Appendix A

A. Preliminaries

We first introduce a general recursion as a minor modification of [19, eq. (83)-(85)]. Consider an $I \times J$ random matrix $\mathbf{A} \in \mathbb{C}^{I\times J}$ with i.i.d. entries $a_{ij} \sim \mathcal{CN}(a_{ij}; 0, \zeta_a/I)$. Define a set $\{\boldsymbol{\theta}_{i_n}(l) \in \mathbb{C}^{k_n} | n = 1, ..., n_I.\ i_n = \sum_{k=0}^{n-1} I_k + 1, ..., \sum_{k=0}^{n} I_k.\ l = 0, ..., L-1.\}$ with $I_0 = 0$ and $\sum_{n=1}^{n_I} I_n = I$. Similarly, define a set $\{\boldsymbol{\varphi}_{j_n}(l) \in \mathbb{C}^{q_n} | n = 1, ..., n_J.\ j_n = \sum_{k=0}^{n-1} J_k + 1, ..., \sum_{k=0}^{n} J_k.\ l = 0, ..., L-1.\}$ with $J_0 = 0$ and $\sum_{n=1}^{n_J} J_n = J$. The recursion below involves the updates of $\mathbf{q}(l), \mathbf{m}_1(l), \mathbf{m}_2(l) \in \mathbb{C}^J$ and $\mathbf{e}_1(l), \mathbf{e}_2(l), \mathbf{v}(l) \in \mathbb{C}^I$. Specifically, given $\{\boldsymbol{\theta}_{i_n}(l)\}$ and $\{\boldsymbol{\varphi}_{j_n}(l)\}$, we have

$$\begin{aligned}
&\mathbf{q}(l+1) = \mathbf{A}^H \mathbf{v}(l) - (\mathbf{m}_1(l), \mathbf{m}_2(l))\, \boldsymbol{\xi}(l) \\
&v_{i_n}(l) = g_{l,n}\left(e_{1i_n}(l), e_{2i_n}(l), \boldsymbol{\theta}_{i_n}(l)\right) \qquad (55a)
\end{aligned}$$

$$\begin{aligned}
&(\mathbf{e}_1(l), \mathbf{e}_2(l)) = \mathbf{A}\,(\mathbf{m}_1(l), \mathbf{m}_2(l)) - \mathbf{v}(l-1)\boldsymbol{\gamma}(l)^T \\
&(m_{1j_n}(l), m_{2j_n}(l)) = f_{l,n}\left(q_{j_n}(l), \boldsymbol{\varphi}_{j_n}(l)\right) \qquad (55b)
\end{aligned}$$

where $\boldsymbol{\xi}(l) = \left(\frac{\zeta_a}{I}\sum_{n,i_n} \frac{\partial v_{i_n}(l)}{\partial e_{1i_n}(l)}, \frac{\zeta_a}{I}\sum_{i} \frac{\partial v_{i_n}(l)}{\partial e_{2i_n}(l)}\right)^T \in \mathbb{R}^2$, $\boldsymbol{\gamma}(l) = \left(\frac{\zeta_a}{I}\sum_{n,j_n} \frac{\partial m_{1j_n}(l)}{\partial q_{j_n}(l)}, \frac{\zeta_a}{I} \frac{\partial m_{2j_n}(l)}{\partial q_{j_n}(l)}\right)^T \in \mathbb{R}^2$, and $\mathbf{v}(-1) = \mathbf{0}$. (55) reduces to [19, eq. (83)-(85)] by letting $n_I = n_J = 1$, and letting $\boldsymbol{\theta}_{i_n}(l)$ and $\boldsymbol{\varphi}_{j_n}(l)$ invariant to recursion number l. We note that in each iteration, both $\mathbf{q}(l)$ and $(\mathbf{e}_1(l), \mathbf{e}_2(l))$ are updated through a linear mixing of $\mathbf{v}(l)$ (or $(\mathbf{m}_1(l), \mathbf{m}_2(l))$) by random Gaussian matrix $\mathbf{A}$ (or $\mathbf{A}^H$) together with a point-wise subtraction. The linear mixing makes $\mathbf{q}(l)$ (or $(\mathbf{e}_1(l), \mathbf{e}_2(l))$) distributed as Gaussian vectors (or matrix) in the large system limit and the point-wise subtraction removes the correlation of the components of $\mathbf{q}(l)$ (or the rows of $(\mathbf{e}_1(l), \mathbf{e}_2(l))$). Furthermore, due to symmetry, the components of $\mathbf{q}(l)$ (or the rows of $(\mathbf{e}_1(l), \mathbf{e}_2(l))$) have the same distribution.

To formally describe the asymptotic properties of $\mathbf{q}(l+1)$ and $(\mathbf{e}_1(l), \mathbf{e}_2(l))$, we introduce some definitions by following [20]. We say that a function $\phi(\cdot) : \mathbb{C}^m \to \mathbb{C}$ is *pseudo-Lipschitz* of order 2, if there exists a constant $c > 0$ such that, for any $\mathbf{x}, \mathbf{y} \in \mathbb{C}^m$: $|\phi(\mathbf{x}) - \phi(\mathbf{y})| \le c\,(1 + \|\mathbf{x}\| + \|\mathbf{y}\|)\,\|\mathbf{x} - \mathbf{y}\|$. We say that the empirical distribution of vector sequences $\mathbf{x}_i, i = 1, ..., N$ (denoted by $\hat{p}_{\mathbf{x}}$) converges weakly to a probability density function $p(\tilde{\mathbf{x}})$ if $\lim_{N\to\infty} \mathbb{E}_{\hat{p}_{\mathbf{x}}}[\psi(x)] = E_{p_{\tilde{\mathbf{x}}}}[\psi(x)]$ for any bounded continuous function $\psi(\cdot)$. Our goal is to characterize the distribution of $\mathbf{q}(l+1)$ and $(\mathbf{e}_1(l), \mathbf{e}_2(l))$ conditioned on the quantities previously calculated and used in (55). To this end, define $\mathfrak{S}_{l_1,l_2}$ as the probability space of $\{\mathbf{q}(l)\}_{l=0}^{l_1-1}$, $\{\mathbf{v}(l)\}_{l=0}^{l_1-1}$, $\{\boldsymbol{\theta}_{i_n}(l)\}_{l=0}^{l_1-1}$, $\{\boldsymbol{\varphi}_{j_n}(l)\}_{l=1}^{l_2}$, $\{\mathbf{e}_1(l), \mathbf{e}_2(l)\}_{l=1}^{l_2}$, and $\{\mathbf{m}_1(l), \mathbf{m}_2(l)\}_{l=0}^{l_2}$. Define matrix $\mathbf{V}_l = [\mathbf{v}(0), ..., \mathbf{v}(l-1)]$ and $\mathbf{M}_l = [\mathbf{m}_1(0), \mathbf{m}_2(0) ..., \mathbf{m}_1(l-1), \mathbf{m}_2(l-1)]$. Then we express $\mathbf{v}(l)$ as $\mathbf{v}(l) = \mathbf{v}_{\parallel}(l) + \mathbf{v}_{\perp}(l)$, where $\mathbf{v}_{\parallel}(l)$ is the orthogonal projection of $\mathbf{v}(l)$ onto the column space of $\mathbf{V}_l$, and $\mathbf{v}_{\perp}(l)$ is a vector in the orthogonal complementary space of the column space of $\mathbf{V}_l$. Furthermore, $\mathbf{v}_{\parallel}(l)$ can be expressed as $\mathbf{v}_{\parallel}(l) = \sum_{i=1}^{l-1} \alpha_i \mathbf{v}(i)$ with α_i representing the i-th projection coefficient. Analogously to the expression of $\mathbf{v}(l)$, let $(\mathbf{m}_1(l), \mathbf{m}_2(l)) = (\mathbf{m}_{1,\parallel}(l), \mathbf{m}_{2,\parallel}(l)) + (\mathbf{m}_{1,\perp}(l), \mathbf{m}_{2,\perp}(l))$ where $(\mathbf{m}_{1,\parallel}(l), \mathbf{m}_{2,\parallel}(l)) = \sum_{i=0}^{l-1} (\mathbf{m}_1(i), \mathbf{m}_2(i))\boldsymbol{\beta}_i$ with $\boldsymbol{\beta}_i \in \mathbb{R}^{2\times 2}$.

The state variables of the recursion (55) are $\tau_q(l) \in \mathbb{R}$ and $\boldsymbol{\Sigma}(l) \in \mathbb{C}^{2\times 2}$ given by

$$\tau_q(l) = \zeta_a \sum_{n=1}^{n_I} \frac{I_n}{I} \mathbb{E}\left[\left|g_{l,n}\left(\sqrt{\boldsymbol{\Sigma}(l)}\mathbf{n}, \tilde{\boldsymbol{\theta}}_n(l)\right)\right|^2\right] \qquad (56a)$$

$$\boldsymbol{\Sigma}(l) = \zeta_a \sum_{n=1}^{n_J} \frac{J_n}{I} \mathbb{E}\Big[f_{l,n}(\sqrt{\tau_q(l)}w_q, \tilde{\boldsymbol{\varphi}}_n(l))^H \times f_{l,n}(\sqrt{\tau_q(l)}w_q, \tilde{\boldsymbol{\varphi}}_n(l))\Big] \qquad (56b)$$

where $\mathbf{n} \sim \mathcal{CN}(\mathbf{n}; 0, \mathbf{I})$, $w_q \sim \mathcal{CN}(w_q; 0, 1)$, and $\boldsymbol{\Sigma}(0) = \lim_{J\to\infty} \frac{1}{J}(\mathbf{m}_1(0), \mathbf{m}_2(0))^H (\mathbf{m}_1(0), \mathbf{m}_2(0))$. The expectation $\mathbb{E}[\cdot]$ is taken over $\mathbf{n}$ and $\tilde{\boldsymbol{\theta}}_n(l)$ in (56a), and w_q and $\tilde{\boldsymbol{\varphi}}_n(l)$ in (56b).

Lemma 1. *Consider the recursion* (55). *Assume that the empirical distributions of $\boldsymbol{\theta}_{i_n}(l)$, $\boldsymbol{\varphi}_{j_n}(l)$, and $(m_{1j_n}(0), m_{2j_n}(0))^T$ respectively converge weakly to the probability distributions of random variables $\tilde{\boldsymbol{\theta}}_n(l)$, $\tilde{\boldsymbol{\varphi}}_n(l)$, and $(\tilde{m}_{1n}(0),\ \tilde{m}_{2n}(0))^T$ with bounded second moments. Further assume that the empirical second moments of those vectors respectively converge to the second moments of corresponding random variables. Assume that $g_{l,n}(\cdot)$ and $f_{l,n}(\cdot)$ are Lipschitz continuous and continuously differentiable almost everywhere with bounded derivatives. We have*

$$\mathbf{q}(l+1)|_{\mathfrak{S}_{l+1,l}} \stackrel{\mathrm{d}}{=} \sum_{i=0}^{l-1} \alpha_i \mathbf{q}(i+1) + \tilde{\mathbf{A}}^H \mathbf{v}_{\perp}(l) + \tilde{\mathbf{M}}_{l+1}\mathbf{o}_{l+1} \qquad (57a)$$

$$(\mathbf{e}_1(l), \mathbf{e}_2(l))|_{\mathfrak{S}_{l,l}} \stackrel{\mathrm{d}}{=} \sum_{i=0}^{l-1} (\mathbf{e}_1(i), \mathbf{e}_2(i))\boldsymbol{\beta}_i + \tilde{\mathbf{A}}(\mathbf{m}_{1,\perp}(l), \mathbf{m}_{2,\perp}(l)) + \tilde{\mathbf{V}}_l(\mathbf{o}_l, \mathbf{o}_l) \qquad (57b)$$

where $\tilde{\mathbf{A}}$ is an independent copy of $\mathbf{A}$; the columns of $\tilde{\mathbf{M}}_l$ (or $\tilde{\mathbf{V}}_l$) form an orthogonal basis of the column space of $\mathbf{M}_l$ (or $\mathbf{V}_l$) with $\tilde{\mathbf{M}}_l^H \tilde{\mathbf{M}}_l = N\mathbf{I}_{l\times l}$ (or $\tilde{\mathbf{V}}_l^H \tilde{\mathbf{V}}_l = J\mathbf{I}_{l\times l}$; $\mathbf{o}_l$ is a vector of length l whose elements converge to zero almost surely as $I, J \to \infty$. For any pseudo-Lipschitz functions $\phi_q(\cdot)$ and $\phi_e(\cdot)$ of order 2, we have

$$\lim_{J_n\to\infty} \frac{1}{J_n} \sum_{j_n} \phi_q\left(q_{j_n}(l), \boldsymbol{\varphi}_{j_n}(l)\right) \stackrel{a.s.}{=} \mathbb{E}\left[\phi_q\left(\sqrt{\tau_q(l-1)}w_q, \tilde{\boldsymbol{\varphi}}_n(l)\right)\right], n = 1, ..., n_J \qquad (58a)$$

$$\lim_{I_n\to\infty} \frac{1}{I_n} \sum_{i_n} \phi_e\left(e_{1i_n}(l), e_{2i_n}(l), \boldsymbol{\theta}_{i_n}(l)\right) \stackrel{a.s.}{=} \mathbb{E}\left[\phi_e\left(\sqrt{\boldsymbol{\Sigma}(l)}\mathbf{n}, \tilde{\boldsymbol{\theta}}_n(l)\right)\right], n = 1, ..., n_I \qquad (58b)$$

where $w_q \sim \mathcal{CN}(w_q; 0, 1)$ is independent of $\tilde{\boldsymbol{\varphi}}_n(l)$ and $\mathbf{n} \sim \mathcal{CN}(\mathbf{n}; 0, \mathbf{I})$ is independent of $\tilde{\boldsymbol{\theta}}_n(l)$; $a \stackrel{a.s.}{=} b$ represent a equals b almost surely.

Equations (57a) and (58a) in Lemma 1 mean that in the asymptotic regime $I, J \to \infty$, $\mathbf{q}(l)$ can be treated as a random Gaussian vector with i.i.d. entries of variance $\tau_q(l)$; (57b) and (58b) mean that $(\mathbf{e}_1(l), \mathbf{e}_2(l))$ can be treated as a random Gaussian matrix consisting of i.i.d. row vectors with covariance matrix $\mathbf{\Sigma}(l)$. The recursion (55) is a straightforward extension of [19, eq. (83)-(85)], where the difference is only the choice of n_I, n_J, $\boldsymbol{\theta}_{i_n}(l)$, and $\boldsymbol{\varphi}_{j_n}(l)$. Correspondingly, Lemma 1 is an extension of [19, Lemma 3]. The proof of Lemma 1 is straightforward by borrowing the methodology in [19]. We note that the key to ensuring Lemma 1 is the point-wise subtraction in the left hand-side of (55) for decorrelation and the Gaussian rotational invariance provided by the random Gaussian matrix $\mathbf{A}$.

B. Proof

The results of Theorem 1 comprise parts 1) and 2) for the models of x_{qtk} and u_{qtn} at Module A, parts 3) and 4) for the models of x_{qtk} and c_{qtn} at Module B, and the SE equations including (41) at Module A, (44) at module B, and (45) and (47)-(49) at super variable nodes. Note that (45) and (47)-(49) at super variable nodes are obtained by using the SE equations at Modules A and B. Thus, it suffices to prove parts 1) and 2), and (41) at module A, and parts 3) and 4), and (44) at module B. We prove by showing that the message passing related to modules A and B are both special cases of recursion (55), as detailed below.

1) State evolution related to module A: The message passing related to module A involves the estimates of x_{qtk} and u_{qtk}. Under the i.i.d. assumptions (in Assumption 1 of Theorem 1) on the decoder outputs $\{b_{x_{qtk}}\}$ (or $\{b_{s_{qn}}\}$), the estimation processes of x_{qtk} and u_{qtk} are independent and identical at different sub-blocks q and time-slot t. In what follows, we focus on the message passing related to module A at one time-slot in a sub-block.

Specifically, for recursion matrix, let $\mathbf{A} = [\mathbf{G}, \mathbf{H}]$ and $\zeta_a = 1$; for recursion vectors, let $\mathbf{m}_2(l) = \mathbf{0}$ and $\mathbf{e}_2(l) = \mathbf{0}$; for recursion parameters, let $n_I = 1$, $I_1 = M$, and $\theta_{i_1} = w_{i_1}$ in (1), and let $n_J = 3$, $(J_1, J_2, J_3) = (N_\mathrm{P}, N - N_\mathrm{P}, K)$, $\boldsymbol{\varphi}_{j_1} = (u_{j_1}, c_{j_1}, p_{j_1})$, $\boldsymbol{\varphi}_{j_2} = (u_{j_2}, c_{j_2}, p_{j_2}, s_{j2})$, and $\boldsymbol{\varphi}_{j_3} = (x_{j_3}, o_{j_3})$. Then the correspondence between (55) and the message passing related to module A is given by

$$\mathbf{q} = (\mathbf{u}^T, \mathbf{x}^T)^T - (\mathbf{d}^T, \mathbf{r}^T)^T \text{ and } \mathbf{e}_1 = \mathbf{w} - (\mathbf{y} - \mathbf{b}) \quad (59a)$$

$$v_{i_1} = g_1(e_{i_1}, w_{i_1}) = e_{i_1} - w_{i_1} \quad (59b)$$

$$m_{1j_1} = \mathbb{E}(u_{j_1} | u_{j_1} - q_{j_1}, p_{j_1}; \tau_d, \tau_p) - u_{j_1} \quad (59c)$$

$$m_{1j_2} = \mathbb{E}[u_{j_2} | u_{j_2} - q_{j_2}, p_{j_2}, s_{j_2} \sim \pi_{j_2}; \tau_d, \tau_p] - u_{j_2} \quad (59d)$$

$$m_{1j_3} = \mathbb{E}[x_{j_3} | x_{j_3} - q_{j_3}, o_{j_3}, x_{j_3} \sim \beta_{j3}; \tau_r, \tau_o] - x_{j_3} \quad (59e)$$

with $\xi_1 = 1$ and $\gamma_1 = -\left(\frac{K}{M}\sum_{j=1}^{K}\frac{v_{x_j}}{\tau_{r_1}} + \frac{N}{M}\sum_{j=1}^{N}\frac{v_{u_j}}{\tau_d}\right) = -\frac{\tau_b}{(\tau_b^\mathrm{P} + \sigma_w^2)}$. Note that $\mathbb{E}(\cdot|\cdot)$ in (59c)-(59e) are treated as functions, e.g., $\mathbb{E}(x_{j3}|\cdot)$ in (59e) are functions of q_{j3} and o_{j3}. These functions are Lipschitz continuous since the partial derivatives of these functions exist and are bounded everywhere. The corresponding functions $\mathrm{var}(\cdot|\cdot)$ are pseudo-Lipschitz of order 2. For example, considering (59e), we have $\mathrm{var}(x_{j3}|\cdot) = \int |x_{j3} - \mathbb{E}(x_{j3}|\cdot)|^2 p(x_{j3}|\cdot) dx_{j3}$. Since $\mathbb{E}(x_{j3}|\cdot)$ is Lipschitz continuous and function $|\cdot|^2$ is pseudo-Lipschitz of order 2, $\mathrm{var}(x_{j3}|\cdot)$ belongs to pseudo-Lipschitz functions of order 2.

Applying (57a) in Lemma 1 for $q_{j_3} = x_{j_3} - r_{1j_3}$ in (59a), we prove part 1) of Theorem 1. Similarly, applying (57a) in Lemma 1 for $q_{j_2} = u_{j_2} - d_{j_2}$, we prove part 2) of Theorem 1. Using (56b), we obtain $\Sigma_{1,1} = \frac{N_\mathrm{P}}{M}\mathbb{E}[\mathrm{var}(u_{j_1}|d_{j_1}, p_{j_1}; \tau_d, \tau_p)] + \frac{N - N_\mathrm{P}}{M}\mathbb{E}[\mathrm{var}(u_{j_2}|d_{j_2}, p_{j_2}, s_{j_2} \sim \pi_{j_2}; \tau_d, \tau_p)] + \frac{K}{M}\mathbb{E}[\mathrm{var}(x_{j3}|r_{j_3}, o_{j_3}, x_{j_3} \sim \beta_{j3}; \tau_r, \tau_o)] = \frac{N}{M}v_u + \frac{K}{M}v_x$. With $\Sigma_{1,1} = \frac{N}{M}v_u + \frac{K}{M}v_x$ and (56a), we obtain $\tau_q = \mathbb{E}|\sqrt{\Sigma_{1,1}}n - w_i|^2 = \frac{N}{M}v_u + \frac{K}{M}v_x + \sigma_w^2 = \tau_r = \tau_d$ in (41).

2) State evolution related to module B: Similarly to the proof in the previous subsection, let $\mathbf{A} = \mathbf{F}$, and $\zeta_a = \zeta$; $n_I = 2$, $(I_1, I_2) = (N - N_\mathrm{P}, N_\mathrm{P})$, $\boldsymbol{\theta}_{i_1} = (d_{i_1} - u_{i_1}, s_{i_1}) = (w_{d_{i_1}}, s_{i_1})$, and $\theta_{i_2} = d_{i_2} - u_{i_2} = w_{d_{i_2}}$; $n_J = 1$, $J_1 = K$, and $\boldsymbol{\varphi}_{j_1} = (x_{j_1}, r_{1j_1}, \alpha_{j_1})$. Then the correspondence between (55) and the message passing related to module B is given by

$$\mathbf{q} = \mathbf{o} - \mathbf{x}, \;\; \mathbf{e}_1 = \mathbf{c}, \text{ and } \mathbf{e}_2 = \mathbf{p} \quad (60a)$$

$$v_{i_1} = \frac{\tau_o}{\tau_p}\left(\mathbb{E}[c_{i_1}|s_{i_1}e_{1i_1} + w_{d_{i_1}}, e_{2i_1}, s_{i_1} \sim \pi_{i_1}; \tau_d, \tau_p] - e_{2i_1}\right) \quad (60b)$$

$$v_{i_2} = \frac{\tau_o}{\tau_p}\left(\mathbb{E}[c_{i_2}|e_{1i_2} + w_{d_{i_2}}, e_{2i_2}; \tau_d, \tau_p] - e_{2i_2}\right) \quad (60c)$$

$$(m_{1j_1}, m_{2j_1}) = (x_j, \mathbb{E}[x_{j_1}|r_{j_1}, x_{j_1} + q_{j_1}, x_{j_1} \sim \alpha_{j_1}; \tau_r, \tau_o]) \quad (60d)$$

with $\xi_2 = \frac{\zeta\tau_o}{\tau_p}(\frac{1}{N}\sum_{i=1}^{N}\frac{v_{c_i} - \tau_p}{\tau_p}) = -1$, $\xi_1 = 1$, $\gamma_2 = \frac{\zeta}{N}\sum_{j=1}^{K}\frac{v_{x_j}}{\tau_o} = \frac{\tau_p}{\tau_o}$, and $\gamma_1 = 0$.

Applying (57a) with $q_{j_1} = o_{j_1} - x_{j_1}$, we prove part 3) of Theorem 1. Applying (57b) with $(e_{1i}, e_{2i}) = (-c_i, p_i)$, we prove part 4) of Theorem 1. Using (56b), we obtain $\mathbf{\Sigma} = \zeta\mathbb{E}[f_1(\sqrt{\tau_o}w_q, \tilde{\boldsymbol{\varphi}}_1)^H f_1(\sqrt{\tau_o}w_q, \tilde{\boldsymbol{\varphi}}_1)] = [\zeta\frac{K}{N}, \zeta\frac{K}{N} - \zeta\frac{K}{N}v_x; \zeta\frac{K}{N} - \zeta\frac{K}{N}v_x, \zeta\frac{K}{N}v_x]$, which yields $p(c_i|p_i) = \mathcal{CN}(c_i; p_i, \zeta\frac{K}{N}v_x)$ and $p_i \sim \mathcal{CN}(p_i; 0, \zeta\frac{K}{N} - \zeta\frac{K}{N}v_x)$. Then we obtain the AWGN model $c_i = p_i + w_{c_i}$ in (43) with $\tau_p = \zeta\frac{K}{N}v_x$ in (44). Using (56a) and [19, eq. (76)], we obtain $\tau_q = \zeta\mathbb{E}\left[|g_1(c_i, p_i, \tilde{\boldsymbol{\theta}}_i)|^2\right] = \zeta\tau_o^2\mathbb{E}[\frac{\partial}{\partial p_i}(\frac{1}{\tau_p}(\frac{N_\mathrm{P}}{N}\mathbb{E}[c_i|d_i, p_i; \tau_d, \tau_p] + \frac{N - N_\mathrm{P}}{N}\mathbb{E}[c_i|d_i, p_i, s_i \sim \pi_i; \tau_d, \tau_p] - p_i))] = \tau_p^2/(\zeta(\tau_p - v_c)) = \tau_o$ in (44), which concludes the proof.

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