqrt{x^2-a^2}=a\tan t$，$dx=a\sec t\tan tdt$，于是

$$\int\sqrt{x^2-a^2}dx=\int a\tan t\cdot a\sec t\tan tdt=a^2\int\sec t\tan^2 tdt=a^2\int\frac{1-\cos^2 t}{\cos^3 t}dt=\frac{a^2}{2}\sec t\tan t+\frac{a^2}{2}\ln\left|\sec t+\tan t\right|-$$

$$a^2\ln\left|\sec t+\tan t\right|+C=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right)+C$$

54. $\int\sqrt{\left(x^2-a^2\right)^3}dx$，设 $x=a\sec t\left(0<t<\frac{\pi}{2}\right)$，则 $\sqrt{x^2-a^2}=a\tan t$，$dx=a\sec t\tan tdt$，于是

$$\int\sqrt{\left(x^2-a^2\right)^3}dx=\int a^3\tan^3 ta\sec t\tan tdt=a^4\int\tan^4 t\sec tdt$$

$$\because\int\tan^4 t\sec tdt=\int\left(\sec^2 t-1\right)^2\sec tdt=\int\sec^5 tdt-2\int\sec^3 tdt+\int\sec tdt$$

$$\int\sec^5 tdt=\frac{1}{4}\sec^3 t\tan t+\frac{3}{8}\left(\sec t+\tan t+\ln\left|\sec t+\tan t\right|\right)+C_1$$

$$\int\sec^3 tdt=\frac{1}{2}\left(\sec t\tan t+\ln\left|\sec t+\tan t\right|\right)+C_2；\int\sec tdt=\ln\left|\sec t+\tan t\right|+C_3$$

$$\therefore\int\tan^4 t\sec tdt=\int\sec^5 tdt-2\int\sec^3 tdt+\int\sec tdt=\frac{1}{4}\sec^3 t\tan t+\frac{3}{8}\left(\sec t+\tan t+\ln\left|\sec t+\tan t\right|\right)-2\cdot\frac{1}{2}(\sec t\tan t+$$

$$\ln\left|\sec t+\tan t\right|)+\ln\left|\sec t+\tan t\right|+C_1-2C_2+C_3=\frac{1}{4}\cdot\frac{x^3}{a^3}\cdot\frac{\sqrt{x^2-a^2}}{a}-\frac{5}{8}\cdot\frac{x}{a}\cdot\frac{\sqrt{x^2-a^2}}{a}+\frac{3}{8}\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C_1-2C_2+C_3$$

$$\int\sqrt{\left(x^2-a^2\right)^3}dx=\frac{x^3}{4}\sqrt{x^2-a^2}-\frac{5a^2}{8}x\sqrt{x^2-a^2}+\frac{3}{8}a^4\ln\left|x+\sqrt{x^2-a^2}\right|+C=\frac{x}{8}\left(2x^2-5a^2\right)\sqrt{x^2-a^2}+\frac{3}{8}a^4\ln\left|x+\sqrt{x^2-a^2}\right|+C$$

55. $\int x\sqrt{x^2-a^2}dx$，设 $x=a\sec t\left(0<t<\frac{\pi}{2}\right)$，则 $\sqrt{x^2-a^2}=a\tan t$，$dx=a\sec t\tan tdt$，于是

$$\int x\sqrt{x^2-a^2}dx=\int a\sec t\cdot a\tan t\cdot a\sec t\tan tdt=a^3\int\tan^2 t\sec^2 tdt=a^3\int\frac{1-\cos^2 t}{\cos^4 t}dt=a^3\int\frac{dt}{\cos^4 t}-a^3\int\frac{dt}{\cos^2 t}$$

$$=a^3\int\sec^2 td\tan t-a^3\tan t=a^3\int\left(1+\tan^2 t\right)d\tan t-a^3\tan t=\frac{a^3}{3}\tan^3 t+C=\frac{1}{3}\sqrt{\left(x^2-a^2\right)^3}+C$$

56. $$\int x^2\sqrt{x^2-a^2}dx=\int\left(x^2-a^2\right)\sqrt{x^2-a^2}dx+a^2\int\sqrt{x^2-a^2}dx=\frac{x}{8}\left(2x^2-5a^2\right)\sqrt{x^2-a^2}+\frac{3}{8}a^4\ln\left|x+\sqrt{x^2-a^2}\right|+\frac{a^2}{2}x\cdot\sqrt{x^2-a^2}$$

$$-\frac{a^4}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C=\frac{x}{8}\left(2x^2-a^2\right)\sqrt{x^2-a^2}-\frac{a^4}{8}\ln\left|x+\sqrt{x^2-a^2}\right|+C$$

57. $\int\frac{\sqrt{x^2-a^2}}{x}dx$，设 $x=a\sec t\left(0<t<\frac{\pi}{2}\right)$，则 $\sqrt{x^2-a^2}=a\tan t$，$dx=a\sec t\tan tdt$，于是当 $x>0$ 时，有

$$\int \frac{\sqrt{x^2-a^2}}{x}dx=\int \frac{a\tan t}{a\sec t}a\sec t\tan tdt=a\int \tan^2 tdt=a\int\left(\sec^2 t-1\right)dt=a\tan t-t+C=\sqrt{x^2-a^2}-\arccos\frac{a}{x}+C$$

当 $x<0$ 时，有 $\int \frac{\sqrt{x^2-a^2}}{x}dx=\sqrt{(-x)^2-a^2}-\arccos\frac{a}{-x}+C$；综上所述，$\int \frac{\sqrt{x^2-a^2}}{x}dx=\sqrt{x^2-a^2}-\arccos\frac{a}{|x|}+C$

58. $\int \frac{\sqrt{x^2-a^2}}{x^2}dx$，设 $x=a\sec t\left(0<t<\frac{\pi}{2}\right)$，则 $\sqrt{x^2-a^2}=a\tan t$，$dx=a\sec t\tan tdt$，有

$$\int \frac{\sqrt{x^2-a^2}}{x^2}dx=\int \frac{a\tan t}{a^2\sec^2 t}a\sec t\tan tdt=\int \frac{\tan^2 t}{\sec t}dt=\int \tan^2 td\sin t$$，令 $u=\sin t$，则

$$\int \tan^2 td\sin t=\int \frac{u^2}{1-u^2}du=-\int du+\int \frac{1}{1-u^2}du=-u+\frac{1}{2}\int \frac{du}{1-u}+\frac{1}{2}\int \frac{du}{1+u}=-u+\frac{1}{2}\ln\frac{1+u}{1-u}+C=-\sin t+\frac{1}{2}\ln\frac{1+\sin t}{1-\sin t}+C$$

$$=-\frac{\sqrt{x^2-a^2}}{x}+\ln\left(x+\sqrt{x^2-a^2}\right)+C$$

（八）含有 $\sqrt{x^2-a^2}\,(a>0)$ 的积分

59. $\int \frac{dx}{\sqrt{a^2-x^2}}=\frac{1}{a}\int \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}}=\arcsin\frac{x}{a}+C$

60. $\int \frac{dx}{\sqrt{\left(a^2-x^2\right)^3}}$，令 $x=a\sin t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $\sqrt{\left(a^2-x^2\right)^3}=a^3\cos^3 t$，$dx=a\cos tdt$，于是

$$\int \frac{dx}{\sqrt{\left(a^2-x^2\right)^3}}=\int \frac{a\cos t}{a^3\cos^3 t}dt=\int \frac{dt}{\cos^2 t}=\frac{1}{a^2}\tan t+C=\frac{x}{a^2\sqrt{a^2-x^2}}+C$$

61. $\int \frac{dx}{\sqrt{a^2-x^2}}$，令 $x=a\sin t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $\sqrt{a^2-x^2}=a\cos t$，$dx=a\cos tdt$，于是

$$\int \frac{dx}{\sqrt{a^2-x^2}}=\int \frac{a\sin t}{a\cos t}a\cos tdt=-a\int \sin tdt=-a\cos t=-\sqrt{a^2-x^2}+C$$

62. $\int \frac{xdx}{\sqrt{\left(a^2-x^2\right)^3}}$，令 $x=a\sin t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $\sqrt{\left(a^2-x^2\right)^3}=a^3\cos^3 t$，$dx=a\cos tdt$，于是

$$\int \frac{xdx}{\sqrt{\left(a^2-x^2\right)^3}}=\int \frac{a\sin t}{a^3\cos^3 t}a\cos tdt=\frac{1}{a}\int \frac{\sin t}{\cos^2 t}dt=\frac{1}{a\cos t}+C=\frac{1}{\sqrt{a^2-x^2}}+C$$

63. $\int \frac{x^2dx}{\sqrt{a^2-x^2}}$，令 $x=a\sin t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $\sqrt{a^2-x^2}=a\cos t$，$dx=a\cos tdt$，于是

$$\int\frac{x^2dx}{\sqrt{a^2-x^2}}=\int\frac{a^2\sin^2 t}{a\cos t}a\cos tdt=a^2\int\sin^2 tdt=\frac{1}{2}a^2t-\frac{a^2}{4}\sin 2t+C=\frac{a^2}{2}\arcsin\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}+C$$

64. $\int\frac{x^2dx}{\sqrt{(a^2-x^2)^3}}$，令 $x=a\sin t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $\sqrt{(a^2-x^2)^3}=a^3\cos^3 t$， $dx=a\cos tdt$，于是

$$\int\frac{x^2dx}{\sqrt{(a^2-x^2)^3}}=\int\frac{a^2\sin^2 t}{a^3\cos^3 t}a\cos tdt=\int\frac{dt}{\cos^2 t}-\int dt=\tan t-t+C=\frac{x}{\sqrt{a^2-x^2}}-\arcsin\frac{x}{a}+C$$

65. $\int\frac{dx}{x\sqrt{a^2-x^2}}$，令 $x=a\sin t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $\sqrt{a^2-x^2}=a\cos t$， $dx=a\cos tdt$，于是

$$\int\frac{dx}{x\sqrt{a^2-x^2}}=\int\frac{a\cos t}{a^2\sin t\cos t}dt=\frac{1}{a}\ln\left|\csc t-\cot t\right|+C=\frac{1}{a}\ln\left|\frac{x}{a}-\frac{\sqrt{a^2-x^2}}{x}\right|+C=\frac{1}{a}\ln\frac{a-\sqrt{a^2-x^2}}{|x|}+C$$

66. $\int\frac{dx}{x^2\sqrt{a^2-x^2}}$，令 $x=a\sin t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $\sqrt{a^2-x^2}=a\cos t$， $dx=a\cos tdt$，于是

$$\int\frac{dx}{x^2\sqrt{a^2-x^2}}=\int\frac{a\cos tdt}{a^2\sin^2 ta\cos t}=\frac{1}{a^2}\int\frac{dt}{\sin^2 t}=-\frac{1}{a^2}\cot t+C=-\frac{\sqrt{a^2-x^2}}{a^2x}+C$$

67. $\int\sqrt{a^2-x^2}dx$，令 $x=a\sin t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $\sqrt{a^2-x^2}=a\cos t$， $dx=a\cos tdt$，于是

$$\int\sqrt{a^2-x^2}dx=\int a\cos t\cdot a\cos tdt=a^2\int\cos^2 tdt=a^2\int\frac{1+\cos 2t}{2}dt=\frac{a^2}{2}t+\frac{a^2}{4}\sin 2t+C=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C$$

68. $\int\sqrt{(a^2-x^2)^3}dx$，令 $x=a\sin t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $\sqrt{(a^2-x^2)^3}=a^3\cos^3 t$， $dx=a\cos tdt$，于是

$$\int\sqrt{(a^2-x^2)^3}dx=\int a^3\cos^3 t\cdot a\cos tdt=a^4\int\cos^4 tdt=a^4\int\frac{(1+\cos 2t)^2}{4}dt=\frac{a^4t}{4}+\frac{a^4}{2}\int\cos 2tdt+\frac{a^4}{4}\int\cos^2 2tdt=$$

$$\frac{a^4}{4}\arcsin\frac{x}{a}+\frac{a^4}{4}\sin 2t+\frac{a^4t}{8}+\frac{a^4}{32}\sin 4t+C=\frac{3}{8}a^4\arcsin\frac{x}{a}+\frac{a^2}{2}x\sqrt{a^2-x^2}+\frac{x\sqrt{a^2-x^2}(a^2-2x^2)}{8}+C=$$

$$\frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2}+\frac{3}{8}a^4\arcsin\frac{x}{a}+C$$

69. $\int x\sqrt{a^2-x^2}dx$，令 $x=a\sin t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $\sqrt{a^2-x^2}=a\cos t$， $dx=a\cos tdt$，于是

$$\int x\sqrt{a^2-x^2}dx=\int a\sin t\cdot a\cos t\cdot a\cos tdt=-a^3\int\cos^2 td\cos t=-\frac{a^3}{3}\cos^3 t+C=-\frac{1}{3}\sqrt{(a^2-x^2)^3}+C$$

70. $\int x^2\sqrt{a^2-x^2}dx$，令 $x=a\sin t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $\sqrt{a^2-x^2}=a\cos t$， $dx=a\cos tdt$，于是

$$\int x^2\sqrt{a^2-x^2}dx=\int a^2\sin^2 t\cdot a\cos t\cdot a\cos tdt=a^4\int\sin^2 t\cos^2 tdt=a^4\int\sin^2 t\left(1-\sin^2 t\right)dt=a^4\int\sin^2 tdt-a^4\int\sin^4 tdt$$

$$=a^4\int\frac{1-\cos 2t}{2}dt-a^4\int\frac{(1-\cos 2t)^2}{4}dt=\frac{1}{2}a^4t-\frac{a^4}{4}\sin 2t-\frac{1}{4}a^4t+\frac{a^4}{4}\sin 2t-\frac{a^4}{4}\int\cos^2 2tdt=\frac{a^4}{4}t-\frac{a^4}{4}\int\frac{1+\cos 4t}{2}dt$$

$$=\frac{a^4}{8}t-\frac{a^4}{32}\sin 4t+C=\frac{a^4}{8}\arcsin\frac{x}{a}-\frac{x}{8}\left(a^2-2x^2\right)\sqrt{a^2-x^2}+C$$

71. $\int\frac{\sqrt{a^2-x^2}}{x}dx$，令 $x=a\sin t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $\sqrt{a^2-x^2}=a\cos t$， $dx=a\cos tdt$，于是

$$\int\frac{\sqrt{a^2-x^2}}{x}dx=\int\frac{a\cos t}{a\sin t}a\cos tdt=a\int\cot t\cos tdt=a\int\frac{1-\sin^2 t}{\sin t}dt=a\int\frac{dt}{\sin t}-a\int\sin tdt=a\ln\left|\csc t-\cot t\right|+a\cos t+C=$$

$$a\ln\left|\frac{a}{x}-\frac{\sqrt{a^2-x^2}}{x}\right|+a\frac{\sqrt{a^2-x^2}}{a}+C=a\ln\frac{a-\sqrt{a^2-x^2}}{|x|}+\sqrt{a^2-x^2}+C$$

72. $\int\frac{\sqrt{a^2-x^2}}{x^2}dx$，令 $x=a\sin t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $\sqrt{a^2-x^2}=a\cos t$， $dx=a\cos tdt$，于是

$$\int\frac{\sqrt{a^2-x^2}}{x^2}dx=\int\frac{a\cos t}{a^2\sin^2 t}a\cos tdt\int\cot^2 tdt=\int\frac{1-\sin^2 t}{\sin^2 t}dt=\int\frac{dt}{\sin^2 t}-\int dt=-\cot-t+C=-\frac{\sqrt{a^2-x^2}}{x}-\arcsin\frac{x}{a}+C$$

（九）含有 $\sqrt{\pm ax^2+bx+c}\,(a>0)$ 的积分

73. $\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}}}$，令 $x+\frac{b}{2a}=t$，则 $dx=dt$

当 $b^2-4ac>0$ 时，则令 $\frac{b^2-4ac}{4a^2}=u^2(u>0)$，则 $\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}}}=\frac{1}{\sqrt{a}}\int\frac{dt}{\sqrt{t^2-u^2}}$

再令 $t=u\sec r$，$dt=u\tan r\sec rdr$，$\sqrt{t^2-u^2}=u\tan r$，于是

$$\frac{1}{\sqrt{a}}\int\frac{dt}{\sqrt{t^2-u^2}}=\frac{1}{\sqrt{a}}\int\frac{u\tan r\sec r}{u\tan r}dr=\frac{1}{\sqrt{a}}\int\sec rdr=\frac{1}{\sqrt{a}}\ln\left|\sec r+\tan r\right|+C_1$$

$$\sec r=\frac{t}{u}=\frac{x+\frac{b}{2a}}{\sqrt{\frac{b^2-4ac}{4a^2}}}=\frac{2ax+b}{\sqrt{b^2-4ac}}\quad;\quad\tan r=\frac{\sqrt{1-\cos^2 r}}{\cos r}=\frac{\sqrt{t^2-u^2}}{u}=2\sqrt{a}\sqrt{ax^2+bx+c}\,\frac{1}{\sqrt{b^2-4ac}}$$

$$\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\ln\left|\frac{2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}}{\sqrt{b^2-4ac}}\right|+C_1=\frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C$$

当$b^2-4ac<0$时，则令$\frac{\sqrt{4ac-b^2}}{2a}=u$，$t=x+\frac{b}{2a}$，$\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}}}=\frac{1}{\sqrt{a}}\int\frac{dt}{\sqrt{t^2+u^2}}$

令$t=u\tan r$，$dt=u\sec^2 rdr$　　$\frac{1}{\sqrt{a}}\int\frac{dt}{\sqrt{t^2+u^2}}=\frac{1}{\sqrt{a}}\int\frac{u\sec^2 rdr}{u\sec r}=\frac{1}{\sqrt{a}}\int\sec rdr=\frac{1}{\sqrt{a}}\ln\left|\sec r+\tan r\right|+C_1$

$$\frac{1}{\sec r}=\frac{u}{\sqrt{t^2+u^2}}=\frac{\sqrt{\frac{4ac-b^2}{4a^2}}}{\sqrt{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}}=\frac{\sqrt{4ac-b^2}}{2a}\frac{1}{\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}}；\quad\tan r=\frac{t}{u}=\frac{x+\frac{b}{2a}}{\sqrt{4ac-b^2}}\cdot 2a=\frac{2ax+b}{\sqrt{4ac-b^2}}$$

$$\therefore\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\ln\left|\frac{2\sqrt{a}\sqrt{ax^2+bx+c}}{\sqrt{4ac-b^2}}+\frac{2ax+b}{\sqrt{4ac-b^2}}\right|+C_1=\frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C$$

综上所述，$\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C$

74. $\int\sqrt{ax^2+bx+c}dx=\int\sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}dx$　　当$b^2-4ac>0$时，令$t=x+\frac{b}{2a}$，$\frac{\sqrt{b^2-4ac}}{2a}=u$

$\int\sqrt{ax^2+bx+c}dx=\sqrt{a}\int\sqrt{t^2-u^2}dt$　，再令$t=u\sec r$，$dt=u\sec r\tan rdr$

$$\sqrt{a}\int\sqrt{t^2-u^2}dt=\sqrt{a}\cdot u^2\int\tan^2 r\sec rdr=\sqrt{a}\cdot u^2\frac{1}{2}\left(\sec r\tan r+\ln\left|\sec r+\tan r\right|\right)-\sqrt{a}\cdot u^2\ln\left|\sec r\tan r\right|+C$$

$$=\frac{1}{2}\sqrt{a}\cdot u^2\sec r\tan r-\frac{1}{2}\sqrt{a}\cdot u^2\ln\left|\sec r+\tan r\right|+C_1=\frac{1}{2}\sqrt{a}\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2\frac{2ax+b}{\sqrt{b^2-4ac}}\cdot 2\sqrt{a}\sqrt{ax^2+bx+c}\cdot\frac{1}{\sqrt{b^2-4ac}}-$$

$$\frac{\sqrt{a}}{2}\frac{b^2-4ac}{4a^2}\ln\left|\frac{2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}}{\sqrt{b^2-4ac}}\right|+C_1=\frac{2ax+b}{4a}\sqrt{ax^2+bx+c}+\frac{4ac-b^2}{8\sqrt{a^3}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C$$

当$b^2-4ac<0$时，令$t=x+\frac{b}{2a}$，$\frac{\sqrt{4ac-b^2}}{2a}=u$；$\int\sqrt{ax^2+bx+c}dx=\sqrt{a}\int\sqrt{t^2+u^2}dt$；再令$t=u\tan r$，

$dt=u\sec^2 rdr$，于是

$$\int\sqrt{ax^2+bx+c}dx=\sqrt{a}\int u^2\sec^3 rdr=\frac{1}{2}\sqrt{a}\cdot u^2\left(\sec r\tan r+\ln\left|\sec r+\tan r\right|\right)=\frac{\sqrt{a}}{2}\cdot\frac{4ac-b^2}{4a^2}\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}\cdot\frac{2a}{\sqrt{4ac-b^2}}\times$$

$$\frac{2ax+b}{\sqrt{4ac-b^2}}+\frac{1}{2}\sqrt{a}\cdot\frac{4ac-b^2}{4a^2}\ln\left|\frac{2\sqrt{a}\sqrt{ax^2+bx+c}}{\sqrt{4ac-b^2}}+\frac{2ax+b}{\sqrt{4ac-b^2}}\right|+C_1=\frac{2ax+b}{4a}\sqrt{ax^2+bx+c}+\frac{4ac-b^2}{8\sqrt{a^3}}\ln$$

$$\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C$$

综上所述，$\int\sqrt{ax^2+bx+c}dx=\frac{2ax+b}{4a}\sqrt{ax^2+bx+c}+\frac{4ac-b^2}{8\sqrt{a^3}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C$

75. $\int\frac{xdx}{\sqrt{ax^2+bx+c}}=\int\frac{xdx}{\sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}}$；令 $t=x+\frac{b}{2a}$，当 $\Delta>0$ 时，令 $u=\sqrt{\frac{4ac-b^2}{4a^2}}$

$$\int\frac{xdx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\int\frac{t-\frac{b}{2a}}{\sqrt{t^2-u^2}}dt=\frac{1}{\sqrt{a}}\int\frac{t}{\sqrt{t^2-u^2}}dt-\frac{b}{2a}\int\frac{dt}{\sqrt{a}\sqrt{t^2-u^2}}=\sqrt{\frac{t^2-u^2}{a}}-\frac{b}{2a\sqrt{a}}\ln\left|t+\sqrt{t^2-u^2}\right|+C_1$$

$$=\frac{1}{\sqrt{a}}\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}-\frac{b}{2a\sqrt{a}}\ln\left|x+\frac{b}{2a}+\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}\right|+C_1=\frac{\sqrt{ax^2+bx+c}}{a}-\frac{b}{2a\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C$$

当 $\Delta<0$ 时，令 $u=\frac{\sqrt{4ac-b^2}}{2a}$，于是 $\int\frac{xdx}{\sqrt{ax^2+bx+c}}=\int\frac{t-\frac{b}{2a}}{\sqrt{a}\sqrt{t^2+u^2}}dt=\int\frac{tdt}{\sqrt{a}\sqrt{t^2+u^2}}-\frac{b}{2a\sqrt{a}}$

$$\int\frac{dt}{\sqrt{t^2+u^2}}=\frac{1}{\sqrt{a}}\sqrt{t^2+u^2}-\frac{b}{2a\sqrt{a}}\ln\left|t+\sqrt{t^2+u^2}\right|+C_1=\sqrt{\frac{x^2+\frac{b}{a}x+\frac{c}{a}}{a}}-\frac{b}{2a\sqrt{a}}\ln\left|x+\frac{b}{2a}+\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}\right|+C_1$$

$$=\frac{\sqrt{ax^2+bx+c}}{a}-\frac{1}{2a\sqrt{a}}\ln\left|2ax+b+\sqrt{ax^2+bx+c}\right|+C$$

综上所述，$\int\frac{xdx}{\sqrt{ax^2+bx+c}}=\frac{\sqrt{ax^2+bx+c}}{a}-\frac{b}{2a\sqrt{a}}\ln\left|2ax+b+\sqrt{ax^2+bx+c}\right|+C$

76. $\int\frac{dx}{\sqrt{c+bx-ax^2}}=\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{\frac{b^2-4ac}{4a^2}-\left(x-\frac{b}{2a}\right)^2}}$；令 $t=x-\frac{b}{2a}$，$u=\sqrt{\frac{4ac+b^2}{4a^2}}$，于是

$$\int\frac{dx}{\sqrt{c+bx-ax^2}}=\frac{1}{\sqrt{a}}\int\frac{dt}{\sqrt{u^2-t^2}}=-\frac{1}{\sqrt{a}}\arcsin\frac{u}{t}+C=\frac{1}{\sqrt{a}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C$$

77. $\int\sqrt{c+bx-ax^2}dx=\sqrt{a}\int\sqrt{\frac{b^2+4ac}{4a^2}-\left(x-\frac{b}{2a}\right)^2}$；令 $x-\frac{b}{2a}=t$；$u=\sqrt{\frac{b^2+4ac}{4a^2}}$；于是

$$\int\sqrt{c+bx-ax^2}dx=\sqrt{a}\int\sqrt{u^2-t^2}dt=\frac{t}{2\sqrt{a}}\sqrt{u^2-t^2}+\frac{u^2}{2\sqrt{a}}\arcsin\frac{t}{u}+C=\frac{1}{2}\left(x-\frac{b}{2a}\right)\sqrt{\frac{c}{a}+\frac{b}{a}x-x^2}\cdot\frac{1}{\sqrt{a}}+$$

$$\frac{b^2+4ac}{8a\sqrt{a}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C=\frac{2ax-b}{4a}\sqrt{c+bx-ax^2}+\frac{b^2+4ac}{8a\sqrt{a}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C$$

78. $\int\frac{xdx}{\sqrt{c+bx-ax^2}}=\frac{1}{\sqrt{a}}\int\frac{xdx}{\sqrt{\frac{b^2+4ac}{4a^2}-\left(x-\frac{b}{2a}\right)^2}}$；令 $t=x-\frac{b}{2a}$；$u=\sqrt{\frac{4ac+b^2}{4a^2}}$；于是

$$\int\frac{xdx}{\sqrt{c+bx-ax^2}}=\frac{1}{\sqrt{a}}\int\frac{t+\frac{b}{2a}}{\sqrt{u^2-t^2}}dt=\frac{1}{\sqrt{a}}\int\frac{t}{\sqrt{u^2-t^2}}dt+\frac{b}{2a\sqrt{a}}\int\frac{1}{\sqrt{u^2-t^2}}dt=-\frac{1}{\sqrt{a}}\sqrt{u^2-t^2}+\frac{b}{2a\sqrt{a}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C$$

$$=-\frac{1}{\sqrt{a}}\sqrt{\frac{c}{a}+\frac{b}{a}x-x^2}+\frac{b}{2a\sqrt{a}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C=-\frac{1}{a}\sqrt{c+bx-x^2}+\frac{b}{2a\sqrt{a}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C$$

（十）含有 $\sqrt{\pm\frac{x-a}{x-b}}$ 或 $\sqrt{(x-a)(b-x)}$ 的积分

79. $\int\sqrt{\frac{x-a}{x-b}}dx$；令 $t=\frac{x-a}{x-b}$；则 $x=\frac{bt-a}{t-1}$；$dx=\frac{a-b}{(t-1)^2}dt$；于是

$$\int\sqrt{\frac{x-a}{x-b}}dx=\int\sqrt{t}\frac{a-b}{(t-1)^2}dt=(a-b)\int\frac{\sqrt{t}}{(t-1)^2}dt=(b-a)\int\sqrt{t}d\frac{1}{t-1}=(b-a)\frac{\sqrt{t}}{t-1}+(a-b)\int\frac{1}{t-1}d\sqrt{t}$$

$$=(b-a)\frac{\sqrt{t}}{t-1}+\frac{a-b}{2}\int\left(\frac{1}{\sqrt{t}-1}-\frac{1}{\sqrt{t}+1}\right)d\sqrt{t}=(b-a)\frac{\sqrt{t}}{t-1}+\frac{b-a}{2}\int\frac{d\left(\sqrt{t}+1\right)}{\sqrt{t}+1}-\frac{b-a}{2}\int\frac{d\left(\sqrt{t}-1\right)}{\sqrt{t}-1}=(b-a)\frac{\sqrt{t}}{t-1}$$

$$+\frac{b-a}{2}\ln\frac{\sqrt{t}+1}{\sqrt{t}-1}+C_1=(b-a)\sqrt{\frac{x-a}{x-b}}\cdot\frac{x-a}{x-b}+\frac{b-a}{2}\ln\left[(b-a)\left(\sqrt{|x-a|}+\sqrt{|x-b|}\right)^2\right]+C_1=(x-b)\sqrt{\frac{x-a}{x-b}}+(b-a)\ln(\sqrt{|x-a|}$$

$$+\sqrt{|x-b|})+C_1=(x-b)\sqrt{\frac{x-a}{x-b}}+(b-a)\ln\left(\sqrt{|x-a|}+\sqrt{|x-b|}\right)+C$$

80. $\int\sqrt{\frac{x-a}{b-x}}dx$；令 $\sqrt{\frac{x-a}{b-x}}=t$；则 $x=\frac{bt+a}{1+t}$；$dx=\frac{b-a}{(1+t)^2}dt$；于是

$$\int\sqrt{\frac{x-a}{b-x}}dx=(b-a)\int\frac{\sqrt{t}}{(1+t)^2}dt=(a-b)\int\sqrt{t}d\frac{1}{1+t}=(a-b)\frac{\sqrt{t}}{1+t}-(a-b)\int\frac{d\sqrt{t}}{1+t}=(a-b)\frac{\sqrt{t}}{1+t}-(a-b)\arcsin\sqrt{t}+C$$

$$=(b-x)\sqrt{\frac{x-a}{b-x}}+(b-a)\arctan\sqrt{\frac{x-a}{b-x}}+C$$

81. $\int\frac{dx}{\sqrt{(x-a)(b-x)}}$；令 $x-a=t$；则 $x=a+t$； $dx=dt$；于是 $\int\frac{dx}{\sqrt{(x-a)(b-x)}}=\int\frac{dt}{\sqrt{t(b-a-t)}}=\int\frac{dt}{\sqrt{-t^2+(b-a)t}}$

$=\int\frac{dt}{\sqrt{\frac{(a-b)^2}{4}-\left(t+\frac{a-b}{2}\right)^2}}$；令 $\frac{b-a}{2}=u>0$； $r=t+\frac{a-b}{2}$ 则

$$\int\frac{dt}{\sqrt{\frac{(a-b)^2}{4}-\left(t+\frac{a-b}{2}\right)^2}}=\int\frac{dr}{\sqrt{u^2-r^2}}=\arcsin\frac{2x-a-b}{b-a}+C$$

82. $\int\sqrt{(x-a)(b-x)}dx$；令 $x-a=t$；则 $x=a+t$； $dx=dt$；于是 $\int\sqrt{(x-a)(b-x)}dx=\int\sqrt{\frac{(a-b)^2}{4}-\left(t+\frac{a-b}{2}\right)^2}dt$

令 $\frac{b-a}{2}=u>0$； $r=t+\frac{a-b}{2}$；则 $\int\sqrt{(x-a)(b-x)}dx=\int\sqrt{u^2-r^2}dr=\frac{r}{2}\sqrt{u^2-r^2}+\frac{u^2}{2}\arcsin\frac{r}{u}+C=\frac{2x-a-b}{4}$

$\sqrt{(x-a)(b-x)}dx+\frac{(b-a)^2}{8}\arcsin\frac{2x-a-b}{b-a}+C$

（十一）含有三角函数的积分

83. $\int\sin xdx=-\cos x+C$　84. $\int\cos xdx=\sin x+C$　85. $\int\tan xdx=\int\frac{\sin x}{\cos x}dx=-\ln|\cos x|+C$

86. $\int\cot xdx=\int\frac{\cos x}{\sin x}dx=\ln|\sin x|+C$

87. $\int\sec xdx=\int\frac{dx}{\cos x}=\int\frac{d\left(x+\frac{\pi}{2}\right)}{2\sin\left(x+\frac{\pi}{2}\right)\cos\left(x+\frac{\pi}{2}\right)}=\int\frac{d\left(\frac{x}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\cos^2\left(\frac{x}{2}+\frac{\pi}{4}\right)}=\int\frac{d\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)}=\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|+C$

$\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\frac{2\sin^2\frac{x}{2}}{\sin x}=\frac{1-\cos x}{\sin x}=\csc x-\cot x$　；　$\int\sec xdx=\ln\left|\csc\left(x+\frac{\pi}{2}\right)-\cot\left(x+\frac{\pi}{2}\right)\right|+C=$

$\ln|\sec x+\tan x|+C$

88. $\int\csc xdx=\int\frac{dx}{\sin x}=\int\frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\int\frac{dx}{\tan\frac{x}{2}\cos^2\frac{x}{2}}=\int\frac{d\tan\frac{x}{2}}{\tan\frac{x}{2}}=\ln\left|\tan\frac{x}{2}\right|+C=\ln|\csc x-\cot x|+C$

89. $\int\sec^2xdx=\tan x+C$　90. $\int\csc^2xdx=-\cot x+C$　91. $\int\sec x\tan xdx=\sec x+C$　92. $\int\csc x\cot xdx=-\csc x+C$

93. $\int \sin^2 xdx = \int \frac{1-\cos 2x}{2}dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$ 94. $\int \cos^2 xdx = \int \frac{1+\cos 2x}{2}dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$

95. $\int \sin^n xdx = -\int \sin^{n-1} xd\cos x = -\cos x\sin^{n-1} x + \int \cos xd\sin^{n-1} xdx = -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\cos^2 xdx$

$= -\cos x\sin^{n-1} x + (n-1)\int (1-\sin^2 x)\sin^{n-2} xdx = -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} xdx - (n-1)\int \sin^n xdx$

$\therefore \int \sin^n xdx = -\frac{1}{n}\cos x\sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} xdx$

96. $\int \cos^n xdx = \int \cos^{n-1} xd\sin x = \sin x\cos^{n-1} x - \int \sin xd\cos^{n-1} x = \sin x\cos^{n-1} x + (n-1)\int \sin^2 x\cos^{n-2} xdx = \sin x\cos^{n-1} x +$

$(n-1)\int (1-\cos^2 x)\cos^{n-2} xdx = \sin x\cos^{n-1} x + (n-1)\int \cos^{n-2} xdx - (n-1)\int \cos^n xdx$

$\therefore \int \cos^n xdx = \frac{1}{n}\sin x\cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} xdx$

97. $\int \frac{dx}{\sin^n x} = \int \sin^{-n} xdx = \frac{1}{n}\int \sin^{-n-1} xdx = \frac{1}{n}\sin^{-n-1} x\cos x + \frac{n+1}{n}\int \sin^{-n-2} xdx$

$\int \frac{dx}{\sin^{n-2} x} = \frac{1}{n-2}\frac{\cos x}{\sin^{n-1}} + \frac{n-1}{n-2}\int \frac{dx}{\sin^n x}$ $\therefore \int \frac{dx}{\sin^n x} = -\frac{1}{n-1}\frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} x}$

98. $\int \frac{dx}{\cos^n x} = \int \cos^{-n} xdx = \int \cos^{-n-1} xd\sin x = \cos^{-n-1} x\sin x - \int \sin xd\cos^{-n-1} x = \cos^{-n-1} x\sin x - (n+1)\int \cos^{-n-2} x\sin^2 xdx =$

$\cos^{-n-1} x\sin x - (n+1)\int \cos^{n-2} xdx + (n+1)\int \cos^{-n} xdx$ $\therefore = \int \frac{dx}{\cos^n x} = -\frac{1}{n}\cos^{-n-1} x\sin x + \frac{n+1}{n}\int \cos^{-n-2} xdx$

将 n 换成 $n-2$ 有 $\int \frac{dx}{\cos^{n-2} x} = -\frac{1}{n-2}\frac{\sin x}{\cos^{n-1} x} + \frac{n-1}{n-2}\int \frac{dx}{\cos^n x}$ $\therefore \int \frac{dx}{\cos^n x} = -\frac{1}{n-1}\frac{\sin x}{\cos^{n-1} x} + \frac{n-1}{n-2}\int \frac{dx}{\cos^{n-2} x}$

99. $\int \cos^m x\sin^n xdx = \int \cos^{m-1} x\sin^n xd\sin x = \frac{1}{n+1}\int \cos^{m-1} xd\sin^{n+1} x = \frac{1}{n+1}\sin^{n+1} x\cos^{m-1} x - \frac{1}{n+1}\int \sin^{n+1} xd\cos^{m-1} x$

$= \frac{1}{n+1}\cos^{m-1} x\sin^{n+1} x + \frac{m-1}{n+1}\int \sin^{n+2} x\cos^{m-2} xdx = \frac{1}{n+1}\cos^{m-1} x\sin^{n+1} x + \frac{m-1}{n+1}\int \sin^n x\cos^{m-2} x(1-\cos^2 x)dx$

$= \frac{1}{n+1}\cos^{m-1} x\sin^{n+1} x + \frac{m-1}{m+1}\int \sin^n x\cos^{m-2} xdx - \frac{m-1}{n+1}\int \sin^n x\cos^n xdx$

$\therefore \left(1+\frac{m-1}{n+1}\right)\int \cos^m x\sin^n xdx = \frac{1}{n+1}\cos^{m-1} x\sin^{n+1} x + \frac{m-1}{n+1}\int \cos^{m-2} x\sin^n xdx$

$\therefore \int \cos^m x\sin^n xdx = \frac{1}{m+1}\cos^{m-1} x\sin^{n+1} x + \frac{m-1}{m+n}\int \cos^{m-2} x\sin^n xdx$

又有 $\int \cos^m x\sin^n xdx = -\int \cos^m x\sin^n xd\cos x = -\frac{1}{m+1}\int \sin^{n-1} xd\cos^{m+1} x = -\frac{1}{m+1}\sin^{n-1} x\cos^{m+1} x + \frac{1}{m+1}\int \cos^{m+1} x$

$d\sin^{n-1} x = -\frac{1}{m+1}\sin^{n-1} x\cos^{m+1} x + \frac{n-1}{m+1}\int \cos^{m+2} x\sin^{n-2} xdx = -\frac{1}{m+1}\sin^{n-1} x\cos^{m+1} x + \frac{n-1}{m+1}\int \cos^m x\sin^{n-2} x(1-\sin^2 x)$

$dx = -\frac{1}{m+1}\sin^{n-1} x\cos^{m+1} x + \frac{n-1}{m+1}\int \cos^m x\sin^{n-2} xdx - \frac{n-1}{m+1}\int \cos^m x\sin^n xdx$

$$\therefore\left(1+\frac{n-1}{m+1}\right)\int\cos^m x\sin^n xdx=-\frac{1}{m+1}\sin^{n-1}x\cos^{m+1}x+\frac{n-1}{m+1}\int\cos^m x\sin^{n-2}xdx=-\frac{1}{m+n}\sin^{n-1}x\cos^{m+1}x+\frac{n-1}{m+1}\times$$

$\int\cos^m x\sin^{n-2}xdx$ ； 因此

$$\int\cos^m x\sin^n xdx=\frac{1}{m+n}\cos^{m-1}x\sin^{n+1}x+\frac{m-1}{m+n}\int\cos^{m-2}x\sin^n xdx=-\frac{1}{m+n}\cos^{m+1}x\sin^{n-1}x+\frac{n-1}{m+n}\int\cos^{m-2}x\sin^n xdx$$

100. $\int\sin ax\cos bxdx=\frac{1}{2}\int[\sin(a-b)x+\sin(a+b)x]dx=-\frac{1}{2(a-b)}\cos(a-b)x-\frac{1}{2(a+b)}\cos(a+b)x+C$

101. $\int\sin ax\sin bxdx=\frac{1}{2}\int[\cos(a+b)x-\cos(a-b)x]dx=\frac{1}{2(a+b)}\sin(a+b)x-\frac{1}{2(a-b)}\sin(a-b)x+C$

102. $\int\cos ax\cos bxdx=\frac{1}{2}\int[\cos(a+b)x+\cos(a-b)x]dx=\frac{1}{2(a+b)}\sin(a+b)x+\frac{1}{2(a-b)}\sin(a-b)x+C$

103.
$$\int\frac{dx}{a+b\sin x}=\int\frac{dx}{a+2\sin\frac{x}{2}\cos\frac{x}{2}}=\int\frac{\left(1+\tan^2\frac{x}{2}\right)dx}{a+a\tan^2\frac{x}{2}+2b\tan\frac{x}{2}}=\int\frac{\left(1+\tan^2\frac{x}{2}\right)\cdot\frac{a}{a^2-b^2}}{\frac{\left[a\left(\tan\frac{x}{2}+\frac{b}{a}\right)\right]^2}{a^2-b^2}+1}dx=\frac{a}{a^2-b^2}\int\frac{\left(1+\tan^2\frac{x}{2}\right)dx}{\left[\frac{a\left(\tan\frac{x}{2}+\frac{b}{a}\right)}{\sqrt{a^2-b^2}}\right]^2+1}$$

$$=\frac{2a}{a^2-b^2}\int\frac{\sec^2\frac{x}{2}d\frac{x}{2}}{1+\left[\frac{a\left(\tan\frac{x}{2}+\frac{b}{a}\right)}{\sqrt{a^2-b^2}}\right]^2}=\frac{2a}{a^2-b^2}\int\frac{1}{1+\left[\frac{a\left(\tan\frac{x}{2}+\frac{b}{a}\right)}{\sqrt{a^2-b^2}}\right]^2}d\tan\frac{x}{2}=\frac{2a}{a^2-b^2}\cdot\frac{\sqrt{a^2-b^2}}{a}\int\frac{d\frac{a\left(\tan\frac{x}{2}+\frac{b}{a}\right)}{\sqrt{a^2-b^2}}}{1+\left[\frac{a\left(\tan\frac{x}{2}+\frac{b}{a}\right)}{\sqrt{a^2-b^2}}\right]^2}$$

$$=\frac{2}{\sqrt{a^2-b^2}}\arctan\frac{a\left(\tan\frac{x}{2}+\frac{b}{a}\right)}{\sqrt{a^2-b^2}}+C\left(a^2>b^2\right)$$

104.
$$\int\frac{dx}{a+b\sin x}=\int\frac{dx}{a+2b\sin\frac{x}{2}\cos\frac{x}{2}}=\int\frac{\left(1+\tan^2\frac{x}{2}\right)}{a+a\tan^2\frac{x}{2}+2b\tan\frac{x}{2}}dx=\int\frac{\frac{a}{b^2-a^2}\left(1+\tan^2\frac{x}{2}\right)}{\left[\frac{a\left(\tan\frac{x}{2}+\frac{b}{a}\right)}{\sqrt{b^2-a^2}}\right]-1}dx=\int\frac{\frac{2a}{b^2-a^2}d\tan\frac{x}{2}}{\left[\frac{a\left(\tan\frac{x}{2}+\frac{b}{a}\right)}{\sqrt{b^2-a^2}}\right]-1}$$

令 $\frac{a\tan\frac{x}{2}+b}{\sqrt{b^2-a^2}}=\sec t$；原式 $=\int\frac{\frac{2a}{b^2-a^2}\frac{\sqrt{b^2-a^2}}{a}\tan t\sec tdt}{\tan^2 t}=\frac{2}{\sqrt{b^2-a^2}}\int\frac{\sec t}{\tan t}dt=\frac{2}{\sqrt{b^2-a^2}}\int\csc tdt=\frac{2}{\sqrt{b^2-a^2}}$

$$\ln\left|\csc t-\cot t\right|+C=\frac{2}{\sqrt{b^2-a^2}}\ln\left|\tan\frac{x}{2}\right|+C=\frac{1}{\sqrt{b^2-a^2}}\ln\left|\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}\right|+C\left(a^2<b^2\right)$$

105. $\int\frac{dx}{a+b\cos x}=\int\frac{dx}{a+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}=\int\frac{1+\tan^2\frac{x}{2}}{(a+b)+(a-b)\tan^2\frac{x}{2}}dx=\frac{1}{a+b}\int\frac{\sec^2\frac{x}{2}dx}{1+\frac{a-b}{a+b}\tan^2\frac{x}{2}}=\frac{2}{a+b}\int\frac{d\tan\frac{x}{2}}{1+\frac{a-b}{a+b}\tan^2\frac{x}{2}}$

$=\frac{2}{a+b}\sqrt{\frac{a+b}{a-b}}\arctan\left(\sqrt{\frac{a+b}{a-b}}\tan\frac{x}{2}\right)+C\left(a^2>b^2\right)$

106. $\int\frac{dx}{a+b\cos x}=\int\frac{dx}{a+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}=\int\frac{1+\tan^2\frac{x}{2}}{(a+b)+(a-b)\tan^2\frac{x}{2}}dx=\int\frac{\sec^2\frac{x}{2}dx}{(a+b)(b-a)\tan^2\frac{x}{2}}=\frac{2}{a+b}\int\frac{d\tan\frac{x}{2}}{1-\frac{b-a}{a+b}\tan^2\frac{x}{2}}$

令 $\sqrt{\frac{b-a}{a+b}}\tan\frac{x}{2}=\sin u\left(0<u<\frac{\pi}{2}\right)$；则 $d\tan\frac{x}{2}=\sqrt{\frac{a+b}{b-a}}\cos udu$；于是原式

$=\frac{2}{a+b}\cdot\sqrt{\frac{a+b}{b-a}}\int\frac{\cos udu}{\cos^2 u}=\frac{2}{a+b}\cdot\sqrt{\frac{a+b}{b-a}}\int\sec udu=\frac{2}{a+b}\cdot\sqrt{\frac{a+b}{b-a}}\ln\left|\sec u+\tan u\right|+C=\frac{2}{a+b}\cdot\sqrt{\frac{a+b}{b-a}}\ln\left|\tan\left(\frac{\pi}{4}+\frac{u}{2}\right)\right|+C$

$=\frac{2}{a+b}\cdot\sqrt{\frac{b+a}{b-a}}\ln\left|\frac{1+\tan\frac{u}{2}}{1-\tan\frac{u}{2}}\right|+C=\frac{2}{a+b}\cdot\sqrt{\frac{b+a}{b-a}}\ln\sqrt{\frac{1+\sin u}{1-\sin u}}+C=\frac{2}{a+b}\cdot\sqrt{\frac{b+a}{b-a}}\ln\left|\frac{\tan\frac{x}{2}+\sqrt{\frac{b+a}{b-a}}}{\tan\frac{x}{2}-\sqrt{\frac{b+a}{b-a}}}\right|+C\left(a^2<b^2\right)$

$\frac{1+\tan\frac{u}{2}}{1-\tan\frac{u}{2}}=\frac{\sin\frac{u}{2}+\cos\frac{u}{2}}{\sin\frac{u}{2}-\cos\frac{u}{2}}=\sqrt{\left(\frac{\sin\frac{u}{2}+\cos\frac{u}{2}}{\sin\frac{u}{2}-\cos\frac{u}{2}}\right)^2}=\sqrt{\frac{1+\sin u}{1-\sin u}}$

107. $\int\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}=\int\frac{1+\tan^2 x}{a^2+b^2\tan^2 x}dx=\int\frac{d\tan x}{a^2+b^2\tan^2 x}=\frac{1}{a^2}\int\frac{d\tan x}{1+\left(\frac{b}{a}\tan x\right)^2}=\frac{1}{a^2}\cdot\frac{a}{b}\arctan\left(\frac{b}{a}\tan x\right)+C$

$=\frac{1}{ab}\arctan\left(\frac{b}{a}\tan x\right)+C$

108. $\int\frac{dx}{a^2\cos^2 x-b^2\sin^2 x}=\int\frac{1+\tan^2 x}{a^2-b^2\tan^2 x}dx=\int\frac{d\tan x}{a^2-b^2\tan^2 x}$ （令 $\frac{b}{a}\tan x=\sin u$） $=\frac{1}{a^2}\int\frac{\frac{a}{b}\cos udu}{\cos^2 u}=\frac{1}{ab}\int\sec udu$

$=\frac{1}{ab}\ln\left|\tan\left(\frac{\pi}{4}+\frac{u}{2}\right)\right|+C=\frac{1}{ab}\ln\left|\tan u+\sec u\right|+C=\frac{1}{2ab}\ln\left|\frac{1+\sin u}{1-\sin u}\right|+C=\frac{1}{2ab}\ln\left|\frac{1+\frac{b}{a}\tan x}{1-\frac{b}{a}\tan x}\right|+C=\frac{1}{2ab}\ln\left|\frac{a+b\tan x}{a-b\tan x}\right|+C$

109. $\int x^2\sin axdx=-\frac{1}{a}\int x^2d\cos ax=-\frac{1}{a}x^2\cos ax+\frac{1}{a}\int\cos axdx^2=-\frac{1}{a}x^2\cos ax+\frac{2}{a}\int x\cos axdx=-\frac{1}{a}x^2\cos ax+$

$\frac{2}{a^2}\int xd\sin ax=-\frac{1}{a}x^2\cos ax+\frac{2x}{a^2}\sin ax+\frac{2}{a^3}\cos ax+C$

110. $\int x\sin axdx=-\frac{1}{a}\int xd\cos ax=-\frac{1}{a}x\cos ax+\frac{1}{a}\int\cos axdx=-\frac{1}{a}x\cos ax+\frac{1}{a^2}\sin ax+C$

111. $\int x\cos axdx=\frac{1}{a}\int xd\sin ax=\frac{x}{a}\sin ax+\frac{1}{a^2}\cos ax+C$

112. $\int x^2\cos axdx=\frac{1}{a}\int x^2d\sin ax=\frac{x^2}{a}\sin ax-\frac{1}{a}\int\sin axdx^2=\frac{1}{a}x^2\sin ax-\frac{2}{a}\int x\sin axdx=\frac{1}{a}x^2\sin ax+\frac{2}{a^2}xd\cos ax$

$=\frac{x^2}{a}\sin ax+\frac{2x}{a^2}\cos ax+\frac{2}{a^2}\int\sin axdx=\frac{x^2}{a}\sin ax+\frac{2x}{a^2}\cos ax-\frac{2}{a^3}\cos ax+C$

（十二）含有反三角函数的积分（其中 $a>0$ ）

113. $\int\arcsin\frac{x}{a}dx=x\arcsin\frac{x}{a}-\int xd\arcsin\frac{x}{a}=x\arcsin\frac{x}{a}-\int x\frac{\frac{1}{a}}{\sqrt{1-\frac{x^2}{a^2}}}dx=x\arcsin\frac{x}{a}-\int\frac{x}{\sqrt{a^2-x^2}}dx=x\arcsin\frac{x}{a}$

$+\sqrt{a^2-x^2}+C$

114. $\int x\arcsin\frac{x}{a}dx=\frac{1}{2}\int\arcsin\frac{x}{a}dx^2=\frac{x^2}{2}\arcsin\frac{x}{a}-\frac{1}{2}\int x^2d\arcsin\frac{x}{a}=\frac{x^2}{2}\arcsin\frac{x}{a}-\frac{1}{2}\int\frac{x^2dx}{\sqrt{a^2-x^2}}=\frac{x^2}{2}\arcsin\frac{x}{a}+$

$\frac{x}{4}\sqrt{a^2-x^2}-\frac{a^2}{4}\arcsin\frac{x}{a}+C=\left(\frac{x^2}{2}-\frac{a^2}{4}\right)\arcsin\frac{x}{a}+\frac{x}{4}\sqrt{a^2-x^2}+C$

115. $\int x^2\arcsin\frac{x}{a}dx=\frac{1}{3}\int\arcsin\frac{x}{a}dx^3=\frac{x^3}{3}\arcsin\frac{x}{a}-\frac{1}{3}\int\frac{x^3dx}{\sqrt{a^2-x^2}}$；令 $x=a\sin u(0<u<\pi/2)$；于是

$\int\frac{x^3dx}{\sqrt{a^2-x^2}}=\int a^3\sin^3udu=a^3\int(\cos^2-1)d\cos u=a^3\int\cos^2ud\cos u-a^3\int d\cos u=\frac{1}{3}a^3\cos^3u-a^3\cos u+C$

$=\frac{1}{3}\sqrt{(a^2-x^2)^3}-a^2\sqrt{a^2-x^2}+C$　　　　$\therefore\int x^2\arcsin\frac{x}{a}dx=\frac{x^3}{3}\arcsin\frac{x}{a}+\frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}+C$

116. $\int\arccos\frac{x}{a}dx=x\arccos\frac{x}{a}-\int xd\arccos\frac{x}{a}=x\arccos\frac{x}{a}+\int\frac{xdx}{\sqrt{a^2-x^2}}=x\arccos\frac{x}{a}-\sqrt{a^2-x^2}+C$

117. $\int x\arccos\frac{x}{a}dx=\frac{1}{2}\int\arccos\frac{x}{a}dx^2=\frac{x^2}{2}\arccos\frac{x}{a}-\frac{1}{2}\int x^2d\arccos\frac{x}{a}=\frac{x^2}{2}\arccos\frac{x}{a}+\frac{1}{2}\int\frac{x^2dx}{\sqrt{a^2-x^2}}=\frac{x^2}{2}\arccos\frac{x}{a}-\frac{x}{4}\times$

$\sqrt{a^2-x^2}+\frac{a^2}{4}\arcsin\frac{x}{a}+C_1=\left(\frac{x^2}{2}-\frac{a^2}{4}\right)\arccos\frac{x}{a}-\frac{x}{4}\sqrt{a^2-x^2}+C$

118. $\int x^2\arccos\frac{x}{a}dx=\frac{1}{3}\int\arccos\frac{x}{a}dx^3=\frac{x^3}{3}\arccos\frac{x}{a}-\frac{1}{3}\int x^3d\arccos\frac{x}{a}=\frac{x^3}{3}\arccos\frac{x}{a}+\frac{1}{3}\int\frac{x^3dx}{\sqrt{a^2-x^2}}=\frac{x^3}{3}\arccos\frac{x}{a}-$

$\frac{1}{9}\left(x^2+2a^2\right)\sqrt{a^2-x^2}+C$

119. $\int \arctan\frac{x}{a}dx = x\arctan\frac{x}{a}-\int xd\arctan\frac{x}{a} = x\arctan\frac{x}{a}-\int\frac{xdx}{1+x^2} = x\arctan\frac{x}{a}-\frac{1}{2}\ln\left(1+x^2\right)+C$

120. $\int x\arctan\frac{x}{a}dx = \frac{1}{2}\int\arctan\frac{x}{a}dx^2 = \frac{x^2}{2}\arctan\frac{x}{a}-\frac{1}{2}\int x^2\frac{\frac{1}{a}}{1+\frac{x^2}{a^2}}dx = \frac{x^2}{2}\arctan\frac{x}{a}-\frac{a}{2}\int\frac{x^2dx}{a^2+x^2} = \frac{x^2}{2}\arctan\frac{x}{a}-\frac{ax}{2}+$

$\frac{a^2}{2}\arctan\frac{x}{a}+C = \frac{x^2+a^2}{2}\arctan\frac{x}{a}-\frac{ax}{2}+C$

121. $\int x^2\arctan\frac{x}{a}dx = \frac{1}{3}\int\arctan\frac{x}{a}dx^3 = \frac{x^3}{3}\arctan\frac{x}{a}-\frac{1}{3}\int x^3d\arctan\frac{x}{a} = \frac{x^3}{3}\arctan\frac{x}{a}-\frac{1}{3}\int x^3\frac{a}{a^2+x^2}dx = \frac{x^3}{3}\arctan\frac{x}{a}-$

$\frac{a}{3}\int\frac{x^3}{a^2+x^2}dx$；令 $x = a\tan u\left(0<u<\frac{\pi}{2}\right)$；则 $dx = a\sec^2 udu$； $a^2+x^2 = a^2\sec^2 u$；于是

$\int\frac{x^3}{a^2+x^2}dx = \int\frac{a^3\tan^3 u}{a^2\sec^2 u}a\sec^2 udu = a^2\int\tan^3 udu = a^2\int\tan u\left(\sec^2 u-1\right)du = a^2\int\sec ud\sec u - a^2\int\frac{\sin u}{\cos u}du =$

$\frac{a^2}{2}\sec^2 u+a^2\ln\left|\cos u\right|+C_1 = \frac{a^2+x^2}{2}+a^2\ln\frac{a}{\sqrt{a^2+x^2}}+C_1$ $\quad\therefore\int x^2\arctan\frac{x}{a}dx = \frac{1}{3}x^3\arctan\frac{x}{a}-\frac{a}{6}x^2+\frac{a^3}{6}\ln\left(a^2+x^2\right)+C$

（十三）含有指数函数的积分

122. $\int a^x dx = \frac{a^x}{\ln a}+C$ 123. $\int e^{ax}dx = \frac{1}{a}e^{ax}+C$ 124. $\int xe^{ax}dx = \frac{1}{a}\int xde^{ax} = \frac{xe^{ax}}{a}-\frac{e^{ax}}{a^2}+C = \frac{1}{a^2}\left(ax-1\right)e^{ax}+C$

125. $\int x^n e^{ax}dx = \frac{1}{a}\int x^n de^{ax} = \frac{1}{a}x^n e^{ax}-\frac{n}{a}\int e^{ax}x^{n-1}dx$ 126. $\int xa^x dx = \int\frac{x}{\ln a}da^x = \frac{xa^x}{\ln a}-\frac{1}{\ln a}\int a^x dx = \frac{xa^x}{\ln a}-\frac{a^x}{\left(\ln a\right)^2}+C$

127. $\int x^n a^x dx = \frac{1}{\ln a}\int x^n da^x = \frac{x^n a^x}{\ln a}-\frac{n}{\ln a}\int a^x x^{n-1}dx$

128. $\int e^{ax}\sin bxdx = \frac{1}{a}e^{ax}\sin bx-\frac{1}{a}\int e^{ax}d\sin bx = \frac{1}{a}\cdot e^{ax}\sin bx-\frac{b}{a}\int e^{ax}\cos bxdx = \frac{1}{a}\cdot e^{ax}\sin bx-\frac{b}{a^2}\int\cos bxde^{ax}$

$= \frac{1}{a}e^{ax}\sin bx-\frac{b}{a^2}e^{ax}\cos bx-\frac{b^2}{a^2}\int e^{ax}\sin bxdx$ $\quad\therefore\int e^{ax}\sin bxdx = \frac{1}{a^2+b^2}e^{ax}\left(a\sin bx-b\cos bx\right)+C$

129. $\int e^{ax}\cos bxdx = \frac{1}{a}\int\cos bxde^{ax} = \frac{1}{a}e^{ax}\cos bx+\frac{b}{a}\int e^{ax}\sin bxdx = \frac{1}{a}e^{ax}\cos bx-\frac{b}{a}\cdot\frac{1}{a^2+b^2}e^{ax}\left(a\sin bx-b\cos bx\right)+C$

$= \frac{1}{a^2+b^2}e^{ax}\left(\frac{a^2+b^2}{a}\cos bx+a\sin bx-\frac{b^2}{a}\cos bx\right)+C = \frac{1}{a^2+b^2}e^{ax}\left(a\cos bx+b\sin bx\right)+C$

130. $\int e^{ax}\sin^n bxdx = \frac{1}{a}\int\sin^n bxde^{ax} = \frac{e^{ax}}{a}\sin^n bx-\frac{1}{a}\int e^{ax}d\sin^n bx = \frac{e^{ax}}{a}\sin^n bx-\frac{b}{a}n\int e^{ax}\sin^{n-1}bx\cos bxdx = \frac{1}{a}e^{ax}\sin^n bx-$

$$\frac{bn}{a^2}\int \sin^{n-1}bx\cos bxde^{ax}=\frac{e^{ax}}{a}\sin^n bx-\frac{bn}{a^2}e^{ax}\cos bx\sin^{n-1}bx+\frac{bn}{a^2}\int e^{ax}d\cos bx\sin^{n-1}bx=\frac{1}{a}e^{ax}\sin^n bx-\frac{bn}{a^2}e^{ax}\cos bx\sin^{n-1}bx+$$

$$\frac{b^2n}{a^2}\int e^{ax}(n-1)\sin^{n-2}bx\cos^2 bxdx-\frac{b^2n}{a^2}\int e^{ax}\sin^n bxdx=\frac{e^{ax}}{a}\sin^n bx-\frac{bn}{a^2}e^{ax}\cos bx\sin^{n-1}bx+\frac{b^2n(n-1)}{a^2}\int e^{ax}\sin^{n-2}bxdx-$$

$$\frac{b^2n(n-1)}{a^2}\int e^{ax}\sin^n bxdx-\frac{b^2n}{a^2}\int e^{ax}\sin^n bxdx=\frac{e^{ax}}{a}\sin^n bx-\frac{bn}{a^2}e^{ax}\cos bx\sin^{n-1}bx+\frac{b^2n(n-1)}{a^2}\int e^{ax}\sin^{n-2}bxdx-$$

$$\frac{b^2n^2}{a^2}\int e^{ax}\sin^n bxdx \qquad \therefore \int e^{ax}\sin^n bxdx=\frac{1}{a^2+b^2n^2}e^{ax}\sin^{n-1}bx(a\sin bx-nb\cos bx)+\frac{n(n-1)b^2}{a^2+b^2n^2}\int e^{ax}\sin^{n-2}bxdx$$

(十四)含有对数函数的积分

131. $\int e^{ax}\cos^n bxdx=\frac{1}{a}\int \cos^n bxde^{ax}=\frac{1}{a}e^{ax}\cos^n bx-\frac{1}{a}\int e^{ax}d\cos^n bx=\frac{1}{a}e^{ax}\cos^n bx+\frac{bn}{a}\int e^{ax}\sin bx\cos^{n-1}bxdx=$

$$\frac{1}{a}e^{ax}\cos^n bx+\frac{bn}{a^2}\int \cos^{n-1}bx\sin bxde^{ax}=\frac{1}{a}e^{ax}\cos^n bx+\frac{bn}{a^2}e^{ax}\cos^{n-1}bx\sin bx-\frac{bn}{a^2}\int e^{ax}d\cos^{n-1}bx\sin bx=$$

$$\frac{1}{a}e^{ax}\cos^n bx+\frac{bn}{a^2}e^{ax}\cos^{n-1}bx\sin bx-\frac{b^2n}{a^2}\int e^{ax}\left[-(n-1)\cos^{n-2}bx\sin^2 bx+\cos^n bx\right]dx=\frac{1}{a}e^{ax}\cos^n bx+\frac{bn}{a^2}e^{ax}\cos^{n-1}$$

$$bx\sin bx-\frac{b^2n}{a^2}\int e^{ax}\cos^n bxdx+\frac{b^2n(n-1)}{a^2}\int e^{ax}\cos^{n-2}bxdx-\frac{b^2n(n-1)}{a^2}\int e^{ax}\cos^n bxdx=\frac{1}{a}e^{ax}\cos^n bx+\frac{bn}{a^2}e^{ax}\cos^{n-1}bx\sin bx-\frac{b^2n}{a^2}$$

$$\int e^{ax}\cos^n bxdx+\frac{b^2n(n-1)}{a^2}\int e^{ax}\cos^{n-2}bxdx-\frac{b^2n}{a^2}\int e^{ax}\cos^n bxdx+\frac{b^2n}{a^2}\int e^{ax}\cos^n bxdx$$

$$\therefore \int e^{ax}\cos^n bxdx=\frac{1}{a^2+b^2n^2}e^{ax}\cos^{n-1}bx(a\cos bx+bn\sin bx)+\frac{b^2n(n-1)}{a^2+b^2n^2}\int e^{ax}\cos^{n-2}bxdx$$

132. $\int \ln xdx=x\ln x-x+C$

133. $\int \frac{dx}{x\ln x}=\int \frac{d\ln x}{\ln x}=\ln|\ln x|+C$

134. $\int x^n\ln xdx=\frac{1}{n+1}\int \ln xdx^{n+1}=\frac{x^{n+1}}{n+1}\ln x-\frac{1}{n+1}\int x^{n+1}d\ln x=\frac{x^{n+1}}{n+1}\ln x-\frac{1}{n+1}\int x^n dx=\frac{x^{n+1}}{n+1}\ln x-\frac{1}{(n+1)^2}x^{n+1}+C$

$$=\frac{x^{n+1}}{n+1}\left(\ln x-\frac{1}{n+1}\right)+C$$

135. $\int (\ln x)^n dx=x(\ln x)^n-\int xd(\ln x)^n=x(\ln x)^n-n\int (\ln x)^{n-1}dx$

136. $\int x^m(\ln x)^n dx=\frac{1}{m+1}\int (\ln x)^n dx^{m+1}=\frac{1}{m+1}x^{m+1}(\ln x)^n-\frac{1}{m+1}\int x^{m+1}d(\ln x)^n=\frac{1}{m+1}x^{m+1}(\ln x)^n-\frac{n}{m+1}\int x^{m+1-1}(\ln x)^{n-1}dx=$

$$\frac{1}{m+1}(\ln x)^n x^{m+1} - \frac{n}{m+1}\int x^m(\ln x)^{n-1}dx$$

(十五)含有双曲函数的积分

137. $\int shxdx = \int \frac{e^x - e^{-x}}{2}dx = \frac{e^x + e^{-x}}{2} + C = chx + C$

138. $\int chxdx = \int \frac{e^x + e^{-x}}{2}dx = \frac{e^x - e^{-x}}{2} + C = shx + C$

139. $\int thxdx = \int \frac{shx}{chx}dx = \int \frac{dchx}{chx} = \ln chx + C$

140. $\int sh^2xdx = \int shxdchx = shxchx - \int chxdshx = shxchx - \int ch^2xdx \qquad \because \int\left(ch^2x - sh^2x\right)dx = \int dx = x$

$$\therefore \int sh^2xdx = \frac{\left(e^x - e^{-x}\right)\left(e^x + e^{-x}\right)}{4} - x + C_1 = -\int sh^2xdx \qquad \therefore \int sh^2xdx = \frac{1}{4}sh2x - \frac{x}{2} + C$$

141. $\int ch^2xdx = \int chxdshx = chxshx - \int sh^2xdx$　由于 $\int\left(ch^2x - sh^2x\right)dx = x + C_1 \qquad -\int sh^2xdx = -\int ch^2xdx + C_2$

$$\therefore \int ch^2xdx = \frac{1}{2}x + \frac{1}{4}sh2x + C$$

（十六）定积分

142. $\int_{-\pi}^{\pi}\cos nxdx = \int_{-\pi}^{\pi}\sin nxdx = 0 \qquad \int_{-\pi}^{\pi}\cos nxdx = \frac{1}{n}\int_{-\pi}^{\pi}\cos nxdnx = \frac{1}{n}\sin nx\Big|_{-\pi}^{\pi} = 0$

$$\int_{-\pi}^{\pi}\sin nxdx = \frac{1}{n}\int_{-\pi}^{\pi}\sin nxdnx = -\frac{1}{n}\cos nx\Big|_{-\pi}^{\pi} = -\frac{1}{n}\left[-1-(-1)\right] = 0$$

143. $\int_{-\pi}^{\pi}\cos mx\sin nxdx = 0 \qquad \int\cos mx\sin nxdx = -\frac{1}{2(m+n)}\cos(m+n)x - \frac{1}{2(n-m)}\cos(n-m)x + C$

$$\int_{-\pi}^{\pi}\cos mx\sin nxdx = \left[-\frac{1}{2(m+n)}\cos(m+n)x - \frac{1}{2(n-m)}\cos(n-m)x\right]_{-\pi}^{\pi} = -\frac{1}{2(m+n)}\cos(m+n)\pi - \frac{1}{2(n-m)}\cos(n-m)\pi$$

$$+\frac{1}{2(m+n)}\cos(m+n)\pi + \frac{1}{2(n-m)}\cos(n-m)\pi = 0$$

144. $\int_{-\pi}^{\pi}\cos mx\cos nxdx = \begin{cases}0(m \neq n)\\ \pi(m = n)\end{cases}$

当 $m \neq n$ 时有 $\int_{-\pi}^{\pi}\cos mx\cos nxdx = \left[\frac{1}{2(m+n)}\sin(m+n)x + \frac{1}{2(m-n)}\sin(m-n)x\right]_{-\pi}^{\pi} = 0$

当 $m = n$ 时有 $\int_{-\pi}^{\pi}\cos mx\cos nxdx = \int_{-\pi}^{\pi}\cos^2 mxdx = \frac{1}{m}\int_{-\pi}^{\pi}\cos^2 mxdmx = \frac{1}{m}\left(\frac{mx}{2} + \frac{1}{4}\sin 2mx\right)_{-\pi}^{\pi} = \pi$

145. $\int_{-\pi}^{\pi}\sin mx\sin nxdx = \begin{cases}0(m \neq n)\\ \pi(m = n)\end{cases}$

当 $m \neq n$ 时 $\int_{-\pi}^{\pi} \sin mx \sin nx dx = \left[-\frac{1}{2(m+n)} \sin(m+n)x + \frac{1}{2(m-n)} \sin(m-n)x \right]_{-\pi}^{\pi} = 0$

当 $m = n$ 时 $\int_{-\pi}^{\pi} \sin mx \sin nx dx = \int_{-\pi}^{\pi} \sin^2 mx dx = \frac{1}{m} \left(\frac{mx}{2} - \frac{1}{4} \sin 2mx \right)_{-\pi}^{\pi} = \frac{1}{2}\pi - \left(-\frac{1}{2}\pi \right) = \pi$

146. $\int_0^{\pi} \sin mx \sin nx dx = \int_0^{\pi} \cos mx \cos nx dx = \begin{cases} 0(m \neq n) \\ \frac{\pi}{2}(m = n) \end{cases}$

当 $m \neq n$ 时 $\int_0^{\pi} \sin mx \sin nx dx = \left[-\frac{1}{2(m+n)} \sin(m+n)x + \frac{1}{2(m-n)} \sin(m-n)x \right]_0^{\pi} = 0$

$\int_0^{\pi} \cos mx \cos nx dx = \left[\frac{1}{2(m+n)} \sin(m+n)x + \frac{1}{2(m-n)} \sin(m-n)x \right]_0^{\pi} = 0$

当 $m = n$ 时 $\int_0^{\pi} \sin mx \sin nx dx = \left[\frac{1}{m} \left(\frac{mx}{2} - \frac{1}{4} \sin 2mx \right) \right]_0^{\pi} = \frac{\pi}{2} - 0 - 0 + 0 = \frac{\pi}{2}$

$\int_0^{\pi} \cos mx \cos nx dx = \left[\frac{1}{m} \left(\frac{mx}{2} + \frac{1}{4} \sin 2mx \right) \right]_0^{\pi} = \frac{\pi}{2} + 0 - 0 - 0 = \frac{\pi}{2}$

$\therefore \int_0^{\pi} \sin mx \sin nx dx = \int_0^{\pi} \cos mx \cos nx dx = \begin{cases} 0(m \neq n) \\ \frac{\pi}{2}(m = n) \end{cases}$

147. $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx \qquad I_n = \frac{n-1}{n} I_{n-2}$

$$\begin{cases} I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} (n\text{为比1大的正奇数}), \quad \mathrm{I}_1 = 1 \\ I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} (\mathrm{n}\text{为正偶数}), \quad \mathrm{I}_0 = \frac{\pi}{2} \end{cases}$$

证： $I_n = -\int_0^{\frac{\pi}{2}} \sin^{n-1} x d\cos x = \left[-\cos x \sin^{n-1} x \right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x dx$ 右端第一项等于零，将第二项里 $\cos^2 x$ 写成 $1-\sin^2 x$。并把积分分成两部分有 $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x dx = (n-1)I_{n-2} - (n-1)I_n$

由此得 $I_n = \frac{n-1}{n} I_{n-2}$，这个等式叫做积分 I_n 关于下标的递推公式，如果把 n 换成 $n-2$ 得 $I_{n-2} = \frac{n-3}{n-4} I_{n-4}$，同样地依次进行下去直到 I_n 的下标递减到零或 1 为止，于是

$$I_{2m} = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdot \frac{2m-5}{2m-4} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} I_0$$

$$I_{2m+1} = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdot \frac{2m-4}{2m-3} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} I_1$$

而 $I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}, I_1 = \int_0^{\frac{\pi}{2}} \sin x dx = 1$；

因此 $I_{2m}=\int_o^{\frac{\pi}{2}}\sin^{2m}xdx=\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdot\frac{2m-5}{2m-4}\cdots\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$

$I_{2m+1}=\int_0^{\frac{\pi}{2}}\sin^{2m+1}xdx=\frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdot\frac{2m-4}{2m-3}\cdots\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}$（其中$m=1,2,3\cdots\cdots$）

下面证 $\int_0^{\frac{\pi}{2}}\sin^n xdx=\int_0^{\frac{\pi}{2}}\cos^n xdx$　设 $x=\frac{\pi}{2}-t$；则 $dx=-dt$；且当 $x=0$ 时 $\mathrm{t}=\frac{\pi}{2};x=\frac{\pi}{2},t=0;$ 于是

$$\int_0^{\frac{\pi}{2}}f(\sin x)dx=-\int_0^{\frac{\pi}{2}}f\left[\sin\left(\frac{\pi}{2}-t\right)\right]dt=\int_0^{\frac{\pi}{2}}f(\cos x)dx$$

令 $f(\sin x)=\sin^n x$　；　$f(\cos x)=\cos^n x$ 即 $f(u)=u^n$ 可以得出

$\int_0^{\frac{\pi}{2}}\sin^n xdx=\int_0^{\frac{\pi}{2}}\cos^n xdx$　　$I_n=\frac{n-1}{n}I_{n-2}$

$$\begin{cases} I_n=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{4}{5}\cdot\frac{2}{3}(n\text{为比1大的正奇数}),\ \mathrm{I}_1=1 \\ I_n=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}(\mathrm{n}\text{为正偶数}),\ \mathrm{I}_0=\frac{\pi}{2} \end{cases}$$