On the Estimation of the SABR Model's Beta Parameter: The Role of Hedging in Determining the Beta Parameter

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On the Estimation of the SABR Model's Beta Parameter: The Role of Hedging in Determining the Beta Parameter

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Abstract

The stochastic-alpha-beta-rho (SABR) model has become the dominant interest rate model used by practitioners. The principal effect of the parameter beta in the model is the effect on the skew, reflecting the belief option traders have about the distribution of the option's underlying. This paper introduces a new method for estimating the beta parameter. The key to the proposed method is that the option pricing model parameters can not only be estimated by calibrating the model to the cross-sectional data such as the implied volatility smile, but can also be estimated by choosing the set of parameters that minimize the hedging error. The proposed method meets the no-arbitrage condition, delivering better hedging performance than the existing fixed beta style calibration method. The advantage of the proposed method is demonstrated via empirical analysis. The method can be easily generalized so that it can be applied to any option-pricing model and may be preferred in applications where hedging performance is the principal goal for using an interest rate model.

Keywords: interest-rate model; SABR model; fixed-beta calibration method; hedging error; beta estimation; volatility cube; backbone; volatility smile

Since its introduction by Hagan et al. (2002), the stochastic-alpha-beta-rho (SABR) model has been implemented by many leading financial institutions for dealing with interest rate derivatives, becoming the de facto standard in the industry. More specifically, it is used in modeling the forward rate in fixed income modeling. The model's last three Greek letters denote the parameters that play a key role in determining the shape of the volatility smile and skewness. The parameter beta controls for skewness, reflecting the belief option traders have about the distribution of the option's underlying (ranging from 0 to 1). When beta is 0, the model is said to be a "normal" SABR model. When beta is 1, the model is referred to as a lognormal SABR model. For beta values between 0 and 1, the model is said to be a constant elasticity of variance. A similar role in terms of its impact on skewness is performed by the parameter rho. Alpha is the stochastic volatility reflecting the level of the volatility smile curve. There is a fourth parameter nu that does not appear in the model's name which is said to be the volatility of volatility controlling the curvature of the smile curve.

The original purpose of introducing the SABR model by Hagan et al is to provide a better risk management solution to the interest rate and foreign exchange markets where the so-called volatility smile phenomenon is significant. Using the SABR model to hedge interest rate caps, Wu (2012) finds that the model provided better hedging performance than the Black (1976) model. Applying the SABR model to the foreign-exchange option market, Yang et al (2015) find that the model fits and predicts market volatility very well.[1] Despite its effectiveness in forecasting volatility, however, most implementations of the SABR model focus on fitting market data such as constructing the interest rate volatility cube (which takes into account the term structure) while the pricing and risk management are based on the Black model. After

[1] However, for the foreign exchange options market, Yang et al (2015) do not find that the SABR model outperforms the Black-Scholes option pricing model (Black and Scholes, 1973) model in terms of hedging performance, despite the well-known drawbacks of that model.

retrieving a fitted volatility from the volatility cube constructed with the SABR model, that estimated value is then used as an input for the pricing of caps, floors, and swaptions using the Black formula. For whatever purpose for which it is used, the implementation of the SABR model usually involves fixing beta in the model and calibrating the remaining parameters to the market volatility, though the model provides considerable flexibility to the choice of the beta parameter.

Unlike the prevailing practice just described, in this paper we propose a new SABR beta parameter estimation method that integrates smile curve fitting and hedging. The new method retains the traditional smile fitting based calibration method while at the same time takes hedging performance into account. It not only provides a good fit to the market implied volatility smile curve but also provides better hedging performance. More importantly, we show that the proposed model for parameter estimation satisfies the usual no-arbitrage condition. The new method is not just applicable to the SABR model but can be generalized to other option pricing models. It is especially useful for hedging focused applications.

In the next section we provide a brief review of the SABR model and beta's role within the model. We then discuss calibration and hedging under the SABR model and introduces the new estimation methodology. The empirical results of the calibration of the new method in comparison to the fixed beta calibration method are then reported.

BETA IN THE SABR MODEL

Instead of maintaining the traditional lognormal assumption on asset prices in stochastic volatility models such as in Hull and White (1987) and Heston (1993), Hagan et al (2002) − developers of the SABR model − formulated their model as a hybrid of the constant elasticity variance (Cox, 1996) and stochastic volatility model. Based on this formulation, they derived the following pricing formula for European call and put options in the form of Black model (Black, 1976):

$$V_{call} = D(t_{set})[fN(d_+) - KN(d_-)] \tag{1}$$

$$V_{put} = V_{call} + D(t_{set})[K - f],$$

with:

$$d_{\pm} = \frac{\log\left(\frac{f}{K}\right) \pm \frac{1}{2}\sigma_B t_{ex}}{\sigma_B\sqrt{t_{ex}}},$$

where *f* and *K* are respectively the underlying forward price and the strike price. t_{ex} is the option's expiration date, and t_{set} is the settlement date of the underlying forward. $D(t_{set})$, the discount factor, is the present value for \$1 paid at t_{set}. σ_B is the volatility. *N*() stands for the cumulative density function of the standard normal random variable.

Hagan et al (2002, p. 89) provide the approximation expression of σ_B :

$$\sigma_B = \sigma_B(f, K; \alpha, \beta, \rho, \nu) = A\frac{z}{\chi(z)}B, \tag{2}$$

where α, β, ρ and ν are the four parameters of the SABR model we specified in the introduction and:

$$A = \frac{\alpha}{(fK)^{(1-\beta)/2}[1 + \frac{(1-\beta)^2}{24}log^2\left(\frac{f}{K}\right) + \frac{(1-\beta)^4}{1920}log^4\left(\frac{f}{K}\right)]}$$

$$B = 1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{(1-\beta)}} + \frac{1}{4}\frac{\alpha\beta\rho\nu}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right]t_{ex}$$

$$z = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log\left(\frac{f}{K}\right)$$

and:

$$\chi(z) = \log\left[\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right].$$

Setting $f = K$, we can get the at-the-money (ATM) volatility formula:

$$\sigma_{ATM} = \sigma_B(f, f; \alpha, \beta, \rho, \nu) = \frac{\alpha}{f^{1-\beta}}\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{(2-2\beta)}}+\frac{1}{4}\frac{\alpha\beta\rho\nu}{f^{(1-\beta)}}+\frac{2-3\rho^2}{24}\nu^2\right]t_{ex}\right\}.$$

Since $\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{(2-2\beta)}}+\frac{1}{4}\frac{\alpha\beta\rho\nu}{f^{(1-\beta)}}+\frac{2-3\rho^2}{24}\nu^2\right]t_{ex}$ is usually much smaller than 1, we can obtain the following approximation

$$\sigma_{ATM} \approx \frac{\alpha}{f^{1-\beta}}. \tag{3}$$

By introducing the beta (β) parameter, the SABR model distinguishes itself from other classic lognormal stochastic volatility models. As a result, choosing a proper beta parameter becomes an issue. Since the SABR model is rich in parameters, the choice of beta is not a hard task for pure smile fitting purposes because given a fixed beta one can always adjust the other parameters to fit the market smile curve well.

Exhibit 1 shows the SABR model calibration result based on the implied volatility curve for the September 2015 Eurodollar futures option on 09/12/2015 (one year before expiry). Market volatility is the lognormal volatility implied from the call option closing price based on the Black model. The SABR smile curve given by equation (2) fits the market smile quite well given different choices of beta. Therefore, in this case, the choice of beta depends more on the user's decision. For example, in a low interest rate environment investors tend to assume rates follow a normal distribution instead of a lognormal distribution so that beta is set to zero.

<PLACE EXHIBIT 1 HERE>

However, the situation is quite different when hedging. Hedging performance relies heavily on the parameter value. Consequently, the choice of beta in hedging matters much more than in smile curve fitting. In fact, it is a key parameter controlling the SABR model's hedging performance for two reasons. First, as a combination of the constant elasticity variance model and stochastic volatility model, the forward price's volatility under the SABR model can be decomposed into two parts. Recalling the forward price process under the SABR model:

$$df = \alpha f^{\beta} dW,$$

and comparing this to the standard lognormal setting (the Black model):

$$df = \sigma_f f dW,$$

the volatility of the forward rate is $\sigma_f = \alpha f^{\beta-1}$ where α is the stochastic part and $f^{\beta-1}$ is the predictable part. f directly affects the predictable part via $f^{\beta-1}$ and also implicitly affects the stochastic part via ρ. The predictable volatility change can be hedged by delta hedging while the stochastic volatility risk can be eliminated only with vega hedging. Given the market volatility σ_f, beta determines how much of the volatility risk can be hedged by delta hedging and vega hedging, respectively.

The second reason is that from equation (3) we know that beta controls the trace of at-the-money volatility (σ_{ATM}) as the underlying forward price (f) changes. For instance, if β is equal to 1, α is approximately equal to σ_{ATM}, which means there is no change in σ_{ATM} as f changes. While if β is smaller than 1 because $f^{1-\beta}$ is an increasing function of f, σ_{ATM} will decrease as f increases. This relationship between f and σ_{ATM} is referred to as the "backbone." Since σ_{ATM} represents the level of the smile curve, the backbone's shape indicates the relationship between the smile curve and the underlying forward rate. In this sense, beta controls how much additional volatility the option trader will take as the forward rate moves. Exhibit 2 shows the shape of the backbone when $\beta = 1$ and $\beta = 0.5$. The smile curves are plotted based on the SABR smile curve formula given by equation (2) with hypothetical forward rate and parameter values.

<PLACE EXHIBIT 2 HERE>

CALIBRATION AND HEDGING UNDER THE SABR MODEL

Calibration

Before we introduce the new methodology, we start with the current market implementation of calibrating the SABR model. The common practice of the calibration involves two steps. In the first step, a value for beta is fixed. The normal practice is to pre-specify a fixed beta. The beta is chosen based on market conditions. For example, in high interest rate environments traders tend to assume the rate is log-normally distributed ($\beta = 1$) while in the low interest rate environments the rate is closer to be normally distributed ($\beta = 0$). In the second step, the rest of the SABR model's parameters are estimated using market data by minimizing the sum of squared errors between the market implied volatility smile curve and the SABR smile curve given by equation (2) given a fixed beta. In this paper, we measure the calibration error in relative terms. That is at any day *t*:

$$(\hat{\alpha}_t, \hat{\rho}_t, \hat{\nu}_t) = \underset{\alpha,\rho,\nu}{\operatorname{argmin}} \sum w_i \varepsilon_{i,t}{}^2, \tag{4}$$

where the relative error $\varepsilon_{i,t} = \frac{\sigma_{B,t}(\alpha,\rho,\nu,K_i) - \sigma_{Mkt,t}(K_i)}{\sigma_{Mkt,t}(K_i)}$, K_i represents the *i-th* strike on the smile curve, $\sigma_{Mkt,t}$ is market implied volatility, and w_i is the weight of the squared error for the *i-th* strike. In order to distinguish with the hedging error in this paper, we name the summation term in equation (4) as sum of squared relative calibration error (SSRCE). The boundary condition for the parameters are $\alpha, \nu \in [0, \infty]$ and $\rho \in [-1,1]$. The most commonly used values for beta are 0 for the normal SABR model, 0.5 for the Cox-Ingersoll-Ross square-root process model,[2] and 1 for the lognormal SABR model.

<u>Hedging</u>

In this section, we derive the dynamics of the option price and Greeks for the SABR model based on a traditional dynamic hedging and partial differential equation approach. From the derivation, we show how the hedging error can be leveraged for the parameter estimation.

We begin by constructing a hedging portfolio with a long position in the option and hedging the change in the option price by taking the exposure of $-\Delta$ units of the underlying forward rate (f) and $-\phi$ units of volatility α.[3] The hedging error is therefore equal to:

$$\text{hedging error} = dC - \Delta df - \phi d\alpha, \tag{5}$$

where the option price dynamics (dC) under the SABR model follows:

$$dC = C_f df + C_\alpha d\alpha + C_t dt + \frac{1}{2}\alpha^2 f^{2\beta} C_{ff} dt + \frac{1}{2}\alpha^2 \nu^2 C_{\alpha\alpha} dt + \alpha^2 \nu f^\beta \rho C_{\alpha f} dt.$$

A cash account is not included in hedging because in theory forwards trading does not have initial costs, although in practice there are margin requirements. For pure theoretical derivation, we ignore this practice in the real markets as done for most derivative pricing models. Combining dC and the partial differential equation under the SABR model given by:

$$C_t + \frac{1}{2}\alpha^2 f^{2\beta} C_{ff} + \frac{1}{2}\alpha^2 \nu^2 C_{\alpha\alpha} + \alpha^2 \nu f^\beta \rho C_{\alpha f} = 0,$$

the option price dynamics can be simplified to:

$$dC = C_f df + C_\alpha d\alpha,$$

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[2] See Cox, Ingersoll, and Ross (1985).

[3] The instantaneous stochastic volatility, α, is usually treated as a parameter in the SABR model. In this paper, we treat α as a market risk factor and use it as a hedging instrument.

where:

$$C_f = \frac{\partial C}{\partial f} + \frac{\partial C}{\partial \sigma_B}\frac{\partial \sigma_B}{\partial f} \tag{6}$$

$$C_\alpha = \frac{\partial C}{\partial \sigma_B}\frac{\partial \sigma_B}{\partial \alpha}.$$

Because we assume the market forward price and volatility dynamics follow the SABR model, when we choose $\Delta = C_f$ and $\phi = C_\alpha$, the "correct" parameter set (β, ρ, ν) should produce a zero hedging error $dC - C_f df - C_\alpha d\alpha = 0$. Any other choices for the parameters will give a non-zero hedging error and thus enable arbitrage opportunity. A good analogy here is the delta hedging in the Black-Scholes model. In that model's framework, the delta hedging error will be zero only when implied volatility (which enables the model price to match the market price) to calculate delta is used. Other volatilities cannot give a zero hedging error.

Based on this fact, we introduce a new way of parameter estimation in the SABR model. The logic behind the new methodology is as follows: Given a delta and vega hedged portfolio, the true parameters of the current market condition should give a hedging error of zero. However, in reality, because the market does not follow the SABR model perfectly and there is also some noise in the market, the hedging error cannot be exactly equal to zero. However, the noise reflects all exogenous factors which are unrelated to the choice of the parameters. Consequently, given the same market condition, the right parameters should produce the least hedging error, namely the best hedging performance. We can get the new estimation methodology if we turn this logic the other way around: The SABR model parameters can be estimated by constructing a hedging portfolio given by equation (5) with the Greeks given by equation (6) and estimating the parameters by minimizing the hedging error.

An important requirement for the new method is that the model can predict the market movement well. Poor prediction performance can result in unreasonable parameter values and indicates that an incorrect model is used. Yang et al. (2015) showed that the SABR model does very well in forecasting volatility for the foreign exchange market. It can predict 90% to 95% of the market volatility movement of the major currency pairs. Combining the analysis described in earlier in this paper, we will implement the new hedging based parameter estimation methodology to the estimation of SABR's beta parameter.

Hedging Error and Hedging Performance

Next, we give the hedging error calculation and hedge performance evaluation metrics we adopted in this paper. Consider a time interval *t*-1 to *t*. At time *t*, for each strike K_i, the option price predicted by the SABR model $C_{pi,t}$ is equal to:

$$C_{pi,t} = C_{ai,t-1} + \Delta_{i,t-1}\delta f_t + \phi_{i,t-1}\delta\alpha_t,$$

where $\Delta_{i,t-1}$ and $\phi_{i,t-1}$ are the delta and vega under the SABR model at *t*-1 given by equation (6)[4], $\delta f_t = f_t - f_{t-1}$ and $\delta\alpha_t = \alpha_t - \alpha_{t-1}$. The absolute hedging error $\zeta_{i,t}$ at time *t* is calculated as the difference between the actual option price $C_{ai,t}$ and the predicted option price $C_{pi,t}$:

$$\zeta_{i,t} = C_{ai,t} - C_{pi,t} = C_{ai,t} - C_{ai,t-1} - \Delta_{i,t-1}\delta f_t - \phi_{i,t-1}\delta\alpha_t,$$

and the relative hedging error $\xi_{i,t}$ at time *t* is calculated as:

$$\xi_{i,t} = \frac{C_{pi,t} - C_{ai,t}}{C_{ai,t}}.$$

To evaluate the hedging performance, Wu (2012) and Yang et al. (2015) calculated the hedging variance ratio (HVR) that defines the percentage of the total variation of the option price

[4] Although Bartlett (2006) provided a refined delta and vega under the SABR model, it only provides better hedging performance when conducting delta hedging and vega hedging separately. For a portfolio that is both delta and vega hedged, the original SABR Greeks (6) given by Hagan et al. (2002) provide the same result as Bartlett's new SABR Greeks.

change that can be explained by the model. The higher the HVR, the better the hedging performance is. In our study, the HVR for each strike is calculated as:

$$HVR_i = 1 - \frac{Var(\zeta_{i,t})}{Var(\delta C_{ai,t})},$$

where $\delta C_{ai,t}$ =$C_{ai,t} - C_{ai,t-1}$.To evaluate the overall hedging performance across all the strikes i, we take a simple average on the HVR of all the strikes:

$$HVR = \frac{1}{N}\sum_{i=1}^{N} HVR_i$$

Methodology

In this paper, we choose two approaches to implement the new method, although many variations are possible. Our objective is to estimate the beta parameter at time *t* with the information on *t*-1 to *t*. On the time interval *t*-1 to *t*, we assume the parameters $\beta, \rho,$ and ν are constant and the stochastic volatility α is approximated by equation (3).

The first approach involves a two-step estimation. In the first step, for any given beta, the parameters α_{t-1} and α_t can be calculated and ρ and ν can be calibrated by fitting the SABR smile given by equation (2) to the market implied volatility smile curve at *t*-1. Given these parameter values, the delta and vega from equation (6) at *t*-1 can be calculated and the relative hedging error $\xi_{i,t}$ from *t*-1 to *t* for each strike K_i is obtained. For the entire option chain with strikes $K_1, K_2 \dots K_N$, we square each option's $\xi_{i,t}$ and sum them to obtain the sum of squared relative hedging error (SSRHE) at time *t* across all the strikes:

$$\text{SSRHE}_t = \sum_{i=1}^{N} z_i {\xi_{i,t}}^2, \tag{7}$$

where *N* is the total number of option strikes for the market implied volatility smile curve and z_i is the weight for the *i-th* strike. The time interval here, $[t-1, t]$, can be customized for different use cases (e.g., 1 day, 1 week, and so on).

In the second step, the optimal beta is calculated. From the first step, SSRHE becomes a function of beta. We then search the optimal beta that minimizes SSRHE given by equation (7). For the SABR model, beta lies between 0 and 1 (including 0 and 1). Therefore, our estimation of beta is defined by:

$$\hat{\beta}_t = \underset{\beta \in [0,1]}{\operatorname{argmin}} \, \mathrm{SSRHE}_t\,(\beta).$$

The searching for optimal beta can be conducted numerically by any optimization software package.

Exhibit 3 shows the typical empirical relationship between beta and SSRHE. As beta goes from 0 to 1, in some cases, panel A in Exhibit 3, the hedging error first decreases and then increases. While in other cases, as can be seen in panel B in Exhibit 3, the hedging error monotonically increases. The former case indicates that the estimated beta lies between 0 and 1 while the latter case indicates the estimated beta is equal to 0.

<PLACE EXHIBIT 3 HERE>

The first two-step approach enables the user to combine hedging performance and volatility smile fitting in the parameter estimation. However, it does not allow the user to make a trade-off between them. For example, some users want to sacrifice hedging performance for better smile fitting, while others seek a better hedging performance at the cost of worse smile fitting. To achieve this, we introduce the second approach that calibrates all the model parameters β, ρ, υ in one step by minimizing the summation of the hedging error (equation (7)) and calibration error (equation (4)). Since we define the errors in percentage term, these two errors are additive. One slight difference needs to be noted is that since we treat α as a time varying risk factor which is approximated by equation (3) while letting β be calibrated instead of fixing it, the calibration error (SSRCE) is now a function of β, ρ, υ instead of α, ρ, υ as in normal practice. We also introduce a weighting parameter ω to control the trade-off between the smile fitting and hedging:

$$(\hat{\beta}_t, \hat{\rho}_t, \hat{\upsilon}_t) = \underset{\beta,\rho,\nu\in\Omega}{\operatorname{argmin}} \;\; \omega SSRHE_t(\beta,\rho,\nu) + (1-\omega) SSRCE_t(\beta,\rho,\nu)\,,$$

or explicitly:

$$(\hat{\beta}_t, \hat{\rho}_t, \hat{\upsilon}_t) = \underset{\beta,\rho,\nu\in\Omega}{\operatorname{argmin}} \;\; \omega \sum_{i=1}^{N} z_i {\xi_{i,t}}^2 + (1-\omega) \sum_{i=1}^{N} w_i {\varepsilon_{i,t}}^2,$$

where parameter ω stands for the weight the user places on the hedging error and Ω is the set that defines the boundary of the parameters $\Omega = \{(\beta,\rho,\nu)|\, 0 \le \beta \le 1, -1 \le \rho \le 1, 0 \le \nu\}$. A natural expectation is that as ω increases, the hedging performance improves.

EMPIRICAL ANALYSIS ON THE ESTIMATION AND HEDGING PERFORMANCE

In this section, we compare our method with the fixed beta calibration method that has been widely adopted in the industry. The estimation and hedging performance evaluation are conducted using the following procedure similar to backtesting in which we estimate the parameters and test the hedging performance with real historical data.

Consider two historical time intervals *t*-1 to *t* and *t* to *t*+1. In the first stage, for *t*-1 to *t*, we estimate the beta and all other parameters by using either the fixed beta method or our new hedging-based estimation method. For the fixed beta method, only the market volatility smile data at time *t* is used. For the new method since we need to calculate hedging error, we use the data at both *t*-1 and *t*. In the second stage, for *t* to *t*+1, we use the parameters estimated in the first stage (*t*-1 to *t*) for calculating option Greeks at time *t* and then calculate the absolute hedging errors ($\zeta_{i,t}$) from *t* to *t*+1. We keep rolling this procedure in the sample range. For example, for an *M*-period historical length, we can get *M*-2 estimated parameters and hedging errors (*t* starts from 2 and ends at *M*-1). Then we calculate HVR based on the absolute hedging error on the whole sample range to evaluate the hedging performance.

Data

The instrument we choose for our empirical analysis is the Eurodollar futures option. The option's underlying is the Eurodollar futures. We choose options on four Eurodollar futures contracts that expire in September of 2012, 2013, 2014, and 2015 (exchange ticker: EDU2, EDU3, EDU4, and EDU5). We choose the daily data from two years to one year prior to the expiration of the futures contract. This is because the short-term (less than one year) contracts' forward rate is too low, giving a near-zero ATM strike. This will affect the smile fitting performance. The strike we choose ranges from 0.125% to 2% with an interval of 0.125% (16 strikes in total).[5]

We conduct the test for both call and put options. The discount factor $D(t_{set})$ in equation (1) is calculated as exp $(-rt_{set})$ in which *r* is the USD LIBOR spot rate with tenor equals to t_{set}. The estimation and hedging are done on a daily basis.

Estimation and Hedging Performance

The fixed beta method is conducted using three beta values: 0, 0.5, and 1. In both the smile calibration error (SSRCE) and hedging error (SSRHE) calculations, we choose the weight function (w_i and z_i) to be $\frac{1}{1+(f-K_i)^2}$, which places most weight on the ATM option. The more out-of-the-money or in-the-money the option is, the less it contributes to the smile calibration error and hedging error. This is because the options around the ATM strike are usually the most

[5] The market quotation convention adopted for Eurodollar futures in price terms is the following: $100 -$ forward rate. We convert all the price-based market quotes to rates. For example, a strike price equal to 99 means that the strike rate is 1%. A put option on price is a call option on the rate. For consistency, the put and call quotes referred in this paper are based on rates.

liquid, thereby providing a more reliable result. For the second approach, we set ω to 0.75, 0.5 and 0.25 to test its impact on the hedging performance.

In Exhibit 4 we can see that in the selected testing period, instead of fixing beta to a certain level, the beta parameter exhibits considerably fluctuation. This indicates that the fixing beta method failed to capture the information from the dynamics of the underlying futures and option's price movement. It only captures the information on a cross-sectional basis (i.e., market volatility smile). As to the two different approaches of the new method, the second approach gives a relatively more stable set of parameters. Since hedging error calculation involves the real market movement of the option price which introduces more noise into the estimation procedure, this indicates that the first approach overweighs the hedging error's contribution to the parameter estimation.

<PLACE EXHIBIT 4 HERE>

As shown in Exhibit 5, in general, the parameter calibrated from the new method gives better hedging performance than the fixed beta method. For call option contracts, the HVR of approach 1 is the largest compared to the other fixed beta methods. For put option contracts, the approach 1 is outperformed when beta is fixed at 1 but the performance is still fairly close and it is still better than when beta is equal to 0 and 0.5. For approach 2, its hedging performance dominates the fixed beta method and the first approach in the new method in almost all cases. This is consistent with our observation in Exhibit 4 in which the second approach yields more stable estimation than the first approach. We also see that for some contracts, the hedging performance increases as we put more weight on the hedging error in the estimation, which is consistent with our expectation. However, in some other cases, this relationship does not hold. The possible explanation could be that as we put more weight on the hedging error, more noise from the market data will be introduced into the estimation while less information from the smile curve is leveraged. An extreme case is when we set ω equal to 1 which means no information from the volatility smile is used. Since the skewness and the curvature of the smile contains additional information than the hedging error for the β, ρ, ν parameters, excluding smile in the parameter estimation might generate inferior estimation results and thus reduce the overall hedging performance. In practice, the choice of the ω parameter value should be analyzed case by case.

<PLACE EXHIBIT 5 HERE>

Overall, the advantage of our new method is quite significant. The result is meaningful for financial institutions focusing on risk management. Fixing a constant beta, or equivalently fixing a constant assumption on the underlying distribution such as a normal or lognormal, does not fully capture market risk. There are ways to avoid fixing a constant beta such as fixing rho instead of beta or using the logarithm on both side of equation (3) and estimating beta using a linear regression between ATM market volatility and the underlying forward rate. However, the idea behind those methods is still based on smile curve fitting which loses information from asset price dynamics. Because of the generality of the new methodology, it can be customized so that it can be applied to any option pricing model.

CONCLUSION

The choice of the beta parameter in the SABR model has been viewed as a minor issue in the model's calibration. The most important discovery reported in this paper is the often-overlooked fact that although the beta parameter in the SABR model does not have a major impact on the fit of the model to market data, it does play a critical role in controlling the model's hedging performance. This role cannot be ignored because beta is the parameter that differentiates the SABR model from other stochastic volatility models.

Based on this observation, we propose a new method for estimating the beta parameter by finding the set of SABR parameters that simultaneously fits the implied volatility smile curve and minimizes hedging error. This new method retains the traditional smile-fitting ways of parameter estimation while at the same time takes the hedging performance as a factor in the model's calibration. Moreover, it is also consistent with the no-arbitrage pricing framework and provides better hedging performance. We apply the new method in two different approaches to the Eurodollar futures option contracts and compare the hedging performance with the fixed beta method that is widely used by practitioners in the market. The empirical test shows that the new hedging-based option parameter estimation methodology outperforms in most cases, especially when applying the second approach. The proposed new method should be employed in instances where good hedging performance is the target of the user. The approach can be easily customized and applied to any option pricing model.

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Exhibit 1. Market volatility and fitted SABR smile curve with different betas (βs) for September 2015 Eurodollar futures call option on 09/12/2014

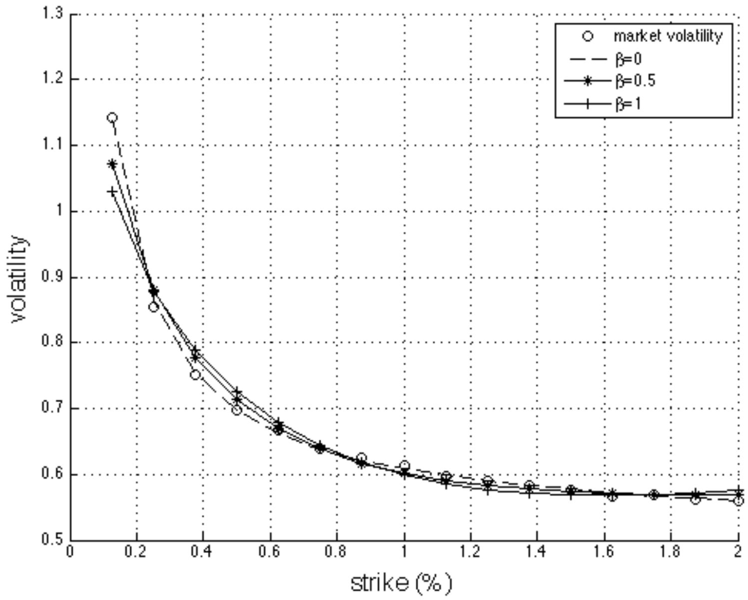
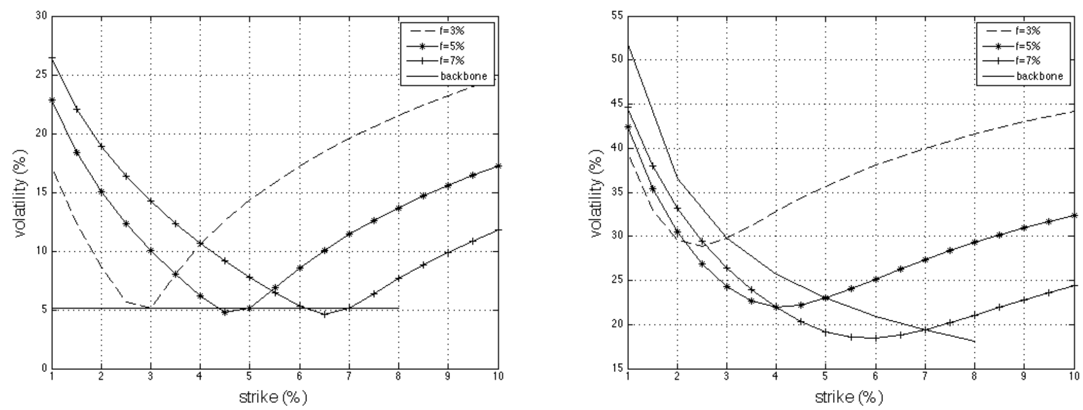


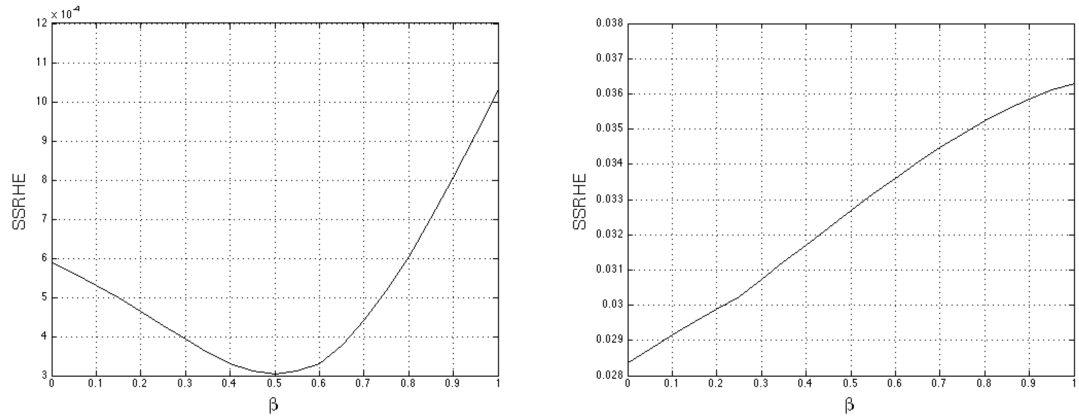
Exhibit 2. Backbone of the SABR model with hypothetical parameter values



A: Backbone for the SABR model ($\beta = 1, \alpha = 0.05, \rho = 0, \upsilon = 1, t_{ex} = 1$)

B: Backbone for the SABR model ($\beta = 0.5, \alpha = 0.05, \rho = 0, \upsilon = 1, t_{ex} = 1$)

Exhibit 3. Empirical relationship between beta (β) and the sum of squared relative hedging error (SSRHE)



A: Estimated β lies between 0 and 1 B: Estimated β equals to 0

Exhibit 4. The daily SABR volatility α and parameters (β, ρ, υ) estimated from the new method with September 2015 Eurodollar futures call option's data from 09/17/2013 to 09/12/2014

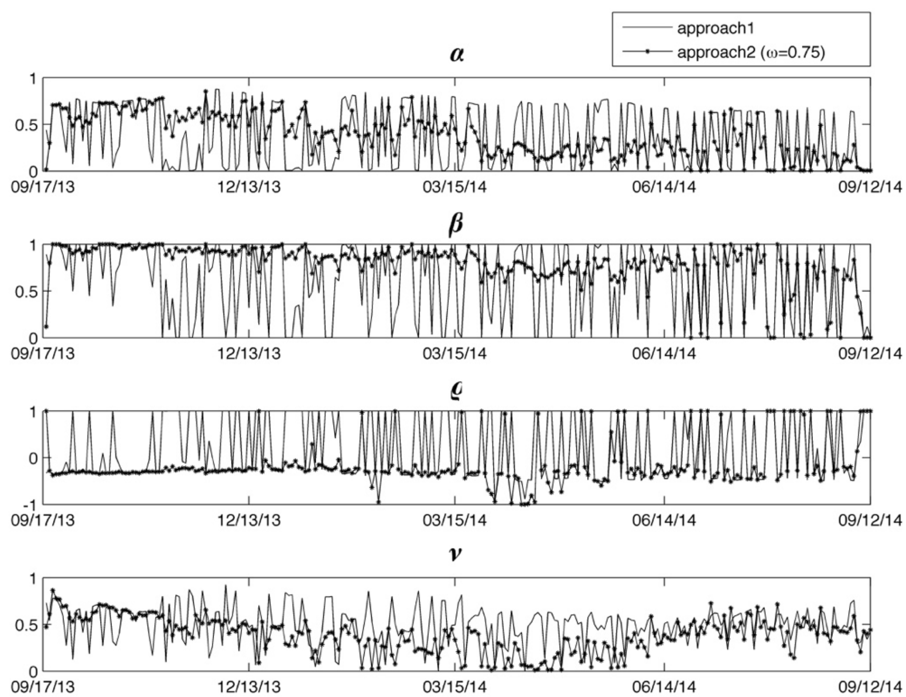


Exhibit 5. Hedging variance ratio for fixed beta method and two approaches in the new method

Call	HVR						
Contract	beta=0	beta=0.5	beta=1	approach 1	approach 2 (ω =0.75)	approach 2 (ω =0.5)	approach 2 (ω =0.25)
EDU2 (obs=251)	0.9902	0.9894	0.9873	0.9906	0.9926	0.9922	0.9920
EDU3 (obs=251)	0.8193	0.9190	0.9372	0.9440	0.9446	0.9436	0.9437
EDU4 (obs=250)	0.7141	0.7670	0.8349	0.8412	0.8574	0.8507	0.8469
EDU5 (obs=250)	0.8606	0.9751	0.9746	0.9747	0.9765	0.9765	0.9766

Put	HVR						
Contract	beta=0	beta=0.5	beta=1	approach 1	approach 2 (ω =0.75)	approach 2 (ω =0.5)	approach 2 (ω =0.25)
EDU2 (obs=251)	0.5379	0.7089	0.8465	0.8351	0.8539	0.8574	0.8554
EDU3 (obs=251)	0.8436	0.9268	0.9373	0.9348	0.9336	0.9379	0.9371
EDU4 (obs=250)	0.8746	0.9251	0.9251	0.9244	0.9270	0.9260	0.9263
EDU5 (obs=250)	0.7605	0.9075	0.9143	0.9006	0.9135	0.9142	0.9139